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A smooth approximation for non-linear second order boundary value problems using composite non-polynomial spline functions

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Abstract. A different amalgamation of non-polynomial splines is used to find the approximate solution of linear and non-linear second order boundary value problems. Cubic spline functions are assembled with exponential and trigonometric functions to develop the different orders of numerical schemes. Free parameter k of the non-polynomial part is also used to form a new scheme, which elevates the accuracy of the solution. Numerical illustrations are given to validate the applicability and feasibility of the present methods and also depicted in the graphs.

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1. Introduction

To demonstrate the basic concept and idea of our technique, we consider the following general non-linear second order two point boundary value problems (BVPs), which arise in a wide variety of engineering applications

$$u^{(2)}(x) = f(x,u),\ -\infty \leqslant a \leqslant x \leqslant b \leqslant \infty \tag{1.1}$$

with the boundary conditions (BCs)

$$u(a) = A_1,\ u(b) = A_2, \tag{1.2}$$

where A_i, i=1, 2 are arbitrary finite real constants and $-\infty < u < \infty$. The function $f(x,u(x))$ is a continuous function of two variables with $f_u \geqslant 0$ on $[a,b]$. DE (1.1) with BC (1.2) has a unique solution, whose existence and uniqueness can be studied in [24]. For the linear case, $f(x,u) = p(x)u + g(x)$ with $p(x)$ and $g(x)$ continuous functions on the interval $[a,b]$.

It is well acknowledged that numerous real-life phenomena in physics and engineering sciences often convert to boundary value problems for second order differential equations such as in heat transfer, optimal control, deflection in cables and plates, vibration of springs, electric circuits and in a number of other scientific applications [19]. Most of the BVPs are essentially solved using numerical approaches as those are not explained enough using existing analytical approaches. Consequently, some useful numerical schemes were being promoted, most notably spline-based' schemes. Spline functions were applied by many authors to establish the accurate and efficient numerical schemes for the solution of boundary value problems [4]. An exploration of the literature on a number of polynomial and non- polynomial spline techniques to solve the second order BVPs can be comprehended as quadratic spline method [8, 26, 32, 42, 49], cubic spline method [2-3, 5, 9-12, 15, 20-23, 27-28, 30-34, 36-38, 40-41, 50], quartic spline method [6, 13-14, 29, 47], quintic spline method [7, 16, 43, 48] and others [39, 46]. Voluminous research work have been contributed to this field but we are mainly concerned on those papers which have implemented non-polynomial splines for the solution of second order BVPs with various types of boundary conditions.

For instance, Rashidinia *et al.* [40] built up a technique based on cubic non-polynomial spline functions of the form

$$T_n = Span\{1, x, sin(\tau x), cos(\tau x)\}, \tag{1.3}$$

They applied their scheme to acquire the numerical solution of the following form of second order two point BVPs

$$-\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] = g(x);\ u(a) = u(b) = 0. \tag{1.4}$$

Here, authors employed direct method to simplify the obtained system and facilitated the smooth approximations to linear second order BVPs. Similar approach was exercised by Islam and Tirmizi [27] to find the approximate solution of the system of two-point second order BVPs with Dirichlet BCs (1.2). They established the consistency equations to attain the desired results and solved linear second order equations to show the feasibility of their method. Khan and Aziz [34] proposed the parametric cubic spline functions with a parameter for attaining approximations to the solutions of the system of BVP. They presented improved results while comparing with some existing methods. Former approach [35] was yet again instituted by Khan in [33] to solve the following second order linear BVPs

$$y^{(2)}(x) = f(x)y(x) + g(x);\ a \leqslant x \leqslant b \tag{1.5}$$

with Dirichlet BCs (1.2). Here, the author developed the method of order four for specific values of parameters, or else his method was of order two. Over again, Zahra et al. [50] used cubic non-polynomial spline function space (1.3) to compute approximation to the solution of above linear BVPs (1.5) but with Neumann BCs. Kalyani and Rao [31] also adopted similar approach demonstrated by [27, 40, 50] to solve the following BVP of second order

$$-\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + v(x)u(x) = g(x);\ u(a) = u(b) = 0. \tag{1.6}$$

They solved many linear and non-linear examples to study the performance of their method. Cubic non-polynomial spline scheme was once more deliberated by Justine and Sulaiman [30] to solve the general linear second order BVPs subject to Dirichlet BCs. To solve the obtained linear system, they used successive over relaxation in conjunction with Gauss-Seidel method. However, to establish the result, here authors considered the total number of iterations, execution times along with maximum absolute error (MAE).

Above, we have summarized numerous contributions that are made to deal with the solution of various types of second order BVPs choosing non-polynomial splines. The present research could contribute remarkably to this field as it includes some novel methods to solve non-linear second order BVPs with significant results. Our method is based on distinctive exponential and trigonometric spline function space given as

$$T_3 = Span\Big\{1, x, e^{kx}, sin(kx)\Big\}$$

$$= Span\Big\{1, x, (\frac{2}{k^2})\left(e^{kx} - kx - 1\right), (\frac{6}{k^3})\left(kx - sin(kx)\right)\Big\}, \qquad (1.7)$$

where k is the frequency of trigonometric and exponential part of the spline function, which can be real or pure imaginary. It follows that if $k \to 0$, T_3 reduces to $Span\big(1, x, x^2, x^3\big)$. In this paper, we have developed different order methods along with a modified k-dependent method based on the angular frequency of the non-polynomial part for smooth approximation of the second order linear and non-linear BVPs. We have solved several examples using our developed methods and also shown comparisons of our results with some known methods like collocation, finite difference, Galerkin, Adomian decomposition and other spline methods. Our spline method solution and comparisons demonstrate that our algorithm performs comparatively better with more precise results.

Now, the paper is organized as follows: section 2 shows the formulation of our schemes and section 3 describes the solution of BVPs using the developed scheme. Section 4 deliberates the convergence of the schemes, while in section 5 some examples are solved using our developed spline methods. Paper is concluded in section 6.

2. Derivation of the method

In this section, we develop a numerical method to approximate the solution of second order BVP (1.1)-(1.2). To do that, we first set a framework of $N + 1$ equally spaced points x_i of an interval [a, b] and divide them into N equal sections such that $x_i = a + ih, i = 0, 1, 2, ..., N$ where $x_0 = a, x_N = b$ and $h = \frac{(b-a)}{N}$. Then, our spline function $P_i(x)$ holds the following structure in every section of the interval

$$P_i(x) = a_i sink(x - x_i) + b_i e^{k(x-x_i)} + c_i(x - x_i) + d_i; i = 0, 1, 2.., N \qquad (2.1)$$

where a_i, b_i, c_i and d_i are constants and k is free parameter, which can be real or purely imaginary and will be used to raise the accuracy of the method. The function $P_i(x)$, which interpolates $S(x)$ at the mesh points x_i and reduces to cubic spline as $k \to 0$, where $S(x)$ is the approximate solution of (1.1). Let $u(x)$ be the exact solution and S_i be an approximation to $u_i = u(x_i)$ obtained by the segment $P_i(x)$ of the spline function passing through the points (x_i, S_i) and (x_{i+1}, S_{i+1}). Then the mixed spline defined by the function $S(x) = P_i(x)$.

Now, we assume

$$P_i(x_i) = S_i, \quad P_i(x_{i+1}) = S_{i+1}, \quad P_i^{(2)}(x_i) = M_i, \quad P_i^{(2)}(x_{i+1}) = M_{i+1},$$

to get the following value of coefficients

$$a_i = \frac{1}{k^2 sin(\theta)}[e^{\theta} M_i - M_{i+1}], \quad b_i = \frac{1}{k^2}[M_i],$$

$$c_i = \frac{S_{i+1} - S_i}{h} + \frac{M_{i+1} + M_i}{k^2 h} - \frac{2e^{\theta} M_i}{k^2 h}, \quad d_i = S_i - \frac{1}{k^2}[M_i],$$

whereby $\theta = kh$ and $i = 0, 1, 2, ..., N$.

Next, use the continuity condition of the first derivative and substitute the value of coefficients a_i, b_i, c_i and d_i. After some algebraic manipulations, we can obtain the following main relation

$$S_{i-1} - 2S_i + S_{i+1} = h^2[\alpha M_{i-1} + \beta M_i + \gamma M_{i+1}]; \ i = 1, 2, ...N - 1, \tag{2.2}$$

where,

$$\alpha = \frac{\theta e^{\theta}\{sin(\theta) + cos(\theta)\} + sin(\theta)(1 - 2e^{\theta})}{\theta^2 sin(\theta)},$$

$$\beta = \frac{2e^{\theta} sin(\theta) - \theta e^{\theta} - \theta\{sin(\theta) + cos(\theta)\}}{\theta^2 sin(\theta)},$$

$$\gamma = \frac{\theta - sin(\theta)}{\theta^2 sin(\theta)}$$

and $M_i = S^{(2)}(x_i) = f(x, u)$, by discretizing the considered DE (1.1) at the nodal point x_i. As $k \to 0$, $\alpha = 1/6, \beta = 4/6$ and $\gamma = 1/6$, our scheme (2.2) reduces to ordinary cubic spline scheme [5] and then, it is evidently second order convergent.

Accordingly, Eq.(2.2) provides a system of $N - 1$ non-linear algebraic equa tions in the $N - 1$ unknowns $S_i, i = 1, 2, ..., N - 1$, which by discretizing can be written as

$$\left(S_{i-1} - \alpha h^2 f(x_{i-1}, S_{i-1})\right) - \left(2S_i + \beta h^2 f(x_i, S_i)\right) + \left(S_{i+1} - \gamma h^2 f(x_{i+1}, S_{i+1})\right) + t_i = 0. \tag{2.3}$$

Then, the local truncation error t_i, $i = 1, 2, ..., N - 1$, can be written as

$$\begin{aligned} t_i = &\{1 - (\alpha + \beta + \gamma)\} h^2 u_i^{(2)} + (\alpha - \gamma) h^3 u_i^{(3)} + \left\{\frac{1}{12} - \frac{1}{2}(\alpha + \gamma)\right\} h^4 u_i^{(4)} \\ &+ \frac{1}{6}(\alpha - \gamma) h^5 u_i^{(5)} + \left\{\frac{1}{360} - \frac{1}{24}(\alpha + \gamma)\right\} h^6 u_i^{(6)} + O(h^7). \end{aligned} \tag{2.4}$$

Thus, our schemes (2.2) and (2.4) give rise to a family of methods of different orders as follows:

2.1. Different order of methods

Case (i): First order method

For $\alpha + \beta + \gamma = 1, \alpha \neq \gamma$. Here,

$$\begin{aligned} t_i &= (\alpha - \gamma) h^3 u_i^{(3)} + O(h^4), \\ \| T \| &= |(\alpha - \gamma)| h^3 M_3, \ M_3 = max|u^{(3)}(x)|. \end{aligned} \tag{2.5}$$

Case (ii): Second order method

For $\alpha + \beta + \gamma = 1, \alpha = \gamma$ and $\alpha + \gamma \neq \frac{1}{6}$. Here,

$$t_i = \left\{\frac{1}{12} - \frac{1}{2}(\alpha + \gamma)\right\} h^4 u_i^{(4)} + O(h^5),$$

$$\| T \|= \left| \frac{1}{12} - \frac{1}{2}(\alpha+\gamma) \right| h^4 M_4, \ M_4 = max|u^{(4)}(x)|. \tag{2.6}$$

Case (iii): Fourth order method

For $\alpha+\beta+\gamma=1, \alpha=\gamma$ and $\alpha+\gamma=\frac{1}{6}$. Here,

$$t_i = \left\{ \frac{1}{360} - \frac{1}{24}(\alpha+\gamma) \right\} h^6 u_i^{(6)} + O(h^7),$$

$$\| T \|= \left| \frac{1}{360} - \frac{1}{24}(\alpha+\gamma) \right| h^6 M_6, \ M_6 = max|u^{(6)}(x)|. \tag{2.7}$$

where $\| \, . \, \|$ represents the ∞ norm in matrix vector.

2.2. Modified k-dependent method

In this section, we will use the parameter k to raise the order of accuracy of the obtained scheme (2.2). To do this, we first rearrange the terms in Eq. (2.4) in the following manner

$$\begin{aligned}
t_i =& h^4 \left[\frac{1}{\theta^2} + \frac{(e^\theta-1)(1-\cos(\theta)) + \sin(\theta)(1+e^\theta)}{\theta^3 \sin(\theta)} \right] (k^2 u_i^{(2)} - u_i^{(4)}) \\
&+ h^5 \left[\frac{e^\theta(\sin(\theta)+\cos(\theta)) - 1}{\theta^3 \sin(\theta)} + \frac{2(1-e^\theta)}{\theta^4} \right] k^2 u_i^{(3)} \\
&+ h^6 \left[\frac{1}{12\theta^2} - \frac{1+e^\theta(\sin(\theta)+\cos(\theta))}{2\theta^3 \sin(\theta)} + \frac{(1+e^\theta)}{\theta^4} \right] k^2 u_i^{(4)} \\
&+ h^6 \left[\frac{(\sin(\theta)+\cos(\theta)) + 1 + e^\theta(\sin(\theta)-\cos(\theta)-1)}{\theta^5 \sin(\theta)} \right] k^2 u_i^{(4)} \\
&+ h^6 \left[\left\{ \frac{1}{360} + \frac{-e^\theta(\sin(\theta)+\cos(\theta))}{24\theta \sin(\theta)} + \frac{(2e^\theta-1)}{24\theta^2} \right\} u_i^{(6)}(\eta_1) + \left\{ \frac{1}{24\theta^2} - \frac{1}{24\theta \sin(\theta)} \right\} u_i^{(6)}(\eta_2) \right] \\
&+ h^7 \left[\frac{e^\theta(\sin(\theta)+\cos(\theta)-1)}{6\theta^3 \sin(\theta)} + \frac{(1-e^\theta)}{3\theta^4} \right] k^2 u_i^{(5)} + \cdots
\end{aligned}$$

Equating the coefficient of the leading term in the above equation to zero, we can get the equation in k_i as

$$k_i^2 = \frac{u_i^{(4)}}{u_i^{(2)}} = \frac{f''(x_i,u_i)}{f(x_i,u_i)} \tag{2.8}$$

For the linear case, $f(x_i,u_i) = p_i u_i + g_i$. Then,

$$k_i^2 = \frac{(p_i'' + p_i^2)u_i + 2p_i' u_i' + p_i g_i + g_i''}{p_i u_i + g_i} \tag{2.9}$$

Thus, from above we see that calculation of k_i requires the approximations for u_i and u_i'. Approximation for u_i can be obtained by means of our developed scheme (2.2) for $k=0$ and for u_i', following steps can be adapted:

(i) Differentiating Eq. (2.1) at $x = x_i$, to get

$$P_i'(x) = \frac{1}{ksin(\theta)} \left\{ (sin(\theta)+e^\theta)M_i - M_{i+1} \right\} + \frac{(S_{i+1}-S_i)}{h} + \frac{1}{k^2 h} \left\{ (1-2e^\theta)M_i + M_{i+1} \right\},$$

(ii) If the limit k going to zero in the above Eq., we obtain

$$P_i'(x) = -\frac{h}{6}f(x_{i+1}, u_{i+1}) - \frac{h}{3}f(x_i, u_i) + \frac{(S_{i+1} - S_i)}{h};\ i = 0, 1, ..., N. \tag{2.10}$$

3. Composite non-polynomial spline solution

To develop the approximation to the solution of BVP (1.1)-(1.2) based on our developed spline method, we write our scheme (2.2) in the following standard matrix form:

$$A_0 S^{(1)} - h^2 B f^{(1)}\left(S^{(1)}\right) = C^{(1)}, \tag{3.1}$$

where A_0 and B are three-band square matrices of order $N-1$, given by

$$A_0 = \begin{bmatrix} -2 & 1 & & & & & & & & \\ 1 & -2 & 1 & & & & & & & \\ & 1 & -2 & 1 & & & & & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & & & & 1 & -2 & 1 \\ & & & & & & & & 1 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} \beta & \gamma & & & & & & & & \\ \alpha & \beta & \gamma & & & & & & & \\ & \alpha & \beta & \gamma & & & & & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & & & & \alpha & \beta & \gamma \\ & & & & & & & & \alpha & \beta \end{bmatrix}$$

Matrix: $f^{(1)}(S^{(1)}) = f(x_i, S_i^{(1)})$, $S^{(1)} = [S_1, S_2, ..., S_{N-1}]^t$ and

$$C^{(1)} = \begin{cases} -A_1 + h^2\alpha f(x_0, A_1), & i = 1, \\ 0, & i = 2, 3, ...N-2, \\ -A_2 + h^2\gamma f(x_N, A_2), & N-1. \end{cases}$$

Likewise,

$$A_0 U^{(1)} - h^2 B f^{(1)}(U^{(1)}) = C^{(1)} + T^{(1)}, \tag{3.2}$$

where the vector $U^{(1)} = u(x_i)$ is the exact solution with truncation error $T^{(1)} = (t_i^{(1)})$, for i =1, 2,...,N-1.

From (3.1) and (3.2), we have

$$[A_0 - h^2 BQ]E^{(1)} = T^{(1)} \tag{3.3}$$

where $E^{(1)} = U^{(1)} - S^{(1)} = [e_1^{(1)}, e_2^{(1)}, ..., e_{N-1}^{(1)}]^T$ and $Q = diag\left(\frac{\partial f_i^{(1)}}{\partial u_i^{(1)}}\right)$, i = 1, 2,....,N-1 is the diagonal matrix of order $N-1$, whereas for the linear case, Q $=diag(f_i^{(1)})$.

Thus, the Eqs. (3.1) - (3.3) demonstrate our scheme, using which one can obtain the approximate solution of non-linear DE (1.1) with the BC (1.2). We shall use Newton's method to obtain the solution of the non-linear system (2.2), which converge to the solution of (1.1)-(1.2) for all sufficiently small values of h [24, 46].

4. Convergence Analysis

Now, we will derive a bound on $\left\|E^{(1)}\right\|$. From Eq. (3.3), we get

$$AE^{(1)} = T^{(1)},$$

where, A $=[A_0 - h^2BQ]$ is a tri-diagonal matrix. The elements of A are given by

$$a_{ij} = \begin{cases} -2 - h^2\beta f_u(x_i, u_i), & i = j, \\ 1 - h^2\alpha f_u(x_i, u_i), & i - j = 1, \\ 1 - h^2\gamma f_u(x_i, u_i), & j - i = 1, \\ 0, & |i - j| > 1. \end{cases}$$

From above, we have

$$\left\|E^{(1)}\right\| \leq \left\|A^{-1}\right\| \left\|T^{(1)}\right\|.$$

(See [24]) $\left\|A^{-1}\right\| \leq (b-a)^2/8h^2$ and so, we can infer the following convergent schemes:

Case 4.1 : First order convergent method

For (α, β, γ) =(75/1920,1755/1920,90/1920), $\left\|T^{(1)}\right\|_\infty = \frac{1}{128}h^3M_3$.

Then from Eq. (2.5), we get

$$\left\|E^{(1)}\right\| \leq K_1 h \cong O(h^1). \tag{4.1}$$

This relation (4.1) shows that the method is first order convergent.

Case 4.2 : Second order convergent method

For $\alpha = \gamma = \frac{3}{38}$ and $\beta = \frac{32}{38}$, $\left\|T^{(1)}\right\|_\infty = \frac{1}{128}h^4M_4$.

Then it follows from (2.6) that

$$\left\|E^{(1)}\right\| \leq K_2 h^2 \cong O(h^2). \tag{4.2}$$

The relation (4.2) confirms second order convergence of the method.

Case 4.3 : Fourth order convergent method

For $\alpha = \gamma = \frac{1}{12}$ and $\beta = \frac{10}{12}$, $\left\|T^{(1)}\right\|_\infty = \frac{1}{240}h^6M_6$.

Then from Eq. (2.7), we have

$$\left\|E^{(1)}\right\| \leq K_3 h^4 \cong O(h^4). \tag{4.3}$$

which confirms fourth order convergence of the method.

5. Numerical Illustration

To illuminate the use of our developed methods, we have considered several linear and non-linear examples of second order BVPs and also compared our results with other existing methods.

Problem 5.1. Consider the linear BVP

$$u^{(2)}(x) = \frac{2}{x^2}u - \frac{1}{x}; \quad 2 < x < 3; \qquad u(2) = u(3) = 0. \tag{5.1}$$

The theoretical (exact) solution of (5.1) is

$$u(x) = \frac{1}{38}(-5x^2 + 19x - \frac{36}{x}). \tag{5.2}$$

Comparing the given Eq. (5.1) with (1.1) at $x = x_i$, we have

$$f(x_i, u_i) = \frac{2}{x_i^2}u_i - \frac{1}{x_i}.$$

Table 1. Absolute error for the solution of Problem 5.1 at different value of x for $N = 8$.

x	Our method for $k = 0$	Our k-based method	Value of k
17/8	2.36×10^{-5}	4.28×10^{-6}	1.0674
18/8	3.66×10^{-5}	6.31×10^{-6}	0.9581
19/8	4.16×10^{-5}	6.86×10^{-6}	0.8623
20/8	4.07×10^{-5}	6.45×10^{-6}	0.7781
21/8	3.52×10^{-5}	5.38×10^{-6}	0.7040
22/8	2.61×10^{-5}	3.87×10^{-6}	0.6387
23/8	1.42×10^{-5}	2.05×10^{-6}	0.5809

For the linear case, $f(x, u) = p(x)u + g(x)$, so $p_i = p(x_i) = 2/x_i^2; g_i = g(x_i) = -1/x_i$ and Eq.(3.1) is changed to $AS = C$, where $A = A_0 - h^2BQ$; $Q = diag(f_i)$. By substituting these values, we get system of linear Eqs. for Problem 5.1 that can be solved using any suitable method. Absolute errors at different point of x are summarized in Table 1 for $k = 0$, i.e. $(\alpha, \beta, \gamma) = (1/6,\ 4/6,\ 1/6)$ and k-based method, when $h = 1/8$. Results indicate that the modified k-dependent method provides better results than the method for $k = 0$. The value of parameter k at different value of x is also listed in Table 1(col. IV).

Table 2 reports the MAE at different value of N for second order schemes together with k-based technique. Table indicates that k-based method is a third order convergent method. Comparison of numerical results with other existing methods is also included in this table. Fourth order method solution when (α, β, γ) = (1/12, 10/12, 1/12) of Problem 5.1 for N=10 is presented in Table 3, along with comparison with Galerkin method.

TABLE 2. Comparison of maximum absolute errors for Problem 5.1.

Our method	N = 4	N = 8	N = 16
Our second order methods			
$(\alpha = \gamma = 3/38, \beta = 32/38)$	5.94×10^{-6}	2.00×10^{-6}	5.37×10^{-7}
$(\alpha = \gamma = 1/13, \beta = 11/13)$	9.88×10^{-6}	3.01×10^{-6}	7.90×10^{-7}
Our method for $k = 0$	1.65×10^{-4}	4.16×10^{-5}	1.04×10^{-5}
Our k-based Method	5.05×10^{-5}	6.86×10^{-6}	8.61×10^{-7}
Quadratic spline [9]	1.60×10^{-4}	2.66×10^{-5}	5.58×10^{-6}
Centered Difference method [10]	2.79×10^{-4}	5.42×10^{-5}	1.19×10^{-5}
Quadratic spline [42]	7.93×10^{-5}	2.06×10^{-5}	5.20×10^{-6}
Cubic spline [10]	5.49×10^{-5}	1.87×10^{-5}	5.07×10^{-6}
Cubic non-poly. spline [33]	2.05×10^{-5}	5.74×10^{-6}	1.47×10^{-6}
Discrete cubic spline [21]	1.77×10^{-5}	5.00×10^{-6}	1.29×10^{-6}

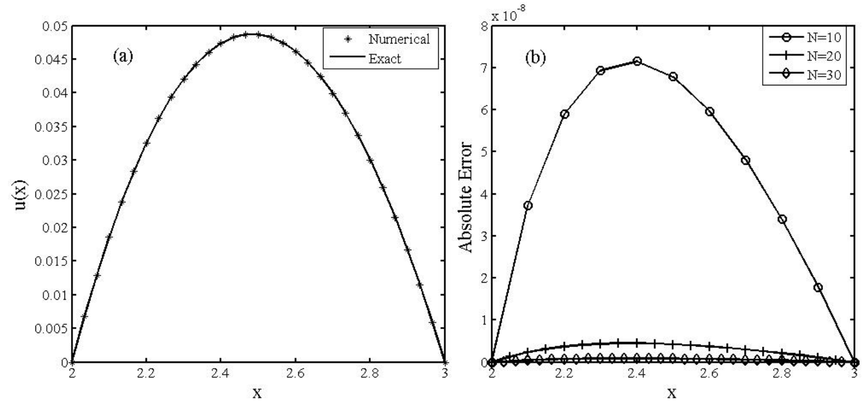

FIGURE 1. [a] Comparison of approximate and exact values for Problem 5.1.
[b] Error graph for Problem 5.1 at different values of N (Table 3).

Problem 5.2. Consider the linear BVP

$$u^{(2)}(x) = 100u; \quad 0 < x < 1; \qquad u(0) = u(1) = 1. \tag{5.3}$$

TABLE 3. Comparison of MAE for the solution of Problem 5.1(Fourth order method)

x	2.1	2.2	2.3	2.4	2.5
Our method	3.73×10^{-8}	5.89×10^{-8}	6.92×10^{-8}	7.15×10^{-8}	6.78×10^{-8}
Galerkin method [25]	2.52×10^{-7}	1.15×10^{-6}	6.73×10^{-7}	6.90×10^{-7}	1.24×10^{-6}
x	2.6	2.7	2.8	2.9	
Our method	5.96×10^{-8}	4.81×10^{-8}	3.39×10^{-8}	1.77×10^{-8}	
Galerkin method [25]	4.51×10^{-7}	7.90×10^{-7}	9.70×10^{-7}	3.17×10^{-7}	

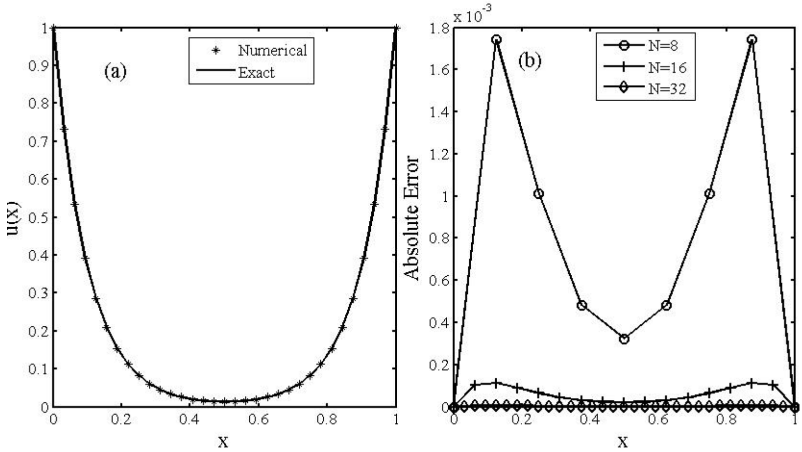

FIGURE 2. [a] Comparison of approximate and exact values for Problem 5.2.
[b] Error graph for Problem 5.2 at different values of N (Table 4).

The theoretical solution of (5.3) is

$$u(x) = \frac{cosh(10x-5)}{cosh\ 5}. \tag{5.4}$$

Problem 5.3. Consider the linear BVP

$$u^{(2)}(x) = u + cos(x), \quad 0 < x < 1; \qquad u(0) = u(1) = 1. \tag{5.5}$$

The theoretical solution of (5.5) is

$$\begin{aligned} u(x) =& \frac{-3cosh(1) + 3sinh(1) + cos(1) + 2}{4sinh(1)} e^{x} \\ &+ \frac{3cosh(1) + 3sinh(1) - cos(1) - 2}{4sinh(1)} e^{-x} - \frac{cos(x)}{2} \end{aligned} \tag{5.6}$$

Maximum absolute errors at the different values of N are tabulated in Table 4 for Problem 5.2 and in Table 5 for Problem 5.3. Fourth order method solution and error graphs at different values of N are also given in Figures 1- 3 respectively for Problems 5.1-5.3.

TABLE 4. Comparison of maximum absolute errors for Problem 5.2

Our method	N = 16	N = 32	N = 20	N = 40
$\alpha=\gamma=3/38, \beta$=32/38	1.95×10^{-4}	7.15×10^{-5}	1.54×10^{-4}	4.75×10^{-5}
$\alpha=\gamma=1/13, \beta$=11/13	3.37×10^{-4}	1.07×10^{-4}	2.47×10^{-4}	7.08×10^{-5}
Our method for $k=0$	6.10×10^{-3}	1.50×10^{-3}	3.90×10^{-3}	9.65×10^{-4}
Our k-based method	1.16×10^{-2}	1.11×10^{-3}	5.40×10^{-3}	5.57×10^{-4}
Our fourth-order method	1.12×10^{-4}	7.28×10^{-6}	4.75×10^{-5}	2.99×10^{-6}
Cubic non-poly. spline [33]	7.22×10^{-4}	2.06×10^{-4}	5.00×10^{-4}	1.34×10^{-4}
Discrete cubic spline [21]	6.18×10^{-4}	1.80×10^{-4}	4.32×10^{-4}	1.17×10^{-4}
Quadratic spline [42]	3.06×10^{-3}	7.58×10^{-4}	—	—
Collocation method [32]	—	—	1.80×10^{-3}	4.70×10^{-4}
Cubic spline [10]	2.27×10^{-3}	6.84×10^{-4}	1.57×10^{-3}	4.53×10^{-4}

TABLE 5. Comparison of maximum absolute errors for the solution of Problem 5.3

x	Our method for $k=0$	Our k-based method	Our fourth order method	Standard Tau-method [45]	Perturbed Tau-method [45]	EADM [17]	EFM [44]
1/8	5.24×10^{-4}	7.13×10^{-6}	8.97×10^{-8}	1.00×10^{-4}	2.10×10^{-4}	4.37×10^{-7}	6.88×10^{-5}
2/8	9.69×10^{-4}	1.17×10^{-5}	1.50×10^{-7}	0	1.10×10^{-4}	8.07×10^{-7}	4.93×10^{-5}
3/8	1.26×10^{-3}	1.43×10^{-5}	1.84×10^{-7}	1.00×10^{-4}	7.51×10^{-5}	1.05×10^{-6}	3.21×10^{-5}
4/8	1.37×10^{-3}	1.50×10^{-5}	1.93×10^{-7}	1.00×10^{-4}	6.25×10^{-5}	1.14×10^{-6}	2.63×10^{-5}
5/8	1.26×10^{-3}	1.39×10^{-5}	1.79×10^{-7}	2.00×10^{-4}	4.31×10^{-5}	1.05×10^{-6}	2.16×10^{-5}
6/8	9.69×10^{-4}	1.11×10^{-5}	1.42×10^{-7}	2.00×10^{-4}	2.43×10^{-5}	8.07×10^{-7}	1.09×10^{-5}
7/8	5.24×10^{-4}	6.56×10^{-6}	8.32×10^{-8}	2.00×10^{-4}	1.13×10^{-5}	4.37×10^{-7}	1.01×10^{-5}

Abbreviations: EADM: Extended Adomian Decomposition Method; EFM: Exponential fitting method

Problem 5.4. Consider the non-linear BVP

$$u^{(2)}(x) = 2(u(x))^3, \quad -1 < x < 0; \qquad u(-1) = 1/2, u(0) = 1/3. \tag{5.7}$$

The theoretical solution of Eq. (5.7) is

$$u(x) = \frac{1}{(x+3)} \tag{5.8}$$

To solve non-linear BVP (Problem 5.4), compare the Eq. (5.7) with (1.1) at $x = x_i$ and we have

$$f(x_i, u_i) = 2(u(x_i))^3;$$

Using Eq. (3.1), we obtain a system of non-linear Eqs. that have been solved using Newton's method. Results are verified with MATLAB builtin solver(*fsolve*) command. Tables 6 and 7 show the maximum absolute errors, in case of k=0, modified k-dependent method and fourth order method solution. Tables clearly indicate that our developed methods produce the better

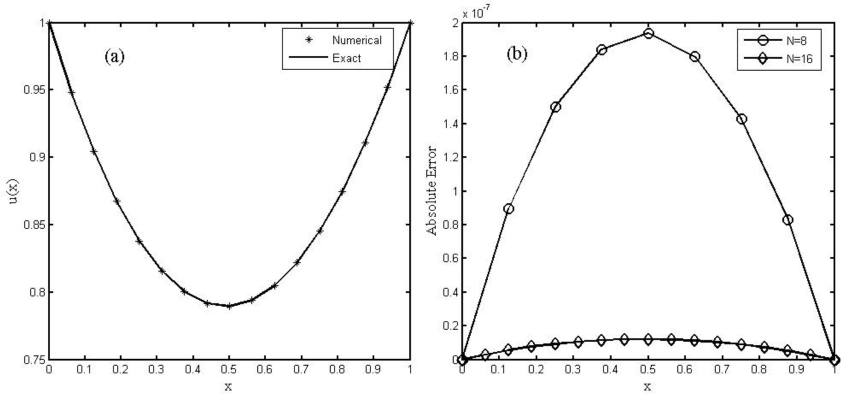

FIGURE 3. [a] Comparison of approximate and exact values for Problem 5.3.
[b] Error graph for Problem 5.3 at different values of N (Table 5).

accuracy than some other specified methods. We have also listed the value of parameter k at different value of x in Table 8.

TABLE 6. Comparison of MAE at N=10 for the solution of Problem 5.4.

Our method for $k = 0$	Our k-based method	Our fourth order method	Quintic spline [7]	Cubic spline[20]	Quartic spline [6]
2.65×10^{-5}	8.08×10^{-6}	3.23×10^{-7}	8.82×10^{-6}	1.68×10^{-5}	4.67×10^{-6}

TABLE 7. Maximum absolute errors at different value of N for Problem 5.4

Our method	N = 4	N = 8	N = 16
Our method for $k = 0$	1.63×10^{-4}	4.13×10^{-5}	1.03×10^{-5}
Our k-based method	1.28×10^{-4}	1.53×10^{-5}	6.83×10^{-6}
Our fourth-order method	2.56×10^{-6}	1.64×10^{-7}	1.08×10^{-8}

TABLE 8. The value of k at different value of x for the solution of Problem 5.4

x	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1
k	1.6499	1.5748	1.5062	1.4433	1.3855	1.3321	1.2827	1.2368	1.1941

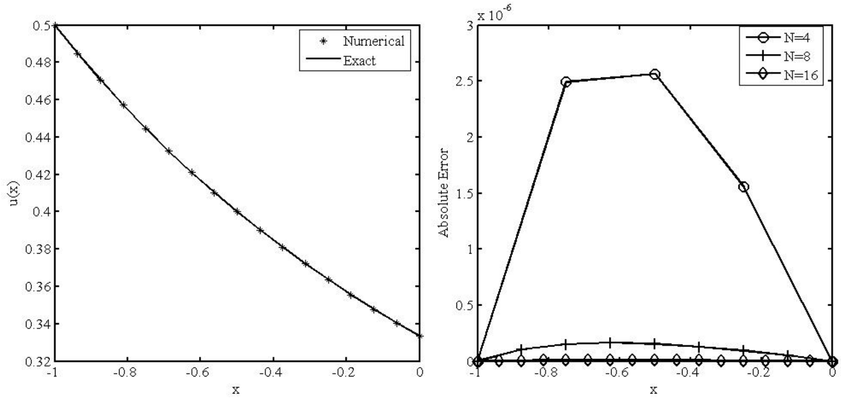

FIGURE 4. [a] Comparison of approximate values and exact values for Problem 5.4.
[b] Error graph for Problem 5.4 at different values of N (Table 7).

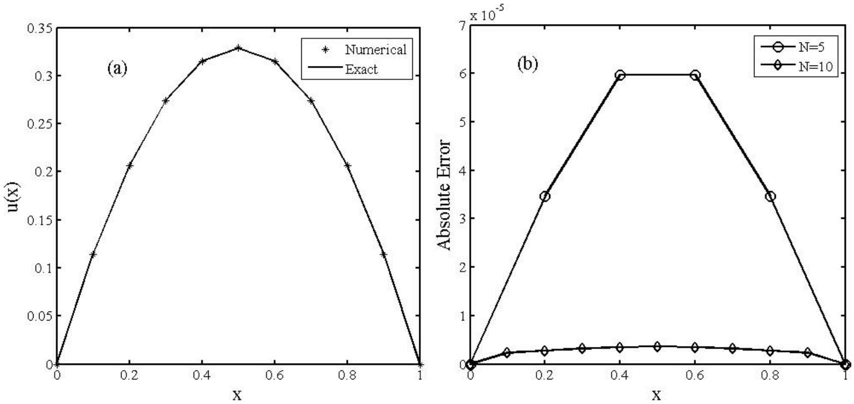

FIGURE 5. [a] Comparison of approximate values and exact values for Problem 5.5.
[b] Error graph for Problem 5.5 at different values of N (Table 9, col. III)

Problem 5.5. Consider the non-linear BVP (Bratu Problem)

$$u^{(2)}(x) + 2e^{u(x)} = 0, \quad 0 < x < 1; \qquad u(0) = u(1) = 0. \tag{5.9}$$

The theoretical solution of (5.9) is

$$u(x) = -2\ln(cosh(1.17878\ (x - 0.5)))/cosh\ (0.589388). \tag{5.10}$$

Table 9. Comparison of MAE for the solution of Problem 5.5 at N=10.

Our method for $k=0$	Our k-based method	Our fourth order method	LGSM [1]	Quintic spline [7]
8.83×10^{-4}	3.56×10^{-5}	3.64×10^{-6}	5.7×10^{-6}	6.22×10^{-6}
B-Spline method [18]	Quartic spline method [6]	Cubic spline[20]	LADM [35]	ADM [22]
5.29×10^{-5}	1.10×10^{-4}	6.26×10^{-4}	1.24×10^{-2}	1.52×10^{-2}

Abbreviations: ADM: Adomian Decomposition Method; LGSM: Lie-group shooting method; LADM: Laplace Adomian Decomposition Method.

Problem 5.6. Consider the non-linear BVP

$$u^{(2)}(x) = \frac{1}{2}(1+x+u)^3, \quad 0<x<1; \qquad u(0)=u(1)=0. \tag{5.11}$$

The theoretical solution of (5.11) is

$$u(x) = \frac{2}{(2-x)} - x - 1. \tag{5.12}$$

The other non-linear BVPs mentioned in Problems 5.5 and 5.6, are also solved just like Problem 5.4 using Newtons method. Obtained results show the efficiency and accuracy of our proposed methods. Maximum absolute errors at the nodal points with a comparison with other methods are summarized in Table 9 for Problem 5.5 and in Table 10 for Problem 5.6, respectively. Figures 4-6 demonstrate the fourth order method solution and error graphs for nonlinear Problems 5.4-5.6 respectively with comparison of errors at the nodal points.

Table 10. Comparison of MAE for Problem 5.6 with Approaching spline method at $N=5$.

x *values*	0	0.2	0.4	0.6	0.8	1
Our method for $k=0$	0	1.30×10^{-3}	2.40×10^{-3}	3.10×10^{-3}	2.80×10^{-3}	0
Our k-based method	0	2.70×10^{-5}	5.25×10^{-5}	7.19×10^{-5}	6.49×10^{-5}	0
Our fourth order method	0	3.80×10^{-5}	7.26×10^{-5}	9.92×10^{-5}	9.96×10^{-5}	0
Approaching spline [31]	0	1.40×10^{-4}	2.60×10^{-4}	3.20×10^{-4}	2.70×10^{-4}	0

6. Conclusion

A unique approach based on a different combination of non-polynomial cubic splines is used to develop various orders methods for solving linear and non-linear second order BVPs. We have also developed a parameter k-based method for smooth approximation of these BVPs. The convergence of the developed method is also established. Competence of the demonstrated technique can also be weighed through comparisons with the literature given in tables, which show that our results are comparatively better with more

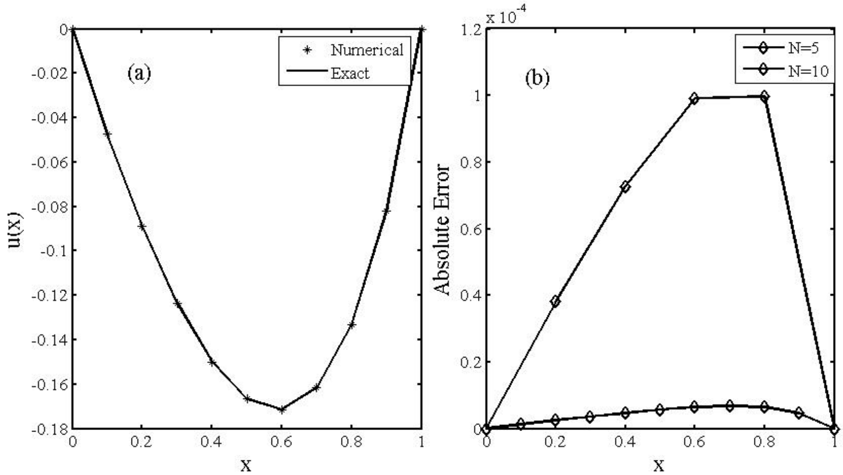

FIGURE 6. [a] Comparison of approximate values and exact values for Problem 5.6.
[b] Error graph for Problem 5.6 at different values of N (Table 10).

precise result. Graphs are plotted at different values of N for all the problems, which clearly show that absolute errors decrease rapidly as step size N increases.

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