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New subclasses of univalent functions on the unit disc in $\mathbb{C}$

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Abstract. In this paper we consider a differential operator $\mathcal{G}_k$ defined on the family of holomorphic normalized functions $\mathcal{H}_0(\mathbb{U})$ that can be used in the construction of new subclasses of univalent functions on the unit disc $\mathbb{U}$. These new subclasses are closely related to the families of convex, respectively starlike functions on $\mathbb{U}$. We study general results related to these new subclasses, such as growth and distortion theorems, coefficients estimates and duality results. We also present examples of functions that belongs to the subclasses defined.

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1. Preliminaries

Let us denote by $\mathbb{U} = \mathcal{U}(0; 1)$ the open unit disc in the complex plane $\mathbb{C}$ and $\mathcal{H}(\mathbb{U})$ the family of all holomorphic functions on the unit disc $\mathbb{U}$. Also, let us denote by $\mathcal{H}_0(\mathbb{U})$ the class of normalized holomorphic functions on $\mathbb{U}$, i.e. $f \in \mathcal{H}(\mathbb{U})$ with $f(0) = 0$ and $f'(0) = 1$. An important class that will be used in our paper is the class of normalized univalent (holomorphic and injective) functions on the unit disc $\mathbb{U}$, denoted by S. For more details about the holomorphic functions and the class of normalized univalent functions, one may consult [2], [3], [5], [10] and [16].

Let us consider $\alpha \in [0, 1)$. In [17] Robertson introduced two important subclasses of the class S, namely the family

$$S^*(\alpha) = \left\{ f \in \mathcal{H}_0(\mathbb{U}) \,:\, \mathfrak{Re}\left[\frac{zf'(z)}{f(z)}\right] > \alpha, \quad z \in \mathbb{U} \right\}$$

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of normalized starlike functions of order α, respectively the family

$$K(\alpha)=\left\{f\in\mathcal{H}_0(\mathbb{U})\,:\,\mathfrak{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]>\alpha,\quad z\in\mathbb{U}\right\}$$

of normalized convex functions of order α. In particular, we obtain that $S^*=S^*(0)$ and $K=K(0)$ are the usual families of starlike, respectively convex functions on the unit disc $\mathbb{U}$. For more details about these families of univalent functions, one may consult [2], [3], [5], or [16].

An important result related to the class S is due to Noshiro and Warschawski (see e.g. [2, Theorem 2.16]) and present a sufficient condition of univalence, as follows:

Theorem 1.1. *Let $f\in\mathcal{H}_0(\mathbb{U})$. If $\mathfrak{Re}f'(z)>0$, for all $z\in\mathbb{U}$, then f is univalent on $\mathbb{U}$.*

Strongly related to the family S is the class of normalized holomorphic functions whose derivative has positive real part (of order α), denoted by

$$\mathcal{R}(\alpha)=\left\{f\in\mathcal{H}_0(\mathbb{U})\,:\,\mathfrak{Re}f'(z)>\alpha,\quad z\in\mathbb{U}\right\},\quad \alpha\in[0,1).$$

In view of Theorem 1.1 it is clear that $\mathcal{R}(\alpha)\subset S$. For more details about the class $\mathcal{R}(\alpha)$ of univalent functions whose derivatives have positive real part, one may consult [6], [11], [12] and [13] . In particular, the class $\mathcal{R}(0)=\mathcal{R}$ was studied by T.H. MacGregor in [13].

In the following sections of this paper we introduce a differential operator $\mathcal{G}_k$ defined on $\mathcal{H}_0(\mathbb{U})$ that is useful in the construction of new subclasses of univalent functions on $\mathbb{U}$ (denoted by E_k, respectively E_k^*) closely related to the class of convex, respectively starlike functions on the unit disc $\mathbb{U}$. An interesting property of these subclasses is that we can obtain coefficient estimates of the form $|a_n|\le\frac{1}{(n-k)!}$, for $n\ge k$, where $k\in\mathbb{N}$ and $a_k,...,a_n$ are the coefficients from the Taylor series expansion of the function $f\in\mathcal{H}_0(\mathbb{U})$.

Remark 1.2. It is important to mention that the operator $\mathcal{G}_k$ can be defined also in the case of several complex variables (see [8]). Although for $n=1$ we have that $E_0(\mathbb{U})=E_1^*(\mathbb{U})=K(\mathbb{U})$, in the case of several complex variables we can prove that $E_1^*(\mathbb{B}^n)\cap K(\mathbb{B}^n)\neq\emptyset$, but $E_1^*(\mathbb{B}^n)\neq K(\mathbb{B}^n)$, where $K(\mathbb{B}^n)$ is the family of convex mappings on the Euclidean unit ball $\mathbb{B}^n$ (for details about univalent mappings in higher dimensions, one may consult [5] and [9]). Another interesting property of E_k^* studied in [8] is related to the Graham-Kohr extension operator (introduced by I. Graham and G. Kohr in [4]).

2. The differential operator $\mathcal{G}_k$

In this section we introduce the differential operator $\mathcal{G}_k$ defined on the family $\mathcal{H}_0(\mathbb{U})$ of normalized holomorphic functions on $\mathbb{U}$. For this operator we present some properties related to the linearity and univalence on the unit disc $\mathbb{U}$ and we discuss about how the convolution product is preserved under the action of the operator $\mathcal{G}_k$.

Definition 2.1. Let $k \in \mathbb{N} = \{0, 1, 2, ...\}$ and let $\mathcal{G}_k : \mathcal{H}_0(\mathbb{U}) \to \mathcal{H}(\mathbb{U})$ be the differential operator defined on the class of normalized holomorphic functions on $\mathbb{U}$, as follows

$$(\mathcal{G}_k f)(z) = \begin{cases} z^k f^{(k)}(z) + a_{k-1}z^{k-1} + ... + a_2 z^2 + a_1 z + a_0, & k \geq 1 \\ f(z) & k = 0, \end{cases} \tag{2.1}$$

for all $f \in \mathcal{H}_0(\mathbb{U})$ and $z \in \mathbb{U}$. Notice that, for $k \geq 1$, $a_0, ..., a_{k-1}$ are the first k coefficients from the Taylor series expansion of the function $f \in \mathcal{H}_0(\mathbb{U})$.

Remark 2.2. In view of the above definition, it is easy to see that the operator $\mathcal{G}_0$ (of order 0) is the identity operator, i.e. $\mathcal{G}_0 f = f$. Another particular form of the operator $\mathcal{G}_k$ is for $k = 1$ (of order 1). In this case, $(\mathcal{G}_1 f)(z) = zf'(z)$, for all $z \in \mathbb{U}$.

Remark 2.3. Let us denote $id : U \to \mathbb{C}$ the identity function on $\mathbb{U}$, given by $id(z) = z$, for all $z \in \mathbb{U}$. Then $\mathcal{G}_k(id) = id$, for all $k \in \mathbb{N}$.

The connection between two differential operators of consecutive orders $k-1$, respectively k, where $k \in \mathbb{N}$ with $k \geq 1$, is given in the following result:

Proposition 2.4. *Let $f \in \mathcal{H}_0(\mathbb{U})$. Then for any $k \in \mathbb{N}^* = \{1, 2, ...\}$ the following relation holds*

$$(\mathcal{G}_k f)(z) = z(\mathcal{G}_{k-1} f)'(z) - (k-1)(\mathcal{G}_{k-1} f)(z) + \sum_{n=0}^{k-1} (k-n) a_n z^n, \quad z \in \mathbb{U}. \tag{2.2}$$

Proof. We prove relation (2.2) by mathematical induction. Assume that

$$P(k): \quad (\mathcal{G}_k f)(z) = z(\mathcal{G}_{k-1} f)'(z) - (k-1)(\mathcal{G}_{k-1} f)(z) + \sum_{n=0}^{k-1} (k-n) a_n z^n$$

is true for a fixed $k \in \mathbb{N}$ with $k \geq 2$. Then

$$z(\mathcal{G}_k f)'(z) - k(\mathcal{G}_k f)(z) = z^{k+1} f^{(k+1)}(z) + \sum_{n=0}^{k-1} (n-k) a_n z^n, \quad z \in \mathbb{U}.$$

Adding $\sum_{n=0}^{k} (k+1-n) a_n z^n$ at the previous equality, we obtain

$$z^{k+1} f^{(k+1)}(z) + \sum_{n=0}^{k-1} (n-k) a_n z^n + \sum_{n=0}^{k} (k+1-n) a_n z^n = (\mathcal{G}_{k+1} f)(z),$$

for all $z \in \mathbb{U}$ and this completes the proof. □

Proposition 2.5. *Let $k \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$ and $f, g \in \mathcal{H}_0(\mathbb{U})$. Then*

$$\mathcal{G}_k(\alpha f + \beta g) = \alpha \mathcal{G}_k f + \beta \mathcal{G}_k g. \tag{2.3}$$

Proof. Let $f, g \in \mathcal{H}_0(\mathbb{U})$ be such that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, for all $z \in \mathbb{U}$, with $a_0 = b_0 = 0$ and $a_1 = b_1 = 1$. Then

$$\begin{aligned} \mathcal{G}_k(\alpha f + \beta g)(z) &= z^k (\alpha f + \beta g)^{(k)}(z) + \sum_{n=0}^{k-1} (\alpha a_n + \beta b_n) z^n \\ &= \alpha(\mathcal{G}_k f)(z) + \beta(\mathcal{G}_k g)(z), \end{aligned}$$

for all $z \in \mathbb{U}$ and $\alpha, \beta \in \mathbb{R}$. □

Remark 2.6. For $f \in \mathcal{H}_0(\mathbb{U})$, we can rewrite (2.1) as

$$(\mathcal{G}_k f)(z) = z + a_2 z^2 + ... + a_{k-1} z^{k-1} + k! a_k z^k + (k+1)! a_{k+1} z^{k+1} + ...$$
$$... + \frac{(k+n)!}{n!} a_{k+n} z^{k+n} + ...,$$

for all $z \in \mathbb{U}$. In other words,

$$(\mathcal{G}_k f)(z) = z + \sum_{n=2}^{\infty} A_n z^n, \quad \text{where} \quad A_n = \begin{cases} a_n, & n \leq k-1 \\ \frac{n!}{(n-k)!} a_n, & n \geq k, \end{cases} \tag{2.4}$$

for all $z \in \mathbb{U}$.

Another interesting property of the operator $\mathcal{G}_k$ is related to the Hadamard (convolution) product (for details, one may consult [2], [3], [5]). Let $f, g \in \mathcal{H}_0(\mathbb{U})$ be given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. We denote by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U} \tag{2.5}$$

the Hadamard (convolution) product of the functions f and g on $\mathbb{U}$ (see e.g. [2], [3], [5]). There is a nice connection between the convolution product of two different operators and the operator applied on a convolution product, as follows in the next result.

Proposition 2.7. *Let $k \in \mathbb{N}$ and $f, g \in \mathcal{H}_0(\mathbb{U})$. Then*

1. $\mathcal{G}_k(f * g) = (\mathcal{G}_k f) * g = f * (\mathcal{G}_k g)$;
2. $(\mathcal{G}_k f) * (\mathcal{G}_k g) = \mathcal{G}_k(\mathcal{G}_k(f * g))$.

Proof. Let $f, g \in \mathcal{H}_0(\mathbb{U})$ be such that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, for all $z \in \mathbb{U}$, with $a_0 = b_0 = 0$ and $a_1 = b_1 = 1$. Then

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,$$

for all $z \in \mathbb{U}$. Moreover, taking into account Remark 2.6, we deduce that

$$\mathcal{G}_k(f * g)(z) = z + \sum_{n=2}^{k-1} a_n b_n z^n + \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n b_n z^n, \quad z \in \mathbb{U}. \tag{2.6}$$

1. First, in view of (2.4) and the definition of the convolution product, we obtain

$$\begin{aligned} \big((\mathcal{G}_k f) * g\big)(z) &= \left(z + \sum_{n=2}^{k-1} a_n z^n + \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n z^n\right) * \left(z + \sum_{n=2}^{\infty} b_n z^n\right) \\ &= z + \sum_{n=2}^{k-1} a_n b_n z^n + \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n b_n z^n \\ &= \mathcal{G}_k(f * g)(z) \end{aligned}$$

for all $z \in \mathbb{U}$. Similarly, we can prove that $\mathcal{G}_k(f * g) = f * (\mathcal{G}_k g)$.

2. For the second part, it is enough to consider relations (2.4) and (2.6). Then

$$\begin{aligned}\mathcal{G}_k(\mathcal{G}_k(f*g))(z) &= z + \sum_{n=2}^{k-1} a_n b_n z^n + \sum_{n=k}^{\infty} \left[\frac{n!}{(n-k)!}\right]^2 a_n b_n z^n \\ &= (\mathcal{G}_k f)(z) * (\mathcal{G}_k g)(z),\end{aligned}$$

for all $z \in \mathbb{U}$ and this completes the proof.

□

Remark 2.8. Notice that we can obtain the second statement from Proposition 2.7 by replacing f with $\mathcal{G}_k h$ (where $h \in \mathcal{H}_0(\mathbb{U})$) and using only the first part of the result. Then

$$(\mathcal{G}_k h) * (\mathcal{G}_k g) = f * (\mathcal{G}_k g) = \mathcal{G}_k(f * g) = \mathcal{G}_k((\mathcal{G}_k h) * g) = \mathcal{G}_k(\mathcal{G}_k(h * g))$$

and this completes the argument.

It is important that we can prove a sufficient condition of univalence for $\mathcal{G}_k$ (in terms of modulus of coefficients a_n), as follows

Proposition 2.9. *Let $k \in \mathbb{N}$ and $f \in \mathcal{H}_0(\mathbb{U})$. Also, let σ_k be defined by*

$$\sigma_k = \begin{cases} \displaystyle\sum_{n=2}^{\infty} \frac{n \cdot n!}{(n-k)!}|a_n|, & k \le 2 \\ \displaystyle\sum_{n=2}^{k-1} n|a_n| + \sum_{n=k}^{\infty} \frac{n \cdot n!}{(n-k)!}|a_n|, & k \ge 3. \end{cases} \tag{2.7}$$

If $\sigma_k \le 1$, then $\mathcal{G}_k f$ is univalent on the unit disc $\mathbb{U}$. In particular, $\mathcal{G}_k f \in S$.

Proof. It is easy to observe that $(\mathcal{G}_k f)(0) = 0$, $(\mathcal{G}_k f)'(0) = 1$ and $\mathcal{G}_k f$ is a holomorphic function on $\mathbb{U}$. In view of relation (2.7), we consider the following two cases:

- If $k \ge 3$, then

$$\begin{aligned}\left|(\mathcal{G}_k f)'(z) - 1\right| &= \left|1 + \sum_{n=2}^{k-1} n a_n z^{n-1} + \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} n a_n z^{n-1} - 1\right| \\ &\le |z| \left(\sum_{n=2}^{k-1} n|a_n| + \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} n|a_n|\right) < 1,\end{aligned}$$

  for all $z \in \mathbb{U}$ and $k \ge 3$. Hence, $(\mathcal{G}_k f)'(z) \in \mathcal{U}(1;1)$, for all $z \in \mathbb{U}$ and this implies that $\mathfrak{Re}(\mathcal{G}_k f)'(z) > 0$, for all $z \in \mathbb{U}$.
- Similarly, for $k \le 2$, we have

$$\begin{aligned}\left|(\mathcal{G}_k f)'(z) - 1\right| &= \left|1 + \sum_{n=2}^{\infty} \frac{n \cdot n!}{(n-k)!} a_n z^{n-1} - 1\right| \\ &\le |z| \sum_{n=2}^{\infty} \frac{n \cdot n!}{(n-k)!} |a_n| < 1,\end{aligned}$$

for all $z \in \mathbb{U}$ and $k \geq 2$. As we seen before, we obtain that $(\mathcal{G}_k f)'(z) \in \mathcal{U}(1;1)$ which implies that $\mathfrak{Re}(\mathcal{G}_k f)'(z) > 0$, for all $z \in \mathbb{U}$.

Finally, according to the univalence criterion given in Theorem 1.1 we deduce that $\mathcal{G}_k f \in S$, for all $k \in \mathbb{N}$ and this completes the proof. □

Remark 2.10. In particular, for $k = 0$, we obtain the well-known univalence condition for a holomorphic function on the unit disc (see for example [5, Exercise 1.1.4]): if $\sum_{n=2}^{\infty} n|a_n| \leq 1$, then f is univalent on $\mathbb{U}$.

3. Subclasses of univalent functions

Using the differential operator $\mathcal{G}_k$ defined above, we can construct some particular subclasses of univalent functions on the unit disc $\mathbb{U}$ in $\mathbb{C}$. These subclasses, denoted here by $E_k^*(\alpha)$, respectively $E_k(\alpha)$, where $\alpha \in [0, 1)$, are related to the classes of starlike, respectively convex functions of order α on $\mathbb{U}$.

3.1. The subclass $E_k^*(\alpha)$

First, we present some general results about the subclass $E_k^*(\alpha)$ and connections of this class with another important classes of univalent functions (for example, the class of starlike functions of order α or the class of univalent functions introduced by Sălăgean in [18]).

Definition 3.1. Let $\alpha \in [0, 1)$ and $k \in \mathbb{N}$. Let $\mathcal{G}_k$ be the differential operator defined by formula (2.1). Then

$$E_k^*(\alpha) = \{f \in S : \mathcal{G}_k f \in S^*(\alpha)\}$$

is the family of normalized univalent functions f on the unit disc such that $\mathcal{G}_k f$ is starlike of order α. In particular, we denote by $E_k^* = E_k^*(0)$.

Remark 3.2. It is clear that $E_0^*(\alpha) = S^*(\alpha)$ is the family of normalized starlike functions of order α on $\mathbb{U}$.

Remark 3.3. Taking into account the definition of starlikeness of order α, we deduce that

$$E_k^*(\alpha) = \left\{f \in S : \mathfrak{Re}\left[\frac{z(\mathcal{G}_k f)'(z)}{(\mathcal{G}_k f)(z)}\right] > \alpha, \quad z \in \mathbb{U}\right\}. \tag{3.1}$$

Indeed, if $f \in S$, then $\mathcal{G}_k f \in \mathcal{H}(\mathbb{U})$, $(\mathcal{G}_k f)(0) = 0$ and $(\mathcal{G}_k f)'(0) = 1$. Together with the condition $\mathfrak{Re}\left[\frac{z(\mathcal{G}_k f)'(z)}{(\mathcal{G}_k f)(z)}\right] > \alpha$, for all $z \in \mathbb{U}$, all the assumptions from the definition of starlikeness of order α are satisfied.

Proposition 3.4. *Let $\alpha \in [0, 1)$. Then $E_1^*(\alpha) = K(\alpha)$.*

Proof. Indeed, according to the previous definition and Remark 2.2, we have that

$$\begin{aligned} E_1^*(\alpha) &= \left\{f \in S : \mathfrak{Re}\left[\frac{z(\mathcal{G}_1 f)'(z)}{(\mathcal{G}_1 f)(z)}\right] > \alpha, \quad z \in \mathbb{U}\right\} \\ &= \left\{f \in S : \mathfrak{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > \alpha, \quad z \in \mathbb{U}\right\} = K(\alpha), \end{aligned}$$

for every $\alpha \in [0, 1)$ and this completes the proof. □

Remark 3.5. As a consequence of the previous two remarks, we obtain that $E_0^* = S^*$ and $E_1^* = K$. It is important to mention here that the second equality is no longer true in the case of several complex variables (see [8]).

Remark 3.6. It is very important to mention here that

$$E_0^*(\alpha) = S_0(\alpha) \quad \text{and} \quad E_1^*(\alpha) = S_1(\alpha),$$

where $S_0(\alpha)$ and $S_1(\alpha)$ are particular forms of the class $S_n(\alpha)$ introduced by Sălăgean in [18] for $\alpha \in [0,1)$. These equalities holds because

$$D^0 f(z) = f(z) = (\mathcal{G}_0 f)(z) \quad \text{and} \quad D^1 f(z) = zf'(z) = (\mathcal{G}_1 f)(z),$$

for all $z \in \mathbb{U}$, where D^n is the differential operator introduced by Sălăgean. However, for $n = k \geq 2$, we have that

$$E_k^*(\alpha) \neq S_n(\alpha),$$

since the Sălăgean differential operator $D^n f$ (see [18]) is different from the operator $\mathcal{G}_k f$, for every $n = k \geq 2$. For example, if $n = 2$, then

$$D^2 f(z) = D(Df(z)) = z^2 f''(z) + zf'(z) \neq z^2 f''(z) + z = (G_2 f)(z),$$

for all $z \in \mathbb{U}$. Hence, the common results from this thesis and the ones obtained by Sălăgean in [18] are only for the particular cases $k = 0$ and $k = 1$ (which are already well-known, as reduces to the classes $S^*(\alpha)$, respectively $K(\alpha)$).

Using a similar argument as in Proposition 2.9, we can prove the following result. We mention here that this result is a general form of the theorem proved by Merkes, Robertson and Scott in [14].

Theorem 3.7. *Let $\alpha \in [0,1)$, $k \in \mathbb{N}$ and $f \in S$. Also, let $\sigma_{k,\alpha}$ be defined by*

$$\sigma_{k,\alpha} = \begin{cases} \displaystyle\sum_{n=2}^{\infty} \frac{(n-\alpha)\cdot n!}{(n-k)!}|a_n|, & k \leq 2 \\[2ex] \displaystyle\sum_{n=2}^{k-1}(n-\alpha)|a_n| + \sum_{n=k}^{\infty} \frac{(n-\alpha)\cdot n!}{(n-k)!}|a_n|, & k \geq 3. \end{cases} \tag{3.2}$$

If $\sigma_{k,\alpha} \leq 1-\alpha$, then $f \in E_k^*(\alpha)$.

Proof. Let $\alpha \in [0,1)$. Using (3.2) and Proposition 2.9, we obtain that $\mathcal{G}_k f$ is a normalized univalent function on $\mathbb{U}$. Morevoer,

$$\begin{aligned} &\left|z(\mathcal{G}_k f)'(z) - (\mathcal{G}_k f)(z)\right| - (1-\alpha)\left|(\mathcal{G}_k f)(z)\right| = \\ &= \left|z + \sum_{n=2}^{\infty} nA_n z^n - z - \sum_{n=2}^{\infty} A_n z^n\right| - (1-\alpha)\left|z + \sum_{n=2}^{\infty} A_n z^n\right| \\ &\leq \sum_{n=2}^{\infty}(n-1)|A_n||z|^n - (1-\alpha)\left(|z| - \sum_{n=2}^{\infty}|A_n||z|^n\right) \\ &\leq |z|\left(\sum_{n=2}^{\infty}(n-\alpha)|A_n| - (1-\alpha)\right) \leq 0, \end{aligned}$$

where A_n is given by relation (2.4). Since $(\mathcal{G}_k f)(z) \neq 0$ for $z \neq 0$ and in view of relation

$$\left|z(\mathcal{G}_k f)'(z) - (\mathcal{G}_k f)(z)\right| - (1-\alpha)\left|(\mathcal{G}_k f)(z)\right| \leq 0,$$

we deduce that

$$\left|\frac{z(\mathcal{G}_k f)'(z)}{(\mathcal{G}_k f)(z)} - 1\right| \leq 1-\alpha, \tag{3.3}$$

for all $z \in \mathbb{U}$. Therefore

$$\mathfrak{Re}\left[\frac{z(\mathcal{G}_k f)'(z)}{(\mathcal{G}_k f)(z)}\right] > \alpha, \quad z \in \mathbb{U}$$

which implies $\mathcal{G}_k f \in S^*(\alpha)$. According to Definition 3.1, we conclude that $f \in E_k^*(\alpha)$ and this completes the proof. □

In the next corollary we present two particular cases of the previous theorem (results proved by Merkes, Robertson and Scott in [14]; see also [5]).

Corollary 3.8. *Let $f \in \mathcal{H}_0(\mathbb{U})$ and $k \in \{0,1\}$.*

1. *If $\sigma_{0,\alpha} = \sum_{n=2}^{\infty}(n-\alpha)|a_n| \leq 1-\alpha$, then $f \in E_0^*(\alpha) = S^*(\alpha)$.*
2. *If $\sigma_{1,\alpha} = \sum_{n=2}^{\infty}(n-\alpha)n|a_n| \leq 1-\alpha$, then $f \in E_1^*(\alpha) = K(\alpha)$.*

Remark 3.9. It is clear that for $\alpha = 0$, we obtain the classical conditions for starlikeness, respectively convexity on the unit disc (see e.g. [2], [3], [5]).

In this subsection we present some results regarding to coefficient estimates and distortion theorems for the class $E_k^*(\alpha)$. For the proof of our first result, we use the coefficient estimates for the class $S^*(\alpha)$ given by Robertson in [17] (see also [5]).

Theorem 3.10. *Let $\alpha \in [0,1)$, $k \in \mathbb{N}$ and $f \in E_k^*(\alpha)$. Then*

$$|a_n| \leq \frac{(n-k)!}{(n-1)! \cdot n!} \prod_{m=2}^{n}(m-2\alpha), \quad n \geq k \geq 2. \tag{3.4}$$

Proof. Let $f \in E_k^*(\alpha)$. Then $f \in S$ and $\mathcal{G}_k f \in S^*(\alpha)$. According to Remark 2.6 and the coefficient bounds for the class $S^*(\alpha)$ given in [17] (see also [5]), we know that

$$|A_n| \leq \frac{1}{(n-1)!} \prod_{m=2}^{n}(m-2\alpha), \tag{3.5}$$

for all $n \geq 2$, where A_n are the coefficients of $\mathcal{G}_k f$ defined by relation (2.4). Since

$$|A_n| = \begin{cases} |a_n|, & n \leq k-1 \\ \frac{n!}{(n-k)!}|a_n|, & n \geq k, \end{cases}$$

we obtain that

$$|a_n| \leq \frac{(n-k)!}{(n-1)! \cdot n!} \prod_{m=2}^{n}(m-2\alpha), \quad n \geq k.$$

Taking into account the product considered in the last relation, we impose the condition $n \geq k \geq 2$ and this completes the proof. □

Corollary 3.11. *Let $k \in \mathbb{N}$ and $f \in E_k^*$. Then*

$$|a_n| \leq \frac{n}{n(n-1)(n-2)\cdot \ldots \cdot (n-k+1)} = \frac{n\cdot(n-k)!}{n!}, \quad n \geq k. \tag{3.6}$$

Proof. In view of Theorem 3.10 for $\alpha = 0$. □

Remark 3.12. As a consequence of the previous Corollary, we obtain the following well-known results (see e.g. [2], [3], [5], [16]):

1. If $k = 0$, then $E_0^* = S^*$ and $|a_n| \leq n$, for all $n \geq 0$.
2. If $k = 1$, then $E_1^* = K$ and $|a_n| \leq 1$, for all $n \geq 1$.

Following the idea presented by Duren in [2] and treated by Goodman in [3] (also by Grigoriciuc in [7]), we can prove a general distortion result for the class E_k^*. In fact, we obtain upper bounds for the m-th derivative of a function $f \in E_k^*$, where $m \in \mathbb{N}$ such that $m \geq k$.

Remark 3.13. Based on [7, Remark 2.5], we have that

$$\frac{1}{(1-r)^k} = \sum_{n=0}^{\infty} \frac{(k+n-1)!}{n!(k-1)!} r^n, \quad k \in \mathbb{N}, \quad r \in [0,1).$$

Theorem 3.14. *Let $k \in \mathbb{N}$. If $f \in E_k^*$, then*

$$\left|f^{(m)}(z)\right| \leq \frac{\big[m + (1-k)|z|\big]\cdot(m-k)!}{(1-|z|)^{m-k+2}}, \tag{3.7}$$

for all $m \geq k$ and $z \in \mathbb{U}$.

Proof. Let $f \in E_k^*$. Then $f \in S$ and $\mathcal{G}_k f \in S^*$. Moreover, for $m \in \mathbb{N}$, we have that

$$f^{(m)}(z) = \sum_{n=0}^{\infty} \frac{(m+n)!}{n!} a_{m+n} z^n, \quad z \in \mathbb{U}. \tag{3.8}$$

Let $r = |z| < 1$. In view of relation (3.6), we obtain

$$\begin{aligned} \left|f^{(m)}(z)\right| = \left|\sum_{n=0}^{\infty} \frac{(m+n)!}{n!} a_{m+n} z^n\right| &\leq \sum_{n=0}^{\infty} \frac{(m+n)!}{n!} |a_{m+n}||z|^n \\ &= \sum_{n=0}^{\infty} \frac{(m+n)(m+n-k)!}{n!} r^n \end{aligned}$$

In view of Remark 3.13 and elementary computations we deduce that

$$\left|f^{(m)}(z)\right| \leq \sum_{n=0}^{\infty} \frac{(m+n)(m+n-k)!}{n!} r^n = \frac{(m-k)!\big[m + r(1-k)\big]}{(1-r)^{m-k+2}},$$

where $r = |z| < 1$. Finally, we conclude that

$$\left|f^{(m)}(z)\right| \leq \frac{\big[m + (1-k)|z|\big]\cdot(m-k)!}{(1-|z|)^{m-k+2}}, \quad z \in \mathbb{U},$$

for all $m \geq k \geq 0$ and this completes the proof. □

Remark 3.15. Obviously, for $k \in \{0,1\}$ we obtain the classical results proved by Goodman in [3].

Based on the previous theorem and the result proved in [7], we propose the following conjecture (already proved for the particular cases $k=0$, $\alpha=0$ and $\alpha=\frac{1}{2}$):

Conjecture 3.16. *Let $\alpha \in [0,1)$ and $m,k \in \mathbb{N}$. If $f \in E_k^*(\alpha)$, then*

$$\left|f^{(m)}(z)\right| \leq \frac{\left[m+(1-k)(1-2\alpha)|z|\right] \cdot B(m-k,\alpha)}{(1-|z|)^{m-k+2-2\alpha}}, \tag{3.9}$$

for all $m \geq k+1$ and $z \in \mathbb{U}$, where

$$B(m-k,\alpha) = \begin{cases} \frac{1}{m}(m-k)!, & \alpha = \frac{1}{2} \\ \dfrac{1}{1-2\alpha}\displaystyle\prod_{j=1}^{m-k}(j-2\alpha), & \alpha \neq \frac{1}{2}. \end{cases} \tag{3.10}$$

Remark 3.17. It is clear that for $k=0$, Conjecture 3.16 reduces to [7, Theorem 3.4]. Moreover, for $\alpha=0$, the previous Conjecture reduces to Theorem 3.14 proved in this section.

Remark 3.18. If $\alpha=\frac{1}{2}$, then (3.9) can be written as

$$\left|f^{(m)}(z)\right| \leq \frac{(m-k)!}{(1-|z|)^{m-k+1}},$$

for all $m \geq k+1$ and $z \in \mathbb{U}$. Following a similar proof as in Theorem 3.14, we obtain that

$$\begin{aligned}\left|f^{(m)}(z)\right| = \left|\sum_{n=0}^{\infty}\frac{(m+n)!}{n!}a_{m+n}z^n\right| &\leq \sum_{n=0}^{\infty}\frac{(m+n)!}{n!}|a_{m+n}||z|^n \\ &\leq \sum_{n=0}^{\infty}\frac{(m+n)!(m+n-k)!r^n}{n!(m+n-1)!(m+n)!}\prod_{j=2}^{m+n}(j-1) \\ &= \frac{(m-k)!}{(1-r)^{m-k+1}},\end{aligned}$$

where $r=|z|<1$. Hence, Conjecture 3.16 is true for $\alpha=\frac{1}{2}$ as we proposed above.

Remark 3.19. The main idea of the results presented in this section is that starting from an index $n \geq k$ we can obtain better estimations for the coefficients a_n of $f \in E_k^*(\alpha)$, respectively upper bounds for the modulus of the m-th derivative of the function $f \in E_k^*(\alpha)$.

3.2. The subclass $E_k(\alpha)$

Similarly as in the previous section, we can use the operator $\mathcal{G}_k$ to define the class $E_k(\alpha)$ of holomorphic functions on the unit disc for which $\mathcal{G}_k f$ is a convex function of order α on $\mathbb{U}$. In the first part, we present some general results for the class $E_k(\alpha)$ related to coefficient estimates and general distortion results. The final part of this section is dedicated to the particular case $k=1$.

In this subsection we introduce the subclass $E_k(\alpha)$ together with some general properties of it.

Definition 3.20. Let $\alpha \in [0,1)$ and $k \in \mathbb{N}$. Let $\mathcal{G}_k$ be the differential operator defined by formula (2.1). Then

$$E_k(\alpha) = \big\{ f \in S \,:\, \mathcal{G}_k f \in K(\alpha) \big\}$$

is the family of normalized univalent functions f on the unit disc such that $\mathcal{G}_k f$ is convex of order α. In particular, we denote by $E_k = E_k(0)$.

Remark 3.21. Taking into account the definition of convexity of order α (see [5], [17], [16]), we deduce that

$$E_k(\alpha) = \left\{ f \in S \,:\, \mathfrak{Re}\left[1 + \frac{z(\mathcal{G}_k f)''(z)}{(\mathcal{G}_k f)'(z)} \right] > \alpha, \quad z \in \mathbb{U} \right\}. \tag{3.11}$$

It is clear that $E_0(\alpha) = K(\alpha)$ is the family of normalized convex functions of order α on $\mathbb{U}$.

Taking into account Theorem 3.7, we can prove a similar criteria for the family $E_k(\alpha)$, as follows

Theorem 3.22. *Let $\alpha \in [0,1)$, $k \in \mathbb{N}$ and $f \in S$. Also, let $\sigma_{k,\alpha}$ be defined by*

$$\sigma_{k,\alpha} = \begin{cases} \displaystyle\sum_{n=2}^{\infty} \frac{n(n-\alpha)\cdot n!}{(n-k)!} |a_n|, & k \le 2 \\[2ex] \displaystyle\sum_{n=2}^{k-1} n(n-\alpha)|a_n| + \sum_{n=k}^{\infty} \frac{n(n-\alpha)\cdot n!}{(n-k)!} |a_n|, & k \ge 3. \end{cases} \tag{3.12}$$

If $\sigma_{k,\alpha} \le 1-\alpha$, then $f \in E_k(\alpha)$.

Proof. Similar to the proof of Theorem 3.7. $\square$

Remark 3.23. If $k=0$, then $E_0(\alpha) = K(\alpha)$ and we obtain the sufficient condition for convexity of order α (one may consult [5] or [14]).

Similar with Theorem 3.10, we can obtain some bounds for the coefficients of a function $f \in E_k(\alpha)$, as follows

Theorem 3.24. *Let $\alpha \in [0,1)$, $k \in \mathbb{N}$ and $f \in E_k(\alpha)$. Then*

$$|a_n| \le \frac{(n-k)!}{n!\cdot n!} \prod_{j=2}^{n} (j-2\alpha), \quad n \ge k \ge 2. \tag{3.13}$$

Proof. Let $f \in E_k(\alpha)$. Then $f \in S$ and $\mathcal{G}_k f \in K(\alpha)$. According to Remark 2.6 and the estimations proved by Robertson in [17] (see also [5]), we deduce that

$$|A_n| \le \frac{1}{n!} \prod_{j=2}^{n} (j-2\alpha), \tag{3.14}$$

for all $n \geq 2$, where A_n are the coefficients of $\mathcal{G}_k f$ defined by relation (2.4). Since

$$|A_n| = \begin{cases} |a_n|, & n \leq k-1 \\ \frac{n!}{(n-k)!}|a_n|, & n \geq k, \end{cases}$$

we obtain that

$$|a_n| \leq \frac{(n-k)!}{n! \cdot n!} \prod_{j=2}^{n} (j - 2\alpha), \quad n \geq k.$$

Taking into account the product considered in the last relation, we impose the condition $n \geq k \geq 2$ and this completes the proof. □

Corollary 3.25. *Let $k \in \mathbb{N}$ and $f \in E_k$. Then*

$$|a_n| \leq \frac{1}{n(n-1)(n-2) \cdot \ldots \cdot (n-k+1)} = \frac{(n-k)!}{n!}, \quad n \geq k. \tag{3.15}$$

Proof. In view of Theorem 3.24 for $\alpha = 0$. □

Remark 3.26. If $k = 0$, then $E_0 = K$ and we obtain the classical result related to the coefficient estimates for convex functions (see e.g. [2]).

Following the remarks presented before Theorem 3.14, we can prove the following general distortion result:

Theorem 3.27. *Let $k \in \mathbb{N}$. If $f \in E_k$, then*

$$\left| f^{(m)}(z) \right| \leq \frac{(m-k)!}{(1-|z|)^{m-k+1}}, \tag{3.16}$$

for all $m \geq k$ and $z \in \mathbb{U}$.

Proof. Let $f \in E_k$. Then $f \in S$ and $\mathcal{G}_k f \in K$. Moreover, for $m \in \mathbb{N}$, we have that

$$f^{(m)}(z) = \sum_{n=0}^{\infty} \frac{(m+n)!}{n!} a_{m+n} z^n, \quad z \in \mathbb{U}.$$

Let $r = |z| < 1$. In view of relation (3.15), we obtain

$$\begin{aligned} \left| f^{(m)}(z) \right| = \left| \sum_{n=0}^{\infty} \frac{(m+n)!}{n!} a_{m+n} z^n \right| &\leq \sum_{n=0}^{\infty} \frac{(m+n)!}{n!} |a_{m+n}| |z|^n \\ &\leq \sum_{n=0}^{\infty} \frac{(m+n)!(m+n-k)! r^n}{n!(m+n)!} r^n \\ &= (m-k)! \sum_{n=0}^{\infty} \frac{(m+n-k)!}{n!(m-k)!} r^n \\ &= (m-k)! \cdot \frac{1}{(1-r)^{m-k+1}}, \end{aligned}$$

according to Remark 3.13. Finally, we conclude that

$$\left| f^{(m)}(z) \right| \leq \frac{(m-k)!}{(1-|z|)^{m-k+1}}, \quad z \in \mathbb{U},$$

for all $m \geq k \geq 0$ and this completes the proof. $\square$

Remark 3.28. It is clear that for $k = 0$ we obtain the result proved by Goodman in [3, Theorem 9, Chapter 8].

3.2.1. The particular case $k = 1$ and $\alpha = 0$. The next section is dedicated to the study of a special form ($k = 1$ and $\alpha = 0$) of the class $E_k(\alpha)$. Because we consider such a particular case, we obtain some nice results and examples related to classical properties of univalent functions on the unit disc. According to Definition 3.20, we have that E_1 is defined by

$$E_1 = \{f \in S : \mathcal{G}_1 f \in K\},$$

where $\mathcal{G}_1 f(z) = zf'(z)$, for all $z \in \mathbb{U}$.

Remark 3.29. In view of the analytical characterization of convexity, we have the following equivalent definition

$$E_1 = \left\{ f \in S : \mathfrak{Re}\left[1 + \frac{z^2 f'''(z) + 2zf''(z)}{f'(z) + zf''(z)}\right] > 0, \quad z \in \mathbb{U} \right\}. \tag{3.17}$$

Indeed, $f \in S$ implies that $\mathcal{G}_1 f \in H(U)$. Moreover, according to the analytical characterization of convexity (see for example [5], [16]), it follows that $(\mathcal{G}_1 f)'(0) \neq 0$ (in fact, $(\mathcal{G}_1 f)'(0) = 1$) and $\mathfrak{Re}\big[1 + \frac{z(\mathcal{G}_1 f)''(z)}{(\mathcal{G}_1 f)'(z)}\big] > 0$, for all $z \in \mathbb{U}$. In view of Definition 2.1, we have that

$$\mathfrak{Re}\left[1 + \frac{z(\mathcal{G}_1 f)''(z)}{(\mathcal{G}_1 f)'(z)}\right] = \mathfrak{Re}\left[1 + \frac{z^2 f'''(z) + 2zf''(z)}{f'(z) + zf''(z)}\right] > 0,$$

for all $z \in \mathbb{U}$, which leads to the definition of E_1 given by (3.17).

Example 3.30. Let $f : \mathbb{U} \to \mathbb{C}$ be given by $f(z) = -\log(1 - z)$, for all $z \in \mathbb{U}$, where log is the principal branch of the complex logarithm. Then $f \in E_1$.

Proof. Indeed, $f \in S$ and $f'(z) = \frac{1}{1-z}$, for all $z \in \mathbb{U}$. Moreover,

$$\mathcal{G}_1 f(z) = zf'(z) = \frac{z}{1 - z}, \quad z \in \mathbb{U}.$$

Then $\mathcal{G}_1 f \in S$ and $\mathcal{G}_1 f(\mathbb{U}) = \{w \in \mathbb{C} : \mathfrak{Re} w > -\frac{1}{2}\}$ is a convex domain in $\mathbb{C}$. Hence, $\mathcal{G}_1 f \in K$ and this completes the proof. $\square$

Next, we present an important result that establishes the connection between classes E_1 and $K(1/2)$. In particular, we obtain that every function from E_1 is also convex. This proof of this result was given by the author and is based on the proof of [5, Theorem 2.3.2] given by Suffridge.

Proposition 3.31. *If $f \in E_1$, then $f \in K(1/2)$. This result is sharp.*

Proof. Let $f \in E_1$. Then $f \in S$ and $\mathcal{G}_1 f \in K$. Taking into account a classical result given by Sheil-Small (see [19]) and Suffridge (see [20]; also, one may consult [5]), we know that

$$\mathcal{G}_1 f \in K \quad \Leftrightarrow \quad \mathfrak{Re}\left[\frac{2z(\mathcal{G}_1 f)'(z)}{(\mathcal{G}_1 f)(z) - (\mathcal{G}_1 f)(\zeta)} - \frac{z + \zeta}{z - \zeta}\right] \geq 0,$$

for all $z, \zeta \in \mathbb{U}$. In particular, for $\zeta = 0$, we obtain that $\mathcal{G}_1 f \in K$ is equivalent to

$$\mathfrak{Re}\left[\frac{z(\mathcal{G}_1 f)'(z)}{(\mathcal{G}_1 f)(z)}\right] \geq 0, \quad z \in \mathbb{U}.$$

In view of (2.1) and the minimum principle for harmonic functions, we deduce that

$$\mathfrak{Re}\left[\frac{z^2 f''(z) + zf'(z)}{zf'(z)}\right] = \mathfrak{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > 0, \tag{3.18}$$

for all $z \in \mathbb{U}$. Hence, according to the definition of the convex functions of order α, we conclude that $f \in K(1/2)$. In order to prove that the result is sharp, it suffices to consider the function $f : \mathbb{U} \to \mathbb{C}$, given by $f(z) = -\log(1-z)$, for all $z \in \mathbb{U}$, where log is the principal branch of the complex logarithm and this completes the proof. □

Remark 3.32. In order to prove that the inclusion $E_1 \subsetneqq K(1/2)$ is strict, we can use the example given in the proof of Theorem 3 from [8]. If we consider $f : \mathbb{U} \to \mathbb{C}$ given by $f(z) = z + \frac{1}{6}z^2$, for all $z \in \mathbb{U}$, then $f \in K(1/2) \setminus E_1$.

Proof. Indeed, the main idea of the proof (cf. [8]) is the following: according to Corollary 3.8 we have that $f \in K(1/2)$. However, it is easy to prove that $\mathcal{G}_1 f \notin K$, where $(\mathcal{G}_1 f)(z) = zf'(z)$, for all $z \in \mathbb{U}$. Hence, $f \notin E_1$ and this completes the proof. □

Proposition 3.33. *If $f \in E_1$, then $f \in \mathcal{R}(1/2)$, i.e. $\mathfrak{Re} f'(z) > 1/2$, for all $z \in \mathbb{U}$.*

Proof. Let $f \in E_1$. Then $f \in S$ and $\mathcal{G}_1 f \in K$, where $(\mathcal{G}_1 f)(z) = zf'(z)$, for all $z \in \mathbb{U}$. In view of a result due to Marx and Strohhäcker (see for example [5]), we have that

$$\frac{1}{2} < \mathfrak{Re}\left[\frac{(\mathcal{G}_1 f)(z)}{z}\right] = \mathfrak{Re}\left[\frac{zf'(z)}{z}\right] = \mathfrak{Re} f'(z),$$

for all $z \in \mathbb{U}$. Hence, $\mathfrak{Re} f'(z) > \frac{1}{2}$, for all $z \in \mathbb{U}$ and this completes the proof. □

Theorem 3.34. *Let $f \in E_1$. Then*

$$\log(1 + |z|) \leq |f(z)| \leq -\log(1 - |z|) \tag{3.19}$$

and

$$\frac{1}{1 + |z|} \leq |f'(z)| \leq \frac{1}{1 - |z|}, \tag{3.20}$$

for all $z \in \mathbb{U}$. All of these estimates are sharp.

Proof. Since $f \in E_1$, we have that $f \in S$ and $\mathcal{G}_1 f \in K$, where $(\mathcal{G}_1 f)(z) = zf'(z)$, for all $z \in \mathbb{U}$. According to distortion theorem for the class K (see e.g. [5], [16]), we know that

$$\frac{r}{1+r} \leq \left|zf'(z)\right| \leq \frac{r}{1-r},$$

where $|z| = r$. Then

$$\frac{1}{1+r} \leq \left|f'(z)\right| \leq \frac{1}{1-r}, \tag{3.21}$$

where $|z| = r < 1$ and we obtain the distortion result for the class E_1. The upper bound in (3.19) follows easily by integrating the upper bound in (3.20) and the lower bound in (3.19) can be obtained using the arguments presented in the proof of Theorem 2.2.8 from [5]. Hence,

$$\log(1+r) \le |f(z)| \le -\log(1-r),$$

where $|z| = r < 1$. The sharpness of all of these estimates is ensured by the function defined in Example 3.30. □

Corollary 3.35. *If $f \in E_1$, then $f(\mathbb{U})$ contains the open disc $\mathcal{U}_{\ln 2}$.*

Proof. The result follows from the lower estimate in relation (3.19) on letting $r \to 1$. □

3.3. Connections between E_k^* and E_k

Based on the Alexander's duality theorem between convex and starlike functions on $\mathbb{U}$ (see [1], [2], [16]), we prove in this section similar duality results for the subclasses E_k^* and E_k.

Lemma 3.36. *Let $k \in \mathbb{N}$ and $f, g \in S$ be such that $g(z) = zf'(z)$, for all $z \in \mathbb{U}$. Then*

$$z(\mathcal{G}_k f)'(z) = (\mathcal{G}_k g)(z), \quad z \in \mathbb{U}. \tag{3.22}$$

Proof. It is clear that for $k = 0$, relation (3.22) reduces to the definition of g. Let us consider $k \ge 1$ and $f, g \in S$ such that $g(z) = zf'(z)$, for all $z \in \mathbb{U}$. By (2.1) we have

$$(\mathcal{G}_k f)(z) = z^k f^{(k)}(z) + a_{k-1} z^{k-1} + \ldots + a_1 z + a_0,$$

for all $z \in \mathbb{U}$, where $a_1 = 1$ and $a_0 = 0$. Then

$$z(\mathcal{G}_k f)'(z) = z^{k+1} f^{(k+1)}(z) + k z^k f^{(k)}(z) + \sum_{n=1}^{k-1} n a_n z^n, \tag{3.23}$$

for all $z \in \mathbb{U}$. According to Leibniz's formula, we deduce that

$$\begin{aligned}(\mathcal{G}_k g)(z) &= z^k g^{(k)}(z) + b_{k-1} z^{k-1} + \ldots + b_2 z^2 + b_1 z + b_0 \\ &= z^{k+1} f^{(k+1)}(z) + k z^k f^{(k)}(z) + b_{k-1} z^{k-1} + \ldots + b_2 z^2 + b_1 z + b_0 \\ &= z^{k+1} f^{(k+1)}(z) + k z^k f^{(k)}(z) + \sum_{n=1}^{k-1} n a_n z^n, \end{aligned} \tag{3.24}$$

for all $z \in \mathbb{U}$. Finally, in view of (3.23) and (3.24) we obtain that

$$z(\mathcal{G}_k f)'(z) = (\mathcal{G}_k g)(z), \quad z \in \mathbb{U}$$

and this completes the proof. □

Based on the previous lemma, we can obtain an Alexander type theorem for the families E_k^* and E_k.

Theorem 3.37. *Let $k \in \mathbb{N}$ and $f, g \in S$. Then $f \in E_k$ if and only if $g \in E_k^*$, where $g(z) = zf'(z)$, for all $z \in \mathbb{U}$.*

Proof. Let $f \in E_k$. According to the definition of the class E_k, we have that
$$f \in E_k \quad \Leftrightarrow \quad f \in S \quad \text{and} \quad \mathcal{G}_k f \in K.$$
Moreover,
$$\mathcal{G}_k f \in K \Leftrightarrow z(\mathcal{G}_k f)'(z) \in S^*,$$
for all $z \in \mathbb{U}$, in view of the Alexander's duality theorem. Using Lemma 3.36 we can rewrite the previous equivalence as
$$\mathcal{G}_k f \in K \Leftrightarrow z(\mathcal{G}_k f)'(z) = (\mathcal{G}_k g)(z) \in S^*,$$
for all $z \in \mathbb{U}$. Then
$$f \in E_k \Leftrightarrow \mathcal{G}_k f \in K \Leftrightarrow \mathcal{G}_k g \in S^* \Leftrightarrow g \in E_k^*,$$
where $g(z) = zf'(z)$, for all $z \in \mathbb{U}$, and this completes the proof. □

Remark 3.38. Since Theorem 3.37 is based on the Alexander's duality theorem, it is clear that for $k = 0$, we have that
$$f \in E_0 = K \quad \Leftrightarrow \quad g \in E_0^* = S^*,$$
where $g(z) = zf'(z)$, for all $z \in \mathbb{U}$.

Remark 3.39. Another interesting remark is that, taking into account Definition 2.1, we can rewrite Theorem 3.37 as follows
$$f \in E_k \Leftrightarrow \mathcal{G}_1 f \in E_k^*, \tag{3.25}$$
for all $k \in \mathbb{N}$, where $\mathcal{G}_1 f$ is given by (2.1).

Theorem 3.40. *Let $k \in \mathbb{N}$. If $f \in E_k$, then $f \in E_k^*(1/2)$.*

Proof. Let $f \in E_k$. Then $f \in S$ and $\mathcal{G}_k f \in K$. According to a result given by Sheil-Small and Suffridge (see e.g. [5]), we know that
$$\mathfrak{Re}\left[\frac{z(\mathcal{G}_k f)'(z)}{(\mathcal{G}_k f)(z)}\right] > 0, \quad z \in \mathbb{U}.$$
Hence, since $f \in S$ and $\mathcal{G}_k f \in S^*(1/2)$, it follows that $f \in E_k^*(1/2)$ and this completes the proof. □

Remark 3.41. It is clear that Theorem 3.40 is a generalization of Proposition 3.31 (where $k = 1$). On the other hand, if $k = 0$, then Theorem 3.40 reduces to [5, Theorem 2.3.2] due to Marx and Strohhäcker.

Finally, we end this section with some questions related to the subclasses E_k and E_k^* studied above. First question is a generalization of Proposition 3.31:

Question 3.42. Is it true that $E_{k+1} \subset E_k$, for all $k \in \mathbb{N}$?

Clearly, a similar question can be formulated also for the subclass E_k^*. Another important property of these subclasses is the compactness. Hence, one may ask

Question 3.43. Is it true that the subclasses E_k and E_k^* are compact in $\mathcal{H}(\mathbb{U})$?

Since E_k^* and E_k are subclasses of the class S, it would be interesting to study also other geometric and analytic properties of them.

3.4. The subclass $E_{\mathbb{N}}$

Let $k \in \mathbb{N}$ and $f \in \bigcap\limits_{k\in\mathbb{N}} E_k$. Then, for every $k \in \mathbb{N}$, we have that $f \in E_k$. Moreover, according to Corollary 3.25, it follows that for every $k \in \mathbb{N}$

$$|a_n| \leq \frac{(n-k)!}{n!}, \quad n \geq k.$$

In particular, for $n = k$ we obtain that $|a_k| \leq \frac{1}{k!}$, for every $k \in \mathbb{N}$. Let us denote by

$$E_{\mathbb{N}} = \left\{ f \in S \,:\, |a_n| \leq \frac{1}{n!},\, n \geq 2 \right\}. \tag{3.26}$$

Then, we obtain the following remark

Remark 3.44. Let $E_{\mathbb{N}}$ be the set defined by (3.26). Then $\bigcap_{k\in\mathbb{N}} E_k \subsetneqq E_{\mathbb{N}}$, i.e. the intersection of all subclasses E_k is included in $E_{\mathbb{N}}$, but it is not equal with $E_{\mathbb{N}}$.

Indeed, we can construct an example of a function $f \in S$ that belongs to the family $E_{\mathbb{N}}$, but not to $\bigcap_{k\in\mathbb{N}} E_k$ (in fact, we prove that $f \notin E_1$), as follows

Example 3.45. Let $f : \mathbb{U} \to \mathbb{C}$ be defined by $f(z) = z + az^2$, for all $z \in \mathbb{U}$, where $a = 1/2$. Then $f \in E_{\mathbb{N}}$, but $f \notin \bigcap_{k\in\mathbb{N}} E_k$.

Proof. It is clear that $f \in S$ and $|a_2| = |a| \leq \frac{1}{2}$. Hence, in view of relation (3.26) we deduce that $f \in E_{\mathbb{N}}$. On the other hand,

$$(\mathcal{G}_1 f)(z) = zf'(z) = z(1+2az) = z + 2az^2, \quad z \in \mathbb{U}, \quad a = 1/2.$$

Let us denote

$$h(z) = 1 + \frac{z(\mathcal{G}_1 f)''(z)}{(\mathcal{G}_1 f)'(z)} = 1 + \frac{4az}{1+4az} = \frac{1+8az}{1+4az},$$

for all $z \in \mathbb{U}$, where $a = 1/2$. Then h is a Möbius function on $\mathbb{U}$ such that $h(0) = 1$, $h(1) = \frac{5}{3}$, $h(i) = \frac{9}{5} + \frac{2}{5}i$ and $h(-i) = \frac{9}{5} - \frac{2}{5}i$. In other words, we obtain that

$$h(\mathbb{U}) = \mathbb{C} \setminus \overline{\mathcal{U}}(7/3, 2/3) = \left\{ x + iy \in \mathbb{C} \,:\, \left(x - \frac{7}{3}\right)^2 + y^2 > \left(\frac{2}{3}\right)^2 \right\},$$

i.e., $h(\mathbb{U})$ is the complementary part of the closed disc $\overline{\mathcal{U}}\left(\frac{7}{3}, \frac{2}{3}\right)$ of center $w_0 = \frac{7}{3}$ and radius $r = \frac{2}{3}$. Moreover, for every point $w \in h(\mathbb{U}) \cap \{x + iy \in \mathbb{C} : x < 0\}$ we have that $\mathfrak{Re}\, w < 0$, i.e. there exists $z_0 \in \mathbb{U}$ such that $\mathfrak{Re}\, h(z_0) < 0$. For example, if $z_0 = -\frac{1}{3}$, then $z_0 \in \mathbb{U}$ and simple computations show that

$$\mathfrak{Re}\, h(z_0) = \mathfrak{Re}\left[1 + \frac{z_0(\mathcal{G}_1 f)''(z_0)}{(\mathcal{G}_1 f)'(z_0)}\right] = -1 < 0.$$

Hence, according to the behavior of the function h on $\mathbb{U}$ and the analytical characterization of convexity (see for example [3] or [5]), we deduce that $\mathcal{G}_1 f \notin K$.

Since $f \in S$, but $\mathcal{G}_1 f \notin K$, we obtain (according to Definition 3.20) that $f \notin E_1$. Now, it is clear that $f \notin \bigcap_{k\in\mathbb{N}} E_k$ and this completes the proof. □

Another interesting example (considered also in [15]) which generates important remarks about the class $E_{\mathbb{N}}$ is the following

Example 3.46. Let $f : \mathbb{U} \to \mathbb{C}$ be given by $f(z) = e^z - 1$, for all $z \in \mathbb{U}$. Then $f \in E_{\mathbb{N}}$.

Proof. Indeed, $f \in S$ and

$$f(z) = e^z - 1 = \sum_{n=0}^{\infty} \frac{z^n}{n!} - 1 = z + \frac{z^2}{2} + \frac{z^3}{6} + ... = z + \sum_{n=2}^{\infty} a_n z^n,$$

where $a_n = \frac{1}{n!}$, for all $n \geq 2$. Hence, $f \in E_{\mathbb{N}}$. $\square$

Taking into account relation (3.8), we can prove the following result

Proposition 3.47. *Let $m \in \mathbb{N}$. If $f \in E_{\mathbb{N}}$, then $\left|f^{(m)}(z)\right| \leq e^{|z|}$, for all $z \in \mathbb{U}$.*

Proof. Let $f \in E_{\mathbb{N}}$ and $|z| = r < 1$. Then $f \in S$ and in view of (3.8) we have that

$$\left|f^{(m)}(z)\right| = \left|\sum_{n=0}^{\infty} \frac{(m+n)!}{n!} a_{m+n} z^n\right| \leq \sum_{n=0}^{\infty} \frac{(m+n)!}{n!} |a_{m+n}||z|^n = \sum_{n=0}^{\infty} \frac{r^n}{n!} = e^r,$$

where $|z| = r < 1$. Hence, we obtain that

$$\left|f^{(m)}(z)\right| \leq e^{|z|},$$

for $z \in \mathbb{U}$ and this completes the proof. $\square$

It is clear that Proposition 3.47 has the following consequence (for the particular case $z = 0$):

Corollary 3.48. *Let $m \in \mathbb{N}$. If $f \in E_{\mathbb{N}}$, then $|f^{(m)}(0)| \leq 1$, for all $m \in \mathbb{N}$.*

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