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Application of Ruscheweyh q-differential operator to analytic functions of reciprocal order

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Abstract. The core object of this paper is to define and study new class of analytic function using Ruscheweyh q-differential operator. We also investigate a number of useful properties such as inclusion relation, coefficient estimates, subordination result,for this newly subclass of analytic functions.

Keywords: Analytic functions, Subordination, Functions with positive real part, Ruscheweyh q-differential operator, reciprocal order.

1. Introduction

Quantum calculus (q-calculus) is simply the study of classical calculus without the notion of limits. The study of q-calculus attracted the researcher due to its applications in various branches of mathematics and physics, see detail [1]. Jackson [2, 3] was the first to give some application of q-calculus and introduced the q-analogue of derivative and integral. Later on Aral and Gupta [5, 6, 7] defined the q-Baskakov Durrmeyer operator by using q-beta function while the author's in [8, 9, 10] discussed the q-generalization of complex operators known as q-Picard and q-Gauss-Weierstrass singular integral operators. Recently, Kanas and Răducanu [11] defined q-analogue of Ruscheweyh differential operator using the concepts of convolution and then studied some of its properties. The application of this differential operator was further studied by Mohammed and Darus [12] and Mahmood and Sokół [13]. The aim of the current paper is to define a new class of analytic functions of reciprocal order involving q-differetial operator.
Let $\mathcal{A}$ be the class of functions having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{M}(\alpha)$ denote a subclass of $\mathcal{A}$ consisting of functions which satisfy the inequality

$$\mathfrak{Re}\frac{zf'(z)}{f(z)} < \alpha \quad (z \in \mathbb{U}),$$

for some $\alpha\,(\alpha > 1)$. And let $\mathcal{N}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions f which satisfy the inequality:

$$\mathfrak{Re}\frac{(zf'(z))'}{f'(z)} < \alpha \quad (z \in \mathbb{U}),$$

for some $\alpha\,(\alpha > 1)$. These classes were studied by Owa et al. [4, 14]. Shams et al. [15] have introduced the k-uniformly starlike $\mathcal{SD}\,(k,\alpha)$ and k-uniformly convex $\mathcal{CD}\,(k,\alpha)$ of order α, for some $k\,(k \geq 0)$ and $\alpha\,(0 \leq \alpha < 1)$. Using these ideas in above defined classes, Junichi et al. [16] introduced the following classes.

Definition 1.1. *Let $f \in \mathcal{A}$. Then f is said to be in class $\mathcal{MD}\,(k,\alpha)$ if it satisfies*

$$\mathfrak{Re}\frac{zf'(z)}{f(z)} < k\left|\frac{zf'(z)}{f(z)} - 1\right| + \alpha \quad (z \in \mathbb{U}),$$

for some $\alpha\,(\alpha > 1)$ and $k\,(k \leq 0)$.

Definition 1.2. *An analytic function f of the form* (1.1) *belongs to the class $\mathcal{ND}\,(k,\alpha)$, if and only if*

$$\mathfrak{Re}\frac{(zf'(z))'}{f'(z)} < k\left|\frac{(zf'(z))'}{f'(z)} - 1\right| + \alpha \quad (z \in \mathbb{U}),$$

for some $\alpha\,(\alpha > 1)$ and $k\,(k \leq 0)$.

If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w, which is analytic in $\mathbb{U}$ with $w\,(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence holds, see [17, 18].

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \quad \Longleftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For two analytic functions

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad (z \in \mathbb{U}),$$

For $t \in \mathbb{R}$ and $q > 0$, $q \neq 1$, the number $[t,q]$ is defined in [13] as

$$[t,q] = \frac{1-q^t}{1-q}, \quad [0,q] = 0.$$

For any non-negative integer n the q-number shift factorial is defined by

$$[n,q]! = [1,q]\,[2,q]\,[3,q]\cdots[n,q]\,, \quad ([0,q]! = 1)\,.$$

We have $\lim\limits_{q\to 1}[n,q] = n$. Throughout in this paper we will assume q to be fixed number between 0 and 1.

The q-derivative operator or q-difference operator for $f \in \mathcal{A}$ is defined as

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \ z \in \mathbb{U}.$$

It can easily be seen that for $n \in \mathbb{N} := \{1, 2, 3, \dots\}$ and $z \in \mathbb{U}$

$$\partial_q z^n = [n, q] z^{n-1}, \quad \partial_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n, q] a_n z^{n-1}.$$

The q-generalized Pochhammer symbol for $t \in \mathbb{R}$ and $n \in \mathbb{N}$ is defined as

$$[t, q]_n = [t, q] [t+1, q] [t+2, q] \cdots [t+n-1, q],$$

and for $t > 0$, let q-gamma function is defined as

$$\Gamma_q(t+1) = [t, q] \Gamma_q(t) \ \text{and} \ \Gamma_q(1) = 1.$$

Definition 1.3. [?] *For a function $f(z) \in \mathcal{A}$, the Ruscheweyh q-differential operator is defined as*

$$\mathfrak{D}_q^{\mu} f(z) = \phi(q, \mu+1; z) * f(z) = z + \sum_{n=2}^{\infty} \Phi_{n-1} a_n z^n, \quad (z \in \mathbb{U} \ \textit{and} \ \mu > -1), \tag{1.2}$$

where

$$\phi(q, \mu+1; z) = z + \sum_{n=2}^{\infty} \Phi_{n-1} z^n, \tag{1.3}$$

and

$$\Phi_{n-1} = \frac{\Gamma_q(\mu+n)}{[n-1, q]! \Gamma_q(\mu+1)} = \frac{[\mu+1, q]_{n-1}}{[n-1, q]!}. \tag{1.4}$$

From (1.2), it can be seen that

$$L_q^0 f(z) = f(z) \ \text{and} \ L_q^1 f(z) = z \partial_q f(z),$$

and

$$L_q^m f(z) = \frac{z \partial_q^m \left(z^{m-1} f(z) \right)}{[m, q]!}, \quad (m \in \mathbb{N}).$$

$$\lim_{q \to 1^-} \phi(q, \mu+1; z) = \frac{z}{(1-z)^{\mu+1}},$$

and

$$\lim_{q \to 1^-} \mathfrak{D}_q^{\mu} f(z) = f(z) * \frac{z}{(1-z)^{\mu+1}}.$$

This shows that in case of $q \to 1^-$, the Ruscheweyh q-differential operator reduces to the Ruscheweyh differential operator $D^{\delta}(f(z))$ (see [19]). From (1.2) the following identity can easily be derived.

$$z \partial \mathfrak{D}_q^{\mu} f(z) = \left(1 + \frac{[\mu, q]}{q^{\mu}} \right) \mathfrak{D}_q^{\mu} f(z) - \frac{[\mu, q]}{q^{\mu}} \mathfrak{D}_q^{\mu} f(z). \tag{1.5}$$

If $q \to 1^-$, then

$$z \left(\mathfrak{D}_q^{\mu} f(z) \right)' = (1+\mu) \mathfrak{D}_q^{\mu} f(z) - \mu \mathfrak{D}_q^{\mu} f(z).$$

Now using the Ruscheweyh q-differential operator, we define the following class.

Definition 1.4. *Let $f \in \mathcal{A}$. Then f is in the class $\mathcal{KD}_q(k, \alpha, \gamma)$ if*

$$\Re\mathfrak{e}\left\{1 + \frac{1}{\gamma}\left(\frac{z\partial_q \mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)} - 1\right)\right\} < k\left|\frac{1}{\gamma}\left(\frac{z\partial_q \mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)} - 1\right)\right| + \alpha,$$

for some $k\,(k \leq 0)$, $\alpha\,(\alpha > 1)$ and for some $\gamma \in \mathbb{C} \setminus \{0\}$.

We note that $\mathcal{LD}_2^0(1, 1, \alpha) = \mathcal{M}(\alpha)$ and $\mathcal{LD}_1^0(1, 1, \alpha) = \mathcal{N}(\alpha)$, the classes introduced by Owa et al. [4, 14]. When we take $\gamma = 1, 2$, $c = 1$, and $a = 1$ the class $\mathcal{KD}_q(k, \alpha, \gamma)$ reduces to the classes $\mathcal{MD}(k, \alpha)$ and $\mathcal{ND}(k, \alpha)$ (see [16]). For $1 < \alpha < 4/3$ the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ were investigated by Uralegaddi et al. [20].

2. Preliminary Results

Lemma 2.1. [21]*For a positive integer t, we have*

$$\sigma \sum_{j=1}^{t} \frac{(\sigma)_{j-1}}{(j-1)!} = \frac{(\sigma)_t}{(t-1)!}. \tag{2.1}$$

Proof. Consider

$$\begin{aligned}
&\sigma \sum_{j=1}^{t} \frac{(\sigma)_{j-1}}{(j-1)!} \\
&= \sigma\left(1 + \frac{\sigma}{1} + \frac{(\sigma)_2}{2!} + \frac{(\sigma)_3}{3!} + \frac{(\sigma)_4}{4!} + \cdots + \frac{(\sigma)_{t-1}}{(t-1)!}\right) \\
&= \sigma(1+\sigma)\left(1 + \frac{\sigma}{2} + \frac{\sigma(\sigma+2)}{2 \times 3} + \cdots + \frac{\sigma(\sigma+2)\cdots(\sigma+t-2)}{2 \times \cdots \times (t-1)}\right) \\
&= \sigma(1+\sigma)\frac{(\sigma+2)}{2}\left(1 + \frac{\sigma}{3} + \cdots + \frac{\sigma(\sigma+3)\cdots(\sigma+t-2)}{3 \times 4 \times \cdots \times (t-1)}\right) \\
&= \sigma(1+\sigma)\frac{(\sigma+2)}{2}\frac{(\sigma+3)}{3}\left(1 + \frac{\sigma}{4} + \cdots + \frac{\sigma(\sigma+4)\cdots(\sigma+t-2)}{4 \times \cdots \times (t-1)}\right) \\
&= \sigma(1+\sigma)\frac{(\sigma+2)}{2}\frac{(\sigma+3)}{3}\frac{(\sigma+4)}{4}\left(1 + \frac{\sigma}{5} + \cdots + \frac{\sigma\cdots(\sigma+t-2)}{5 \times 6 \times \cdots \times (t-1)}\right) \\
&= \sigma(1+\sigma)\frac{(\sigma+2)}{2}\frac{(\sigma+3)}{3}\frac{(\sigma+4)}{4}\cdots\left(1 + \frac{\sigma}{t-1}\right) \\
&= \sigma(1+\sigma)\frac{(\sigma+2)}{2}\frac{(\sigma+3)}{3}\frac{(\sigma+4)}{4}\cdots\left(\frac{\sigma+(t-1)}{t-1}\right) \\
&= \frac{(\sigma)_t}{(t-1)!}.
\end{aligned}$$

□

3. Main Results

With the help of the definition of $\mathcal{KD}_q(k,\alpha,\gamma)$, we prove the following results.

Theorem 3.1. *If* $f(z)\in\mathcal{KD}_q(k,\alpha,\gamma)$, *then*

$$f(z)\in\mathcal{KD}_q\left(0,\frac{\alpha-k}{1-k},\gamma\right).$$

Proof. Because $k\le 0$, we have

$$\begin{aligned}\mathfrak{Re}\left\{1+\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right)\right\} &< k\left|\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right)\right|+\alpha,\\ &\le k\mathfrak{Re}\left(\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right)\right)+\alpha-k,\end{aligned}$$

which implies that

$$(1-k)\,\mathfrak{Re}\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right)<\alpha-k.$$

After simplification, we obtain

$$\mathfrak{Re}\left[1+\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right)\right]<\frac{\alpha-k}{1-k},(k\le 0,\ \alpha>1\text{ and }). \tag{3.1}$$

This completes the proof. □

Theorem 3.2. *If* $f(z)\in\mathcal{KD}_q(k,\alpha,\gamma)$ *and if* $f(z)$ *has the form* (1.1), *then*

$$|a_n|\le\frac{(\sigma)_{n-1}}{(n-1)!\Phi_{n-1}}, \tag{3.2}$$

where

$$\sigma=\frac{2|\gamma|(\alpha-1)}{q(1-k)}. \tag{3.3}$$

Proof. Let us define a function

$$p(z)=\frac{(\alpha-k)-(1-k)\left[1+\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right)\right]}{\alpha-1}. \tag{3.4}$$

Then $p(z)$ is analytic in $\mathbb{U}$, $p(0)=1$ and $\mathfrak{Re}\{p(z)\}>0$ for $z\in\mathbb{U}$. We can write

$$\left[1+\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right)\right]=\frac{(\alpha-k)-(\alpha-1)p(z)}{1-k} \tag{3.5}$$

If we take $p(z)=1+\sum\limits_{n=1}^{\infty}p_nz^n$, then (3.5) can be written as

$$z\partial_q\mathfrak{D}_q^\mu f(z)-\mathfrak{D}_q^\mu f(z)=-\frac{\gamma(\alpha-1)}{1-k}\left(\mathfrak{D}_q^\mu f(z)\right)\left(\sum_{n=1}^{\infty}p_nz^n\right).$$

this implies that

$$\left[\sum_{n=2}^{\infty}q\,[n-1]\,\Phi_{n-1}a_nz^n\right]=-\frac{\gamma(\alpha-1)}{1-k}\left(\sum_{n=1}^{\infty}\Phi_{n-1}a_nz^n\right)\left(\sum_{n=1}^{\infty}p_nz^n\right).$$

Using Cauchy product $\left(\sum_{n=1}^{\infty} x_n\right) \cdot \left(\sum_{n=1}^{\infty} y_n\right) = \sum_{j=1}^{\infty} \sum_{k=1}^{j} x_k y_{k-j}$, we obtain

$$q\,[n-1]\,\Phi_{n-1} a_n z^n = -\frac{\gamma(\alpha-1)}{1-k} \sum_{n=2}^{\infty} \left(\sum_{j=1}^{n-1} \Phi_{j-1} a_j p_{n-j} \right) z^n.$$

Comparing the coefficients of nth term on both sides, we obtain

$$a_n = \frac{-\gamma(\alpha-1)}{q\,[n-1]\,\Phi_{n-1}\,(1-k)} \sum_{j=1}^{n-1} \Phi_{j-1} a_j p_{n-j}.$$

By taking absolute value and applying triangle inequality, we get

$$|a_n| \leq \frac{|\gamma|\,(\alpha-1)}{q\,[n-1]\,\Phi_{n-1}\,(1-k)} \sum_{j=1}^{n-1} \Phi_{j-1}\,|a_j|\,|p_{n-j}|\,.$$

Applying the coefficient estimates $|p_n| \leq 2$ $(n \geq 1)$ for Caratheodory functions [17], we obtain

$$\begin{aligned} |a_n| &\leq \frac{2\,|\gamma|\,(\alpha-1)}{q\,[n-1]\,\Phi_{n-1}(1-k)} \sum_{j=1}^{n-1} \Phi_{j-1}\,|a_j| \\ &= \frac{\sigma}{[n-1]\,\Phi_{n-1}} \sum_{j=1}^{n-1} \psi_{j-1}\,|a_j|\,, \end{aligned} \tag{3.6}$$

where $\sigma = 2|\gamma|(\alpha-1)/q(1-k)$. To prove (3.2) we apply mathematical induction. So for $n = 2$, we have from (3.6)

$$|a_2| \leq \frac{\sigma}{\Phi_1} = \frac{(\sigma)_{2-1}}{[2-1]!\Phi_{2-1}}, \tag{3.7}$$

which shows that (3.2) holds for $n = 2$. For $n = 3$, we have from (3.6)

$$|a_3| \leq \frac{\sigma}{[3-1]\,\Phi_{3-1}} \left\{1 + \Phi_1\,|a_2|\right\},$$

using (3.7), we have

$$|a_3| \leq \frac{\sigma}{[2]\,\Phi_2}\,(1+\sigma) = \frac{(\sigma)_{3-1}}{[3-1]\,\Phi_{3-1}},$$

which shows that (3.2) holds for $n = 3$. Let us assume that (3.2) is true for $n \leq t$, that is,

$$|a_t| \leq \frac{(\sigma)_{t-1}}{[t-1]!\Phi_{t-1}} \quad j = 1, 2, \ldots, t. \tag{3.8}$$

Using (3.6) and (3.8), we have

$$\begin{aligned} |a_{t+1}| &\leq \frac{\sigma}{t\Phi_t}\sum_{j=1}^{t}\Phi_{j-1}|a_j| \\ &\leq \frac{\sigma}{t\Phi_t}\sum_{j=1}^{t}\psi_{j-1}\frac{(\sigma)_{j-1}}{[j-1]!\Phi_{j-1}} \\ &= \frac{\sigma}{t\Phi_t}\sum_{j=1}^{t}\frac{(\sigma)_{j-1}}{[j-1]!}. \end{aligned}$$

Applying (2.1), we have

$$\begin{aligned} |a_{t+1}| &\leq \frac{1}{t\Phi_t}\frac{(\sigma)_t}{[t-1]!} \\ &= \frac{1}{\Phi_t}\frac{(\sigma)_t}{[t]!}. \end{aligned}$$

Consequently, using mathematical induction, we have proved that (3.2) holds true for all n, $n \geq 2$. This completes the proof. □

Theorem 3.3. *If a function $f \in \mathcal{KD}_q(k,\alpha,\gamma)$, then*

$$\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)} \prec 1+2(\alpha_1-1)-\frac{2(\alpha_1-1)}{1-z} \quad (z\in\mathbb{U}), \tag{3.9}$$

$$\alpha_1 = \frac{\alpha-k}{1-k}. \tag{3.10}$$

Proof. If $f(z) \in \mathcal{KD}_q(k,\alpha,\gamma)$, then by (3.1)

$$\mathfrak{Re}\left\{1+\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)}-1\right)\right\} < \alpha_1. \tag{3.11}$$

Then there exists a Schwarz function $w(z)$ such that

$$\frac{\alpha_1-\left\{1+\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)}-1\right)\right\}}{\alpha_1-1} = \frac{1+w(z)}{1-w(z)}, \tag{3.12}$$

and

$$\mathfrak{Re}\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0, \quad (z\in\mathbb{U}).$$

Therefore, from (3.12), we obtain

$$\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)} = 1+\gamma(\alpha_1-1)\left(1-\frac{1+w(z)}{1-w(z)}\right).$$

This gives

$$\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)} = 1+2\gamma(\alpha_1-1)-\frac{2\gamma(\alpha_1-1)}{1-w(z)}$$

and hence

$$\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)} \prec 1+2\gamma(\alpha_1-1)-\frac{2\gamma(\alpha_1-1)}{1-z} \quad (z\in\mathbb{U}).$$

which was required in (3.9). □

Theorem 3.4. *If function* $f \in \mathcal{KD}_q(k,\alpha,\gamma)$, *then we have*

$$\frac{1-[1+2\gamma(\alpha_1-1)]\,r}{1-r} \le \mathfrak{Re}\left\{\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)}\right\} \le \frac{1+[1+2\gamma(\alpha_1-1)]\,r}{1+r}, \tag{3.13}$$

for $|z|=r<1$ *and* α_1 *is defined by* (3.10).

Proof. By the virtue of Theorem (3.3), let us take the function $\phi(z)$ defined by

$$\phi(z) = 1+2\gamma(\alpha_1-1) - \frac{2\gamma(\alpha_1-1)}{1-z} \quad (z\in\mathbb{U}).$$

Letting $z=re^{i\theta}(0\le r<1)$, we see that

$$\mathfrak{Re}\phi(z) = 1+2\gamma(\alpha_1-1) + \frac{2\gamma(1-\alpha_1)(1-r\cos\theta)}{1+r^2-2r\cos\theta}.$$

Let us define

$$\psi(t) = \frac{1-rt}{1+r^2-2rt} \quad (t=\cos\theta).$$

Since $\psi'(t) = \frac{r(1-r^2)}{(1+r^2-2rt)^2} \ge 0$, because $r<1$. Therefore we get

$$1+2\gamma(\alpha_1-1) - \frac{2\gamma(\alpha_1-1)}{1-r} \le \mathfrak{Re}\phi(z) \le 1+2\gamma(\alpha_1-1) - \frac{2\gamma(\alpha_1-1)}{1+r}.$$

After simplification, we have

$$\frac{1-[1+2\gamma(\alpha_1-1)]\,r}{1-r} \le \mathfrak{Re}\phi(z) \le \frac{1+[1+2\gamma(\alpha_1-1))]\,r}{1+r}.$$

Since we note that $\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)} \prec \phi(z), (z\in\mathbb{U})$ by Theorem 3.3 and $\phi(z)$ is analytic in $\mathbb{U}$, we proved the inequality (3.13). □

Theorem 3.5. *If* $f\in\mathcal{A}$ *satisfies*

$$\left|\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)} - 1\right| < \frac{(\alpha-1)|\gamma|}{(1-k)} \quad z\in\mathbb{U}, \tag{3.14}$$

for some $k\,(k\le 0)$, $\alpha\,(\alpha>1)$ *and* $\gamma\in\mathbb{C}\setminus\{0\}$. *Then* $f\in\mathcal{KD}_q(k,\alpha,\gamma)$.

Proof.

$$\begin{aligned}
&\left|\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)}-1\right|<\frac{(\alpha-1)|\gamma|}{(1-k)}\\
\Rightarrow\quad&\left|\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)}-1\right)\right|<\frac{\alpha-1}{1-k}\\
\Rightarrow\quad&(1-k)\left|\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)}-1\right)\right|+1<\alpha\\
\Rightarrow\quad&\left|\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)}-1\right)\right|+1<k\left|\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)}-1\right)\right|+\alpha\\
\Rightarrow\quad&\mathfrak{Re}\left\{1+\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)}-1\right)\right\}+1<k\left|\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)}-1\right)\right|+\alpha\\
\Rightarrow\quad&f\in\mathcal{LD}_b^k(a,c,\beta)
\end{aligned}$$

□

Corollary 3.6. *Let $f\in\mathcal{A}$ be of the form* (1.1) *and satisfies*

$$\left|\frac{\sum_{n=2}^{\infty}[n-1]\,\Phi_{n-1}a_nz^{n-1}}{1+\sum_{n=2}^{\infty}\Phi_{n-1}a_nz^{n-1}}\right|<\frac{(\alpha-1)|\gamma|}{q(1-k)}\quad z\in\mathbb{U},\tag{3.15}$$

for some $k\,(k\le 0)$, $\beta\,(\beta>1)$ and for some $b\in\mathbb{C}\setminus\{0\}$. Then $f\in\mathcal{KD}_q(k,\alpha,\gamma)$..

Proof. We have

$$\mathfrak{D}_q^{\mu}f(z)=z+\sum_{n=2}^{\infty}\Phi_{n-1}a_nz^n$$

and by (1.5)

$$z\partial\mathfrak{D}_q^{\mu}f(z)=z+\sum_{n=2}^{\infty}[n]\,\Phi_{n-1}a_nz^n.$$

Therefore, (3.14) follows immediately (3.15). □

Theorem 3.7. *Let $f\in\mathcal{A}$ be of the form* (1.1) *and satisfies*

$$\sum_{n=2}^{\infty}([n-1]+y)\,|\Phi_{n-1}||a_n|<y\quad z\in\mathbb{U},\tag{3.16}$$

for some $k\,(k\le 0)$, $\beta\,(\beta>1)$ and for some $b\in\mathbb{C}\setminus\{0\}$ and where

$$y=\frac{(\alpha-1)|\gamma|}{q(1-k)}>0.$$

Then $f\in\mathcal{KD}_q(k,\alpha,\gamma)$.

Proof. We have

$$\begin{aligned}
& \sum_{n=2}^{\infty}([n-1]+y)\,|\Phi_{n-1}||a_n| < y \\
\Rightarrow\quad & \sum_{n=2}^{\infty}([n-1]+y)\,|\Phi_{n-1}||a_n| < y - y\sum_{n=2}^{\infty}|\Phi_{n-1}||a_n| \\
\Rightarrow\quad & 0 < y - y\sum_{n=2}^{\infty}|\Phi_{n-1}||a_n| \\
\Rightarrow\quad & 0 < y - y\sum_{n=2}^{\infty}|\Phi_{n-1}||a_n||z^{n-1}| \\
\Rightarrow\quad & 0 < y\left|1+\sum_{n=2}^{\infty}\Phi_{n-1}a_n z^{n-1}\right| \qquad (3.17)
\end{aligned}$$

We have

$$\begin{aligned}
& \sum_{n=2}^{\infty}([n-1]+y)\,|\Phi_{n-1}||a_n| < y \\
\Rightarrow\quad & \sum_{n=2}^{\infty}([n-1]+y)\,|\Phi_{n-1}||a_n||z^{n-1}| < y \\
\Rightarrow\quad & \sum_{n=2}^{\infty}[n-1]\,|\Phi_{n-1}||a_n||z^{n-1}| < y - y\sum_{n=2}^{\infty}|\Phi_{n-1}||a_n||z^{n-1}| \\
\Rightarrow\quad & \left|\sum_{n=2}^{\infty}[n-1]\,\Phi_{n-1}a_n z^{n-1}\right| < y\left|1+\sum_{n=2}^{\infty}\Phi_{n-1}a_n z^{n-1}\right| \\
\Rightarrow\quad & \left|\frac{\sum_{n=2}^{\infty}[n-1]\,\Phi_{n-1}a_n z^{n-1}}{1+\sum_{n=2}^{\infty}\Phi_{n-1}a_n z^{n-1}}\right| < y,
\end{aligned}$$

because of (3.17). By (3.15) it follows $f \in \mathcal{LD}_b^k(a,c,\beta)$. □

Competing interests

The authors declare that they have no competing interests.

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