Differential Subordinations and Superordinations. Applications

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Based partially on two joint works with E. N. Cho (Busan, Korea), H. M. Srivastava (Victoria, Canada), and respectively with J. K. Prajapat (Kishangarh, India).

Outline

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Subordinations

Definition 1.1.

- Let denote by $H(\mathrm{U})$ **the space of all analytical functions** in the unit disk $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$, and let

$$\mathcal{B} = \{w \in H(\mathrm{U}) : w = 0,\ |w(z)| < 1, z \in \mathrm{U}\}.$$

  the class of **Schwarz functions**.
- If $f, g \in H(\mathrm{U})$, we say that the function f is **subordinate** to g, or g is **superordinate** to f, written $f(z) \prec g(z)$, if there exists a function $w \in \mathcal{B}$, such that $f(z) = g(w(z))$, for all $z \in \mathrm{U}$.

Remarks 1.1.

1. If $f(z) \prec g(z)$, then $f(0) = g(0)$ and $f(\mathrm{U}) \subseteq g(\mathrm{U})$.
2. If $f(z) \prec g(z)$, then $f(\overline{\mathrm{U}}_r) \subseteq g(\overline{\mathrm{U}}_r)$, where $\mathrm{U}_r = \{z \in \mathbb{C} : |z| < r\}$, $r < 1$, and the equality holds if and only if $f(z) = g(\lambda z)$, $|\lambda| = 1$.
3. Let $f, g \in H(\mathrm{U})$, and suppose that the function g is **univalent** in U. Then,

$$f(z) \prec g(z) \quad \Leftrightarrow \quad f(0) = g(0) \text{ and } f(\mathrm{U}) \subseteq g(\mathrm{U}).$$

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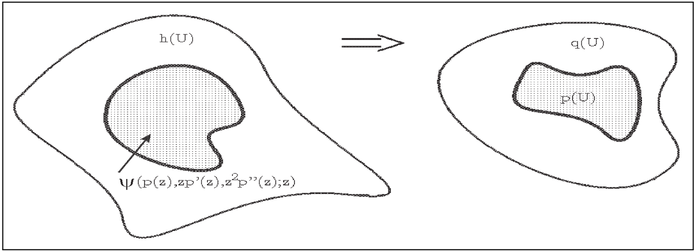
► Let $\psi : \mathbb{C}^3 \times \mathrm{U} \to \mathbb{C}$ and let $h, q \in H_u(\mathrm{U})$. The heart of the *differential subordination theory* deals with the following implication, where $p \in H(\mathrm{U})$:

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \Rightarrow p(z) \prec q(z). \tag{1.1}$$



Problem 1. Given the $h, q \in H_u(\mathrm{U})$ functions, find a class of *admissible functions* $\Psi[h, q]$ such that, if $\psi \in \Psi[h, q]$, then (1.1) holds.
Problem 2. Given the ψ and the $h \in H_u(\mathrm{U})$ functions, find a *dominant* $q \in H_u(\mathrm{U})$ so that (1.1) holds. Moreover, find *the best dominant*.
Problem 3. Given ψ and the dominant $q \in H_u(\mathrm{U})$, find the largest class of $h \in H_u(\mathrm{U})$ functions so that (1.1) holds.

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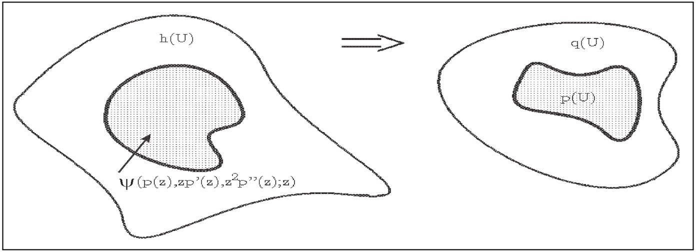

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► Let $\varphi : \mathbb{C}^3 \times \mathrm{U} \to \mathbb{C}$ and let $h, q \in H_u(\mathrm{U})$. The heart of the *differential superordination theory* deals with the following implication, where $p \in H(\mathrm{U})$:

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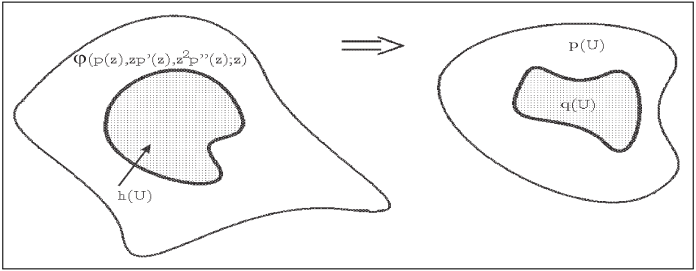

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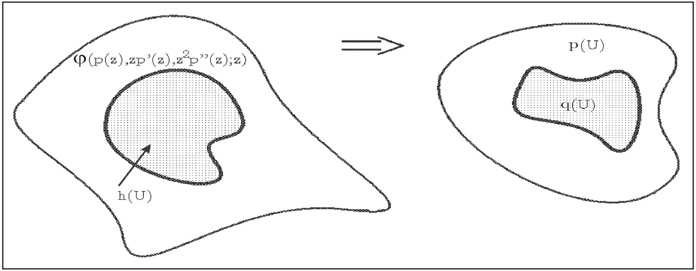

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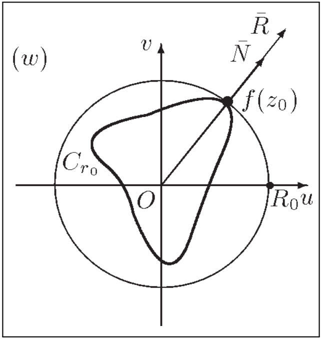
Lemma 1.1. [Miller, Mocanu 1981, Lemma 1], [Miller, Mocanu 2000]

Let $q \in \mathcal{Q}$ with $q(0) = a$ and let the function $p \in H[a, n]$, $p(z) \not\equiv a$ and $n \geq 1$. If $p(z) \not\prec q(z)$ then there exist the points $z_0 = r_0 e^{i\theta_0}$ and $\zeta_0 \in \partial\mathrm{U} \setminus E(q)$ and a number $m \geq n \geq 1$ such that $p(\mathrm{U}(0; r_0)) \subset q(\mathrm{U})$ and

$$\begin{aligned} &(i) \quad p(z_0) = q(\zeta_0) \\ &(ii) \quad z_0 p'(z_0) = m\zeta_0 q'(\zeta_0) \\ &(iii) \quad \mathrm{Re}\,\frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m\,\mathrm{Re}\left(\frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right). \end{aligned}$$



Subordination-preserving operators

Definition 1.2.

Let $K \subset H(\mathrm{U})$, and let $\mathrm{I} : K \to H(\mathrm{U})$ be an operator. We say that the operator I **preserves the subordination**, if

$$f(z) \prec g(z) \Rightarrow \mathrm{I}(f)(z) \prec \mathrm{I}(g)(z). \tag{1.3}$$

1. In 1935, G. M. Goluzin [Goluzin 1935] considered the operator $\mathrm{I} : \{f \in H(\mathrm{U}) : f(0) = 0\} \to H(\mathrm{U})$ defined by

$$\mathrm{I}(f)(z) = \int_0^z \frac{f(t)}{t}\, dt,$$

and he showed that if the function g is **convex** in U, then (1.3) holds.

2. In 1970, T. Suffridge [Suffridge 1970] generalized the above result by proving that the implication (1.3) holds even that the function g is **starlike** in U.

3. In 1981, S. S. Miller and P. T. Mocanu [Miller, Mocanu 1981] generalized these results proving that the operator $\mathrm{I} : \{f \in H(\mathrm{U}) : f(0) = 0\} \to H(\mathrm{U})$ defined by

$$\mathrm{I}(f)(z) = \left[\int_0^z \frac{f^\beta(t)}{t}\, dt\right]^{\frac{1}{\beta}},$$

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$$[zF(z)]' \prec [zG(z)]', \quad \text{where} \quad F(0) = G(0),$$

showed that this implies

$$F(rz) \prec G(rz) \quad \text{for} \quad r \leq \frac{1}{5}.$$

Denoting $f(z) = [zF'(z)]'$ and $g(z) = [zG'(z)]'$, this result could be rewritten as

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In 1984, S. S. Miller, P. T. Mocanu and M. O. Reade [Miller, Mocanu, Reade 1984] considered the integral operator $\mathrm{I}_{\beta,\gamma} : K \to H(\mathrm{U})$, $K \subset H(\mathrm{U})$, defined by

$$\mathrm{I}_{\beta,\gamma}(f)(z) = \left[\frac{1}{z^{\gamma}} \int_0^z f^{\beta}(t) t^{\gamma-1}\, dt\right]^{\frac{1}{\beta}}. \tag{1.4}$$

If $\beta, \gamma \in \mathbb{C}$ with $\operatorname{Re}\beta > 0$ and $\operatorname{Re}\gamma \geq 0$, let $K = K_{\beta,\gamma}$ where

$$K_{\beta,\gamma} = \begin{cases} H(\mathrm{U}), & \text{if} \quad \beta = 1, \gamma \neq 0 \\ \{f \in H(\mathrm{U}) : f(0) = 0\}, & \text{if} \quad \beta = 1, \gamma = 0 \\ \left\{f \in H(\mathrm{U}) : f(z) = z^j h(z), h(z) \neq 0, z \in \mathrm{U}, j \geq 1\right\}, & \text{if} \quad \frac{1}{\beta} \in \mathbb{N} \setminus \{1\} \\ \left\{f \in H(\mathrm{U}) : f(0) = 0, f'(0) \neq 0, \operatorname{Re}\left[\beta \dfrac{zf'(z)}{f(z)} + \gamma\right] > 0, z \in \mathrm{U}\right\}, \text{in rest.} & \end{cases}$$

They proved the following two results with some important consequences:

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Theorem 1.1. [Miller, Mocanu, Reade 1984]

Let $f \in K_{\beta,0}$ with $\beta > 0$, and let g be a starlike function in U of the form $g(z) = b_1 z + b_2 z^2 + \cdots\,,\ z \in \mathrm{U}$.
If the operator $\mathrm{I} = \mathrm{I}_{\beta,0} : K_{\beta,0} \to H(\mathrm{U})$ is defined by

$$\mathrm{I}(f)(z) = \mathrm{I}_{\beta,0}(f)(z) = \left[\int_0^z \frac{f^{\beta}(t)}{t}\,dt\right]^{\frac{1}{\beta}},$$

then $\mathrm{I}(g)$ is a univalent function in U, and

$$f(z) \prec g(z) \Rightarrow \mathrm{I}(f)(z) \prec \mathrm{I}(g)(z).$$

Theorem 1.2. [Miller, Mocanu, Reade 1984]

Let $\beta, \gamma \in \mathbb{C}$*, with* $\operatorname{Re}\beta > 0, \operatorname{Re}\gamma \geq 0$ *and let*

$$\delta = \min\{\operatorname{Re}\gamma, \delta_0\}, \quad \textit{where} \quad \delta_0 = \frac{1}{2}\frac{|\beta+\gamma| - |\beta-\overline{\gamma}|}{|\beta+\gamma| + |\beta-\overline{\gamma}|} = \frac{2\operatorname{Re}\beta\operatorname{Re}\gamma}{(|\beta+\gamma| + |\beta-\overline{\gamma}|)^2}.$$

If $f, g \in K_{\beta,\gamma}$ *cu* $g'(0) \neq 0$ *and*

$$\operatorname{Re}\left[(\beta-1)\frac{zg'(z)}{g(z)} + 1 + \frac{zg''(z)}{g'(z)}\right] > -\delta, \ z \in \mathrm{U},$$

then

$$f(z) \prec g(z) \Rightarrow \mathrm{I}_{\beta,\gamma}(f)(z) \prec \mathrm{I}_{\beta,\gamma}(g)(z).$$

Generalized integral operators

Now, let consider the integral operator $\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta} : \mathcal{K} \to H(\mathrm{U})$, with $\mathcal{K} \subset H(\mathrm{U})$, defined by

$$\text{(2.1)} \qquad \mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}[f](z) = \left[\frac{\beta+\gamma}{z^{\gamma}\phi(z)} \int_0^z f^{\alpha}(t)\varphi(t)t^{\delta-1}\,\mathrm{d}\,t \right]^{1/\beta},$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $\phi, \varphi \in H(\mathrm{U})$ (all powers are principal ones).

We generalized these previous results, in the sense of giving sufficient conditions on the g_1 and g_2 functions and on the α, β, γ and δ parameters, such that the next **sandwich-type result** holds:

$$z\varphi(z)\left[\frac{g_1(z)}{z}\right]^{\alpha} \prec z\varphi(z)\left[\frac{f(z)}{z}\right]^{\alpha} \prec z\varphi(z)\left[\frac{g_2(z)}{z}\right]^{\alpha}$$

implies

$$z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}[g_1](z)}{z}\right]^{\beta} \prec z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}[f](z)}{z}\right]^{\beta} \prec z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}[g_2](z)}{z}\right]^{\beta}.$$

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Generalized integral operators

Now, let consider the integral operator $\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta} : \mathcal{K} \to H(\mathrm{U})$, with $\mathcal{K} \subset H(\mathrm{U})$, defined by

(2.1) $$\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}[f](z) = \left[\frac{\beta+\gamma}{z^{\gamma}\phi(z)}\int_0^z f^{\alpha}(t)\varphi(t)t^{\delta-1}\,\mathrm{d}\,t\right]^{1/\beta},$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $\phi, \varphi \in H(\mathrm{U})$ (all powers are principal ones).

We generalized these previous results, in the sense of giving sufficient conditions on the g_1 and g_2 functions and on the α, β, γ and δ parameters, such that the next **sandwich-type result** holds:

$$z\varphi(z)\left[\frac{g_1(z)}{z}\right]^{\alpha} \prec z\varphi(z)\left[\frac{f(z)}{z}\right]^{\alpha} \prec z\varphi(z)\left[\frac{g_2(z)}{z}\right]^{\alpha}$$

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Preliminary results and tools

To prove our main results, we will need the following definitions and lemmas presented in this subsection.

Definition 2.1.

Let $c \in \mathbb{C}$ with $\operatorname{Re} c > 0$, let $n \in \mathbb{N}^*$ and let

$$C_n = C_n(c) = \frac{n}{\operatorname{Re} c}\left[|c|\sqrt{1 + 2\operatorname{Re}\left(\frac{c}{n}\right)} + \operatorname{Im} c\right].$$

If R is the univalent function $R(z) = \dfrac{2C_n z}{1 - z^2}$, then the **open door function** $R_{c,n}$ is defined by

$$R_{c,n}(z) = R\left(\frac{z + b}{1 + \overline{b}z}\right),\ z \in \mathrm{U},$$

where $b = R^{-1}(c)$.

Remarks 2.1.

1. Remark that $R_{c,n}$ is univalent in U, $R_{c,n}(0) = c$ and $R_{c,n}(\mathrm{U}) = R(\mathrm{U})$ is the complex plane slit along the half-lines $\operatorname{Re} w = 0$, $\operatorname{Im} w \geq C_n$ and $\operatorname{Re} w = 0$, $\operatorname{Im} w \leq -C_n$.
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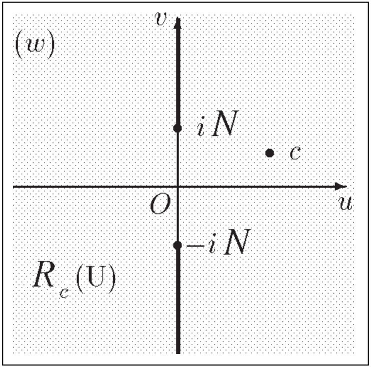

The R_c function

Definition 2.2.

A function $L(z;t) : \mathrm{U} \times [0,+\infty) \to \mathbb{C}$ is called **a subordination (or a Loewner) chain** if $L(\cdot;t)$ is analytic and univalent in U for all $t \geq 0$, and $L(z;s) \prec L(z;t)$ when $0 \leq s \leq t$.

The next well-known lemma gives a sufficient condition so that the $L(z;t)$ function will be a subordination chain.

Lemma 2.1. [Pommerenke 1975, p. 159]

Let $L(z;t) = a_1(t)z + a_2(t)z^2 + \ldots$, with $a_1(t) \neq 0$ for all $t \geq 0$ and $\lim_{t\to+\infty} |a_1(t)| = +\infty$. Suppose that $L(\cdot;t)$ is analytic in U for all $t \geq 0$, $L(z;\cdot)$ is continuously differentiable on $[0,+\infty)$ for all $z \in \mathrm{U}$. If $L(z;t)$ satisfies

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] > 0,\ z \in \mathrm{U},\ t \geq 0.$$

and

$$|L(z;t)| \leq K_0 \, |a_1(t)|\,,\ |z| < r_0 < 1,\ t \geq 0$$

for some positive constants K_0 and r_0, then $L(z;t)$ is a subordination chain.

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Remark 2.1.

We emphasize that in the previous lemma **both of the conditions are essential**. For example, considering the function

$$L(z;t) = \exp\left[(1+t)\pi z\right] - 1,\ z \in \mathrm{U},\ t \geq 0,$$

it is easy to check that

$$\mathrm{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] = 1 + t \geq 1,\ z \in \mathrm{U},\ t \geq 0,$$

while for any $t_0 \geq 0$ the function $L(z;t_0)$ is not univalent in U.

As in [Miller, Mocanu 2000], let denote by $\mathcal{Q}$ the set of functions f that are analytic and injective on $\overline{\mathrm{U}} \setminus E(f)$, where

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Sandwich-type results for a class of convex integral operators

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we denote

$$H[a, n] = \{f \in H(\mathrm{U}) : f(z) = a + a_n z^n + \dots\}.$$

First we need to determine the subset $\mathcal{K} \subset H(\mathrm{U})$ such that the integral operator $\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}$ given by (2.1) will be well-defined. If we choose in the **Integral Existence Theorem** [Miller, Mocanu 1989, Miller, Mocanu 1991] the correspondent functions $\Phi \equiv \phi \in H[1, 1]$ and $\phi \equiv \varphi \in H[1, 1]$, with $\phi(z)\varphi(z) \neq 0$ for all $z \in \mathrm{U}$, then we get the set $\mathcal{K}$ where the integral operator $\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}$ is well-defined.

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If $F = \mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}[f]$, then $f \in \mathcal{K}^{\varphi}_{\alpha,\delta}$ implies $F \in A$, $\dfrac{F(z)}{z} \neq 0$, $z \in \mathrm{U}$, and

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Theorem 2.1. [B 2012]

Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *with* $\beta \neq 0$, $1 < \beta + \gamma \leq 2$, $\alpha + \delta = \beta + \gamma$. *Let* $g_1, g_2 \in \mathcal{K}^{\varphi}_{\alpha,\delta}$, *and for* $\alpha \neq 1$ *suppose in addition that* $g_k(z)/z \neq 0$ *for* $z \in \mathrm{U}$ *and* $k = 1, 2$. *Suppose that the next two conditions are satisfied*

$$\operatorname{Re}\left[1 + \frac{zu_k''(z)}{u_k'(z)}\right] > \frac{1 - (\beta + \gamma)}{2}, \; z \in \mathrm{U}, \; \textit{for } k = 1, 2,$$

where $u_k(z) = z\varphi(z)\left[\frac{g_k(z)}{z}\right]^{\alpha}$ *and* $k = 1, 2$.

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$$z\varphi(z)\left[\frac{g_1(z)}{z}\right]^{\alpha} \prec z\varphi(z)\left[\frac{f(z)}{z}\right]^{\alpha} \prec z\varphi(z)\left[\frac{g_2(z)}{z}\right]^{\alpha}$$

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Moreover, the functions $z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}[g_1](z)}{z}\right]^{\beta}$ *and* $z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}[g_2](z)}{z}\right]^{\beta}$ *are respectively the best subordinant and the best dominant.*

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This theorem generalize Theorem 3.2. from [B 2002-2], that may be obtained for the special case $\alpha = \beta$, $\phi \equiv 1$ and $\varphi \equiv 1$.
For the case $\alpha = \beta = 1$, $\phi \equiv 1$ and $\varphi \equiv 1$, the result was obtained in [Miller, Mocanu 2000, Corollary 6.1], where the authors assumed that $\operatorname{Re}\gamma \geq 0$ and g_1, g_2 are convex functions.

Because the assumption that the functions $z\varphi(z)\left[\frac{f(z)}{z}\right]^{\alpha}$ and $z\phi(z)\left[\frac{A^{\phi;\varphi}_{\alpha,\beta,\gamma,\delta}[f](z)}{z}\right]^{\beta}$ need to be univalent in U is difficult to be checked, we will replace this by another condition, that is more easy to be verified.

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Corollary 2.1. [B 2012]

Let $\alpha,\beta,\gamma,\delta\in\mathbb{C}$ *with* $\beta\neq 0$, $1<\beta+\gamma\leq 2$, $\alpha+\delta=\beta+\gamma$. *Let* $f, g_1, g_2\in\mathcal{K}^{\varphi}_{\alpha,\delta}$, *and for* $\alpha\neq 1$ *suppose in addition that* $f(z)/z\neq 0$, $g_k(z)/z\neq 0$ *for* $z\in\mathrm{U}$ *and* $k=1,2$. *Suppose that the next three conditions are satisfied*

$$\operatorname{Re}\left[1+\frac{zu_k''(z)}{u_k'(z)}\right]>\frac{1-(\beta+\gamma)}{2},\ z\in\mathrm{U},\ \textit{for}\ k=1,2,3,$$

where $u_k(z)=z\varphi(z)\left[\frac{g_k(z)}{z}\right]^{\alpha}$, $k=1,2$ *and* $u_3(z)=z\varphi(z)\left[\frac{f(z)}{z}\right]^{\alpha}$.

Then

$$z\varphi(z)\left[\frac{g_1(z)}{z}\right]^{\alpha}\prec z\varphi(z)\left[\frac{f(z)}{z}\right]^{\alpha}\prec z\varphi(z)\left[\frac{g_2(z)}{z}\right]^{\alpha}$$

implies

$$z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}[g_1](z)}{z}\right]^{\beta}\prec z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}[f](z)}{z}\right]^{\beta}\prec z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}[g_2](z)}{z}\right]^{\beta}.$$

Moreover, the functions $z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}[g_1](z)}{z}\right]^{\beta}$ *and* $z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}[g_2](z)}{z}\right]^{\beta}$ *are respectively the best subordinant and the best dominant.*

New improvement of some sandwich-type results

We denote the class $\mathcal{D}$ by

$$\mathcal{D} := \{\varphi \in H(\mathrm{U}) : \varphi(0) = 1,\ \varphi(z) \neq 0,\ z \in \mathrm{U}\},$$

and let recall the integral operator $\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta} : \mathcal{K} \to H(\mathrm{U})$, with $\mathcal{K} \subset H(\mathrm{U})$, defined by (2.1), i.e.

$$\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}[f](z) = \left[\frac{\beta+\gamma}{z^{\gamma}\phi(z)} \int_0^z f^{\alpha}(t)\varphi(t)t^{\delta-1}\,\mathrm{d}\,t\right]^{1/\beta},$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $\phi, \varphi \in H(\mathrm{U})$ (all powers are principal ones).

$$(f \in \mathcal{K},\ \alpha, \gamma, \delta \in \mathbb{C},\ \beta \in \mathbb{C} \setminus \{0\},\ \alpha + \delta = \beta + \gamma,\ \mathrm{Re}(\alpha + \delta) > 0,\ \phi, \varphi \in \mathcal{D}).$$

As it was shown in Lemma 2.2, the above integral operator is well-defined on the set $\mathcal{K} = \mathcal{K}^{\varphi}_{\alpha,\delta}$ defined by (2.2).

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$$\mathcal{D} := \{\varphi \in H(\mathrm{U}) : \varphi(0) = 1,\ \varphi(z) \neq 0,\ z \in \mathrm{U}\},$$

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As it was shown in Lemma 2.2, the above integral operator is well-defined on the set $\mathcal{K} = \mathcal{K}_{\alpha,\delta}^{\varphi}$ defined by (2.2).

Theorem 2.2. [Cho, B, Srivastava 2012]

Let $f, g_k \in \mathcal{K}^{\varphi}_{\alpha,\delta}$, $k = 1, 2$, *where* $\mathcal{K}^{\varphi}_{\alpha,\delta}$ *is defined by* (2.2). *Suppose also that*

$$\operatorname{Re}\left(1 + \frac{z\nu_k''(z)}{\nu_k'(z)}\right) > -\rho,\ z \in \mathrm{U}, \quad \textit{where} \quad \nu_k(z) := z\varphi(z)\left[\frac{g_k(z)}{z}\right]^{\alpha},\ k = 1, 2, \quad \textit{and}$$

$$\rho = \frac{1 + |\beta + \gamma - 1|^2 - |1 - (\beta + \gamma - 1)^2|}{4\operatorname{Re}(\beta + \gamma - 1)}, \quad \textit{with} \quad \operatorname{Re}(\beta + \gamma - 1) > 0.$$

If $z\varphi(z)\left[f(z)/z\right]^{\alpha}$ *is univalent in* U *and* $z\phi(z)\left[\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}[f](z)/z\right]^{\beta} \in \mathcal{Q}$, *where* $\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}$ *is the integral operator defined by* (2.1), *then the subordination relation*

$$z\varphi(z)\left[\frac{g_1(z)}{z}\right]^{\alpha} \prec z\varphi(z)\left[\frac{f(z)}{z}\right]^{\alpha} \prec z\varphi(z)\left[\frac{g_2(z)}{z}\right]^{\alpha}$$

implies that

$$z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}[g_1](z)}{z}\right]^{\beta} \prec z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}[f](z)}{z}\right]^{\beta} \prec z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}[g_2](z)}{z}\right]^{\beta}.$$

Moreover, the functions $z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}[g_1](z)}{z}\right]^{\beta}$ *and* $z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}[g_2](z)}{z}\right]^{\beta}$ *are the best subordinant and the best dominant, respectively.*

If we take in the previous theorem (or is some of its sides) the parameters α, β, γ and δ with the restrictions $\phi(z) = \varphi(z) = 1$, $\alpha = \beta$, $\gamma = \delta$, and $1 < \beta + \gamma \leq 2$, then we have the previously obtained results [B 1997, B 2002-2]. Taking $\beta + \gamma = 2$ in Theorem 2.2, we have the following result:

Corollary 2.2. [Cho, B, Srivastava 2012]

Let $f, g_k \in \mathcal{K}^{\varphi}_{\alpha,2-\alpha}$, $k = 1, 2$, *where* $\mathcal{K}^{\varphi}_{\alpha,2-\alpha}$ *is defined by* (2.2), *with* $\delta = 2 - \alpha$. *Suppose that*

$$\operatorname{Re}\left(1 + \frac{z\nu_k''(z)}{\nu_k'(z)}\right) > -\frac{1}{2},\ z \in \mathrm{U}, \quad \text{where} \quad \nu_k(z) := z\varphi(z)\left[\frac{g_k(z)}{z}\right]^{\alpha},\ k = 1, 2.$$

If $z\varphi(z)[f(z)/z]^{\alpha}$ *is univalent functions in* U *and* $z\phi(z)\left(\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,2-\beta,2-\alpha} f(z)/z\right)^{\beta} \in \mathcal{Q}$, *where the integral operator* $\mathrm{A}_{\alpha,\beta,1-\beta,1-\delta}$ *is defined by* (2.1), *with* $\gamma = 1 - \beta$ *and* $\delta = 1 - \alpha$, *then the following subordination relation*

$$z\varphi(z)\left[\frac{g_1(z)}{z}\right]^{\alpha} \prec z\varphi(z)\left[\frac{f(z)}{z}\right]^{\alpha} \prec z\varphi(z)\left[\frac{g_2(z)}{z}\right]^{\alpha}$$

implies that

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Moreover, the functions $z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,1-\beta,1-\alpha}[g_1](z)}{z}\right]^{\beta}$ *and* $z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,1-\beta,1-\alpha}[g_2](z)}{z}\right]^{\beta}$ *are the best subordinant and the best dominant, respectively.*

If we take in the previous theorem (or is some of its sides) the parameters α, β, γ and δ with the restrictions $\phi(z) = \varphi(z) = 1$, $\alpha = \beta$, $\gamma = \delta$, and $1 < \beta + \gamma \leq 2$, then we have the previously obtained results [B 1997, B 2002-2]. Taking $\beta + \gamma = 2$ in Theorem 2.2, we have the following result:

Corollary 2.2. [Cho, B, Srivastava 2012]

Let $f, g_k \in \mathcal{K}^{\varphi}_{\alpha,2-\alpha}$, $k = 1, 2$, where $\mathcal{K}^{\varphi}_{\alpha,2-\alpha}$ is defined by (2.2)*, with $\delta = 2 - \alpha$. Suppose that*

$$\operatorname{Re}\left(1 + \frac{z\nu_k''(z)}{\nu_k'(z)}\right) > -\frac{1}{2},\ z \in \mathrm{U}, \quad \textit{where} \quad \nu_k(z) := z\varphi(z)\left[\frac{g_k(z)}{z}\right]^{\alpha},\ k = 1, 2.$$

If $z\varphi(z)[f(z)/z]^{\alpha}$ is univalent functions in U and $z\phi(z)\left(\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,2-\beta,2-\alpha} f(z)/z\right)^{\beta} \in \mathcal{Q}$, where the integral operator $\mathrm{A}_{\alpha,\beta,1-\beta,1-\delta}$ is defined by (2.1)*, with $\gamma = 1 - \beta$ and $\delta = 1 - \alpha$, then the following subordination relation*

$$z\varphi(z)\left[\frac{g_1(z)}{z}\right]^{\alpha} \prec z\varphi(z)\left[\frac{f(z)}{z}\right]^{\alpha} \prec z\varphi(z)\left[\frac{g_2(z)}{z}\right]^{\alpha}$$

implies that

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Moreover, the functions $z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,1-\beta,1-\alpha}[g_1](z)}{z}\right]^{\beta}$ and $z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,1-\beta,1-\alpha}[g_2](z)}{z}\right]^{\beta}$ are the best subordinant and the best dominant, respectively.

Taking $\beta+\gamma=2+i$ in Theorem 2.2, we are easily led to the following result:

Corollary 2.3. [Cho, B, Srivastava 2012]

Let $f, g_k \in \mathcal{K}^{\varphi}_{\alpha,2+i-\alpha}$, $k=1,2$, *where* $\mathcal{K}^{\varphi}_{\alpha,2+i-\alpha}$ *is defined by* (2.2), *with* $\delta=2+i-\alpha$. *Suppose also that*

$$\operatorname{Re}\left(1+\frac{z\nu_k''(z)}{\nu_k'(z)}\right)>-\frac{3-\sqrt{5}}{4},\ z\in\mathrm{U},\quad \text{where}\quad \nu_k(z):=z\varphi(z)\left[\frac{g_k(z)}{z}\right]^{\alpha},\ k=1,2.$$

If $z(f(z)/z)^{\alpha}\varphi(z)$ *is univalent functions in* U *and* $z(\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,2+i-\beta,\delta}\,f(z)/z)^{\beta}\phi(z)\in\mathcal{Q}$, *where the integral operator* $\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,2+i-\beta,2+i-\alpha}$ *is defined by* (2.1), *with* $\gamma=2+i-\beta$ *and* $\delta=2+i-\alpha$, *then the subordination relation*

$$z\varphi(z)\left[\frac{g_1(z)}{z}\right]^{\alpha}\prec z\varphi(z)\left[\frac{f(z)}{z}\right]^{\alpha}\prec z\varphi(z)\left[\frac{g_2(z)}{z}\right]^{\alpha}$$

implies that

$$z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,2+i-\beta,2+i-\alpha}[g_1](z)}{z}\right]^{\beta}\prec z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,2+i-\beta,2+i-\alpha}[f](z)}{z}\right]^{\beta}\prec z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,2+i-\beta,2+i-\alpha}[g_2](z)}{z}\right]^{\beta}.$$

Moreover, the functions $z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,2+i-\beta,2+i-\alpha}[g_1](z)}{z}\right]^{\beta}$ *and* $z\phi(z)\left[\frac{\mathrm{A}^{\phi,\varphi}_{\alpha,\beta,2+i-\beta,2+i-\alpha}[g_2](z)}{z}\right]^{\beta}$ *are the best subordinant and the best dominant, respectively.*

Generalized Srivastava-Attiya operator

Definition 2.3.

- The **generalized hypergeometric** function ${}_qF_s$ is defined by

$$ {}_qF_s(z) = {}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \frac{z^n}{n!}, \ z \in \mathrm{U}, $$

where $\alpha_j \in \mathbb{C}$ $(j = 1, \dots, q)$, $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $\mathbb{Z}_0^- = \{0, -1, \dots\}$ $(j = 1, \dots, s)$, $q \le s + 1$, $q, s \in \mathbb{N}_0$, where $(\alpha)_k$ is the **Pochhammer symbol** defined by

$$ (\alpha)_0 = 1, \ (\alpha)_k = \alpha(\alpha + 1) \dots (\alpha + k - 1), \ (k \in \mathbb{N}). $$

- The **general Hurwitz-Lerch Zeta function** $\phi(z, s, a)$ is defined by (*cf., e.g.* [Srivastava, Choi 2001, p. 21 *et seq.*])

$$ \phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(a + n)^s} = \frac{1}{a^s} + \frac{z}{(1 + a)^s} + \frac{z^2}{(2 + a)^s} + \dots, $$

with $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|z| < 1$, and $\mathrm{Re}\, s > 1$ when $|z| = 1$.

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► A generalization of the above defined Hurwitz-Lerch Zeta function $\phi(z,s,b)$ was studied by Garg *et al.* [Garg, Jain, Kalla 2009] in the following form [Garg, Jain, Kalla 2009, p. 27, Eq.(1.4)] (see also [Srivastava, Saxena, Pogany, Saxena 2011]):

$$\Phi_{\lambda,\mu;\nu}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu)_n}{(\nu)_n n!}\frac{z^n}{(n+a)^s},$$

with $\lambda,\mu,s\in\mathbb{C}$, $\nu,a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ when $|z|<1$, and $\mathrm{Re}(s+\nu-\lambda-\mu)>1$ when $|z|=1$.

► Motivated by earlier investigation by Srivastava and Attiya [Srivastava, Attiya 2007], Prajapat and Goyal [Prajapat, Goyal 2009], we introduced the linear operator

$$\mathcal{J}_{\lambda,\mu;\nu}^{s,a}:\mathcal{A}\to\mathcal{A},\quad \mathcal{A}:=\{f\in H[a,1]: f(0)=0,\ f'(0)=1\},$$

which is defined by means of the following **Hadamard (or convolution) product**, that is

$$\mathcal{J}_{\lambda,\mu;\nu}^{s,a}(f)(z)=\mathcal{G}_{\lambda,\mu;\nu}^{s,a}(z)*f(z),\ z\in\mathrm{U}, \tag{2.3}$$

where $\lambda,\mu,s\in\mathbb{C}$, $\nu,a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ and $f\in\mathcal{A}$, while the function $\mathcal{G}_{\lambda,\mu;\nu}^{s,a}$ is defined by

$$\mathcal{G}_{\lambda,\mu;\nu}^{s,a}(z)=\frac{\nu(1+a)^s}{\lambda\,\mu}\left[\Phi_{\lambda,\mu;\nu}(z,s,a)-a^{-s}\right] \tag{2.4}$$

$$=z+\sum_{n=2}^{\infty}\frac{(\lambda+1)_{n-1}(\mu+1)_{n-1}}{(\nu+1)_{n-1}\,n!}\left(\frac{1+a}{n+a}\right)^s z^n,\ z\in\mathrm{U}.$$

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$$\begin{aligned} \mathcal{G}^{s,a}_{\lambda,\mu;\nu}(z) &= \frac{\nu(1+a)^s}{\lambda\,\mu}\left[\Phi_{\lambda,\mu;\nu}(z, s, a) - a^{-s}\right] \\ &= z + \sum_{n=2}^{\infty} \frac{(\lambda+1)_{n-1}(\mu+1)_{n-1}}{(\nu+1)_{n-1}\, n!}\left(\frac{1+a}{n+a}\right)^s z^n,\ z \in \mathrm{U}. \end{aligned} \tag{2.4}$$

Now, by using (2.4) in (2.3), we get

$$\mathcal{J}^{s,a}_{\lambda,\mu;\nu} f(z) = z + \sum_{n=2}^{\infty} \frac{(\lambda+1)_{n-1}(\mu+1)_{n-1}}{(\nu+1)_{n-1}\, n!} \left(\frac{1+a}{n+a}\right)^{s} a_n z^n,\ z \in \mathrm{U}. \tag{2.5}$$

Theorem 2.3. [Prajapat, B 2012]

Let $f, g_k \in \mathcal{A}$ $(k = 1, 2)$, $a \geq 0$, *and* $\operatorname{Re}\left(1 + \frac{z\varphi_k''(z)}{\varphi_k'(z)}\right) > -\rho$, $z \in \mathrm{U}$, *with* $\varphi_k(z) = \mathcal{J}^{s,a}_{\lambda,\mu,\nu} g_k(z)$ $(k = 1, 2)$, *where* $\rho = 0$ *if* $a = 0$ *and*

$$\rho = \rho(a) = \begin{cases} a/2, & \text{if} \quad 0 < a \leq 1, \\ 1/(2a), & \text{if} \quad a > 1. \end{cases} \tag{2.6}$$

Suppose that the function $\mathcal{J}^{s,a}_{\lambda,\mu,\nu} f$ *is univalent in* U, *and* $\mathcal{J}^{s+1,a}_{\lambda,\mu,\nu} f \in H[0,1] \cap \mathcal{Q}$.
Then, the double subordination

$$\mathcal{J}^{s,a}_{\lambda,\mu,\nu} g_1(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu} f(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu} g_2(z) \tag{2.7}$$

implies

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Moreover, the functions $\mathcal{J}^{s+1,a}_{\lambda,\mu,\nu} g_1$ *and* $\mathcal{J}^{s+1,a}_{\lambda,\mu,\nu} g_2$ *are, respectively the best subordinant and best dominant of (2.7).*

Now, by using (2.4) in (2.3), we get

$$\mathcal{J}^{s,a}_{\lambda,\mu;\nu} f(z) = z + \sum_{n=2}^{\infty} \frac{(\lambda+1)_{n-1}(\mu+1)_{n-1}}{(\nu+1)_{n-1}\, n!} \left(\frac{1+a}{n+a}\right)^{s} a_n z^n,\ z \in \mathrm{U}. \tag{2.5}$$

Theorem 2.3. [Prajapat, B 2012]

Let $f, g_k \in \mathcal{A}$ $(k = 1, 2)$, $a \geq 0$, *and* $\mathrm{Re}\left(1 + \frac{z\varphi_k''(z)}{\varphi_k'(z)}\right) > -\rho$, $z \in \mathrm{U}$, *with* $\varphi_k(z) = \mathcal{J}^{s,a}_{\lambda,\mu,\nu} g_k(z)$ $(k = 1, 2)$, *where* $\rho = 0$ *if* $a = 0$ *and*

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Moreover, the functions $\mathcal{J}^{s+1,a}_{\lambda,\mu,\nu} g_1$ *and* $\mathcal{J}^{s+1,a}_{\lambda,\mu,\nu} g_2$ *are, respectively the best subordinant and best dominant of* (2.7).

Theorem 2.4. [Prajapat, B 2012]

Let $f, g_k \in \mathcal{A}$ $(k = 1, 2)$, $\lambda > 0$ *and*

$$\operatorname{Re}\left(1 + \frac{z\psi_k''(z)}{\psi_k'(z)}\right) > -\tau,\ z \in \mathrm{U},$$

with $\psi_k(z) = \mathcal{J}^{s,a}_{\lambda+1,\mu,\nu} g_k(z)$ $(k = 1, 2)$, *where* $\tau = 0$ *if* $\lambda = 0$ *and*

$$\tau = \tau(\lambda) = \begin{cases} \lambda/2, & \text{if} \quad 0 < \lambda \leq 1, \\ 1/(2\lambda), & \text{if} \quad \lambda > 1. \end{cases}$$

Suppose that the function $\mathcal{J}^{s,a}_{\lambda+1,\mu,\nu} f$ *is univalent in* U, *and* $\mathcal{J}^{s,a}_{\lambda,\mu,\nu} f \in H[0, 1] \cap \mathcal{Q}$. *Then, the double subordination*

$$\mathcal{J}^{s,a}_{\lambda+1,\mu,\nu} g_1(z) \prec \mathcal{J}^{s,a}_{\lambda+1,\mu,\nu} f(z) \prec \mathcal{J}^{s,a}_{\lambda+1,\mu,\nu} g_2(z) \tag{2.8}$$

implies

$$\mathcal{J}^{s,a}_{\lambda,\mu,\nu} g_1(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu} f(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu} g_2(z).$$

Moreover, the functions $\mathcal{J}^{s,a}_{\lambda,\mu,\nu} g_1$ *and* $\mathcal{J}^{s,a}_{\lambda,\mu,\nu} g_2$ *are, respectively the best subordinant and best dominant of* (2.8).

Theorem 2.5. [Prajapat, B 2012]

Let $f, g_k \in \mathcal{A}$ $(k = 1, 2)$, $\nu > 0$ *and*

$$\operatorname{Re}\left(1 + \frac{z\vartheta_k''(z)}{\vartheta_k'(z)}\right) > -\sigma,\ z \in \mathrm{U},$$

with $\vartheta_k(z) = \mathcal{J}^{s,a}_{\lambda,\mu,\nu} g_k(z)$ $(k = 1, 2)$, *where* $\sigma = 0$ *if* $\nu = 0$ *and*

$$\sigma = \sigma(\nu) = \begin{cases} \nu/2, & \text{if} \quad 0 < \nu \leq 1, \\ 1/(2\nu), & \text{if} \quad \nu > 1. \end{cases}$$

Suppose that the function $\mathcal{J}^{s,a}_{\lambda,\mu,\nu} f$ *is univalent in* U, *and* $\mathcal{J}^{s,a}_{\lambda,\mu,\nu+1} f \in H[0, 1] \cap \mathcal{Q}$.
Then, the double subordination

$$\mathcal{J}^{s,a}_{\lambda,\mu,\nu} g_1(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu} f(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu} g_2(z) \tag{2.9}$$

implies

$$\mathcal{J}^{s,a}_{\lambda,\mu,\nu+1} g_1(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu+1} f(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu+1} g_2(z).$$

Moreover, the functions $\mathcal{J}^{s,a}_{\lambda,\mu,\nu+1} g_1$ *and* $\mathcal{J}^{s,a}_{\lambda,\mu,\nu+1} g_2$ *are, respectively the best subordinant and best dominant of* (2.9).

As an interesting application, let define the linear operator $\mathcal{S}_a^m f : \mathcal{A} \to \mathcal{A}$ $(m \in \mathbb{N}_0,\ a \geq 0)$ by

$$\mathcal{S}_a^0 f(z) = f(z), \quad \mathcal{S}_a^{m+1}(z) = \frac{1}{a+1}\left[a\mathcal{S}_a^m(z) + z\left(\mathcal{S}_a^m(z)\right)'\right],\ (m \in \mathbb{N}).$$

For

$$\mathrm{I}_d(z) = \frac{z}{1-z},$$

denoting $s_{m,a}(z) \equiv \mathcal{S}_a^m \mathrm{I}_d(z)$, then the explicit form of the function $s_{m,a}$ is given by

$$s_{m,a}(z) = z + \sum_{n=2}^{\infty}\left(\frac{n+a}{1+a}\right)^m z^n,\ z \in \mathrm{U}.$$

If we take $s = m$ $(m = \mathbb{N}_0)$ and $g(z) = z\left(s_{m,a}(z)\right)'$ in the second subordination part of Theorem 2.3, we obtain the following special case:

Theorem 2.6. [Prajapat, B 2012]

Let $f \in \mathcal{A}$, $a \geq 0$ and $m \in \mathbb{N}_0$. Suppose that

$$\operatorname{Re} \frac{{}_4F_3(\lambda+1, \mu+1, 2, 2; \nu+1, 1, 1; z)}{{}_3F_2(\lambda+1, \mu+1, 2; \nu+1, 1; z)} > -\rho, \ z \in \mathrm{U},$$

where $\rho = 0$ if $a = 0$, and ρ is given by (2.6) if $a > 0$. Then the subordination condition

$$\mathcal{J}_{\lambda,\mu,\nu}^{m,a} f(z) \prec z\, {}_2F_1(\lambda+1, \mu+1; \nu+1; z) \tag{2.10}$$

implies

$$\mathcal{J}_{\lambda,\mu,\nu}^{m+1,a} f(z) \prec z\, {}_3F_2(\lambda+1, \mu+1, a+1; \nu+1, a+2; z).$$

Moreover, the function $z\, {}_3F_2(\lambda+1, \mu+1, a+1; \nu+1, a+2; z)$ is the best dominant of (2.10).

Further, setting $\lambda = \nu$ and $\mu = 1$ in the above theorem, we get:

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implies

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Moreover, the function $z\,{}_3F_2(\lambda+1, \mu+1, a+1; \nu+1, a+2; z)$ is the best dominant of (2.10).

Further, setting $\lambda = \nu$ and $\mu = 1$ in the above theorem, we get:

Corollary 2.4. [Prajapat, B 2012]

Let $f \in \mathcal{A}$, $a \geq 0$ *and* $m \in \mathbb{N}_0$. *Suppose that*

$$\operatorname{Re} \frac{{}_3F_2(2,2,2;1,1;z)}{{}_2F_1(2,2;1;z)} > -\rho, \ z \in \mathrm{U},$$

where $\rho = 0$ *if* $a = 0$, *and* ρ *is given by* (2.6) *if* $a > 0$.
Then, the subordination condition

$$J_{m,a}f(z) \prec \frac{z}{(1-z)^2} \tag{2.11}$$

implies

$$J_{m+1,a}f(z) \prec {}_2F_1(a+1,2;a+2;z).$$

Moreover, the function ${}_2F_1(a+1,2;a+2;z)$ *is the best dominant of* (2.11). *Here,* $J_{m,a} \equiv \mathcal{J}^{m,a}_{\lambda,1;\lambda}$ *is the already mentioned Srivastava–Attiya integral operator.*

We conclude by remarking that in view of the generalized operator defined by the (2.5) and expressed in term of convolution (2.3) involving arbitrary coefficients, the main results would lead additional new results.
In fact, by appropriately selecting the arbitrary parameters in (2.5), the results presented in this paper would find further applications which incorporate generalized form of linear operators. These considerations can fruitfully be worked out and we skip the details in this regards.

Corollary 2.4. [Prajapat, B 2012]

Let $f \in \mathcal{A}$, $a \geq 0$ *and* $m \in \mathbb{N}_0$. *Suppose that*

$$\operatorname{Re} \frac{{}_3F_2(2,2,2;1,1;z)}{{}_2F_1(2,2;1;z)} > -\rho, \ z \in \mathrm{U},$$

where $\rho = 0$ *if* $a = 0$, *and* ρ *is given by* (2.6) *if* $a > 0$.
Then, the subordination condition

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