Stud. Univ. Babeş-Bolyai Math. ??(??), No. ??, 1–4

Generalized result on the global existence of positive solutions for a parabolic reaction diffusion model with a full diffusion matrix

Nabila Barrouk and Salim Mesbahi

Abstract. In this paper, we study the global existence in time of solutions for a parabolic reaction diffusion model with a full matrix of diffusion coefficients on a bounded domain. The technique used is based on compact semigroup methods and some estimates. Our objective is to show, under appropriate hypotheses, that the proposed model has a global solution with a large choice of nonlinearities.

Mathematics Subject Classification (2010): 35K57, 37L05, 35K55, 35K40.

Keywords: global existence, compact semigroups, reaction diffusion systems.

1. Introduction

Diffusion reaction systems have been among the most active and developed mathematical subjects for a long time, especially in recent years. The great interest of mathematicians in the study of this type of problems is due to its great importance in all fields of science and technology, where we find many applications in physics, chemistry, environment, biology and other disciplines. Examples include combustion problems, gas dynamics, population dynamics, industrial catalytic processes, chemistry in interstellar media, transport of contaminants in the environment, flame spread, spread of epidemics, pattern formation. We guide the reader to Britton [5], Fife [6], Murray [21], [22] and Pierre [24] where he finds many detailed mathematical models in biology, ecology, and others, and the reader can also find examples and other models in the references mentioned in this article and the references therein.

To study reaction diffusion systems, we need a variety of different methods and techniques in many areas of mathematics, such as numerical analysis, semigroup theory, fixed point methods in appropriate spaces, and many others.

In the works of Mesbahi and Alaa [1], [2], [17] and [18], we find new developed methods based on truncation functions, fixed point theorems and compactness, etc.

Other techniques based mainly on invariant regions and Lyapunov functional have been developed by several authors, in some cases, allow to obtain the global existence of their reaction diffusion systems. The reader can see this technique in Kouachi's works, such as [11] and [12].

There is also another very powerful method that relies on compact semigroups, which is the method we will use in this work. For a better understanding, we send the reader to the works of Moumeni and Barrouk [19] and [20].

In recent years, particular attention has been paid to the reaction diffusion systems of two equations with diffusion coefficients and specific reaction functions. This is due to its broad applications in various sciences, particularly in biology and engineering.

In this paper, we study the existence and uniqueness of solutions for a parabolic reaction diffusion model with homogeneous boundary conditions of Neumann or Dirichlet. To answer these questions, we use a technique based on compact semigroups. To get a more complete survey the reader is refered to Lions [14], Pazy [23], Rothe [25] and Smoller [26].

We are therefore interested in the global existence in time of solutions for the following parabolic reaction diffusion model with homogeneous Neumann or Dirichlet boundary conditions

$$\frac{\partial u}{\partial t} - a\Delta u - b\Delta v = f(u,v) \quad , \text{ in } \mathbb{R}^+ \times \Omega \tag{1.1}$$

$$\frac{\partial v}{\partial t} - c\Delta u - d\Delta v = g(u,v) \quad , \text{ in } \mathbb{R}^+ \times \Omega \tag{1.2}$$

with the following boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \quad \text{or} \quad u = v = 0 \quad , \text{ on } \mathbb{R}^+ \times \partial\Omega \tag{1.3}$$

and the initial data

$$u(0,x) = u_0(x) \in L^1(\Omega) \quad , \quad v(0,x) = v_0(x) \in L^1(\Omega) \tag{1.4}$$

where Ω is an open bounded domain of class C^1 in $\mathbb{R}^n$, with boundary $\partial\Omega$, and $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative on $\partial\Omega$. The diffusion coefficients a, b, c and d are supposed to be positive such that $a \leq d$, and $(b+c)^2 \leq 4ad$, which ensures the parabolicity of the system and implies that the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is positive definite, that is the eigenvalues λ_1 and λ_2 $(\lambda_1 < \lambda_2)$ of its transposed are positive.

Several authors have studied the problem proposed in the diagonal case, i.e. where $b = c = 0$, see for example Alikakos [3], Masuda [15], Haraux and Youkana [7].

In [19] and [20], Moumeni and Barrouk obtained a global existence result of solutions for reaction diffusion systems with a diagonal and triangular matrix of diffusion coefficents. By combining the compact semigroup methods and some L^1 estimates, we show the global existence of solutions for a large class of nonlinearities f and g.

In [12], Kouachi and Youkana have generalized the method of Haraux and Youkana in [7] to the triangular case, i.e. when $b = 0$.

In the same direction, Kouachi [11] has proved the global existence of solutions for two-component reaction diffusion systems with a general full matrix of diffusion coefficients, nonhomogeneous boundary conditions and polynomial growth conditions on the nonlinear terms and he obtained in [12] the global existence of solutions for the same system with homogeneous Neumann boundary conditions.

Mebarki and Moumeni [16] consider the problem (1.1)-(1.4) with $b > 0$ and $c > 0$, where the function f and g are assumed to satisfy

$$\sup\left\{\left|f\left(r,s\right)\right|,\left|g\left(r,s\right)\right|\right\} \leq C\left(r+s+1\right)^{m},\ \forall r,s \geq 0$$

and by adopting the Lyapunov method combined with some L^p estimates, they established a result of global existence of the solution.

The system (1.1)-(1.2) is a mathematical model describing various chemical and biological phenomena. In this case the components $u\left(t,x\right)$ and $v\left(t,x\right)$ represent chemical concentrations or biological population densities of wells. The reader can find models similar to this in Britton [5], Fife [6], Murray [21], [22] and the references therein.

The rest of this paper is organized as follows : In the next section, we present some hypotheses on our problem and then state the main result. In the third section, we provide a result on local existence and another on compactness, they are necessary to fully understand the content of this work. We give in the fourth section some results concerning the approximate problem. The last section is devoted to prove the main result.

2. Formulation of the main result

2.1. Assumptions

We consider the problem (1.1)-(1.4) where we assume the following hypotheses :

The Initial data are assumed in the following region

$$\Sigma = \left\{(u_0, v_0) \in \mathbb{R}^2,\ \text{such that}\ \frac{a-\lambda_2}{c}v_0 \leq u_0 \leq \frac{a-\lambda_1}{c}v_0\right\} \tag{2.1}$$

and

$$\begin{aligned} f\left(\frac{a-\lambda_1}{c}\xi_2, \xi_2\right) &\leq \frac{a-\lambda_1}{c}g\left(\frac{a-\lambda_1}{c}\xi_2, \xi_2\right) \\ \frac{a-\lambda_2}{c}g\left(\frac{a-\lambda_2}{c}\xi_2, \xi_2\right) &\leq f\left(\frac{a-\lambda_2}{c}\xi_2, \xi_2\right) \end{aligned} \tag{2.2}$$

for all $(\xi_1,\xi_2)\in\Sigma$.

There exist nonnegative constants C, C_1 and C_2 independent of (ξ_1,ξ_2) such that

$$g(\xi_1,\xi_2)\le C\xi_2 \quad, \text{ for all } (\xi_1,\xi_2)\in\Sigma \tag{2.3}$$

$$-f(\xi_1,\xi_2)+\frac{a-\lambda_1}{c}g(\xi_1,\xi_2)\le C_1\left(\frac{\lambda_2-\lambda_1}{c}\xi_2\right)^{C_2}, \quad \text{for all } (\xi_1,\xi_2)\in\Sigma \tag{2.4}$$

and

$$f(\xi_1,\xi_2)-\frac{a-\lambda_2}{c}g(\xi_1,\xi_2)\le C_1\left(\frac{\lambda_2-\lambda_1}{c}\xi_2\right)^{C_2}, \quad \text{for all } (\xi_1,\xi_2)\in\Sigma \tag{2.5}$$

2.2. The main result

Multiplying equation (1.2) one time through by $\frac{a-\lambda_1}{c}$ and subtracting equation (1.1) and another time by $-\frac{a-\lambda_2}{c}$ and adding equation (1.1) we get

$$\frac{\partial w}{\partial t}-\lambda_1\Delta w=F(w,z) \quad, \text{ in } \mathbb{R}^+\times\Omega \tag{2.6}$$

$$\frac{\partial z}{\partial t}-\lambda_2\Delta z=G(w,z) \quad, \text{ in } \mathbb{R}^+\times\Omega \tag{2.7}$$

$$\frac{\partial w}{\partial \eta}=\frac{\partial z}{\partial \eta}=0 \quad \text{or} \quad w=z=0 \quad, \text{ on } \mathbb{R}^+\times\partial\Omega \tag{2.8}$$

$$w(0,x)=w_0(x) \quad \text{and} \quad z(0,x)=z_0(x) \quad, \text{ in } \Omega \tag{2.9}$$

where

$$\begin{aligned} w(t,x)&=-u(t,x)+\frac{a-\lambda_1}{c}v(t,x)\\ z(t,x)&=u(t,x)-\frac{a-\lambda_2}{c}v(t,x)\end{aligned} \tag{2.10}$$

and

$$\begin{aligned} F(w,z)&=-f(u,v)+\frac{a-\lambda_1}{c}g(u,v)\\ G(w,z)&=f(u,v)-\frac{a-\lambda_2}{c}g(u,v)\end{aligned}$$

Suppose that the hypotheses (2.1)-(2.5) are satisfied, then the problem (2.6)-(2.9) satisfies the following hypotheses :

$$w_0,z_0 \text{ are nonnegative functions in } L^1(\Omega) \tag{2.11}$$

$$F(0,z)\ge 0 \quad, \quad G(w,0)\ge 0 \quad, \text{ for all } w,z\ge 0 \tag{2.12}$$

There exist nonnegative constants C, C_1 and C_2 independent of (w,z) such that

$$F(w,z)+G(w,z)\le C(w+z) \quad, \text{ for all } (w,z)\in\mathbb{R}_+^2 \tag{2.13}$$

$$\begin{cases} F(w,z)\le C_1(w+z)^{C_2} \text{ for all } (w,z)\in\mathbb{R}_+^2\\ G(w,z)\le C_1(w+z)^{C_2} \text{ for all } (w,z)\in\mathbb{R}_+^2\end{cases} \tag{2.14}$$

The existence of global solutions for the system (2.6)-(2.9) is equivalent to the existence of (w, z) illustrated by the following main Theorem :

Theorem 2.1. *Assume that the hypotheses (*2.11*)*-(2.14) *are satisfied, then there exists a positive global solution (w, z) of the problem (*2.6*)-(*2.9*) in the following sense :*

$$\begin{cases} w, z \in C\left([0, +\infty[\,, L^1(\Omega)\right) \\ F(w, z)\,,\ G(w, z) \in L^1(Q_T) \ \text{ where } Q_T =]0, T[\times \Omega \text{ for all } T > 0 \\ w(t) = S_1(t)\, w_0 + \int_0^t S_1(t-s)\, F(w(s), z(s))\, ds\ ,\ \forall t \in [0, T[\\ z(t) = S_2(t)\, z_0 + \int_0^t S_2(t-s)\, G(w(s), z(s))\, ds\ ,\ \forall t \in [0, T[\end{cases} \tag{2.15}$$

where $S_1(t)$ and $S_2(t)$ are contraction semigroups in $L^1(\Omega)$ generated, respectively, by $\lambda_1 \Delta$ and $\lambda_2 \Delta$.

To prove this Theorem, we will use the results which we will present in the following section :

3. Preliminaries

3.1. Local existence

Theorem 3.1. *Let Ω is an open bounded domain in $\mathbb{R}^n$, and $X = L^1(\Omega) \cap H^2(\Omega)$. The operator A defined by*

$$\begin{cases} D(A) = \left\{ u \in L^1(\Omega) \cap H^2(\Omega)\ ,\ \dfrac{\partial u}{\partial \eta} = 0 \ \text{ or } \ u = 0 \quad \text{on } \partial\Omega \right\} \\ Au = \Delta u \ \ ,\ \text{for all } u \in D(A) \end{cases}$$

is m-dissipative in $L^1(\Omega) \cap H^2(\Omega)$.

An important result of functional analysis which ensures the local existence of the solution is the following Lemma :

Lemma 3.2. *Let A be a m-dissipative operator of dense domain in a Banach space X and $S(t)$ a contraction semigroup generated by A, F a locally Lipchitz function, so $\forall u_0 \in X$, there exists $T_{\max} = T(u_0)$ such that the problem*

$$\begin{cases} u \in C([0, T], D(A)) \cap C^1([0, T], X) \\ \dfrac{du}{dt} - Au = F(u(s)) \\ u(0) = u_0 \end{cases} \tag{3.1}$$

admits a unique solution u verifying

$$u(t) = S(t)\, u_0 + \int_0^t S(t-s)\, F(u(s))\, ds \quad ,\quad \forall t \in [0, T_{\max}]$$

3.2. Compactness result

In this section, we will give a compactness result of operator L defining the solution of the problem (3.1) in the case where the initial value equals zero, i.e. $u(0) = 0$, with

$$L(F)(t) = u(t) = \int_0^t S(t-s) F(u(s))\, ds \quad , \quad \forall t \in [0,T]$$

Theorem 3.3. *If for all $t > 0$, the operator $S(t)$ is compact, then L is compact of $L^1([0,T], X)$ in $L^1([0,T], X)$.*

Proof. **Step 1.** To show that $S(\lambda) L : F \to S(\lambda) L(F)$ is compact in $L^1([0,T], X)$, it suffices to prove that the set $\{S(\lambda) L(F)(t);\ \|F\|_1 \leq 1\}$ is relatively compact in $L^1([0,T], X)$, $\forall t \in [0,T]$.

Since $S(t)$ is compact then, the application $t \to S(t)$ is continuous of $]0, +\infty[$ in $\mathcal{L}(X)$, therefore

$$\forall \varepsilon > 0,\ \forall \delta > 0,\ \exists \eta > 0,\ \forall 0 \leq h \leq \eta,\ \forall t \geq \delta,\ \|S(t+h) - S(t)\|_{\mathcal{L}(X)} \leq \varepsilon$$

By choosing $\lambda = \delta$, we have for $0 \leq t \leq T - h$

$$\begin{aligned} & S(\lambda) u(t+h) - S(\lambda) u(t) \\ = & \int_0^{t+h} S(\lambda + t + h - s) F(u(s))\, ds - \int_0^t S(\lambda + t - s) F(u(s))\, ds \\ = & \int_t^{t+h} S(\lambda + t + h - s) F(u(s))\, ds + \\ & \int_0^t (S(\lambda + t + h - s) - S(\lambda + t - s)) F(u(s))\, ds \end{aligned}$$

We obtain then

$$\|S(\lambda) u(t+h) - S(\lambda) u(t)\|_X \leq \int_t^{t+h} \|F(u(s))\|_X\, ds + \varepsilon \int_0^t \|F(u(s))\|_X\, ds$$

We define $v(t)$ by

$$v(t) = \begin{cases} u(t) & \text{if } 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

therefore

$$\|S(\lambda) v(t+h) - S(\lambda) v(t)\|_1 \leq (h + \varepsilon T) \|F(u(s))\|_1$$

which implies that $\{S(\lambda) v,\ \|F\|_1 \leq 1\}$ is equi-integrable, then $\{S(\lambda) L(F)(t),\ \|F\|_1 \leq 1\}$ is relatively compact in $L^1([0,T], X)$, which means that $S(\lambda) L$ is compact.

Step 2. We prove that $S(\lambda) L$ converges to L when λ tends to 0 in $L^1([0,T], X)$. We have

$$S(\lambda) u(t) - u(t) = \int_0^t S(\lambda + t - s) F(u(s))\, ds - \int_0^t S(t-s) F(u(s))\, ds$$

So, for $t \geq \delta$, we have

$$\begin{aligned} \|S(\lambda) u(t) - u(t)\| &\leq \int_{\delta}^{t} \|S(\lambda + s) - S(s)\|_{\pounds(X)} \|F(u(s))\| \, ds \\ &\quad + 2 \int_{t-\delta}^{t} \|F(u(s))\| \, ds \end{aligned}$$

We choose $0 < \lambda < \eta$, then

$$\|S(\lambda) u(t) - u(t)\| \leq \varepsilon \int_{\delta}^{t} \|F(u(s))\| \, ds + 2 \int_{t-\delta}^{t} \|F(u(s))\| \, ds$$

and for $0 \leq t < \delta$, we have

$$\|S(\lambda) u(t) - u(t)\| \leq 2 \int_{0}^{t} \|F(u(s))\| \, ds$$

Since $F \in L^1(0, T, X)$, we obtain

$$\|S(\lambda) u(t) - u(t)\| \leq (\varepsilon T + 2\delta) \|F(u(s))\|_1$$

So if $\lambda \to 0$ then $S(\lambda) u \to u$ in $L^1([0, T], X)$.

The operator L is a uniform limit with compact linear operator between two Banach spaces, then L is compact in $L^1([0, T], X)$. □

Remark 3.4. The semigroup $S(t)$ generated by the operator Δ is compact in $L^1(\Omega)$.

Proof. See Pazy [23]. □

4. Approximating problem

For all $n > 0$, we define the functions w_{n_0} and z_{n_0} by

$$w_{n_0} = \min\{w_0, n\} \quad \text{and} \quad z_{n_0} = \min\{z_0, n\}$$

It is clear that w_{n_0} and z_{n_0} verify (2.11), i.e.

$$w_{n_0} \text{ and } z_{n_0} \text{ are nonnegative functions in } L^1(\Omega)$$

Now, we suppose the following problem

$$\begin{cases} \dfrac{\partial w_n}{\partial t} - \lambda_1 \Delta w_n = F(w_n, z_n) & \text{in } [0, T[\times \Omega \\ \dfrac{\partial z_n}{\partial t} - \lambda_2 \Delta z_n = G(w_n, z_n) & \text{in } [0, T[\times \Omega \\ \dfrac{\partial w_n}{\partial \eta} = \dfrac{\partial z_n}{\partial \eta} = 0 \ \text{ or } \ w_n = z_n = 0 & \text{on } [0, T[\times \partial\Omega \\ w_n(0, x) = w_{n_0}(x) \quad , \quad z_n(0, x) = z_{n_0}(x) & \text{in } \Omega \end{cases} \tag{4.1}$$

4.1. Local existence of the solution of problem (4.1)

We transform the system (4.1) into a first order system in the Banach space $X = L^1(\Omega) \times L^1(\Omega)$, we obtain

$$\begin{cases} \dfrac{\partial \omega_n}{\partial t} = A\omega_n + \Psi(\omega_n) & , \ t > 0 \\ \omega_n(0) = \omega_{n_0} = (w_{n_0}, z_{n_0}) \in X \end{cases} \tag{4.2}$$

Here $\omega_n = \mathrm{col}(w_n, z_n)$, the operator A is defined as follows

$$A = \begin{pmatrix} \lambda_1 \Delta & 0 \\ 0 & \lambda_2 \Delta \end{pmatrix}$$

where

$$D(A) := \{\omega_n = \mathrm{col}(w_n, z_n) \in X : \mathrm{col}(\Delta w_n, \Delta z_n) \in X\}$$

and the function Ψ is defined by

$$\Psi(\omega_n(t)) = \mathrm{col}(F(\omega_n(t)), G(\omega_n(t)))$$

Therefore, the system (4.2) can be returned to the form of the system (3.1), thus, if (w_n, z_n) is a solution of (4.2) then it checks the integral equations

$$\begin{cases} w_n(t) = S_1(t) w_{n_0} + \int_0^t S_1(t-s) F(w_n(s), z_n(s)) \, ds \\ z_n(t) = S_2(t) z_{n_0} + \int_0^t S_2(t-s) G(w_n(s), z_n(s)) \, ds \end{cases} \tag{4.3}$$

where $S_1(t)$ and $S_2(t)$ are the contraction semigroups generated, respectively, by $\lambda_1 \Delta$ and $\lambda_2 \Delta$.

Theorem 4.1. *There exist $T_M > 0$ and (w_n, z_n) a local solution of (4.2) for all $t \in [0, T_M]$.*

Proof. We know that $S_1(t)$, $S_2(t)$ are contraction semigroups and that Ψ is locally Lipschitz in ω_n, then there exists $T_M > 0$ such that (w_n, z_n) is a local solution of (4.2) on $[0, T_M]$. □

4.2. Positivity of the solution of problem (4.1)

Lemma 4.2. *Let (w_n, z_n) be a solution of problem (4.1), then the region*

$$\Sigma = \left\{ (w_{n_0}, z_{n_0}) \in \mathbb{R}^2 \ \text{such that} \ w_{n_0} \geq 0, \ z_{n_0} \geq 0 \right\} = \mathbb{R}^+ \times \mathbb{R}^+$$

is invariant for system (4.1).

Proof. Let $\bar{w}_n(t,x) = 0$ in $]0, T[\times \Omega$, then $\dfrac{\partial \bar{w}_n}{\partial t} = 0$ and $\Delta \bar{w}_n = 0$.

According to (4.1), we have

$$\frac{\partial w_n}{\partial t} - \lambda_1 \Delta w_n - F(w_n, z_n) = 0 \geq \frac{\partial \bar{w}_n}{\partial t} - \lambda_1 \Delta \bar{w}_n - F(\bar{w}_n, z_n)$$

and

$$w_n(0,x) = w_{n_0}(x) \geq 0 = \bar{w}_n(0,x)$$

By comparison we get

$$w_n(t,x) \geq \bar{w}_n(t,x)$$

which gives us $w_n(t,x) \geq 0$. In the same way, we get $z_n(t,x) \geq 0$. □

4.3. Global existence of the solution of problem (4.1)

To prove the global existence of the solution of problem (4.1), it suffices to find an estimate of the solution for all $t \geq 0$, according to Haraux and Kirane [8], Henry [9] and Rothe [25]. For this, we give the following Lemma :

Lemma 4.3. *Let (w_n, z_n) be a solution of the problem (4.1), then there exists $M(t)$ which only depends on t, such that, for all $0 \leq t \leq T_M$, we have*

$$\|w_n + z_n\|_{L^1(\Omega)} \leq M(t)$$

Proof. From (4.1), it comes

$$\frac{\partial}{\partial t}(w_n + z_n) - \Delta(\lambda_1 w_n + \lambda_2 z_n) = F(w_n, z_n) + G(w_n, z_n)$$

and by taking into account of (2.13), we have

$$\frac{\partial}{\partial t}(w_n + z_n) - \Delta(\lambda_1 w_n + \lambda_2 z_n) \leq \hat{C}(w_n + z_n)$$

By Integration on Ω and by applying Green's formula, we find

$$\frac{\partial}{\partial t}\int_\Omega (w_n + z_n)\,dx \leq C\int_\Omega (w_n + z_n)\,dx$$

which give

$$\frac{\frac{\partial}{\partial t}\int_\Omega (w_n + z_n)\,dx}{\int_\Omega (w_n + z_n)\,dx} \leq C$$

By integrating on $[0,t]$, we get

$$\log\left(\int_\Omega (w_n + z_n)\,dx\Big|_0^t\right) \leq Ct$$

which implies

$$\frac{\int_\Omega (w_n + z_n)\,dx}{\int_\Omega (w_{n_0} + z_{n_0})\,dx} \leq \exp(Ct)$$

and for $w_{n_0} \leq w_0$, $z_{n_0} \leq z_0$, we have

$$\int_\Omega (w_n + z_n)\,dx \leq \exp(Ct)\,.\int_\Omega (w_0 + z_0)\,dx$$

Since w_n and z_n are positive, we get

$$\|w_n + z_n\|_{L^1(\Omega)} \leq M(t) \quad , \; 0 \leq t \leq T_M$$

with

$$M(t) = \exp(Ct)\,.\,\|w_0 + z_0\|_{L^1(\Omega)}$$

□

We can conclude from this estimate that the solution (w_n, z_n) given by the Theorem 4.1 is a global solution.

Now, we give the following Lemma which shows the existence of an estimate of the solution (w_n, z_n) of the problem (4.1) in $L^1(Q)$.

Lemma 4.4. *For any solution* (w_n, z_n) *of* (4.1)*, there exists a constant* $K(t)$ *depends only on* t*, such that*

$$\|w_n + z_n\|_{L^1(Q_T)} \leq K(t) . \|w_0 + z_0\|_{L^1(\Omega)}$$

Proof. To prove this Lemma, we will use some of the results demonstrated in the works of Bonafede and Schmitt [4] and Hollis et *al.* [10].

We introduce $\theta \in C_0^\infty(Q_T)$, $\theta \geq 0$, and $\Phi \in C^{1,2}(Q_T)$ a nonnegative solution of the following system

$$\begin{cases} -\dfrac{\partial \Phi}{\partial t} - d_1 \Delta \Phi = \theta & \text{on } Q_T \\ \dfrac{\partial \Phi}{\partial \eta} = 0 & \text{on } [0,T] \times \partial\Omega \\ \Phi(T, \cdot) = 0 & \text{on } \Omega \end{cases} \tag{4.4}$$

According to Ladyzenskaya et *al.* [13], the system (4.4) has a unique nonnegative solution. Moreover, for all $q \in]1, +\infty[$, there exists a nonnegative constant c independent of θ, such that,

$$\|\Phi\|_{L^q(Q_T)} \leq c \|\theta\|_{L^q(Q_T)}$$

According to Bonafede and Schmitt [4], we have

$$\int_{Q_T} S_1(t) w_{n_0}(x) \left(-\frac{\partial \Phi}{\partial t} - d_1 \Delta \Phi\right) dxdt = \int_\Omega w_{n_0}(x) \Phi(0, x) dx$$

and

$$\begin{aligned} &\int_{Q_T} \left(\int_0^t S_1(t-s) F(w_n, z_n) ds\right) \left(-\frac{\partial \Phi}{\partial t} - d_1 \Delta \Phi\right) dxdt \\ = \; &\int_{Q_T} F(w_n, z_n) \Phi(s, x) dxds \end{aligned}$$

where from

$$\int_{Q_T} (S_1(t) w_{n_0}(x)) \theta dxdt = \int_\Omega w_{n_0}(x) \Phi(0, x) dx \tag{4.5}$$

and

$$\int_{Q_T} \left(\int_0^t S_1(t-s) F(w_n, z_n) ds\right) \theta dxdt = \int_{Q_T} F(w_n, z_n) \Phi(s, x) dxds \tag{4.6}$$

We multiply the first equation of (4.3) by θ, we integrate on Q_T, and using (4.5) and (4.6), we obtain

$$\begin{aligned} \int_{Q_T} w_n \theta dxdt &= \int_{Q_T} S_1(t) w_{n_0}(x) \theta dxdt \\ &\quad + \int_{Q_T} \left(\int_0^t S_1(t-s) F(w_n, z_n) ds\right) \theta dxdt \\ &= \int_\Omega w_{n_0}(x) \Phi(0, x) dx + \int_{Q_T} F(w_n, z_n) \Phi(s, x) dxds \end{aligned}$$

We also find

$$\int_{Q_T} z_n \theta dxdt = \int_{\Omega} z_{n_0}(x)\Phi(0,x)\,dx + \int_{Q_T} G(w_n, z_n)\Phi(s,x)\,dxds$$

and therefore

$$\begin{aligned}\int_{Q_T}(w_n+z_n)\theta dxdt &= \int_{\Omega}(w_{n_0}(x)+z_{n_0}(x))\Phi(0,x)\,dx \\ &\quad + \int_{Q_T}(F(w_n,z_n)+G(w_n,z_n))\Phi(s,x)\,dxds \\ &\leq \int_{\Omega}(w_0(x)+z_0(x))\Phi(0,x)\,dx \\ &\quad + \int_{Q_T} C(w_n+z_n)\Phi(s,x)\,dxds\end{aligned}$$

Using Holder's inequality, we deduce

$$\begin{aligned}\int_{Q_T}(w_n+z_n)\theta dxdt &\leq \|w_0+z_0\|_{L^1(\Omega)}\cdot\|\Phi(0,.)\|_{L^\infty(Q_T)} \\ &\quad + C\|w_n+z_n\|_{L^1(Q_T)}\cdot\|\Phi\|_{L^\infty(Q_T)} \\ &\leq k_1\left(\|w_0+z_0\|_{L^1(\Omega)}+\|w_n+z_n\|_{L^1(Q_T)}\right)\|\theta\|_{L^\infty(Q_T)}\end{aligned}$$

where $k_1 = \max\{c, cC\}$.

Since θ is arbitrary in $C_0^\infty(Q_T)$, this implies

$$\|w_n+z_n\|_{L^1(Q_T)} \leq k_1\left(\|w_0+z_0\|_{L^1(\Omega)}+\|w_n+z_n\|_{L^1(Q_T)}\right)$$

If we take $k = \dfrac{k_1(t)}{1-k_1(t)}$, we find

$$\|w_n+z_n\|_{L^1(Q_T)} \leq k(t)\,.\,\|w_0+z_0\|_{L^1(\Omega)}$$

□

5. Proof of the main result (Theorem 2.1)

We are now ready to prove the main result of this work:

Proof of theorem 2.1. We define the application L by

$$L:(w_0,h)\mapsto S_d(t)\,w_0+\int_0^t S_d(t-s)\,h(s)\,ds$$

where $S_d(t)$ is the contraction semigroup generated by the operator $d\Delta$. According to the previous Theorem 3.3 and as $S_d(t)$ is compact, then the application L is the addition of two compact applications in $L^1(Q)$, which shows that L is also compact from $L^1(Q_T)\times L^1(Q_T)$ in $L^1(Q_T)$.

Therefore, there is a subsequence (w_{n_j}, z_{n_j}) of (w_n, z_n) and (w,z) of $L^1(Q_T)\times L^1(Q_T)$, such that

$$(w_{n_j}, z_{n_j}) \to (w,z)$$

Let us now show that $\left(w_{n_j}, z_{n_j}\right)$ is a solution of (4.3). We have

$$\left\{\begin{array}{l} w_{n_j}(t,x) = S_1(t)\, w_{n_0} + \int_0^t S_1(t-s)\, F\left(w_{n_j}(s), z_{n_j}(s)\right) ds \\ z_{n_j}(t,x) = S_2(t)\, z_{n_0} + \int_0^t S_2(t-s)\, G\left(w_{n_j}(s), z_{n_j}(s)\right) ds \end{array}\right. \tag{5.1}$$

It suffices to show that (w, z) satisfies (2.15). It is clear that if $j \to +\infty$, we have the following limits

$$F\left(w_{n_j}, z_{n_j}\right) \to F(w,z) \quad \text{and} \quad G\left(w_{n_j}, z_{n_j}\right) \to G(w,z) \ , \text{a.e.} \tag{5.2}$$

and

$$w_{n_0} \to w_0 \ \ , \ \ z_{n_0} \to z_0$$

Thus, to show that (w, z) satisfies (2.15), we have to show that

$$F\left(w_{n_j}, z_{n_j}\right) \to F(w,z) \quad \text{and} \ \ G\left(w_{n_j}, z_{n_j}\right) \to G(w,z) \ \text{ in } L^1(Q_T)$$

We integrate the two equations of (4.1) on Q_T taking into account that

$$-\lambda_1 \int_{Q_T} \Delta w_{n_j} dxdt = 0 \quad \text{and} \quad -\lambda_2 \int_{Q_T} \Delta z_{n_j} dxdt = 0$$

we have

$$\begin{aligned} \int_\Omega w_{n_j} dx - \int_\Omega w_{n_0} dx &= \int_{Q_T} F\left(w_{n_j}, z_{n_j}\right) dxdt \\ \int_\Omega z_{n_j} dx - \int_\Omega z_{n_0} dx &= \int_{Q_T} G\left(w_{n_j}, z_{n_j}\right) dxdt \end{aligned}$$

which give

$$-\int_{Q_T} F\left(w_{n_j}, z_{n_j}\right) dxdt \le \int_\Omega w_0 dx \tag{5.3}$$

$$-\int_{Q_T} G\left(w_{n_j}, z_{n_j}\right) dxdt \le \int_\Omega z_0 dx \tag{5.4}$$

We denote

$$\begin{aligned} N_n &= C_1\left(w_{n_j} + z_{n_j}\right)^{C_2} - F\left(w_{n_j}, z_{n_j}\right) \\ M_n &= C_1\left(w_{n_j} + z_{n_j}\right)^{C_2} - G\left(w_{n_j}, z_{n_j}\right) \end{aligned}$$

According to (2.13), it is clear that N_n and M_n are positive. From (5.3) and (5.4), we obtain

$$\begin{aligned} \int_{Q_T} N_n dxdt &\le C_1 \int_{Q_T} \left(w_{n_j} + z_{n_j}\right)^{C_2} dxdt + \int_\Omega w_0 dx \\ \int_{Q_T} M_n dxdt &\le C_1 \int_{Q_T} \left(w_{n_j} + z_{n_j}\right)^{C_2} dxdt + \int_\Omega z_0 dx \end{aligned}$$

The Lemma 4.4 gives us

$$\int_{Q_T} N_n dxdt < +\infty \quad \text{and} \quad \int_{Q_T} M_n dxdt < +\infty$$

which implies

$$\int_{Q_T} \left|F\left(w_{n_j}, z_{n_j}\right)\right| dxdt \le C_1 \int_{Q_T} \left(w_{n_j} + z_{n_j}\right)^{C_2} dxdt + \int_{Q_T} N_n dxdt < +\infty$$

and

$$\int_{Q_T} \left|G\left(w_{n_j}, z_{n_j}\right)\right| dxdt \le C_1 \int_{Q_T} \left(w_{n_j} + z_{n_j}\right)^{C_2} dxdt + \int_{Q_T} M_n dxdt < +\infty$$

The functions

$$\begin{aligned} \varphi_n &= N_n + C_1 \left(w_{n_j} + z_{n_j}\right)^{C_2} \\ \psi_n &= M_n + C_1 \left(w_{n_j} + z_{n_j}\right)^{C_2} \end{aligned}$$

are from $L^1(Q_T)$ and positive, moreover

$$\left|F\left(w_{n_j}, z_{n_j}\right)\right| \le \varphi_n \quad \text{and} \quad \left|G\left(w_{n_j}, z_{n_j}\right)\right| \le \psi_n \text{ a.e.}$$

We combine this result with (5.2) and by applying Lebesgue's dominated convergence Theorem, we obtain

$$F\left(w_{n_j}, z_{n_j}\right) \to F(w, z) \quad \text{and} \quad G\left(w_{n_j}, z_{n_j}\right) \to G(w, z) \quad \text{in } L^1(Q_T)$$

By passing to the limit of (5.1) when $j \to +\infty$ in $L^1(Q_T)$, we find

$$\begin{cases} w(t) = S_1(t) w_0 + \int_0^t S_1(t-s) F(w(s), z(s)) ds \\ z(t) = S_2(t) z_0 + \int_0^t S_2(t-s) G(w(s), z(s)) ds \end{cases}$$

which implies that (w, z) satisfies (2.15). Therefore (w, z) is a solution of (2.6)-(2.9). □

We conclude by (2.10) the existence in time of solutions of the reaction diffusion system (1.1)-(1.4).

References

[1] N. Alaa, S. Mesbahi, W. Bouarifi, *Global existence of weak solutions for parabolic triangular reaction diffusion systems applied to a climate model*, An. Univ. Craiova Ser. Mat. Inform., Vol. 42 (1) (2015), 80-97.

[2] N. Alaa, S. Mesbahi, A. Mouida, W. Bouarifi, *Existence of solutions for quasilinear elliptic degenerate systems with L^1 data and nonlinearity in the gradient*, Electron. J. Diff. Equ., Vol. 2013 (2013), No. 142, 1-13.

[3] N.D. Alikakos, *L^p-bounds of solutions of reaction diffusion equations*, Comm. Partial Differential Equations, 4 (1979), 827-868.

[4] S. Bonafede, D. Schmitt, *Triangular reaction diffusion systems with integrable initial data*, Nonlinear analysis, 33 (1998), 785-801.

[5] N.F. Britton, *Reaction diffusion equations and their applications to Biology*, Academic Press, London, (1986).

[6] P.C. Fife, *Mathematical aspects of reacting and diffusing systems*, Lecture Notes in Biomath., 28, Springer-Verlag, Berlin, New York, (1979).

[7] A. Haraux, A. Youkana, *On a result of K. Masuda concerning reaction diffusion equations*, Tohoku Math. J., 40 (1988), 159-163.

[8] A. Haraux, M. Kirane, *Estimations C^1 pour des problèmes paraboliques semilineaires*, Ann. Fac. Sci. Toulouse Math., 5 (1983), 265-280.

[9] D. Henry, *Theory of semilinear parabolic equations*, Lecture notes in Math, 840, Springer-Verlag, New York (1984).

[10] S.L. Hollis, R. H. Martin, M. Pierre, *Global existence and boundedness in reaction diffusion systems*, SIAM J. Math. Anal., 18: (1987), 744-761.

[11] S. Kouachi, *Invariant regions and global existence of solutions for reaction diffusion systems with full matrix of diffusion coefficients and nonhomogeneous boundary conditions*, Georgian Math. J., 11 (2004), 349-359.

[12] S. Kouachi, A. Youkana, *Global existence for a class of reaction diffusion systems*, Bull. Polish. Acad. Sci. Math., 49 (3), (2001).

[13] O.A. Ladyzenskaya, V. A Solonnikov, N. N Uralceva, *Linear and quasilinear equations of parabolic type*, Trans. Math. Monographs., vol. 23, AMS; Providence, R. I, (1968).

[14] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod : Gauthier-Villars, 1969.

[15] K. Masuda, *On the global existence and asymptotic behaviour of solution of reaction diffusion equations*, Hokkaido Math, J., 12: (1983), 360-370.

[16] M. Mebarki, A. Moumeni, *Global solution of system reaction diffusion with full matrix*, Global Journal of Mathematical Analysis, (2015), 04-25.

[17] S. Mesbahi, N. Alaa, *Mathematical analysis of a reaction diffusion model for image restoration*, An. Univ. Craiova Ser. Mat. Inform., Vol. 42 (1) (2015), 70-79.

[18] S. Mesbahi, N. Alaa, *Existence result for triangular reaction diffusion systems with L^1 data and critical growth with respect to the gradient*, Mediterr. J. Math., 10 (2013), 255-275.

[19] A. Moumeni, N. Barrouk, *Existence of global solutions for systems of reaction diffusion with compact result*, IJPAM., 102 (2) (2015), 169-186.

[20] A. Moumeni, N. Barrouk, *Triangular reaction diffusion systems with compact result*, GJPAM., 11 (6) (2015), 4729-4747.

[21] J. D. Murray, *Mathematical Biology I : An Introduction*, volume I, Springer-Verlag, 3rd edition, 2003.

[22] J. D. Murray, *Mathematical Biology II : Spatial Models and Biochemical Applications*, volume II, Springer-Verlag, 3rd edition, 2003.

[23] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

[24] M. Pierre, *Global existence in reaction diffusion systems with dissipation of mass : a survey*, Milan J. Math., Vol. 78 (2) (2010), 417-455.

[25] F. Rothe, *Global existence of reaction diffusion systems, Lecture Notes in Math*, 1072, Springer-Verlag, Berlin, 1984.

[26] J. Smoller, *Shock waves and reaction difussion systems*, Springer-Verlag, New York, (1983).

Nabila Barrouk
Mathematics, Dynamics and Modeling Laboratory,
Department of Mathematics and Computer Science,
Faculty of Sciences and Technology,
Mohamed Cherif Messaadia University,
Souk Ahras, 41000.
Algeria.
e-mail: n.barrouk@univ-soukahras.dz

Salim Mesbahi
Fundamental and Numerical Mathematics Laboratory,
Department of Mathematics,
Faculty of Sciences,
Ferhat Abbas University,
Setif, 19000.
Algeria.
e-mail: salimbra@gmail.com