Stud. Univ. Babeş-Bolyai Math. 62(2017), No. 2, 163–173
DOI: 10.24193/subbmath.2017.2.03

An inequality of Ostrowski-Grüss type for double integrals

Hüseyin Budak and Mehmet Zeki Sarıkaya

Abstract. In this study, we establish Ostrowski-Grüss type involving functions of two independent variables for double integrals. Cubature formula is also provided.

Mathematics Subject Classification (2010): 26D15.

Keywords: Ostrowski-Grüss type inequality, double integrals, two independent variables.

1. Introduction

In 1935, G. Grüss [7] proved the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx\right| \leq \frac{1}{4}(\Phi_1-\varphi_1)(\Phi_2-\varphi_2), \tag{1.1}$$

provided that f and g are two integrable function on $[a,b]$ satisfying the condition

$$\varphi_1 \leq f(x) \leq \Phi_1 \text{ and } \varphi_2 \leq g(x) \leq \Phi_2 \text{ for all } x \in [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible.

In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives [9]:

Theorem 1.1 (Ostrowski inequality). *Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) whose derivative $f' : (a,b) \to R$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, we have the inequality*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \tag{1.3}$$

for all $x \in [a,b]$. *The constant* $\frac{1}{4}$ *is the best possible.*

In 1882, P. L. Čebyšev [2] gave the following inequality:

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty , \tag{1.4}$$

where $f,g : [a,b] \to \mathbb{R}$ are absolutely continuous function, whose first derivatives f' and g' are bounded,

$$\begin{aligned} & T(f,g) & (1.5) \\ = & \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \end{aligned}$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|p\|_\infty = ess\sup_{t\in[a,b]} |p(t)|$.

The following result of Grüss type was proved by Dragomir and Fedotov [4]:

Theorem 1.2. *Let* $f,u : [a,b] \to \mathbb{R}$ *be such that* u *is L-Lipshitzian on* $[a,b]$, *i.e,*

$$|u(x) - u(y)| \leq L|x-y| \quad \text{for all } x \in [a,b], \tag{1.6}$$

f *is Riemann integrable on* $[a,b]$ *and there exist the real numbers* m, M *so that*

$$m \leq f(x) \leq M \quad \text{for all } x \in [a,b]. \tag{1.7}$$

Then we have the inequality,

$$\left|\int_a^b f(x)du(x) - \frac{u(b)-u(a)}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2}L(M-m)(b-a).$$

From [8], if $f : [a,b] \to \mathbb{R}$ is differentiable on (a,b) with the first derivative f' integrable on $[a,b]$, then Montgomery identity holds:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \int_a^b P(x,t)f'(t)dt, \tag{1.8}$$

where $P(x,t)$ is the Peano kernel defined by

$$P(x,t) = \begin{cases} \frac{t-a}{b-a}, & a \leq t \leq x \\ \frac{t-b}{b-a}, & x < t \leq b. \end{cases}$$

In [5], Dragomir and Wang proved following Ostrowski-Grüss type inequality using the inequality (1.1) and Montgomery identity (1.8):

Theorem 1.3. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differantiable mapping in* I° *and let* $a,b \in I^\circ$ *with* $a < b$. *If* $f \in L_1[a,b]$ *and*

$$\varphi_3 \leq f'(x) \leq \Phi_3, \ \forall x \in [a,b],$$

then we have the following inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right| \le \frac{1}{4}(b-a)(\Phi_3-\varphi_3), \tag{1.9}$$

for all $x\in[a,b]$.

Barnett and Dragomir established following Ostrowski inequality for double integrals in [1]:

Theorem 1.4. *Let* $f:[a,b]\times[c,d]\to\mathbb{R}$ *be a continuous on* $[a,b]\times[c,d]$, $f_{xy}=\frac{\partial^2 f}{\partial x\partial y}$ *exists on* $(a,b)\times(c,d)$, *and is bounded, i.e.,*

$$\|f_{xy}\|_\infty=\sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right|<\infty$$

then we have the inequality

$$\begin{aligned}&\left|\int_a^b\int_c^d f(t,s)dsdt-\left[(b-a)\int_c^d f(x,s)ds\right.\right. \\ &\left.\left.+(d-c)\int_a^b f(t,y)dt-(b-a)(d-c)f(x,y)\right]\right| \\ \le&\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\left[\frac{1}{4}(d-c)^2+\left(y-\frac{c+d}{2}\right)^2\right]\|f_{xy}\|_\infty\end{aligned} \tag{1.10}$$

for all $(x,y)\in[a,b]\times[c,d]$.

In [1], the inequality (1.10) is established by the use of integral identity involving Peano kernels. In [10], Pachpatte obtained an inequality in the view (1.10) by using elementary analysis. The interested reader is also refered to ([1], [6], [10],[11],[13]-[15]) for Ostrowski type inequalities in several independent variables.

Recently, Sarikaya and Kiris have proved the following Grüss type inequality for double integrals in [12]:

Theorem 1.5. *Let* $f,g:[a,b]\times[c,d]\to\mathbb{R}$ *be two functions defined and integrable on* $[a,b]\times[c,d]$. *Then for*

$$\varphi\le f(x,y)\le\Phi \text{ and } \gamma\le g(x,y)\le\Gamma. \text{ for all } (x,y)\in[a,b]\times[c,d]$$

we have

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx\right. \tag{1.11}$$

$$\left.-\left(\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right)\left(\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(x,y)dydx\right)\right|$$

$$\leq \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma).$$

Moreover, Cerone and Dragomir [3] extended Gruss type inequalities for Lebesgue integrals on measurable spaces. This includes domaind from the plane provided in [12].

In this work, using the inequality (1.11), we will obtain an Ostrowski-Grüss type inequality for functions of two independent variables.

2. Main results

First, we give the following notations to simplify the presentation of some intervals.

$$\begin{aligned}\Delta_1 &= [a,x]\times[c,y],\ \Delta_2=[a,x]\times[y,d],\\ \Delta_3 &= [x,b]\times[c,y],\ \Delta_4=[x,b]\times[y,d].\end{aligned}$$

Theorem 2.1. *Let $f:\Delta:[a,b]\times[c,d]\to\mathbb{R}$ be a continuous on Δ, $f_{xy}=\frac{\partial^2 f}{\partial x\partial y}$ exists on Δ°. If f integrable and*

$$\varphi\leq f_{xy}(x,y)\leq\Phi,\ \forall(x,y)\in\Delta$$

then we have the following inequality

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt-\left[\frac{1}{(d-c)}\int_c^d f(x,s)ds\right.\right. \tag{2.1}$$

$$\left.+\frac{1}{(b-a)}\int_a^b f(t,y)dt-f(x,y)\right]$$

$$\left.-\frac{f(b,d)-f(b,c)-f(a,d)+f(a,c)}{(b-a)(d-c)}\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)\right|$$

$$\leq \frac{1}{4}(P-p)(\Phi-\varphi)$$

where

$$P=\max\{(x-a)(y-c),(b-x)(d-y)\}$$

and

$$p = \min\left\{(x-a)(y-d),(x-b)(y-c)\right\}$$

for all $(x,y)\in\Delta$.

Proof. Define the kernel $p(x,t;y,s)$ by

$$p(x,t;y,s):=\begin{cases}(t-a)(s-c), & \text{if } (t,s)\in[a,x]\times[c,y]\\ (t-a)(s-d), & \text{if } (t,s)\in[a,x]\times(y,d]\\ (t-b)(s-c), & \text{if } (t,s)\in(x,b]\times[c,y]\\ (t-b)(s-d), & \text{if } (t,s)\in(x,b]\times(y,d].\end{cases}$$

Then, we have

$$\int_a^b\int_c^d p(x,t;y,s)f_{ts}(t,s)dsdt \tag{2.2}$$

$$= \int_a^x\int_c^y(t-a)(s-c)f_{ts}(t,s)dsdt+\int_a^x\int_y^d(t-a)(s-d)f_{ts}(t,s)dsdt$$

$$+\int_x^b\int_c^y(t-b)(s-c)f_{ts}(t,s)dsdt+\int_x^b\int_y^d(t-b)(s-d)f_{ts}(t,s)dsdt$$

$$= I_1+I_2+I_3+I_4.$$

Let us calculate the integrals I_1, I_2, I_3 and I_4. Firstly, we have the equality

$$I_1=\int_a^x\int_c^y(t-a)(s-c)f_{ts}(t,s)dsdt \tag{2.3}$$

$$=\int_a^x(t-a)\left[(y-c)f_t(t,y)-\int_c^y f_t(t,s)ds\right]dt$$

$$=(y-c)\int_a^x(t-a)f_t(t,y)dt-\int_c^y\left(\int_a^x(t-a)f_t(t,s)dt\right)ds$$

$$=(y-c)\left[(x-a)f(x,y)-\int_a^x f(t,y)dt\right]-\int_c^y\left[(x-a)f(x,s)-\int_a^x f(t,s)dt\right]ds$$

$$=(x-a)(y-c)f(x,y)-(y-c)\int_a^x f(t,y)dt-(x-a)\int_c^y f(x,s)ds+\int_a^x\int_c^y f(t,s)dsdt.$$

Also, similar computations we have the equalities

$$I_2 = \int_a^x \int_y^d (t-a)(s-d) f_{ts}(t,s)dsdt \tag{2.4}$$

$$= (x-a)(d-y)f(x,y) - (d-y)\int_a^x f(t,y)dt - (x-a)\int_y^d f(x,s)ds + \int_a^x \int_y^d f(t,s)dsdt,$$

$$I_3 = \int_x^b \int_c^y (t-b)(s-c) f_{ts}(t,s)dsdt \tag{2.5}$$

$$= (b-x)(y-c)f(x,y) - (y-c)\int_x^b f(t,y)dt - (b-x)\int_c^y f(x,s)ds + \int_x^b \int_c^y f(t,s)dsdt,$$

and

$$I_4 = \int_x^b \int_y^d (t-b)(s-d) f_{ts}(t,s)dsdt \tag{2.6}$$

$$= (b-x)(d-y)f(x,y) - (d-y)\int_x^b f(t,y)dt - (b-x)\int_y^d f(x,s)ds + \int_x^b \int_y^d f(t,s)dsdt.$$

If we substitute the equalities (2.3)-(2.6) in (2.2), then we have

$$\int_a^b \int_c^d p(x,t;y,s) f_{ts}(t,s)dsdt \tag{2.7}$$

$$= (b-a)\,(d-c)\,f(x,y) - (b-a)\int_c^d f(x,s)ds - (d-c)\int_a^b f(t,y)dt + \int_a^b \int_c^d f(t,s)dsdt.$$

Applying Theorem 1.5 to mappings $p(x,.;y,.)$ and $f_{ts}(.,.)$, we establish

$$\begin{aligned} &\Bigg| \frac{1}{(b-a)\,(d-c)} \int_a^b \int_c^d p(x,t;y,s) f_{ts}(t,s)dsdt \\ &- \left(\frac{1}{(b-a)\,(d-c)} \int_a^b \int_c^d p(x,t;y,s)dsdt \right) \\ &\times \left(\frac{1}{(b-a)\,(d-c)} \int_a^b \int_c^d f_{ts}(t,s)dsdt \right) \Bigg| \\ \leq\ & \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma). \end{aligned} \tag{2.8}$$

where

$$\begin{aligned}\Gamma &= \sup_{(t,s)\in\Delta} p(x,t;y,s) \qquad (2.9)\\ &= \max\left\{\sup_{(t,s)\in\Delta_1}(t-a)(s-c), \sup_{(t,s)\in\Delta_2}(t-a)(s-d),\right.\\ &\left.\sup_{(t,s)\in\Delta_3}(t-b)(s-c), \sup_{(t,s)\in\Delta_4}(t-b)(s-d)\right\}\\ &= \max\{(x-a)(y-c),(b-x)(d-y)\} = P,\end{aligned}$$

and

$$\begin{aligned}\gamma &= \inf_{(t,s)\in\Delta} p(x,t;y,s) \qquad (2.10)\\ &= \min\left\{\inf_{(t,s)\in\Delta_1}(t-a)(s-c), \inf_{(t,s)\in\Delta_2}(t-a)(s-d),\right.\\ &\left.\inf_{(t,s)\in\Delta_3}(t-b)(s-c), \inf_{(t,s)\in\Delta_4}(t-b)(s-d)\right\}\\ &= \min\{(x-a)(y-d),(x-b)(y-c)\} = p.\end{aligned}$$

Also, we have the equalities

$$\begin{aligned}&\int_a^b\int_c^d p(x,t;y,s)dsdt \qquad (2.11)\\ &= \int_a^x\int_c^y (t-a)(s-c)dsdt + \int_a^x\int_y^d (t-a)(s-d)dsdt\\ &+\int_x^b\int_c^y (t-b)(s-c)dsdt + \int_x^b\int_y^d (t-b)(s-d)dsdt\\ &= \frac{(x-a)^2(y-c)^2}{4} - \frac{(x-a)^2(d-y)^2}{4}\\ &-\frac{(b-x)^2(y-c)^2}{4} + \frac{(b-x)^2(d-y)^2}{4}\\ &= \frac{\left[(x-a)^2-(b-x)^2\right]\left[(y-c)^2-(d-y)^2\right]}{4}\\ &= (b-a)(d-c)\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)\end{aligned}$$

and

$$\int_a^b \int_c^d f_{ts}(t,s)dsdt = f(b,d) - f(b,c) - f(a,d) + f(a,c). \tag{2.12}$$

If we put the equalities (2.7) and (2.9)-(2.12) in (2.8), then we obtain the desired inequality (2.1). □

Corollary 2.2. *With the assumptions in Theorem* 2.1, *if* $|f_{xy}(x,y)| \leq M$ *for all* $(x,y) \in [a,b] \times [c,d]$ *and some positive constant* M, *then we have*

$$\begin{aligned} &\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt \right. \\ &- \left[\frac{1}{(d-c)} \int_c^d f(x,s)ds + \frac{1}{(b-a)} \int_a^b f(t,y)dt - f(x,y) \right] \\ &\left. - \frac{f(b,d) - f(b,c) - f(a,d) + f(a,c)}{(b-a)(d-c)} \left(x - \frac{a+b}{2} \right) \left(y - \frac{c+d}{2} \right) \right| \\ \leq\ & \frac{1}{2}(P-p)M \end{aligned}$$

where

$$P = \max\{(x-a)(y-c), (b-x)(d-y)\}$$

and

$$p = \min\{(x-a)(y-d), (x-b)(y-c)\}$$

for all $(x,y) \in [a,b] \times [c,d]$.

Corollary 2.3. *Under assumptions of Theorem* 2.1 *with* $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$, *we have the following inequality*

$$\begin{aligned} &\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt - \left[\frac{1}{(d-c)} \int_c^d f\left(\frac{a+b}{2}, s \right) ds \right. \right. \\ &\left. \left. + \frac{1}{(b-a)} \int_a^b f\left(t, \frac{c+d}{2} \right) dt - f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right] \right| \\ \leq\ & \frac{1}{8}(b-a)(d-c)(\Phi - \varphi). \end{aligned}$$

Corollary 2.4. *Under assumption of Theorem* 2.1 *with* $x = b$ *and* $y = d$, *we get the inequality*

$$\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt - \left[\frac{1}{(d-c)} \int_c^d f(b,s)ds + \frac{1}{(b-a)} \int_a^b f(t,d)dt - f(b,d) \right] - \frac{f(b,d) - f(b,c) - f(a,d) + f(a,c)}{4} \right|$$
$$\leq \frac{1}{4}(b-a)(d-c)(\Phi - \varphi).$$

3. Applications for cubature formulae

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < ... < x_n = b$, and $J_m : c = y_0 < y_1 < ... < y_m = d$, $h_i := x_{i+1} - x_i$ $(i = 0, ..., n-1)$, and $l_j := y_{j+1} - y_j$ $(j = 0, ..., m-1)$,

$$\upsilon(h) := \max\{h_i | \ i = 0, ..., n-1\},$$
$$\mu(l) := \max\{l_j | \ j = 0, ..., m-1\}.$$

Then, the following theorem holds.

Theorem 3.1. *Let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be as in Theorem* 2.1 *and* $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$, $\eta_j \in [y_j, y_{j+1}]$ $(j = 0, ..., m-1)$ *be intermediate points. Then we have the cubature formula:*

$$\int_a^b \int_c^d f(t,s)dsdt \tag{3.1}$$
$$= \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i \int_{y_j}^{y_{j+1}} f(\xi_i, s)ds + \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} l_j \int_{x_i}^{x_{i+1}} f(t, \eta_j)dt$$
$$- \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i l_j f(\xi_i, \eta_j)$$
$$+ \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} [f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j)]$$
$$\times \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) \left(\eta_j - \frac{y_j + y_{j+1}}{2} \right)$$
$$+ R(\xi, \eta, I_n, J_m, f).$$

where the remainer term $R(\xi,\eta,I_n,J_m,f)$ satisfies the estimation

$$|R(\xi,\eta,I_n,J_m,f)| \leq \frac{1}{4}\upsilon(h)\mu(l)\max_{i,j}\left(P_{ij}-p_{ij}\right)\left(\Phi-\varphi\right) \tag{3.2}$$

where

$$P_{ij}=\max\left\{\left(\xi_i-x_i\right)\left(\eta_j-y_j\right),\left(x_{i+1}-\xi_i\right)\left(y_{j+1}-\eta_j\right)\right\},$$

and

$$p_{ij}=\min\left\{\left(\xi_i-x_i\right)\left(\eta_j-y_{j+1}\right),\left(\xi_i-x_{i+1}\right)\left(\eta_j-y_j\right)\right\}.$$

Proof. Aplying Theorem 2.1 on the bidimentional interval $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$, we get

$$\begin{aligned}
&\Bigg|\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} f(t,s)dsdt \\
&-\left[h_i\int_{y_j}^{y_{j+1}} f(\xi_i,s)ds+l_j\int_{x_i}^{x_{i+1}} f(t,\eta_j)dt-h_il_jf(\xi_i,\eta_j)\right] \\
&-\left[f\left(x_{i+1},y_{j+1}\right)-f\left(x_{i+1},y_j\right)-f\left(x_i,y_{j+1}\right)+f\left(x_i,y_j\right)\right] \\
&\times\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)\left(\eta_j-\frac{y_j+y_{j+1}}{2}\right)\Bigg| \\
\leq\ &\frac{1}{4}h_il_j\left(P_{ij}-p_{ij}\right)\left(\Phi_{ij}-\varphi_{ij}\right)
\end{aligned} \tag{3.3}$$

where

$$\Phi_{ij}:=\sup_{(t,s)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]}|f_{ts}(t,s)|,\quad \varphi_{ij}:=\inf_{(t,s)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]}|f_{ts}(t,s)|$$

for all $i=0,1,...,n-1$; $j=0,1,...,m-1$.

Summing the inequality (3.3) over i from 0 to $n-1$ and j from 0 to $m-1$ and using the generalized triangle inequality, we get

$$\begin{aligned}
|R(\xi,\eta,I_n,J_m,f)| &\leq \frac{1}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}h_il_j\left(P_{ij}-p_{ij}\right)\left(\Phi_{ij}-\varphi_{ij}\right) \\
&\leq \frac{1}{4}\upsilon(h)\mu(l)\max_{i,j}\left(P_{ij}-p_{ij}\right)\max_{ij}\left(\Phi_{ij}-\varphi_{ij}\right)\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}1 \\
&= \frac{nm}{4}\upsilon(h)\mu(l)\max_{i,j}\left(P_{ij}-p_{ij}\right)\left(\Phi-\varphi\right).
\end{aligned}$$

This completes the proof. □

References

[1] Barnett, N.S., Dragomir, S.S., *An Ostrowski type inequality for double integrals and applications for cubature formulae*, Soochow J. Math., **27**(2001), no. 1, 109-114.

[2] Čebyšev, P.L., *Sur less expressions approximatives des integrales definies par les autres prises entre les memes limites,* Proc. Math. Soc. Charkov, **2**(1882), 93-98.

[3] Cerone, P., Dragomir, S.S., *A refinement of the Grüss inequality and applications*, Tamkang J. Math., **38**(2007), no. 1, 37-49.

[4] Dragomir, S.S., Fedotov, I., *An inequality of Grüss type for Riemann-Stieltjes integral and applications for special means,* Tamkang J. of Math., **29**(1998), no. 4, 287-292.

[5] Dragomir, S.S., Wang, S., *An inequality of Ostrowski-Grüss' type and its applications to the estimation of error bounds for some special means and for some sumerical quadrature rules*, Computers Math. Applic., **33**(1997), no. 11, 15-20.

[6] Dragomir, S.S., Barnett, N.S., Cerone, P., *An n-dimensional version of Ostrowski's inequality for mappings of Hölder type*, RGMIA Res. Pep. Coll., **2**(1999), no. 2, 169-180.

[7] Grüss, G., *Über das maximum des absoluten Betrages von* $\frac{1}{b-a}\int\limits_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int\limits_a^b f(x)dx\int\limits_a^b g(x)dx$, Math. Z., **39**(1935), 215-226.

[8] Mitrinovic, D.S., Pecaric, J.E., Fink, A.M., *Inequalities involving functions and their integrals and derivatives*, Kluwer Academic Publishers, Dordrecht, 1991.

[9] Ostrowski, A.M., *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert*, Comment. Math. Helv., **10**(1938), 226-227.

[10] Pachpatte, B.G., *On a new Ostrowski type inequality in two independent variables*, Tamkang J. Math., **32**(2001), no. 1, 45-49.

[11] Pachpatte, B.G., *A new Ostrowski type inequality for double integrals*, Soochow J. Math., **32**(2006), no. 2, 317-322.

[12] Sarikaya, M.Z., Kiris, M.E., *On Čebysev-Grüss type inequalities for double integrals*, T.J.M.M., **7**(2015), no. 1, 75-83.

[13] Sarikaya, M.Z., *On the Ostrowski type integral inequality*, Acta Math. Univ. Comenianae, **79**(2010), no. 1, 129-134.

[14] Sarikaya, M.Z., *On the Ostrowski type integral inequalityfor double integrals*, Demostratio Mathematica, **45**(2012), no. 3, 533-540.

[15] Ujević, N., *Some double integral inequalities and applications*, Appl. Math. E-Notes, **7**(2007), 93-101.

Hüseyin Budak
Düzce University, Faculty of Science and Arts
Department of Mathematics
Konuralp Campus, Düzce, Turkey
e-mail: `hsyn.budak@gmail.com`

Mehmet Zeki Sarıkaya
Düzce University, Faculty of Science and Arts
Department of Mathematics
Konuralp Campus, Düzce, Turkey
e-mail: `sarikayamz@gmail.com`