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# Harmonic close-to-convex mappings associated with Sălăgean $q$-differential operator

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**Abstract.** In this paper, we define a new subclass $\mathcal{W}(n,\alpha,q)$ of analytic functions and a new subclass $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ of harmonic functions $f = h+\overline{g} \in \mathcal{H}^0$ associated with Sălăgean $q$-differential operator. We prove that a harmonic function $f = h+\bar{g}$ belongs to the class $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ if and only if the analytic functions $h+\epsilon g$ belong to $\mathcal{W}(n,\alpha,q)$ for each $\epsilon$ ($|\epsilon| = 1$), and using a method by Clunie and Sheil-Small, we determine a sufficient condition for the class $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ to be close-to-convex. We provide sharp coefficient estimates, sufficient coefficient condition, and convolution properties for such functions classes. We also determine several conditions of partial sums of $f \in \mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$.

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**Keywords:** Sălăgean $q$-differential operator, analytic functions, harmonic functions, partial sums.

## 1. Introduction

Quantum calculus is the calculus without use of the limits. The history of quantum calculus dates back to the studies of Leonhard Euler (1707-1783) and Carl Gustav Jacobi (1804-1851). Later, geometrical interpretation of the $q$-calculus has been applied in studies of quantum groups. The great interest to quantum calculus is due to its applications in various branches of mathematics and physics; as for example, in quantum mechanics, analytic number theory, sobolev spaces, group representation theory, theta functions, gamma functions, operator theory and several other areas. For the definitions and properties of $q$-calculus, one may refer to the books [5] and

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[14]. Jackson [10, 11] was the first who gave some applications of $q$-calculus by introducing the $q$-analogues of derivative and integral. The $q-$derivative (or $q-$difference operator) of a function $h$, defined on a subset of $\mathbb{C}$, is given by

$$(D_q h)(z) = \begin{cases} \frac{h(z)-h(qz)}{(1-q)z}, & z \neq 0 \\ \\ h'(0), & z = 0, \end{cases}$$

where $q \in (0,1)$. Note that $\lim_{q\to 1^-}(D_q h)(z) = h'(z)$ if $h$ is differentiable at $z$ ([10]).

For a function $h(z) = z^k$ $(k \in \mathbb{N})$, we observe that

$$D_q z^k = [k]_q z^{k-1},$$

where

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + q^2 + \ldots q^{k-1}$$

is the $q-$number of $k$. Clearly, $\lim_{q\to 1^-}[k]_q = k$. For more details, one may refer to [14] and references therein.

Connection of $q$-calculus with geometric function theory was first introduced by Ismail *et al.* [9]. Recently, $q$-calculus is involved in the theory of analytic functions [7, 8, 21]. But research on $q$-calculus in connection with harmonic functions is fairly new and not much published (see [12, 23, 22, 28]).

Let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ denote an open disk with $r > 0$. The open unit disk will be denoted by $\mathbb{D}_1 = \mathbb{D}$. Let $\mathcal{H}$ denote the class of complex-valued functions $f = u + iv$ which are harmonic in the open unit disk $\mathbb{D}$, where $u$ and $v$ are real-valued harmonic functions in $\mathbb{D}$. Functions $f \in \mathcal{H}$ can also be expressed as $f = h + \overline{g}$, where $h$ the analytic and $g$ the co-analytic parts of $f$, respectively. A subclass of functions $f = h + \overline{g} \in \mathcal{H}$ with the additional condition $g'(0) = 0$ is denoted by $\mathcal{H}^0$. According to the Lewy's Theorem [15], every harmonic function $f = h + \overline{g} \in \mathcal{H}$ is locally univalent and sense preserving in $\mathbb{D}$ if and only if the Jacobian of $f$, given by $J_f(z) = |h'(z)|^2 - |\, g'(z)|^2$, is positive in $\mathbb{D}$. This case is equivalent to the existence of an analytic function $\omega(z) = g'(z)/h'(z)$ in $\mathbb{D}$, which is called as the dilatation of $f$ such that

$$|\omega(z)| < 1 \quad \text{for all} \quad z \in \mathbb{D}.$$

Clunie and Sheil-Small [3] introduced the class of all univalent, sense preserving harmonic functions $f = h + \overline{g}$, denoted by $\mathcal{S}_{\mathcal{H}}$, with the normalized conditions $h(0) = 0 = g(0)$ and $h'(0) = 1$ . If the function $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$, then

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad (z \in \mathbb{D})\,. \tag{1.1}$$

A subclass of functions $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$ with the condition $g'(0) = 0$ is denoted by $\mathcal{S}_{\mathcal{H}}^0$. Further, the subclass of functions $f$ in $\mathcal{S}_{\mathcal{H}}$ $\left(\mathcal{S}_{\mathcal{H}}^0\right)$, denoted by $\mathcal{K}_{\mathcal{H}}$ $\left(\mathcal{K}_{\mathcal{H}}^0\right)$ consists of functions $f$ that map the unit disk $\mathbb{D}$ onto a convex region, the subclass $\mathcal{S}_{\mathcal{H}}^*$ $\left(\mathcal{S}_{\mathcal{H}}^{*0}\right)$ consists of functions $f$ that are starlike, and the subclass $\mathcal{C}_{\mathcal{H}}^*$ $\left(\mathcal{C}_{\mathcal{H}}^{*0}\right)$ consists of functions $f$ which are close-to-convex. Also, if $g(z) \equiv 0$, the class $\mathcal{S}_{\mathcal{H}}$ reduces to the class $\mathcal{S}$ of univalent functions in the class $\mathcal{A}$. Here, $\mathcal{A}$ is the class of all analytic functions of the form $h(z) = z + \sum_{k=2}^{\infty} a_k z^k$. For more details, we refer [4].

Let $f \in \mathcal{S}$ and be given by $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then the $l^{th}$ section (partial sum) of $f$ is defined by

$$s_l(f)(z) = \sum_{k=0}^{l} a_k z^k, \quad (l \in \mathbb{N})$$

where $a_0 = 0$ and $a_1 = 1$. For a harmonic function $f = h + \overline{g} \in \mathcal{H}$, where $h$ and $g$ of the form (1.1), the sequences of sections (partial sums) of $f$ is defined by

$$s_{i,j}(f)(z) = s_i(h)(z) + \overline{s_j(g)(z)},$$

where $s_i(h)(z) = \sum_{k=1}^{i} a_k z^k$ and $s_j(g)(z) = \sum_{k=1}^{j} b_k z^k$, $i, j \geq 1$ with $a_1 = 1$.

In [32], it is noted that the partial sums of univalent functions is univalent in the disk $\mathbb{D}_{1/4}$. Starlikeness and convexity of the partial sums of univalent functions was discussed in [29, 30].

The convolution or Hadamard product of two analytic functions

$$f_1(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{and} \quad f_2(z) = \sum_{k=0}^{\infty} b_k z^k$$

is defined by

$$(f_1 * f_2)(z) = \sum_{k=0}^{\infty} a_k b_k z^k, \quad (z \in \mathbb{D}).$$

The convolution of two harmonic functions $f = h + \overline{g}$ and $F = H + \overline{G}$ is defined by

$$(f * F)(z) = (h * H)(z) + \overline{(g * G)(z)}, \quad (z \in \mathbb{D}).$$

In 2013, Li and Ponnusamy [16] investigated properties of functions given by

$$\mathcal{P}_{\mathcal{H}}^0 = \{f = h + \overline{g} \in \mathcal{H}^0 : \Re(h'(z)) > |g'(z)|, \quad z \in \mathbb{D}\}$$

The class $\mathcal{P}_{\mathcal{H}}^0$ is harmonic analogue of the class $\mathcal{R} = \{f \in \mathcal{S} : \Re(f'(z)) > 0, \ z \in \mathbb{D}\}$ introduced by MacGregor [20]. It is known that a harmonic function $f = h + \overline{g}$ belongs to the class $\mathcal{P}_{\mathcal{H}}^0$ if and only if the analytic function $h + \epsilon g$ belongs to $\mathcal{R}$ for each $\epsilon$ ($|\epsilon| = 1$).

In 1977, Chichra [2] studied the class $\mathcal{W}(\alpha)$ consisting of functions $f \in \mathcal{A}$ such that $\Re(f'(z) + \alpha z f''(z)) > 0$ for $\alpha \geq 0$ and $z \in \mathbb{D}$. Later, Nagpal and Ravichandran [24] studied the following class

$$\mathcal{W}_{\mathcal{H}}^0 = \{f = h + \overline{g} \in \mathcal{H}^0 : \Re(h'(z) + zh''(z)) > |g'(z) + zg''(z)|, \ z \in \mathbb{D}\},$$

which is harmonic analogue of $\mathcal{W}(1)$. Recently, Ghosh and Vasudevarao [6] defined the class $\mathcal{W}_{\mathcal{H}}^0(\alpha)$ for $\alpha \geq 0$ by

$$\mathcal{W}_{\mathcal{H}}^0(\alpha) = \{f = h + \overline{g} \in \mathcal{H}^0 : \Re(h'(z) + \alpha z h''(z)) > |g'(z) + \alpha z g''(z)|, \ z \in \mathbb{D}\}.$$

In [2], Chichra also studied the class $\mathcal{G}(\alpha)$ of an analytic function $f$ for $\alpha \geq 0$ such that

$$\Re\left[(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right] > 0$$

for $|z| < r$ with $r \in (0,1]$. In 2018, Liu ang Yang [19] defined the class

$$\mathcal{G}^k_{\mathcal{H}}(\alpha) = \left\{ f = h + \bar{g} \in \mathcal{H}^0 : \Re\big((1-\alpha)\frac{h(z)}{z} + \alpha h'(z)\big) > \big|(1-\alpha)\frac{g(z)}{z} + \alpha g'(z)\big| \right\},$$

where $\alpha \geq 0$, $k \geq 1$ and $|z| < r$ with $r \in (0,1]$.

For an analytic function $h \in \mathcal{A}$, let the Sălăgean $q$-differential operator be defined by ([7]);

$$\mathcal{D}^0_q h(z) = h(z), \quad \mathcal{D}^1_q h(z) = zD_q h(z), ..., \quad \mathcal{D}^n_q h(z) = zD_q(\mathcal{D}^{n-1}_q h(z)),$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Making use of $h$ given by (1.1), and simple calculations yield

$$\mathcal{D}^n_q h(z) = h(z) * \mathcal{F}_{q,n}(z) = z + \sum_{k=2}^{\infty} [k]^n_q a_k z^k, \quad (z \in \mathbb{D}) \tag{1.2}$$

where

$$\mathcal{F}_{q,n}(z) = z + \sum_{k=2}^{\infty} [k]^n_q z^k,$$

and $[k]^n_q = \left(\frac{1-q^k}{1-q}\right)^n$, $q \in (0,1)$. The operator (1.2) easily reduces to the well-known Sălăgean differential operator as $q \to 1^-$ (see [27]).

For a harmonic function $f = h + \overline{g}$ given by (1.1) and the operator $\mathcal{D}^n_q$ defined by (1.2), the harmonic Sălăgean $q$-differential operator is defined by ([12]);

$$\begin{aligned} \mathcal{D}^n_q f(z) &= \mathcal{D}^n_q h(z) + (-1)^n \overline{\mathcal{D}^n_q g(z)} \\ &= z + \sum_{k=2}^{\infty} [k]^n_q a_k z^k + (-1)^n \overline{\sum_{k=1}^{\infty} [k]^n_q b_k z^k}. \end{aligned}$$

As $q \to 1^-$, the operator $\mathcal{D}^n_q f$ reduces to the Sălăgean differential operator $\mathcal{D}^n f$ for a harmonic function $f = h + \bar{g}$ ([13]).

Motivated by the Sălăgean $q$-differential operator, we define a new subclass $\mathcal{W}(n, \alpha, q)$ of analytic functions as follows:

**Definition 1.1.** An analytic function $f \in \mathcal{A}$ is in the class $\mathcal{W}(n, \alpha, q)$ if it satisfies the condition

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}^n_q f(z) + \alpha \mathcal{D}^{n+1}_q f(z)}{z}\right) > 0, \tag{1.3}$$

where $\mathcal{D}^n_q f(z)$ is the Sălăgean $q$-differential operator defined by (1.2), and where $\alpha \geq 0$, $n \in \mathbb{N}_0$, $q \in (0,1)$ and $|z| < r$ with $0 < r \leq 1$.

**Remark 1.2.** i) Letting $q \to 1^-$, $n = 0$ we get the class $\mathcal{W}(0, \alpha, q) := \mathcal{G}(\alpha)$ introduced by Chichra [2].

ii) Letting $q \to 1^-$, $n = 1$ we get the class $\mathcal{W}(1, \alpha, q) := \mathcal{W}(\alpha)$ introduced by Chichra [2].

iii) Letting $q \to 1^-$, $n = 1$, $\alpha = 0$ we get the class $\mathcal{W}(1, 0, q) := \mathcal{R}$ introduced by MacGregor [20].

Making use of the harmonic Sălăgean $q$-differential operator, we also define the class $\mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$ of harmonic functions as follows:

**Definition 1.3.** A harmonic function $f = h + \overline{g} \in \mathcal{H}^0$ with $h(0) = g(0) = g'(0) = h'(0) - 1 = 0$ is in the class $\mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$ if it satisfies the condition

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}^n_q h(z) + \alpha\mathcal{D}^{n+1}_q h(z)}{z}\right) > \left|\frac{(1-\alpha)\,\mathcal{D}^n_q g(z) + \alpha\mathcal{D}^{n+1}_q g(z)}{z}\right|, \tag{1.4}$$

where $\mathcal{D}^n_q f(z)$ is the harmonic Sălăgean $q$-differential operator, and where $\alpha \geq 0$, $n \in \mathbb{N}_0$, $q \in (0, 1)$ and $|z| < r$ with $0 < r \leq 1$.

**Remark 1.4.** i) Letting $q \to 1^-$, $n = 0$ we get the class $\mathcal{W}^0_{\mathcal{H}}(0, \alpha, q) := \mathcal{G}^1_{\mathcal{H}}(\alpha)$ introduced by Liu ang Yang [19].

ii) Letting $q \to 1^-$, $n = 1$ we get the class $\mathcal{W}^0_{\mathcal{H}}(1, \alpha, q) := \mathcal{W}^0_{\mathcal{H}}(\alpha)$ introduced by Ghosh and Vasudevarao [6].

iii) Letting $q \to 1^-$, $n = 1$, $\alpha = 1$ we get the class $\mathcal{W}^0_{\mathcal{H}}(1, 1, q) := \mathcal{W}^0_{\mathcal{H}}$ introduced by Nagpal and Ravichandran in [24].

iv) Letting $q \to 1^-$, $n = 1$, $\alpha = 0$ we get the class $\mathcal{W}^0_{\mathcal{H}}(1, 0, q) := \mathcal{P}^0_{\mathcal{H}}$ introduced by Li and Ponnusamy [16].

In this paper, we define a new subclass $\mathcal{W}(n, \alpha, q)$ of analytic functions and a new subclass $\mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$ of harmonic functions $f = h + \overline{g} \in \mathcal{H}^0$ associated with Sălăgean $q$-differential operator. In Section 2, we prove that a harmonic function $f \in \mathcal{H}^0$ belongs to the class $\mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$ if and only if the analytic functions $h + \epsilon g$ belong to $\mathcal{W}(n, \alpha, q)$ for each $\epsilon$ with $|\epsilon| = 1$, and by a method of Clunie and Sheil-Small, we obtain a sufficient condition for the class $\mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$ to be close-to-convex. We also provide sharp coefficient estimates and sufficient coefficient condition for such functions classes. In Section 3, we examine that the class $\mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$ is closed under convex combinations and convolutions of its members. In Section 4, we determine several conditions of partial sums of $f \in \mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$.

## 2. Coefficient bounds

Clunie and Sheil-Small proved the following result, which gives a sufficient condition for a harmonic function $f$ to be close-to-convex.

**Lemma 2.1.** [3] *If $h$ and $g$ are analytic in $\mathbb{D}$ satisfies $|g'(0)| < |h'(0)|$ and the function $f_\epsilon = h + \epsilon g$ is close-to-convex for all complex number $\epsilon$ with $|\epsilon| = 1$, then $f = h + \overline{g}$ is close-to-convex..*

**Theorem 2.2.** *A harmonic mapping $f = h + \overline{g}$ is in $\mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$ if and only if the analytic function $f_\epsilon = h + \epsilon g$ belongs to $\mathcal{W}(n, \alpha, q)$ for each complex number $\epsilon$ with $|\epsilon| = 1$.*

*Proof.* If $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$, then for each complex number $\epsilon$ with $|\epsilon| = 1$

$$\begin{aligned}
&\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n f_\epsilon(z) + \alpha\mathcal{D}_q^{n+1} f_\epsilon(z)}{z}\right)\\
&= \Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n (h(z)+\epsilon g(z)) + \alpha\mathcal{D}_q^{n+1}(h(z)+\epsilon g(z))}{z}\right)\\
&= \Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1} h(z) + \epsilon\big((1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)\big)}{z}\right)\\
&> \Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1} h(z)}{z}\right) - \left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)}{z}\right| > 0,
\end{aligned}$$

thus $f_\epsilon = h + \epsilon g \in \mathcal{W}(n,\alpha,q)$ for each $\epsilon$ with $|\epsilon| = 1$.

Conversely, if $f_\epsilon = h + \epsilon g \in \mathcal{W}(n,\alpha,q)$, then

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1} h(z) + \epsilon\big((1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)\big)}{z}\right) > 0, \;\; (z \in \mathbb{D}_r)$$

or equivalently

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1} h(z)}{z}\right) > -\Re\left(\frac{\epsilon\big((1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)\big)}{z}\right), (z \in \mathbb{D}_r).$$

Since $|\epsilon| = 1$ is arbitrary, for an appropriate choice of $\epsilon$ we obtain

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1} h(z)}{z}\right) > \left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)}{z}\right|, \;\; (z \in \mathbb{D}_r)$$

Hence, $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$. □

**Theorem 2.3.** *The functions in the class* $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ *are close-to-convex in* $\mathbb{D}$.

*Proof.* Let $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$, and let $f_\epsilon = h + \epsilon g \in \mathcal{W}(n,\alpha,q)$ where $|\epsilon| = 1$. By the method used by Ponnusammy *et al.* [25, Theorem 1.3], if $f_\epsilon \in \mathcal{W}(n,\alpha,q)$, then $q$-derivative of $f_\epsilon$ is positive; that is, $\Re\{\mathcal{D}_q^n f_\epsilon\} > 0$, and hence $f_\epsilon$ is analytic and close-to-convex function. Therefore,

$$\begin{aligned}
&\Re\{\mathcal{D}_q^n f_\epsilon\} =\\
&\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1} h(z) + \epsilon\big((1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)\big)}{z}\right)\\
&> \left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)}{z}\right| + \Re\left(\frac{\epsilon\big((1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)\big)}{z}\right)\\
&\geq \left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)}{z}\right| - \left|\frac{\epsilon\big((1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)\big)}{z}\right| = 0,
\end{aligned}$$

showing that $f_\epsilon$ is analytic and close-to-convex function. Thus according to Lemma 2.1 and Theorem 2.2, it follows that the harmonic function $f \in \mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ is also close-to-convex in $\mathbb{D}$. □

We now establish the sharp coefficient bounds for functions in the class $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$.

**Theorem 2.4.** *Let $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$ be of the form (1.1) with $b_1 = 0$. Then for any $k \geq 2$*

$$|b_k| \leq \frac{1}{[k]^n_q(1 + \alpha([k]_q - 1))}. \tag{2.1}$$

*The result is sharp when $f$ is given by $f(z) = z + \frac{1}{[k]^n_q(1+\alpha([k]_q-1))}\overline{z}^k$.*

*Proof.* Let $f \in \mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$. Then

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}^n_q h(z) + \alpha \mathcal{D}^{n+1}_q h(z)}{z}\right) > \left|\frac{(1-\alpha)\,\mathcal{D}^n_q g(z) + \alpha \mathcal{D}^{n+1}_q g(z)}{z}\right|$$

and

$$\frac{(1-\alpha)\,\mathcal{D}^n_q g(z) + \alpha \mathcal{D}^{n+1}_q g(z)}{z} = \sum_{k=2}^{\infty} [k]^n_q(1 + \alpha([k]_q - 1)) b_k z^{k-1}.$$

Using the series expansion of $g$, we derive

$$\begin{aligned} r^{k-1}[k]^n_q(1 + \alpha([k]_q - 1))|b_k| &\leq \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{(1-\alpha)\,\mathcal{D}^n_q g(re^{i\theta}) + \alpha \mathcal{D}^{n+1}_q g(re^{i\theta})}{re^{i\theta}}\right| d\theta \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \Re\left(\frac{(1-\alpha)\,\mathcal{D}^n_q h(re^{i\theta}) + \alpha \mathcal{D}^{n+1}_q h(re^{i\theta})}{re^{i\theta}}\right) d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \Re\left(1 + [k]^n_q\big(1 + \alpha([k]_q - 1)\big) a_k r^{k-1}\right) d\theta \\ &= 1. \end{aligned}$$

Letting $r \to 1^-$ gives the desired bound. □

**Remark 2.5.** (i) When $q \to 1^-$, $n = 0$ we get the result by Liu ang Yang [19, Corollary 3.2].

(ii) When $q \to 1^-$, $n = 1$ we get the result by Ghosh and Vasudevarao [6, Theorem 4.2].

**Theorem 2.6.** *Let $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$ be of the form (1.1) with $b_1 = 0$. Then for any $k \geq 2$*

(i) $|a_k| + |b_k| \leq \frac{2}{[k]^n_q(1+\alpha([k]_q-1))}$

(ii) $||a_k| - |b_k|| \leq \frac{2}{[k]^n_q(1+\alpha([k]_q-1))}$

(iii) $|a_k| \leq \frac{2}{[k]^n_q(1+\alpha([k]_q-1))}$

*The results are sharp and the equality is held for the function*

$$f(z) = z + \sum_{k=2}^{\infty} \frac{2}{[k]^n_q(1 + \alpha([k]_q - 1))} z^k.$$

*Proof.* Suppose that $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$, then from Theorem 2.2 $f_\epsilon = h + \epsilon g \in \mathcal{W}(n, \alpha, q)$ for $\epsilon$ with $|\epsilon| = 1$. Thus for any $|\epsilon| = 1$, we have

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}^n_q(h(z) + \epsilon g(z)) + \alpha \mathcal{D}^{n+1}_q(h(z) + \epsilon g(z))}{z}\right) > 0, \quad |z| < r.$$

Then there exists an analytic function $p$ of the form $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ with $\Re(p(z)) > 0$ in $\mathbb{D}$ such that

$$\frac{(1-\alpha)\,\mathcal{D}_q^n(h(z)+\epsilon g(z)) + \alpha \mathcal{D}_q^{n+1}(h(z)+\epsilon g(z))}{z} = p(z). \tag{2.2}$$

Comparing coefficients on both sides of (2.2), we have

$$[k]_q^n(1+\alpha([k]_q-1))(a_k+\epsilon b_k) = p_{k-1}, \quad k \geq 2. \tag{2.3}$$

Since $|p_k| \leq 2$ for $k \geq 1$ and $\epsilon$ ($|\epsilon| = 1$) is arbitrary, from (2.3) we get

$$[k]_q^n(1+\alpha([k]_q-1))(|a_k|+|b_k|) \leq 2,$$

which proves (i). The last two inequalities are consequences of the first inequality. □

**Remark 2.7.** (i) When $q \to 1^-$, $n = 0$ we get the result by Liu ang Yang [19, Corollary 3.4].

(ii) When $q \to 1^-$, $n = 1$ we get the result by Ghosh and Vasudevarao [6, Theorem 4.3].

The following result gives a sufficient condition for a function to be belong to $\mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$.

**Theorem 2.8.** *Let $f = h + \overline{g} \in \mathcal{H}^0$ be of the form (1.1) with $b_1 = 0$. If*

$$\sum_{k=2}^{\infty} [k]_q^n(1+\alpha([k]_q-1))(|a_k|+|b_k|) \leq 1, \tag{2.4}$$

*then $f \in \mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$.*

*Proof.* Let $f = h + \overline{g} \in \mathcal{H}^0$. Using the series representation of $h$ given by (1.1), we get

$$\begin{aligned}
\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha \mathcal{D}_q^{n+1} h(z)}{z}\right) &= \Re\left(1 + \sum_{k=2}^{\infty} [k]_q^n(1+\alpha([k]_q-1)a_k z^{k-1}\right) \\
&> 1 - \sum_{k=2}^{\infty} [k]_q^n(1+\alpha([k]_q-1)|a_k| \\
&\geq \sum_{k=2}^{\infty} [k]_q^n(1+\alpha([k]_q-1)|b_k| \\
&> \left|\sum_{k=2}^{\infty} [k]_q^n(1+\alpha([k]_q-1)b_k z^{k-1}\right| \\
&= \left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha \mathcal{D}_q^{n+1} g(z)}{z}\right|,
\end{aligned}$$

therefore $f \in \mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$. □

**Remark 2.9.** When $q \to 1^-$, $n = 1$ we get the result by Ghosh and Vasudevarao [6, Theorem 4.5].

## 3. Convex combinations and convolutions

In this section, we prove that the class $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ is closed under convex combinations and convolutions of its members.

**Theorem 3.1.** *The class $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ is closed under convex combinations.*

*Proof.* Suppose $\mathcal{D}^n_q f_i = \mathcal{D}^n_q h_i + (-1)^n\overline{\mathcal{D}^n_q g_i} \in \mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ for $i = 1,2,...,k$ and $\sum_{i=1}^k t_i = 1$ $(0 \le t_i \le 1)$. The convex combination of functions $\mathcal{D}^n_q f_i$ can be written as

$$\mathcal{D}^n_q f(z) = \sum_{i=1}^k t_i \mathcal{D}^n_q f_i(z) = \mathcal{D}^n_q h(z) + (-1)^n \mathcal{D}^n_q \overline{g(z)}$$

where $\mathcal{D}^n_q h(z) = \sum_{i=1}^k t_i \mathcal{D}^n_q h_i(z)$ and $\mathcal{D}^n_q g(z) = \sum_{i=1}^k t_i \mathcal{D}^n_q g_i(z)$. Then $h$ and $g$ both are analytic in $\mathbb{D}$ with $h(0) = g(0) = h'(0) - 1 = g'(0) = 0$. A simple computation yields

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}^n_q h(z) + \alpha\mathcal{D}^{n+1}_q h(z)}{z}\right) = \Re\left(\sum_{i=1}^k t_i \frac{(1-\alpha)\,\mathcal{D}^n_q h_i(z) + \alpha\mathcal{D}^{n+1}_q h_i(z)}{z}\right)$$

$$> \left|\sum_{i=1}^k t_i \frac{(-1)^n\,(1-\alpha)\,\mathcal{D}^n_q g_i(z) + (-1)^{n+1}\alpha\mathcal{D}^{n+1}_q g_i(z)}{z}\right|$$

$$\ge \left|\frac{(1-\alpha)\,\mathcal{D}^n_q g(z) + \alpha\mathcal{D}^{n+1}_q g(z)}{z}\right|.$$

This shows that $f \in \mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ □

A sequence $\{c_k\}_{k=0}^{\infty}$ of non-negative real numbers is said to be a convex null sequence if $c_k \to 0$ as $k \to \infty$, and

$$c_0 - c_1 \ge c_1 - c_2 \ge c_2 - c_3 \ge ... \ge c_{k-1} - c_k \ge ... \ge 0.$$

To prove the convolution results, we need the following lemmas.

**Lemma 3.2.** [31] *Let $\{c_k\}_{k=0}^{\infty}$ be a convex null sequence. Then the function*

$$s(z) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k z^k$$

*is analytic, and $\Re(s(z)) > 0$ in $\mathbb{D}$.*

**Lemma 3.3.** [31] *Let the function $p$ be analytic in $\mathbb{D}$ with $p(0) = 1$ and $\Re(p(z)) > 1/2$ in $\mathbb{D}$. Then for any analytic function $F$ in $\mathbb{D}$, the function $p * F$ takes values in the convex hull of the image of $\mathbb{D}$ under $F$.*

Using Lemmas 3.2 and 3.3, we prove the following lemma.

**Lemma 3.4.** *Let $F \in \mathcal{W}(n,\alpha,q)$, then $\Re\left(\frac{F(z)}{z}\right) > \frac{1}{2}$.*

*Proof.* Suppose $F \in \mathcal{W}(n,\alpha,q)$ be given by $F(z) = z + \sum_{k=2}^{\infty} A_k z^k$, then

$$\Re\left(1 + \sum_{k=2}^{\infty} [k]_q^n (1 + \alpha([k]_q - 1)) A_k z^{k-1}\right) > 0,$$

which is equivalent to $\Re(p(z)) > 1/2$ in $\mathbb{D}$, where

$$p(z) = 1 + \frac{1}{2}\sum_{k=2}^{\infty} [k]_q^n (1 + \alpha([k]_q - 1)) A_k z^{k-1}.$$

Now consider a sequence $\{c_k\}_{k=0}^{\infty}$ defined by

$$c_0 = 1 \quad \text{and} \quad c_{k-1} = \frac{2}{[k]_q^n (1 + \alpha([k]_q - 1))} \quad \text{for} \quad k \geq 2.$$

It can be easily seen that the sequence $\{c_k\}_{k=0}^{\infty}$ is convex null sequence and using Lemma 3.2, the function

$$s(z) = 1 + \sum_{k=2}^{\infty} \frac{2}{[k]_q^n (1 + \alpha([k]_q - 1)} z^{k-1}$$

is analytic with $\Re(s(z)) > \frac{1}{2}$ in $\mathbb{D}$. Hence

$$\frac{F(z)}{z} = p(z) * \left(1 + \sum_{k=2}^{\infty} \frac{2}{[k]_q^n (1 + \alpha([k]_q - 1)} z^{k-1}\right)$$

$$= \left(1 + \frac{1}{2}\sum_{k=2}^{\infty} [k]_q^n (1 + \alpha([k]_q - 1)) A_k z^{k-1}\right) * \left(1 + \sum_{k=2}^{\infty} \frac{2}{[k]_q^n (1 + \alpha([k]_q - 1)} z^{k-1}\right)$$

and making use of Lemma 3.3 we observe that $\Re\left(\frac{F(z)}{z}\right) > \frac{1}{2}$ for $z \in \mathbb{D}$. □

**Lemma 3.5.** *Let $F_1$ and $F_2$ belong to $\mathcal{W}(n,\alpha,q)$. Then $F = F_1 * F_2 \in \mathcal{W}(n,\alpha,q)$.*

*Proof.* Let $F_1(z) = z + \sum_{k=2}^{\infty} A_k z^k$ and $F_2(z) = z + \sum_{k=2}^{\infty} B_k z^k$. Then the convolution of $F_1$ and $F_2$ is given by

$$F(z) = (F_1 * F_2)(z) = z + \sum_{k=2}^{\infty} A_k B_k z^k.$$

To prove that $F \in \mathcal{W}(n,\alpha,q)$, we have to show that

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n F(z) + \alpha \mathcal{D}_q^{n+1} F(z)}{z}\right) > 0,$$

which is equivalent to

$$\Re\left(1 + \sum_{k=2}^{\infty} [k]_q^n (1 + \alpha([k]_q - 1)) A_k B_k z^{k-1}\right) > 0$$

or

$$\Re\left(1 + \frac{1}{2}\sum_{k=2}^{\infty} [k]_q^n (1 + \alpha([k]_q - 1)) A_k B_k z^{k-1}\right) > \frac{1}{2}. \tag{3.1}$$

Since $F_1 \in \mathcal{W}(n,\alpha,q)$ we have

$$\Re\left(1+\frac{1}{2}\sum_{k=2}^{\infty}[k]_q^n(1+\alpha([k]_q-1))A_k z^{k-1}\right)>\frac{1}{2}$$

and by Lemma 3.4, $F_2 \in \mathcal{W}(n,\alpha,q)$ implies $\Re\left(\frac{F_2(z)}{z}\right)>\frac{1}{2}$ in $\mathbb{D}$ or

$$\Re\left(1+\frac{1}{2}\sum_{k=2}^{\infty}[k]_q^n(1+\alpha([k]_q-1))B_k z^{k-1}\right)>\frac{1}{2}.$$

By applying Lemma 3.3, we conclude we have (3.1). Hence, $F=F_1*F_2\in\mathcal{W}(n,\alpha,q)$. ☐

Now using Lemma 3.5, we prove that the class $\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ is closed under convolutions of its members.

**Theorem 3.6.** *If $f_1$ and $f_2$ belong to $\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$, then $f_1*f_2\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$.*

*Proof.* Let $f_1=h_1+\overline{g}_1$ and $f_2=h_2+\overline{g}_2$ be two functions in $\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$. Then the convolution of $f_1$ and $f_2$ is defined as $f_1*f_2=h_1*h_2+\overline{g_1*g_2}$. In order to prove that $f_1*f_2\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$, we need to prove that $F=h_1*h_2+\epsilon(g_1*g_2)\in\mathcal{W}(n,\alpha,q)$ for each $\epsilon$ ($|\epsilon|=1$). By Lemma 3.5, the class $\mathcal{W}(n,\alpha,q)$ is closed under convolutions for each $\epsilon$ ($|\epsilon|=1$), $h_i+\epsilon g_i\in\mathcal{W}(n,\alpha,q)$ for $i=1,2$. Then both $F_1$ and $F_2$ given by

$$F_1=(h_1-g_1)*(h_2-\epsilon g_2)$$

and

$$F_2=(h_1+g_1)*(h_2+\epsilon g_2)$$

belong to $\mathcal{W}(n,\alpha,q)$. Since $\mathcal{W}(n,\alpha,q)$ is closed under convex combinations, then the function

$$F=\frac{1}{2}(F_1+F_2)=h_1*h_2+\epsilon(g_1*g_2)$$

belongs to $\mathcal{W}(n,\alpha,q)$. Thus $\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ is closed under convolution. ☐

## 4. Partial sums

In this section, we examine sections (partial sums) of functions in the class $\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$.

**Theorem 4.1.** *Let $f=h+\overline{g}\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ with $\alpha\geq 0$. Then for each $\epsilon$ ($|\epsilon|=1$) and $|z|<1/2$, we have*

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_3(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_3(g))}{z}\right)>\frac{1}{4}.$$

*Proof.* Let $f=h+\overline{g}\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$. Then by Theorem 2.2, $h+\epsilon g\in\mathcal{W}(n,\alpha,q)$ for $\epsilon$ ($|\epsilon|=1$), so $\Re f_\epsilon(z)>0$, where

$$f_\epsilon(z)=\frac{(1-\alpha)\,\mathcal{D}_q^n(h(z)+\epsilon g(z))+\alpha\mathcal{D}_q^{n+1}(h(z)+\epsilon g(z))}{z}=1+\sum_{k=1}^{\infty}p_k z^k.$$

Moreover

$$\begin{aligned}&\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_3(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_3(g))}{z}\\&=1+[2]_q^n(1+\alpha([2]_q-1)(a_2+\epsilon b_2)z+[3]_q^n(1+\alpha([3]_q-1)(a_3+\epsilon b_3)z^2\\&=1+p_1z+p_2z^2.\end{aligned}$$

It is easy to see that

$$|2p_2-p_1^2|\leq 4-|p_1|^2$$

Let $2p_2-p_1^2=p$. Then $p_2=p/2+p_1^2/2$ and $|p|\leq 4-|p_1|^2$. Also, let $p_1z=\gamma+i\beta$ and $\sqrt{pz}=\eta+i\delta$ where $\beta,\gamma,\delta,\eta$ are real numbers. Then for $|z|<1/2$

$$\gamma^2+\beta^2=|p_1|^2|z|^2\leq\frac{|p_1|^2}{4}$$

and

$$\delta^2=|p||z|^2-\eta^2\leq\frac{|p|}{4}-\eta^2\leq\frac{4-|p_1|^2}{4}-\eta^2\leq 1-(\gamma^2+\beta^2)-\eta^2$$

so that

$$\begin{aligned}&\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_3(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_3(g))}{z}\right)\\&=\Re(1+p_1z+p_2z^2)\\&=\Re(1+p_1z+\frac{p}{2}z^2+\frac{p_1^2}{2}z^2)\\&=1+\gamma+\left(\frac{\eta^2}{2}-\frac{\delta^2}{2}\right)+\left(\frac{\gamma^2}{2}-\frac{\beta^2}{2}\right)\\&=1+\gamma+\frac{\eta^2}{2}-\frac{1-\gamma^2-\beta^2-\eta^2}{2}+\frac{\gamma^2}{2}-\frac{\beta^2}{2}\\&=\frac{1}{4}+\left(\gamma+\frac{1}{2}\right)^2+\eta^2\geq\frac{1}{4},\end{aligned}$$

which gives the result. □

**Theorem 4.2.** *Let $f=h+\overline{g}\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$, where $h$ and $g$ given by (1.1) with $b_1=0$. Then for each $j\geq 2$, $s_{1,j}(f)\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ for $|z|<1/2$.*

*Proof.* Let $f=h+\overline{g}\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$. It is clear that

$$s_{1,j}(f)(z)=s_1(h)(z)+\overline{s_j(g)(z)}=z+\sum_{k=2}^{j}\overline{b_kz^k}$$

It follows from Theorem 2.4 that for all $|z| < 1/2$,

$$\begin{aligned}
&\left|\frac{(1-\alpha)\,\mathcal{D}_q^n s_j(g)(z) + \alpha\mathcal{D}_q^{n+1} s_j(g)(z)}{z}\right| \\
&= \left|\sum_{k=2}^{j} [k]_q^n (1+\alpha([k]_q - 1)) b_k z^{k-1}\right| \\
&\leq \sum_{k=2}^{j} [k]_q^n (1+\alpha([k]_q - 1)) |b_k||z^{k-1}| \\
&\leq \sum_{k=2}^{j} |z|^{k-1} = \frac{|z|(1-|z|^{j-1})}{1-|z|} < \frac{|z|}{1-|z|} \\
&< 1 = \Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n s_1(h)(z) + \alpha\mathcal{D}_q^{n+1} s_1(h)(z)}{z}\right).
\end{aligned}$$

This implies that $s_{1,j}(f) \in \mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z| < 1/2$. □

**Theorem 4.3.** *Let $f = h + \overline{g} \in \mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$, where $h$ and $g$ given by (1.1) with $b_1 = 0$, and let $i$ and $j$ satisfy of the following conditions:*

(i) $3 \leq i < j$,
(ii) $i = j \geq 2$,
(iii) $i = 3$ *and* $j = 2$.

*Then $s_{i,j}(f) \in \mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z| < 1/2$.*

*Proof.* Let $\vartheta_i(h)(z) = \sum_{k=i+1}^{\infty} a_k z^k$ and $\vartheta_j(g)(z) = \sum_{k=j+1}^{\infty} b_k z^k$. Then

$$h = s_i(h) + \vartheta_i(h) \quad \text{and} \quad g = s_j(g) + \vartheta_j(g).$$

To prove $s_{i,j}(f) \in \mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$, it suffices to prove that $s_i(h) + \epsilon s_j(g) \in \mathcal{W}(n,\alpha,q)$ for $\epsilon$ ($|\epsilon| = 1$). If $f \in \mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$, then

$$\begin{aligned}
&\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n (s_i(h)+\epsilon s_j(g)) + \alpha\mathcal{D}_q^{n+1}(s_i(h)+\epsilon s_j(g))}{z}\right) \\
&= \Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n (h+\epsilon g) + \alpha\mathcal{D}_q^{n+1}(h+\epsilon g)}{z}\right. \\
&\qquad \left. - \frac{(1-\alpha)\,\mathcal{D}_q^n (\vartheta_i(h)+\epsilon\vartheta_j(g)) + \alpha\mathcal{D}_q^{n+1}(\vartheta_i(h)+\epsilon\vartheta_j(g))}{z}\right) \\
&\geq \Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n (h+\epsilon g) + \alpha\mathcal{D}_q^{n+1}(h+\epsilon g)}{z}\right) \\
&\qquad - \left|\frac{(1-\alpha)\,\mathcal{D}_q^n (\vartheta_i(h)+\epsilon\vartheta_j(g)) + \alpha\mathcal{D}_q^{n+1}(\vartheta_i(h)+\epsilon\vartheta_j(g))}{z}\right|.
\end{aligned} \tag{4.1}$$

By assumption, we see that

$$\frac{(1-\alpha)\,\mathcal{D}_q^n(h+\epsilon g)+\alpha\mathcal{D}_q^{n+1}(h+\epsilon g)}{z}\prec\frac{1+z}{1-z},$$

where $\prec$ is the subordination symbol. From the last relation, we conclude that

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(h+\epsilon g)+\alpha\mathcal{D}_q^{n+1}(h+\epsilon g)}{z}\right)\geq\frac{1-|z|}{1+|z|}. \tag{4.2}$$

**Case (i):** $3\leq i<j$

Applying Theorems 2.4 and 2.6, we observe that

$$\begin{aligned}&\left|\frac{(1-\alpha)\,\mathcal{D}_q^n(\vartheta_i(h)+\epsilon\vartheta_j(g))+\alpha\mathcal{D}_q^{n+1}(\vartheta_i(h)+\epsilon\vartheta_j(g))}{z}\right|\\&=\left|\sum_{k=i+1}^{j}[k]_q^n(1+\alpha([k]_q-1))a_kz^{k-1}+\sum_{k=j+1}^{\infty}[k]_q^n(1+\alpha([k]_q-1))(a_k+\epsilon b_k)z^{k-1}\right|\\&\leq\sum_{k=i+1}^{j}2|z|^{k-1}+\sum_{k=j+1}^{\infty}2|z|^{k-1}=2\frac{|z|^i}{1-|z|}\end{aligned} \tag{4.3}$$

Using (4.1), (4.2) and (4.3), we obtain

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_i(h)+\epsilon s_j(g))+\alpha\mathcal{D}_q^{n+1}(s_i(h)+\epsilon s_j(g))}{z}\right)\geq\frac{1-|z|}{1+|z|}-2\frac{|z|^i}{1-|z|}. \tag{4.4}$$

For $4\leq i<j$ and $|z|=1/2$, the inequality (4.4) gives

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_i(h)+\epsilon s_j(g))+\alpha\mathcal{D}_q^{n+1}(s_i(h)+\epsilon s_j(g))}{z}\right)\geq\frac{1}{3}-\frac{1}{4}>0.$$

Since $\Re\left(\frac{(1-\alpha)\mathcal{D}_q^n(s_i(h)+\epsilon s_j(g))+\alpha\mathcal{D}_q^{n+1}(s_i(h)+\epsilon s_j(g))}{z}\right)$ is harmonic, it assumes its minimum value on the circle $|z|=1/2$. Hence, if $4\leq i<j$ then $s_{i,j}(f)\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z|<1/2$.

If $i=3<j$, then in view of Theorem 2.4 and Theorem 4.1, we attain

$$\begin{aligned}&\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_j(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_j(g))}{z}\right)\\&=\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_3(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_3(g))}{z}\right.\\&\qquad\left.+\,\epsilon\sum_{k=4}^{j}[k]_q^n(1+\alpha([k]_q-1))b_kz^{k-1}\right)\\&\geq\frac{1}{4}-\sum_{k=4}^{j}[k]_q^n(1+\alpha([k]_q-1))|b_kz^{k-1}|\\&\geq\frac{1}{4}-\frac{|z|^3}{1-|z|}\end{aligned}$$

so that

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_j(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_j(g))}{z}\right)>0$$

for $|z|<1/2$, and thus $s_{3,j}(f)\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z|<1/2$.

**Case (ii):** $i=j\geq 2$

If $i=j\geq 4$, then the inequality (4.4) gives $s_{i,j}(f)\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z|<1/2$. For $i=j=2$, $s_{2,2}(f)(z)=z+a_2z^2+\overline{b_2z^2}$. Using Theorem 2.6, we get

$$\begin{aligned}&\Re\big(1+[2]_q^n(1+\alpha([2]_q-1))(a_2+\epsilon b_2)z\big)\\ &\geq 1-[2]_q^n(1+\alpha([2]_q-1))|a_2+\epsilon b_2||z|\\ &\geq 1-2|z|>0\end{aligned}$$

in $|z|<1/2$.

If $i=j=3$, then Theorem 4.1 shows that $s_{3,3}(f)\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z|<1/2$. Therefore, we prove that for $i=j\geq 2$, $s_{i,j}(f)\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z|<1/2$.

**Case (iii):** $i=3$ and $j=2$.

In view of Theorems 2.4 and 4.1, we have

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_2(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_2(g))}{z}\right)$$

$$=\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_3(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_3(g))}{z}-\epsilon[3]_q^n(1+\alpha([3]_q-1))b_3z^2\right)$$

$$\geq\frac{1}{4}-|z|^2=\frac{1}{4}-\frac{1}{2^2}=0$$

for $|z|<1/2$. Thus $s_{3,2}(f)\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z|<1/2$. □

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