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Application of Hayman's theorem to directional differential equations with analytic solutions in the unit ball

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Abstract. In this paper, we investigate analytic solutions of higher order linear non-homogeneous directional differential equations whose coefficients are analytic functions in the unit ball. We use methods of theory of analytic functions in the unit ball having bounded L-index in direction, where $L : \mathbb{B}^n \to \mathbb{R}_+$ is a continuous function such that $L(z) > \frac{\beta|\mathbf{b}|}{1-|z|}$ for all $z \in \mathbb{B}^n$, $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ be a fixed direction, $\beta > 1$ is some constant. Our proofs are based on application of inequalities from analog of Hayman's theorem for analytic functions in the unit ball. There are presented growth estimates of their solutions which contains parameters depending on the coefficients of the equations. Also we obtained sufficient conditions that every analytic solution of the equation has bounded L-index in the direction. The deduced results are also new in one-dimensional case, i.e. for functions analytic in the unit disc.

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1. Introduction

B. Lepson [16] introduced a concept of entire function of bounded index as another approach in analytic theory of differential equations. The first paper was closely connected to linear differential equation of infinite order with constant coefficients. But the functions of bounded index have interesting properties [21, 20, 19]: some regular behavior, uniform distribution of zeros, growth estimates, etc. There are many

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approaches to generalize the concept for wider classes of entire functions of single variable because every entire function of bounded index is a function of exponential type. Perhaps, the most successful approach is a concept of function of bounded l-index, proposed by A. Kuzyk and M. Sheremeta [14], where $l : \mathbb{C} \to \mathbb{R}_+$ is a continuous function. For these functions there are known existence theorems [10, 12]: *for an entire function f there exists a function $l : \mathbb{C} \to \mathbb{R}_+$ such that f has bounded l-index if and only if the function f has bounded multiplicities of zeros.*

In the last years, analytic functions of several variables having bounded index are intensively investigated. Main objects of investigations are such function classes: entire functions of several variables [4, 5, 17, 18], functions analytic in a polydisc [2], in a ball [6, 7].

For entire functions and analytic function in a ball there were proposed two approaches to introduce a concept of index boundedness in a multidimensional complex space. They generate so-called functions of bounded L-index in a direction and functions of bounded $\mathbf{L}$-index in joint variables.

In this research, we will consider the first approach, i.e. analytic functions in the unit ball of bounded L-index in direction. A connection between these two approaches is investigated in [8]. We will consider an application of these functions to study properties of analytic solutions of a linear higher order non-homogeneous differential equation with directional derivatives of the following form:

$$g_0(z)\partial_{\mathbf{b}}^{p}F(z) + g_1(z)\frac{\partial^{p-1}F(z)}{\partial\mathbf{b}^{p-1}} + \cdots + g_p(z)F(z) = h(z). \tag{1.1}$$

Also, we estimated asymptotic behavior of modulus of analytic functions in the unit ball by the function L and the L-index in the direction $\mathbf{b}$.

The linear PDE's (1.1) can easily be turned into ODEs by a suitable change of directional derivative in the direction $\mathbf{b}$ into a canonical direction along a coordinate axis. The cross-terms will vanish and hence an ODE. But the coefficients of the ODE depend on z if we consider a slice $z+t\mathbf{b}$, $z \in \mathbb{B}^n$, $t \in \mathbb{C}$, $|t| < \frac{1-|z|}{|\mathbf{b}|}$. Therefore, all one-dimensional results need uniform estimates in z. Let us consider an entire function $F(z_1, z_2) = \cos\sqrt{z_1 z_2} = \sum_{n=0}^{\infty} \frac{(-1)^n (z_1 z_2)^{2n}}{n!}$. It is known [4] that for fixed $z^0 = (z_1^0, z_2^0)$ and for every $\mathbf{b} = (b_1, b_2) \in \mathbb{C}^2 \setminus \{0\}$ the function $F(z_1^0 + tb_1, z^0 + tb_2)$ has bounded index as a function of variable $t \in \mathbb{C}$. But the function $F(z_1, z_2)$ has unbounded index in any direction $\mathbf{b}$ because the indexes of the function $F(z_1^0 + tb_1, z^0 + tb_2)$ are not uniformly bounded in (z_1^0, z_2^0). This fact was proved with the application of differential equations in [4]. It shows that uniform estimates play an important role and the bounded index in direction can not be simply reduced to the one-dimensional bounded index.

Our results are generalizations of earlier obtained results for entire functions of bounded L-index in a direction [3]. But now we consider the function L of more general form than in [3]. There was considered only $L(z) = l(|z|)$, where $l : \mathbb{R}_+ \to \mathbb{R}_+$, and $z \in \mathbb{C}^n$. In addition to Hayman's theorem, L-index boundedness in direction of analytic solutions of partial differential equations can be established by the so-called logarithmic criterion (see [9])). This approach requires that all coefficients of the equations have bounded L-index in the direction. For entire functions of one variable having bounded l-index the similar results were deduced in [15, 11].

Let us introduce some notations and definitions.

Note that the positivity and continuity of the function L are weak restrictions to deduce constructive results. Thus, we assume additional restrictions by the function L.

Let $\mathbf{0} = (0, \ldots, 0)$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a given direction, $\mathbb{R}_+ = (0, +\infty)$, $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$, $L :\ \mathbb{B}^n \to \mathbb{R}_+$ be a continuous function such that for all $z \in \mathbb{B}^n$

$$L(z) > \frac{\beta|\mathbf{b}|}{1-|z|},\ \beta = \mathrm{const} > 1. \tag{1.2}$$

Analytic function $F : \mathbb{B}^n \to \mathbb{C}$ is called [7] a function of *bounded L-index in a direction* $\mathbf{b}$ if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and every $z \in \mathbb{B}^n$ the following inequality is valid

$$\frac{|\partial_{\mathbf{b}}^m F(z)|}{m! L^m(z)} \le \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)} : 0 \le k \le m_0\right\}, \tag{1.3}$$

where $\partial_{\mathbf{b}}^0 F(z) = F(z), \partial_{\mathbf{b}} F(z) = \sum\limits_{j=1}^{n} \frac{\partial F(z)}{\partial z_j} b_j$, $\partial_{\mathbf{b}}^k F(z) = \partial_{\mathbf{b}}\Big(\partial_{\mathbf{b}}^{k-1} F(z)\Big)$, $k \ge 2$.

The least such integer $m_0 = m_0(\mathbf{b})$ is called the *L-index in the direction* $\mathbf{b}$ *of the analytic function F and is denoted by* $N_{\mathbf{b}}(F, L) = m_0$. If $n = 1$, $\mathbf{b} = 1$, $L = l$, $F = f$, then $N(f, l) \equiv N_1(f, l)$ is called the l-index of the function f. In the case $n = 1$ and $\mathbf{b} = 1$ we obtain the definition of an analytic function in the unit disc of bounded l-index. The definition is a generalization of concept of bounded L-index in direction introduced and considered for entire functions of several variables in [5]. The primary definition of bounded index for entire function of one variable was defined by B. Lepson [16].

Let $\mathbb{D} = \{t \in \mathbb{C} : |t| < 1\}$, $L : \mathbb{B}^n \to \mathbb{R}_+$ be a continuous function. For $z \in \mathbb{B}^n$ we denote $D_z = \{t \in \mathbb{C} : |t| \le \frac{1-|z|}{|\mathbf{b}|}\}$,

$$\lambda_{\mathbf{b}}(\eta) = \sup_{z\in\mathbb{B}^n} \sup_{t_1,t_2\in D_z} \left\{\frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})} : |t_1 - t_2| \le \frac{\eta}{\min\{L(z+t_1\mathbf{b}), L(z+t_2\mathbf{b})\}}\right\}.$$

The notation $Q_{\mathbf{b}}(\mathbb{B}^n)$ stands for a class of positive continuous functions $L : \mathbb{B}^n \to \mathbb{R}_+$, satisfying (1.2) and

$$(\forall \eta \in [0, \beta]) :\ \lambda_{\mathbf{b}}(\eta) < +\infty. \tag{1.4}$$

If $n = 1$ then $Q(\mathbb{D}) \equiv Q_1(\mathbb{B}^1)$ and $\lambda(\eta) \equiv \lambda_1(\eta)$.

Let D be an arbitrary bounded domain in $\mathbb{B}^n$ such that $\mathrm{dist}(D, \mathbb{B}^n) > 0$. If inequality (1.3) holds for all $z \in D$ instead $\mathbb{B}^n$, then the analytic function $F : \mathbb{B}^n \to \mathbb{C}$ is called *a function of bounded L-index in the direction* $\mathbf{b}$ *in the domain D*. The least such integer m_0 is called the *L-index in the direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *in domain D* and is denoted by $N_{\mathbf{b}}(F, L, D) = m_0$. The notation $\overline{D}$ stands for a closure of the domain D.

Lemma 1.1 ([1]). *Let D be an arbitrary bounded domain in $\mathbb{B}^n$ such that*

$$d = \mathrm{dist}(D, \mathbb{B}^n) = \inf_{z\in D}(1-|z|) > 0,\ \beta > 1,\ \mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$$

be an arbitrary direction. If $L\colon \mathbb{B}^n \to \mathbb{R}_+$ *is continuous function such that* $L(z) \geq \frac{\beta|\mathbf{b}|}{d}$, *and* $F : \mathbb{B}^n \to \mathbb{C}$ *is analytic function such that* $(\forall z^0 \in \overline{D})\colon \quad F(z^0 + t\mathbf{b}) \not\equiv 0$, *then* $N_{\mathbf{b}}(F, L, D) < \infty$.

Below we present an analog of Hayman's Theorem [13]. The theorem helps to investigate boundedness L-index in direction of analytic solutions of differential equations. At the end of the paper, we will present a scheme of this application.

Theorem 1.2 ([7]). *Let* $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$. *An analytic function* $F : \mathbb{B}^n \to \mathbb{C}$ *is of bounded* L*-index in the direction* $\mathbf{b}$ *if and only if there exist* $p \in \mathbb{Z}_+$ *and* $C > 0$ *such that for every* $z \in \mathbb{B}^n$

$$\left|\frac{\partial_{\mathbf{b}}^{p+1} F(z)}{L^{p+1}(z)}\right| \leq C \max\left\{\left|\frac{\partial_{\mathbf{b}}^{k} F(z)}{L^{k}(z)}\right| :\ 0 \leq k \leq p\right\}. \tag{1.5}$$

Using Lemma 1.1, we yield the following corollary with this criterion.

Corollary 1.3. *Let* $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, G *be a domain compactly embedded in* $\mathbb{B}^n$ *such that* $d = \operatorname{dist}(D, \mathbb{B}^n) = \inf_{z\in D}(1 - |z|) > 0$ *and for all* $z \in G \quad L(z) \geq \frac{\beta|\mathbf{b}|}{d}$. *An analytic function* $F : \mathbb{B}^n \to \mathbb{C}$ *has bounded* L*-index in the direction* $\mathbf{b}$ *if and only if there exist* $p \in \mathbb{Z}_+$ *and* $C > 0$ *such that for all* $z \in \mathbb{B}^n \setminus G$ *the following relation holds*

$$\frac{|\partial_{\mathbf{b}}^{p+1} F(z)|}{L^{p+1}(z)} \leq C \max\left\{\frac{|\partial_{\mathbf{b}}^{k} F(z)|}{L^{k}(z)} :\ 0 \leq k \leq p\right\}. \tag{1.6}$$

2. Auxiliary lemmas

We denote $a^+ = \max\{a, 0\}$. Set $u(r) = u(z^0, \theta, r) = L(z^0 + re^{i\theta}\mathbf{b})$. Let $W_{\mathbf{b}}(\mathbb{B}^n)$ be a class of positive continuous function $L : \mathbb{B}^n \to \mathbb{R}_+$ satisfying all following conditions:

1) for all $z \in \mathbb{B}^n \quad L(z) > \frac{\beta|\mathbf{b}|}{1-|z|}$, where $\beta = \text{const} > 1$;
2) for every $z^0 \in \mathbb{B}^n$ and every $\theta \in [0, 2\pi]$ the function $u(r, z^0, \theta)$ be a continuously differentiable function of real variable $r \in [0, r_0)$, where
$$r_0 = \min\{s \in \mathbb{R}_+ :\ |z^0 + se^{i\theta}\mathbf{b}| = 1\};$$
3) for every $z^0 \in \mathbb{B}^n$, $\theta \in [0, 2\pi]$ one has
$$\left(\frac{d}{ds}\frac{1}{L(z^0 + sre^{i\theta}\mathbf{b})}\Big|_{s=1}\right)^+ \to 0 \text{ as } |z^0 + re^{i\theta}\mathbf{b}| \to 1, \text{ i.e. } r \to r_0.$$

The conditions 2) and 3) together can be replaced by some strict condition $\partial_{\mathbf{b}}(1/L(Re^{i\Theta})) \to 0$ as $|R| \to 1$, where $Re^{i\Theta} = (r_1e^{i\theta_1}, \ldots, r_ne^{i\theta_n})$, $|R| < 1$, $\theta_j \in [0, 2\pi]$, and $L(Re^{i\Theta})$ is positive continuously differentiable function in each variable r_j, $j \in \{1, \ldots, n\}$. Moreover, condition 3) is equivalent to $\frac{(-u'_r(z^0,\theta,r))^+}{L^2(z^0+re^{i\theta}\mathbf{b})} \to 0$ as $r \to r_0$. Beside, condition 1) yields that $\int_0^r L(z^0 + xe^{i\theta}\mathbf{b})dx \to +\infty$ as $|z^0 + re^{i\theta}\mathbf{b}| \to 1$.

First, we prove the following two lemmas. For entire functions of bounded L-index in direction they were obtained in [3].

Lemma 2.1. *Let $L \in W_{\mathbf{b}}(\mathbb{B}^n)$, $F : \mathbb{B}^n \to \mathbb{C}$ be an analytic function such that $\exists R \in [0,1)$ $\forall z \in \mathbb{B}^n$ $|z| < R$ one has $F(z+t\mathbf{b}) \not\equiv 0$. If there exist numbers $p \in \mathbb{Z}_+$, $C > 0$ such that for all $z \in \mathbb{B}^n$, $|z| \geq R$, the inequality*

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L^{p+1}(z)} \leq C \max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z)|}{L^{k}(z)} : 0 \leq k \leq p\right\}, \tag{2.1}$$

holds then for every $z^0 \in \mathbb{B}^n$ and for every $\theta \in [0, 2\pi]$

$$\varlimsup_{|z^0+re^{i\theta}\mathbf{b}|\to 1} \frac{\ln|F(z^0+re^{i\theta}\mathbf{b})|}{\int_0^r L(z^0+xe^{i\theta}\mathbf{b})dx} \leq \max\{1, C\}.$$

Proof. Let $\theta \in [0, 2\pi]$, $z^0 \in \mathbb{B}^n$ be fixed and $x \in [0, r_0)$ be such that $|z^0 + xe^{i\theta}\mathbf{b}| \geq R$. We define

$$\Omega_{z^0}(x) = \max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z^0+xe^{i\theta}\mathbf{b})|}{L^{k}(z^0+xe^{i\theta}\mathbf{b})} : 0 \leq k \leq p\right\}.$$

The function $\frac{|\partial_{\mathbf{b}}^{k}F(z^0+xe^{i\theta}\mathbf{b})|}{L^{k}(z^0+xe^{i\theta}\mathbf{b})}$ is continuously differentiable by real $x \in [0, r^*]$, outside the zero set of function $|\partial_{\mathbf{b}}^{k}F(z^0+xe^{i\theta}\mathbf{b})|$ because $L \in W_{\mathbf{b}}(\mathbb{B}^n)$. Thus, the function $\Omega_{z^0}(x)$ is a continuously differentiable function on $[0, r^*]$, apart from, possibly, a countable set. For absolutely continuous functions $h_1, h_2, \ldots, h_k$ and $h(x) := \max\{h_j(z) : 1 \leq j \leq k\}$, $h'(x) \leq \max\{h_j'(x) : 1 \leq j \leq k\}$, $x \in [a,b]$ (see [21, Lemma 4.1, p. 81]). The function $\Omega_{z^0}(x)$ is absolutely continuous. Therefore,

$$\Omega_{z^0}'(x) \leq \max\left\{\frac{d}{dx}\left(\frac{1}{L^k(z^0+xe^{i\theta}\mathbf{b})} \cdot \left|\partial_{\mathbf{b}}^{k}F(z^0+xe^{i\theta}\mathbf{b})\right|\right) : 0 \leq k \leq p\right\}$$

except on a countable set of points.

Using the inequality $\frac{d}{dx}|\varphi(x)| \leq \left|\frac{d}{dx}\varphi(x)\right|$, which holds for complex-valued function of real argument except at the points $x = t$ such that $\varphi(t) = 0$, in view of (2.1) we obtain

$$\begin{aligned}\Omega_{z^0}'(x) \leq \max_{0\leq k\leq p}\Big\{|e^{i\theta}|\frac{1}{L^k(z^0+xe^{i\theta}\mathbf{b})}\left|\partial_{\mathbf{b}}^{k+1}F(z^0+xe^{i\theta}\mathbf{b})\right| - \left|\partial_{\mathbf{b}}^{k}F(z^0+xe^{i\theta}\mathbf{b})\right| \times \\ \times\frac{k\cdot u_x'(z^0,\theta,x)}{L^{k+1}(z^0+xe^{i\theta}\mathbf{b})}\Big\} \leq \\ \leq \max_{0\leq k\leq p}\Big\{\frac{1}{L^{k+1}(z^0+xe^{i\theta}\mathbf{b})}\left|\partial_{\mathbf{b}}^{k+1}F(z^0+xe^{i\theta}\mathbf{b})\right| L(z^0+xe^{i\theta}\mathbf{b}) - \\ -\frac{1}{L^k(z^0+xe^{i\theta}\mathbf{b})}\left|\partial_{\mathbf{b}}^{k}F(z^0+xe^{i\theta}\mathbf{b})\right|\frac{ku_x'(z^0,\theta,x)}{L(z^0+xe^{i\theta}\mathbf{b})}\Big\} \leq \\ \leq \Omega_{z^0}(x)\left(CL(z^0+xe^{i\theta}\mathbf{b}) + p\frac{(-u_x'(z^0,\theta,x))^+}{L(z^0+xe^{i\theta}\mathbf{b})}\right).\end{aligned}$$

From condition 3) in the definition of the class $W_{\mathbf{b}}(\mathbb{B}^n)$ we have

$$u_x'(z^0,\theta,x) = o(L^2(z^0+xe^{i\theta}\mathbf{b})) \text{ as } x \to r_0,$$

then

$$\Omega'_{z^0}(x) \le \Omega_{z^0}(x)(\max\{1,C\}L(z^0+xe^{i\theta}\mathbf{b})+p\varepsilon L(z^0+xe^{i\theta}\mathbf{b})) \le \Omega_{z^0}(x)\times$$
$$\times L(z^0+xe^{i\theta}\mathbf{b})(\max\{1,C\}+p\varepsilon) \le \Omega_{z^0}(x)L(z^0+xe^{i\theta}\mathbf{b})\max\{1,C\}(1+p\varepsilon)$$

for all $\varepsilon>0$ and for all $x\in[x_0(z^0,\theta,\varepsilon),r_0)$ outside a countable set of points for given $z^0\in\mathbb{B}^n$ and $\theta\in[0,2\pi]$ Hence, there exists $r_1\ge x_0(z^0,\theta,\varepsilon)$ such that

$$\Omega_{z^0}(r)\le\Omega_{z^0}(r_1)\cdot\exp\left\{(1+p\varepsilon)\max\{1,C\}\int_{r_1}^{r}L(z^0+xe^{i\theta}\mathbf{b})dx\right\}$$

for every $r\in[r_1,r_0)$. From the definition of $\Omega_{z^0}(x)$ for $k=0$ we obtain that

$$|F(z^0+re^{i\theta}\mathbf{b})|\le\Omega_{z^0}(r_1)\cdot\exp\left\{(1+p\varepsilon)\max\{1,C\}\int_{0}^{r}L(z^0+xe^{i\theta}\mathbf{b})dx\right\},$$

$$\ln|F(z^0+re^{i\theta}\mathbf{b})|\le\ln\Omega_{z^0}(r_1)+(1+p\varepsilon)\max\{1,C\}\int_{0}^{r}L(z^0+xe^{i\theta}\mathbf{b})dx,$$

$$\frac{\ln|F(z^0+re^{i\theta}\mathbf{b})|}{\int_0^r L(z^0+xe^{i\theta}\mathbf{b})dx}\le\frac{\ln\Omega_{z^0}(r_1)}{\int_0^r L(z^0+xe^{i\theta}\mathbf{b})dx}+(1+p\varepsilon)\max\{1,C\}.$$

From this inequality for all $z^0\in\mathbb{B}^n$ and $\theta\in[0,2\pi]$ we obtain that

$$\varlimsup_{|z^0+re^{i\theta}\mathbf{b}|\to1}\frac{\ln|F(z^0+re^{i\theta}\mathbf{b})|}{\int\limits_0^r L(z^0+xe^{i\theta}\mathbf{b})dx}\le\max\{1,C\}. \qquad \square$$

Lemma 2.2. *Let $L\in W_{\mathbf{b}}(\mathbb{B}^n)$, $F:\mathbb{B}^n\to\mathbb{C}$ be an analytic function such that $\exists R\in[0,1)$ $\forall z\in\mathbb{B}^n$ $|z|<R$ one has $F(z+t\mathbf{b})\not\equiv0$. If there exist numbers $p\in\mathbb{Z}_+$, $C>0$ such that for all $z\in\mathbb{B}^n$, $|z|\ge R$, the inequality*

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{(p+1)!L^{p+1}(z)}\le C\max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z)|}{k!L^k(z)}:0\le k\le p\right\}, \tag{2.2}$$

holds then for all $z^0\in\mathbb{B}^n$ $\varlimsup\limits_{|z^0+re^{i\theta}\mathbf{b}|\to1}\frac{\ln|F(z^0+re^{i\theta}\mathbf{b})|}{\int\limits_0^r L(z^0+xe^{i\theta}\mathbf{b})dx}\le(p+1)\max\{1,C\}$.

Proof. Let $\theta\in[0,2\pi]$, $z^0\in\mathbb{B}^n$ be fixed and $x\in\mathbb{R}_+$ be such that $|z^0+xe^{i\theta}\mathbf{b}|\ge R$. We denote

$$\Omega_{z^0}(x)=\max\left\{\frac{1}{k!L^k(z^0+xe^{i\theta}\mathbf{b})}\cdot\left|\partial_{\mathbf{b}}^kF(z^0+xe^{i\theta}\mathbf{b})\right|:0\le k\le p\right\}.$$

As in Lemma 2.1 the function $\Omega_{z^0}(x)$ is continuously differentiable because $L\in W_{\mathbf{b}}(\mathbb{B}^n)$ and

$$\Omega'_{z^0}(x)\le\max\left\{\frac{d}{dx}\left(\frac{1}{k!L^k(z^0+xe^{i\theta}\mathbf{b})}\left|\partial_{\mathbf{b}}^kF(z^0+xe^{i\theta}\mathbf{b})\right|\right):0\le k\le p\right\}$$

except a countable set of points. Applying the inequality $\frac{d}{dx}|\varphi(x)|\le\left|\frac{d}{dx}\varphi(x)\right|$, which holds for complex-valued function of real argument outside a countable set of points, in view of (2.2) we obtain

$$\Omega'_{z^0}(x) \leq \max\Big\{|e^{i\theta}|\frac{1}{k!L^k(z^0+xe^{i\theta}\mathbf{b})}\left|\partial_{\mathbf{b}}^{k+1}F(z^0+xe^{i\theta}\mathbf{b})\right| -$$
$$-\left|\partial_{\mathbf{b}}^{k}F(z^0+xe^{i\theta}\mathbf{b})\right|\frac{ku'_x(z^0,\theta,x)}{k!L^{k+1}(z^0+xe^{i\theta}\mathbf{b})} : 0\leq k\leq p\Big\} \leq$$
$$\leq \max\left\{\frac{1}{(k+1)!L^{k+1}(z^0+xe^{i\theta}\mathbf{b})}\left|\partial_{\mathbf{b}}^{k+1}F(z^0+xe^{i\theta}\mathbf{b})\right|(k+1)L(z^0+xe^{i\theta}\mathbf{b}) -\right.$$
$$\left.-\frac{1}{L^k(z^0+xe^{i\theta}\mathbf{b})k!}\left|\partial_{\mathbf{b}}^{k}F(z^0+xe^{i\theta}\mathbf{b})\right|\frac{ku'_x(z^0,\theta,x)}{L(z^0+xe^{i\theta}\mathbf{b})} : 0\leq k\leq p\right\} \leq$$
$$\leq \Omega_{z^0}(x)\left(\max\{1,C\}L(z^0+xe^{i\theta}\mathbf{b})(p+1)+p\frac{(-u'_x(z^0,\theta,x))^+}{L(z^0+xe^{i\theta}\mathbf{b})}\right).$$

But we have that $L \in W_{\mathbf{b}}(\mathbb{B}^n)$, i.e. $\frac{(-u'_x(z^0,\theta,x))^+}{L^2(z^0+xe^{i\theta}\mathbf{b})} \to 0$ as $|z^0+xe^{i\theta}\mathbf{b}| \to 1$. Therefore,

$$\Omega'_{z^0}(x) \leq \Omega_{z^0}(x)(\max\{1,C\}L(z^0+xe^{i\theta}\mathbf{b})(p+1)+p\varepsilon L(z^0+xe^{i\theta}\mathbf{b})) \leq$$
$$\leq \Omega_{z^0}(x)L(z^0+xe^{i\theta}\mathbf{b})(\max\{1,C\}(p+1)+p\varepsilon) \leq \Omega_{z^0}(x)L(z^0+xe^{i\theta}\mathbf{b})\times$$
$$\times\max\{1,C\}(p+1)\left(1+\frac{p}{p+1}\varepsilon\right)$$

for all $\varepsilon > 0$ and for all $x \geq x_0(z^0,\theta,\varepsilon)$, except a countable set of points at given z^0 and θ. Thus, there exists $r_1 \geq x_0(z^0,\theta,\varepsilon)$ that for $r > r_1$ we have

$$\Omega_{z^0}(r) \leq \Omega_{z^0}(r_1)\cdot\exp\left\{(1+\varepsilon)\max\{1,C\}(p+1)\int_{r_1}^{r} L(z^0+xe^{i\theta}\mathbf{b})dx\right\}.$$

Be definition of $\Omega_{z^0}(x)$ at $k=0$ we obtain

$$|F(z^0+re^{i\theta}\mathbf{b})| \leq \Omega_{z^0}(r_0)\exp\left\{(1+\varepsilon)\max\{1,C\}(p+1)\int_{0}^{r} L(z^0+xe^{i\theta}\mathbf{b})dx\right\}.$$

Therefore,

$$\ln|F(z^0+re^{i\theta}\mathbf{b})| \leq \ln\Omega_{z^0}(r_0)+(1+\varepsilon)\max\{1,C\}\int_0^r L(z^0+xe^{i\theta}\mathbf{b})dx.$$

Dividing of the inequality by $\int\limits_0^r L(z^0+xe^{i\theta}\mathbf{b})dx$, we obtain

$$\frac{\ln|F(z^0+e^{i\theta}r\mathbf{b})|}{\int\limits_0^r L(z^0+xe^{i\theta}\mathbf{b})dx} \leq \frac{\ln\Omega_{z^0}(r_0)}{\int\limits_0^r L(z^0+xe^{i\theta}\mathbf{b})dx}+(1+\varepsilon)\max\{1,C\}(p+1).$$

Thus, for all $z \in \mathbb{B}^n$ and $\theta \in [0,2\pi]$ we obtain an estimate

$$\varlimsup_{|z^0+re^{i\theta}\mathbf{b}|\to 1}\frac{\ln|F(z^0+e^{i\theta}r\mathbf{b})|}{\int\limits_0^r L(z^0+xe^{i\theta}\mathbf{b})dx} \leq \max\{1,C\}(p+1). \qquad \square$$

Remark 2.3. Note that condition (2.2) means that

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L^{p+1}(z)} =$$

$$= (p+1)!\cdot\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{(p+1)!L^{p+1}(z)} \le (p+1)!C\max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z)|}{k!L^{k}(z)} : 0\le k\le p\right\}\le$$

$$\le (p+1)!C\cdot\max\left\{\frac{1}{k!} : 0\le k\le p\right\}\cdot\max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z)|}{L^{k}(z)} : 0\le k\le p\right\}\le$$

$$\le (p+1)!C\max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z)|}{L^{k}(z)} : 0\le k\le p\right\}.$$

Hence, by Lemma 2.1 we have

$$\varlimsup_{|z^0+re^{i\theta}\mathbf{b}|\to 1}\frac{\ln|F(z^0+e^{i\theta}r\mathbf{b})|}{\int\limits_0^r L(z^0+xe^{i\theta}\mathbf{b})dx}\le\max\{1, C(p+1)!\}.$$

Since $c(p+1)! > c(p+1)$ for $p>1$, we see that Lemma 2.2 does not imply growth estimate of Lemma 2.1. Clearly, Lemma 2.1 does not imply Lemma 2.2 as well. Therefore, we need both Lemma 2.1 and Lemma 2.2.

3. Growth and bounded L-index in direction of analytic solutions of partial differential equations

Using proved lemmas we formulate and prove propositions that provide growth estimates of analytic solutions of the partial differential equation

$$g_0(z)\partial_{\mathbf{b}}^{p}F(z)+g_1(z)\frac{\partial^{p-1}F(z)}{\partial\mathbf{b}^{p-1}}+\cdots+g_p(z)F(z)=h(z). \tag{3.1}$$

Let us denote $QW_{\mathbf{b}}(\mathbb{B}^n)=Q_{\mathbf{b}}(\mathbb{B}^n)\cap W_{\mathbf{b}}(\mathbb{B}^n)$.

Theorem 3.1. *Let $L\in QW_{\mathbf{b}}(\mathbb{B}^n)$, functions g_0, g_1,.., g_p, and h be analytic in the unit ball and there exists $R\in[0,1)$ such that for all $z\in\mathbb{B}^n$, $|z|\ge R$, the following conditions hold*

1) $|g_j(z)|\le m_jL^j(z)|g_0(z)|$ *for* $1\le j\le p$;
2) $|\partial_{\mathbf{b}}g_j(z)|< M_j\cdot L^{j+1}(z)|g_0(z)|$ *for* $0\le j\le p$;
3) $|\partial_{\mathbf{b}}h(z)|\le M\cdot L(z)\cdot|h(z)|$,

where m_j and M are nonnegative constants and M_j are positive constants. If an analytic function $F:\mathbb{B}^n\to\mathbb{C}$ satisfies equation (3.1) and $\forall z\in\mathbb{B}^n$, $|z|<R$,

$$F(z+t\mathbf{b})\not\equiv 0$$

then F has bounded L-index in the direction $\mathbf{b}$ and for all $z^0\in\mathbb{B}^n$, $\theta\in[0,2\pi]$

$$\varlimsup_{|z^0+re^{i\theta}\mathbf{b}|\to 1}\frac{\ln|F(z^0+e^{i\theta}r\mathbf{b})|}{\int\limits_0^r L(z^0+te^{i\theta}\mathbf{b})dt}\le\max\{1, C\}, \tag{3.2}$$

where

$$C = \sum_{j=1}^{p} M_j + (M+1)\sum_{j=1}^{p} m_j + M.$$

Proof. First, we note that the second condition of the theorem with $j = 0$ implies that

$$g_0(z) \neq 0 \text{ for } z \in \mathbb{B}^n,\ |z| \geq R.$$

Taking into account that the function $F(z)$ satisfies equation (3.1), we calculate the derivative in the direction $\mathbf{b}$ for this equation

$$g_0(z)\partial_{\mathbf{b}}^{p+1}F(z) + \sum_{j=0}^{p} \partial_{\mathbf{b}} g_j(z)\cdot \partial_{\mathbf{b}}^{p-j}F(z) + \sum_{j=1}^{p} g_j(z)\partial_{\mathbf{b}}^{p-j+1}F(z) + \\ + \sum_{j=1}^{p} g_j(z)\partial_{\mathbf{b}}^{p-j+1}F(z) = \partial_{\mathbf{b}} h(z). \tag{3.3}$$

Using the third condition of the theorem, we obtain

$$|\partial_{\mathbf{b}} h(z)| \leq ML(z)h(z) \leq ML(z)\sum_{j=0}^{p} |g_j(z)| \left|\partial_{\mathbf{b}}^{p-j}F(z)\right|. \tag{3.4}$$

By (3.3)

$$\partial_{\mathbf{b}}^{p+1}F(z) = \frac{1}{g_0(z)}\left(\partial_{\mathbf{b}} h(z) - \sum_{j=0}^{p} \partial_{\mathbf{b}} g_j(z)\cdot \partial_{\mathbf{b}}^{p-j}F(z) - \sum_{j=1}^{p} g_j(z)\partial_{\mathbf{b}}^{p-j+1}F(z)\right). \tag{3.5}$$

Putting in the first condition of the theorem $m_0 = 1$, from (3.5) in view of the second condition we obtain

$$\left|\partial_{\mathbf{b}}^{p+1}F(z)\right| \leq \frac{1}{g_0(z)}\Bigg(ML(z)\sum_{j=0}^{p} |g_j(z)| \left|\partial_{\mathbf{b}}^{p-j}F(z)\right| + \sum_{j=0}^{p} |\partial_{\mathbf{b}} g_j(z)| \times \\ \times \left|\partial_{\mathbf{b}}^{p-j}F(z)\right| + \sum_{j=1}^{p} |g_j(z)| \left|\partial_{\mathbf{b}}^{p-j+1}F(z)\right|\Bigg) \leq \\ \leq ML(z)\sum_{j=0}^{p} m_j L^j(z) \left|\partial_{\mathbf{b}}^{p-j}F(z)\right| + \\ + \sum_{j=0}^{p} M_j L^{j+1}(z) \left|\partial_{\mathbf{b}}^{p-j}F(z)\right| + \sum_{j=1}^{p} m_j L^j(z) \left|\partial_{\mathbf{b}}^{p-j+1}F(z)\right|.$$

Dividing this inequality by $L^{p+1}(z)$, we obtain

$$\frac{1}{L^{p+1}(z)}\left|\partial_{\mathbf{b}}^{p+1}F(z)\right| \le M\sum_{j=0}^{p} m_j\frac{1}{L^{p-j}(z)}\left|\partial_{\mathbf{b}}^{p-j}F(z)\right| + \sum_{j=0}^{p} M_j\frac{1}{L^{p-j}(z)}\times$$

$$\times\left|\partial_{\mathbf{b}}^{p-j+1}F(z)\right| + \sum_{j=1}^{p} m_j\frac{1}{L^{p-j+1}(z)}\left|\partial_{\mathbf{b}}^{p-j+1}F(z)\right| \le \Bigg(M\sum_{j=0}^{p} m_j+$$

$$+\sum_{j=0}^{p} M_j + \sum_{j=1}^{p} m_j\Bigg)\max\left\{\frac{1}{L^{k}(z)}\left|\partial_{\mathbf{b}}^{k}F(z)\right| : 0\le k\le p\right\} =$$

$$= \left((M+1)\sum_{j=1}^{p} m_j + \sum_{j=0}^{p} M_j + M\right)\max\left\{\frac{1}{L^{k}(z)}\left|\partial_{\mathbf{b}}^{k}F(z)\right| : 0\le k\le p\right\}$$

for all $z\in\mathbb{B}^n$, $|z|\ge R$.

Thus, by Lemma 2.1 estimate (3.2) holds, and by Corollary 1.3 the analytic function $F(z)$ is of bounded L-index in the direction b. □

In the case when equation (3.1) is homogeneous ($h(z)\equiv 0$), the previous theorem can be simplified.

Theorem 3.2. *Let $L\in QW_{\mathbf{b}}(\mathbb{B}^n)$, functions g_0, g_1,.., g_p be analytic in the unit ball and there exists $R\in[0,1)$ such that for all $z\in\mathbb{B}^n$, $|z|\ge R$, one has $|g_j(z)|\le m_jL^j(z)|g_0(z)|$ for $1\le j\le p$, where m_j are some nonnegative constants. If an analytic function $F:\mathbb{B}^n\to\mathbb{C}$ satisfies equation (3.1) with $h(z)\equiv 0$ and $\forall z\in\mathbb{B}^n$, $|z|<R$, $F(z+t\mathbf{b})\not\equiv 0$ then $F(z)$ is of bounded L-index in the direction $\mathbf{b}$ and for all $z^0\in\mathbb{B}^n$, $\theta\in[0,2\pi]$*

$$\varlimsup_{|z^0+re^{i\theta}\mathbf{b}|\to 1}\frac{\ln|F(z^0+e^{i\theta}r\mathbf{b})|}{\int_0^r L(z^0+te^{i\theta}\mathbf{b})dt} \le \max\left\{1,\sum_{j=1}^{p} m_j\right\}. \tag{3.6}$$

Proof. Equation (3.1) implies $g_0(z)\partial_{\mathbf{b}}^{p}F(z) = -\sum_{j=1}^{p} g_j(z)\partial_{\mathbf{b}}^{p-j}F(z)$. Then

$$|g_0(z)|\left|\partial_{\mathbf{b}}^{p}F(z)\right| \le \sum_{j=1}^{p}|g_j(z)|\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|.$$

Dividing the obtained inequality by $g_0(z)L^p(z)$ and using assumptions of the theorem by the functions $g_j(z)$, we obtain

$$\frac{1}{L^{p}(z)}\left|\partial_{\mathbf{b}}^{p}F(z)\right| \le \sum_{j=1}^{p}\left|\frac{g_j(z)}{g_0(z)}\right|\frac{1}{L^{p}(z)}\left|\partial_{\mathbf{b}}^{p-j}F(z)\right| \le \sum_{j=1}^{p}\frac{m_j}{L^{p-j}(z)}\times$$

$$\times\left|\partial_{\mathbf{b}}^{p-j}F(z)\right| \le \sum_{j=1}^{p} m_j\max\left\{\frac{1}{L^{k}(z)}\left|\partial_{\mathbf{b}}^{k}F(z)\right| : 0\le k\le p-1\right\}.$$

Thus, all conditions of Corollary 1.3 are obeyed. Hence, the function F is of bounded L-index in the direction $\mathbf{b}$ and by Lemma 2.1 estimate (3.6) is true. □

Moreover, using Corollary 1.3 and Lemma 2.2 we can complement two previous Theorems 3.1 and 3.2 by propositions, that contain growth estimates, which can sometimes be better than (3.6) and (3.2). Two following theorems have similar proofs as in Theorems 3.1 and 3.2.

Theorem 3.3. *Let $L \in QW_{\mathbf{b}}(\mathbb{B}^n)$, functions g_0, $g_1, .., g_p$, and h be analytic in the unit ball and there exists $R \in [0,1)$ such that for all $z \in \mathbb{B}^n$, $|z| \geq R$, the following conditions hold*

1. $|g_j(z)| \leq m_j L^j(z)|g_0(z)|$ *for* $1 \leq j \leq p$;
2. $|\partial_{\mathbf{b}} g_j(z)| < M_j \cdot L^{j+1}(z)|g_0(z)|$ *for* $0 \leq j \leq p$;
3. $|\partial_{\mathbf{b}} h(z)| \leq M \cdot L(z) \cdot |h(z)|$,

where m_j and M are nonnegative constants and M_j are positive constants. If an analytic function $F : \mathbb{B}^n \to \mathbb{C}$ is a solution of equation (3.1) and $\forall z \in \mathbb{B}^n$, $|z| < R$, $F(z + t\mathbf{b}) \not\equiv 0$ then $F(z)$ is of bounded L-index in the direction $\mathbf{b}$ and for all $z^0 \in \mathbb{B}^n$, $\theta \in [0, 2\pi]$

$$\overline{\lim_{|z^0+re^{i\theta}\mathbf{b}|\to 1}} \frac{\ln |F(z^0 + e^{i\theta} r\mathbf{b})|}{\int_0^r L(z^0 + te^{i\theta}\mathbf{b})dt} \leq \max\{p+1, 2(M+2)M^*\}, \tag{3.7}$$

where $M^* = \max\{1, m_j, M_j\}$.

Proof. First, we note that the second condition of this theorem when $j = 0$ implies that $g_0(z) \neq 0$ for $z \in \mathbb{B}^n$, $|z| \geq R$, because in this case we have

$$\left|\frac{\partial g_0(z)}{\partial \mathbf{b}}\right| < M_0 L(z) g_0(z).$$

Since the function $F(z)$ satisfies the equation (3.1), then we calculate a derivative of this equation in the direction $\mathbf{b}$:

$$g_0(z)\partial_{\mathbf{b}}^{p+1}F(z) + \sum_{j=0}^{p} \frac{\partial_{\mathbf{b}} g_j(z)}{.} \partial_{\mathbf{b}}^{p-j}F(z) + \sum_{j=1}^{p} g_j(z)\partial_{\mathbf{b}}^{p-j+1}F(z) + \\ + \sum_{j=1}^{p} g_j(z)\partial_{\mathbf{b}}^{p-j+1}F(z) = \frac{\partial h(z)}{\partial \mathbf{b}}. \tag{3.8}$$

Using the third condition of this theorem, we obtain

$$|\partial_{\mathbf{b}} h(z)| \leq ML(z)|h(z)| \leq ML(z) \sum_{j=0}^{p} |g_j(z)| \left|\partial_{\mathbf{b}}^{p-j}F(z)\right|.$$

From (3.8) it follows

$$\partial_{\mathbf{b}}^{p+1}F(z)=\frac{1}{g_0(z)}\left(\partial_{\mathbf{b}}h(z)-\sum_{j=0}^{p}\partial_{\mathbf{b}}g_j(z)\cdot\partial_{\mathbf{b}}^{p-j}F(z)-\sum_{j=1}^{p}g_j(z)\partial_{\mathbf{b}}^{p-j+1}F(z)\right). \tag{3.9}$$

Putting in the first condition of this theorem $m_0=1$, with (3.9) in view of the second condition we obtain

$$\begin{gathered}
\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|\le\frac{1}{|g_0(z)|}\left(ML(z)\sum_{j=0}^{p}|g_j(z)|\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+\sum_{j=0}^{p}|\partial_{\mathbf{b}}g_j(z)|\times\right.\\
\left.\times\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+\sum_{j=1}^{p}|g_j(z)|\left|\partial_{\mathbf{b}}^{p-j+1}F(z)\right|\right)\le M\sum_{j=0}^{p}\frac{|g_j(z)|}{L^j(z)|g_0(z)|}\times\\
\times L^{j+1}(z)\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+\sum_{j=0}^{p}|\partial_{\mathbf{b}}g_j(z)|\frac{1}{|g_0(z)|L^{j+1}(z)|}L^{j+1}(z)\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+\\
+\sum_{j=1}^{p}\frac{|g_j(z)|}{|g_0(z)|}\frac{1}{L^j(z)}L^j(z)\left|\partial_{\mathbf{b}}^{p-j+1}F(z)\right|\le M\sum_{j=0}^{p}m_jL^{j+1}(z)\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+\\
+\sum_{j=0}^{p}M_jL^{j+1}(z)\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+\sum_{j=1}^{p}m_jL^j(z)\left|\partial_{\mathbf{b}}^{p-j+1}F(z)\right|\le\\
\le M^*\left((M+1)\sum_{j=0}^{p}L^{j+1}(z)\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+\sum_{j=1}^{p}L^j(z)\left|\partial_{\mathbf{b}}^{p-j+1}F(z)\right|\right)=\\
=M^*\left((M+1)\sum_{j=0}^{p}L^{j+1}(z)\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+\sum_{j=0}^{p-1}L^{j+1}(z)\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+\right.\\
\left.+L^{p+1}(z)|F(z)|\right)\le M^*\left((M+2)\sum_{j=0}^{p}L^{j+1}(z)\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|\right).
\end{gathered}$$

We divide the obtained inequality by $(p+1)!L^{p+1}(z)$

$$\begin{gathered}
\frac{1}{(p+1)!L^{p+1}(z)}\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|\le M^*(M+2)\sum_{j=0}^{p}\frac{1}{(p-j)!L^{p-j}(z)}\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|\times\\
\times\frac{(p-j)!}{(p+1)!}\le\frac{2M^*(M^*+2)}{(p+1)}\max\left\{\frac{1}{k!L^k(z)}\left|\partial_{\mathbf{b}}^{k}F(z)\right|:0\le k\le p\right\},
\end{gathered}$$

because

$$\sum_{j=0}^{p}\frac{(p-j)!}{(p+1)!}\leq\frac{0!+1!+2!+3!+\cdots+p!}{(p+1)!}=\frac{2\cdot 1!+2!+3!+4!+\cdots+p!}{(p+1)!}=$$
$$=\frac{2\cdot 2!+2!+3!+4!+\cdots+p!}{(p+1)!}\leq\frac{2\cdot 3!+4!+5!+\cdots+p!}{(p+1)!}\leq$$
$$\leq\frac{2\cdot 4!+5!+\cdots+p!}{(p+1)!}\leq\frac{2\cdot 5!+\cdots p!}{(p+1)!}\leq\frac{2p!}{(p+1)!}=\frac{2}{p+1}. \tag{3.10}$$

Hence, by Corollary 1.3 the function F has bounded L-index in the direction $\mathbf{b}$, because

$$\frac{1}{L^{p+1}(z)}\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|\leq M^*(M+2)\sum_{j=0}^{p}\frac{1}{L^{p-j}(z)}\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|\leq$$
$$\leq 2M^*(M+2)\max\left\{\frac{1}{L^k(z)}\left|\partial_{\mathbf{b}}^{k}F(z)\right|:0\leq k\leq p\right\}.$$

And by Lemma 2.2 corresponding estimate (3.7) holds. □

Theorem 3.4. *Let $L\in QW_{\mathbf{b}}(\mathbb{B}^n)$, functions g_0, g_1, .., g_p be analytic in the unit ball and there exists $R\in[0,1)$ such that for all $z\in\mathbb{B}^n$, $|z|\geq R$, one has*

$$|g_j(z)|\leq m_jL^j(z)|g_0(z)| \text{ for } 1\leq j\leq p,$$

where m_j are some nonnegative constants. If an analytic function $F:\mathbb{B}^n\to\mathbb{C}$ satisfies equation (3.1) with $h(z)\equiv 0$ and $\forall z\in\mathbb{B}^n$, $|z|<R$, $F(z+t\mathbf{b})\not\equiv 0$ then $F(z)$ is of bounded L-index in the direction $\mathbf{b}$ and for all $z^0\in\mathbb{B}^n$, $\theta\in[0,2\pi]$

$$\overline{\lim_{|z^0+re^{i\theta}\mathbf{b}|\to 1}}\frac{\ln|F(z^0+e^{i\theta}r\mathbf{b})|}{\int_0^r L(z^0+te^{i\theta}\mathbf{b})dt}\leq\max\{p,2M^*\}, \tag{3.11}$$

where $M^*=\max\{1,m_j\}$.

Proof. The proof of this theorem is similar to the proofs of Theorems 3.2 and 3.3. In particular, from equation (3.1) with $h(z)\equiv 0$ it follows that

$$g_0(z)\partial_{\mathbf{b}}^{p}F(z)=-\left(\sum_{j=1}^{p}g_j(z)\partial_{\mathbf{b}}^{p-j}F(z)\right),$$

then

$$|g_0(z)|\,|\partial_{\mathbf{b}}^{p}F(z)|\leq\sum_{j=1}^{p}|g_j(z)|\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|. \tag{3.12}$$

Dividing the obtained inequality by $|g_0(z)|L^p(z)$ and using the conditions of this theorem for functions $g_j(z)$, we obtain

$$\frac{1}{L^p(z)}\,|\partial_{\mathbf{b}}^p F(z)| \le \sum_{j=1}^p \left|\frac{g_j(z)}{g_0(z)}\right| \frac{1}{L^p(z)}\left|\partial_{\mathbf{b}}^{p-j} F(z)\right| \le \sum_{j=1}^p \frac{m_j}{L^{p-j}(z)}\times$$
$$\times\left|\partial_{\mathbf{b}}^{p-j} F(z)\right| \le \sum_{j=1}^p m_j \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{L^k(z)} : 0\le k\le p-1\right\}.$$

Thus, by Corollary 1.3 the function F is of bounded L-index in the direction $\mathbf{b}$. We show that conditions of Lemma 2.2 are satisfied. Dividing inequality (3.12) by $p!L^p(z)$, we obtain

$$\frac{|\partial_{\mathbf{b}}^p F(z)|}{p!L^p(z)} \le \sum_{j=1}^p \left|\frac{g_j(z)}{g_0(z)}\right| \frac{1}{p!L^p(z)}\left|\partial_{\mathbf{b}}^{p-j} F(z)\right| \le \sum_{j=1}^p \frac{m_j}{L^{p-j}(z)}\frac{(p-j)!}{p!}\frac{1}{(p-j)!}\times$$
$$\times\left|\partial_{\mathbf{b}}^{p-j} F(z)\right| \le \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)} : 0\le k\le p-1\right\}\sum_{j=1}^p m_j\frac{(p-j)!}{p!} \le$$
$$\le M^*\sum_{j=1}^p \frac{(p-j)!}{p!}\max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)} : 0\le k\le p-1\right\} \le$$
$$\le \frac{2M^*}{p}\max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{K!L^k(z)} : 0\le k\le p-1\right\}.$$

In the proof of this estimate we used an inequality (3.10), which was obtained in the proof of Theorem 3.3. Thus, by Lemma 2.2 the corresponding estimate (3.11) holds. □

Remark 3.5. The conditions in Theorems 3.1-3.4 imposed by the coefficients of equations are easy satisfied because there is some freedom to choose a function L. In the worst case we can choose the function L as an iterated exponential function in this form $A\exp_k(\frac{1}{1-|z|})$, where $\exp_k(t)=\exp(\exp_{k-1}(t))$, $A>0$ is sufficiently big number and k is integer number depending on the growth of coefficients $g_j(z)$ and $h(z)$. For example, in Theorem 3.1 this number k can be chosen such that $(|g_j(z)|/|g_0(z)|)^{1/j} = O(\exp_k(\frac{1}{1-|z|})$ for $1\le j\le p$, $(|\partial_{\mathbf{b}} g_j(z)|/|g_0(z)|)^{1/(j+1)} = O(\exp_k(\frac{1}{1-|z|})$ for $0\le j\le p$ and $|\partial_{\mathbf{b}} h(z)|/|h(z)| = O(\exp_k(\frac{1}{1-|z|})$ as $|z|\to 1-0$.

Example 3.6. Let us consider the following third order partial differential equation from [7]:

$$\partial_{\mathbf{b}}^3 F = 2(\pi b_1 z_2 + \pi b_2 z_1)\partial_{\mathbf{b}}^2 F +$$
$$+2\left(\frac{(\pi b_1 z_2+\pi b_2 z_1)^2}{\cos^2 \pi z_1 z_2} + 2\pi b_1 b_2 \tan \pi z_1 z_2\right)\partial_{\mathbf{b}} F - \frac{4\pi b_1 b_2(\pi b_1 z_2 + \pi b_2 z_1)}{\cos^2 \pi z_1 z_2} F \quad (3.13)$$

where $\mathbf{b} = (b_1, b_2) \in \mathbb{C}^2 \setminus \{0\}$. It is easy to check that conditions of Theorem 3.2 are satisfied for this equation and for the functions

$$L(z_1, z_2) = \frac{|b_1 z_2 + b_2 z_1| + 1}{(1 - |z|)|\frac{1}{2} - z_1 z_2|}$$

in the unit ball. Therefore, by Theorem 3.2 every analytic solution of equation (3.13) has bounded L-index in the direction $\mathbf{b}$ and its growth is described by estimate (3.6). Namely, the function $F(z_1, z_2) = \tan(\pi z_1 z_2)$ has the bounded L-index in this direction because the function F is analytic solution in $\mathbb{B}^2$ of equation (3.13).

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