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Redacţia: 3400 Cluj-Napoca, str. M. Kogalniceanu nr. 1 • Telefon: 194315

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PROPERTIES OF A NEW CLASS OF ABSOLUTELY SUMMING OPERATORS

CRISTINA ANTONESCU

Abstract. In [1] there was been introduced a new class of absolutely summing operators and there was been obtained some of its properties and, also, the relations with the known classes of absolutely summing operators.

In this article we go on with the study of the properties of this new operator class.

1. Preliminaries

We shall just refer briefly to those notions and results which are necessary for the proofs.

Let E, F be Banach spaces over the field Γ, where Γ is the set of the real or of the complex numbers. In the sequel we shall use the following notations:

1) $L(E,F) := \{T : E \to F : \ T \text{ is linear and bounded}\}$.

2) $E^* := L(E,\Gamma)$.

3) $U_E := \{x \in E : \|x\| \leq 1\}$.

4) For $a \in E^*$ and $x \in E$, let $\langle x, a\rangle := a(x)$.

5) Let $a \in E^*$ and $y \in F$. We denote by $a \otimes y$ the following operator

$$a \otimes y : E \to F, \ \ (a \otimes y)(x) = \langle x, a\rangle \cdot y, \ \text{for all } x \in E.$$

6) We denote by l_∞ the set of all real number sequences, $\{x_n\}_n$, with the property

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$$\|x\|_\infty := \sup_{n \text{ natural}} |x_n| < \infty.$$

7) We denote by c_0 the set of all real number sequences, $\{x_n\}_n$, with the property

$$\lim_{n\to\infty} |x_n| = 0.$$

8) We denote by l_p, $0 < p < \infty$, the set of all real number sequences, $\{x_n\}_n$, with the property

$$\|x\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} < \infty.$$

Definition 1. ([7])

For $x = \{x_n\}_n \in l_\infty$, let
$s_n(x) := \inf\{\sigma \geq 0 : card\{i : |x_i| \geq \sigma\} < n\}$.

Remark 1. ([7])

If the sequence $x = \{x_n\}_n \in l_\infty$ is ordered such that $|x_n| \geq |x_{n+1}|$, for any natural n, then

$$s_n(x) = |x_n|.$$

Proposition 2. ([7])

The numbers $s_n(x)$ have the following properties:

1. $\|x\|_\infty = s_1(x) \geq s_2(x) \geq \ldots \geq 0$, for all $x = \{x_n\}_n \in l_\infty$,

2. $s_{n+m-1}(x+y) \leq s_n(x) + s_m(y)$, for all $x = \{x_i\}_i \in l_\infty, y = \{y_i\}_i \in l_\infty$, and $n, m \in \{1, 2, ...\}$, where $x + y = \{x_i + y_i\}_i$,

3. $s_{n+m-1}(x \cdot y) \leq s_n(x) \cdot s_m(y)$, for all $x = \{x_i\}_i \in l_\infty, y = \{y_i\}_i \in l_\infty$, and $n, m \in \{1, 2, ...\}$, where $x \cdot y = \{x_i \cdot y_i\}_i$,

4. If $x = \{x_i\}_i \in l_\infty$ and $card\{i : x_i \neq 0\} < n$ then $s_n(x) = 0$.

Let us remark the similarity between the properties of the sequence $s_n(x)$, where $x = \{x_n\}_n \in l_\infty$, and the axioms from the definition of an **additive and multiplicative** s**-scale,** an s**-scale** being a rule, $s : T \to \{s_n(T)\}_n$, which assigns to every linear and bounded operator a scalar sequence with the following properties:

1. $\|T\| = s_1(T) \geq s_2(T) \geq ... \geq 0$, for all $T \in L(E, F)$,

2. $s_{n+m-1}(T + S) \leq s_n(T) + s_m(S)$, for all $T, S \in L(E, F)$ and $n, m \in \{1, 2, ...\}$,

3. $s_{n+m-1}(T \circ S) \leq s_n(T) \cdot s_m(S)$, for all $T \in L(F, F_0), S \in L(E, F)$ and $n, m \in \{1, 2, ...\}$,

4. $s_n(T) = 0$, $\dim T < n$,

5. $s_n(I_E) = 1$, if $\dim E \geq n$, where $I_E(x) = x$, for all $x \in E$.

We call $s_n(T)$ the n-th $s-$**number** of the operator T.

For properties, examples of $s-$**numbers** and relations between diferent s-numbers it can be seen [3], [4], [5], [6], [7].

We continue by giving some basic facts about the classical real interpolation method, called the K-method.

For those interested to find an introduction on interpolation theory we recommend, for example, [2], [9].

Definition 3. ([2], [8])

For a compatible couple (X_0, X_1), in the sense of the interpolation theory, of normed or quasi-normed spaces, and $t > 0$ consider the functional:

$$K(t, x) := \inf \{ \|x_0 \mid X_0\| + t \cdot \|x_1 \mid X_1\| : x = x_0 + x_1, x_i \in X_i, i = 0, 1 \}.$$

Let $0 < \theta < 1$ and $0 < q \leq \infty$. The **interpolation space** $(X_0, X_1)_{\theta,q}$ is defined as follows:

$$(X_0, X_1)_{\theta,q} := \left\{ x = x_0 + x_1, x_i \in X_i, i = 0, 1 : \left(\int_0^\infty \left[t^{-\theta} \cdot K(t, x) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\},$$

if $q < \infty$, and

$$(X_0, X_1)_{\theta,\infty} := \left\{ x = x_0 + x_1, x_i \in X_i, i = 0, 1 : \sup_{t>0} t^{-\theta} \cdot K(t, x) < \infty \right\}.$$

The operator classes $P_{p,q,\gamma}$, introduced in [1], are closed related to the Lorentz-Zygmund sequence spaces. For that we shall recall here a few things about these sequence spaces and the Lorentz-Zygmund operator ideals.

Definition 4. ([7])

Let $0 < p, q < \infty$ and $-\infty < \gamma < \infty$. The **Lorentz-Zygmund sequence spaces** are defined as follows

$$l_{p,q,\gamma} := \left\{ x = \{x_i\}_i \in c_0 : \sum_{i=1}^\infty \left[i^{\frac{1}{p} - \frac{1}{q}} \cdot (1 + \log i)^\gamma \cdot s_i(x) \right]^q < \infty \right\}.$$

These are quasi-normed spaces, with the quasi-norm

$$\|x\|_{p,q,\gamma} := \left(\sum_{i=1}^{\infty} \left[i^{\frac{1}{p}-\frac{1}{q}} \cdot (1+\log i)^{\gamma} \cdot s_i(x) \right]^q \right)^{\frac{1}{q}}.$$

Definition 5. ([7], [8])

Let E, F be Banach spaces, s an additive s-number and $0 < p < \infty$, $0 < q < \infty$, $-\infty < \gamma < \infty$. We introduce the following operator classes:

$$L^{(s)}_{p,q,\gamma}(E,F) :=$$
$$\left\{ T \in L(E,F) : \ \|T\|^{(s)}_{p,q,\gamma} := \left(\sum_{n=1}^{\infty} \left[n^{\frac{1}{p}} \cdot (1+logn)^{\gamma} \cdot s_n(T) \right]^q \cdot n^{-1} \right)^{\frac{1}{q}} < \infty \right\},$$

and for $q = \infty$

$$L^{(s)}_{p,\infty,\gamma}(E,F) :=$$
$$\left\{ T \in L(E,F) : \ \|T\|^{(s)}_{p,\infty,\gamma} := \sup_n n^{\frac{1}{p}} \cdot (1+logn)^{\gamma} \cdot s_n(T) < \infty \right\}.$$

We denote by $L^{(s)}_{p,q,\gamma} := \bigcup_{E,F \text{ Banach spaces}} L^{(s)}_{p,q,\gamma}(E,F)$.

Remark 2. ([7], [8])

Let s be an additive s-number and $0 < p < \infty$, $0 < q \leq \infty$, $-\infty < \gamma < \infty$, then $\left(L^{(s)}_{p,q,\gamma}, \|\cdot\|^{(s)}_{p,q,\gamma} \right)$ is a quasi-normed operator ideal.

We are giving now an interpolation result obtained by classical methods

Proposition 6. ([8])

Let E, F be Banach spaces, $0 < p_0 < p_1 < \infty$, $0 < q_0, q_1, q \leq \infty$, $0 < \gamma_0, \gamma_1 < \infty$ and $0 < \theta < 1$. Then

$$\left(L^{(s)}_{p_0,q_0,\gamma_0}(E,F), L^{(s)}_{p_1,q_1,\gamma_1}(E,F) \right)_{\theta,q} \subseteq L^{(s)}_{p,q,\gamma}(E,F), \text{ where } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

and $\gamma = (1-\theta) \cdot \gamma_0 + \theta \cdot \gamma_1$.

At the end of this section we shall remind the construction of the operator classes $P_{p,q,\gamma}$ and one of the properties proved in [1].

Definition 7. ([4])

Let E be a Banach space and I an index set. An E-valued family $\{x_i\}_{i \in I}$ is said to be **absolutely r-summable** if $\{\|x_i\|\} \in l_r(I)$. The set of these families is denoted by $[l_r(I), E]$.

For $\{x_i\}_{i\in I} \in [l_r(I), E]$ we define:

$$\|\{x_i\} \mid [l_r(I), E]\| := \left(\sum_i \|x_i\|^r\right)^{1/r}.$$

If there is no risk of confusion, then we use the shortened symbol $\|\{x_i\} \mid l_r\|$. Moreover,we write $[l_r, E]$ instead of $[l_r(N), E]$.

Proposition 8. ([4])

$[l_r(I), E]$ is a Banach space.

Definition 9. ([4])

Let E be a Banach space and I an index set. An E-valued family $\{x_i\}_{i\in I}$ is said to be **weakly r-summable if** $\{\langle x_i, a\rangle\} \in l_r(I)$ for all $a \in E^*$.

The set of these families is denoted by $[w_r(I), E]$.For $\{x_i\}_{i\in I} \in [w_r(I), E]$ we define:

$$\|(x_i) \mid [w_r(I), E]\| = sup\left\{\left(\sum_i |\langle x_i, a\rangle|^r\right)^{1/r} : a \in U_{E^*}\right\}.$$

If there is no risk of confusion, then we use the shortened symbol $\|\{x_i\} \mid w_r\|$. Moreover,we write $[w_r, E]$ instead of $[w_r(N), E]$.

Proposition 10. ([4])

$[w_r(I), E]$ is a Banach space.

Remark 3. ([1])

Using the Lorentz-Zygmund sequence spaces, $l_{p,q,\gamma}$, we can define, in a similar way, the spaces $[l_{p,q,\gamma}(I), E]$ and $[w_{p,q,\gamma}(I), E]$.

Definition 11. ([1])

Let E, F be Banach spaces and $0 < p_1, p_2 < \infty$, $1 \leq q_2 \leq q_1 < \infty$, $-\infty < \gamma_1, \gamma_2 < \infty$. An operator $T \in L(E, F)$ is called **absolutely** $(p_{12}, q_{12}, \gamma_{12})$ **−summing** if there exists a constant $c \geq 0$ such that

$$\left(\sum_{i=1}^{n}\left[i^{\frac{1}{p_1}-\frac{1}{q_1}} \cdot (1+\log i)^{\gamma_1} \cdot \|Tx_i\|\right]^{q_1}\right)^{\frac{1}{q_1}} \leq$$

$$\leq c \cdot \sup_{a \in U_{E^*}} \left(\sum_{i=1}^{n}\left[i^{\frac{1}{p_2}-\frac{1}{q_2}} \cdot (1+\log i)^{\gamma_2} \cdot |\langle x_i, a\rangle|\right]^{q_2}\right)^{\frac{1}{q_2}}$$, for every finite family of elements $x_1, \ldots x_n \in E$. The set of these operators is denoted by $P_{p_{12},q_{12},\gamma_{12}}(E, F)$.

For $T \in P_{p_{12},q_{12},\gamma_{12}}(E, F)$ we define $\pi_{p_{12},q_{12},\gamma_{12}}(T) := \inf c$, the infimum being taken over all constants $c \geq 0$ for which the above inequality holds.

Theorem 12. ([1])

$P_{p_{12},q_{12},\gamma_{12}}$ is an injective Banach operator ideal.

2. **Results**

We start by giving a result concerning the "lexicografic order" of the Lorentz-Zygmund sequence spaces.

Proposition 13.

We have the following inclusion:

$$l_{p,q_0,\gamma} \subseteq l_{p,q_1,\gamma}, \text{ where } 0 < p < \infty,\ 0 < q_0 < q_1 \leq \infty,\ \gamma > 0.$$

Proof. We whall need the following result, established by N. Tiţa, in [8], for the operator ideal case.

Proposition 14.

Let $0 < p < \infty$, $0 < q \leq \infty$ and $0 < \gamma < \infty$ then

$$\{x_n\}_n \in l_{p,q,\gamma} \Leftrightarrow \left\{2^{\frac{n-1}{p}} \cdot s_{2^{n-1}}(x)\right\}_n \in l_{r,q}, \text{ where } \gamma = \frac{1}{r} - \frac{1}{q}.$$

Moreoever there are the constants c and $\overline{c}$, which depend on p, q, γ, such that:

$$c \cdot \left\| \left\{2^{\frac{n-1}{p}} \cdot s_{2^{n-1}}(x)\right\}_n \right\|_{r,q} \leq \|x\|_{p,q,\gamma} \leq \overline{c} \cdot \left\| \left\{2^{\frac{n-1}{p}} \cdot s_{2^{n-1}}(x)\right\}_n \right\|_{r,q}.$$

We start now our proof.

Let $\xi = \{\xi_n\}_n \in l_{p,q_0,\gamma} \Leftrightarrow \left\{2^{\frac{n-1}{p}} \cdot s_{2^{n-1}}(\xi)\right\}_n \in l_{r,q_0}$, where $\gamma = \frac{1}{r} - \frac{1}{q_0}$.

Let $q_1 > q_0$ and r_1 such that $\gamma = \frac{1}{r_1} - \frac{1}{q_1}$. It follows that

$\frac{1}{r} - \frac{1}{q_0} = \frac{1}{r_1} - \frac{1}{q_1} \Leftrightarrow \frac{1}{r_1} = \frac{1}{r} + \left(\frac{1}{q_1} - \frac{1}{q_2}\right) \Rightarrow \frac{1}{r_1} < \frac{1}{r} \Rightarrow r_1 > r.$

From the "lexicografic orderliness" of the Lorentz sequence spaces, [4], [7], we know that $l_{r,q_0} \subseteq l_{r_1,q_1}$.

So $\left\{2^{\frac{n-1}{p}} \cdot s_{2^{n-1}}(\xi)\right\}_n \in l_{r,q_1} \Leftrightarrow \xi = \{\xi_n\}_n \in l_{p,q_1,\gamma}$.

In conclusion $l_{p,q_0,\gamma} \subseteq l_{p,q_1,\gamma}$, for $0 < p < \infty$, $0 < q_0 < q_1 \leq \infty$, $\gamma > 0$. □

Proposition 15.

Let $0 < p_0 < p_1 < \infty$, $0 < q_0, q_1, q \leq \infty$, $0 < \gamma_0, \gamma_1 < \infty$ and $0 < \theta < 1$. Then $(l_{p_0,q_0,\gamma_0}, l_{p_1,q_1,\gamma_1})_{\theta,q} \subseteq l_{p,q,\gamma}$, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\gamma = (1-\theta) \cdot \gamma_0 + \theta \cdot \gamma_1$.

Proof. If we take account of the similarity between the properties of the sequences $\{s_n(T)\}_n$, where s is an additive s-scale, $T \in L(E, F)$, and $\{s_n(x)\}_n$, where $x = \{x_n\}_n \in l_\infty$, the proof of the above inclusion will have the same course like the proof of the Proposition 6.

We shall consider $q < \infty$, the proof of the case $q = \infty$ being similar.

From the Proposition 13 it follows that we have the inclusion

$(l_{p_0,q_0,\gamma_0}, l_{p_1,q_1,\gamma_1})_{\theta,q} \subseteq (l_{p_0,\infty,\gamma_0}, l_{p_1,\infty,\gamma_1})_{\theta,q}$.

So it will be enough to prove the relation

$(l_{p_0,\infty,\gamma_0}, l_{p_1,\infty,\gamma_1})_{\theta,q} \subseteq l_{p,q,\gamma}$.

Let $x = \{x_n\}_n \in (l_{p_0,\infty,\gamma_0}, l_{p_1,\infty,\gamma_1})_{\theta,q}$. We shall consider the arbitrary decomposition

$x = x^0 + x^1$, where $x^i = \{x_n^i\}_n \in l_{p_i,\infty,\gamma_i}$, $i \in \{0,1\}$.

Let $i \in \{0,1\}$ then

$$x^i = \{x_n^i\}_n \in l_{p_i,\infty,\gamma_i} \Leftrightarrow$$

$$\Leftrightarrow \|x^i\|_{p_i,\infty,\gamma_i} = \sup_n \left[n^{\frac{1}{p_i}} \cdot (1+logn)^{\gamma_i} \cdot s_n(x^i)\right] < \infty \Rightarrow$$

$$\Rightarrow s_n(x^i) \le n^{-\frac{1}{p_i}} \cdot (1+logn)^{-\gamma_i} \cdot \|x^i\|_{p_i,\infty,\gamma_i},$$ for any natural n.

We shall evaluate $\|x\|_{p,q,\gamma}$.

$$\left(\|x\|_{p,q,\gamma}\right)^q = \sum_{n=1}^{\infty}\left[n^{\frac{1}{p}} \cdot (1+logn)^{\gamma} \cdot s_n(x)\right]^q \cdot \frac{1}{n} =$$

$$= \sum_{n=1}^{\infty}\left[(2\cdot n-1)^{\frac{1}{p}-\frac{1}{q}} \cdot (1+log(2\cdot n-1))^{\gamma} \cdot s_{2\cdot n-1}(x)\right]^q +$$

$$+\sum_{n=1}^{\infty}\left[(2\cdot n)^{\frac{1}{p}-\frac{1}{q}} \cdot (1+log(2\cdot n))^{\gamma} \cdot s_{2\cdot n}(x)\right]^q \le$$

$$\le \sum_{n=1}^{\infty}\left[(2\cdot n-1)^{\frac{1}{p}-\frac{1}{q}} \cdot (1+log(2\cdot n-1))^{\gamma} \cdot s_{2\cdot n-1}(x)\right]^q +$$

$$+\sum_{n=1}^{\infty}\left[(2\cdot n)^{\frac{1}{p}-\frac{1}{q}} \cdot (1+log(2\cdot n))^{\gamma} \cdot s_{2\cdot n-1}(x)\right] \le$$

$$\le c(p,q,\gamma)\cdot \sum_{n=1}^{\infty}\left[n^{\frac{1}{p}-\frac{1}{q}} \cdot (1+logn)^{\gamma} \cdot s_{2\cdot n-1}(x)\right]^q \le$$

$$\le c(p,q,\gamma)\cdot \sum_{n=1}^{\infty}\left[n^{\frac{1}{p}-\frac{1}{q}} \cdot (1+logn)^{\gamma} \cdot (s_n(x^0)+s_n(x^1))\right]^q \le$$

$$\le c(p,q,\gamma)\cdot$$

$$\sum_{n=1}^{\infty}\left[n^{\frac{1}{p}-\frac{1}{q}}(1+logn)^{\gamma-\gamma_0} n^{-\frac{1}{p_0}}\left(\|x^0\|_{p_0,\infty,\gamma_0} + n^{\frac{1}{p_0}-\frac{1}{p_1}}(1+logn)^{\gamma_0-\gamma_1}\|x^1\|_{p_1,\infty,\gamma_1}\right)\right]^q.$$

The decomposition $x = x^0 + x^1$ being arbitrary and taking account of the K-functional's definition

$K\left(x, n^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+logn)^{\gamma_0-\gamma_1}, l_{p_0,\infty,\gamma_0}, l_{p_1,\infty,\gamma_1}\right)$ we obtain that

$$\|x^0\|_{p_0,\infty,\gamma_0} + n^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+logn)^{\gamma_0-\gamma_1} \cdot \|x^1\|_{p_1,\infty,\gamma_1} \le$$

$$\le K\left(x, n^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+logn)^{\gamma_0-\gamma_1}, l_{p_0,\infty,\gamma_0}, l_{p_1,\infty,\gamma_1}\right).$$

So $\left(\|x\|_{p,q,\gamma}\right)^q \le$

$$\le c\cdot \sum_{n=1}^{\infty}\left[n^{\frac{1}{p}-\frac{1}{p_0}}(1+logn)^{\gamma-\gamma_0} K\left(x, n^{\frac{1}{p_0}-\frac{1}{p_1}}(1+logn)^{\gamma_0-\gamma_1}\right)\right]^q \cdot \frac{1}{n} \le$$

$$\le c_1 \cdot \int_1^{\infty}\left[t^{\frac{1}{p}-\frac{1}{p_0}}(1+logt)^{\gamma-\gamma_0} K\left(x, t^{\frac{1}{p_0}-\frac{1}{p_1}}(1+logt)^{\gamma_0-\gamma_1}\right)\right]^q \frac{dt}{t} =$$

$$= c_1 \cdot \int_1^{\infty}\left[t^{\frac{1-\theta}{p_0}+\frac{\theta}{p_1}-\frac{1}{p_0}}(1+logt)^{(1-\theta)\gamma_0+\theta\gamma_1-\gamma_0} K\left(x, t^{\frac{1}{p_0}-\frac{1}{p_1}}(1+logt)^{\gamma_0-\gamma_1}\right)\right]^q \frac{dt}{t} =$$

$$= c_1 \cdot \int_1^\infty \left[t^{-\theta\left(\frac{1}{p_0}-\frac{1}{p_1}\right)} (1+logt)^{-\theta(\gamma_0-\gamma_1)} K\left(x, t^{\frac{1}{p_0}-\frac{1}{p_1}} (1+logt)^{\gamma_0-\gamma_1}\right)\right]^q \frac{dt}{t} =$$

$$= c_1 \cdot \int_1^\infty \left[\left(t^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+logt)^{\gamma_0-\gamma_1}\right)^{-\theta} \cdot K\left(x, t^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+logt)^{\gamma_0-\gamma_1}\right)\right]^q \frac{dt}{t}.$$

Let now define $f : (1,\infty) \to (0,\infty)$, $f(t) = t^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+logt)^{\gamma_0-\gamma_1}$.

$$f'(t) \cdot t = \left(\frac{1}{p_0} - \frac{1}{p_1}\right) \cdot t^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+logt)^{\gamma_0-\gamma_1} +$$

$$+ (\gamma_0 - \gamma_1) \cdot t^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+logt)^{\gamma_0-\gamma_1} \cdot \frac{1}{1+logt} \cdot c_2 =$$

$$= t^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+logt)^{\gamma_0-\gamma_1} \cdot \left(\frac{1}{p_0} - \frac{1}{p_1} + (\gamma_0-\gamma_1) \cdot \frac{1}{1+logt} \cdot c_2\right) \le c_3 \cdot f(t).$$

Hence we obtain $\left(\|x\|_{p,q,\gamma}\right)^q \le$

$$\le c_1 \cdot \int_1^\infty \left[\left(t^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+logt)^{\gamma_0-\gamma_1}\right)^{-\theta} \cdot K\left(x, t^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+logt)^{\gamma_0-\gamma_1}\right)\right]^q \frac{dt}{t} =$$

$$= c_1 \cdot \int_1^\infty \left[f(t)^{-\theta} \cdot K(x, f(t))\right]^q \cdot f'(t) \cdot \frac{1}{f'(t) \cdot t} dt \le c_3 \cdot \int_1^\infty \left[f(t)^{-\theta} \cdot K(x, f(t))\right]^q \cdot$$

$$\frac{1}{f(t)} \cdot f'(t) \cdot dt =$$

$$= c_3 \cdot \int_0^\infty \left[s^{-\theta} \cdot K(x,s)\right]^q \cdot \frac{ds}{s} < \infty.$$

(We have made the following change of variable $f(t) = s$.)

In conclusion $x \in l_{p,q,\gamma}$. □

Proposition 16.

Let I be any infinite index set. An operator $T \in L(E,F)$ is absolutely $(p_{12}, q_{12}, \gamma_{12})$ −summing if and only if $T(I) : \{x_i\}_{i\in I} \to \{Tx_i\}_i$ defines a linear and bounded operator from $[w_{p_2,q_2,\gamma_2}(I), E]$ into $[l_{p_1,q_1,\gamma_1}(I), F]$. When this is so, then

$$\pi_{p_{12},q_{12},\gamma_{12}}(T) = \|T(I) : [w_{p_2,q_2,\gamma_2}(I), E] \to [l_{p_1,q_1,\gamma_1}(I), F]\|.$$

Proof. It is similar to the proof for the similar result for absolutely (r,s) −summing operators, see Proposition 1.2.2 from [3].

Suppose that $T \in P_{p_{12},q_{12},\gamma_{12}}(E,F)$ and $(x_i)_{i\in I} \in [w_{p_2,q_2,\gamma_2}(I), E]$. Then we have

$$\left(\sum_i \left[i^{\frac{1}{p_1}-\frac{1}{q_1}} \cdot (1+\log i)^{\gamma_1} \cdot \|Tx_i\|\right]^{q_1}\right)^{\frac{1}{q_1}} \le$$

$\le \pi_{p_{12},q_{12},\gamma_{12}}(T) \cdot \sup_{a\in U_{E^*}} \left(\sum_i \left[i^{\frac{1}{p_2}-\frac{1}{q_2}} \cdot (1+\log i)^{\gamma_2} \cdot |\langle x_i, a\rangle|\right]^{q_2}\right)^{\frac{1}{q_2}}$, for all F, $F \in \mathbf{F}(I)$. Passing to the limit I yields

$$\left\|(Tx_i)_{i\in I} \mid [l_{p_1,q_1,\gamma_1}(I), F]\right\| \le \pi_{p_{12},q_{12},\gamma_{12}}(T)\cdot \left\|(x_i)_{i\in I} \mid [w_{p_2,q_2,\gamma_2}(I), E]\right\|.$$

This proves that

$$\|T(I) : [w_{p_2,q_2,\gamma_2}(I), E] \to [l_{p_1,q_1,\gamma_1}(I), F]\| \le \pi_{p_{12},q_{12},\gamma_{12}}(T).$$

The reverse inequality is obvious. □

Theorem 17. *(interpolation theorem)*

Let E, F be Banach spaces. If $0 < p_1 < p_3 < \infty$, $0 < p_2 < \infty$, $0 < q_1, q_2, q_3, q_4 < \infty$, $0 < \gamma_1, \gamma_3 < \infty$ and $0 < \theta < 1$, then

$\left(P_{p_{12},q_{12},\gamma_{12}}(E,F), P_{p_{32},q_{32},\gamma_{32}}(E,F)\right)_{\theta,q_4} \subseteq P_{p_{42},q_{42},\gamma_{42}}(E,F)$, where $\frac{1}{p_4} = \frac{1-\theta}{p_1} + \frac{\theta}{p_3}$ and $\gamma_4 = (1-\theta)\cdot\gamma_1 + \theta\cdot\gamma_3$.

Proof. We use the idea from the proof of the **interpolation theorem** for the **absolutely** (p,q) **−summing operators.** This theorem can be found in [4], Proposition 1.2.6.

Let $\{x_i\}_i \in [w_{p_2,q_2,\gamma_2}, F]$. We define the operator

$X : T \in L(E,F) \to \{Tx_i\}_i$. From the Proposition 16 it follows that, for $T \in P_{p_{12},q_{12},\gamma_{12}}(E,F)$, $\{Tx_i\}_i \in [l_{p_1,q_1,\gamma_1}, F]$ and, for $T \in P_{p_{32},q_{32},\gamma_{32}}(E,F)$, $\{Tx_i\}_i \in [l_{p_3,q_3,\gamma_3}, F]$.

So for

$T \in \left(P_{p_{12},q_{12},\gamma_{12}}(E,F), P_{p_{32},q_{32},\gamma_{32}}(E,F)\right)_{\theta,q_4} \Rightarrow$

$\Rightarrow \{Tx_i\}_i \in \left([l_{p_1,q_1,\gamma_1}, F], [l_{p_3,q_3,\gamma_3}, F]\right)_{\theta,q_4} \subseteq [l_{p_4,q_4,\gamma_4}, F]$. We have applied the Proposition 15.

In conclusion

$X : T \in \left(P_{p_{12},q_{12},\gamma_{12}}(E,F), P_{p_{32},q_{32},\gamma_{32}}(E,F)\right)_{\theta,q_4} \to \{Tx_i\}_i \in [l_{p_4,q_4,\gamma_4}, F]$.

Hence the assertion follows from the Proposition 16. □

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"Dr. I. Meşotă" Secondary School, 2200 Braşov, Romania
E-mail address: adiant@fx.ro

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PERTURBATION ANALYSIS OF MONOTONE GENERALIZED EQUATIONS

ANDRÁS DOMOKOS

Abstract. Our goal is to establish new methods and results in reflexive Banach spaces to the theory of local stability of the solutions of some non-compact generalized equations, including parametric variational inequalities. The continuity of the projections of a fixed point onto a family of nonempty, closed, convex sets will be also studied using these methods. The results from this paper generalize results proved in finit dimensional spaces and Hilbert spaces.

0. Introduction

Stability topics for parametric variational inequalities were studied in many papers in finite or in infinite dimensional Hilbert spaces [1, 3, 4, 8, 11]. The proofs in those papers are closely connected with the Hilbert spaces' properties (for example, the nonexpansivity of the metric projection onto a closed, convex set).

Our method is independent from the above mentioned properties and also from compactness assumptions (for compact perturbation of monotone operators see [7].

Papers [3, 11] use the strong-monotonicity condition in finite dimensional spaces. We will replace this condition by a weaker one, φ-uniform-monotonicity (used also in [1]). In Banach spaces this is a weaker and more usefull condition, (see Proposition 1.1, Examples 5.1 and 5.3) that the strong-monotonicity.

We will discuss also some aspects with respect to a consistency condition. Consistency conditions are frequently used in the theory of implicit function theorems [1, 2, 4]. Our condition is a generalization of those used in [1, 4]. We will show that the

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consistency condition is satisfied under reasonable conditions, as pseudo-continuity or lower-semicontinuity (see Corollary 2.1, Examples 5.1 and 5.2). The continuity of the projection of a fixed point onto a family of nonempty, closed, convex sets implies the consistency of the normal cone operator. A result similar to the Hölder continuity of the projections of a fixed point onto a pseudo-Lipschitz continuous family of closed convex sets [11] holds for uniformly-convex Banach spaces. We will use a generalization of the metric projection operator introduced by E. Zarantonello [12].

We will denote by Ω, Λ, W topological spaces and by X a reflexive Banach space. Throughout this paper we will work with the fixed points $x_0 \in X$, $\omega_0 \in \Omega$, $\lambda_0 \in \Lambda$, $w_0 \in W$ and with their neighborhoods $X_0 = B(x_0, r)$ (the closed ball centered at x_0 and radius r) of x_0, Ω_0 of ω_0, Λ_0 of λ_0, and W_0 of w_0. We need a single-valued mapping $f : X_0 \times \Omega_0 \to X^*$, a set-valued mapping $F : X_0 \times W_0 \rightsquigarrow X^*$ and an other set-valued mapping $C : \Lambda_0 \rightsquigarrow X$ with nonempty, closed, convex values.

Let us consider the following parametric variational inequality

$$\begin{cases} \text{find} \quad x \in C(\lambda) \quad \text{such that} \\ \langle f(x,\omega) \,,\, y - x \rangle \ \geq \ 0 \,, \quad \text{for all} \quad y \in C(\lambda) \end{cases} \tag{$VI(\omega,\lambda)$}$$

and the equivalent generalized equation

$$0 \ \in \ f(x,\omega) \ + \ N_{C(\lambda)}(x) \,, \tag{$GE(\omega,\lambda)$}$$

where

$$N_{C(\lambda)}(x) \ = \ \{x^* \in X^* \ : \ \langle x^* \,,\, y - x \rangle \leq 0 \,, \quad \text{for all} \quad y \in C(\lambda)\}$$

is the normal cone to the set $C(\lambda)$ at the point x.

The normal cone mappings $N_{C(\lambda)} \ : \ X \rightsquigarrow X^*$ are maximal-monotone, because the sets $C(\lambda)$ are nonempty, closed and convex.

In a reflexive Banach space we can introduce equivalent norms for which the space is strictly-convex with strictly-convex dual or locally uniform-convex with locally uniform-convex dual. The continuity and monotonicity properties from this paper remain the same when we use these equivalent norms, so we can use them when we need better properties for the duality mapping.

1. Preliminary results

In this section we present basic definitions and results.

Definition 1.1. [1] *The mappings $F(\cdot, w)$ are said φ-uniformly-monotone for all $w \in W_0$, if there exists an increasing function $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$, with $\varphi(r) > 0$ when $r > 0$, such that for all $w \in W_0$, $x_1, x_2 \in Dom\, F(\cdot, w)$, $x_1^* \in F(x_1, w)$, $x_2^* \in F(x_2, w)$ hold*

$$\langle x_1^* - x_2^* \, , \, x_1 - x_2 \rangle \, \geq \, \varphi\left(\|x_1 - x_2\|\right) \|x_1 - x_2\| \, .$$

If the function φ is defined as $\varphi(r) = ar$, with $a > 0$, then the mappings $F(\cdot, w)$ are said strongly-monotone with constant a.

The following proposition shows that φ-uniform-monotonicity is a natural one in uniformly-convex spaces.

Proposition 1.1. [10] *A Banach space X is uniformly-convex if and only if for each $R > 0$ there exists an increasing function $\varphi_R : \mathbf{R}_+ \to \mathbf{R}_+$, with $\varphi_R(r) > 0$ when $r > 0$, such that the normalized duality mapping $J : X \rightsquigarrow X^*$, defined by*

$$J(x) \, = \, \left\{ \, x^* \in X^* \; : \, \langle x^* \, , \, x \rangle \, = \, \|x\|^2 \, , \, \|x\| = \|x^*\| \, \right\} \; ,$$

is φ_R-uniformly-monotone in $B(0, R)$.

Definition 1.2. *Let $A, B \subset X$. The Hausdorff distance between A, B is defined as*

$$H(A, B) \, = \, \max\{e(A, B) \, , \, e(B, A)\} \; ,$$

where

$$e(A, B) \, = \, \sup_{a \in A} \inf_{b \in B} \|a - b\| \, .$$

Definition 1.3. *Let (Λ, d) be a metric space.*

a) The set-valued mapping C is pseudo-continuous at $(\lambda_0, x_0) \in Graph\, C$ if there exist neighborhoods $V \subset \Lambda_0$ of λ_0, $U \subset X_0$ of x_0 and there exists a function $\beta : \mathbf{R}_+ \to \mathbf{R}_+$ continuous at 0, with $\beta(0) = 0$, such that

$$C(\lambda_0) \cap U \, \subset \, C(\lambda) \, + \, \beta\left(d(\lambda, \lambda_0)\right) \, B(0, 1) \tag{1.1}$$

and

$$C(\lambda) \cap U \subset C(\lambda_0) + \beta(d(\lambda, \lambda_0))\, B(0,1) \tag{1.2}$$

for all $\lambda \in V$.

b) The set-valued mapping C *is pseudo-continuous around* $(\lambda_0, x_0) \in Graph\, C$ *if there exist neighborhoods* $V \subset \Lambda_0$ *of* λ_0, $U \subset X_0$ *of* x_0 *and there exists a function* $\beta : \mathbf{R}_+ \to \mathbf{R}_+$ *continuous at* 0, *with* $\beta(0) = 0$, *such that*

$$C(\lambda_1) \cap U \subset C(\lambda_2) + \beta(d(\lambda_1, \lambda_2))\, B(0,1) \tag{1.3}$$

for all $\lambda_1, \lambda_2 \in V$.

c) If the function β *is defined as* $\beta(r) = Lr$, *with* $L \geq 0$, *then we say that* C *is pseudo-Lipschitz continuous at* (λ_0, x_0) *(resp. around* (λ_0, x_0)*).*

d) The set-valued mapping C *is pseudo-continuous on the set* $\Lambda_1 \subset \Lambda_0$, *if it is pseudo-continuous at each point* $(\lambda, x) \in Graph\, C$, $\lambda \in \Lambda_1$.

Remark 1.1. If the set-valued mapping $C(\cdot) \cap X_0$ is continuous with respect to the Hausdorff distance at λ_0 (resp. in a neighborhood of λ_0), then the set-valued mapping C is pseudo-continuous at (λ_0, x_0) (resp. around (λ_0, x_0)).

In [11] it is proved the following theorem:

Theorem 1.1. *In the case of* $\Omega \subset \mathbf{R}^m$, $\Lambda \subset \mathbf{R}^p$, $X = \mathbf{R}^n$, *let us suppose:*

(i) x_0 *is a solution of* $VI(\omega_0, \lambda_0)$;

(ii) there exists $l > 0$ *such that*

$$||f(x_1, \omega_1) - f(x_2, \omega_2)|| \leq l\,(||x_1 - x_2|| + ||\omega_1 - \omega_2||) ,$$

for all $x_1, x_2 \in X_0$, $\omega_1, \omega_2 \in \Omega_0$;

(iii) the mappings $f(\cdot, \omega)$ *are strongly-monotone with a constant* $a > 0$, *for all* $\omega \in \Omega_0$;

(iv) the set-valued mapping C *is pseudo-Lipschitz continuous around* $(\lambda_0, x_0) \in Graph\, C$.

Then there exist constants l_{ω_0}, $l_{\lambda_0} \geq 0$ *and there exist neighborhoods* $\Omega' \subset \Omega_0$ *of* ω_0, $\Lambda' \subset \Lambda_0$ *of* λ_0 *such that:*

a) for every $(\omega, \lambda) \in \Omega' \times \Lambda'$ *there exists a unique solution* $x(\omega, \lambda)$ *of* $VI(\omega, \lambda)$*;*
b) for all (ω_1, λ_1), $(\omega_2, \lambda_2) \in \Omega' \times \Lambda'$ *we have*

$$||x(\omega_1, \lambda_1) - x(\omega_2, \lambda_2)|| \leq l_{\omega_0}||\omega_1 - \omega_2|| + l_{\lambda_0}||\lambda_1 - \lambda_2||^{\frac{1}{2}} .$$

The Hölder continuity with respect to λ it is the consequence of the following result :

Proposition 1.2. [11] *Let* $\Omega \subset \mathbf{R}^p$ *and* $X = \mathbf{R}^n$. *Let us assume that the set-valued mapping* C *is pseudo-Lipschitz continuous around* (λ_0, x_0).
Then there exist neghborhoods $\Omega' \subset \Omega_0$ *of* ω_0 *and* $X' \subset X_0$ *of* x_0 *and there exists a constant* $l' > 0$ *such that*

$$\left\| P_{C(\lambda_1)\cap X_0}(x) - P_{C(\lambda_2)\cap X_0}(x) \right\| \leq l'||\lambda_1 - \lambda_2||^{\frac{1}{2}} ,$$

for all λ_1, $\lambda_2 \in \Lambda'$ *and* $x \in X'$.

We denoted by $P_{C(\lambda)\cap X_0}(x)$ the metric projection of the point x onto the set $C(\lambda) \cap X_0$, i.e. the unique point in $C(\lambda) \cap X_0$ with minimal distance to x.
The continuity of $C(\cdot) \cap X_0$, with respect to the Hausdorff distance, at λ_0 is assumed in [3] and the continuity of $P_{C(\lambda)\cap X_0}$ at λ_0 is proved.

2. An implicit function theorem for monotone mappings

In this section we will show that Theorem 4.3 of [1] remains true when we suppose X a reflexive Banach space (Theorem 2.1). We will use this theorem to study the stability of the solutions of $VI(\omega, \lambda)$, using only the consistency of the normal cone operator which is a weaker property then the continuity of the projections of a fixed point onto a family of nonempty, closed, convex sets.

Lemma 2.1. [5] *Let* $T : X \rightsquigarrow X^*$ *be a maximal-monotone set-valued mapping. For all integers* $k \geq 1$ *we define the following single-valued mappings:*

$$P_k = (J + kT)^{-1} : X^* \to X .$$

If a sequence (x_k)*, with* $x_{k+1} = P_k(Jx_k)$ *is bounded, then there exists* $\overline{x} \in X$ *such that* $0 \in T(\overline{x})$ *and* (x_k) *has a subsequence weakly converging to* $\overline{x}$.

Remark 2.1. From $x_{k+1} = P_k(Jx_k)$ we have

$$\frac{1}{k}(Jx_k - Jx_{k+1}) \in T(x_{k+1}) ,$$

so $x_{k+1} \in D(T)$.

If $D(T)$ is bounded, then (x_k) is also bounded and T has a zero in $D(T)$.

If $T_1 = T + N_{B(0,\varepsilon)}$ is maximal-monotone for an $\varepsilon > 0$, then there exists $x \in B(0,\varepsilon)$ such that $0 \in T(x) + N_{B(0,\varepsilon)}$. If $||x|| < \varepsilon$, then $0 \in T(x)$.

Lemma 2.2. *Let $T : X \rightsquigarrow X^*$ be a maximal-monotone map. We suppose that there exist $0 < \delta < \varepsilon$ such that $D(T) \cap intB(0,\varepsilon) \neq \emptyset$ and $\langle y, x\rangle > 0$, for all $x \in X$, with $\delta \leq ||x|| \leq \varepsilon$ and for all $y \in T(x)$.*

Then there exists $\overline{x} \in B(0,\delta)$, such that $0 \in T(\overline{x})$.

Proof. Because of $D(T) \cap intB(0,\varepsilon) \neq \emptyset$, $T_1 = T + N_{B(0,\varepsilon)}$ is a maximal-monotone mapping with $D(T_1) = B(0,\varepsilon)$.

If $x \in B(0,\varepsilon)$ and $y_1 \in T_1(x)$, then there exist $y \in T(x)$, $n \in N_{B(0,\varepsilon)}(x)$ with $n = 0$ or $n = \lambda J(x)$, $\lambda > 0$, such that $y_1 = y + n$.

Then $\langle y_1, x\rangle = \langle y + n, x\rangle \geq \langle y, x\rangle$ and hence the assumptions of Lemma 2.2 are also satisfied by T_1.

Let us denote

$$P_k(x) = (J + kT_1)^{-1}(Jx) , \; x_1 = 0 , \; x_{k+1} = P_k(x_k) .$$

We will prove that $||x_k|| \leq \delta$, for all $k \geq 1$.

Let us suppose the contrary and let k_0 be the first integer such that $||x_{k_0}|| \leq \delta$ and $||x_{k_0+1}|| > \delta$. Then

$$Jx_{k_0} \in Jx_{k_0+1} + k_0 T_1(x_{k_0+1}) ,$$

so $x_{k_0+1} \in D(T_1) = B(0,\varepsilon)$ and there exists $u_{k_0+1} \in T_1(x_{k_0+1})$, such that

$$Jx_{k_0} = Jx_{k_0+1} + k_0 u_{k_0+1} .$$

Then

$$||Jx_{k_0}||||x_{k_0+1}|| \geq \langle x_{k_0+1}, Jx_{k_0}\rangle = \langle x_{k_0+1}, Jx_{k_0+1}\rangle + k_0\langle x_{k_0+1}, u_{k_0+1}\rangle >$$

$$> \|x_{k_0+1}\|^2 .$$

Hence $\|x_{k_0}\| = \|Jx_{k_0}\| > \|x_{k_0+1}\| > \delta$, which is a contradiction.

So, $(x_k) \subset B(0,\delta)$ and using Lemma 2.1 together with the weakly-compactness of $B(0,\delta)$, we can found $\overline{x} \in B(0,\delta)$, such that $0 \in T_1(\overline{x})$.

But $N_{B(0,\varepsilon)}(\overline{x}) = \{0\}$, so $0 \in T(\overline{x})$.

Remark 2.2. Let us fix an $x_0 \in X$.

If we use Lemma 2.2 for $F(x) = T(x + x_0)$ we get:

Let $0 < \delta < \varepsilon$ be such that $D(T) \cap int B(x_0, \varepsilon) \neq \emptyset$ and $\langle y, x\rangle > 0$, for all $x \in X$ with $\delta \leq \|x\| \leq \varepsilon$ and for all $y \in T(x + x_0)$.

Then there exists $\overline{x} \in B(x_0,\delta)$, such that $0 \in T(\overline{x})$.

Definition 2.1. *Let $F : X \times W \rightsquigarrow X^*$ be a set-valued map and let $y_0 \in F(x_0, w_0)$. We say that F is consistent with respect to w at (x_0, w_0, y_0) in a neighborhood W_0 of w_0, if there exists a function $\beta : W_0 \to \mathbf{R}_+$, continuous at w_0, with $\beta(w_0) = 0$, such that for all $w \in W_0$ there exists $(x_w, y_w) \in GraphT(\cdot, w)$, satisfying $\|x_w - x_0\| \leq \beta(w)$ and $\|y_w - y_0\| \leq \beta(w)$.*

Remark 2.3. For example, F is consistent in w at (x_0, w_0, y_0), if $F(\cdot, w_0)$ has a continuous selection trough $(x_0, y_0) \in \mathrm{Graph}F(\cdot, w_0)$.

In [4] it is used a stronger assumption ($x_w = x_0$ for all $w \in W_0$), but in the study of the parametric variational inequalities this form cannot be used. We will show also that the normal cone operator is consistent if the projection operator is continuous with respect to the parameter λ.

The following theorem is the generalization of the Theorem 4.3 of [1], in the case of reflexive Banach spaces. We will suppose that X is renormed strictly-convex with strictly-convex dual.

Theorem 2.1. *Let us assume that:*

i) $0 \in F(x_0, w_0)$;

ii) F is consistent with respect to w at $(x_0, w_0, 0)$ in W_0;

iii) the set-valued mappings $F(\cdot, w)$ are maximal-monotone and φ-uniformly-monotone for all $w \in W_0$.

Then there exists a neighborhood W_1 of w_0 and a unique mapping $x : W_1 \to X_0$, continuous at w_0, such that $x(w_0) = x_0$ and $0 \in F(x(w), w)$, for all $w \in W_1$.

Proof. Let us fix $0 < \varepsilon < \varepsilon_1$ such that $B(x_0, \varepsilon_1) \subset X_0$. Let $W' \subset W_0$ be a neighborhood of w_0, such that $\beta(w) < \varepsilon_1$, for all $w \in W'$.

Let $0 < \delta \leq \varepsilon$ and $w \in W'$ be choosen arbitrarily.

Then $D(F(\cdot, w)) \cap int B(x_0, \varepsilon_1) \neq \emptyset$, because from assumption *ii)* , there exists $(x_w, y_w) \in \text{Graph} F(\cdot, w)$, such that $||x_w - x_0|| \leq \beta(w) < \varepsilon_1$ and $||y_w|| \leq \beta(w)$.

Let us choose $x \in X$, with $\delta \leq ||x|| \leq \varepsilon$ and $y \in F(x + x_w, w)$. Then

$$\varphi(\,||x||\,)||x|| \leq \langle y - y_w, x\rangle = \langle y, x\rangle - \langle y_w, x\rangle$$

and hence

$$\langle y, x\rangle \geq \varphi(\delta)\delta - \varepsilon\beta(w)\ .$$

Let us denote $M_w = \{\delta > 0 \ : \ \delta\varphi(\delta) > \varepsilon\beta(w)\}$.

We can see that $M_w \neq \emptyset$ for all w in a neighborhood $W_1 \subset W'$ of w_0 and $\inf M_w \to 0$, when $w \to w_0$. So, we can choose a selection $\delta(w) \in M_w$, such that $\delta(w) \to 0$, when $w \to w_0$.

Using Remark 2.2 we can find, for all $w \in W_1$, a solution $x(w) \in B(x_w, \delta(w))$ of [3] and this solution is unique because of the φ-uniform-monotonicity of $F(\cdot, w)$. We have also

$$||x(w) - x_0|| \leq ||x(w) - x_w|| + ||x_w - x_0|| \leq \delta(w) + \beta(w) \to 0\ ,$$

when $w \to w_0$.

Remark 2.4. In the case when F is a single valued mapping the assumptions of Theorem 2.1 can be written as:

i) $0 = F(x_0, w_0)$;

ii) the mapping F is continuous at (x_0, w_0);

iii) the mappings $F(\cdot, w)$ are hemicontinuous and φ-uniformly-monotone on X_0, for *all* $w \in W_0$.

In the following two corollaries we will study the continuity of the solutions in a neighborhood of the fixed parameter ω_0.

Corollary 2.1. *Let* (W, d) *be a metric space.*
If we replace assumption ii) of Theorem 2.1 by:
ii') the set-valued mappings $F(x, \cdot)$ *are pseudo-continuous on* W_0*, for all* $x \in X_0$*, then the mapping* x *is continuous in a neighborhood of* w_0.

Proof. We will show that the pseudo-continuity at $(w_0, 0) \in \operatorname{Graph} F(x_0, \cdot)$ implies the consistency condition ii). Indeed, there exist neighborhoods U of 0_{X^*}, V of w_0 and a function $\beta_0 : \mathbf{R}_+ \to \mathbf{R}_+$continuous at 0, with $\beta_0(0) = 0$, such that

$$F(x_0, w_0) \cap U \subset F(x_0, w) + \beta_0(d(w, w_0))B(0, 1)$$

and

$$F(x_0, w) \cap U \subset F(x_0, w_0) + \beta_0(d(w, w_0))B(0, 1)$$

for all $w \in V$. Hence, for $0 \in F(x_0, w_0) \cap U$ and for all $w \in V$, there exists $z_w \in F(x_0, w)$ such that $\|z_w\| \leq \beta_0(d(w, w_0))$.
Now we use Theorem 2.1 to obtain a neighborhood W_1 of w_0 and a unique mapping $x : W_1 \to X_0$ continuous at w_0, with $x(w_0) = x_0$ and $0 \in F(x(w), w)$ for all $w \in W_1$. The continuity of the mapping x at w_0 implies that there exists an open neighborhood $W_1' \subset W_1$ of w_0, such that $x(w) \in \operatorname{int} X_0$ for all $w \in W_1'$.
If we choose $\overline{w} \in W_1'$ arbitrarily, a constant $\overline{r} > 0$, such that $B(x(\overline{w}), \overline{r}) \subset X_0$ and we use the pseudo-continuity of $F(x(\overline{w}), \cdot)$ at $(\overline{w}, 0)$, which implies the consistency at $(x(\overline{w}), \overline{w}, 0)$, then we can use Theorem 2.1 to find a neighborhood $\overline{W} \subset W_1$ of $\overline{w}$ and a unique mapping $\overline{x} : \overline{W} \to B(x(\overline{w}), \overline{r})$ continuous at $\overline{w}$, such that $\overline{x}(\overline{w}) = x(\overline{w})$, and $0 \in F(\overline{x}(w), w)$ for all $w \in \overline{W}$. The uniqueness of the mappings x and $\overline{x}$ implies that they coincide on $\overline{W}$, so we have proved the continuity of the mapping x at $\overline{w}$. Because $\overline{w}$ has been choosen arbitrarily, the continuity holds on W_1'.

Corollary 2.2. *Let (W,d) be a metric space. Let us suppose that in Theorem 2.1 we replace assumption ii) by:*

ii_L) there exists a constant $L > 0$ and for each $x \in X_0$ there exists a neighborhood U_x of 0_{X^*} such that

$$F(x,w_1) \cap U_x \subset F(x,w_2) + Ld(w_1,w_2)B(0,1)$$

for all $w_1, w_2 \in W_0$,

and assumption iii) by:

iii_L) the set-valued mappings $F(\cdot,w)$ are maximal-monotone and strongly-monotone with a constant $a > 0$, for all $w \in W_0$.

Then the mapping x is Lipschitz-continuous, with the constant $\frac{L}{a}$, in a neighborhood of w_0.

Proof. Using Corollary 2.1 we obtain a neighborhood W_1' of w_0, such that the mapping x is continuous on W_1'.

Let us choose $w_1, w_2 \in W_1'$ arbitrarily. Assumption ii_L) implies that for $x(w_1) \in X_0$ there exists a neighborhood U of 0_{X^*} such that

$$F(x(w_1),z) \cap U \subset F(x(w_1),t) + Ld(z,t)B(0,1)$$

for all $z, t \in W_0$.

Hence for $0 \in F(x(w_1),w_1) \cap U$ there exists $z_2 \in F(x(w_1),w_2)$ such that $||z_2|| \leq Ld(w_1,w_2)$. Then

$$a||x(w_1) - x(w_2)||^2 \leq \langle z_2 - 0 \, , \, x(w_1) - x(w_2) \rangle \leq$$

$$\leq ||z_2|| ||x(w_1) - x(w_2)|| \, .$$

So,

$$||x(w_1) - x(w_2)|| \leq \frac{L}{a} d(w_1,w_2) \, .$$

In the followings we will apply Theorem 2.1 in the study of $VI(\omega,\lambda)$. We suppose the consistency of the normal cone operator instead of the continuity, with respect to parameters, of the projections. The adventage of this approache is that the assumptions a-d) of Theorem 2.2 are independent from the geometrical properties of the reflexive Banach space X. We will show also that, in locally-uniform convex Banach spaces with locally-uniform convex dual, the consistency is a weaker property that the continuity of the projection. Assumption *iii)* of Theorem 1.1, due to Proposition 1.2, implies the continuity of the projections which it is supposed also in [1] and [3].

Theorem 2.2. *Let us suppose that:*

a) $0 \in f(x_0, \omega_0) + N_{C(\lambda_0)}(x_0)$;

b) f *is continuous on* $X_0 \times \Omega_0$;

c) the mapping $N(x, \lambda) = N_{C(\lambda)\cap X_0}(x)$ *is consistent with respect to* λ *at* $(x_0, \lambda_0, -f(x_0, \omega_0))$ *in* Λ_0;

d) the mappings $f(\cdot, \omega)$ *are* φ*-uniformly-monotone on* X_0, *for all* $\omega \in \Omega_0$.

Then there exist neighborhoods Ω_1 *and* Λ_1 *of* ω_0 *and* λ_0 *and a unique mapping* $x : \Omega_1 \times \Lambda_1 \to X_0$, *continuous at* (ω_0, λ_0), *such that* $x(\omega_0, \lambda_0) = x_0$ *and* $0 \in f(x(\omega, \lambda), \omega) + N_{C(\lambda)}(x(\omega, \lambda))$.

Proof. Let us denote $W = \Omega \times \Lambda$ and $F(x, w) = F(x, \omega, \lambda) = f(x, \omega) + N_{C(\lambda)\cap X_0}(x)$. The mappings $F(\cdot, w)$ are maximal-monotone. These mappings are also φ-uniformly-monotone on X_0 as a sum of a φ-uniformly-monotone and a monotone mapping.
Assumption *c)* implies the existence of a function $\beta_1 : \Lambda_0 \to \mathbf{R}_+$, continuous at λ_0, with $\beta_1(\lambda_0) = 0$, such that for all $\lambda \in \Lambda_0$ there exists $(x_\lambda, n_\lambda) \in \mathrm{Graph} N(\cdot, \lambda)$ such that $||x_\lambda - x_0|| \leq \beta_1(\lambda)$ and $||n_\lambda + f(x_0, \omega_0)|| \leq \beta_1(\lambda)$.
Let us choose $(\omega, \lambda) \in \Omega_0 \times \Lambda_0$.
We denote $x_{\omega,\lambda} = x_\lambda$ and $y_{\omega,\lambda} = n_\lambda + f(x_\lambda, \omega)$.
Then $y_{\omega,\lambda} \in F(x_{\omega,\lambda}, \omega, \lambda)$ and

$$||y_{\omega,\lambda}|| \leq ||n_\lambda + f(x_0, \omega_0)|| + ||f(x_\lambda, \omega) - f(x_0, \omega_0)|| \leq$$

$$\leq \beta_1(\lambda) + ||f(x_\lambda, \omega) - f(x_0, \omega_0)|| = \beta(\omega, \lambda) \ .$$

Using the continuity of f, we get $\beta(\omega, \lambda) \to 0$, when $(\omega, \lambda) \to (\omega_0, \lambda_0)$, hence the assumptions of Theorem 2.1 are satisfied and the existence and continuity at (ω_0, λ_0) of the solutions of $0 \in F(x, w)$ are proved.
When (ω, λ) is close enough to (ω_0, λ_0), then $x(\omega, \lambda) \in int X_0$ and hence $N_{C(\lambda)}(x(\omega, \lambda)) =$

$N_{C(\lambda)\cap X_0}(x(\omega,\lambda))$, so the proof is complete.

Remark 2.5. We observe that if in Theorem 2.2 we suppose the same type of pseudo-continuities for the set-valued mapping N, as in the previous corollaries for F, then we obtain same continuities for the mapping x.

Definition 2.2. [12] *Let $C \subset X$ be a nonempty, closed, convex set. The projection onto C is the mapping $P_C : X^* \to X$ defined by*

$$P_C(x^*) = (J + N_C)^{-1}(x^*) .$$

In the case when X is a Hilbert space, this is the metric projection onto C.
Let X be a locally-uniform convex, reflexive Banach space, with X^* locally-uniform convex. In this case the normalized duality mapping is continuous from the strong topology of X to the strong topology of X^*.
Let us define the mapping $P : X^* \times \Lambda \to X$ by

$$P(x^*, \lambda) = P_{C(\lambda)\cap X_0}(x^*) .$$

The following result shows that the continuity of the projection with respect to a parameter implies the consistency of the normal cone operator.

Proposition 2.1. *If for an $n_0 \in N(x_0, \lambda_0)$, $P(Jx_0 + n_0, \cdot)$ is continuous at λ_0, then N is consistent with respect to λ at (x_0, λ_0, n_0) in a neighborhood of λ_0.*

Proof. Let us take $x_\lambda = P(Jx_0 + n_0, \lambda)$. Because of $x_0 = P(Jx_0 + n_0, \lambda_0)$ and the continuity of P hold $||x_\lambda - x_0|| \to 0$, when $\lambda \to \lambda_0$.
We have also

$$Jx_0 + n_0 \in Jx_\lambda + N(x_\lambda, \lambda)$$

and hence

$$Jx_0 + n_0 - Jx_\lambda \in N(x_\lambda, \lambda) .$$

We can take

$$y_\lambda = Jx_0 + n_0 - Jx_\lambda ,$$

$$\beta(\lambda) = \max\{\,||x_\lambda - x_0||\,,\,||Jx_\lambda - Jx_0||\,\}$$

and the consistency of N is proved.

3. Continuity of the projection with respect to a parameter

In this section, assuming the pseudo-continuity of the set-valued mapping C, we will show that the continuity, with respect to a parameter, of the projection operator holds in a uniformly-convex Banach space. We cannot obtain the same type of Hölder-continuity as in Proposition 1.2 because, as will be shown, that holds only in Hilbert-spaces.

Proposition 3.1. *Let (Λ, d) be a metric space.*
Let $A : X_0 \to X^*$ be a continuous, φ-uniformly-monotone mapping. Let us suppose that the set-valued mapping C is pseudo-continuous at $(\lambda_0, x_0) \in Graph\,C$ and $0 \in A(x_0) + N_{C(\lambda_0)}(x_0)$.
Then there exist a neighborhood $V \subset \Lambda_0$ of λ_0, a function $\beta_1 : \mathbf{R}_+ \to \mathbf{R}_+$ continuous at 0, with $\beta_1(0) = 0$, and a constant $s > 0$ such that the generalized equation

$$0 \in A(x) + N_{C(\lambda)}(x) \tag{3.1}$$

has a unique solution $x(\lambda) \in B(x_0, s)$ for all $\lambda \in V$ and also hold

$$\varphi\left(||x(\lambda) - x_0||\right)\,||x(\lambda) - x_0|| \leq \beta_1\left(d(\lambda, \lambda_0)\right)\,. \tag{3.2}$$

Proof. We choose a constant $0 < s < r$ such that the pseudo-continuity of C can be written as:
- there exist a function $\beta : \mathbf{R}_+ \to \mathbf{R}_+$ continuous at 0, with $\beta(0) = 0$, and a neighborhood $V' \subset \Lambda_0$ of λ_0 such that

$$C(\lambda_0) \cap B(x_0, s) \subset C(\lambda) + \beta\left(d(\lambda, \lambda_0)\right) B(0, 1)$$

and

$$C(\lambda) \cap B(x_0, s) \subset C(\lambda_0) + \beta\left(d(\lambda, \lambda_0)\right) B(0, 1)$$

for all $\lambda \in V'$.

Using the continuity of β at 0, we can choose $\varepsilon > 0$ such that $B(\lambda_0, \varepsilon) \subset V'$ and $\beta(d(\lambda, \lambda_0)) \leq s$, for all $\lambda \in B(\lambda_0, \varepsilon)$.

Let us define $V_\varepsilon = B(\lambda_0, \varepsilon)$.

Let $\lambda \in V_\varepsilon$ be choosen arbitrarly. Then the inclusion

$$x_0 \in C(\lambda_0) \cap B(x_0, s) \subset C(\lambda) + \beta(d(\lambda, \lambda_0))\, B(0,1)$$

implies the existence of an $u_\lambda \in C(\lambda)$ such that

$$\|x_0 - u_\lambda\| \leq \beta(d(\lambda, \lambda_0)) \leq s .$$

This means that $C(\lambda) \cap B(x_0, s)$ is nonempty for all $\lambda \in V$. Corollary 32.35 of [13] shows that the generalized equation

$$0 \in A(x) + N_{C(\lambda) \cap B(x_0, s)}(x)$$

has a unique solution $x(\lambda) \in C(\lambda) \cap B(x_0, s)$. So

$$\langle A(x(\lambda)),\, u - x(\lambda)\rangle \geq 0$$

for all $u \in C(\lambda) \cap B(x_0, s)$.

The pseudo-continuity of the set-valued mapping C implies that for $x(\lambda)$ there exists an element $u_0 \in C(\lambda_0)$ such that $\|x(\lambda) - u_0\| \leq \beta(d(\lambda, \lambda_0))$.

Using the φ-uniform-monotonicity of A we obtain

$$\begin{aligned}
\varphi(\|x(\lambda) - x_0\|)\, \|x(\lambda) - x_0\| &\leq \langle A(x(\lambda)) - A(x_0),\, x(\lambda) - x_0\rangle \leq \\
&\leq \langle A(x(\lambda)) - A(x_0),\, x(\lambda) - x_0\rangle + \langle A(x_0),\, u_0 - x_0\rangle + \\
&\quad + \langle A(x(\lambda)),\, u_\lambda - x(\lambda)\rangle = \\
&= -\langle A(x(\lambda)),\, u_\lambda - x_0\rangle + \langle A(x_0),\, u_0 - x(\lambda)\rangle \leq \\
&\leq \|A(x(\lambda))\|\, \|u_\lambda - x_0\| + \|A(x_0)\|\, \|u_0 - x(\lambda)\| \leq \\
&\leq 2M\beta(d(\lambda, \lambda_0)) ,
\end{aligned}$$

where $M = \sup\{\|A(x)\| : x \in B(x_0, s)\}$ is finite, because a continuous, monotone mapping is bounded on the interior of its domain. The inequality

$$\varphi(\|x(\lambda) - x_0\|)\,\|x(\lambda) - x_0\| \leq 2M\beta(d(\lambda, \lambda_0))$$

means that $x(\lambda) \to x_0$, when $\lambda \to \lambda_0$.

We can choose a neighborhood $V \subset V_\varepsilon$ of λ_0, such that $\|x(\lambda) - x_0\| < s$, for all $\lambda \in V$. This means that for $\lambda \in V$

$$N_{C(\lambda)}(x(\lambda)) = N_{C(\lambda)\cap B(x_0,s)}(x(\lambda))$$

and hence $x(\lambda)$ is a solution of the problem (3.1).

The inequality (3.2) is satisfied with $\beta_1(r) = 2M\beta(r)$.

Corollary 3.1. *If in Proposition 3.1 we suppose that the set-valued mapping C is pseudo-continuous around (λ_0, x_0), then*

$$\varphi(\|x(\lambda_1) - x(\lambda_2)\|)\,\|x(\lambda_1) - x(\lambda_2)\| \leq \beta_1(d(\lambda_1, \lambda_2)),$$

for all $\lambda_1, \lambda_2 \in V$.

Proof. Let us choose the constant $0 < s < r$ and the neighborhood $V' \subset \Lambda_0$ such that

$$C(\lambda_1) \cap B(x_0, s) \subset C(\lambda_2) + \beta(d(\lambda_1, \lambda_2))B(0, 1)$$

for all $\lambda_1, \lambda_2 \in V'$.

As in the proof of the Proposition 3.1 we obtain the neighborhood V of λ_0 and the solution $x(\lambda)$ of (3.1), for all $\lambda \in V$.

Let us choose $\lambda_1, \lambda_2 \in V$. For $x(\lambda_1) \in C(\lambda_1) \cap B(x_0, s)$ there exists $u_2 \in C(\lambda_2)$, such that

$$\|x(\lambda_1) - u_2\| \leq \beta(d(\lambda_1, \lambda_2)).$$

For $x(\lambda_2) \in C(\lambda_2) \cap B(x_0, s)$ there exists $u_1 \in C(\lambda_1)$ such that

$$\|x(\lambda_2) - u_1\| \leq \beta(d(\lambda_1, \lambda_2)).$$

Then

$$\varphi(\|x(\lambda_1) - x(\lambda_2)\|)\,\|x(\lambda_1) - x(\lambda_2)\| \leq$$

$$\leq \langle A(x(\lambda_1)) - A(x(\lambda_2)), x(\lambda_1) - x(\lambda_2)\rangle \leq$$

$$\leq \langle A(x(\lambda_1)) - A(x(\lambda_2)), x(\lambda_1) - x(\lambda_2)\rangle + \langle A(x(\lambda_1)), u_1 - x(\lambda_1)\rangle +$$

$$+ \langle A(x(\lambda_2)), u_2 - x(\lambda_2)\rangle =$$

$$= \langle A(x(\lambda_1)), u_1 - x(\lambda_2)\rangle - \langle A(x(\lambda_2)), x(\lambda_1) - u_2\rangle \leq$$

$$\leq \|A(x(\lambda_1))\|\,\|u_1 - x(\lambda_2)\| + \|A(x(\lambda_2))\|\,\|x(\lambda_1) - u_2\| \leq$$

$$\leq 2M\beta(d(\lambda_1,\lambda_2)) = \beta_1(d(\lambda_1,\lambda_2)) .$$

Corollary 3.2. *Let (Λ, d) be a metric space and let X be a uniformly-convex Banach space. If the set-valued mapping C is pseudo-continuous at (λ_0, x_0) (resp. around (λ_0, x_0)), then $P(\cdot, x^*)$ is continuous at λ_0 (resp. in a neighborhood of λ_0) for all $x^* \in X^*$.*

Proof. Let us choose $r, R > 0$ such that $x_0 \in \operatorname{int} B(0,R)$, $B(x_0,r) \subset B(0,R)$. Let us fix an element $x^* \in X^*$. We use Proposition 3.1 (resp. Corollary 3.1) in the case of the mapping $A : B(x_0,r) \to X^*$, defined by $A(x) = J(x) - x^*$, which is φ_R-uniformly-monotone due to Proposition 1.1. In this way we obtain a neighborhood V of λ_0 and a unique mapping $x : V \to X_0$ continuous at λ_0 (resp. in a neighborhood of λ_0), such that for all $\lambda \in V$ we have

$$0 \in J(x(\lambda)) - x^* + N_{C(\lambda)}(x(\lambda)) ,$$

which means that $x(\lambda) = P(x^*, \lambda)$.

Remark 3.1. If we suppose that C is pseudo-continuous around (λ_0, x_0), then $P(\cdot, x^*)$ is continuous in a neighborhood of λ_0.

We observe that in a uniformly-convex Banach space, which is not a Hilbert space, we cannot prove the Hölder-continuity of Proposition 1.2, even when C is pseudo-Lipschitz continuous. The Hölder-continuity holds only in a Hilbert space because only in this case is the normalized duality mapping strongly-monotone.

4. Parametric variational inequalities

In this section we generalize Theorem 1.1, on the continuity of the solutions of $VI(\omega, \lambda)$, in the case of reflexive Banach spaces. The continuity of the projection operator or the consistency of the normal cone operator will be not supposed. A consequence of Theorem 4.1 is that all the results from [1], [3], [8], [11] remain true in reflexive Banach spaces.

We suppose that X is renormed strictly-convex with strictly-convex dual and let (Λ, d) be a metric space.

Theorem 4.1. *Let us suppose that:*

a) $0 \in f(x_0, \omega_0) + N_{C(\lambda_0)}(x_0)$*;*

b) f *is continuous on* $X_0 \times \Omega_0$*;*

c) the mappings $f(\cdot, \omega)$ *are* φ*-uniformly-monotone on* X_0*, for all* $\omega \in \Omega_0$*;*

d) the set-valued mapping C *is pseudo-continuous at* (λ_0, x_0)*.*

Then there exist neighborhoods Ω' *of* ω_0*,* Λ' *of* λ_0 *and a unique mapping* $x : \Omega' \times \Lambda' \to X_0$ *continuous at* (ω_0, λ_0)*, such that* $x(\omega_0, \lambda_0) = x_0$ *and*

$$0 \in f(x(\omega, \lambda), \omega) + N_{C(\lambda)}(x(\omega, \lambda))$$

for all $\omega \in \Omega'$*,* $\lambda \in \Lambda'$*.*

Proof. We choose the positive constants ε, r small enough to

$$\beta(d(\lambda, \lambda_0)) \leq r ,$$

$$C(\lambda_0) \cap B(x_0, r) \subset C(\lambda) + \beta(d(\lambda, \lambda_0))B(0,1)$$

and

$$C(\lambda) \cap B(x_0, r) \subset C(\lambda_0) + \beta(d(\lambda, \lambda_0))B(0,1) ,$$

for all $\lambda \in B(\lambda_0, \varepsilon)$.

As in the previous proofs, for all $\lambda \in B(\lambda_0, \varepsilon)$, the set $C(\lambda) \cap B(x_0, r)$ is nonempty. Hence, for all $(\omega, \lambda) \in \Omega_0 \times B(\lambda_0, \varepsilon)$ there exists a unique element $x(\omega, \lambda)$ such that

$$0 \in f(x(\omega, \lambda), \omega) + N_{C(\lambda)}(x(\omega, \lambda)) .$$

Using Proposition 3.1 we deduce that for all $\omega \in \Omega_0$ the mappings $x(\omega, \cdot)$ are continuous at λ_0 and Theorem 2.2 implies that $x(\cdot, \lambda_0)$ is continuous at ω_0.
The above continuities shows that the mapping x is continuous at (ω_0, λ_0).

Remark 4.1. As in the previous section, if we suppose that the set-valued mapping C is pseudo-continuous around (λ_0, x_0), then the solution mapping x is continuous in a neighborhood of (ω_0, λ_0).

5. Applications

The reason of the following examples is to show that Theorem 2.1 is usefull in the study of the continuity of the solutions of parametric integral equations and evolution differential inclusions. We will also show that the consistency condition appear under well-known assumptions and it is important because some of the mappings are not defined everywhere, they have only dense domains.

Example 5.1. Let $(a, b) \subset \mathbf{R}$ be an open interval, let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, let $\lambda_0 \in \mathbf{R}$, let Λ_0 be a neighborhood of λ_0 and let $u_0 \in L^p(a, b)$.
We suppose that the mappings $F : (a, b) \times \mathbf{R} \times \Lambda_0 \to \mathbf{R}$ and $K : (a, b) \times (a, b) \to \mathbf{R}$ satisfy the following conditions:

(F_1) *the mappings* $F(\cdot, r, \lambda)$ *are measurable for all* $r \in \mathbf{R}$ *and* $\lambda \in \Lambda_0$;

(F_2) *the mappings* $F(x, \cdot, \lambda)$ *are continuous a.e.* $x \in (a, b)$ *and* $\lambda \in \Lambda_0$;

(F_3) *for each* $\lambda \in \Lambda_0$ *there exist* $g_\lambda \in L^q(a, b)$ *and* $c_\lambda > 0$ *such that*

$$|F(x, r, \lambda)| \leq g_\lambda(x) + c_\lambda |r|^{p-1}$$

for all $r \in \mathbf{R}$ *and* $x \in (a, b)$;

(F_4) *there exists a constant* $d > 0$ *such that*

$$(F(x, r_1, \lambda) - F(x, r_2, \lambda))(r_1 - r_2) \geq d|r_1 - r_2|^p ,$$

for all $x \in (a, b)$, $r_1, r_2 \in \mathbf{R}$, $\lambda \in \Lambda_0$;

(F_5) *the mappings* $F(\cdot, u_0(\cdot), \lambda)$ *converge uniformly to* $F(\cdot, u_0(\cdot), \lambda_0)$ *on (a,b), when*

$\lambda \to \lambda_0$;

(K_1) *there exist constants* $c_1, c_2 > 0$, $s, t > 1$ *such that*

$$\left(\int_a^b |K(x,y)|^s \, dx\right)^{\frac{1}{s}} \leq c_1 \,, \quad a.e. \quad y \in (a,b) \,,$$

$$\left(\int_a^b |K(x,y)|^t \, dy\right)^{\frac{1}{t}} \leq c_2 \,, \quad a.e. \quad x \in (a,b) \,,$$

$$p \geq s \,, \quad \left(1 - \frac{s}{p}\right) p \leq t \,;$$

(K_2) *for all* $u \in L^p(a,b)$, $u \neq 0$ *we have*

$$\int_a^b \int_a^b K(x,y) u(x) u(y) \, dx \, dy > 0 \,.$$

Remark 5.1. Assumptions (F_3) and (F_4) do not exclude. We can take, for example, $F(x,r,\lambda) = x\lambda + |r|^{p-2} r$.

Assumptions (F_1), (F_2), (F_3) imply that ([9]) for all $\lambda \in \Lambda_0$ the mappings $H(\cdot,\lambda) : L^p(a,b) \to L^q(a,b)$, defined by $H(u,\lambda)(x) = F(x,u(x),\lambda)$, are well-defined and continuous.

Assumption (F_4) implies the φ-uniform-monotonicity of the mappings $H(\cdot,\lambda)$, with $\varphi(r) = dr^{p-1}$. This means that the strong-monotonicity is satisfied locally when $1 < p \leq 2$ and is not satisfied when $2 < p$.

Assumption (K_1) implies that ([9]) the mapping $G : L^q(a,b) \to L^p(a,b)$, defined by

$$G(v)(x) = \int_a^b K(x,y) v(y) \, dx \,,$$

is well-defined and continuous (not nescesarrily compact).

Assumption (K_2) implies the strict-monotonicity of G.

Let us consider the following parametric Hammerstein integral equation:

$$u(x) + \int_a^b K(x,y) F(y, u(y), \lambda) \, dy = \omega(x) \,. \tag{5.1}$$

Proposition 5.1. *Let us consider that assumptions* $(F_1)-(F_5)$, $(K_1)-(K_2)$ *are satisfied and there exist* $u_0, \omega_0 \in L^p(a,b)$ *such that*

$$u_0(x) + \int_a^b K(x,y)F(y,u_0(y),\lambda_0)\,dy = \omega_0(x) .$$

Then there exists a unique mapping $u : L^p(a,b)\times\Lambda_0 \to L^p(a,b)$ *such that* $u(\omega,\lambda)$ *is a solution of (5.1) for all* $(\omega,\lambda)\in L^p(a,b)\times\Lambda_0$, $u(\omega_0,\lambda_0)=u_0$ *and the mapping* u *is continuous at* (ω_0,λ_0).

Proof. The existence of the solutions $u(\omega,\lambda)$ is proved in [9]. We will prove now the continuity.

Equation (5.1) can be written as

$$u + G\circ H(u,\lambda) = \omega$$

or equivalently

$$0 \in H(u,\lambda) - G^{-1}(\omega - u) .$$

We define the mapping $T : L^p(a,b)\times L^p(a,b) \to L^q(a,b)$ by

$$T(\omega,u) = -G^{-1}(\omega-u) .$$

The mappings $T(\omega,\cdot)$ are linear, continuous, maximal-monotone and strictly-monotone and hence G^{-1} is linear, continuous, maximal-monotone on $Dom\,G^{-1}$. We observe also that in this case $Dom\,G^{-1}$ is dense in $L^p(a,b)$ ([12]).

Let us fix $\omega\in L^p(a,b)$. Then we can choose $u_\omega\in L^p(a,b)$ such that $||u_\omega - u_0|| \le ||\omega-\omega_0||$ and $\omega - u_\omega \in Dom\,G^{-1}$. Hence $G^{-1}(\omega-u_\omega)\to G^{-1}(\omega_0-u_0)$, when $\omega\to\omega_0$, so we proved the consistency of T with respect to ω at $(u_0,\omega_0,T(\omega_0,u_0))$.

Assumption (F_5) implies that the mapping $H(u_0,\cdot)$ is continuous at λ_0 and using the continuity of the mappings $H(\cdot,\lambda)$ we conclude that H is continuous at (u_0,λ_0). This continuity together with the consistency of T implies that $H+T$ is consistent with respect to (ω,λ) at $(u_0,\omega_0,\lambda_0,0)$.

Now we can use Theorem 2.1 for the mapping $H+T$ to get the desired continuity.

Example 5.2. Let H be a Hilbert space. Let us consider the following problem:

$$\begin{cases} u' + A(u,\lambda) \ni f \\ u(0) = 0 \end{cases} \tag{5.2}$$

in the case when $T > 0$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^q(0,T;H)$, $\lambda_0 \in \mathbf{R}$, Λ_0 is a neighborhood of λ_0, $A : L^p(0,T;H) \times \Lambda_0 \to L^q(0,T;H)$.

Definition 5.1. *Let X and Z be topological spaces. A set-valued mapping $F : X \to Z$ is called lower-semicontinuous at $(x_0, z_0) \in Graf F$, if for all neighborhood Z_0 of z_0 there exists a neighborhood X_0 of x_0 such that $F(x) \cap Z_0 \neq \emptyset$, for all $x \in X_0$.*

Lemma 5.1. [13] *The linear mapping $L : L^p(0,T;H) \to L^q(0,T;H)$, defined by $L(u) = u'$ and $Dom\, L = \{u \in W^{1,p}(0,T;H) \ : \ u(0) = 0\}$, is maximal-monotone.*

Proposition 5.2. *Let us suppose that:*

a) there exist $u_0 \in Dom\, L$ and $v_0 \in A(u_0, \lambda_0)$ such that $u_0' + v_0 - f = 0$;

b) the set-valued mapping A is lower-semicontinuous at $((u_0, \lambda_0)\ ,\ v_0)$;

c) the set-valued mappings $A(\cdot, \lambda)$ are maximal-monotone and φ-uniformly-monotone for all $\lambda \in \Lambda_0$.

Then there exist a neighborhood Λ' of λ_0 and a unique mapping $u : \Lambda' \to L^p(0,T;H)$ such that $u(\lambda_0) = u_0$, $u(\lambda)$ is the unique solution for each $\lambda \in \Lambda'$ for (5.2) and u is continuous at λ_0.

Proof. Let us denote $X = L^p(0,T;H)$. Assumption *b)* implies that for all $\varepsilon > 0$ there exists $\eta > 0$ such that, for all $(u, \lambda) \in X \times \Lambda_0$, with $\|u - u_0\| < \eta$, $\|\lambda - \lambda_0\| < \eta$, hold $A(u, \lambda) \cap B(v_0, \varepsilon) \neq \emptyset$.

Hence for all $\varepsilon > 0$ there exists $v_{u,\lambda} \in A(u, \lambda)$ such that $\|v_{u,\lambda} - v_0\| \leq \varepsilon$.

Let us consider the sequence $(\varepsilon_n)_{n \in \mathbf{N}}$, $\varepsilon_n = \frac{1}{n}$, and a corresponding sequence $(\eta_n)_{n \in \mathbf{N}}$ converging to 0 such that $B(\lambda_0, \eta_1) \subset \Lambda_0$.

Let us choose arbitrarily $\lambda \in B(\lambda_0, \eta_1)$. Then for all $u \in B(u_0, \eta_1)$ we have $A(u, \lambda) \cap B(u_0, 1) \neq \emptyset$, so $B(u_0, \eta_1) \subset \operatorname{Dom} A(\cdot, \lambda)$. In this way we can see that $L + A(\cdot, \lambda)$ is maximal-monotone and as a sum between a monotone and another φ-uniformly-monotone mapping, is φ-uniformly-monotone.

Let η_{n_λ} be the smallest number in the sequence $(\eta_n)_{n\in\mathbf{N}}$ for which $\lambda \in B(\lambda_0, \eta_{n_\lambda})$. Then there exists $v_\lambda \in A(u_0, \lambda)$ such that $\|v_\lambda - v_0\| \leq \frac{1}{n_\lambda}$. Hence

$$\|L(u_0) + v_\lambda - f\| = \|L(u_0) + v_\lambda - L(u_0) - v_0\| = \|v_\lambda - v_0\| \leq \frac{1}{n_\lambda} .$$

We define the function $\beta : B(\lambda_0, \eta_1) \to \mathbf{R}_+$ by

$$\beta(\lambda) = \max\left\{\eta_{n_\lambda}, \frac{1}{n_\lambda}\right\} .$$

Using this function β we conclude that the mapping $L + A - f$ is consistent with respect to λ at $(u_0, \lambda_0, 0)$ in $B(\lambda_0, \eta_1)$. The conclusion of this proposition follows now from Theorem 2.1.

Example 5.3. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain, $p, q \in \mathbf{R}_+$ such that $2 \leq p < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and let $\lambda \in \mathbf{R}_+$.
We denote $X = W_0^{1,p}(\Omega)$ and

$$a(u, v, \lambda) = \int_\Omega \left(\sum_{p=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} + \lambda uv\right) dx ,$$

$$F_f(v) = \int_\Omega f(x)v(x)\, dx .$$

Let us consider the following problem:

- for $f \in L^q(\Omega)$ and $\lambda \in \mathbf{R}_+$, find $u \in X$ such that

$$a(u, v, \lambda) = F_f(v) , \quad \text{for all } v \in X . \tag{5.3}$$

Let us define the mapping $A : \mathbf{R}_+ \times X \to X^*$ by

$$A(\lambda, u)(v) = a(u, v, \lambda) = \int_\Omega \left(\sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} + \lambda uv\right) dx . \tag{5.4}$$

Proposition 5.3. [13] *For all $\lambda \in \mathbf{R}_+$ and $u \in X$, the mapping $A(\lambda, u)$ is well-defined and the mappings $A(\lambda, \cdot) : X \to X^*$ are continuous, φ-uniformly-monotone, with $\varphi(r) = c_1 r^{p-1}$.*

Proposition 5.4. *For all $\lambda \in \mathbf{R}_+$ and $f \in L^q(\Omega)$, the problem (2.3) has a unique solution $u(\lambda, f) \in W_0^{1,p}(\Omega)$ and these solutions are continuous in λ and f.*

Proof. Using the surjectivity and the φ-uniform-monotonicity of the mappings $A(\cdot,\lambda)$, we deduce that for all $f \in L^q(\Omega)$ and $\lambda \in \mathbf{R}_+$ there exists a unique element $u(\lambda, f) \in X$ such that $A(u,\lambda) = F_f$.

For all $\lambda_1, \lambda_2 \in \mathbf{R}_+$ and $u, v \in X$ hold

$$(A(u,\lambda_1) - A(u,\lambda_2)\,,\,v) \;=\; \int_\Omega (\lambda_1 - \lambda_2) uv\,dx \;\leq$$

$$\leq\; |\lambda_1 - \lambda_2|\,\|u\|_{L^2}\,\|v\|_{L^2} \;\leq\; |\lambda_1 - \lambda_2| c \|u\|\,\|v\|\,.$$

Hence

$$\|A(u,\lambda_1) - A(u,\lambda_2)\| \;\leq\; c|\lambda_1 - \lambda_2|\,\|u\|\,,$$

which means that the mappings $A(u,\cdot)$ are continuous on $\mathbf{R}_+$.

Let us fix $\lambda_0 \in \mathbf{R}_+$ and $f_0 \in L^q(\Omega)$.

Theorem 2.1 implies the existence of a neighborhood $\Lambda_0 \times U_0$ of (λ_0, f_0) and of a unique mapping $u_0 : \Lambda_0 \times U_0 \to X$ continuuous at (λ_0, f_0), such that $u_0(\lambda, f)$ is the unique solution of the problem (2.3) for all $(\lambda, f) \in \Lambda_0 \times U_0$. The uniqueness of the solutions implies that the mappings u_0 and u coincide on $\Lambda_0 \times U_0$. Hence the continuity of u at (λ_0, f_0) is proved.

(λ_0, f_0) being choosed arbitrarily, the continuty holds for all $(\lambda, f) \in \Lambda_0 \times U_0$.

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BABEŞ -BOLYAI UNIVERSITY, DEPT. OF APPLIED MATHEMATICS, 3400 CLUJ-NAPOCA, STR. M. KOGALNICEANU 1, ROMANIA

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ON THE GOURSAT PROBLEM FOR HYPERBOLIC FUNCTIONAL-DIFFERENTIAL EQUATIONS

LUBOMIR P. GEORGIEV

It is known that in many problems of nonlinear fields theory, plasma physics and etc. (cf. [1]) arise hyperbolic functional-differential equations with so-called 'distributed' deviations (cf. [2]). The main purpose of the present paper is to formulate conditions under which there exist solutions of the Goursat problem, characteristic of functional-differential equations with 'concentrated' deviations (particular case of distributed deviations), using the fixed point theorems, proved by Angelov [3].

Typical in this respect is the following simple example, where the discontinuity of the initial function generates the discontinuity of the solution:

$$\begin{cases} u_{xy}(x,y) = k(x,y)u_{xy}(x-1,y-1), & (x,y) \in \mathbb{R}^2_+ = \{(x,y):\ x>0,\ y>0\} \\ u(x,y) = \varphi(x,y), & (x,y) \in A_0 \cup B_0, \end{cases} \tag{1}$$

where

$$u_{xy} = \frac{\partial^2 u}{\partial x \partial y}; \quad A_0 = \{(x,y):\ x \geq -1,\ -1 \leq y \leq 0\},$$

$$B_0 = \{(x,y):\ -1 \leq x \leq 0,\ y \geq -1\};$$

$$\varphi(x,y) = \begin{cases} 1, & (x,y) \in \mathbb{R}^x_+ \cup \mathbb{R}^y_+, \\ & \mathbb{R}^x_+ = \{(x,y) : x \geq 0,\ y = 0\},\ \mathbb{R}^y_+ = \{(x,y):\ x = 0,\ y \geq 0\} \\ 0, & (x,y) \in A_0 \cup B_0 \setminus (\mathbb{R}^x_+ \cup \mathbb{R}^y_+) \end{cases}$$

$$k(x,y) = 1 - \frac{1}{1+n},\ (x,y) \in A_n \cup B_n\ (n = 1,2,\ldots),$$
$$A_n = \{(x,y):\ x \geq n-1,\ n-1 \leq y \leq n\},\ B_n = \{(x,y):\ n-1 \leq x \leq n,\ y \geq n-1\}.$$

Integrating the above equation we have

$$u(x,y) - k(x,y)u(x-1,y-1) = C_1(x) + C_2(y).$$

Then the conditions

$$u(0,0) - k(0,0)u(-1,-1) = C_1(0) + C_2(0),$$

$$u(x,0) - k(x,0)u(x-1,-1) = C_1(x) + C_2(0),$$

$$u(0,y) - k(0,y)u(-1,y-1) = C_1(0) + C_2(0)$$

imply $C_1(x) + C_2(y) = 1$, so that we obtain the problem

$$u(x,y) = \begin{cases} k(x,y)u(x-1,y-1) + 1, & (x,y) \in \mathbb{R}^2_+ \\ \varphi(x,y), & (x,y) \in A_0 \cup B_0 \end{cases}$$

It is quite obvious that when $(x,y) \in A_n \cup B_n$ then $(x-1,y-1) \in A_{n-1} \cup B_{n-1}$ and we can construct immediately the following solution

$$u(x,y) = \begin{cases} 0, \quad (x,y) \in (A_0 \cup B_0) \setminus (\mathbb{R}^x_+ \cup \mathbb{R}^y_+) \\ 1, \quad (x,y) \in (A_1 \cup B_1) \setminus (\mathbb{R}^{x+1}_+ \cup \mathbb{R}^{y+1}_+) \\ \left(1 - \frac{1}{3}\right) + 1, \quad (x,y) \in (A_2 \cup B_2) \setminus (\mathbb{R}^{x+2}_+ \cup \mathbb{R}^{y+2}_+) \\ \dots \\ \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) \dots \left(1 - \frac{1}{n+1}\right) + \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{5}\right) \dots \left(1 - \frac{1}{n+1}\right) + \dots + \\ \qquad + \left(1 - \frac{1}{n+1}\right) + 1 = \frac{n(n+3)}{2(n+1)}, \quad (x,y) \in (A_n \cup B_n) \setminus (\mathbb{R}^{x+n}_+ \cup \mathbb{R}^{y+n}_+) \\ \dots \end{cases}$$

where $\mathbb{R}^{x+n}_+ = \{(x,y) : x \geq n, \ y = n\}$, $\mathbb{R}^{y+n}_+ = \{(x,y) : x = n, \ y \geq n\}$, $n = 0, 1, 2, \dots$

The fixed point technique for operators in metric spaces has been very well developed (cf. [4]), but the above example shows that the hyperbolic functional-differential equations of neutral type (following the terminology introduced in [5]) possesses solutions with locally essentially bounded mixed derivative u_{xy}. (We note the known results [6]-[8], where only continuous solutions have been obtained with restrictions on the deviations of retarded type.) Moreover the example shows:

1. the Goursat problem allows L^∞_{loc}-solutions so that it cannot formulate as an operator equation in Banach or metric space.

2. the operator defined by the right-hand side (even in the linear case) will be not a global contraction because of $esssup\{k(x,y) : x \geq 0, \ y \geq 0\} = 1$.

That is why, we shall use the fixed point theorems from [3].

Let X be a Hausdorff sequentially complete uniform space with uniformity defined by a saturated family of pseudometrics $\{\rho_\alpha(x,y)\}_{\alpha\in\mathcal{A}}$, $\mathcal{A}$ being an index set.

Let $\Phi = \{\Phi_\alpha(t) : \alpha \in \mathcal{A}\}$ be a family of functions $\Phi_\alpha(t) : [0,\infty) \to [0,\infty)$ with the properties

1) $\Phi_\alpha(t)$ is monotone non-decreasing and continuous from the right on $[0,\infty)$;

2) $\Phi_\alpha(t) < t$, $\forall\, t > 0$,

and $j : \mathcal{A} \to \mathcal{A}$ is a mapping on the index set $\mathcal{A}$ into itself, where $j^0(\alpha) = \alpha$, $j^k(\alpha) = j(j^{k-1}(\alpha))$, $k \in \mathbb{N}$.

Definition. The map $T : Y \to Y$ is said to be Φ-*contraction* on Y if

$$\rho_\alpha(Tx,Ty) \le \Phi_\alpha(\rho_{j(\alpha)}(x,y))$$

for every $x, y \in Y$ and $\alpha \in \mathcal{A}$, $Y \subset X$.

Theorem 1. (theorem 2 from [3]) *Let us suppose*

1. the operator $T : X \to X$ is a Φ-contraction;

2. for each $\alpha \in \mathcal{A}$ there exists a Φ-function $\overline{\Phi}_\alpha(t)$ such that

$$\sup\{\Phi_{j^k(\alpha)}(t) : \ k = 0,1,2,\dots\} \le \overline{\Phi}_\alpha(t)$$

and $\overline{\Phi}_\alpha(t)/t$ is non-decreasing;

3. there exists an element $x_0 \in X$ such that $\rho_{j^k(\alpha)}(x_0, Tx_0) \le p(\alpha) < \infty$ $(k = 0,1,2,\dots)$.

Then T has at least one fixed point in X.

Theorem 2. (theorem 3 from [3]) *If, in addition, we suppose that*

4. the sequence $\{\rho_{j^k(\alpha)}(x,y)\}_{k=0}^{\infty}$ is bounded for each $\alpha \in \mathcal{A}$ and $x, y \in X$, i.e.

$$\rho_{j^k(\alpha)}(x,y) \le q(x,y,\alpha) < \infty \quad (k = 0,1,2,\dots).$$

Then the fixed point of T is unique.

Consider the general Goursat problem for hyperbolic functional-differential equation:

$$\begin{aligned} &u_{xy}(x,y) = F(x,y,u(\Delta,\tau),u_x(\alpha,\beta),u_y(\theta,\kappa),u_{xy}(\mu,\nu)),\ (x,y)\in\mathbb{R}^2_+ \\ &u(x,y)=\psi(x,y),\ u_x(x,y)=\psi_x(x,y),\ u_y(x,y)=\psi_y(x,y), \\ &u_{xy}(x,y)=\psi_{xy}(x,y),\ (x,y)\in\mathbb{R}^2\setminus\mathbb{R}^2_+, \end{aligned} \tag{2}$$

where $F(x,y,z_1,z_2,z_3,z_4)$, $\Delta=\Delta(x,y)$, $\tau=\tau(x,y)$, $\alpha=\alpha(x,y)$, $\beta=\beta(x,y)$, $\theta=\theta(x,y)$, $\kappa=\kappa(x,y)$, $\mu=\mu(x,y)$, $\nu=\nu(x,y)$ and $\psi(x,y)$ are given functions.

We set

$v(x,y)=u_{xy}(x,y)$, when $(x,y)\in\mathbb{R}^2_+$,

$\varphi(x,y)=\psi_{xy}(x,y)$, when $(x,y)\in\mathbb{R}^2\setminus\mathbb{R}^2_+$

and after standard calculations, we obtain

$$u(x,y)=\varphi_0(x,y)+\int_0^x\int_0^y v(\xi,\eta)d\eta d\xi,$$

$$u_x(x,y)=\varphi_1(x)+\int_0^y v(x,\eta)d\eta,$$

$$u_y(x,y)=\varphi_2(y)+\int_0^x v(\xi,y)d\xi,$$

where

$$\varphi_0(x,y)=\psi(0,y)+\psi(x,0)-\psi(0,0),$$

$$\varphi_1(x)=\psi_x(x,0),\quad \varphi_2(y)=\psi_y(0,y),$$

so that the problem (2) corresponds the following problem

$$v(x,y)=\begin{cases} F(x,y,\overline{\varphi}_0+\int_0^\Delta\int_0^\tau v(\xi,\eta)d\eta d\xi,\overline{\varphi}_1+\int_0^\beta v(\alpha,\eta)d\eta,\overline{\varphi}_2+ \\ \quad +\int_0^\theta v(\xi,\kappa)d\xi, v(\mu,\nu)),(x,y)\in\mathbb{R}^2_+ \\ \varphi(x,y),(x,y)\in\mathbb{R}^2\setminus\mathbb{R}^2_+, \end{cases} \tag{3}$$

where $\overline{\varphi}_0=\varphi_0(\Delta(x,y),\tau(x,y))$, $\overline{\varphi}_1=\varphi_1(\alpha(x,y))$, $\overline{\varphi}_2=\varphi_2(\kappa(x,y))$.

Definition. The function $u(x,y)$ is said to be a solution (in generalized sense) of problem (2) if the function $v(x,y)$ is a solution of problem (3).

In what follows, we look for a solution of (3), belonging to L^∞_{loc}.

We say that the function $G:\mathbb{R}^2\to\mathbb{R}^2$ has the property (M) if inverse image of every set with null measure is measurable.

Let us suppose:

(A1) ψ is absolutely continuous;

$\psi(x,0), \psi(0,y), \psi_x(x,0), \psi_y(0,y)$ are continuous and $\varphi=\psi_{xy}\in L^{\infty}_{loc}(\mathbb{R}^2\backslash\mathbb{R}^2_+)$.

(A2) The functions $\Delta,\tau,\alpha,\beta,\theta,\kappa,\mu,\nu:\mathbb{R}^2_+\to\mathbb{R}$ are measurable, have the property (M) (without Δ and τ) and map bounded sets into bounded sets.

(A3) $\forall\ (x,y)\in\mathbb{R}^2_+$ for which $(\Delta(x,y),\tau(x,y))\in\mathbb{R}^2_+$ (or $(\alpha(x,y),\beta(x,y))\in\mathbb{R}^2_+$, or $(\theta(x,y),\kappa(x,y))\in\mathbb{R}^2_+$ is fulfilled $\Delta(x,y)+\tau(x,y)\le x+y$ (respectively $\alpha(x,y)+\beta(x,y)\le x+y$, or $\theta(x,y)+\kappa(x,y)\le x+y$);

$\exists\delta_0>0$ such that $\forall\ (x,y)\in\mathbb{R}^2_+:\ (\mu(x,y),\nu(x,y))\in\mathbb{R}^2_+$ is fulfilled $\mu(x,y)+\nu(x,y)\le x+y-\delta_0$.

(A4) The function $F(x,y,z_1,z_2,z_3,z_4)\ :\ \mathbb{R}^2_+\times\mathbb{R}^4\to\mathbb{R}$ satisfies the Caratheodory condition (measurable in x and y and continuous in $z_1,\dots,z_4$) and the conditions:

$$|F(x,y,z_1,z_2,z_3,z_4)|\le\Omega_1(x,y,|z_1|,|z_2|,|z_3|,|z_4|)$$

$$|F(x,y,z_1,z_2,z_3,z_4)-F(x,y,\bar z_1,\bar z_2,\bar z_3,\bar z_4)|\le$$

$$\le\Omega_2(x,y,|z_1-\bar z_1|,|z_2-\bar z_2|,|z_3-\bar z_3|,|z_4-\bar z_4|),$$

where the functions $\Omega_{1,2}(x,y,t_1,\dots,t_4):\mathbb{R}^2_+\times\overline{\mathbb{R}}^4_+\to[0,\infty)$ ($\overline{\mathbb{R}}^n_+=[0,\infty)\times\cdots\times[0,\infty)$ - n times) satisfy the Caratheodory condition, $\Omega_1(\cdot,\cdot,t_1,\dots,t_4)\in L^{\infty}_{loc}(\mathbb{R}^2_+)$, $\Omega_2(x,y,t_1,\dots,t_4)$ is non-decreasing in $t_1,\dots,t_4$ and

$\exists\ \omega\in L^{\infty}(\mathbb{R}^2_+)$ such that $\forall\ t\ge0\ \Omega_2(\cdot,\cdot,t,t,t,t)\le t\omega(\cdot,\cdot)$ a.e. in $\mathbb{R}^2_+$.

Let $\mathcal{A}$ be the set of all compact sets $K\subset\mathbb{R}^2$. Denote by $K_+=K\cap\mathbb{R}^2_+$, we define the map $j:\mathcal{A}\to\mathcal{A}$:

$$j(K)=\begin{cases}K, & K_+=\emptyset\\ K_{\Delta\tau}\cup K_{\alpha\beta}\cup K_{\theta\kappa}\cup K_{\mu\nu}, & K_+\neq\emptyset\end{cases}$$

where $K_{\Delta\tau}=K_\Delta\times K_\tau$, $K_{\alpha\beta}=K_\alpha\times K_\beta$, $K_{\theta\kappa}=K_\theta\times K_\kappa$,

$$K_{\mu\nu}=\overline{\{(\mu(x,y),\nu(x,y)):\ (x,y)\in K\}},\quad (\overline{A}\overset{def}{=}clA),$$

$$K_\Delta = \begin{cases} [\Delta_{inf}, \Delta_{sup}], & \Delta_{inf} < 0 < \Delta_{sup} \\ [0, \Delta_{sup}], & \Delta_{inf} \ge 0 \\ [\Delta_{inf}, 0], & \Delta_{sup} \le 0 \end{cases}$$

$$K_\tau = \begin{cases} [\tau_{inf}, \tau_{sup}], & \tau_{inf} < 0 < \tau_{sup} \\ [0, \tau_{sup}], & \tau_{inf} \ge 0 \\ [\tau_{inf}, 0], & \tau_{sup} \le 0 \end{cases}$$

$$K_\beta = \begin{cases} [\beta_{inf}, \beta_{sup}], & \beta_{inf} < 0 < \beta_{sup} \\ [0, \beta_{sup}], & \beta_{inf} \ge 0 \\ [\beta_{inf}, 0], & \beta_{sup} \le 0 \end{cases}$$

$$K_\theta = \begin{cases} [\theta_{inf}, \theta_{sup}], & \theta_{inf} < 0 < \theta_{sup} \\ [0, \theta_{sup}], & \theta_{inf} \ge 0 \\ [\theta_{inf}, 0], & \theta_{sup} \le 0 \end{cases}$$

$$K_\alpha = \overline{\alpha(K)}, \quad K_\kappa = \overline{\kappa(K)}$$

$$(\Delta_{inf} = \inf\{\Delta(x,y) : (x,y) \in K_+\}, \ldots, \theta_{sup} = \sup\{\theta(x,y) : (x,y) \in K_+\}).$$

It is obvious that $j(K)$ is compact set and $j^l(K)$ can be defined inductively: $j^l(K) = j(j^{l-1}(K))$ for all $l \in \mathbb{N}$.

Now we assume:

(A5) $\forall\, K \in \mathcal{A}\ \exists\ \widehat{K} \in \mathcal{A} :\ j^l(K) \subset \widehat{K}\ \forall\, l = 0, 1, 2, \ldots$

We prove the following existence-uniqueness result:

Theorem 3. *If conditions (A1)-(A5) hold true, then there exists a unique solution $v(x,y) \in L^\infty_{loc}(\mathbb{R}^2)$ of problem (3).*

Proof. Let X be the uniform sequentially complete Hausdorff space consisting of all functions, belonging to $L^\infty_{loc}(\mathbb{R}^2)$, which equal $\varphi(x,y)$ for a.e. $(x,y) \in \mathbb{R}^2 \backslash \mathbb{R}^2_+$, with a saturated family $P = \{\rho_K :\ K \in \mathcal{A}\}$ of pseudometrics

$$\rho_K(f,g) = esssup\{e^{-\lambda(|x|+|y|)}|f(x,y) - g(x,y)| :\ (x,y) \in K\},$$

where K runs over all compact subsets of $\mathbb{R}^2$ (with some $\lambda > 0$).

The operator $T : X \to X$ is defined by the formula:

$$T(f)(x,y) = \begin{cases} F(x,y,\overline{\varphi}_0 + \int_0^{\Delta}\int_0^{\tau} f(\xi,\eta)d\eta d\xi, \overline{\varphi}_1 + \int_0^{\beta} f(\alpha,\eta)d\eta, \overline{\varphi}_2 + \\ \quad + \int_0^{\theta} f(\xi,\kappa)d\xi, f(\mu,\nu)), (x,y) \in \mathbb{R}^2_+ \\ \varphi(x,y), (x,y) \in \mathbb{R}^2 \setminus \mathbb{R}^2_+, \end{cases}$$

The measurability of $T(f)(x,y)$ follows from the fact that $\alpha, \beta, \theta, \kappa, \mu, \nu$ have the property (M).

$T(f) \in L^{\infty}_{loc}(\mathbb{R}^2)$ because of conditions A1, A4. Consequently $T(f) \in X$.

Let $K \subset \mathbb{R}^2$ be any fixed compact set. Of $K_+ = \emptyset$ then $T(f) - T(g) = 0$ for all $f, g \in X$ a.e. in K. Let $K_+ \neq \emptyset$. For a.e. $(x,y) \in K \cap (\mathbb{R}^2 \setminus \mathbb{R}^2_+)$ we have $T(f) - T(g) = 0$.

For a.e. $(x,y) \in K_+$ we obtain (by means of (A4)):

$$|T(f)(x,y) - T(g)(x,y)| \leq$$

$$\leq \Omega_2(x,y, |\int_0^{\Delta}\int_0^{\tau}(f(\xi,\eta) - g(\xi,\eta))d\eta d\xi|, |\int_0^{\beta}(f(\alpha,\eta) - g(\alpha,\eta))d\eta|,$$
$$|\int_0^{\theta}(f(\xi,\kappa) - g(\xi,\kappa))d\xi|, |f(\mu,\nu) - g(\mu,\nu)|)$$

If $(\Delta(x,y), \tau(x,y)) \notin \mathbb{R}^2_+$ then

$$\int_0^{\Delta}\int_0^{\tau}(f(\xi,\eta) - g(\xi,\eta))d\eta d\xi = 0$$

and respectively if $(\alpha(x,y), \beta(x,y)) \notin \mathbb{R}^2_+$ then

$$\int_0^{\beta}(f(\alpha,\eta) - g(\alpha,\eta))d\eta = 0,$$

if $(\theta(x,y), \kappa(x,y)) \notin \mathbb{R}^2_+$ then

$$\int_0^{\theta}(f(\xi,\kappa) - g(\xi,\kappa))d\xi = 0,$$

if $(\mu(x,y), \nu(x,y)) \notin \mathbb{R}^2_+$ then $f(\mu,\nu) - g(\mu,\nu) = 0$.

For positive values of $\Delta(x,y), \tau(x,y)$; $\alpha(x,y), \beta(x,y)$; $\theta(x,y), \kappa(x,y)$; $\mu(x,y)$, $\nu(x,y)$ we obtain as follows:

$$|\int_0^{\Delta}\int_0^{\tau}(f(\xi,\eta) - g(\xi,\eta))d\eta d\xi| \leq \int_0^{\Delta}\int_0^{\tau}|f(\xi,\eta) - g(\xi,\eta)|d\eta d\xi \leq$$

$$\leq esssup\{e^{-\lambda(\xi+\eta)}|f(\xi,\eta)-g(\xi,\eta)| : 0\leq\xi\leq\Delta_{sup}, 0\leq\eta\leq\tau_{sup}\}\int_0^\Delta\int_0^\tau e^{\lambda(\xi+\eta)}d\eta d\xi =$$

$$=\lambda^{-2}\rho_{K_{\Delta\tau}}(f,g)(e^{\lambda\Delta}-1)(e^{\lambda\tau}-1)\leq\lambda^{-2}e^{\lambda(\Delta+\tau)}\rho_{K_{\Delta\tau}}\leq\lambda^{-2}e^{\lambda(x+y)}\rho_{K_{\Delta\tau}}(f,g)(\text{cf.(A3)}).$$

$$|\int_0^\beta (f(\alpha,\eta)-g(\alpha,\eta))d\eta| \leq \int_0^\beta |f(\alpha,\eta)-g(\alpha,\eta)|d\eta \leq$$

$$\leq esssup\{e^{-\lambda(\alpha+\eta)}|f(\alpha,\eta)-g(\alpha,\eta)| :\ 0\leq\eta\leq\beta_{sup}\}e^{\lambda\alpha}\int_0^\beta e^{\lambda\eta}d\eta \leq$$

$$\leq\lambda^{-1}\rho_{K_{\alpha\beta}}(f,g)e^{\lambda\alpha}(e^{\lambda\beta}-1)\leq\lambda^{-1}e^{\lambda(\alpha+\beta)}\rho_{K_{\alpha\beta}}(f,g)\leq\lambda^{-1}e^{\lambda(x+y)}\rho_{K_{\alpha\beta}}(f,g)\ (\text{cf.(A3)}).$$

In the same way we prove (by means of (A3)) that

$$|\int_0^\theta (f(\xi,\kappa)-g(\xi,\kappa))d\xi| \leq \lambda^{-1}e^{\lambda(x+y)}\rho_{K_{\theta\kappa}}(f,g).$$

$$|f(\mu,\nu)-g(\mu,\nu)| \leq e^{\lambda(\mu+\nu)}esssup\{e^{-\lambda(p+q)}|f(p,q)-g(p,q)| : (p,q)\leq K_{\mu\nu}\}\leq$$

$$\leq e^{\lambda(x+y-\delta_0)}\rho_{K_{\mu\nu}}(f,g)\leq\lambda^{-\delta_0}e^{\lambda(x+y)}\rho_{K_{\mu\nu}}(f,g)\ (\text{cf.(A3)}).$$

Let $\lambda > 1$. Chosing γ so that

$$\lambda^{-\gamma} = \max\{\lambda^{-1},\lambda^{-\delta_0}\} = \begin{cases}\lambda^{-1}, & \delta_0\geq 1\\ \lambda^{-\delta_0}, & 0<\delta_0\leq 1\end{cases}$$

we obtain (since $\Omega_2(x,y,t_1,\ldots,t_4)$ is non-decreasing in $t_1,\ldots,t_4$)

$$|T(f)(x,y)-T(g)(x,y)|\leq$$

$$\leq\Omega_2(x,y,\lambda^{-\gamma}e^{\lambda(x+y)}\rho_{j(K)}(f,g),\lambda^{-\gamma}e^{\lambda(x+y)}\rho_{j(K)}(f,g),$$

$$\lambda^{-\gamma}e^{\lambda(x+y)}\rho_{j(K)}(f,g),\lambda^{-\gamma}e^{\lambda(x+y)}\rho_{j(K)}(f,g))\leq$$

$$\leq\lambda^{-\gamma}e^{\lambda(x+y)}\rho_{j(K)}(f,g)\omega(x,y)\quad(\text{cf.(A4)}).$$

Define (for $t\geq 0$)

$$\Phi_K(t) = \begin{cases}0, & \text{if}\quad K_+ = \emptyset\\ t\lambda^{-\gamma}||\omega||_{L^\infty(K_+)}, & \text{if}\quad K_+\neq\emptyset\end{cases}$$

We can find and fix λ so that $\lambda^\gamma > ||\omega||_{L^\infty(\mathbb{R}^2_+)}$. Consequently $\Phi_K(t) < t$ $\forall\ t>0$, $\forall\ K\in\mathcal{A}$ and $\Phi_K(t)$ is continuous non-decreasing in $[0,\infty)$. On the other hand

$$\rho_K(T(f),T(g)) = \rho_{\overline{K}_+}(T(f),T(g))\leq\Phi_k(\rho_{j(K)}(f,g)),$$

i.e. $T : X\to X$ is a Φ-contraction.

$\forall\ K \in \mathcal{A}$ we set $\overline{\Phi}_K = \Phi_{\widehat{K}}$ (recall that (A5) assures an existence of such a $\widehat{K}$ that $j^l(K) \subset \widehat{K}$ $(l = 0, 1, 2, \ldots)$) and so $\sup\{\Phi_{j^l(K)}(t) :\ l = 0, 1, 2, \ldots\} \leq \overline{\Phi}_K(t)$, $\dfrac{\overline{\Phi}_K(t)}{t} = const\ \Rightarrow$ non-decreasing.

Hence condition 1, 2 of theorem 1 is fulfilled.

We choose the element $f_0 \in X$:

$$f_0(x, y) = \begin{cases} 0, & \text{a.e. on } \mathbb{R}^2_+ \\ \varphi(x, y), & (x, y) \in \mathbb{R}^2 \setminus \mathbb{R}^2_+. \end{cases}$$

Then for any integer $l \geq 0$ we have

$$\rho_{j^l(K)}(f_0, T(f_0)) \leq \rho_{\widehat{K}}(f_0, T(f_0)) = \rho_{\overline{\widehat{K}_+}}(f_0, T(f_0)) =$$

$$= esssup\{e^{-\lambda(x+y)}|F(x, y, \overline{\varphi}_0, \overline{\varphi}_1, \overline{\varphi}_2, 0)| : (x, y) \in \overline{\widehat{K}_+}\} < \infty$$

(i.e. condition 3 of theorem 1 is fulfilled).

Besides $\rho_{j^l(K)}(f, g) \leq \rho_{\overline{\widehat{K}_+}}(f, g)$ for abitrary $f, g \in X$, i.e. condiyion 4 of theorem 2 is also fulfilled.

All conditions of theorems 1 and 2 are satisfied. Therefore the problem (3) has a unique solution $v \in L^\infty_{loc}(\mathbb{R}^2)$.

We are going to formulate conditions for the existence and uniqueness of a solution of (3) belonging to $L^p_{loc}(\mathbb{R}^2)$ for some $p \in (1, \infty)$:

(A1') The initial function ψ is absolutely continuous;

$\psi(x, 0), \psi(0, y), \psi_x(x, 0), \psi_y(0, y)$ are continuous and $\varphi = \psi_{xy} \in L^p_{loc}(\mathbb{R}^2 \setminus \mathbb{R}^2_+)$.

(A4') The function $F(x, y, z_1, z_2, z_3, z_4)\ :\ \mathbb{R}^2_+ \times \mathbb{R}^4 \to \mathbb{R}$ satisfies the Caratheodory condition (measurable in x and y and continuous in $z_1, \ldots, z_4$) and the conditions:

$$|F(x, y, z_1, z_2, z_3, z_4)| \leq a(x, y) + b(|z_1| + |z_2| + |z_3| + |z_4|)$$

$$|F(x, y, z_1, z_2, z_3, z_4) - F(x, y, \overline{z}_1, \overline{z}_2, \overline{z}_3, \overline{z}_4)| \leq$$

$$\leq \omega_1(x, y)|z_1 - \overline{z}_1| + \omega_2|z_2 - \overline{z}_2| + \omega_3|z_3 - \overline{z}_3| + \omega_4|z_4 - \overline{z}_4|,$$

where $a(\cdot, \cdot) \in L^p_{loc}(\mathbb{R}^2_+)$, $b = const \geq 0$, $\omega_1(\cdot, \cdot) \in L^p(\mathbb{R}^2_+)$, $\omega_{2,3}(\cdot) \in L^p(\mathbb{R}^1_+)$, $\omega_4 = const \geq 0$

(A6) The transformations

$$\left| \begin{array}{l} u = \alpha(x,y) \\ v = y \end{array} \right. \qquad \left| \begin{array}{l} u = x \\ v = \kappa(x,y) \end{array} \right. \qquad \left| \begin{array}{l} u = \mu(x,y) \\ v = \nu(x,y) \end{array} \right.$$

are admissible, sufficiently smooth and $\alpha^*_u, \kappa^*_v, \dfrac{D(\mu^*,\nu^*)}{D(u,v)} \in L^\infty(\mathbb{R}^2_+)$, where

$$(\alpha^*(\alpha(x,y),y),y) = (x,y), \quad (x,\kappa^*(x,\kappa(x,y))) = (x,y),$$

$$(\mu^*(\mu(x,y),\nu(x,y)),\nu^*(\mu(x,y),\nu(x,y))) = (x,y).$$

Theorem 4. *If conditions (A1'), (A2), (A3), (A4'), (A5), (A6) hold true, then there exists a unique solution $v(x,y) \in L^p_{loc}(\mathbb{R}^2)$ of problem (3).*

Proof. Let X be the space consisting of all functions, belonging to $L^p_{loc}(\mathbb{R}^2)$, which equal $\varphi(x,y)$ a.e. $(x,y) \in \mathbb{R}^2 \setminus \mathbb{R}^2_+$, with a saturated family P of pseudometrics

$$\rho_K(f,g) = \left(\int_K \int e^{-\lambda(|x|+|y|)} |f(x,y) - g(x,y)|^p dxdy \right)^{\frac{1}{p}} \qquad (K \in \mathcal{A}),$$

where $\mathcal{A}$ is the family of all compact sets in $\mathbb{R}^2$, $\lambda > 0$.

The map $j : \mathcal{A} \to \mathcal{A}$ and the operator $T : X \to X$ are defined as in the proof of theorem 3.

For any $K \in \mathcal{A}$, $f, g \in X$ we have $T(f)(x,y) - T(g)(x,y) = 0$, for a.e. $(x,y) \in K \setminus K_+$;

If $(x,y) \in K_+ \neq \emptyset$, then

$$|T(f)(x,y) - T(g)(x,y)|^p \leq$$

$$\leq \Bigg(\omega_1(x,y) \left| \int_0^\Delta \int_0^\tau (f(\xi,\eta) - g(\xi,\eta)) d\eta d\xi \right| + \omega_2(y) \left| \int_0^\beta (f(\alpha,\eta) - g(\alpha,\eta)) d\eta \right| +$$

$$+ \omega_3(x) \left| \int_0^\theta (f(\xi,\kappa) - g(\xi,\kappa)) d\xi \right| + \omega_4 (f(\mu,\nu) - g(\mu,\nu)) \Bigg|\Bigg)^p \leq$$

$$\leq 4^{p-1} \Bigg(\omega_1^p(x,y) \left| \int_0^\Delta \int_0^\tau (f(\xi,\eta) - g(\xi,\eta)) d\eta d\xi \right|^p + \omega_2^p(y) \left| \int_0^\beta (f(\alpha,\eta) - g(\alpha,\eta)) d\eta \right|^p +$$

$$+ \omega_3^p(x) \left| \int_0^\theta (f(\xi,\kappa) - g(\xi,\kappa)) d\xi \right|^p + \omega_4^p |f(\mu,\nu) - g(\mu,\nu)|^p \Bigg).$$

If $(\Delta,\tau), (\alpha,\beta), (\theta,\kappa), (\mu,\nu) \notin \mathbb{R}^2_+$, then $T(f)(x,y) - T(g)(x,y) = 0$.

If $(\Delta,\tau)\in\mathbb{R}^2_+$, then (with $\frac{1}{p}+\frac{1}{q}=1$)

$$\left|\int_0^\Delta\int_0^\tau (f(\xi,\eta)-g(\xi,\eta))d\eta d\xi\right|^p\leq$$

$$\leq\left(\int_0^\Delta\int_0^\tau |f(\xi,\eta)-g(\xi,\eta)|d\eta d\xi\right)^p=$$

$$=\left(\int_0^\Delta\int_0^\tau e^{\frac{\lambda}{p}(\xi+\eta-\xi-\eta)}|f(\xi,\eta)-g(\xi,\eta)|d\eta d\xi\right)^p\leq$$

$$\leq\left(\int_0^\Delta\int_0^\tau e^{\frac{q\lambda}{p}(\xi+\eta)}d\eta d\xi\right)^{\frac{p}{q}}\int_0^\Delta\int_0^\tau e^{-\lambda(\xi+\eta)}|f(\xi,\eta)-g(\xi,\eta)|^p d\eta d\xi\leq$$

$$\leq\left(\frac{p-1}{\lambda}\right)^{2(p-1)}e^{\lambda(\Delta+\tau)}\rho^p_{K_{\Delta\tau}}(f,g)\leq$$

$$\leq\left(\frac{p-1}{\lambda}\right)^{2(p-1)}e^{\lambda(x+y)}\rho^p_{K_{\Delta\tau}}(f,g)\quad\text{(cf.(A3))}.$$

If $(\alpha,\beta)\in\mathbb{R}^2_+$, then

$$\left|\int_0^\beta (f(\alpha,\eta)-g(\alpha,\eta))d\eta\right|^p\leq$$

$$\leq\left(\int_0^\beta |f(\alpha,\eta)-g(\alpha,\eta)|d\eta\right)^p=\left(\int_0^\beta e^{\frac{\lambda}{p}(\alpha+\eta-\alpha-\eta)}|f(\alpha,\eta)-g(\alpha,\eta)|d\eta\right)^p\leq$$

$$\leq\left(\int_0^\beta e^{\frac{q\lambda}{p}(\alpha+\eta)}d\eta\right)^{\frac{p}{q}}\int_0^\beta e^{-\lambda(\alpha+\eta)}|f(\alpha,\eta)-g(\alpha,\eta)|^p d\eta\leq$$

$$\leq\left(\frac{p-1}{\lambda}\right)^{p-1}e^{\lambda(\alpha+\beta)}\int_0^\beta e^{-\lambda(\alpha+\eta)}|f(\alpha,\eta)-g(\alpha,\eta)|^p d\eta\leq$$

$$\leq\left(\frac{p-1}{\lambda}\right)^{p-1}e^{\lambda(x+y)}\int_0^\beta e^{-\lambda(\alpha+\eta)}|f(\alpha,\eta)-g(\alpha,\eta)|^p d\eta\quad\text{(cf.(A3))}.$$

In the same way (by means of (A3)) we obtain: if $(\theta,\kappa)\in\mathbb{R}^2_+$, then

$$\left|\int_0^\theta (f(\xi,\kappa)-g(\xi,\kappa))d\xi\right|^p\leq\left(\frac{p-1}{\lambda}\right)^{p-1}e^{\lambda(x+y)}\int_0^\theta e^{-\lambda(\xi+\kappa)}|f(\xi,\kappa)-g(\xi,\kappa)|^p d\xi.$$

Hence

$$\int_K\int e^{-\lambda(|x|+|y|)}|T(f)(x,y)-T(g)(x,y)|^p dxdy\leq$$

$$\leq 4^{p-1}\left(\left(\frac{p-1}{\lambda}\right)^{2(p-1)}\rho^p_{K_{\Delta r}}(f,g)\int_{K_+}\int\omega_1^p(x,y)dxdy+\right.$$

$$+\left(\frac{p-1}{\lambda}\right)^{p-1}\int_{K_+}\int\omega_2^p(y)\int_0^\beta e^{-\lambda(\alpha+\eta)}|f(\alpha,\eta)-g(\alpha,\eta)|^p d\eta dxdy+$$

$$+\left(\frac{p-1}{\lambda}\right)^{p-1}\int_{K_+}\int\omega_3^p(x)\int_0^\theta e^{-\lambda(\xi+\kappa)}|f(\xi,\kappa)-g(\xi,\kappa)|^p d\xi dxdy+$$

$$\left.+\omega_4^p e^{-\lambda\delta_0}\int_{K_+}\int e^{-\lambda(\mu+\nu)}|f(\mu,\nu)-g(\mu,\nu)|^p dxdy\right).$$

Denote $K_y=\{y:(x,y)\in K_+\}$, $K_x=\{x:(x,y)\in K_+\}$. Consequently

$$\int_{K_+}\int\omega_2^p(y)\int_0^\beta e^{-\lambda(\alpha+\eta)}|f(\alpha,\eta)-g(\alpha,\eta)|^p d\eta dxdy\leq$$

$$\leq\int_{K_y}\omega_2^p(v)\int_{K_\alpha}\int_{K_\beta}|\alpha_u^*(u,v)|e^{-\lambda(u+\eta)}|f(u,\eta)-g(u,\eta)|^p d\eta dudv\leq$$

$$\leq||\alpha_u^*||_{L^\infty(\mathbb{R}^2_+)}\rho^p_{K_{\alpha\beta}}(f,g)\int_{K_y}\omega_2^p(v)dv$$

and similarly

$$\int_{K_+}\int\omega_3^p(x)\int_0^\theta e^{-\lambda(\xi+\kappa)}|f(\xi,\kappa)-g(\xi,\kappa)|^p d\xi dxdy\leq$$

$$\leq||\kappa_v^*||_{L^\infty(\mathbb{R}^2_+)}\rho^p_{K_{\theta\kappa}}(f,g)\int_{K_x}\omega_3^p(u)du.$$

$$\int_{K_+}\int e^{-\lambda(\mu+\nu)}|f(\mu,\nu)-g(\mu,\nu)|^p dxdy=$$

$$=\int_{K_{\mu\nu}}\int\left|\frac{D(\mu^*,\nu^*)}{D(u,v)}\right|e^{-\lambda(u+v)}|f(u,v)-g(u,v)|^p dudv\leq$$

$$\leq\left\|\frac{D(\mu^*,\nu^*)}{D(u,v)}\right\|_{L^\infty(\mathbb{R}^2_+)}\rho^p_{K_{\mu\nu}}(f,g).$$

Thus we receive the estimate

$$\rho^p_K(T(f),T(g))\leq 4^{p-1}\rho^p_{j(K)}(f,g)\left(\left(\frac{p-1}{\lambda}\right)^{2p-2}||\omega_1||^p_{L^p(K_{\Delta r})}+\right.$$

$$\left.+\left(\frac{p-1}{\lambda}\right)^{p-1}C_\alpha||\omega_2||^p_{L^p(K_y)}+\left(\frac{p-1}{\lambda}\right)^{p-1}C_\kappa||\omega_3||^p_{L^p(K_x)}+\lambda^{-\delta_0}C_{\mu\nu}\omega_4^p\right),$$

where $C_\alpha=||\alpha_u^*||_{L^\infty(\mathbb{R}^2_+)}$, $C_\kappa=||\kappa_v^*||_{L^\infty(\mathbb{R}^2_+)}$, $C_{\mu\nu}=\left\|\frac{D(\mu^*,\nu^*)}{D(u,v)}\right\|_{L^\infty(\mathbb{R}^2_+)}$.

Define

$$\Phi_K(t) =$$

$$\begin{cases} 0, & K_+ = \emptyset \\ t\sqrt[p]{\left(\frac{2p-2}{\lambda}\right)^{2p-2}\|\omega_1\|^p_{L^p(K_{\Delta\tau})} + \left(\frac{4p-4}{\lambda}\right)^{p-1} C_\alpha \|\omega_2\|^p_{L^p(K_y)} + \left(\frac{4p-4}{\lambda}\right)^{p-1} C_\kappa \|\omega_3\|^p_{L^p(K_x)} + \frac{(4\omega_4)^p C_{\mu\nu}}{4\lambda^{\delta_0}}}, & K_+ \neq \emptyset \end{cases}$$

Then $\rho_K(T(f),T(g)) \leq \Phi_K(\rho_{j(K)}(f,g))\ \forall\ K \in \mathcal{A},\ \forall\ f,g \in X$.

We can find and fix $\lambda > 1$ so that

$$\left(\frac{2p-2}{\lambda}\right)^{2p-2} \|\omega_1\|^p_{L^p(\mathbb{R}^2_+)} + \left(\frac{4p-4}{\lambda}\right)^{p-1} C_\alpha \|\omega_2\|^p_{L^p(\mathbb{R}^1_+)} +$$
$$+ \left(\frac{4p-4}{\lambda}\right)^{p-1} C_\kappa \|\omega_3\|^p_{L^p(\mathbb{R}^1_+)} + \frac{(4\omega_4)^p C_{\mu\nu}}{4\lambda^{\delta_0}} < 1$$

for example

$$\lambda > \max\{2^q(p-1)\|\omega_1\|^{q/2}_{L^p(\mathbb{R}^2_+)}, 4^q C_\alpha^{q/p}(p-1)\|\omega_2\|^q_{L^p(\mathbb{R}^1_+)},$$
$$4^q C_\kappa^{q/p}(p-1)\|\omega_3\|^q_{L^p(\mathbb{R}^1_+)}, C_{\mu\nu}^{1/\delta_0}(4\omega_4)^{p/\delta_0}\}.$$

Consequently $\Phi_K(t) < t$, $\Phi_K(t)/t = const$ and T is a Φ-contraction.

K_+ is bounded set $\Rightarrow$ $\Delta(K_+), \tau(K_+), \alpha(K_+), \kappa(K_+)$ are bounded sets too, so (A1') implies $\exists\ C_K = const \geq 0$:

$$|F(x,y,\varphi_0(\Delta,\tau),\varphi_1(\alpha),\varphi_2(\kappa),0)| \leq a(x,y) + bC_K \in L^p_{loc}(\mathbb{R}^2_+)\ (\text{cf.}(\text{A1}')).$$

We choose the element $f_0 \in X$:

$$f_0(x,y) = \begin{cases} 0, & \text{a.e. on } \mathbb{R}^2_+ \\ \varphi(x,y), & (x,y) \in \mathbb{R}^2 \setminus \mathbb{R}^2_+. \end{cases}$$

Then

$$T(f_0) = \begin{cases} F(x,y,\varphi_0(\Delta,\tau),\varphi_1(\alpha),\varphi_2(\kappa),0), & \text{a.e. on } \mathbb{R}^2_+ \\ \varphi(x,y), & (x,y) \in \mathbb{R}^2 \setminus \mathbb{R}^2_+. \end{cases} \Rightarrow T(f_0) \in X$$

and consequently

$$\|T(f)\|_{L^p(K)} \leq \|T(f) - T(f_0)\|_{L^p(K)} + \|T(f_0)\|_{L^p(K)} \leq$$
$$\leq (\max_{(x,y)\in K} e^{\lambda(|x|+|y|)})^{\frac{1}{p}} \rho_K(T(f),T(f_0)) + \|T(f_0)\|_{L^p(K)} \leq$$
$$\leq c(K,\lambda,p)\rho_{j(K)}(f,f_0) + \|T(f_0)\|_{L^p(K)},\ \forall\ f \in X.$$

But $\rho_{j(K)}(f, f_0) \le \|f\|_{L^p(j(K)\cap\mathbb{R}^2_+)} \Rightarrow T(f) \in X$.

Besides the estimates $\rho_{j^l(K)}(f_0, T(f_0)) \le \rho_{\widehat{K}}(f_0, T(f_0))$, $\rho_{j^l(K)}(f,g) \le \rho_{\widehat{K}}(f,g)$ for any integer $l \ge 0$, $\forall\ f,g \in X$ (cf. (A5)) show that conditions 3 of theorem 1 and 4 of theorem 2 are fulfilled. Using once again (A5), we check that condition 2 of theorem 1 is also fulfilled, which completes the proof of theorem 4.

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UNIVERSITY OF MINING AND GEOLOGY "ST.I. RILSKI" DEPARTMENT OF MATHEMATICS, 1100 SOFIA, BULGARIA

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GLOBAL EXISTENCE AND ESTIMATES FOR SOLUTIONS OF CERTAIN HIGHER ORDER DIFFERENTIAL EQUATIONS

B.G. PACHPATTE

Abstract. In this paper results on the global existence and estimates for solutions of general higher order differential equations are established. The main tools employed in our analysis are based on the applications of the Leray-Schauder alternative and certain integral inequalities which provide explicit bounds on the unknown functions.

1. Introduction

Let $r_i(t) > 0$, $i = 1, 2, \ldots, n-1$ and $x(t)$ be sufficiently smooth functions on $I = [t_0, T]$, $t_0 \geq 0$, $T > 0$. Then for $x(t)$ the r-derivatives are defined as follows

$$D_r^{(0)} x = x,$$

$$D_r^{(k)} x = r_k (D_r^{(k-1)} x)', \quad k = 1, 2, \ldots, n-1, \quad \left(' = \frac{d}{dt} = D \right),$$

$$D_r^{(n)} x = (D_r^{(n-1)} x)'.$$

In this paper we consider the n-th order ($n > 1$) differential equation of the form

$$(P) \qquad D_r^{(n)} y = f(t, D_r^{(0)} y, D_r^{(1)} y, \ldots, D_r^{(n-1)} y),$$

with the given initial conditions

$$(P_0) \qquad D_r^{(m)} y(t_0) = c_m, \quad m = 0, 1, 2, \ldots, n-1,$$

where $f : I \times R^n \to R$ is a continuous function, R denotes the set of real numbers and c_m are given real constants. We define $B = C^{n-1}(I) = C^{n-1}(I, R)$ to be the Banach space of functions u such that $D_r^{(n-1)}u$ is continuous on I endowed with norm

$$\|u\| = \max\{|D_r^{(0)}u|_0, |D_r^{(1)}u|_0, \ldots, |D_r^{(n-1)}u|_0\},$$

where $|u|_0 = \max\{|u(t)| : t \in I\}$. In the past two decades there has been a great deal of interest in the study of oscillatory and asymptotic behavior of the solutions of equations of the form (P) and its various special versions. We choose to refer here the papers by Fink and Kusano [3], Kusano and Trench [4], Pachpatte [7,8], Philos [14,15], Philos and Staikos [16] and Trench [17,18] and the references given therein.

As noted by Kusano and Trench [4,p.381], it seems that very little is known about the global existence and various other properties of the solutions of such equations in the literature. The main purpose of this paper is to establish some results concerning the global existence and estimates for solutions of (P_0) which in turn contains in the special cases a number of higher order differential equations studied by many authors by using different techniques. The main tools employed in our analysis are based on the applications of the Leray-Schauder alternative and certain integral inequalities which provide explicit bounds on the unknown functions.

2. Global existence of solutions

In order to obtain our result on the global existence of solutions of $(P)-(P_0)$, we need the following theorem, which is a version of the topological transversality theorem given by A. Granas in [2,p.61].

Theorem G. *Let B be a convex subset of a normed linear space E and assume $0 \in B$. Let $F : B \to B$ be a completely continuous operator and let*

$$U(F) = \{x \in B : x = \lambda Fx \text{ for some } 0 < \lambda < 1\}.$$

Then either $U(F)$ is unbounded or F has a fixed point.

For our purposes, for any integers m and k with $0 \leq m \leq k \leq n-1$, we introduce the function R_{mk} which is defined on I by

$$R_{mk}(t) = \begin{cases} 1, \text{ if } m = k, \\ \displaystyle\int_{t_0}^{t=s_m} \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots x \\ \displaystyle x \int_{t_0}^{s_{k-1}} \frac{1}{r_k(x_k)} ds_k \ldots ds_{m+2} ds_{m+1}, \text{ if } m < k. \end{cases} \tag{2.1}$$

In particular, for any integer k with $0 \leq k \leq n-1$, we put

$$R_k(t) = R_{0k}(t), \quad t \geq t_0.$$

The following theorem constitute the main result of this section.

Theorem 1. *Suppose that there exists a function $a \in C(I, R_+)$, $R_+ = [0, \infty)$ such that*

$$|f(t, D_r^{(0)}y, D_r^{(1)}y, \ldots, D_r^{(n-1)}y)| \leq a(t) H \left(\sum_{m=0}^{n-1} |D_r^{(m)}y| \right), \tag{2.2}$$

where $H : R_+ \to (0, \infty)$ is a continuous nondecreasing function. Then the problem $(P) - (P_0)$ has a solution y in B provided that T satisfies

$$int_{t_0}^{T} M(s) ds < \int_c^{\infty} \frac{ds}{H(s)}, \tag{2.3}$$

where

$$c = \sum_{m=0}^{n-1} \left[|c_m| + \sum_{k=m+1}^{n-1} |c_k| R_{mk}(T) \right], \tag{2.4}$$

in which $R_{mk}(T)$ is defined by taking $t = T$ in (2.1) and

$$M(t) = \sum_{m=0}^{n-1} \frac{1}{r_{m+1}(t)} \int_{t_0}^{t} \frac{1}{r_{m+2}(s_{m+2})} \int_{t_0}^{s_{m+2}} \frac{1}{r_{m+3}(s_{m+3})} \cdots \times \tag{2.5}$$

$$\times \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{t_0}^{s_{n-1}} a(s) ds ds_{n-1} \cdots \times ds_{m+3} ds_{m+2},$$

for $t \in I$.

Proof. First we establish the a-priori bounds for the problem $(P)_\lambda - (P_0)$, $\lambda \in (0,1)$, where

$$(P)_\lambda \qquad D_r^{(n)} y = \lambda f(t, D_r^{(0)} y, D_r^{(1)} y, \ldots, D_r^{(n-1)} y).$$

Let $y(t)$ be a solution of $(P)_\lambda - (P_0)$. Then $y(t)$ and its derivatives can be written as

$$D_r^{(m)} y(t) = c_m + \sum_{k=m+1}^{n-1} c_k R_{mk}(t) + \lambda \int_{t_0}^{t} \frac{1}{r_{m+1}(s_{m+1})} \times \tag{2.6}$$

$$\times \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{t_0}^{s_{n-1}} \overline{f}(y(s)) ds ds_{n-1} \ldots ds_{m+2} ds_{m+1},$$

for $0 \leq m \leq n-1$, where

$$\overline{f}(y(t)) = f(t, D_r^{(0)} y(t), D_r^{(1)} y(t), \ldots, D_r^{(n-1)} y(t)), \tag{2.7}$$

and $R_{mk}(t)$ is defined by (2.1). From (2.6) and using the condition (2.2) we have

$$\sum_{m=0}^{n-1} |D_r^{(m)} y(t)| \leq c + \sum_{m=0}^{n-1} \int_{t_0}^{t} \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \times \tag{2.8}$$

$$\times \cdots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{t_0}^{s_{n-1}} a(s) \times H\left(\sum_{m=0}^{n-1} |D_r^{(m)} y(s)| \right) ds ds_{n-1} \ldots ds_{m+2} ds_{m+1},$$

where c is defined by (2.4). Define a function $u(t)$ by the right side of (2.8). Then

$$\sum_{m=0}^{n-1} |D_r^{(m)} y(t)| \leq u(t), \quad u(t_0) = c,$$

and

$$u'(t) \leq \sum_{m=0}^{n-1} \frac{1}{r_{m+1}(t)} \int_{t_0}^{t} \frac{1}{r_{m+2}(s_{m+2})} \int_{t_0}^{s_{m+2}} \frac{1}{r_{m+3}(s_{m+3})} \times \tag{2.9}$$

$$\times \cdots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{t_0}^{s_{n-1}} a(s) H(u(s)) ds ds_{n-1} \times ds_{m+3} ds_{m+2} \leq M(t) H(u(t)),$$

for $t \in I$. From (2.9) it follows that

$$\frac{u'(t)}{H(u(t))} \leq M(t). \tag{2.10}$$

The integration of (2.10) from t_0 to t and the use of the change of variable and the condition (2.3) give

$$\int_c^{u(t)} \frac{ds}{H(s)} = \int_{t_0}^{t} \frac{u'(s)}{H(u(s))} ds \leq \int_{t_0}^{t} M(s)ds \leq \int_{t_0}^{T} M(s)ds < \int_c^{\infty} \frac{ds}{H(s)}. \qquad (2.11)$$

From (2.11) it follows that $u(t)$ must be bounded on I, i.e. there is a positive constant α independent of $\lambda \in (0,1)$ such that $u(t) \leq \alpha$ and hence $\sum_{m=0}^{n-1} |D_r^{(m)} y(t)| \leq \alpha$ for $t \in I$. Thus we have $|D_r^{(m)} y(t)| \leq \alpha,\ t \in I$ for $0 \leq m \leq n-1$, and consequently $\|y\| \leq \alpha$.

In the next step we rewrite the problem $(P) - (P_0)$ as follows. If $y(t) = z(t) + e(t)$, where $e(t) = c_0 + \sum_{k=1}^{n-1} c_k R_k(t),\ t \in I$, then it is easy to see that $z(t_0) = y(t_0) - e(t_0) = 0$,

$$z(t) = \int_{t_0}^{t} \frac{1}{r_1(s_1)} \int_{t_0}^{s_1} \frac{1}{r_2(s_2)} \times \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \times \qquad (2.12)$$
$$\times \int_{t_0}^{s_{n-1}} f^*(z(s) + e(s)) ds ds_{n-1} \ldots ds_2 ds_1,$$

if and only if $y(t)$ satisfies $(P) - (P_0)$, where we have used the notation $f^*(z(s) + e(s))$ for

$$f(s, D_r^{(0)}(z(s) + e(s)), D_r^{(1)}(z(s) + e(s)), \ldots, D_r^{(0)}(z(s) + e(s))).$$

Define $F : B_0 \to B_0,\ B_0 = \{z \in B :\ z(t_0) = 0\}$ by

$$Fz(t) = \int_{t_0}^{t} \frac{1}{r_1(s_1)} \int_{t_0}^{s_1} \frac{1}{r_2(s_2)} \cdots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \times \qquad (2.13)$$
$$\times \int_{t_0}^{s_{n-1}} f^*(z(s) + e(s)) ds ds_{n-1} \ldots ds_2 ds_1,$$

for $t \in I$. Then F is clearly continuous. Now we shall prove that F is completely continuous.

Let $\{w_k\}$ be a bounded sequence in B_0, i.e. $\|w_k\| \leq \beta$ for all k, where β is a positive constant. From (2.13) and using condition (2.2) and setting $M^* = \sup\{M(t) : t \in I\}$ and $e^* = \sup\{|D_r^{(m)} e(t)| :\ t \in I,\ 0 \leq m \leq n-1\}$, we have

$$|D_r^{(m)}(Fw_k(t))| \leq \int_{t_0}^{t} \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \times \qquad (2.14)$$

$$\times \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{t_0}^{s_{n-1}} |f^*(w_k(s)+e(s))| ds ds_{n-1} \dots ds_{m+2} ds_{m+1} \le$$
$$\le M^*(H(n(\beta+e^*))(T-t_0) = L,$$

for $0 \le m \le n-1$, where $L \ge 0$ is a constant. Hence from (2.14) we obtain $\|Fw_k\| \le L$. This means that $\{Fw_k\}$ is uniformly bounded.

Now we shall that the sequence $\{Fw_k\}$ is equicontinuous. Let $t_0 \le t_1 \le t_2 \le T$. Then from (2.13) and using the condition (2.2), and letting $\{w_k\}$, M^*, e^* as defined above, we observe that

$$|D_r^{(m)}(Fw_k(t_2)) - D_r^{(m)}(Fw_k(t_1))| \le \tag{2.15}$$
$$\le \int_{t_1}^{t_2} \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \times$$
$$\times \int_{t_0}^{s_{n-1}} |f^*(w_k(s)+e(s))| ds ds_{n-1} \dots ds_{m+2} ds_{m+1} \le$$
$$\le \int_{t_1}^{t_2} \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \times$$
$$\times \int_{t_0}^{s_{n-1}} a(s) H\left(\sum_{m=0}^{n-1} [|D_r^{(m)} w_k(s)| + |D_r^{(m)} e(s)|]\right) ds ds_{n-1} \dots ds_{m+2} ds_{m+1} \le$$
$$\le \int_{t_1}^{t_2} \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \times$$
$$\times \int_{t_0}^{s_{n-1}} M^* H(n(\beta+e^*)) ds ds_{n-1} \dots ds_{m+2} ds_{m+1} \le \int_{t_1}^{t_2} M^* H(n(\beta+e^*)) ds.$$

From (2.15) we conclude that $\{Fw_k\}$ is equicontinuous and hence by Arzela-Ascoli theorem the operator F is completely continuous.

Moreover, the set $U(F) = \{z \in B_0 : z = \lambda F z, \lambda \in (0,1)\}$ is bounded. Since for every z in $U(F)$ the function $y(t) = z(t) + e(t)$ is a solution of $(P)_\lambda - (P_0)$, for which we have proved that $\|y\| \le \alpha$ and hence $\|z\| \le \alpha_e^*$. By applying Theorem G, the operator F has a fixed point in B_0. This means that the problem $(P) - (P_0)$ has a solution $y(t)$ in B. The proof is complete.

Remark 1. We note that our Theorem 1 extends the well known theorem of Wintner [20] on the global existence of solution of Cauchy problem for first order differential equation, to higher order differential equations $(P)-(P_0)$. For some recent extensions of Wintner's theorem, see [1,5-10,12,13]. Further we note that out Theorem 1 contains in the special cases the global existence of solutions of the following differential equations

(P_1) $(r(t)y^{(n-1)}(t))' = f(t, r(t)y(t), r(t)y'(t), r(t)y''(t), \ldots, r(t)y^{(n-1)}(t))$,

(P_2) $(r(t)y'(t))^{(n-1)} = f(t, y(t), r(t)y'(t), (r(t)y'(t))', \ldots, (r(t)y'(t))^{(n-2)})$,

(P_3) $y^{(n)}(t) = f(t, y(t), y'(t), \ldots, y^{(n-1)}(t))$,

with some suitable given initial conditions, and studied by many authors in the literature with different viewpoints, see [3,4,14-19].

3. Estimates on the solutions

In this section we obtain estimates on the solutions of $(P)-(P_0)$ which can be used to study the various properties of the solutions of equations $(P)-(P_0)$, by using the integral inequalities given in [11, Theorem 3.3.1, p.222 and Theorem 1.3.2, p.13].

The following theorem deals with the estimates on the solution and their derivatives of the problem $(P)-(P_0)$.

Theorem 2. *Suppose that the function f in (P) satisfies*

$$|f(t, D_r^{(0)}y, D_r^{(1)}y, \ldots, D_r^{(n-1)}y| \le L\left(t, \sum_{m=0}^{n-1} |D_r^{(m)}y|\right), \tag{3.1}$$

for $t \in I$, where $L : I \times R_+ \to R_+$ be a continuous function such that

$$(L) \qquad 0 \le L(t,u) - L(t,v) \le w(t,v)(u-v),$$

for $t \in I$ and $u \ge v \ge 0$, where $w : I \times R_+ \to R_+$ is a continuous function. If $y(t)$ is a solution of $(P)-(P_0)$ on I, then

$$\sum_{m=0}^{n-1} |D_r^{(m)}y(t)| \le a(t) + b(t)\int_{t_0}^{t} L(s, a(s)) \exp\left(\int_s^t w(\sigma, a(\sigma))b(\sigma)d\sigma\right) ds, \tag{3.2}$$

where

$$a(t) = \sum_{m=0}^{n-1}\left[|c_m| + \sum_{k=m+1}^{n-1} |c_k R_{mk}(t)|\right], \tag{3.3}$$

$$b(t)=\sum_{m=0}^{n-1}\int_{t_0}^{t}\frac{1}{r_{m+1}(s_{m+1})}\int_{t_0}^{s_{m+1}}\frac{1}{r_{m+2}(s_{m+2})}\cdots\int_{t_0}^{s_{n-2}}\frac{1}{r_{n-1}(s_{n-1})}\times \times ds_{n-1}\dots ds_{m+2}ds_{m+1}, \tag{3.4}$$

for $t\in I$ and $R_{mk}(t)$ is defined by (2.1).

Proof. If $y(t)$ is a solution of $(P)-(P_0)$, then $y(t)$ and its derivatives can be written as

$$D_r^{(m)}y(t)=c_m+\sum_{k=m+1}^{n-1}c_kR_{mk}(t)+\int_{t_0}^{t}\frac{1}{r_{m+1}(s_{m+1})}\times \tag{3.5}$$

$$\times\int_{t_0}^{s_{m+1}}\frac{1}{r_{m+2}(s_{m+2})}\cdots\int_{t_0}^{s_{n-2}}\frac{1}{r_{n-1}(s_{n-1})}\int_{t_0}^{s_{n-1}}\overline{f}(y(s))ds\,ds_{n-1}\dots ds_{m+2}ds_{m+1},$$

for $0\le m\le n-1$, where $R_{mk}(t)$ and $\overline{f}(y(t))$ are defined by (2.1) and (2.7) respectively. From (3.5), (3.1), (3.3) and (3.4) we observe that

$$\sum_{m=0}^{n-1}|D_r^{(m)}y(t)|\le a(t)+b(t)\int_{t_0}^{t}L\left(s,\sum_{m=0}^{n-1}|D_r^{(m)}y(s)|\right)ds. \tag{3.6}$$

Now an application of Theorem 3.2.1 given in [11,p.222] to (3.6) yields the desired inequality in (3.2). The proof is complete.

Our next theorem deals with a slight variant of Theorem 2 which can be used more conveniently in certain applications.

Theorem 3. *Suppose that the function f in (P) satisfies the condition (3.1). If $y(t)$ is a solution of $(P)-(P_0)$ existing on I, then*

$$\sum_{m=0}^{n-1}|D_r^{(m)}y(t)-\psi^{(m)}(t)|\le \tag{3.7}$$

$$\le q(t)+b(t)\left(\int_{t_0}^{t}q(s)w\left(s,\sum_{m=0}^{n-1}|\psi^{(m)}(s)|\right)\exp\left(\int_{s}^{t}b(\sigma)w\left(\sigma,\sum_{m=0}^{n-1}|\psi^{(m)}(\sigma)\right)d\sigma\right)ds\right),$$

for $t\in I$, where

$$\psi^{(m)}(t)=c_m+\sum_{k=m+1}^{n-1}c_kR_{mk}(t), \tag{3.8}$$

$$q(t)=b(t)\int_{t_0}^{t}L\left(s,\sum_{m=0}^{n-1}|\psi^{(m)}(s)|\right)ds, \tag{3.9}$$

for $t \in I$, $R_{mk}(t)$ and $b(t)$ are defined by (2.1) and (3.4) respectively.

Proof. If $y(t)$ is a solution of $(P)-(P_0)$, then $y(t)$ and its derivatives can be written as

$$D_r^{(m)}y(t) = \psi^{(m)}(t) + \int_{t_0}^{t} \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \times \tag{3.10}$$

$$\times \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{t_0}^{s_{n-1}} \overline{f}(y(s))ds ds_{n-1} \ldots ds_{m+2} ds_{m+1},$$

where $\overline{f}(y(t))$ is defined by (2.7). From (3.10), (3.1), (3.9), and the condition (L) we observe that

$$\sum_{m=0}^{n-1} |D_r^{(m)}y(t) - \psi^{(m)}(t)| \leq \sum_{m=0}^{n-1} \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \times \tag{3.11}$$

$$\times \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{t_0}^{s_{n-1}} |\overline{f}(y(s))ds ds_{n-1} \ldots ds_{m+2} ds_{m+1} \leq$$

$$\leq b(t) \int_{t_0}^{t} L\left(s, \sum_{m=0}^{n-1} |D_r^{(m)}y(s)|\right) ds \leq$$

$$\leq b(t) \int_{t_0}^{t} \left[L\left(s, \sum_{m=0}^{n-1} |D_r^{(m)}y(s) - \psi^{(m)}(s)| + \sum_{m=0}^{n-1} |\psi^{(m)}(s)|\right) - \right.$$

$$\left. -L\left(s, \sum_{m=0}^{n-1} |\psi^{(m)}(s)|\right) + L\left(s, \sum_{m=0}^{n-1} |\psi^{(m)}(s)|\right)\right] ds \leq$$

$$\leq q(t) + b(t) \int_{t_0}^{t} w\left(s, \sum_{m=0}^{n-1} |\psi^{(m)}(s)|\right) \sum_{m=0}^{n-1} |D_r^{(m)}y(s) - \psi^{(m)}(s)| x ds.$$

Now an application of Theorem 1.3.2 given in [11,p.13] yields the required inequality in (3.7). The proof is complete.

Another useful variant of Theorem 2 which deals with the bounds on the solution $y(t)$ of $(P)-(P_0)$ and its derivatives is given in the following theorem.

Theorem 4. *Suppose that the function f in (P) satisfies the condition*

$$|f(t, D_r^{(0)}y, D_r^{(1)}y, \ldots, D_r^{(n-1)}y)| \leq h(t)\left(\sum_{m=0}^{n-1} |D_r^{(m)}y|\right), \tag{3.12}$$

for $t \in I$, *where* $h : I \to R_+$ *is a continuous function. If* $y(t)$ *is a solution of* $(P)-(P_0)$, *then*

$$\sum_{m=0}^{n-1} |D_r^{(m)} y(t)| \leq a(t) + b(t) \int_{t_0}^{t} h(s)a(s) \exp\left(\int_s^t h(\sigma)b(\sigma)d\sigma\right) ds, \tag{3.13}$$

for $t \in I$, *where* $a(t)$ *and* $b(t)$ *are defined by (3.3) and (3.4) respectively.*

Proof. Let $y(t)$ be a solution of $(P)-(P_0)$ for $t \in I$, then the solution $y(t)$ and its derivatives can be written as in (3.5). From (3.5), (3.12), (3.3) and (3.4) we have

$$\sum_{m=0}^{n-1} |D_r^{(m)} y(t)| \leq a(t) + b(t) \int_{t_0}^{t} h(s) \left(\sum_{m=0}^{n-1} |D_r^{(m)} y(s)|\right) ds. \tag{3.14}$$

Now an application of Theorem 1.3.2 given in [11,p.13] yields the desired bound in (3.13). The proof is complete.

Our next result deals with the dependency of solutions of equations (P) on initial values.

Theorem 5. *Let* $y_1(y)$ *and* $y_2(t)$ *be the solutions of* $(P)-(P_0)$ *with the given initial conditions*

$$D_r^{(m)} y_1(t_0) = c_m, \tag{3.15}$$

and

$$D_r^{(m)} y_2(t_0) = d_m, \tag{3.16}$$

respectively, for $m = 0, 1, 2, \ldots, n-1$, *where* c_m, d_m *are given real constants. Suppose that the function* f *in* (P) *satisfies the condition*

$$|f(t, D_r^{(0)} y_1, D_r^{(1)} y_1, \ldots, D_r^{(n-1)} y_1) - f(t, D_r^{(0)} y_2, D_r^{(1)} y_2, \ldots, D_r^{(n-1)} y_2)| \leq \tag{3.17}$$

$$\leq h(t) \left(\sum_{m=0}^{n-1} |D_r^{(m)} y_1 - D_r^{(m)} y_2|\right),$$

for $t \in I$, *where* $h : I \to R_+$ *is a continuous function. Then*

$$\sum_{m=0}^{n-1} |D_r^{(m)} y_1(t) - D_r^{(m)} y_2(t)| \leq \tag{3.18}$$

$$\leq A(t) + b(t) \int_{t_0}^{t} h(s) A(s) \exp\left(\int_{s}^{t} h(\sigma) b(\sigma) d\sigma\right) ds,$$

where

$$A(t) = \sum_{m=0}^{n-1} \left[|c_m - d_m| + \sum_{k=m+1}^{n-1} |c_k - d_k| R_{mk}(t) \right], \tag{3.19}$$

for $t \in I$, $R_{mk}(t)$ *and* $b(t)$ *are defined by (2.1) and (3.4) respectively.*

Proof. In view of the facts that $y_1(t)$ and $y_2(t)$ are the solutions of (P) with the given initial conditions (3.15) and (3.16) respectively, we have

$$D_r^{(m)} y_1(t) - D_r^{(m)} y_2(t) = (c_m - d_m) + \sum_{k=m+1}^{n-1} (c_k - d_k) R_{mk}(t) + \tag{3.20}$$

$$+ \int_{t_0}^{t} \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \times$$

$$\times \int_{t_0}^{s_{n-1}} (\overline{f}(y_1(s)) - \overline{f}(y_2(s))) ds ds_{n-1} \ldots ds_{m+2} ds_{m+1},$$

where $R_{mk}(t)$ and $\overline{f}(y(t))$ are defined by (2.1) and (2.7) respectively. From (3.20), (3.17), (3.19) and (3.4) we observe that

$$\sum_{m=0}^{n-1} |D_r^{(m)} y_1(t) - D_r^{(m)} y_2(t)| \leq \tag{3.21}$$

$$\leq A(t) + b(t) \int_{t_0}^{t} h(s) \left(\sum_{m=0}^{n-1} |D_r^{(m)} y_1(s) - D_r^{(m)} y_2(s)| \right) ds.$$

Now an application of Theorem 1.3.2 given in [11,p.13] yields the required inequality in (3.18) and hence the proof is complete.

We next consider the following differential equations

(Q_1) $$D_r^{(n)} y = f(t, D_r^{(0)} y, D_r^{(1)} y, \ldots, D_r^{(n-1)} y, \mu),$$

(Q_2) $$D_r^{(n)} y = f(t, D_r^{(0)} y, D_r^{(1)} y, \ldots, D_r^{(n-1)} y, \mu_0),$$

with the given initial conditions in (P_0), where $f : I \times R^n \times R \to R$ is a continuous function and μ, μ_0 are real parameters.

The following theorem shows the dependency of solutions of equations $(Q_1) - (P_0)$ and $(Q_2) - (P_0)$ on pure parameters.

Theorem 6. *Suppose that*

$$|f(t, D_r^{(0)}y, D_r^{(1)}y, \ldots, D_r^{(n-1)}y, \mu - f(t, D_r^{(0)}\overline{y}, D_r^{(1)}\overline{y}, \ldots, D_r^{(n-1)}\overline{y}, \mu)| \leq \tag{3.22}$$

$$\leq h(t)\left(\sum_{m=0}^{n-1} |D_r^{(m)}y - D_r^{(m)}\overline{y}|\right),$$

$$|f(t, D_r^{(0)}y, D_r^{(1)}y, \ldots, D_r^{(n-1)}y, \mu) - \tag{3.23}$$

$$-f(t, D_r^{(0)}y, D_r^{(1)}y, \ldots, D_r^{(n-1)}y, \mu_0)| \leq g(t)|\mu - \mu_0|.$$

where $h, g : I \to R_+$ *are continuous functions. If* $y_1(t)$ *and* $y_2(t)$ *are solutions of* $(Q_1) - (P_0)$ *and* $(Q_2) - (P_0)$, *then*

$$\sum_{m=0}^{n-1} |D_r^{(m)}y_1(t) - D_r^{(m)}y_2(t)| \leq \tag{3.24}$$

$$\leq \overline{A}(t) + b(t)\int_{t_0}^{t} h(s)\overline{A}(s)\exp\left(\int_s^t h(\sigma)b(\sigma)d\sigma\right)ds,$$

for $t \in I$, *where*

$$\overline{A}(t) = |\mu - \mu_0|b(t)\int_{t_0}^{t} g(s)ds, \tag{3.25}$$

for $t \in I$ *and* $b(t)$ *is defined by (3.4).*

Proof. Let $z(t) = y_1(t) - y_2(t)$ for $t \in I$. As in the proof of Theorem 5, from the hypothesis we observe that

$$D_r^{(m)}z(t) = D_r^{(m)}y_1(t) - D_r^{(m)}y_2(t) = \tag{3.26}$$

$$= \int_{t_0}^{t} \frac{1}{r_{m+1}(s_{m+1})}\int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})}\cdots\int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \times$$

$$\times \int_{t_0}^{s_{n-1}} \{f(s, D_r^{(0)}y_1(s), D_r^{(1)}y_1(s), \ldots, D_r^{(n-1)}y_1(s), \mu) -$$

$$-f(s, D_r^{(0)}y_2(s), D_r^{(1)}y_2(s), \ldots, D_r^{(n-1)}y_2(s), \mu) +$$

$$+f(s, D_r^{(0)}y_2(s), D_r^{(1)}y_2(s), \ldots, D_r^{(n-1)}y_2(s), \mu) -$$

$$-f(s, D_r^{(0)}y_2(s), D_r^{(1)}y_2(s), \ldots, D_r^{(n-1)}y_2(s), \mu_0)\}ds\,ds_{n-1}\ldots ds_{m+2}ds_{m+1}.$$

From (3.26), (3.22), (3.23), (3.25) and (3.4) we observe that

$$\sum_{m=0}^{n-1} |D_r^{(m)} z(t)| \leq \int_{t_0}^{t} \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \times \tag{3.27}$$

$$\times \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{t_0}^{s_{n-1}} \left\{ h(s) \left(\sum_{m=0}^{n-1} |D_r^{(m)} y_1(s) - D_r^{(m)} y_2(s)| \right) + \right.$$

$$\left. + g(s)|\mu - \mu_0| \right\} ds ds_{n-1} \ldots ds_{m+2} ds_{m+1} \leq$$

$$\leq b(t) \int_{t_0}^{t} \left\{ h(s) \left(\sum_{m=0}^{n-1} |D_r^{(m)} z(s)| \right) + g(s)|\mu - \mu_0| \right\} ds =$$

$$= \overline{A}(t) + b(t) \int_{t_0}^{t} h(s) \left(\sum_{m=0}^{n-1} |D_r^{(m)} z(s)| \right) ds$$

Now an application of Theorem 1.3.2 given in [11,p.13] yields the required inequality in (3.24) and the proof is complete.

Remark 2. We note that the results obtained in this paper can be very easily extended to the more general integrodifferential equation of the form

$$(Q) \qquad D_r^{(n)} y = f(t, D_r^{(0)} y, D_r^{(1)} y, \ldots, D_r^{(n-1)} y,$$

$$\int_{t_0}^{t} g(t, s, D_r^{(0)} y(s), D_r^{(1)} y(s), \ldots, D_r^{(n-1)} y(s)) ds),$$

with the given initial conditions in (P_0), under some suitable conditions on the functions involved in (Q) and by using the suitable general versions of the inequalities given in Chapters 1 and 3 in [11]. For similar results, see references [7-10,12,13].

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DEPARTMENT OF MATHEMATICS, MARATHWADA UNIVERSITY, AURANGABAD 431 004, (MAHARASHTRA), INDIA

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ANALYSIS OF SOME NEUTRAL DELAY DIFFERENTIAL EQUATIONS

RADU PRECUP

Abstract. The paper is devoted to the study of the neutral differential equation with delay $x'(t) = f(t, x(t), x(\theta(t)), x'(\theta(t)))$. Our analysis is concerned with the existence, uniqueness and monotone iterative approximation of the nondecreasing global solutions of the initial-value problem. We use fixed point theorems (Schauder, Krasnoselskii, Leray-Schauder) and monotone iterative techniques.

1. Introduction

In this paper, we are concerned with the following nonlinear neutral delay equation

$$x'(t) = f(t, x(t), x(\theta(t)), x'(\theta(t))), \tag{1.1}$$

where $-\tau \leq \theta(t) \leq t$ for some $\tau \geq 0$.

Equations of this type arise when modelling biological, physical, etc., processes whose growth rate at any moment of time t is determined not only by the present state, but also by past states and the past growth rate. For example, such models are described by K. Gopalsamy [4] and Y. Kuang [8], from population dynamics, and by R.D. Driver [3], in connection with the two-body problem.

Basic theory and much literature on differential equations with delay, including the neutral ones, can be found in the monographs by V. Lakshmikantham, L. Wen, B. Zhang [9], V. Kolmanovskii, A. Myshkis [7], D. Bainov, D.P. Mishev [1] and J. Hale [5].

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Recently, T.A. Burton [2] established an analogue of the Peano *local* existence theorem for the Cauchy problem (1.1)-(1.2), where

$$x(t) = \phi(t), \quad -\tau \leq t \leq 0. \tag{1.2}$$

Motivated by the above paper, this article deals with the *global* solvability (on a given interval $[0, T]$) of the Cauchy problem (1.1)-(1.2).

We shall assume that f is nonnegative and continuous, θ is continuous, $\phi \in C^1[-\tau, 0]$ and satisfies the *sewing condition*

$$\phi'(0) = f(0, \phi(0), \phi(\theta(0)), \phi'(\theta(0))). \tag{1.3}$$

We shall look for nondecreasing solutions $x \in C^1[0, T]$ with $x(t) \in [a, R]$ and $x'(0) = b$, where $a = \phi(0)$, $b = \phi'(0)$ and $a < R \leq \infty$. In case that $R = \infty$, all intervals of the form $[c, R]$ should be interpreted as $[c, \infty)$ and all inequalities of the form $c \leq R$, as $c < \infty$.

Let

$$K = \{x \in C^1[0, T]; \quad a \leq x \text{ on } [0, T]\}$$

and

$$K_R = \{x \in K; \quad x \leq R \text{ on } [0, T]\}.$$

Clearly, K is a closed convex set of $C^1[0, T]$ and (1.1)-(1.2) is equivalent to the fixed point problem $A(x) = x$ for the map $A: K_R \to K$,

$$A(x)(t) = a + \int_0^t f(s, x(s), \tilde{x}(\theta(s)), \tilde{x}'(\theta(s)))\, ds, \quad 0 \leq t \leq T, \tag{1.4}$$

where $\tilde{x}(t) = \phi(t)$ on $[-\tau, 0)$ and $\tilde{x}(t) = x(t)$ on $[0, T]$. Obviously, each fixed point x of A also satisfies $x(0) = a$ and $x'(0) = b$ and so, its prolongation by ϕ is a function in $C^1[-\tau, T]$.

Notice that the dependence of $f(t, x, y, z)$ on the *neutral variable* z is the cause that A is not completely continuous. This is why one tries to represent A as a sum of a completely continuous mapping and a contraction. This happens when f admits the decomposition

$$f(t,x,y,z) = f_0(t,x,y) + f_1(t,x,y,z), \tag{1.5}$$

with f_0 continuous and f_1 satisfying the Lipschitz condition

$$|f_1(t,x,y,z) - f_1(t,\overline{x},\overline{y},\overline{z})| \leq \alpha|x-\overline{x}| + \beta|y-\overline{y}| + \gamma|z-\overline{z}| \tag{1.6}$$

for $\alpha, \beta \geq 0$ and $0 \leq \gamma < 1$. Then A can be represented as $A = A_0 + A_1$, where

$$A_0(x)(t) = a + \int_0^t f_0(s, x(s), \widetilde{x}(\theta(s)))\, ds$$

and

$$A_1(x)(t) = \int_0^t f_1(s, x(s), \widetilde{x}(\theta(s)), \widetilde{x}'(\theta(s)))\, ds.$$

The mapping A_0 is completely continuous by the Ascoli-Arzelá theorem, while A_1 is a contraction with respect to a suitable norm on $C^1[0,T]$ as shows the following lemma.

Lemma 1.1. *Suppose* $0 \leq \gamma < 1$. *Then, for each* $\eta > \max\{(\alpha+\beta)/(1-\gamma), \alpha+\beta+\gamma\}$, A_1 *is a contraction on* K_R *with respect to the norm*

$$||x||_{1,\eta} = \max\left\{||x||_{0,\eta}, ||x'||_{0,\eta}\right\}$$

on $C^1[0,T]$, *where*

$$||x||_{0,\eta} = \max_{[0,T]}(|x(t)|\exp(-\eta t)).$$

Proof. Let $x, y \in K_R$. Using $\theta(t) \leq t$, we obtain

$$|A_1(x)(t) - A_1(y)(t)| \leq \alpha \int_0^t |x(s) - y(s)|\, ds$$

$$+\beta \int_0^t |\widetilde{x}(\theta(s)) - \widetilde{y}(\theta(s))|\, ds + \gamma \int_0^t |\widetilde{x}'(\theta(s)) - \widetilde{y}'(\theta(s))|\, ds$$

$$\leq \alpha \int_0^t |x(s) - y(s)| \exp(-\eta s) \exp(\eta s)\, ds$$

$$+\beta \int_0^t |\widetilde{x}(\theta(s)) - \widetilde{y}(\theta(s))| \exp(-\eta\theta(s)) \exp(\eta\theta(s))\, ds$$

$$+\gamma \int_0^t |\widetilde{x}'(\theta(s)) - \widetilde{y}'(\theta(s))| \exp(-\eta\theta(s)) \exp(\eta\theta(s))\, ds$$

$$\leq \left[(\alpha+\beta)\eta^{-1} \|x-y\|_{0,\eta} + \gamma\eta^{-1} \|x'-y'\|_{0,\eta}\right] \exp(\eta t).$$

It follows that

$$\|A_1(x) - A_1(y)\|_{0,\eta} \leq (\alpha+\beta+\gamma)\eta^{-1} \|x-y\|_{1,\eta}.$$

Similarly

$$\left|A_1(x)'(t) - A_1(y)'(t)\right| \leq \alpha |x(t) - y(t)|$$

$$+\beta |\widetilde{x}(\theta(t)) - \widetilde{y}(\theta(t))| + \gamma |\widetilde{x}'(\theta(t)) - \widetilde{y}'(\theta(t))|$$

$$\leq \alpha \int_0^t |x'(s) - y'(s)|\, ds + \beta \int_0^{\theta(t)} |\widetilde{x}'(s) - \widetilde{y}'(s)|\, ds$$

$$+\gamma |\widetilde{x}'(\theta(t)) - \widetilde{y}'(\theta(t))| \leq (\alpha+\beta) \int_0^t |x'(s) - y'(s)|\, ds$$

$$+\gamma |\widetilde{x}(\theta(t)) - \widetilde{y}(\theta(t))| \leq \left[(\alpha+\beta)\eta^{-1} + \gamma\right] \|x'-y'\|_{0,\eta} \exp(\eta t).$$

Hence

$$\left\|A_1(x)' - A_1(y)'\right\|_{0,\eta} \leq \left[(\alpha+\beta)\eta^{-1} + \gamma\right] \|x-y\|_{1,\eta}.$$

Therefore

$$\|A_1(x) - A_1(y)\|_{1,\eta} \leq L \|x-y\|_{1,\eta}, \tag{1.7}$$

where

$$L = \max\left\{(\alpha+\beta+\gamma)\eta^{-1}, (\alpha+\beta)\eta^{-1} + \gamma\right\}. \tag{1.8}$$

□

There is a remarkable case when in spite of the neutral variable, we still can work with completely continuous mappings: the case when the *step method* applies. We are in this case if

$$\theta(t) < t \quad \text{on } (0,T] \text{ and } \inf\{t > 0;\ \theta(t) > 0\} > 0. \tag{1.9}$$

By using the step method, the solving of (1.1)-(1.2) is reduced to that of a finite number of Cauchy problems for equations without deviated arguments. To explain this, let $t_0 = 0$ and

$$t_n = \inf\{t \in (t_{n-1}, T];\ \theta(t) > t_{n-1}\}, \quad n = 1, 2, ..., \tag{1.10}$$

where we set $t_n = T$ in case that the infimum is taken over the empty set. Obviously, (t_n) is a bounded nondecreasing sequence and if $t_m = T$ for some m, then $t_n = T$ for all $n \geq m$. In addition, if $t_n < T$, then

$$\theta(t_n) = t_{n-1} \text{ and } \theta(t) \leq t_{n-1} \text{ for } t_{n-1} \leq t \leq t_n. \tag{1.11}$$

The second inequality in (1.9) implies $t_0 < t_1 \leq T$, while the first one assures the strict monotonicity $t_{n-1} < t_n$ whenever $t_{n-1} < T$, and also the existence of a $k \geq 1$ with $t_{k-1} < t_k = T$. Indeed, otherwise, we should have $t_0 < t_1 < ... < t_n < ... < T$. If we denote $t_* = \lim_{n\to\infty} t_n$, then $0 < t_* \leq T$ and $\theta(t_*) = t_*$, which contradicts (1.9). Thus, there exists a finite partition of $[0,T]$, say

$$0 = t_0 < t_1 < ... < t_{k-1} < t_k = T.$$

A solution to (1.1)-(1.2) will be defined step by step, on each subinterval $[-\tau, t_n]$, $n = 1, 2, ..., k$. Denote $x_0 = \phi$ and let $x_{n+1} \in C^1[-\tau, t_{n+1}]$ be a prolongation of $x_n \in C^1[-\tau, t_n]$ by a solution of the following problem

$$\begin{cases} x'(t) = f(t, x(t), x_n(\theta(t)), x_n'(\theta(t))), & t_n \leq t \leq t_{n+1}, \\ x(t_n) = a_n, \end{cases} \tag{1.12}$$

where $a_n = x_n(t_n)$, $n = 0, 1, ..., k-1$. It is clear that x_k will represent a solution of (1.1)-(1.2). Thus, at each step n, we have to solve (1.12), or equivalently, to find a fixed point of the completely continuous mapping $A_n : C[t_n, t_{n+1}] \to C[t_n, t_{n+1}]$,

$$A_n(x)(t) = a_n + \int_{t_n}^{t} f(s, x(s), x_n(\theta(s)), x_n'(\theta(t)))\, ds. \qquad (1.13)$$

Organization of the paper

In Section 2, we discuss the initial value problem for (1.1) in case that the step method applies. In Section 3, the same problem is studied when the step method does not apply. In Section 4, we obtain minimal and maximal solutions to the Cauchy problem. We use fixed point theorems (Schauder, Krasnoselskii, Leray-Schauder) and monotone iterative techniques.

Notice that by a somewhat similar approach, we discussed in [6] the initial value problem for a delay integral equation modelling infectious disease (see also [11]). The results are new and they improve and complement the existing literature (see [10] for example, for related topics).

We finish this introductory section by some abstract existence principles.

Fixed point theory

Theorem 1.2. *(Schauder) Let X be a Banach space and $D \subset X$ nonempty bounded closed convex. Suppose $A : D \to D$ is compact (i.e. continuous with $A(D)$ relatively compact). Then A has at least one fixed point.*

Theorem 1.3. *(Krasnoselskii) Let X be a Banach space and $D \subset X$ nonempty bounded closed convex. Suppose $A_0 : D \to X$ is compact, $A_1 : D \to X$ is a contraction and that $A_0(x) + A_1(y) \in D$ for all $x, y \in D$. Then $A_0 + A_1$ has at least one fixed point.*

Theorem 1.4. *(Leray-Schauder) Let X be a Banach space, $K \subset X$ closed convex and $U \subset K$ bounded open in K. Suppose $A : \bar{U} \to K$ is compact and*

$$(1-\lambda)x_0 + \lambda A(x) \neq x \quad \textit{for all } x \in \partial U \textit{ and } \lambda \in [0,1],$$

for some $x_0 \in U$. Then A has at least one fixed point in U.

2. Existence via the step method

Let us list our assumptions:

(a1) $\theta \in C[0,T]$, $-\tau \leq \theta$ and (1.9) (*step condition*) holds.

(a2) $\phi \in C^1[-\tau, 0]$ and (1.3) (*sewing condition*) is satisfied.

(a3) $f(t,x,y,z)$ is nonnegative and continuous on $D = [0,T]\times[a,R]\times[m,M]\times[m',\infty)$, where $a < R \leq \infty$, $m = \min_{[-\tau,0]}\phi(t)$, $M = \max\{R, \max_{[-\tau,0]}\phi(t)\}$ and $m' = \min\{0, \min_{[-\tau,0]}\phi'(t)\}$.

(a4) $f(t,x,y,z) \leq \alpha(t)\beta(x)\gamma(y,z)$ on D, where α, β, γ are continuous, $\alpha \geq 0$, $\beta > 0$, $\gamma \geq 0$ and

$$\sup_{[m,M]\times[m',\infty)} \gamma(y,z) \cdot \int_0^T \alpha(t)\,dt < \int_0^R \frac{du}{\beta(u)} \tag{2.1}$$

(*Wintner type condition*).

We make the convention that when the left side in (2.1) equals ∞, then the right side is ∞ too.

Theorem 2.1. *Suppose (a1)-(a4) are satisfied. Then (1.1)-(1.2) has at least one solution $x \in C^1[-\tau,T]$ with $a \leq x < R$ and $x' \geq 0$ on $[0,T]$.*

Proof. First we prove that for each $x \in C^1[-\tau, t_n]$ with $a \leq x \leq R$ and satisfying (1.1) and (1.2) on $[0,t_n]$, there exists $R_n \in [a,R)$ depending only on the restriction of x to $[-\tau, t_{n-1}]$, such that $x \leq R_n$ on $[0,t_n]$.

Indeed, by (a4), we have

$$x'(t) \leq \alpha(t)\beta(x(t))\gamma(x(\theta(t)), x'(\theta(t))), \quad 0 \leq t \leq t_n.$$

Divide by $\beta(x(t))$ and integrate from 0 to t_n to obtain

$$\int_a^{x(t_n)} \frac{du}{\beta(u)} = \int_0^{t_n} \frac{x'(t)}{\beta(x(t))}dt \leq M_n \int_0^{t_n} \alpha(t)\,dt,$$

where $M_n = \max_{[0,t_{n-1}]}\gamma(x(\theta(t)), x'(\theta(t)))$. By (2.1), this implies

$$\int_a^{x(t_n)} \frac{du}{\beta(u)} \leq M_n \int_0^{t_n} \alpha(t)\,dt < \int_a^R \frac{du}{\beta(u)}.$$

Thus $x(t_n) \le R_n < R$, where

$$M_n \int_0^{t_n} \alpha(t)\,dt = \int_a^{R_n} \frac{du}{\beta(u)}. \tag{2.2}$$

Since x is nondecreasing on $[0,t_n]$, we have $x(t) \le x(t_n) \le R_n$ for all $t \in [0,t_n]$, as claimed.

Now suppose we have already defined $x_n \in C^1[-\tau,t_n]$, a solution of (1.1)-(1.2) on $[-\tau,t_n]$, with $a \le x_n \le R$ and $x_n' \ge 0$ on $[0,t_n]$. Then $x_n \le R_n < R$ and

$$\int_a^{a_n} \frac{du}{\beta(u)} \le M_n \int_0^{t_n} \alpha(t)\,dt, \tag{2.3}$$

where R_n is given by (2.2), with $M_n = \max_{[0,t_{n-1}]} \gamma(x_n(\theta(t)), x_n'(\theta(t)))$.

Next we try to extend x_n to a solution $x_{n+1} \in C^1[-\tau,t_{n+1}]$ satisfying $a \le x_{n+1} \le R$ and $x_{n+1}' \ge 0$ on $[0,t_{n+1}]$. Let R_{n+1} be given by (2.2), for $M_{n+1} = \max_{[0,t_n]} \gamma(x_n(\theta(t)), x_n'(\theta(t)))$. It is clear that $M_n \le M_{n+1}$ and $R_n \le R_{n+1} < R$. Choose a finite $R' \in (R_{n+1}, R]$ and define

$$K_n = \{x \in C[t_n,t_{n+1}];\ a \le x\}, \quad U_n = \{x \in K_n;\ x < R'\}$$

and

$$A_n : \overline{U}_n \to K_n, \quad A_n(x)(t) = a_n + \int_{t_n}^{t} f(s, x(s), x_n(\theta(s)), x_n'(\theta(s)))\,ds.$$

Obviously, $K_n \subset C[t_n,t_{n+1}]$ is closed and convex, $U_n \subset K_n$ is bounded and open in K_n, the constant function a_n belongs to U_n (because $a_n \le R_n < R'$) and A_n is completely continuous. Also, if x is a fixed point of A_n, then $x(t_n) = a_n$, $x'(t_n) = x_n'(t_n)$ and the prolongation x_{n+1} of x_n by x will represent a solution of (1.1)-(1.2) on $[-\tau,t_{n+1}]$ satisfying $a \le x_{n+1} \le R$ and $x_{n+1}' \ge 0$ on $[0,t_{n+1}]$.

The existence of a fixed point of A_n will follow by the Leray-Schauder principle if the boundary condition

$$x \ne (1-\lambda)a_n + \lambda A_n(x) \quad \text{for all } x \in \partial U_n,\ \lambda \in (0,1) \tag{2.4}$$

holds. To check it, suppose $x \in \overline{U}_n$ satisfies $x = (1-\lambda)\, a_n + \lambda A_n(x)$ for some $\lambda \in (0,1)$. Then, $x(t_n) = a_n$ and

$$x'(t) = \lambda f(t, x(t), x_n(\theta(t)), x_n'(\theta(t))) \quad \text{for all } t \in [t_n, t_{n+1}].$$

As above, we obtain

$$\int_{a_n}^{x(t_{n+1})} \frac{du}{\beta(u)} \le M_{n+1} \int_{t_n}^{t_{n+1}} \alpha(t)\, dt.$$

Taking into account (2.3) and $M_n \le M_{n+1}$, we deduce

$$\int_{a}^{x(t_{n+1})} \frac{du}{\beta(u)} \le M_{n+1} \int_{0}^{t_{n+1}} \alpha(t)\, dt.$$

Hence $x(t_{n+1}) \le R_{n+1}$ and consequently, $x \le R_{n+1} < R'$ on $[t_n, t_{n+1}]$. Thus, $x \notin \partial U_n$ and (2.4) is proved. □

Remark 2.1. *The conclusion of Theorem 2.1 remains true if instead of (a4) the following condition is satisfied:*

(a4') $f(t,x,y,z) \le \beta(x)\, \delta(t,y,z)$ on D, where $\beta > 0, \delta \ge 0$ and

$$T \cdot \sup_{[0,T]\times[m,M]\times[m',\infty)} \delta(t,y,z) < \int_a^R \frac{du}{\beta(u)}. \tag{2.5}$$

Remark 2.2. *Suppose $R = \infty$ and that in (a4'), $\beta(u) = u + c$, where $c \ge 0$. In this case, (2.5) trivially holds since its right side equals to infinity. Moreover, a fixed point of A_n follows directly by Schauder's fixed point theorem. Indeed, if $R = \infty$, the map A_n can be defined on the entire K_n and $A_n(K_n) \subset K_n$. In addition, for $\eta > 0$ and $x \in K_n$, we have*

$$0 \le A_n(x)(t) \le a_n + \widetilde{M}_n \int_{t_n}^{t} (x(s) + c)\, ds$$

$$= a_n + c\widetilde{M}_n (t - t_n) + \widetilde{M}_n \int_{t_n}^{t} x(s)\, ds$$

$$\le \tilde{a}_n + \widetilde{M}_n \eta^{-1} \|x\|_{0,\eta} \exp(-\eta t), \quad t_n \le t \le t_{n+1},$$

where $\widetilde{M}_n = \max_{[t_n,t_{n+1}]} \delta(t, x_n(\theta(t)), x_n'(\theta(t)))$ *and* $\widetilde{a}_n = a_n + c\widetilde{M}_n(t_{n+1} - t_n)$. *In consequence,*

$$\|A_n(x)\|_{0,\eta} \le \widetilde{a}_n + \widetilde{M}_n \eta^{-1} \|x\|_{0,\eta} \quad (x \in K_n).$$

Thus, if we choose $\eta > \widetilde{M}_n$ *and* $R' \ge \widetilde{a}_n / \left(1 - \widetilde{M}_n \eta^{-1}\right)$, *then Schauder's theorem applies on* $\{x \in K_n;\ \|x\|_{0,\eta} \le R'\}$.

Remark 2.3. *Suppose* $R = \infty$ *and that a more restrictive condition than (a4') holds, namely*

(a4") $|f(t,x,y,z) - f(t,\bar{x},y,z)| \le L(t,y,z)|x - \bar{x}|$ *on* D,

where L *is continuous and nonnegative.*

From (a4"),

$$f(t,x,y,z) \le L(t,y,z)(x-a) + f(t,a,y,z) \le \delta(t,y,z)\,x,$$

where $\delta(t,y,z) = \max\{L(t,y,z), f(t,a,y,z)/a\}$. *Hence we are in the frame of Remark 2.2. In addition, the initial value problem has a unique solution and at each step, the unique fixed point of* A_n *can be obtained by means of the contraction principle. Indeed, for* $\eta > 0$ *and* $x, y \in K_n$, *we have*

$$|A_n(x)(t) - A_n(y)(t)| \le \int_{t_n}^{t} L(s, x_n(\theta(s)), x_n'(\theta(s)))\,|x(s) - y(s)|\,ds$$

$$\le \overline{M}_n \int_{t_n}^{t} |x(s) - y(s)|\,ds \le \overline{M}_n \eta^{-1} \|x - y\|_{0,\eta} \exp(-\eta t),$$

where $\overline{M}_n = \max_{[t_n,t_{n+1}]} L(t, x_n(\theta(t)), x_n'(\theta(t)))$. *Now our claim follows if we choose* $\eta > \overline{M}_n$.

3. Existence without the step condition

The assumptions for this section are as follows:

(A1) $\theta \in C[0,T]$ and $-\tau \le \theta(t) \le t$.

(A2) = (a2).

(A3) $f(t,x,y,z)$ is nonnegative on D and admits the decomposition (1.5), where f_0, f_1 are continuous and f_1 satisfies the Lipschitz condition (1.6) for some $\alpha, \beta \geq 0$ and $\gamma \in [0,1)$.

(A4) $f(t,x,y,z) \leq \alpha(t)\beta(x)$ on D, where α, β are continuous, $\alpha \geq 0$, $\beta > 0$ and

$$\int_0^T \alpha(t)\,dt < \int_a^R \frac{du}{\beta(u)}. \tag{3.1}$$

Theorem 3.1. *Suppose (A1)-(A4) are satisfied. Then (1.1)-(1.2) has at least one solution $x \in C^1[-\tau,T]$ with $a \leq x \leq R$ and $x' \geq 0$ on $[0,T]$. Moreover, any such solution satisfies*

$$x(t) \leq R_*, \quad 0 \leq t \leq T, \tag{3.2}$$

where $R_* < R$ is so that

$$\int_0^T \alpha(t)\,dt = \int_a^{R_*} \frac{du}{\beta(u)}. \tag{3.3}$$

Proof. With the notations of Section 1, the mapping $A : K_R \to K$ is the sum $A_0 + A_1$, where A_0 is completely continuous and A_1 is a contraction with respect to a suitable norm on $C^1[0,T]$.

We claim that (3.2) holds for each solution $x \in K_R$ to

$$x = (1-\lambda)a + \lambda A(x) \quad (\lambda \in [0,1]). \tag{3.4}$$

Once the claim is satisfied the result follows from the Leray-Schauder principle applied to $A : \overline{U} \to K$, where $U = \{x \in K;\ x < R' \text{ on } [0,T]\}$ and R' is any number such that $R_* < R' \leq R$.

To prove the claim, let $x \in K_R$ be any solution of (3.4). Then

$$x'(t) = \lambda f(t, x(t), x(\theta(t)), x'(\theta(t))) \leq \alpha(t)\beta(x(t)) \quad \text{on } [0,T].$$

It follows that

$$\int_a^{x(t)} \frac{du}{\beta(u)} = \int_0^t \frac{x'(s)}{\beta(x(s))}ds \leq \int_0^t \alpha(s)\,ds.$$

This together with (3.3) implies (3.2). □

Suppose now that instead of (A4), the following condition holds.

(A4') $|f_0(t,x,y)| \leq \alpha_0 x + \beta_0 |y| + \delta$ on D, where α_0, β_0 and δ are nonnegative constants.

Theorem 3.2. *Suppose (A1)-(A3), (A4') are satisfied and* $R=\infty$. *Then (1.1)-(1.2) has at least one solution* $x \in C^1[-\tau,T]$ *such that* $a \leq x$ *and* $x' \geq 0$ *on* $[0,T]$.

Proof. Since $R=\infty$, we may define $A: K \to K$ and, as above, $A = A_0 + A_1$, where A_0 is completely continuous and A_1 is a contraction with respect to the norm $||.||_{1,\eta}$ on $C^1[0,T]$, for $\eta > \max\{(\alpha+\beta)/(1-\gamma), \alpha+\beta+\gamma\}$.

We claim that there exists η sufficiently large and a finite $R'>0$ such that

$$x, y \in K, \ ||x||_{1,\eta} \leq R', \ ||y||_{1,\eta} \leq R' \text{ imply } ||A_0(x) + A_1(y)||_{1,\eta} \leq R'. \tag{3.5}$$

Once the claim is proved the result is a consequence of the Krasnoselskii fixed point theorem.

To establish (3.5) we need the following estimates:

$$|A_0(x)(t)| \leq a + \alpha_0 \int_0^t x(s)\,ds + \beta_0 \int_0^t \tilde{x}(\theta(s))\,ds + \delta t$$

$$= a + \alpha_0 \int_0^t x(s)\,ds + \beta_0 \int_{(0<\theta(s))} x(\theta(s))\,ds + \beta_0 \int_{(\theta(s)<0)} \phi(\theta(s))\,ds + \delta t$$

$$\leq a + (\alpha_0 + \beta_0)\eta^{-1} ||x||_{0,\eta} \exp(\eta t) + \beta_0 ||\phi||_0 T + \delta T$$

$$= (\alpha_0 + \beta_0)\eta^{-1} ||x||_{0,\eta} \exp(\eta t) + c_0'.$$

Also

$$\left|A_0(x)'(t)\right| \leq \alpha_0 x(t) + \beta_0 \tilde{x}(\theta(t)) + \delta$$

$$= \alpha_0 \int_0^t x'(s)\,ds + \beta_0 \int_0^{\theta(t)} \tilde{x}'(s)\,ds + (\alpha_0 + \beta_0)a + \delta$$

$$\leq (\alpha_0 + \beta_0)\eta^{-1} ||x'||_{0,\eta} \exp(\eta t) + \beta_0 ||\phi'||_0 T + (\alpha_0 + \beta_0)a + \delta$$

$$= (\alpha_0 + \beta_0)\eta^{-1} ||x'||_{0,\eta} \exp(\eta t) + c_0''.$$

Thus

$$\|A_0(x)\|_{1,\eta} \leq (\alpha_0 + \beta_0)\,\eta^{-1}\,\|x\|_{1,\eta} + c_0.$$

This together with (1.7) yields

$$\|A_0(x) + A_1(y)\|_{1,\eta} \leq (\alpha_0 + \beta_0)\,\eta^{-1}\,\|x\|_{1,\eta} + L\,\|y\|_{1,\eta} + c,$$

where L is given by (1.8). It is clear that if η is sufficiently large, then $(\alpha_0 + \beta_0)\,\eta^{-1} + L < 1$ and we may find $R' > 0$ such that (3.5) holds. □

4. Minimal and maximal solutions

Theorem 4.1. *Suppose (a1)-(a3) are satisfied and $w \in C^1[0,T]$, $a \leq w \leq R$, is an upper solution, i.e.*

$$w'(t) \geq f\left(t, w(t), \tilde{w}(\theta(t)), \tilde{w}'(\theta(t))\right), \quad 0 \leq t \leq T. \tag{4.1}$$

In addition assume that

$$f(t, x_1, y_1, z_1) \leq f(t, x_2, y_2, z_2) \tag{4.2}$$

for $x_1 \leq x_2 \leq w(t)$, $y_1 \leq y_2 \leq \tilde{w}(\theta(t))$ and $z_1 \leq z_2 \leq \tilde{w}'(\theta(t))$. Then we may define $\underline{x}_0 = \bar{x}_0 = \phi$,

$$\underline{x}_{n+1}(t) = \begin{cases} \underline{x}_n(t) & \text{on } [-\tau, t_n] \\ \lim\limits_{j\to\infty} u_{nj}(t) & \text{on } [t_n, t_{n+1}] \end{cases} \tag{4.3}$$

and

$$\bar{x}_{n+1}(t) = \begin{cases} \bar{x}_n(t) & \text{on } [-\tau, t_n] \\ \lim\limits_{j\to\infty} v_{nj}(t) & \text{on } [t_n, t_{n+1}], \end{cases} \tag{4.4}$$

where $u_{n0}(t) \equiv a$, $v_{n0}(t) = w(t)$, $u_{nj} = \underline{A}_n(u_{nj-1})$, $v_{nj} = \bar{A}_n(v_{nj-1})$,

$$\underline{A}_n(x)(t) = \underline{x}_n(t_n) + \int_{t_n}^{t} f\left(s, x(s), \underline{x}_n(\theta(s)), \underline{x}_n'(\theta(s))\right) ds,$$

$$\bar{A}_n(x)(t) = \bar{x}_n(t_n) + \int_{t_n}^{t} f\left(s, x(s), \bar{x}_n(\theta(s)), \bar{x}_n(\theta(s))\right) ds$$

$(t \in [t_n, t_{n+1}])$, $j = 1, 2, ..., n = 0, 1, ..., k-1$. *Moreover,* $\underline{x} = \underline{x}_k$ *and* $\bar{x} = \bar{x}_k$ *are the minimal and maximal solutions of (1.1)-(1.2) satisfying* $a \leq x \leq w$ *on* $[0, T]$,

$$a \leq \underline{x} \leq \bar{x} \leq w, \quad 0 \leq \underline{x}' \leq \bar{x}' \leq w' \quad \text{on } [0, T],$$

$$u_{n0} \leq u_{n1} \leq ... \leq u_{nj} \leq ... \quad \text{on } [t_n, t_{n+1}]$$

$$v_{n0} \geq v_{n1} \geq ... \geq v_{nj} \geq ... \quad \text{on } [t_n, t_{n+1}]$$

and

$$u_{nj}(t) \to \underline{x}(t), \quad v_{nj}(t) \to \bar{x}(t) \quad \text{as } j \to \infty$$

uniformly on $[t_n, t_{n+1}]$ $(n = 0, 1, ..., k-1)$.

Proof. Suppose we have already defined $\underline{x}_n$ and $\bar{x}_n$ such that

$$a \leq \underline{x}_n \leq \bar{x}_n \leq w \text{ and } 0 \leq \underline{x}_n' \leq \bar{x}_n' \leq w' \text{ on } [0, t_n]. \tag{4.5}$$

First we prove that

$$a \leq u_{nj} \leq v_{nj} \leq w \quad \text{on } [t_n, t_{n+1}], \tag{4.6}$$

by induction after j. For $j = 0$, (4.6) trivially holds. Assume (4.6) is true for some j. Then, using also (4.2), we easily see that

$$a \leq \underline{A}_n(a) \leq \underline{A}_n(u_{nj}) \leq \underline{A}_n(v_{nj}) \leq \bar{A}_n(v_{nj}) \leq \bar{A}_n(w) \leq w,$$

which shows that (4.6) also holds for $j+1$. Thus (4.6) is true for all $j \geq 0$. Since $\underline{A}_n$ and $\bar{A}_n$ are completely continuous, the sequences $(u_{nj})_{j \geq 0}$ and $(v_{nj})_{j \geq 0}$ will contain convergent subsequences. Due to their monotonicity, the entire sequences will converge on $[t_n, t_{n+1}]$, which justifies (4.3) and (4.4). Also, by (4.6), $a \leq \underline{x}_{n+1} \leq \bar{x}_{n+1} \leq w$ on $[0, t_{n+1}]$. Then

$$0 \leq \underline{x}_{n+1}'(t) = f(t, \underline{x}_{n+1}(t), \underline{x}_n(\theta(t)), \underline{x}_n'(\theta(t)))$$

$$\leq f(t, \bar{x}_{n+1}(t), \bar{x}_n(\theta(t)), \bar{x}_n'(\theta(t))) = \bar{x}_{n+1}'(t)$$

$$\leq f(t, w(t), \widetilde{w}(\theta(t)), \widetilde{w}'(\theta(t))) \leq w'(t).$$

Hence (4.5) also holds for $j+1$. □

The next result is about the equality $\underline{x} = \bar{x}$ in Theorem 3.1.

Theorem 4.2. *Suppose the assumptions of Theorem 3.1 are satisfied. In addition assume $a > 0$ and that there exists a function $\chi : [a_w, 1) \to \mathbf{R}$, where $a_w = a \min_{[0,T]} 1/w(t)$, such that for all $\rho \in [a_w, 1)$, $t \in [0,T]$, $x \in [a, w(t)]$, $y \in [m, M]$ and $z \in \{m', \infty)$, one has*

$$1 \geq \chi(\rho) > \rho \quad \text{and} \quad f(t, \rho x, y, z) \geq \chi(\rho) f(t, x, y, z). \tag{4.7}$$

Then $\underline{x} = \bar{x}$ is the unique solution of (1.1)-(1.2) satisfying $a \leq x \leq w$ on $[0,T]$.

Proof. We show succesively that $\underline{x}_n = \bar{x}_n$ for $n = 0, 1, ..., k$. For $n = 0$, this trivially holds. Assume $\underline{x}_n = \bar{x}_n$ for some n. Then $\underline{A}_n = \bar{A}_n$. Clearly, the restrictions of $\underline{x}_{n+1}$, $\bar{x}_{n+1}$to $[t_n, t_{n+1}]$ represent the minimal and maximal fixed point of $B_n := \bar{A}_n$ satisfying $a \leq x \leq w$ on $[t_n, t_{n+1}]$. To show that $\underline{x}_{n+1} = \bar{x}_{n+1}$ on $[t_n, t_{n+1}]$, let $\rho_0 = \min_{[t_n, t_{n+1}]}(\underline{x}_{n+1}(t) / \bar{x}_{n+1}(t))$. We have $\rho_0 \in [a_w, 1]$. We claim that $\rho_0 = 1$. Assume $\rho_0 < 1$. Since $\underline{x}_{n+1}(t) \geq \max\{a, \rho_0 \bar{x}_{n+1}(t)\} = \rho_0 \max\{a/\rho_0, \bar{x}_{n+1}(t)\} \geq a$ on $[t_n, t_{n+1}]$, we get

$$\underline{x}_{n+1} = B_n(\underline{x}_{n+1}) \geq B_n(\rho_0 \max\{a/\rho_0, \bar{x}_{n+1}\})$$

$$\geq \chi(\rho_0) B_n(\max\{a/\rho_0, \bar{x}_{n+1}\}) \geq \chi(\rho_0) B_n(\bar{x}_{n+1}) = \chi(\rho_0) \bar{x}_{n+1},$$

on $[t_n, t_{n+1}]$. It follows $\rho_0 \geq \chi(\rho_0)$, a contradiction. Therefore $\rho_0 = 1$ and so $\underline{x}_{n+1} = \bar{x}_{n+1}$ on $[t_n, t_{n+1}]$. □

Remark 4.1. *For example, we may take $\chi(\rho) = \rho^\alpha$, where $\alpha \in (0,1)$, in case that $f(t,x,y,z)$ is of the form $x^\alpha g(t,y,z)$. Also, $\chi(\rho) = \log(1 + a\rho) / \log(1+a)$ for $f(t,x,y,z) = g(t,y,z) \log(1+x)$.*

Theorem 4.3. *Suppose (A1)-(A3) are satisfied and that $w \in C^1[0,T]$, $a \leq w \leq R$, is an upper solution. In addition assume that (4.2) holds. Denote*

$$U_0(t) \equiv a, \; V_0(t) = w(t), \; U_{n+1} = A(U_n) \; \text{and} \; V_{n+1} = A(V_n)$$

($t \in [0,T]$), $n = 0, 1, \ldots$ Then

$$a = U_0 \le U_1 \le \ldots \le U_n \le \ldots \le V_n \le \ldots \le V_1 \le V_0 = w, \tag{4.8}$$

$$0 \le U_1' \le \ldots \le U_n' \le \ldots \le V_n' \le \ldots \le V_1' \le w' \tag{4.9}$$

on $[0,T]$. Also, the following limits exist

$$\underline{x}(t) = \lim_{n\to\infty} U_n(t), \quad \bar{x}(t) = \lim_{n\to\infty} V_n(t) \tag{4.10}$$

uniformly on $[0,T]$. Moreover, $\underline{x}$, $\bar{x}$ are the minimal and maximal solutions to (1.1)-(1.2) in K_R satisfying $x \le w$ on $[0,T]$.

Proof. From $a \le w$ we see that $a \le A(a) \le A(w) \le w$ and $0 \le A(a)' \le A(w)' \le w'$, i.e. $U_0 \le U_1 \le V_1 \le V_0$ and $0 \le U_1' \le V_1' \le w'$. Further, (4.8) and (4.9) follow succesively. Let α be the Kuratowski measure of noncompactness on the space $C^1[0,T]$ endowed with the norm $\|.\|_{1,\eta}$. Since

$$(U_n)_{n\ge1} = A\left((U_n)_{n\ge0}\right),$$

A_0 is completely continuous and A_1 is a contraction, we have

$$\alpha\left((U_n)_{n\ge1}\right) = \alpha\left(A\left((U_n)_{n\ge0}\right)\right) \le \alpha\left(A_0\left((U_n)_{n\ge0}\right)\right)$$

$$+\alpha\left(A_1\left((U_n)_{n\ge0}\right)\right) = \alpha\left(A_1\left((U_n)_{n\ge0}\right)\right) \le L\alpha\left((U_n)_{n\ge0}\right),$$

where L is given by (1.8). Recall that $L < 1$. In consequence, $\alpha\left((U_n)_{n\ge0}\right) = 0$. Thus $(U_n)_{n\ge0}$ contains a convergent subsequence. By the monotonicity, the entire sequence (U_n) will converge. Similarly, (V_n) is convergent. □

Remark 4.2. *Let (A1)-(A2) be satisfied. In addition assume that the following condition holds instead of (A3):*

(A3') $f(t,x,y,z)$ is nonnegative and continuous on D.

Then Theorem 3.2 is still true with the meaning that $\underline{x}$ and $\bar{x}$ are weak solutions of (1.1), i.e. $\underline{x}$, $\bar{x} \in AC[0,T]$ (are absolutely continuous) and satisfy (1.1) *almost everywhere on* $[0,T]$. *Indeed, by (4.8), (4.9) and the Beppo-Levi theorem, there exist* $\underline{x}, \underline{y} \in L^1(0,T)$ *with*

$$U_n(t) \to \underline{x}(t), \quad U_n'(t) \to \underline{y}(t) \quad \text{on } [0,T],$$

$$U_n \to \underline{x} \quad \text{and} \quad U_n' \to \underline{y} \quad \text{in } L^1(0,T).$$

From

$$U_n(t) = a + \int_0^t U_n'(s)\,ds$$

we then derive

$$\underline{x}(t) = a + \int_0^t \underline{y}(s)\,ds,$$

which shows that $\underline{x} \in AC[0,T]$ *and* $\underline{x}'(t) = \underline{y}(t)$ *for a.e.* $t \in [0,T]$. *Letting* $n \to \infty$ *in*

$$U_{n+1}'(t) = f\left(t, U_n(t), \widetilde{U}_n(\theta(t)), \widetilde{U}_n'(\theta(t))\right),$$

we obtain

$$\underline{y}(t) = f\left(t, \underline{x}(t), \widetilde{\underline{x}}(\theta(t)), \widetilde{\underline{y}}(\theta(t))\right) \quad \text{for all } t \in [0,T],$$

i.e.

$$\underline{x}'(t) = f\left(t, \underline{x}(t), \widetilde{\underline{x}}(\theta(t)), \widetilde{\underline{x}}'(\theta(t))\right) \quad \text{a.e. } t \in [0,T].$$

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DEPARTMENT OF APPLIED MATHEMATICS, FACULTY OF MATHEMATICS AND INFORMATICS, UNIVERSITY "BABEŞ-BOLYAI", 3400 CLUJ, ROMANIA
E-mail address: r.precup@math.ubbcluj.ro

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ON THE UNIVALENCE OF FUNCTIONS RELATED TO HYPERGEOMETRIC FUNCTIONS

IRINEL RADOMIR

Abstract. In lucrare se studiază univalenţa unei clase de funcţii exprimată prin intermediul funcţiei hipergeometrice.

1. Introduction

Let A be the class of function f which are analytic in the unit disk $U = \{ z \in C : |z| < 1 \}$ with $f(0) = 0$ and $f'(0) = 1$. In this note we improve the result from [2] using another univalence criterion.

2. Preliminaries

Theorem 2.1. ([2]). *Let $f \in A$ and let α be a complex number, $Re\alpha > 0$. If*

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \qquad (\forall)\ z \in U \tag{1}$$

then for all complex numbers β with $Re\beta \geq Re\alpha$, the function

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u)du\right]^{1/\beta} \tag{2}$$

is analytic and univalent in U.

3. Main results

It is easy to prove the following:

Lemma 3.1. *Let α, γ be complex numbers and let the function*

$$E(\alpha,\ \gamma, z, \bar{z}) = \frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\gamma\frac{z}{1-z}\right|, \qquad |z| < 1. \tag{3}$$

If $0 < Re\alpha < 1$, *then*

$$E(\alpha,\ \gamma, z, \bar{z}) \leq \frac{|\gamma|}{Re\alpha}, \qquad (\forall) z \in U. \tag{4}$$

If $Re\alpha \geq 1$, *then*

$$E(\alpha,\ \gamma, z, \bar{z}) \leq 2|\gamma|, \qquad ,(\forall) z \in U. \tag{5}$$

Theorem 3.1. *Let* α, β, γ *be complex numbers. If*

$$|\gamma| \leq Re\alpha < 1\ , \tag{6}$$

$$|\gamma| \leq \frac{1}{2} \ \ and \ \ Re\alpha \geq 1\ , \tag{7}$$

$$Re\beta \geq Re\alpha\ , \tag{8}$$

then the function

$$F_\beta(z) = z \cdot [\ F(\ \beta,\ \gamma,\ \beta + 1,\ z\ )]^{1/\beta} \tag{9}$$

is analytic and univalent in U, *where by* $F(a, b, c, z)$ *we denoted the hypergeometric function.*

Proof. If in (2) we make the change $u = tz$, we obtain

$$F_\beta(z) = z \cdot \left[\beta \int_0^1 t^{\beta - 1} f'(tz) dt\right]^{1/\beta}. \tag{10}$$

In the following we consider the function

$$f(z) = \int_0^z (1 - u)^{-\gamma} du \tag{11}$$

For this function we obtain

$$\frac{z f''(z)}{f'(z)} = \gamma \frac{z}{1 - z} \ and$$

$$\frac{1 - |z|^{2Re\alpha}}{Re\alpha} \left|\frac{z f''(z)}{f'(z)}\right| = \frac{1 - |z|^{2Re\alpha}}{Re\alpha} \left|\gamma \frac{z}{1 - z}\right|.$$

According to Lemma 3.1 we deduce that the condition (1) from Theorem 2.1 , for the function (11) is verified in the cases

$$(i) \qquad |\gamma| \leq Re\alpha < 1\ ;$$

$$(ii) \qquad |\gamma| \leq \frac{1}{2} \ and \ Re\alpha \geq 1 .$$

Replacing in (10) the function f defined by (11) we obtain

$$F_\beta(z) = z \cdot \left[\beta \int_0^1 t^{\beta-1}(1-tz)^{-\gamma}dt\right]^{1/\beta} =$$

$$= z \cdot [\, F(\,\beta,\ \gamma,\ \beta+1,\ z\,)\,]^{1/\beta} .$$

where by $F(a,b,c,z)$ we noted the hypergeometric function.

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DEPARTMENT OF MATHEMATICS, "TRANSILVANIA" UNIVERSITY, BRAŞOV, ROMANIA

STUDIA UNIV. "BABEŞ-BOLYAI", MATHEMATICA, Volume XLIV, Number 3, September 1999

FIBER PICARD OPERATORS THEOREM AND APPLICATIONS

IOAN A. RUS

Abstract. In this paper we study the following problem: Let (X_k, d_k), $k = \overline{0,p}$, $p \geq 1$, be metric spaces and $A_k : X_0 \times \cdots \times X_k \to X_k$, $k = \overline{0,p}$ be operators. We suppose that

(a) the operators A_k are continuous, $k = \overline{0,p}$;

(b) the operators $A_0, A_k(x_0, \ldots, x_{k-1}, \cdot)$, $k = \overline{1,p}$ are (weakly) Picard operators.

Establish conditions which imply that the operator

$$B_p : X_0 \times \cdots \times X_p \to X_0 \times \cdots \times X_p$$

$$B_p(x_0, \ldots, x_p) := (A_0(x_0), A_1(x_0, x_1), \ldots, A_p(x_0, \ldots, x_p)),$$

is a (weakly) Picard operator.

1. Introduction

Let (X, d) be a metric space and $A : X \to X$ an operator. In this paper we shall use the following notations:

$P(X) := \{Y \subset X \mid Y \neq \emptyset\}$,

$F_A := \{x \in X \mid A(x) = x\}$ - the fixed point set of A,

$I(A) := \{Y \in P(X) \mid A(Y) \subset Y\}$.

Definition 1.1 (Rus [9], [11]). An operator $A : X \to X$ is weakly Picard operator (WPO) if the sequence

$$(A^n(x))_{n \in N}$$

converges, for all $x \in X$, and the limit (which may depend on x) is a fixed point of A.

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Definition 1.2 (Rus [9], [11]). If A is WPO, then we consider the operator A^{∞} defined by

$$A^{\infty} : X \to X, \quad A^{\infty}(x) := \lim_{n\to\infty} A^n(x).$$

We remark that, $A^{\infty}(X) = F_A$.

Definition 1.3 (Rus [9], [11]). If A is WPO and $F_A = \{x^*\}$, then by definition the operator A is a Picard operator.

Example 1.1. Let (X,d) be a complete metric space and $A : X \to X$ such that

$$d(A^2(x), A(x)) \leq ad(x, A(x))$$

for all $x \in X$ and for some $a \in]0,1[$. Then A is weakly Picard operator (see [8], [9], [11]).

Example 1.2. Let (X,d) be a complete metric space and $A, B : X \to X$ such that

$$d(A(x), B(y)) \leq a[d(x, A(x)) + d(y, B(y))]$$

for all $x, y \in X$ for some $a \in \left]0, \frac{1}{2}\right[$. Then A and B are Picard operators.

Example 1.3. $X = C[0,1]$, $d(x,y) = \|x-y\|_C$,

$$A(x)(t) = x(0) + \int_0^t K(t,s)x(s)ds, \quad t \in [0,1]$$

where $K \in C([0,1]\times[0,1])$. Then A is WPO.

For other examples see [13], [10], [1], [2], [20],...

We have the following characterization theorem for WPO.

Theorem 1.1. *Let (X,d) be a metric space and $A : X \to X$ an WPO. Then there exist $X_i \in I(A)$, $i \in I$, such that*

(i) $X = \bigcup_{i\in I} X_i$, $X_i \cap X_j = \emptyset$, $i \neq j$,

(ii) $A\big|_{X_i}$ *is a Picard operator,* $i \in I$.

Proof. Let $x \in F_A$. Let X_x be the domain of attraction of x. It is clear that

$$X = \bigcup_{x\in F_A} X_x$$

is a partition of X and that $X_x \in I(A)$. By the definition of X_x, we have that

$$F_A \cap X_x = \{x\}.$$

In this paper we study the following problem:

Problem 1.1. Let (X, d) and (Y, ρ) be the metric spaces and $A = (B, C) : X \times Y \to X \times Y$ a triangular operator, i.e.

$$A(x, y) = (B(x), C(x, y)), \quad x \in X, \quad y \in Y.$$

We suppose that the operators $B : X \to X$, $C(x, \cdot) : Y \to Y$, $x \in X$, are Picard operators. Establish conditions which imply that the operator A is Picard operator.

If the operators, $B : X \to X$, $C(x, \cdot) : Y \to Y$, $x \in X$, are WPO, establish conditions which imply that the operator A is WPO.

2. Fiber Picard operators theorem

The following result is given by M.W. Hirsch and C.C. Pugh ([5], 1970):

Theorem 2.1 (Fiber contraction theorem). *Let (X, d) be a metric space and $B : X \to X$ be an operator having an atractive fixed point $p \in X$. Let (Y, ρ) be a metric space and $C : X \times Y \to Y$ an operator such that*

(i) there exists $\lambda \in [0, 1[$, such that the operator $C(x, \cdot)$ is a λ-contraction for all $x \in X$;

(ii) the operator $A : X \times Y \to X \times Y$, $A(x, y) := (B(x), C(x, y))$ is continuous.

Let $q \in Y$ be a fixed point for $C(p, \cdot)$.

Then (p, q) is an attractive fixed point for A.

For some generalization of this theorem see [10]-[15], [18] and [19].

We have

Theorem 2.2. *Let (X, d) and (Y, ρ) be two metric space and $A = (B, C)$ a triangular operator. We suppose that*

(i) (Y, ρ) is a complete metric space;

(ii) the operator $B : X \to X$ is WPO;

(iii) there exists $\alpha \in [0, 1[$, such that $C(x, \cdot)$ is an α-contraction, for all $x \in X$;

(iv) if $(x^*, y^*) \in F_A$, then $C(\cdot, y^*)$ is continuous in x^*.

Then the operator A is WPO.

If B is Picard operator, then A is Picard operator.

Proof. Let $(x,y) \in X \times Y$. Let y^* the unique fixed point of $C(B^\infty(x),\cdot)$. We shall prove that $A^n(x,y) \to (B^\infty(x), y^*)$ as $n \to \infty$. Let $A^n(x,y) = (x_n, y_n)$. Then

$$x_n = B^n(x), \quad y_n = C(x_{n-1}, y_{n-1}).$$

The proof that $y_n \to y^*$ as $n \to \infty$ is similarly with the proof given in [5] for the Theorem 1.

Remark 2.1. The proof that $y_n \to y^*$ as $n \to \infty$ follows, also, from the following

Lemma 2.1 (see [13]). *Let (X,d) be a complete metric space and A_n, $A : X \to X$, $n \in N$, some operators. We suppose that*

(a) the sequence $(A_n)_{n\in N}$ pointiwse converges to A;

(b) there exist $\alpha \in [0,1[$ such that the operators A_n and A, $n \in N$, are α-contractions.

Then the sequence $(A_n \circ A_{n-1} \circ \cdots \circ A_0)_{n\in N}$ poinwise converges to A^∞.

Remark 2.2. In the proof of Lemma 2.2 on uses the following

Lemma 2.2 (see [13], [14] and [15]). *Let $a_n, b_n \in R_+$, $n \in N$. We suppose that*

(a) $a_n \to 0$ as $n \to \infty$;

(b) $\sum_{k=0}^{\infty} b_k < +\infty$.

Then

$$\sum_{k=0}^{n} a_k b_{n-k} \to 0 \text{ as } n \to \infty.$$

Remark 2.3. For to have a generalization of the Theorem 2.2, we need suitable generalization for Lemma 2.1 and Lemma 2.2. For some generalization of these Lemmas, see [15] and [19].

Remark 2.4. By induction, from the Theorem 2.2 we have

Theorem 2.3 (see [13]). *Let (X_k, d_k), $k = \overline{0,p}$, $p \geq 1$, be some metric spaces. Let*

$$A_k : X_0 \times \cdots \times X_k \to X_k, \quad k = \overline{0,p}$$

be some operators. We suppose that:

(a) the spaces (X_k, d_k), $k = \overline{1,p}$ are complete metric spaces;

(b) the operator A_0 is WPO;

(c) there exist $\alpha_k \in [0,1[$ such that the operators $A_k(x_0,\dots,x_{k-1},\cdot)$ are α_k-contractions;

*(d) if $(x_0^*,\dots,x_p^*) \in F_{B_p}$, $B_p = (A_0,\dots,A_p)$, then the operators $A_k(\cdot,\dots,\cdot,x_k^*)$,
$k=\overline{1,p}$, are continuous in $(x_0^*,x_1^*,\dots,x_{k-1}^*)$.*

Then the operator B_p is WPO.

Remark 2.5. The next conjecture is in connection with our results.

Discrete Markus-Yamabe Conjecture (see [3], [6], [1]). Let A be a C^1 function from R^n into itself such that $A(0)=0$ and for any $x \in R^n$, $JA(x)$ (the Jacobian matrix of A at x) has all its eigenvalues with modulus less than one. Then A is a Picard function.

From the fiber Picard operators theorem we have

Theorem 2.4. *Let $A : R^n \to R^n$ be a C^1 triangular function, $A=(A_1,\dots,A_n)$.*

If there exists $\alpha \in]0,1[$ such that

$$\left|\frac{\partial A_i}{\partial x_i}\right| \le \alpha, \quad i=\overline{1,n}.$$

Then the function A is Picard function.

A. Cima, A. Gasull, F. Mañosas prove that the Discrete Markus-Yamabe Conjecture ([3], 1999) is a theorem for A provided

$$\left|\frac{\partial A_i}{\partial x_j}\right| < 1, \quad j=\overline{1,i}, \quad i=\overline{1,n}.$$

3. Applications

The fiber Picard operators theorem is very useful for proving solutions of operatorial equations to be differentiable with respect to parameters (see [17], [12], [13], [14], [15], [20], [18]). For example:

- (J. Sotomayor) differentiability with respect to initial data for the solution of differential equations

$$x\prime = f(t,x), \quad x(t_0)=x_0, \quad f:\Omega \to R^n, \quad \Omega \subset R^{n+1};$$

- (I.A. Rus [12]) differentiability with respect to λ for the solution of the integral equation

$$x(t) = 1 + \lambda \int_t^1 x(s)x(s-t)ds, \quad t \in [0,1],$$

where $\lambda \in R$;

- (A. Tămăşan) differentiability with respect to lag function for pantograph equation

$$x\prime(t) = f(t), x(t), x(\lambda t)), \quad t > 0; \quad 0 < \lambda < 1, \quad x(0) = 0.$$

In what follow we apply the fiber Picard operators theorem to study the following integral equations modelling population growth in a periodic environment (see [10], [7])

$$x(t) = \int_{t-\tau}^{t} f(s, x(s); \lambda)ds \tag{1}$$

where $f \in C(R \times [\alpha, \beta] \times J, [m, M]$, with $\tau, \alpha, \beta, m, M \in R_+^*$ and $J \subset R$ a compact interval.

Let

$$X_\omega := \{x \in C(R \times J, [\alpha, \beta])|\ x(t+\omega, \lambda) = x(t, \lambda),$$

$$\text{for all } t \in R,\ \lambda \in J\},\ \omega > 0.$$

We consider on X_ω the metric $d(x,y) := ||x-y||_C$. We have

Theorem 3.1. *We suppose that*

(a) $0 < m < M$, $0 < \alpha < \beta$; $\alpha \leq m\tau$, $\beta \geq M\tau$;

(b) $m \leq f(t, u; \lambda) \leq M$, *for* $t \in R$, $u \in [\alpha, \beta]$, $\lambda \in J$;

(c) $f(t+\omega, u; \lambda) = f(t, u; \lambda)$, $t \in R$, $u \in [\alpha, \beta]$, $\lambda \in J$;

(d) there exists $l(t)$, *such that*

$$|f(t, u; \lambda) - f(t, v; \lambda)| \leq l(t)|u - v|$$

for all $t \in R$, $u, v \in [\alpha, \beta]$;

(e) there exists $q \in]0,1[$ *such that*

$$\int_{t-\tau}^{t} l(s)ds \leq q, \text{ for all } t \in R.$$

Then

(i) the equation (1) has in X_ω a unique solution x^*;

(ii) for all $x_0 \in X_\omega$, the sequence defined by

$$x_{n+1}(t,\lambda) = \int_{t-\tau}^{t} f(s, x_n(s,\lambda))ds$$

converges uniformly to x^*;

(iii) if $f(t,\cdot,\cdot) \in C^1$, then $x^*(t,\cdot) \in C^1(J)$.

Proof. (i)+(ii). We consider the operator

$$B : X_\omega \to C(R \times J), \quad B(x)(t,\lambda) := \int_{t-\tau}^{t} f(s, x(s,\lambda))ds.$$

From (a) and (c) we have that $X_\omega \in I(B)$. From (d) it follows that B is a contraction.

By the contraction principle we have that B is a Picard operator.

(iii). Let we prove that there exists $\frac{\partial x^*}{\partial \lambda}$ and $\frac{\partial x^*}{\partial \lambda} \in C(R \times J)$.

If we suppose that there exists $\frac{\partial x^*}{\partial \lambda}$, then from

$$x(t,\lambda) = \int_{t-\tau}^{t} f(s, x(s,\lambda);\lambda)ds$$

we have

$$\frac{\partial x(t,\lambda)}{\partial \lambda} = \int_{t-\tau}^{t} \frac{\partial f(s,x(s,\lambda);\lambda)}{\partial x} \cdot \frac{\partial x(s,\lambda)}{\partial \lambda} ds + \int_{t-\tau}^{t} \frac{\partial f(s,x(s,\lambda);\lambda)}{\partial \lambda} ds.$$

This relation suggest us to consider the following operator

$$A : X_\omega \times Y_\omega \to X_\omega \times Y_\omega$$

defined by

$$A = (B, C), \quad A(x,y) = (B(x), C(x,y)),$$

where

$$C(x,y)(t,\lambda) := \int_{t-\tau}^{t} \frac{\partial f(s,x(s,\lambda);\lambda)}{\partial x} y(s,\lambda)ds + \int_{t-\tau}^{t} \frac{\partial f(s,x(s,\lambda);\lambda)}{\partial \lambda} ds$$

and $Y_\omega := \{y \in C(R \times J) |\ y(t+\omega,\lambda) = y(t,\lambda),\ t \in R,\ \lambda \in J\}$.

Now we are in the condition of the fiber Picard operators theorem. From this theorem, the operator A is a Picard operator and the sequences

$$x_{n+1} = B(x_n)$$

and

$$y_{n+1} = C(x_n, y_n)$$

converge uniformly to $(x^*, y^*) \in F_A$, for all $x_0 \in X_\omega$, $y_0 \in Y_\omega$.

If we take $x_0 \in X_\omega$, $y_0 \in Y_\omega$ such that $y_0 = \frac{\partial x_0}{\partial \lambda}$, then we have that $y_n = \frac{\partial x_n}{\partial \lambda}$, for all $n \in N$.

So

$$x_n \xrightarrow{unif.} x^* \text{ as } n \to \infty,$$

$$\frac{\partial x_n}{\partial \lambda} \xrightarrow{unif.} y^* \text{ as } n \to \infty.$$

Using a Weierstrass argument we conclude that x^* is differentiable and $y^* = \frac{\partial x^*}{\partial \lambda}$.

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Department of Applied Mathematics, Faculty of Mathematics and Informatics, University "Babeş-Bolyai", 3400 Cluj, Romania

FIBER φ–CONTRACTIONS

MARCEL-ADRIAN ŞERBAN

Abstract. This paper containes some conditions for proving that if an operator is fiber Picard operator then this operator is Picard operator. The result obtained is used for proving the differentiability with respect some parameters.

1. Introduction

Let X be a nonempty set and $A : X \to X$ an operator. We note by:

$$P(X) := \{Y \subset X \,|Y \neq \emptyset\}$$

$$F_A := \{x \in X \,|A(x) = x\} \quad - \quad \text{the fixed point set of A.}$$

Definition 1.1. (I.A. Rus [6]). Let (X,d) be a metric space. An operator $A : X \to X$ is (uniformly) Picard operator if there exists $x^* \in X$ such that:

(a) $F_A = \{x^*\}$,

(b) the sequence $(A^n(x))_{n\in N}$ converges (uniformly) to x^*, for all $x \in X$.

Definition 1.2. (I.A. Rus [6]). Let (X,d) be a metric space. An operator $A : X \to X$ is (uniformly) weakly Picard operator if:

(a) the sequence $(A^n(x))_{n\in N}$ converges (uniformly), for all $x \in X$,

(b) the limit (which may depend on x) is a fixed point of A.

If A is weakly Picard operator then we consider the following operator:

$$A^\infty : X \to X,$$

$$A^\infty(x) = \lim_{n\to\infty} A^n(x).$$

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In this paper we consider the following class of operators:

$$A : X_1 \times ... \times X_p \to X_1 \times ... \times X_p$$

$$(x_1, ..., x_p) \mapsto (A_1(x_1), A_2(x_1, x_2), ..., A_p(x_1, ..., x_p)),$$

where (X_1, d_i), $i = \overline{1,p}$, are metrical spaces and $A_k : X_1 \times ... \times X_k \to X_k$, $k = \overline{1,p}$, are such that the operators

$$A_k(x_1, ..., x_{k-1}, \cdot) : X_k \to X_k$$

are weakly Picard operators, for all $x_i \in X_i$, $i = \overline{1,k}$, $k = \overline{1,p}$.

The aim of this paper is to give an answer of Problem 4.2 from I. A. Rus [5]. We replace the condition that $A_k(x_1, ..., x_{k-1}, \cdot)$ is $\alpha-$contraction with $A_k(x_1, ..., x_{k-1}, \cdot)$ is φ_k-contraction and we give the conditions for φ_k to obtain that operator Λ is a Picard operator.

2. Comparison functions and (c)-comparison function

Definition 2.1.(I.A. Rus [5]). A function $\varphi : \Re_+ \to \Re_+$ is called comparison function if:

(a) φ is monotone increasing: $t_1 \leq t_1 \Longrightarrow \varphi(t_1) \leq \varphi(t_2)$, $\qquad t_1, t_1 \in \Re_+$.

(b) $(\varphi^n(t))_{n \in N}$ converges to 0, as $n \to \infty$, for each t.

We are interested in finding that comparison functions which satisfies the condition:

$$\sum_{k=0}^{\infty} \varphi^k(t) < \infty. \tag{1}$$

V. Berinde in [2] gave a necessary and sufficient result for the convergence of the series of decreasing positive terms.

Theorem 2.1.(V. Berinde [2]). *A series* $\sum\limits_{k=0}^{\infty} u_k$ *of decreasing positive terms converges if and only if there exists a convergent series of nonnegative terms* $\sum\limits_{k=0}^{\infty} v_k$ *such that:*

$$\frac{u_{n+1}}{u_n + v_n} \leq \alpha < 1, for \quad n \geq n_0, \tag{2}$$

is satisfied.

Using this result we obtain which comparison functions satisfy the condition (1).

Corollary 2.1.(V. Berinde [3]) *Let $\varphi : \Re_+ \to \Re_+$ be a comparison function. The series $\sum_{k=0}^{\infty} \varphi^k(t)$, $t \in \Re_+$, is convergent if and only if there exists a number α, $0 < \alpha < 1$, and there exists a convergent series of nonnegative terms $\sum_{i=0}^{\infty} v_k$ such that:*

$$\frac{\varphi^{k+1}(t)}{\varphi^k(t) + v_k} \leq \alpha \leq 1, \quad for \quad k \geq k_0. \tag{3}$$

Remark 2.1. If $\sum_{i=0}^{\infty} v_i$ is a convergent series of nonnegative terms and $0 < \alpha$ then also $\sum_{i=0}^{\infty} \alpha v_i$ is a convergent series of nonnegative terms, so we can write the condition (3) in equivalent form:

$$\varphi^{k+1}(t) \leq \alpha \varphi^k(t) + v'_k, \tag{4}$$

where $0 < \alpha < 1$, $\sum_{i=0}^{\infty} v'_k$ is a convergent series of nonnegative terms.

By Corollary 2.1 and Remark 2.1 we obtain a new class of comparison functions.

Definition 2.2.(V. Berinde [1], [2]) A function $\varphi : \Re_+ \to \Re_+$ is called (c)-comparison function if the following condition hold:

(a) φ is monotone increasing: $t_1 \leq t_1 \Longrightarrow \varphi(t_1) \leq \varphi(t_2)$, $\qquad t_1, t_1 \in \Re_+$.

(b) there exist two numbers k_0, α, $0 < \alpha < 1$, and a convergent series of nonnegative terms $\sum_{i=0}^{\infty} v_k$ such that:

$$\varphi^{k+1}(t) \leq \alpha \varphi^k(t) + v_k,$$

for each t and $k \geq k_0$.

Theorem 2.2. (V. Berinde [1], [3]) *If $\varphi : \Re_+ \to \Re_+$ is a* (c)-comparison function then:

(i) $\varphi(t) < t$, for each $t > 0$;

(ii) φ is continuous in 0;

(iii) the series $\sum_{k=0}^{\infty} \varphi^k(t)$ converges for each $t \in \Re_+$;

(iv) the sum of the series (1), s(t), is monotone increasing and continuous in 0;

(v) $(\varphi^n(t))_{n\in N}$ converges to 0, as $n\to\infty$, for each t.

Example 2.1. The function $\varphi:\Re_+\to\Re_+$, $\varphi(t)=\frac{t}{t+1}$ is a comparison function, but is not a (c)-comparison function.

Example 2.2. The function $\varphi:\Re_+\to\Re_+$, $\varphi(t)=\alpha t$, $0<\alpha<1$, is a (c)-comparison function.

Example 2.3. The function $\varphi:\Re_+\to\Re_+$,

$$\varphi(t)=\begin{cases} at & t\in[0;2a] \\ bt+c & t>2a \end{cases}$$, where $0\le a<1$, $a-\frac{c}{2a}\le b\le 1$ and $c<0$ is a comparison function.

Example 2.4. For the function from Example 2.3, if $a=\frac{1}{2}$, $b=1$, $c=-\frac{1}{3}$, we obtain a (c)-comparison function.

Definition 2.3. (I.A. Rus [8]) Let (X,d) be a metric space and $\varphi:\Re_+\to\Re_+$ is a comparison function. A mapping $f:X\to X$ is a $\varphi-$contraction if:

$$d(f(x),f(y))\le\varphi(d(x,y)),$$

for every $x,y\in X$.

3. Fiber Picard operators problem

We'll start with a result which generalize Lemma 3.2 from I.A. Rus [5].

Lemma 3.1. *Let $\alpha_n\in\Re_+$, $n\in N$, and $\varphi:\Re_+\to\Re_+$ such that:*

(i) $\alpha_n\to 0$ as $n\to\infty$;

(ii) φ is a (c)-comparison function.

Then $\sum\limits_{k=0}^{\infty}\varphi^{n-k}(\alpha_k)\to 0$ as $n\to\infty$.

Proof. We split the partial sum of the series in two parts:

$$s_n=\sum_{k=0}^{n}\varphi^{n-k}(\alpha_k)=\sum_{k=0}^{[\frac{n}{2}]}\varphi^{n-k}(\alpha_k)+\sum_{k=[\frac{n}{2}]+1}^{n}\varphi^{n-k}(\alpha_k).$$

For the first part of partial sum we have:

$$\sum_{k=0}^{[\frac{n}{2}]}\varphi^{n-k}(\alpha_k)\le\sum_{k=0}^{[\frac{n}{2}]}\varphi^{n-k}(\max_{n\in N}\alpha_k)\to 0$$

as $n \to \infty$, because of the fact that φ is a (c)-comparison function and the point (iii) from Theorem 2.2.

For the second part of the partial sum we have:

$$\sum_{k=\left[\frac{n}{2}\right]+1}^{n} \varphi^{n-k}(\alpha_k) \leq \sum_{k=\left[\frac{n}{2}\right]+1}^{n} \varphi^{n-k}(\max_{j\leq n} \alpha_j) \leq s(\max_{j\leq n} \alpha_j).$$

Using the continuity of s in 0, (Theorem 2.2, (iv)), and the fact that $\max_{j\leq n} \alpha_j \to 0$ as $n \to \infty$ we deduce that the second part also tends to 0 as $n \to \infty$. □

Considering the open problem 3.1 from I.A. Rus [5], we'll give the following result: **Lemma 3.2.** *Let (X,d) be a complete metric space, $\varphi : \Re_+ \to \Re_+$ a (c)-comparison function and A_n, $A : X \to X$, $n \in N$, operators such that:*

(i) φ is subadditive: $\varphi(t_1 + t_2) \leq \varphi(t_1) + \varphi(t_2)$, $\quad \forall t_1, t_2 \in \Re_+$;

(ii) the sequence $(A_n)_{n\in N}$ pointwise converges to A;

(iii) A_n and A, $n \in N$, are φ-contractions.

Then the sequence $(A_n \circ A_{n-1} \circ ... \circ A_0)_{n\in N}$ pointwise converges to A^∞.

Proof. From (iii) we deduce that there exists a unique $x^* \in F_A$, so $A^\infty(x) = x^*$, for all $x \in X$. Let $x \in X$. We have:

$$d((A_n \circ A_{n-1} \circ \cdots \circ A_0)(x), x^*) \leq$$

$$\leq d((A_n \circ A_{n-1} \circ \cdots \circ A_0)(x), (A_n \circ A_{n-1} \circ \cdots \circ A_0)(x^*)) +$$

$$+ d((A_n \circ A_{n-1} \circ \cdots \circ A_0)(x^*), A_n(x^*)) + d(A_n(x^*), x^*) \leq \cdots \leq$$

$$\leq \varphi^{n+1}(d(x, x^*)) + \varphi^n(d(A_0(x^*), x^*)) + \varphi^{n-1}(d(A_1(x^*), x^*)) + \cdots + d(A_n(x^*), x^*).$$

Let $\alpha_k := d(A_k(x^*), x^*)$. It is obvious that $\alpha_k \to 0$ as $k \to \infty$ and the proof of the theorem follows from Lemma 3.1. □

Lemma 3.3. *Let (X,d) and (Y,ρ) be two metric spaces, $x_n, x^* \in X$, $\varphi : \Re_+ \to \Re_+$ a (c)-comparison function and $f : X \times Y \to Y$ an operator such that:*

(i) $x_n \to x^*$ as $n \to \infty$;

(ii) φ is subadditive;

(iii) the operator $f(\cdot, y) : X \to X$ is continuous for all $y \in Y$;

(iv) $f(x, \cdot) : Y \to Y$ *is* $\varphi-$*contraction for all* $x \in X$;

(v) (Y,ρ) *is a complete metric space.*

Then the sequence defined by: $y_{n+1} = f(x_n, y_n)$, $y_1 = y$, $n \in N$ *converges to* y^*, *the unique fixed point of* $f(x^*, \cdot)$, *for all* $y \in Y$.

Proof. The proof is a simple application of Lemma 3.3 with $A_n : Y \to Y$, $A_n(y) = f(x_n, y)$, $A : Y \to Y$, $A(y) = f(x^*, y)$. □

The main result of this paper is related to the open problem 4.1, (I.A. Rus [5]). This result is an answer of open problem 4.2, (I.A. Rus [5]), which generalize the Theorem 4.1.

Theorem 3.1. *Let* (X_k, d_k), $k = \overline{0,p}$, $p \geq 1$, *be some metric spaces. Let*

$$A_k : X_0 \times ... \times X_k \to X_k, \qquad k = \overline{0,p},$$

be some operators such that:

(i) *the spaces* (X_k, d_k), $k = \overline{1,p}$, *are complete metric spaces;*

(ii) *the operator* A_0 *is (weakly) Picard operator;*

(iii) *there exist* $\varphi_k : \Re_+ \to \Re_+$ *subadditive (c)-comparison functions such that the operators* $A_k(x_0, ..., x_{k-1}, \cdot)$ *are* φ_k-*contractions,* $k = \overline{1,p}$;

(iv) *the operators* A_k *are continuous with respect to* $(x_0, ..., x_{k-1})$ *for all* $x_k \in X_k$, $k = \overline{1,p}$.

Then the operator $B_p = (A_0, ..., A_p)$ *is (weakly) Picard operator. Moreover if* A_0 *is a Picard operator and*

$$F_{A_0} = \{x_0^*\}, \qquad F_{A_1(x_0^*,\cdot)} = \{x_1^*\}, ..., F_{A_p(x_0^*,...,x_{p-1}^*,\cdot)} = \{x_p^*\}$$

then

$$F_{B_p} = \{(x_1^*, x_2^*, ..., x_p^*).$$

Proof. We prove this theorem by induction respect to $p \in N^*$. First we consider the case of $p = 1$.

Let $x_0 \in X_0$ and $x_1 \in X_1$. We show that

$$B_1^n(x_0, x_1) \to (A_0^\infty(x_0), x_1^*(x_0))$$

as $n \to \infty$, where $x_1^*(x_0)$ is a unique fixed point of $A_1(A_0^\infty(x_0), \cdot)$ It si easy to check that

$$B_1^n(x_0, x_1) = (A_0^n(x_0), y_n),$$

where $y_0 = x_1$, $y_1 = A_1(x_0, y_0)$, ..., $y_{n+1} = A_1(A_0^n(x_0), y_n)$, ...

Using again Lemma 3.3 we obtain the proof in the case $p = 1$.

Now we suppose that the statemant of the theorem is true for the $p \leq k$ and we prove the theorem for the $p = k + 1$. We remark that $B_{k+1} = (B_k, A_{k+1})$, where B_k is (weakly) Picard operator, so we are in the case $p = 1$ and thus the proof is complete. □

*Remark 3.1.*The Lemma 3.2, Lemma 3.3, Theorem 4.1 from I.A. Rus [5] can be obtained using φ as in Example 2.2.

4. **Application**

We consider the folowing integral equation:

$$x(t) = g(t) + \lambda \cdot \int_a^b K(t, s, x(s))ds, \qquad t \in [a; b]. \tag{5}$$

*Theorem 4.1. S*uppose that the following conditions hold:

(i) $g \in C[a; b]$, $K \in C([a; b] \times [a; b] \times \Re)$;

(ii) there exists $L_K > 0$ such that: $|K(t, s, u) - K(t, s, v)| \leq L_k |u - v|$, for all $t, s \in [a; b]$, $u, v \in \Re$;

(iii) $\lambda_0 L_K (b - a) < 1$, where $\lambda_0 \in \Re_+^*$.

Then

(a) the equation (5) has a unique solution $x^*(\cdot, \lambda)$ in $C([a; b])$, for all $\lambda \in [-\lambda_0; \lambda_0]$;

(b) for all $x_0 \in C([a;b])$ the sequence $(x_n)_{n\in N}$ defined by

$$x_{n+1}(t;\lambda) = g(t) + \lambda \cdot \int_a^b K(t,s,x_n(s))ds,$$

converges uniformly to x^*, for all $t,s \in [a;b]$, $\lambda \in [-\lambda_0;\lambda_0]$;

(c) we have the estimation:

$$\|x_n - x^*\|_C \leq \frac{\alpha^n}{1-\alpha} \cdot \|x_1 - x_0\|_C ,$$

where $\alpha = \lambda_0 L_K(b-a)$;

(d) the function $x^* : [a;b] \times [-\lambda_0;\lambda_0] \to \Re\ (t,\lambda) \longmapsto x^*(t;\lambda)$ is continuous;

(e) if $K(t,s,\cdot) \in C^1(\Re)$, for all $t,s \in [a;b]$, then

$$x^*(t;\cdot) \in C^1([-\lambda_0;\lambda_0])$$

for all $t \in [a;b]$.

Proof. We consider the Banach space $X := (C([a;b] \times [-\lambda_0;\lambda_0]), \|\cdot\|_C)$, where $\|\cdot\|_C$ is Chebyshev norm, and the operator defined by

$$A_0 : X \to X,$$

$$A_0(x)(t;\lambda) = g(t) + \lambda \cdot \int_a^b K(t,s,x(s;\lambda))ds,$$

for all $t,s \in [a;b]$, $\lambda \in [-\lambda_0;\lambda_0]$.

Using (ii) we obtain:

$$\|A_0(x) - A_0(y)\|_C \leq \lambda_0 L_K(b-a) \cdot \|x-y\|_C \tag{6}$$

for all $x,y \in X$, so A_0 is a φ-contraction, where $\varphi(t) = \alpha t$ is a (c)-comparison function because of (iii). From Theorem 3, (V. Berinde, [2]) we conclude (a), (b), (c), (d).

We'll prove that there exists $\frac{\partial x^*}{\partial \lambda}$. If we formally derivate the relation (5) respect to λ we obtain:

$$\frac{\partial x(t;\lambda)}{\partial \lambda} = \int_a^b K(t,s,x(s;\lambda))ds + \lambda \cdot \int_a^b \frac{\partial K(t,s,x(s;\lambda))}{\partial \lambda} \cdot \frac{x(s;\lambda)}{\partial \lambda} ds.$$

This relation sugest to consider the following operator:

$$A_1 : X \times X \to X,$$

$$A_1(x,y)(t;\lambda) = \int_a^b K(t,s,x(s;\lambda))ds + \lambda \cdot \int_a^b \frac{\partial K(t,s,x(s;\lambda))}{\partial \lambda} \cdot y(s;\lambda)ds.$$

We estimate that:

$$\|A_1(x,y_1) - A_1(x,y_2)\|_C \leq \lambda_0 L_K (b-a) \cdot \|y_1 - y_2\|_C ,$$

for all $x \in X$. If we take the operator

$$B : X \times X \to X \times X, \qquad B = (A_0, A_1)$$

then we are in the conditions of Theorem 3.1, thus B is a Picard operator and the sequences

$$x_{n+1}(t;\lambda) := g(t) + \lambda \cdot \int_a^b K(t,s,x_n(s))ds$$

$$y_{n+1}(t;\lambda) := \int_a^b K(t,s,x_n(s;\lambda))ds + \lambda \cdot \int_a^b \frac{\partial K(t,s,x_n(s;\lambda))}{\partial \lambda} \cdot y_n(s;\lambda)ds$$

converges uniformly (with respect to $t \in [a;b]$, $\lambda \in [-\lambda_0;\lambda_0]$) to $(x^*,y^*) \in F_B$, for all $x_0, y_0 \in X$. But for fixed $x_0, y_0 \in X$ we have that $y_1 = \frac{\partial x_1}{\partial \lambda}$ and by induction we prove that $y_n = \frac{\partial x_n}{\partial \lambda}$, so we have:

$$x_n \overset{unif.}{\to} x^* \quad \text{as} \quad n \to \infty,$$

$$\frac{\partial x_n}{\partial \lambda} \overset{unif.}{\to} y^*$$

as $n \to \infty$.

These imply that there exists $\frac{\partial x^*}{\partial \lambda}$ and $\frac{\partial x^*}{\partial \lambda} = y^*$. □

Remark 4.1. If $K(t,s,\cdot) \in C^m(\Re)$ then $x^*(t;\cdot) \in C^m([-\lambda_0;\lambda_0])$.

Remark 4.2. For other examples of integral equations where Theorem 3.1 is used see I. A. Rus [4], [5], M. A. Şerban [9].

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"Babeş-Bolyai" University, str. Kogălniceanu, nr. 1, 3400 Cluj-Napoca
E-mail address: mserban@math.ubbcluj.ro

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SOME APPROXIMATION IDEALS

NICOLAE TIŢA

Abstract. We consider some approximation ideals of operators on operator spaces. The method used is similar to that from [8], [10], [11], in the case of the classical Banach spaces, or [2], [7] for the case of Hilbert spaces.

1. Introduction

The theory of the approximation ideals is well known for the case of linear and bounded operators on Hilbert, or Banach, spaces [2], [6], [7], [8], [11].

Here we consider the special case of the completely bounded operators on operator spaces. For these notions it can be seen [1], [3], [5].

We begin by recalling some definitions.

An operator space E , in short O.S, is a Banach space, or a normed space before completion, given with an isometric embedding $J : E \to L(H)$, where $L(H)$ is the space of all linear and bounded operators $T : H \to H$, H being a Hilbert space. We shall identify often E with $J(E)$ and so we shall say that an O.S is a (closed) subspace of $L(H)$.

If $E \subset L(H)$ is an operator space then $M_n \otimes E$ can be identified with the space of all $n \times n$ matrices having entries in E, that it will be denoted by $M_n(E)$.

Clearly $M_n(E)$ can be seen as an o.s. embedded in $L(H^n)$, where

$H^n = H \otimes ... \otimes H$ (number of H is n).

Let us denote by $\|\cdot\|_n$ the norm induced by $L(H^n)$ on $M_n(E)$, in the particular case $n = 1$ we get the norm of E. Taking the natural embedding $M_n(E) \to M_{n+1}(E)$ we can consider $M_n(E)$ included in $M_{n+1}(E)$, and $\|\cdot\|_n$ induced by $\|\cdot\|_{n+1}$.

Thus we may consider $\bigcup_n M_n(E)$ a normed space equipped with it's natural norm $\|\cdot\|_\infty$.

We denote by $K[E]$ the completion of $\bigcup_n M_n(E)$. If we denote by $K_0 = \bigcup_n M_n$, the case $E = C$, then the completion of K_0 coincides isometrically with the C^*-algebra, $C(l_2)$, of all compact operators on the space l_2.

It is easy to check that $\bigcup_n M_n(E)$ can be identified isometrically with $K_0 \otimes E$.

The basic idea of o.s. is that the norm of the Banach space E is replaced by a sequence of norms $\{\|\cdot\|_n\}_n$ on $\{M_n(E)\}_n$ or by a single norm $\|\cdot\|_\infty$ on the space $K[E]$.

Definition 1. Let $E_1 \subset L(H_1)$ and $E_2 \subset L(H_2)$ be operator spaces,

$u : E_1 \to E_2$ be a linear map and

$u_n : (x_{ij}) \in M_n(E_1) \to (u(x_{ij})) \in M_n(E_2)$. We say that u is **completely bounded, c.b.,** if $\sup_n \|u_n\| < \infty$ and we define $\|u\|_{c.b.} := \sup_n \|u_n\|$.

Definition 2. (equivalent) u is **completely bounded** if the maps u_n can be extended to a single bounded map $u_\infty : K[E_1] \to K[E_2]$ and we have $\|u\|_{c.b.} = \|u_\infty\|$.

Definition 3. $c.b.(E_1, E_2) := \{u : E_1 \to E_2 : \ u \text{ is c.b.}\}$. We shall consider the $c.b.(E_1, E_2)$ equipped with $\|\cdot\|_{c.b.}$.

Remark 1. The similar definition of the uniform norm for the bounded operators can be written as follows:

$$\|u\|_{c.b.} = \sup \{\|u_\infty\| : x \in K[E], \|x\| \leq 1\}.$$

Remark 2. Likewise the case of an isomorphism between two Banach spaces, we say that two o.s. E_1, E_2 are **completely isomorphic**, **completely isometric,** if there is an c.b. isomorphism $u : E_1 \to E_2$ with c.b. inverse and in addition $\|u\|_{c.b.} = \|u^{-1}|_{u(E_1)}\|_{c.b.} = 1$

Let $E_1 \subset L(H_1)$ and $E_2 \subset L(H_2)$ be operator spaces. There is an embedding $J : E_1 \otimes E_2 \to L(H_1 \otimes H_2)$ defined by $J(x_1 \otimes x_2)(h_1 \otimes h_2) = x_1(h_1) \otimes x_2(h_2)$.

We denote by $E_1 \otimes_{\min} E_2$ the completion of $E_1 \otimes E_2$ equipped with the norm $x \to \|Jx\|$.

Obviously J can be extend to an isometric embedding. So we can see $E_1 \otimes_{\min} E_2$ as an o.s. embedded into $L(H_1 \otimes_\sigma H_2)$. This space is called the **minimal, spatial, tensor product** of E_1 and E_2. ($H_1 \otimes_\sigma H_2$ is the hilbertian tensor product, [7], [11].)

If $E \subset L(H)$ is an o.s. then $M_n \otimes_{\min} E$ can be identified with the space $M_n(E)$ and $K[E]$ can be identified isometrically with $K \otimes_{\min} E$. Thus, for any linear map $u : E_1 \to E_2$ we have $\|u\|_{c.b.} = \|I \otimes u : K \otimes_{\min} E_1 \to K \otimes_{\min} E_2\| = \|I \otimes u : K \otimes_{\min} E_1 \to K \otimes_{\min} E_2\|_{c.b.}$. More generaly it can be shown that, for any o.s. $F \subset L(\widetilde{H})$, we have $\|I_F \otimes u : F \otimes_{\min} E_1 \to F \otimes_{\min} E_2\| \leq \|u\|_{c.b.}$. Further on, if $v : F_1 \to F_2$ is another c.b. map, we obtain

$\|u \otimes v : F_1 \otimes_{\min} E_1 \to F_2 \otimes_{\min} E_2\|_{c.b.} \leq \|v\|_{c.b.} \cdot \|u\|_{c.b.}$.

This relation will be very useful in the sequel.

For others properties of the minimal tensor product it can be seen the papers [1], [5], etc.

2. Approximation numbers of completely bounded operators

Definition 4. Let $u : E \to F$ be a completely bounded map, $u \in c.b.(E,F)$. The **approximation numbers**, $a_n^{c.b.}(u)$ will be defined as follows

$$a_n^{c.b.}(u) := \inf\left\{\|u - a\|_{c.b.} : a \in c.b.(E,F), rank(a) < n\right\},\ n = 1, 2, \ldots$$

Remark 3. From this definition it results that $\|u\|_{c.b.} = a_1^{c.b.}(u) \geq a_2^{c.b.}(u) \geq \ldots \geq 0$.

Proposition 1. *The approximation numbers $a_n^{c.b.}(u)$ verify the following inequalities:*

1. $\sum_{n=1}^{k} a_n^{c.b.}(u_1 + u_2) \leq 2 \cdot \sum_{n=1}^{k} \left(a_n^{c.b.}(u_1) + a_n^{c.b.}(u_2)\right)$, for $k = 1, 2, \ldots$
2. $\sum_{n=1}^{k} a_n^{c.b.}(u_1 \circ u_2) \leq 2 \cdot \sum_{n=1}^{k} \left(a_n^{c.b.}(u_1) \cdot a_n^{c.b.}(u_2)\right)$, for $k = 1, 2, \ldots$

Proof. 1) Let $\varepsilon > 0$. There are a_i, $i = 1, 2$, such that $rank\,(a_i) < n$ and

$$\|u_i - a_i\|_{c.b.} \le a_n^{c.b.}(u_i) + \tfrac{\varepsilon}{2}.$$

We obtain:

$$a_{2\cdot n-1}^{c.b.}(u_1 + u_2) \le \|(u_1 + u_2) - (a_1 + a_2)\|_{c.b.} \le$$
$$\le \|u_1 - a_1\|_{c.b.} + \|u_2 - a_2\|_{c.b.} \le$$
$$\le a_n^{c.b.}(u_1) + a_n^{c.b.}(u_2) + \varepsilon.$$

Since ε is arbitrary it follows that:

$$a_{2\cdot n-1}^{c.b.}(u_1 + u_2) \le a_n^{c.b.}(u_1) + a_n^{c.b.}(u_2).$$

Further on it results:

$$\sum_{n=1}^{k} a_n^{c.b.}(u_1 + u_2) \le \sum_{n=1}^{k} a_{2\cdot n-1}^{c.b.}(u_1 + u_2) + \sum_{n=1}^{k} a_{2\cdot n}^{c.b.}(u_1 + u_2) \le$$
$$\le 2 \cdot \sum_{n=1}^{k} a_{2\cdot n-1}^{c.b.}(u_1 + u_2) \le 2 \cdot \sum_{n=1}^{k} \left(a_n^{c.b.}(u_1) + a_n^{c.b.}(u_2)\right).$$

2) We consider also a_i, $i = 1, 2$, such that $rank\,(a_i) < n$ and

$$\|u_i - a_i\|_{c.b.} \le a_n^{c.b.}(u_i) + \tfrac{\varepsilon}{2}.$$

We obtain:

$$a_n^{c.b.}(u_1 \circ u_2) \le \|(u_1 \circ u_2) - [u_1 \circ a_2 + a_1 \circ (u_2 - a_2)]\|_{c.b.} =$$
$$= \|(u_1 - a_1) \circ (u_2 - a_2)\|_{c.b.} \le \left(a_n^{c.b.}(u_1) + \tfrac{\varepsilon}{2}\right) \cdot \left(a_n^{c.b.}(u_2) + \tfrac{\varepsilon}{2}\right).$$

Since ε is arbitrary it follows that:

$$a_{2\cdot n-1}^{c.b.}(u_1 \circ u_2) \le a_n^{c.b.}(u_1) \cdot a_n^{c.b.}(u_2).$$

Likewise the 1) results 2). □

Remark 4. For the case of the linear and bounded operators between Banach spaces the above inequalities are known, [8], [11].

In the sequel we deduce an inequality for the case of the c.b. operator $u_1 \otimes_{\min} u_2$ using a similar method with that from [9], [10], used for the classical case of the bounded operators on Banach spaces.

Proposition 2. *The approximation numbers $a_n^{c.b.}(u_1 \otimes_{\min} u_2)$ verify the inequalities:*

$$\sum_{n=1}^{k} \frac{a_n^{c.b.}(u_1 \otimes_{\min} u_2)}{n} \le 6 \cdot \sum_{n=1}^{k} \frac{a_n^{c.b.}(u_1) \cdot \|u_2\|_{c.b.} + a_n^{c.b.}(u_2) \cdot \|u_1\|_{c.b.}}{n}, \text{ for } k = 1, 2, \dots.$$

Proof. Let $\varepsilon > 0$. There are a_i, $i = 1,2$, such that $rank\,(a_i) < n$ and

$\|u_i - a_i\|_{c.b.} \leq a_n^{c.b.}(u_i) + \frac{\varepsilon}{2}$.

We obtain:

$$a_{n^2}^{c.b.}(u_1 \otimes_{\min} u_2) \leq \|u_1 \otimes_{\min} u_2 - a_1 \otimes_{\min} a_2\|_{c.b.} =$$
$$= \|(u_1 - a_1) \otimes_{\min} u_2 - a_1 \otimes_{\min} (u_2 - a_2)\|_{c.b.} \leq$$
$$\leq \|u_1 - a_1\|_{c.b.} \cdot \|u_2\|_{c.b.} + \|a_1\|_{c.b.} \cdot \|u_2 - a_2\|_{c.b.} \leq$$
$$\leq \left(a_n^{c.b.}(u_1) + \tfrac{\varepsilon}{2}\right) \cdot \|u_2\|_{c.b.} + \|a_1 - u_1 + u_1\|_{c.b.} \cdot \left(a_n^{c.b.}(u_2) + \tfrac{\varepsilon}{2}\right) \leq$$
$$\leq \left(a_n^{c.b.}(u_1) + \tfrac{\varepsilon}{2}\right) \cdot \|u_2\|_{c.b.} + \left(\|a_1 - u_1\|_{c.b.} + \|u_1\|_{c.b.}\right) \cdot \left(a_n^{c.b.}(u_2) + \tfrac{\varepsilon}{2}\right) \leq$$
$$\leq \left(a_n^{c.b.}(u_1) + \tfrac{\varepsilon}{2}\right) \cdot \|u_2\|_{c.b.} + 2 \cdot \|u_1\|_{c.b.} \cdot \left(a_n^{c.b.}(u_2) + \tfrac{\varepsilon}{2}\right).$$

Since ε is arbitrary we obtain:

$$a_{n^2}^{c.b.}(u_1 \otimes_{\min} u_2) \leq 2 \cdot \left(a_n^{c.b.}(u_1) \cdot \|u_2\|_{c.b.} + a_n^{c.b.}(u_2) \cdot \|u_1\|_{c.b.}\right).$$

Taking account that the sequence of the aproximation numbers is decreasing we can writte:

$$\sum_{n=1}^{k} \frac{a_n^{c.b.}(u_1 \otimes_{\min} u_2)}{n} \leq \sum_{n=1}^{j} (2 \cdot n + 1) \frac{a_{n^2}^{c.b.}(u_1 \otimes_{\min} u_2)}{n^2}, \text{ where } j^2 \leq k < (j+1)^2.$$

Now we obtain:

$$\sum_{n=1}^{k} \frac{a_n^{c.b.}(u_1 \otimes_{\min} u_2)}{n} \leq \sum_{n=1}^{j} (2 \cdot n + 1) \frac{a_{n^2}^{c.b.}(u_1 \otimes_{\min} u_2)}{n^2} \leq 3 \cdot \sum_{n=1}^{j} n \cdot \frac{a_{n^2}^{c.b.}(u_1 \otimes_{\min} u_2)}{n^2} \leq$$
$$\leq 6 \cdot \sum_{n=1}^{j} \frac{a_n^{c.b.}(u_1) \cdot \|u_2\|_{c.b.} + a_n^{c.b.}(u_2) \cdot \|u_1\|_{c.b.}}{n} \leq 6 \cdot \sum_{n=1}^{k} \frac{a_n^{c.b.}(u_1) \cdot \|u_2\|_{c.b.} + a_n^{c.b.}(u_2) \cdot \|u_1\|_{c.b.}}{n}.$$

This finishes the proof. □

Remark 5. By means of these approximation numbers we can define special approximation ideals in $c.b.(E,F)$.

3. Special approximation ideals

Definition 5. Let $x = \{x_1, x_2, \ldots\}$ be a real sequence and let $card\,(x)$ be $card\,\{i \in N : x_i \neq 0\}$.

Let K be the set of all real sequences $x \in l_\infty$ having the following two properties:

1. $card\,(x) < n\,(x)$, $n\,(x)$ is a natural number

2. $x_1 \geq x_2 \geq ... \geq x_{n(x)} \geq 0$.

A function $\Phi : K \to R$ is called a **symmetric norming function** if:

1. $\Phi(x) > 0$ if $x \in K$ and $x \neq 0$;
2. $\Phi(\alpha \cdot x) = \alpha \cdot \Phi(x)$, for every $\alpha \geq 0$ and $x \in K$;
3. $\Phi(x + y) \leq \Phi(x) + \Phi(y)$, for every $x, y \in K$;
4. $\Phi(\{1, 0, 0...\}) = 1$;
5. If $x, y \in K$ and $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$, for every $k = 1, 2, ...$, then $\Phi(x) \leq \Phi(y)$.

Remark 6. The above definition can be extend on the whole space l_∞ taking

$\Phi(x) := \lim_{n\to\infty} \Phi(\{x_1^*, ..., x_n^*, 0, 0...\})$, where $x^* = \{x_i^*\}_{i\in N}$ is the sequence $\{|x_i|\}_{i\in N}$ rearranged in decreasing order.

Definition 6. In the sequel we shall consider a subclass of $c.b.(E, F)$ which is defined as follows:

$$\Phi - c.b.(E, F) := \left\{u \in c.b.(E, F) : ||u||_\Phi^{c.b.} := \Phi\left(\left\{a_n^{c.b.}(u)\right\}_n\right) < \infty\right\}.$$

Remark 7. We prove that this class has similar properties with the similar classes defined for linear and bounded operators. (For the case of the Hilbert spaces it can be seen [2], [7] and for the case of the Banach spaces it can be seen [8], [9], [10], [11].)

Proposition 3. $\left(\Phi - c.b., ||.||_\Phi^{c.b.}\right)$ *is a quasi-normed operator ideal.*

Proof. 1. Any unidimensional operator , $u \in c.b.(E, F)$, belongs to

$\Phi - c.b.(E, F)$ because, in this case, the sequence $\left\{a_n^{c.b.}(u)\right\} = \{||u||_{c.b.}^{\cdot}, 0, 0, ...\}$ and hence $\Phi\left(\left\{a_n^{c.b.}(u)\right\}_n\right) = ||u||_{c.b.}^{\cdot} < \infty$.

2. If $u_1, u_2 \in c.b.(E, F)$ then $u_1 + u_2 \in c.b.(E, F)$. This results from the proposition 5 (1). and from the properties of Φ, as follows:

$\Phi\left(\left\{a_n^{c.b.}(u_1 + u_2)\right\}_n\right) \leq 2 \cdot \Phi\left(\left\{a_n^{c.b.}(u_1) + a_n^{c.b.}(u_2)\right\}_n\right) \leq$
$\leq 2 \cdot \left(\Phi\left(\left\{a_n^{c.b.}(u_1)\right\}_n\right) + \Phi\left(\left\{a_n^{c.b.}(u_2)\right\}_n\right)\right)$.

3. If $v \in c.b.(E, E)$, $u \in \Phi - c.b.(E, F)$ and $w \in c.b.(F, F)$ then

$w \circ u \circ v \in \Phi - c.b.(E,F)$.

From the proposition 5 (2) and from the definition of $a_n^{c.b.}(u)$ it follows that $a_n^{c.b.}(w \circ u \circ v) \leq \|w\|_{c.b.} \cdot a_n^{c.b.}(u) \cdot \|v\|_{c.b.}$ and hence
$\Phi\left(\left\{a_n^{c.b.}(w \circ u \circ v)\right\}_n\right) \leq \|w\|_{c.b.} \cdot \Phi\left(\left\{a_n^{c.b.}(u)\right\}_n\right) \cdot \|v\|_{c.b.}$. □

Remark 8. We present now some properties similar to the properties of the classical approximation ideals L_Φ, [8], [11].

Lemma 1. *The approximation numbers $a_n^{c.b.}(u)$ verify the inequalities:*

$$\sum_{n=1}^{k} a_{2\cdot n-1}^{c.b.}(u) \leq \sum_{n=1}^{k} a_n^{c.b.}(u) \leq 2 \cdot \sum_{n=1}^{k} a_{2\cdot n-1}^{c.b.}(u)\,, k = 1, 2, \ldots.$$

Proof. The first inequality is a consequence of the fact that the sequence $\left\{a_n^{c.b.}(u)\right\}_n$ is decreasing.

The second inequality results as follows:
$\sum_{n=1}^{k} a_n^{c.b.}(u) \leq \sum_{n=1}^{2\cdot k} a_n^{c.b.}(u) \leq \sum_{n=1}^{k} a_{2\cdot n-1}^{c.b.}(u) + \sum_{n=1}^{k} a_{2\cdot n}^{c.b.}(u) \leq$
$\leq 2 \cdot \sum_{n=1}^{k} a_{2\cdot n-1}^{c.b.}(u)$. □

Corollary 1. $\|u\|_{\Phi}^{c.b.} := \Phi\left(\left\{a_{2\cdot n-1}^{c.b.}(u)\right\}_n\right)$ *is a quasi-norm equivalent*

with $\|u\|_{\Phi}^{c.b.} = \Phi\left(\left\{a_n^{c.b.}(u)\right\}_n\right)$.

Remark 9. Since $\sum_{n=1}^{k} \left(a_n^{c.b.}(u)\right)^p \leq \sum_{n=1}^{k} \left(\frac{1}{n} \cdot \sum_{i=1}^{n} a_i^{c.b.}(u)\right)^p \leq$

$\leq c(p) \cdot \sum_{n=1}^{k} \left(a_n^{c.b.}(u)\right)^p,\ 1 < p < \infty, k = 1, 2, \ldots,$
see the Hardy inequality [11], it follows that

$$\Phi\left(\left\{\left(a_n^{c.b.}(u)\right)^p\right\}_n\right) \leq \Phi\left(\left\{\left(\frac{1}{n} \cdot \sum_{i=1}^{n} a_i^{c.b.}(u)\right)^p\right\}_n\right) \leq c(p) \cdot \Phi\left(\left\{\left(a_n^{c.b.}(u)\right)^p\right\}_n\right)$$

and hence $\|u\|_{\Phi_{(p)}}^{c.b.}$ is equivalent with $\overline{\|u\|_{\Phi_{(p)}}^{c.b.}} := \Phi_{(p)}\left(\left\{\frac{1}{n} \cdot \sum_{i=1}^{n} a_i^{c.b.}(u)\right\}_n\right)$, where $\Phi_{(p)}(\{x_i\}) = \Phi(\{x_i^p\})^{\frac{1}{p}}$ is a symmetric norming function, [2], [6], [7], [11], for $1 < p < \infty$.

Because $a_n^{c.b.}(u) \leq \left(a_1^{c.b.}(u) \cdot \ldots \cdot a_n^{c.b.}(u)\right)^{\frac{1}{n}} \leq \frac{1}{n} \cdot \sum_{i=1}^{n} a_i^{c.b.}(u)$ it follows, also, that

$$\left\{a_n^{c.b.}(u)\right\}_n \in l_{\Phi_{(p)}} \text{ if } g_n(u) := \left\{\left(a_1^{c.b.}(u) \cdot \ldots \cdot a_n^{c.b.}(u)\right)^{\frac{1}{n}}\right\}_n \in l_{\Phi_{(p)}},$$
$$\{x_n\} \in l_{\Phi_{(p)}} \Longleftrightarrow \Phi_{(p)}(\{x_n\}) < \infty.$$

Remark 10. If we consider the special case of the function

$$\overline{\Phi}_{(p)} : \left(\left\{a_n^{c.b.}(u)\right\}\right) \to \Phi\left(\left\{\frac{\left(a_n^{c.b.}(u)\right)^p}{n}\right\}\right)^{\frac{1}{p}},$$

where $1 \leq p < \infty$, the classes $\overline{\Phi}_{(p)} - c.b.(E,F)$ are tensor product stable.

Proposition 4. *If* $u_k \in \overline{\Phi}_{(p)} - c.b.(E_k, F_k)$, $k = 1, 2.$, *then*

$$u_1 \otimes_{\min} u_2 \in \overline{\Phi}_{(p)} - c.b.(E_1 \otimes_{\min} E_2, F_1 \otimes_{\min} F_2).$$

Proof. It is similar to that for the classical approximation ideals [10], [11].

First we remark that, using the relation

$$a_{n^2}^{c.b.}(u_1 \otimes_{\min} u_2) \leq 2 \cdot \left(a_n^{c.b.}(u_1) \cdot \|u_2\|_{c.b.} + a_n^{c.b.}(u_2) \cdot \|u_1\|_{c.b.}\right)$$

we can obtain

$$\sum_{n=1}^{k} \frac{\left(a_n^{c.b.}(u_1 \otimes_{\min} u_2)\right)^p}{n} \leq c(p) \cdot \sum_{n=1}^{k} \frac{\left(a_n^{c.b.}(u_1) \cdot \|u_2\|_{c.b.}\right)^p + \left(a_n^{c.b.}(u_2) \cdot \|u_1\|_{c.b.}\right)^p}{n},$$

for $k = 1, 2, \ldots$, see Proposition 6 for $p = 1$.

Now taking into account the properties of the functions Φ it follows that

$$\Phi\left(\left\{\frac{\left(a_n^{c.b.}(u_1 \otimes_{\min} u_2)\right)^p}{n}\right\}\right) \leq$$
$$c(p) \cdot \Phi\left(\left\{\frac{\left(a_n^{c.b.}(u_1) \cdot \|u_2\|_{c.b.}\right)^p}{n} + \frac{\left(a_n^{c.b.}(u_2) \cdot \|u_1\|_{c.b.}\right)^p}{n}\right\}\right).$$

Hence

$$\overline{\Phi}_{(p)}\left(\left\{a_n^{c.b.}\left(u_1\otimes_{\min}u_2\right)\right\}\right)\leq$$
$$\leq c_1(p)\cdot\left(\overline{\Phi}_{(p)}\left(\left\{a_n^{c.b.}(u_1)\right\}\right)\cdot\|u_2\|_{c.b.}+\overline{\Phi}_{(p)}\left(\left\{a_n^{c.b.}(u_2)\right\}\right)\cdot\|u_1\|_{c.b.}\right)\leq$$
$$\leq 2\cdot c_1(p)\cdot\overline{\Phi}_{(p)}\left(\left\{a_n^{c.b.}(u_1)\right\}\right)\cdot\overline{\Phi}_{(p)}\left(\left\{a_n^{c.b.}(u_2)\right\}\right),$$
$$c_1(p)=c(p)^{\frac{1}{p}} \text{ and } \|u_k\|_{c.b}\leq\overline{\Phi}_{(p)}\left(\left\{a_n^{c.b.}(u_k)\right\}\right),\ k=1,2.$$

The proof is fulfiled. □

Remark 11. The above result remains true if we consider **the maximal tensor product.**

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DEPARTMENT OF MATH., FAC. OF SCIENCE, "TRANSILVANIA" UNIV., 2200 BRAŞOV, ROMANIA