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Redacţia: 3400 Cluj-Napoca, str. M. Kogalniceanu nr. 1 • Telefon: 194315

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SOME APPLICATIONS OF ORE'S GENERALIZED THEOREMS IN THE FORMATION THEORY

RODICA COVACI

Abstract. Ore's theorems [9] are a powerful tool in the formation theory of finite solvable groups. In [4] we obtained a generalization of some of these theorems on finite π-solvable groups, where π is an arbitrary set of primes. The present paper applies Ore's generalized theorems to prove the existence and conjugacy of covering subgroups in finite π-solvable groups.

1. Preliminaries

All groups considered in the paper are finite. We denote by π an arbitrary set of primes and by π' the complement to π in the set of all primes.

Definition 1.1. 1. A class $\underline{X}$ of groups is a *homomorph* if $\underline{X}$ is closed under homomorphisms.

2. A group G is *primitive* if G has a *stabilizer*, i.e. a maximal subgroup H with $\mathrm{core}_G H = 1$, where $\mathrm{core}_G H = \cap \{ H^g / g \in G \}$.
3. A homomorph $\underline{X}$ is a *Schunck class* if $\underline{X}$ is *primitively closed*, i.e. if any group G, all of whose primitive factor groups are in $\underline{X}$, is itself in $\underline{X}$.

4. If $\underline{X}$ is a class of groups and G is a group, a subgroup E of G is called an $\underline{X}$-*covering subgroup* of G if: (i) $E \in \underline{X}$; (ii) $E \leq V \leq G$, $V_0 < V$, $V/V_0 \in \underline{X}$ imply $V = E V_0$.

Definition 1.2. a) A group G is π-*solvable* if every chief factor of G is either a solvable π-group or a π'-group. When π is the set of all primes, we obtain the notion of

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solvable group.

b) A class $\underline{X}$ of groups is said to be *π-closed* if:

$G/O_{\pi'}(G) \in \underline{X} \Rightarrow G \in \underline{X}$,

where $O\pi'(G)$ denotes the largest normal π'-subgroup of G. We shall call *π-homomorph* a π-closed homomorph and *π-Schunck class a* π-closed Schunck class.

Let $\underline{X}$ be a homomorph. The following properties given in [8] are also true for any finite group:

Proposition 1.3. *If E is an $\underline{X}$-covering subgroup of G and $E \leq H \leq G$, then E is an $\underline{X}$-covering subgroup of H.*

Proposition 1.4. *Let E be an $\underline{X}$-covering subgroup of G and N a normal subgroup of G. Then EN/N is an $\underline{X}$-covering subgroup of G/N.*

Proposition 1.5. *If N is a normal subgroup of G, E^*/N is an $\underline{X}$-covering subgroup of G/N and E is an $\underline{X}$-covering subgroup of E^*, then E is an $\underline{X}$-covering subgroup of G.*

Finally, we shall use a result of R. Baer [1] which we give below:

Theorem 1.6. *A solvable minimal normal subgroup of a group is abelian.*

2. Ore's generalized theorems

In [3] we gave some properties of finite primitive groups, among which we remind the following:

Proposition 2.1. *If G is a primitive group and W is a stabilizer of G, then for any minimal normal subgroup M of G we have $MW = G$.*

In [4] we proved the following theorems generalizing Ore's theorems from [9]:

Theorem 2.2. *Let G be a primitive π-solvable group. If G has a minimal normal subgroup which is a solvable π-group, then G has one and only one minimal normal subgroup.*

Corollary 2.3. *If G is a primitive π-solvable group, then G has at most one minimal normal subgroup which is a solvable π-group.*

Corollary 2.4. *If a primitive π-solvable group G has a minimal normal subgroup which is a solvable π-group, then G has no minimal normal subgroups which are π'-groups.*

Theorem 2.5. *If G is a primitive π-solvable group and N is a minimal normal subgroup of G which is a solvable π-group, then $C_G(N) = N$.*

Theorem 2.6. *Let G be a π-solvable group such that:*
(i) there is a minimal normal subgroup M of G which is a solvable π-group and $C_G(M) = M$;
(ii) there is a minimal normal subgroup L/M of G/M such that L/M is a π'-group.
Then G is primitive.

Theorem 2.7. *If G is a π-solvable group satisfying (i) and (ii) from 2.6., then any two stabilizers W and W^* of G are conjugate in G.*

Theorem 2.8. *If G is a primitive π-solvable group, $V < G$ such that there is a minimal normal subgroup M of G which is a solvable π-group and $MV = G$, then V is a stabilizer of G.*

3. Existence and conjugacy of covering subgroups in finite π-solvable groups

We give here a new proof of the existence and conjugacy theorems of covering subgroups in finite π-solvable groups [2]. The proof from [2] is based on some R. Baer's theorems (see [1]). According to the importance of Ore's theorems in the formation theory of finite solvable groups, we put the question if Ore's generalized theorems could not be used to prove the existence and conjugacy theorems of covering subgroups in finite π-solvable groups. The answer is affirmative as we show below.

Theorem 3.1. *If $\underline{X}$ is a π-homomorph and G is a π-solvable group, then any two $\underline{X}$-covering subgroups of G are conjugate in G.*

Proof. By induction on $|G|$. Let E and F be two $\underline{X}$-covering subgroups of G. If $G \in \underline{X}$, using 1.1.d) we obtain $E = F = G$ and so E and F are conjugate in G. Let now $G \notin \underline{X}$. If N is a minimal normal subgroup of G, by 1.4. we have that EN/N and FN/N are $\underline{X}$-covering subgroups of G/N. By the induction, EN/N and FN/N are conjugate in G/N and so $EN/N = (FN/N)^{xN}$, where $x \in G$. But this imply $EN = F^x N$. We distinguish two cases:

1) There is a minimal normal subgroup M of G such that $EM \neq G$. We put $N = M$. By 1.3., E and F^x are $\underline{X}$-covering subgroups of EM, hence by the induction E and F^x are conjugate in EM and so E and F are conjugate in G.

2) For any minimal normal subgroup N of G we have $EN = G = FN$. We prove that any minimal normal subgroup N of G is a solvable π-group. Indeed, since G is π-solvable, N is either a solvable π-group or a π'-group. Supposing that N is a π'-group, we obtain $N \leq O\pi'(G)$. From

$G/ O\pi'(G) \cong (G/N)/(O\pi'(G)/N)$

and

$G/N = EN/N \cong E/E \cap N \in \underline{X}$

it follows $G/ O\pi'(G) \in \underline{X}$. By the π-closure of $\underline{X}$ we obtain the contradiction $G \in \underline{X}$.. Thus N is a solvable π-group and by 1.6. N is abelian.

Now E is a stabilizer of G. Indeed, E is a maximal subgroup of G since $E \neq G$ ($E \in \underline{X}$ but $G \notin \underline{X}$) and if $E \leq H < G$ then $E = H$, because otherwise let $h \in H - E$, $h = en$, $e \in E$, $n \in N$ and $n = e^{-1}h \in N \cap H = 1$ (N being abelian) which means the contradiction $h = e \in E$. Further $core_G E = 1$, for supposing $core_G E \neq 1$ we have a minimal normal subgroup M of G with $M \leq core_G E$, hence $G = EM = E\ core_G E = E$, in contradiction with $E \in \underline{X}$ and $G \notin \underline{X}$. So E is a stabilizer of G and G is a primitive π-solvable group. Since $F < G$

($F \in \underline{X}$ but $G \notin \underline{X}$) and since, for any minimal normal subgroup N of G, N is a solvable π-group and $FN = G$, applying 2.8. we obtain that F is also a stabilizer of G.

By 2.2., G has one and only one minimal normal subgroup N. By 2.5., $C_G(N) = N$. So condition (i) from 2.6. is valid. Further, we shall prove below that condition (ii) from 2.6. is also true. Indeed, let us suppose that (ii) is not valid. It means that there is not a minimal normal subgroup L/N of G/N such that L/N is a π'-group. G/N being π-solvable, we deduce that any minimal normal subgroup L/N of G/N is a solvable π-group. But N being a solvable π-group, it follws that L is a solvable π-group. L being normal in G, we have two possibilities, both leading to a contradiction:

a) L is a minimal normal subgroup of G. But G having one and only one minimal normal subgroup N, we deduce that L = N, a contradiction with $L/N \neq 1$.

b) L is not a minimal normal subgroup of G. Then N < L, hence

$G = EN < EL \leq G$,

a contradiction.

We proved that G is a π-solvable group satisfying conditions (i) and (ii) from 2.6. Then, by theorem 2.7., we obtain that the two stabilizers E and F are conjugate in G. □

Theorem 3.2. *Let $\underline{X}$ be a π-homomorph. $\underline{X}$ is a Schunck class if and only if any π-solvable group G has $\underline{X}$-covering subgroups.*

Proof. Let $\underline{X}$ be a π-Schunck class. We prove by induction on |G| that any π-solvable group G has $\underline{X}$-covering subgroups. Two cases are considered:

1) There is a minimal normal subgroup M of G such that $G/M \notin \underline{X}$. By the induction, G/M has an $\underline{X}$-covering subgroup H^*/M. Since $G/M \notin \underline{X}$ we have $H^* < G$. By the induction, H^* has an $\underline{X}$-covering subgroup H. Applying now 1.5., H is an $\underline{X}$-covering subgroup of G.

2) For any minimal normal subgroup M of G we have $G/M \in \underline{X}$. Two possibilities can be considered again:

a) G is not primitive. Let G/K be a primitive factor of G. Since $K \neq 1$, there is a minimal normal subgroup M of G such that $M \subseteq K$. We have $G/M \in \underline{X}$. Hence

$G/K \cong (G/M)/(K/M) \in \underline{X}$.

By the primitively closure of $\underline{X}$, $G \in \underline{X}$. So G is its own $\underline{X}$-covering subgroup.

b) G is primitive. Let S be a stabilizer of G. If $G \in \underline{X}$, then G is its own $\underline{X}$-covering subgroup. Let now $G \notin \underline{X}$. We shall prove that S is an $\underline{X}$-covering subgroup of G. First

$S \in \underline{X}$. Indeed, let M be a minimal normal subgroup of G. Since G is primitive and S is a stabilizer of G, by 2.1. we have MS = G. On the other side, G being π-solvable, M is either a solvable π-group or a π'-group. But if we suppose that M is a π'-group we have

$$M \leq O_{\pi'}(G)$$

and

$$G/O_{\pi'}(G) \cong (G/M)/(O_{\pi'}(G)/M) \in \underline{X},$$

hence by the π-closure of $\underline{X}$ we deduce that $G \in \underline{X}$, a contradiction. Thus M is a solvable π-group. Applying 1.6., M is abelian. This and G = MS lead to $M \cap S = 1$. Then

$$S \cong S/1 = S/M \cap S \cong MS/M = G/M \in \underline{X}.$$

So $S \in \underline{X}$. Further if $S \leq V \leq G$, $V_0 < V$, $V/V_0 \in \underline{X}$, we shall prove that $V = SV_0$. Because S is a maximal subgroup of G, two possibilities can happen: V = S or V = G. If V = S, we have $V = VV_0 = S\ V_0$. If V = G, we notice that V_0 is a normal subgroup of G and $V_0 \neq 1$ (else, $G = V \cong V/1 = V/V_0 \in \underline{X}$, a contradiction). Then let M_0 be a minimal normal subgroup of G such that $M_0 \subseteq V_0$. Applying 2.1., M_0S = G. Hence

$V = G = M_0S = V_0S = SV_0$.

Conversely, let $\underline{X}$ be a π-homomorph such that any π-solvable group has $\underline{X}$-covering subgroups. We prove that $\underline{X}$ is primitively closed. Suppose that $\underline{X}$ is not primitively closed and let G be a π-solvable group of minimal order with respect to the conditions: any primitive factor of G is in $\underline{X}$ but $G \notin \underline{X}$. Let M be a minimal normal subgroup of G. By the minimality of G we have $G/M \in \underline{X}$. G being π-solvable, G has an $\underline{X}$-covering subgroup H. From $H \leq G = G$, $M < G$, $G/M \in \underline{X}$ follows G = MH. By the π-closure of $\underline{X}$, M is a solvable π-group and so by 1.6. M is abelian. From this and from G=MH we obtain $M \cap H = 1$. Like in the proof of theorem 3.1., we obtain that H is a maximal subgroup of G. Two cases are possible:

1) G is primitive. Then $G \cong G/1$ is a primitive factor of G and by the choice of G, we obtain $G \cong G/1 \in \underline{X}$, in contradiction with $G \notin \underline{X}$. So this case leads to a

contradiction.

2) G is not primitive. Then $core_G H \neq 1$, else H is a stabilizer of G and G is primitive. By the minimality of G we have $G/core_G H \in \underline{X}$. By 1.4., $H/core_G H$ is an $\underline{X}$-covering subgroup of $G/core_G H$. It follows that $H/core_G H = G/core_G H$, hence $H = G$, in contradiction with $H \in \underline{X}$ but $G \notin \underline{X}$. This case leads also to a contradiction.

It follows that $\underline{X}$ is primitively closed and so $\underline{X}$ is a Schunck class. □

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"BABEŞ-BOLYAI" UNIVERSITY, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, 3400 CLUJ-NAPOCA, ROMANIA.

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ON α-TYPE UNIFORMLY CONVEX FUNCTIONS

IOANA MAGDAŞ

Abstract. We determine necessary and sufficient condition for a function f with negative coefficients to be n-uniformly starlike of type α and we obtain a conection between the class of all such functions $UT_n(\alpha)$ and the class of the functions n-starlike of order α and type β with negative coefficients $T_n(\alpha,\beta)$. Distortion bounds and extreme points are also obtained.

1. Introduction

Denote by S the family of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are analytic and univalent in the unit disk $U = \{z : |z| < 1\}$ and by S^*, respectively $S^c(\alpha)$ the usual class of starlike functions, respectively convex functions of order α, $\alpha \geq 0$.

Definition 1. A function f is said to be uniformly convex in U if f is in S^c and has the property that for every circular arc γ contained in U, with center ζ also in U, the arc $f(\zeta)$ is a convex arc.

Let be UCV or US^c denote the class of all such functions.

Goodman gave the following two-variable analytic characterizations of this class, then Ma and Minda [1] and Rønning [2] independently found a one variable characterization for US^c.

Theorem A. *Let f have the form (1). Then the following are equivalent:*

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(i) $f \in US^c$

(ii) $\operatorname{Re}\left\{1 + \frac{(z-\zeta)f''(z)}{f'(z)}\right\} \geq 0$ *for all pairs* $(z,\zeta) \in U \times U$

(iii) $\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right|$, *for all* $z \in U$

(iv) $1 + \frac{zf''(z)}{f'(z)} \prec q$, *where* $q(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$ *is a Riemann mapping function from* U *to* $\Omega = \{w = u+iv : \; v^2 < 2u-1\} = \{w : \; \operatorname{Re} w > |w-1|\}$.

Note that Ω is the interior of a parabola in the right half-plane which is symmetric about the real axis and has vertex at $(1/2, 0)$.

Denote by T the subclass of S consisting of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \; (n \in \mathbb{N} \setminus \{0,1\}), \; z \in U \tag{2}$$

and denote by $T^*(\alpha)$ and $T^c(\alpha)$ the class of functions of the form (2) that are, respectively, starlike of order α and convex of order α, $\alpha \in [0,1)$, and denote by $UT^c = US^c \cap T$ the class of functions uniformly convex with negative coefficients.

Definition 2. A function f of the form (1) is said to be uniformly convex of α-type, $\alpha \geq 0$ if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \alpha \left|\frac{zf''(z)}{f'(z)}\right|, \tag{3}$$

for all $z \in U$.

We let $US^c(\alpha)$ denote the class of all such functions.

Note that $US^c(0) = S^c$, $US^c(1) = US^c$ and $US^c(\alpha) \subset US^c$ for $\alpha > 1$.

Remark. A function f of the form (1) is in $US^c(\alpha)$ if and only if $1 + zf''(z)/f'(z) \in D$ for all $z \in U$, where D is:

i) for $\alpha > 1$ bounded by the ellipse

$$\frac{\left(u - \frac{\alpha^2}{\alpha^2-1}\right)^2}{\frac{\alpha^2}{(\alpha^2-1)^2}} + \frac{v^2}{\frac{1}{\alpha^2-1}} = 1$$

ii) for $\alpha = 1$ bounded by the parabola

$$v^2 = 2u - 1$$

iii) for $\alpha \in (0,1)$ bounded by the positive branch of the hyperbole

$$\frac{\left(u + \frac{\alpha^2}{1-\alpha^2}\right)^2}{\frac{\alpha^2}{(1-\alpha^2)^2}} - \frac{v^2}{\frac{1}{1-\alpha^2}} = 1$$

iv) for $\alpha = 0$ the half-plane $u \geq 0$

In conclusion $US^c(\alpha) \subset S^c(\alpha/(\alpha+1))$ for $\alpha \geq 0$.

In [5] is defined $UT^c(\alpha) = US^c(\alpha) \cap T$ and it is given a coefficient characterization for this class.

Theorem A. *Let f have the form (1) and $\alpha \geq 0$. f is in $UT^c(\alpha)$ if and only if*

$$\sum_{j=2}^{\infty} j[j(\alpha+1) - \alpha]a_j \leq 1, \tag{4}$$

hence $UT^c(\alpha) = T^c(\alpha/(\alpha+1))$.

Sălăgean [4] introduced the differential operator

$$D^n : A \to A, \quad n \in \mathbb{N},\ A = \{f \in H(U) :\ f(0) = f'(0) - 1 = 0\}$$

defined by $D^0f(z) = f(z)$, $D^1f(z) = Df(z) = zf'(z)$, $D^nf(z) = D(D^{n-1}f(z))$, for $n \geq 2$ and it is easy to prove that

$$D^nf(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j. \tag{5}$$

He also defined the class $S_n(\alpha, \beta)$ of n-starlike functions of order α and type β by

$$S_n(\alpha, \beta) = \{f \in A : \ |J(f, n, \alpha; z)| < \beta\}, \quad \alpha \in [0, 1), \ \beta \in (0, 1], \ n \in \mathbb{N}$$

where

$$J(f, n, \alpha; z) = \frac{D^{n+1}f(z) - D^nf(z)}{D^{n+1}f(z) + (1 - 2\alpha)D^nf(z)}, \quad z \in U. \tag{6}$$

Denote by $T_n(\alpha, \beta) = S_n(\alpha, \beta) \cap T$ the class of functions n-starlike of order α and type β with negative coefficients.

Sălăgean [4] gave a coefficient characterization for this class.

Theorem B. *Let f have the form (2), $\alpha \in [0, 1)$, $\beta \in (0, 1]$. f is in $T_n(\alpha, \beta)$ if and only if*

$$\sum_{j=2}^{\infty} j^n[j - 1 + \beta(j + 1 - 2\alpha)]a_j \leq 2\beta(1 - \alpha). \tag{7}$$

The result is exactly and the extremal functions are

$$f_j(z) = z - \frac{2\beta(1 - \alpha)}{j^n[j - 1 + \beta(j + 1 - 2\alpha)]} z^j, \quad j \in \mathbb{N}_2 = \mathbb{N} \setminus \{0, 1\}. \tag{8}$$

Definition 3. A function f of the form (1) is said to be n-uniformly starlike of type α, $\alpha \geq 0$ and $n \in \mathbb{N}$ if

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^nf(z)}\right\} \geq \alpha \left|\frac{D^{n+1}f(z)}{D^nf(z)} - 1\right| \tag{9}$$

for all $z \in U$.

We let $US_n(\alpha)$ denote the class of all such functions.

Note that $US_0(1) = S_p$ introduced in [3], $US_1(1) = US^c$ and because $US_n(\alpha) \subset S_n(0, 1) \subset S^*$ follow that the uniformly functions of type α are univalents.

Remark. f is in $US_n(\alpha)$ if and only if $D^{n+1}f(z)/D^n f(z) \in D$ for all $z \in U$.

Denote by $UT_n(\alpha) = US_n(\alpha) \cap T$ the class of n-uniformly starlike functions of type α with negative coefficients.

We will give a coefficient characterization for this class.

2. Main results

Theorem 1. *Let f have the form (2), $\alpha \geq 0$, $n \in \mathbb{N}$. Then f is in $UT_n(\alpha)$ if and only if*

$$\sum_{j=2}^{\infty} j^n[j(\alpha+1)-\alpha]a_j \leq 1. \tag{10}$$

The result is exactly and the extremal functions are

$$f_j(z) = z - \frac{1}{j^n[j(\alpha+1)-\alpha]}z^j, \quad j \in \mathbb{N}_2 = \mathbb{N} \setminus \{0,1\}.$$

Proof. Assume that $f \in UT_n(\alpha)$, then

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} \geq \alpha\left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right| \tag{11}$$

for all $z \in U$.

For $z \in [0,1)$ the inequality become

$$\frac{1-\sum_{j=2}^{\infty} j^{n+1}a_j z^{j-1}}{1-\sum_{j=2}^{\infty} j^n a_j z^{j-1}} \geq \alpha\left|\frac{\sum_{j=2}^{\infty} j^n(j-1)a_j z^{j-1}}{1-\sum_{j=2}^{\infty} j^n a_j z^{j-1}}\right|. \tag{12}$$

Since $UT_n(\alpha) \subset T_n(0,1)$ we have:

$$\sum_{j=2}^{\infty} j^{n+1}a_j < 1$$

then

$$\sum_{j=2}^{\infty} j^n a_j z^{j-1} < 1.$$

Inequality (13) reduce to

$$1-\sum_{j=2}^{\infty} j^{n+1}a_j z^{j-1} \geq \alpha\sum_{j=2}^{\infty} j^n(j-1)a_j z^{j-1}$$

and letting $z \to 1^-$ along the real axis, we obtain the desired inequality

$$\sum_{j=2}^{\infty} j^n[j(\alpha+1)-\alpha]a_j \leq 1.$$

Conversely we assume the inequality (11) and it suffices to show that:

$$\alpha\left|\frac{D^{n+1}f(z)}{D^nf(z)}-1\right| - \operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^nf(z)}-1\right\} \leq 1.$$

We have

$$\alpha\left|\frac{D^{n+1}f(z)}{D^nf(z)}-1\right| - \operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^nf(z)}-1\right\} \leq (\alpha+1)\left|\frac{D^{n+1}f(z)}{D^nf(z)}-1\right| \leq$$

$$\leq (\alpha+1)\frac{\sum\limits_{j=2}^{\infty} j^n(j-1)a_j|z|^{j-1}}{1-\sum\limits_{j=2}^{\infty} j^n a_j|z|^{j-1}} \leq (\alpha+1)\frac{\sum\limits_{j=2}^{\infty} j^n(j-1)a_j}{1-\sum\limits_{j=2}^{\infty} j^n a_j} \leq 1$$

according to (11), and the proof is complete. □

Remark. For $n=1$ we obtain the Theorem A.

Corollary 1. *Let f have the form (2). If f is in $UT_n(\alpha)$ then*

$$a_j \leq \frac{1}{j^n[j(\alpha+1)-\alpha]}, \quad j \in \mathbb{N}_2. \tag{13}$$

Corollary 2. *For $\alpha \geq 0$ and $n \in \mathbb{N}$, $UT_n(\alpha) = T_n(\alpha/\alpha+1, 1)$.*

Proof. Replacing α with $\alpha/\alpha+1$, β with 1 in the necessary and sufficient coefficient conditions in Theorem B, we obtain the corresponding coefficient condition of Corollary 2. □

Theorem 2. *If $f \in UT_n(\alpha)$, $\alpha \geq 0$ then*

$$\begin{array}{l} r-\dfrac{1}{2^n(\alpha+2)}r^2 \leq |f(z)| \leq r+\dfrac{1}{2^n(\alpha+2)}r^2 \\ 1-\dfrac{1}{2^{n-1}(\alpha+2)}r \leq |f'(z)| \leq 1+\dfrac{1}{2^{n-1}(\alpha+2)}r^2 \end{array}, \quad |z|=r.$$

The results are the best possible.

Let f and g be two functions of the form (2)

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j \quad \text{and} \quad g(z) = z - \sum_{j=2}^{\infty} b_j z^j$$

then we define the (modified) Hadamard product or convolution of f and g by

$$(f * g)(z) = z - \sum_{j=2}^{\infty} a_j b_j z^j.$$

Theorem 3. *If $f, g \in UT_n(\alpha)$, $\alpha \geq 0$ then $f * g \in UT_n\left(\frac{\rho}{1-\rho}\right)$, where*

$$\rho = 1 - \frac{1}{2^{2n}(\alpha+2)^2 - 1}.$$

This result is sharp.

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"BABEŞ-BOLYAI" UNIVERSITY, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, 3400 CLUJ-NAPOCA, ROMANIA.

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NOTE ON SPREADS AND PARTIAL SPREADS

DĂNUŢ MARCU

Abstract. The aim of this note is to give an answer to a question of [1].

1. Introduction

In this note, we show the existence of a spread, which is not a dual spread, thus answering to a question in [1]. We also obtain some related results on spreads and partial spreads.

Let $\mathbf{P} = PG(2t-1, F)$ be a projective space of odd dimension $(2t-1,\ t \geq 2)$ over the field F. In accordance with [1], we use the following definitions. A partial spread S of $\mathbf{P}$ is a collection of $(t-1)$-dimensional projective subspaces of $\mathbf{P}$, which are pairwise disjoint. S is maximal, if it is not properly contained in any other partial spread. In particular, if every point of $\mathbf{P}$ is contained in some member of S, then S is a spread. If each $(2t-2)$-dimensional projective subspace of $\mathbf{P}$ contains exactly one member of S, then S is called a dual spread.

2. Main results

In the sequel, $|S|$ will denote the number of subspaces in S.

Theorem 1. *If F is finite, then S is a spread if and only if S is a dual spread.*

Proof. Suppose that S is a spread, which is not a dual spread of $\mathbf{P}$. Let δ be any correlation of $\mathbf{P}$ (for the existence of such a δ, see [2, p.41]). Then, S^δ, the image of S under δ, ia a partial spread, which is not a spread. But, $|S^\delta| = |S|$ and F is finite. So, we obtain a contradiction. Similarly, every dual spread is a spread. □

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For simplicity, we now specialize to the case $t = 2$ and we assume that F is commutative, to facilitate the notion of regulus.

We say that a spread S is regular provided that, for every line l of $\mathbf{P}$ which is not in S, the lines of S meeting l form a regulus R of $\mathbf{P}$.

Not al spreads are regular. We can obtain a new non-regular spread S' from S, by the process of replacing some regulus R by its opposite regulus R'. If S' can be obtained from a regular spread S by finitely many iterations of such a process, then S is called subregular.

Theorem 2. *Every regular spread S of |bfP is a dual spread.*

Proof. Let π be any plane of $\mathbf{P}$. Then, π contains at most one line of S. To show that there must be one, let l be any line of π, which is not in S. The lines of S, meeting l, form a regulus R. Let p and q be any two lines of the opposite regulus R', different from l. Then, p and q meet π in distinct point P and Q, not on l. The line PQ of π meets l and, hence, meets three lines of R'. Thus, PQ is a line of R, that is, of S. □

A straightforward extension of this argument yields the following

Theorem 3. *Let S be a spread, which is a dual spread. Suppose that S contains a regulus R. Then, the spread S', obtained from S by replacing the regulus R by its opposite regulus R', is also a dual spread.*

Corollary 1. *Every subregular spread is a dual spread.*

Theorem 4. *There exists a spread S of $\mathbf{P}$, such that S is not a dual spread and no four lines of S are contained in a regulus.*

Proof. Let F be infinite and countable. Choose any plane π and list the points in $\pi(P_1, P_2, P_3, \ldots)$ and the points not in $\pi(Q_1, Q_2, Q_3, \ldots)$. Through P_1, construct the line $l_1 = P_1Q_1$. Suppose that $l_1, l_2, \ldots, l_n$ have been constructed, such that:

(a) no l_i is in π,

(b) no two l_i intersect and

(c) no four l_i are in a regulus.

We now show that l_{n+1} can be constructed in such a way, that (a)-(c) are satisfied also by $\{l_1, l_2, \ldots, l_{n+1}\}$.

If n is odd, let $X = P_j$ be the first pint in π, which is on none of the lines $l_1, l_2, \ldots, l_n$ and $Y = Q_k$ the first point not in π, such that:

(d) Y is on none of the n planes Xl_i, $i = 1, 2, \ldots, n$ and

(e) XY does not belong to any one of the (n_3) reguli determined by $l_1, l_2, \ldots, l_n$.

Then, put $l_{n+1} = XY = P_j Q_k$.

If n is even, let $X = Q_s$ be the first point not in π, which is on none of the l_i, $i = 1, 2, \ldots, n$ and $Y = P_t$ the first point in π, such that (d) and (e) are satisfied. Then, put $l_{n+1} = XY = Q_s P_t$.

Clearly, $l_1, l_2, \ldots, l_{n+1}$ satisfy the conditions (a)-(c). Furthermore, our construction guarantees that each point of $\mathbf{P}$ is on a line of S. Thus, the theorem is proved. □

There is an interesting consequence of the Theorem 4, that is,

Corollary 2. *Maximal partial spreads W, which are not spreads, exist in* $\mathbf{P}$.

Proof. Consider the image W of S, under any correlation of $\mathbf{P}$. □

Remark. the above corollary is also true if F is finite (for an example in $PG(3,4)$, see [3]).

We end this note with the following

Conjecture. *There exist such maximal partial spreads W, with $q^2 - q + 1 \leq |W| \leq q^2 - q + 2$ in $PG(3,q)$, for any q.*

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STR. PASULUI 3, SECT.2, 70241 BUCHAREST, ROMANIA

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FUZZY SYSTEMATIC SPACES

M.R. MOLAEI

Abstract. In this paper fuzzy systematic spaces are considered. In terms of definitions of fuzzy cover and fuzzy sheaf it will be defined fuzzy cohomology in the sense of Čech.

1. Introduction

In philosophy [1,2,3] there are some definitions of systems. These definitions help us to define fuzzy systematic spaces, which are suitable spaces for description of language [6]. It will be shown that on a consistent fuzzy systematic space the notion of fuzzy cover is acquired, so we can define fuzzy sheaf cohomology in the sence of Čech.

Fuzzy relations have been studied by Zadeh [8], Kaufman [5], Rosenfeld [7] and in this paper in the one hand the fuzzy relations are used and in the other hand a fuzzy systematic space is explained.

Definition 1.1. A fuzzy system for the fuzzy set X is a collection $S = \{R_\gamma\}$, $\gamma \in \Gamma$ which satisfies the following conditions:

(*i*) $R_\gamma \subset X \times Y_\gamma$ are fuzzy relations;

(*ii*) for every $x \in X$ there exist $\gamma \in \Gamma$ and y_γ such that $(x, y_\gamma) \in R_\gamma$.

A fuzzy systematic space is an order pair (X, S).

Example 1.2. Let $(C, \subset)$ be a Site [4], for all $a \in C$ define $R_a = \{(a,b) : b \subset a\}$ and $\mu_{R_a}(a,b) := \max\{\mu_C(a), \mu_C(b)\}$. Then $(C, \{R_a\})$ is a fuzzy systematic space. (μ is a membership function.)

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Definition 1.3. Let (X, S) be a fuzzy systematic space. Then an object $R \in S$ is called a speciality of S if for all $\gamma \in \Gamma$, $R \cap R_\gamma \in S$.

Definition 1.4. A consistent fuzzy systematic space is a fuzzy systematic space with the following condition:

For all R_1 and R_2 belong to S, $R_1 \cap R_2 \in S$ and $\mu_{R_1|_{R_1 \cap R_2}} = \mu_{R_2|_{R_1 \cap R_2}}$

An inconsistent fuzzy systematic space is a fuzzy systematic space which is not consistent.

Theorem 1.5. *Let (X, S) be a fuzzy systematic space. Then there are subsets, X_c, X_I of X and S_c, S_I of S such as:*
(i) $X = X_c \cup X_I$;
(ii) $S = S_c \cup S_I$, and $S_c \cap S_I = \emptyset$;
(iii) (X_c, S_c) is a consistent fuzzy systematic space and (X_I, S_I) is an inconsistent fuzzy systematic space that has no speciality.

Proof. Let

$$\begin{aligned} S &= \{R_\gamma\} \text{ put} \\ S_c &= \{R_\gamma : (\forall \beta \in \Gamma)(R_\gamma \cap R_\beta \in S)\}; \\ S_I &= S - S_c, \\ X_I &= \{x \in X : (x, y) \in R_\gamma \text{ for some } R_\gamma \in S_c\}; \\ X_I &= \{x \in X : (x, y) \in R_\gamma \text{ for some } R_\gamma \in S_I\}. \blacksquare \end{aligned}$$

□

Definition 1.6. If $S' \subset S$, then the fuzzy systematic space (X, S') is called a sub-systematic space of (X, S).

2. Fuzzy Sheaf

There must be a definition of fuzzy Grothendieck topology on S, define as a map:

"Element of $S \longmapsto A$ subset of the powerset"

with the following conditions:

(a) $\{R\} \in G(R)$ for all $R \in S$;

(b) If $\{R_i\} \in G(R)$ then $R_i \subset R$ and $\mu_{R_i} = \mu_{R|_{R_i}}$.

A fuzzy cover for $R \in S$ is an element of $G(R)$.

Definition 2.1. A subset U of the fuzzy system S is a fuzzy cover for S if for all $R \in S, U$ contains at least one fuzzy cover of R.

A fuzzy cover U' is a finer fuzzy cover than U, if for all $R' \in U'$ there exists $R \in U$ so that $R' \subset R$ and $\mu_{R'}$ be equal to μ_R on R'.

We define a fuzzy presheaf P on a fuzzy systematic space (X, S) as a map;

$$\begin{aligned} \text{Object in } \; S \; &\longrightarrow \; \text{Fuzzy abelian groups} \\ R \; &\longmapsto \; \Gamma(R, P) \end{aligned}$$

together with restriction maps

$$\rho_{RF} : \Gamma(F, P) \longrightarrow \Gamma(R, P) \quad \text{if} \quad R \subset F$$

that satisfies the following properties;

(i) The restrictions are fuzzy group homomorphisms;

(ii) If $R \subset F$ and $G \subset R$ then $\rho_{FF} = id$, $\rho_{GR} o \rho_{RF} = \rho_{GF}$.

A fuzzy presheaf P on systematic space (X, S) is a sheaf if satisfies the following conditions:

(iii) For every fuzzy cover $\{R_i\}$ of R and $a, b \in \Gamma(R, P)$, if $a|_{R_i} = b|_{R_i}$ for all i, then $a = b$;

(iv) For every fuzzy cover $\{R_i\}$ of R, if $a_i \in \Gamma(R_i, P)$ and $a_i|_{R_j} = a_j|_{R_i}$ for all i, j, then there exists $a \in \Gamma(R, P)$ such that $a|_{R_i} = a_i$.

Example 2.2. Suppose that X is an n-dimensional complex fuzzy manifold

$$S = \{R_{UV} = U \times V : U \; \text{ and } \; V \; \text{ are charts of } \; X\}$$

and $\mu_{R_{UV}}(u, v) = \max\{\mu_X(u), \mu_X(v)\}$ for all $(u, v) \in U \times V$;

(a) If $\Gamma(U \times V, B)$ be the fuzzy group of bounded holomorphic functions on chart $U \times V$ of $X \times X$ then the map, $R_{UV} \longrightarrow \Gamma(U \times V, B)$ is a fuzzy presheaf that is not a fuzzy sheaf.

(b) If $\Gamma(U \times V, O(m))$ be the fuzzy group of homogeneous holomorphic functions of degree m on chart $U \times V$ then the map $R_{UV} \longrightarrow \Gamma(U \times V, O(m))$ is a fuzzy sheaf.

3. Fuzzy Cohomology

Now we define fuzzy cohomology in the sence of Čech for a consistent fuzzy systematic space.

Suppose that U is a fuzzy cover for a consistent fuzzy systematic space (X, S). A q-simplex is a $q+1$ tuple of elements of U. For $\delta = (R_0, R_1, \ldots, R_q)$ define $|\delta| = R_0 \cap R_1 \cap \cdots \cap R_q$. A q-cochain with respect to U with coefficients in a fuzzy sheaf P is a map;

$$\begin{array}{rcl} \{\delta : \delta \text{ is a } q-\text{simplex}\} & \xrightarrow{f} & U\Gamma(\delta, P) \\ \delta & \longmapsto & f(\delta) \end{array}$$

If $\delta = (R_{i_0}, R_{i_1}, \ldots, R_{i_q})$ then we denote $f(\delta)$ by $f_{i_0 i_1 \ldots i_q}$ and $\{f_{i_0 i_1 \ldots i_q} \in \Gamma(R_{i_0} \cap R_{i_1} \cap \cdots \cap R_{i_q}, P)\}$ is a called q-cochain. The set of these q-cochains is denoted by $C^q(U, P)$. The coboundary operator is:

$$\{f_{i_0 i_1 \ldots i_q}\} \xmapsto{\delta_{q+1}} \{\rho_{[i_0 f_{i_1} \cdots i_{q+1}]}\}$$

where ρ_{i_0} is restriction to R_{i_0}.

As usual we define $Z^q(U, P) = Ker\delta_{q+1}$ and $B^q(U, P) = I_m\delta_q$. The set $H^q(U, P) = Z^q(U, P)/B^q(U, P)$ is called the fuzzy Čech cohomology of P with respected to U and the set $H^q(S, P) = \lim_{U} \text{ind} H^q(U, P)$ is called the fuzzy Čech cohomology of (X, S) with cofficients in the fuzzy sheaf P, where limind is the inductive limit.

Theorem 3.1. *Let (X, S) be a fuzzy systematic space and suppose that there exists $R \in S$ so that R is the maximal element of S with respected to the inclusion. Then $H^0(S, P) = \Gamma(R, P)$.*

Proof. Let $d \in H^0(S, P) = (U^0 H^0(U, P)/ \sim)$ that $\sim$ is the usual equivalence relation. So $d = [(g_i)]$ where $g_i \in \Gamma(R_i, P)$ for some fuzzy cover V of S. By the condition $\delta g = 0$, we have;

$$(\delta g)_{ij} = g_j - g_i = 0 \quad \text{on} \quad R_i \cap R_j.$$

Therefore there is a global section $g \in \Gamma(R, P)$ which agrees with the g_i locally.■ □

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FACULTY OF MATHEMATICS, SHAHID BAHONAR UNIVERSITY OF KERMAN KERMAN, IRAN, P.O.Box 133-76135
E-mail address: mrmolaei@arg3.uk.ac.ir

SOME QUALITATIVE PROPERTIES OF THE SOLUTIONS TO QUASI-LINEAR DIFFERENTIAL INCLUSIONS

MARIAN MUREŞAN

Abstract. The aim of the present paper is to introduce some recent results on the existence and on some qualitative properties of the solutions to some differential inclusions of evolution.

1. Introduction

By a study of some qualitative properties of the set of solutions to a differential inclusion (similarly to the case of a differential equation) we mean a study of one or several aspects in connection with: the existence of a solution, dependence on the initial value, parameter and/or right hand side, connectedness of the set of solutions, relaxation, periodicity and/or stability of the solution(s), etc. without a straight access to the solution(s).

Let us recall some well-known facts from the theory of differential equations. Consider $I = [0,T]$, $0 < T$, $X = \mathbb{R}^n$. We have the following result

Theorem 1 ([23], p. 10). *Let $x, f \in X$; $f(t,x)$ is continuous on $I \times \{x \mid \mid x - x_0 \mid \leq b\}$; M is a bound for $\mid f(t,x) \mid$ on $I \times \{x \mid \mid x - x_0 \mid \leq b\}$; $\alpha = \min\{a, b/M\}$. Then*

$$x' = f(t,x), \quad x(0) = x_0 \tag{1}$$

possesses at least one solution $x = x(t)$ on $[0, \alpha]$.

Remarks. In this case each solution is *continuously differentiable* on the interval $]0, \alpha[$. The solution is not unique. For uniqueness it is necessary an extra assumption, e.g. a Kamke condition. The above theorem fails in an infinite dimensional spaces, [6].

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Adding a Lipschitz condition in respect to the second variable, the local existence is guaranteed.

When function f is not longer continuous, but it is continuous in x for almost all t and measurable in t for all x (it is a *Carathéodory function*), then we have

Theorem 2 ([21], p. 4). *For $t \in I$, $|x - x_0| \leq b$ let f be a Carathéodory function and $|f(t,x)| \leq m(t)$, the function m being summable. Then on the closed interval $[0,d]$, where $d > 0$, there exists a solution of the initial value problem* (1). *In this case d satisfies: $d \in (0,T]$, $\phi(t) := \int_0^t m(s)\,ds$, $\phi(t+d) \leq b$.*

Now a solution is an *absolutely continuous* function on I. The assumptions are weaker, the conclusions are weaker, too.

These two situations may be encountered similarly in the case of more general Banach spaces. We have to notice that in a general Banach space X an absolutely continuous function defined on I is not almost everywhere differentiable on $]0,T[$. But if the Banach space X is *reflexive*, thanks to a theorem of Komura, [6, p. 16], we know that every X-valued absolutely continuous function x on I is a.e. differentiable on $]0,T[$ and $x(t) = x(0) + \int_0^t (dx/ds)(s)\,ds$, $t \in I$. Here the integral is considered as a *Bochner integral*, [18] or [17].

Now we take a look to the case of linear differential equations when $X = \mathbb{R}^n$. Consider the following two systems of ordinary differential equations

$$x'(t) = A(t)x(t) \tag{2}$$

$$x'(t) = A(t)x(t) + f(t), \quad x(0) = x_0, \tag{3}$$

where A is a $n \times n$ matrix and f is a function, both continuous on I. If Y is the fundamental matrix of equation (2), then the solution of equation (3) is given by

$$x(t) = Y(t,0)x_0 + \int_0^t Y(t,s)f(s)\,ds, \tag{4}$$

where $Y(t) = Y(t,0)$. Obviously, this solution is continuously differentiable on I. If $f \in \mathcal{L}^1(I)$, then x is absolutely continuous on I.

If X is a Banach space, the function defined by (4) is said to be the *mild solution* of equation (3), if it exists. The study of the equations of the form (3) in infinite dimensional spaces is performed, e.g., [6], [34], [47], [48]; applications [34], [41].

Now we turn for a while to observe some very elementary properties of the differential inclusions. For the beginning we remark that a differential inclusion is a differential equation whose the right-hand side is a set-valued function (multifunction, correspondence, etc.), [21], [3], [16], [31].

Consider the following differential inclusion $x' \in \{-1,+1\}$, $x(0) = 0$, $t \in [0,1]$. We see that if we require that the solution to be continuously differentiable, then the set of solutions is very poor; but if we permit to a solution to be absolutely continuous, then the set of solutions is rich enough. In this case one can construct a sequence of solutions x_n converging uniformly to the constant function $x = 0$. But $x = 0$ is not a solution, hence the set of solutions is not always a closed set.

A classical way to obtain a differential inclusion, [3], [4], is that, starting from a dynamical system $x'(t) = f(t, x(t), u(t))$, $x(t_0) = x_0$, "controlled" by the parameters $u \in U$, to define $F(t, x(t)) = \{f(t, x(t), u(t))\}_{u \in U}$. For definitions of the solution of a differential inclusion we refer to [21]. The coincidence of the sets of solutions was studied for the first time by Wazewski in [57]. The set-valued functions and differential inclusions are useful tools not only in control problems, but also in economical problems, [31].

Let Z be a linear topological space. We will use the following notations: $\mathrm{P}(Z) = \{A \subset Z \mid A \neq \emptyset\}$, $\mathrm{C}(Z) = \{A \in \mathrm{P}(Z) \mid A \text{ closed}\}$, $\mathrm{CCo}(Z) = \{A \in \mathrm{C}(Z) \mid A \text{ convex}\}$, $\mathrm{KCo}(Z) = \{A \in \mathrm{P}(Z) \mid A \text{ compact and convex}\}$.

The *Hausdorff-Pompeiu metric* of the sets $A, B \in \mathrm{C}(X)$ ((X, ρ) is a metric space) is defined by $\mathrm{D}(A,B) = \max\{\mathrm{d}(A,B), \mathrm{d}(B,A)\}$ where $\mathrm{d}(A,B) = \sup\{\mathrm{d}(x,B) \mid x \in A\}$. Several properties of the Hausdorff-Pompeiu metric may by found in [11], [8].

Let I be the interval $I = [0,T]$, $T > 0$ fixed, and X a Banach space. A family of bounded linear operators $\mathcal{U}(t,s)$, on X, $0 \le s \le t \le T$, depending on two parameters is said to be an *evolution system*, [48], if there are fulfilled the following two conditions $\mathcal{U}(s,s)=1$, $\mathcal{U}(t,r)\mathcal{U}(r,s)=\mathcal{U}(t,s)$ for $0\leq s\leq r\leq t\leq T$; $(t,s)\rightarrow\mathcal{U}(t,s)$ is strongly continuous for $0\leq s\leq t\leq T$, where by strongly continuity is meant that $\lim_{t\searrow s}\mathcal{U}(t,s)x=x$, for all $x\in X$.

By a *Cauchy problem for a quasi-linear differential inclusion* we mean

$$\frac{dx(t)}{dt}\in A(t,x(t))x(t)+F(t,x(t)),\quad \text{a.e. } t\in I,\quad \text{and } x(0)=x_0, \tag{CP}$$

where $A(t,w)$ is a linear operator from X to X depending on $t\in I$ and $w\in X$, and F is a set-valued map.

If operator A depends on t and w, the differential inclusion in (CP) is said to be *quasi-linear*, if A depends only on t, the differential inclusion is said to be *semi-linear*, and if A depends neither on t nor on w, the differential inclusion is said to be *linear*.

We are interested to study the *mild solutions* of the (CP), i.e. continuous functions having the following representation

$$x(t)=\mathcal{U}_x(t,0)a+\int_0^t\mathcal{U}_x(t,s)f(s)\,ds\quad t\in I,\ f\in S^1_{F(\cdot,x(\cdot))},$$

where $S^1_{F_x}=S^1_{F_x(\cdot)}=S^1_{F(\cdot,x(\cdot))}$ is the set of Bochner integrable selections from $F(\cdot,x(\cdot))$.

We use the following assumptions:

(X_1) X is a separable Banach space;

(X_2) X satisfies (X_1) and, moreover, it is reflexive;

(A) For every $u\in C(I,X)$ the family of linear operators $\{A(t,u)\mid t\in I\}$ generates a unique strongly continuous evolution system $\mathcal{U}_u(t,s)$, $0\leq s\leq t\leq T$;

(U_1) If $u\in C(I,X)$, the evolution system $\mathcal{U}_u(t,s)$, $0\leq s\leq t\leq T$ satisfies

(i) there exists a $c_1\geq 0$ with $\|\mathcal{U}_u(t,s)\|\leq c_1$ for $0\leq s\leq t\leq T$, uniformly in u;

(ii) there exists a $c_2 \geq 0$ such that for any u, $v \in C(I,X)$ and any $w \in X$ we have

$$\|\mathcal{U}_u(t,s)w - \mathcal{U}_v(t,s)w\| \leq c_2\|w\| \int_s^t \|u(\tau) - v(\tau)\|\, d\tau;$$

(U_2) If $u \in C(I,X)$ and $0 \leq s \leq t \leq T$, then $\mathcal{U}_u(t,s)$ is a compact operator, i.e. it transforms bounded sets in relatively compact sets. In this case, [48, p. 48], $\mathcal{U}_u(t,s)$ is continuous in the uniform operatorial topology.

(U_3) If $t, t+\delta \in I$, $\delta > 0$, then $\lim_{\delta\to 0}\mathcal{U}_u(t+\delta,t) = 1$, uniformly in u and t.

Remarks. If operator A does not depend on w, but it depends on t, then the assumption (A) reads as follows: $\{A(t) \mid t \in I\}$ generates a unique strongly continuous evolution system $\mathcal{U}(t,s)$, $0 \leq s \leq t \leq T$. In this case we take $c_2 = 0$ (in (ii) from (U_1)). The dependence that is used in (U_1) (ii) was inspired by [48, p. 202], [32, lemma 2.2].

In connection with the multifunction F we will use the following assumptions:

(F_1) $F : I \times X \to \mathrm{C}(X)$ and for any $x \in X$, $F(\cdot, x)$ is measurable;

(F_2) $F : I \times X \to \mathrm{CCo}(X)$ and for any $x \in X$, $F(\cdot, x)$ is measurable;

(F_3) F satisfies (F_1) and for any $t \in I$, $F(t,\cdot) : X \to \mathrm{C}(X)$ is lower semicontinuous from X in $\mathrm{C}(X)$ and it is upper semicontinuous from X in $\mathrm{C}(w{-}X)$, where $w{-}X$ is X endowed with the weak topology;

(F_4) F satisfies (F_1), it is product-measurable and for all $t \in I$, $F(t,\cdot) : X \to \mathrm{C}(X)$ is upper semicontinuous;

(F_5) F satisfies (F_1) and, moreover, it is $k(t)$-Lipschitz, i.e. exists $k \in \mathcal{L}^1(I, \mathbb{R}_+)$ such that for almost all $t \in I$ and for all x, $y \in X$, $\mathrm{D}(F(t,x), F(t,y)) \leq k(t)\|x-y\|$, D being the Hausdorff-Pompeiu metric;

(F_6) F is integrably bounded by a function $m \in \mathcal{L}^1(I, \mathbb{R}_+)$, that is, for all $x \in C(I,X)$ and $t \in I$ we have $F(t,x(t)) \subset m(t)B$, B is the closed unit ball in X;

(F_7) the function $t \mapsto \mathrm{d}(0, F(t,0))$ is integrable on I.

By an *inclusion of evolution* we mean an inclusion of the following form

$$\frac{dx(t)}{dt} \in A(t, x(t))x(t) + F(t, x(t)), \quad \text{a.e. } t \in I.$$

Remark. The evolution inclusions have been investigated in a series of works: [48], [47], [50], [2], [44], [45], [1], etc. A different approach of evolution inclusions is used in [5], [28]. Their approach is based on the Galerkin approximations (e.g. [15], [33]).

2. Existence of solutions

We need the next assumption (M_1) If A depends on w, then for any $t \in I$, $M_b(t)$ is relatively compact in X, where $b = (||x_0|| + ||m||_1)c_1$,

$$M_b = \{x \in C(I, X) \mid x(0) = x_0,\ ||x|| \le b\}, \quad M_b(t) = \{y(t) \mid y \in \psi(x), x \in M_b\},$$

$$\psi(x) = \{y \mid y(t) = \mathcal{U}_x(t,0)x_0 + \int_0^t \mathcal{U}_x(t,s)f(s)\,ds,\ x(0) = x_0,\ f \in S^1_{F_x}\}.$$

Based on fixed point techniques we have proved the following two existence theorems

Theorem 3 ([40]). *If there are satisfied the following assumptions (X_2), (A), (U_1), (F_2), (F_{5-6}) and if* $0 < c_3 < 1$, ($c_3 = c_2T(||x_0|| + ||m||_1) + c_1||k||_1$), *then there exists a mild solution of the problem (CP) in M_b.*

Theorem 4 ([40]). *Suppose there are satisfied the following assumptions*

(i) (X_2), (A), (U_1), (U_3), (F_2), (F_{5-6});

(ii) (M_1) *or* (U_2);

(iii) (F_3) *or* (F_4).

Then there exists a mild solution in M_b of the (CP) problem.

Theorem 3 uses a multivalued version of the Banach fixed point principle, while theorem 4 is based on the Bohnenblust-Karlin fixed point theorem, [16], [49]. More refined existence results may be obtained by the method developed in [19], [20].

3. Filippov-Gronwall theorems

By a Filippov-Gronwall theorem we understand a result which from the existence of a function or a solution of a differential equation ensures the existence of a solution of an other differential equation or inclusion provided a "closeness" condition is satisfied.

First results on this topic have been published by Filippov in [19], [20]. Later there were published more and more papers in connection with Filippov's results, let us mention just a few of them: [25], [29], [42], [26], [3], [16] and [59]. In the case of linear evolution inclusions such a problem has been investigated in [22] and [55]. Tolstonogov, in [55], studied also the case when A is an m-dissipative operator.

In the sequel we consider a new Cauchy problem

$$\frac{dy(t)}{dt} = A(t, y(t))y(t) + g(t), \quad g \in \mathcal{L}^1(I,X), \text{ a.e. } t \in I, \quad \text{and } y(0) = y_0. \tag{5}$$

(S_1) Suppose that problem (5) has a mild solution $y(t) = \mathcal{U}_y(t,0)y_0 + \int_0^t \mathcal{U}_y(t,s)g(s)\, ds$, $t \in I$.

It is shown that if the initial values and the nonlinear parts to (CP) and (5) are sufficiently close and (S_1) it holds, then problem (CP) has a mild solution x whose distance to y does not exceed a certain value.

We consider problems (CP) and (5) under the assumptions (X_1), (A), (F_5) and (F_6). Denote $\delta = \|x_0 - y_0\|$, $p = c_2(\|x_0\| + \|m\|_1)$, $k_\varepsilon(t) = k(t) + \varepsilon$, $\varepsilon > 0$, $K(t) = \int_0^t [p + k_\varepsilon(s)]\, ds$, $E(t) = \exp(K(t))$, $t \in I$. Moreover, we admit assumption (S_1) and let $\gamma(t) = \mathrm{d}(g(t), F(t, y(t)))$, $t \in I$. Based [40, lemma 2.15] we have that $\gamma \in \mathcal{L}^1$ and then consider $n(t) = [\delta + \int_0^t (\gamma(s) + \varepsilon)\, ds]$, $t \in I$.

Theorem 5 ([38]). *Suppose that the following assumptions are fulfilled:* (X_1), (A), (U_1), (F_5), (F_6), (F_7) *and* (S_1). *Then problem (CP) has a mild solution* $x \in C(I, X)$ *such that*

$$\|x(t) - y(t)\| \le n(t)E(t) = c_1 \left[\delta E(t) + E(t) \int_0^t (\gamma(s) + \varepsilon)\, ds\right], \quad t \in I, \tag{6}$$

$$\|f(t) - g(t)\| \le \gamma(t) + \varepsilon + n(t) k_\varepsilon(t) E(t) \text{ a.e. } t \in I. \tag{7}$$

The method of the proof (as in [19], [20], [22], [55]) consists in constructing two convergent sequences $(x_n)_{n\geq 1} \subset C(I, X)$ and $(f_n)_{n\geq 1} \subset \mathcal{L}^1(I, X)$ such that x, the limit of $(x_n)_{n\geq 1}$ in the uniform topology from $C(I, X)$, is the mild solution of the problem (CP) and it satisfies (6). f, the limit of the sequence $(f_n)_{n\geq 1}$ in $\mathcal{L}^1(I, X)$, satisfies (7) and appears in the formula of x.

Remark. If problem (CP) is linear, we get a result from [22]. Obviously, in this case the assumption (F_6) is useless and $c_2 = 0$ implies that $p = 0$.

Theorem 6 ([38]). *Suppose there are satisfied all the assumptions of theorem 5. Then problem (CP) has a mild solution $x \in C(I,X)$ such that*

$$\|x(t) - y(t)\| \leq c_1 \left[\delta E(t) + \int_0^t \frac{E(t)}{E(s)}(\gamma(s) + \varepsilon)\, ds\right], \quad t \in I \tag{8}$$

$$\|f(t) - g(t)\| \leq \gamma(t) + \varepsilon + k_\varepsilon(t) c_1 \left[\delta E(t) + \int_0^t \frac{E(t)}{E(s)}(\gamma(s) + \varepsilon)\, ds\right], \quad a.e.\ t \in I.$$

Remark. It is obvious now, comparing (6) and (8), that the estimations in theorem 6 are better than the estimations in theorem 5.

Theorem 7 ([38]). *Suppose there are satisfied all the assumptions of the theorem 6 with the only change that instead of the function n we consider $\overline{n}(t) = c_1 \left[\delta + \int_0^t 2\gamma(s)\, ds\right]$, K is replaced by $\overline{K}(t) = \int_0^t [p + 2c_1 k(s)]\, ds$, and E is replaced by $\overline{E}(t) = \exp(\overline{K}(t))$, $t \in I$. Then problem (CP) has a mild solution $x \in C(I,X)$ such that*

$$\|x(t) - y(t)\| \leq c_1 \left[\delta \overline{E}(t) + \int_0^t \frac{\overline{E}(t)}{\overline{E}(s)} 2\gamma(s)\, ds\right], \quad on\ I$$

$$\|f(t) - g(t)\| \leq 2\gamma(t) + 2k(t) c_1 \left[\delta \overline{E}(t) + \int_0^t \frac{\overline{E}(t)}{\overline{E}(s)} 2\gamma(s)\, ds\right], \quad a.e.\ t \in I.$$

Remark. In [55] it is considered only the case of linear inclusions. This fact implies that the evolution system $\mathcal{U}$ depends only on t. Thus condition (ii) in (U_1) is useless and we may take $c_2 = 0$ (hence $p = 0$). The assumption (ii) in (U_1) gives us the dependence way of the evolution system $\mathcal{U}_u$ in respect to the function $u \in C(I,X)$. This kind of dependence may be met in the so called "hyperbolic" case, [48], [41, p 272]. Theorem 7 contains, for the case of linear inclusion, theorem 3.2 in [55].

4. Continuous dependence results

It is useful to know the manner of dependence of the set of solutions of an initial value problem upon the initial value, a parameter or the right hand side of an equation or inclusion. The continuous dependence results are widely used in numerical methods, too. For ODE dependence results may be found in many books, for instance [23].

Recent dependence results on differential inclusions may be found in: [43], [51], [59], [55], [13] and [14].

The uniqueness of the mild solution of the Cauchy problem for quasi-linear equations follows from

Theorem 8 ([40]). *Let $f, g \in \mathcal{L}^1(I, X)$, $\chi = \|f - g\|_1$ and $\delta = \|x_0 - y_0\|$ such that there are satisfied all the assumptions of theorem 5 taking f instead of F. Denote by x and y two mild solutions of the quasi-linear equations corresponding to f, x_0, respectively g, y_0. Then the following estimation holds*

$$\|x(t) - y(t)\| \leq c_1(\chi + \delta) \exp\left[c_2(\min\{\|x_0\|, \|y_0\|\} + \min\{\|f\|_1, \|g\|_1\})t\right], \quad t \in I.$$

Corollary 1 ([40]). *If the assumptions of the above theorem are satisfied, then*

$$\|x - y\|_{C(I,X)} \leq c_1(\delta + \chi) \exp\left[c_2(\min\{\|x_0\|, \|y_0\|\} + \min\{\|f\|_1, \|g\|_1\})T\right].$$

Corollary 2 ([40]). *If the assumptions of the above theorem are satisfied and if $\delta = \chi = 0$, then $x = y$, hence the mild solution of problem* (5) *is unique.*

Remark. If in problem (5) the operator A is linear, then the uniqueness problem is discussed, for instance, in [48, p. 106].

Let us denote by $\mathcal{S}(x_0)$ the set of mild solutions of (CP). For problem (CP) there holds a Lipschitz dependence upon the initial values:

Theorem 9 ([38]). *Suppose there are satisfied all the assumptions of theorem 7 and, moreover, that* (5) *is a differential inclusion having F instead of g. Then*

$$D(\overline{\mathcal{S}(x_0)}, \overline{\mathcal{S}(y_0)}) \leq L\|x_0 - y_0\|,$$

where $L = c_1\overline{E}(T)$.

Corollary 3 ([38]). *If in the inclusion (CP) A does not depend on w, then we obtain theorem 4.1 in* [55].

Corollary 4 ([38]). *If in the above mentioned theorem we consider* $A \equiv 0$, *then we get corollary 1,* [3, p. 121], *and if we suppose, moreover, that F is single-valued, then we obtain estimation* (4), *in* [3, p. 119].

Remark. Under the assumptions of the above mentioned theorem the set-valued map $x_0 \mapsto \mathcal{S}(x_0)$ is globally Lipschitz. This set-valued map is studied under various assumptions, for instance, in [55] and [59].

Now we are interested in dependence of the set of solutions upon a parameter.

Definition. Suppose that assumptions (X_1) and (A) are satisfied. Then a function $x(\cdot,\xi) : I \times X \to X$ is said to be a *mild solution* of the problem (CP) with $a = \xi$ if there exists $f(\cdot,\xi) \in \mathcal{L}^1(I,X)$ such that

$$f(t,\xi) \in F(t,x(t,\xi)), \qquad \text{a.e. on } I,$$

$$x(t,\xi) = \mathcal{U}_{x(\cdot,\xi)}(t,0)\xi + \int_0^t \mathcal{U}_{x(\cdot,\xi)}(t,s)f(s,\xi)\,ds, \quad \text{for each } t \in I.$$

We need the following hypothesis (which is (S_1) with $g \equiv 0$):

(S_1'). The next problem

$$\frac{dx(t)}{dt} = A(t,x(t))x(t), \quad \text{a.e. } t \in I, \quad \text{and } x(0) = \xi,$$

has a mild solution, let it be $x_0(t,\xi) = \mathcal{U}_{x_0(\cdot,\xi)}(t,0)\xi$, for all $t \in I$.

Theorem 10 ([38]). *Suppose the following assumptions are satisfied:* (X_1), (A), (U_1), (F_5), (F_6), (F_7) *and* (S_1'). *Denote by* $\mathcal{S}(\xi)$ *the set of solutions of the problem (CP), the initial value being equal to* ξ, $(a = \xi)$. *Then there exists a function* $x(\cdot,\cdot) : I \times X \to X$ *such that*

$$x(\cdot,\xi) \in \mathcal{S}(\xi) \quad \text{for each } \xi \in X,$$

$$\xi \to x(\cdot,\xi) \quad \text{is continuous from } X \text{ in } C(I,X).$$

Partially, the method is the same to the method used in the proof of the theorem 6. Here we do not search integrable selections, but based on theorem 3.1 in [24] and proposition 2.2 in [14] at each iteration we choose a continuous selection.

Remark. If the differential inclusion in (CP) is linear, then we recover a result from [51] or theorem 3.3 in [52]. If the differential inclusion in (CP) is semi-linear, then $c_2 = 0$ and the assumption (F_6) are unnecessary.

5. Connectedness of the set of solutions

The connectedness and the arcwise connectedness of the set of solutions is a topic discussed in several papers such as [51], [52], [53], [54] and [56].

In [56] it is studied problem (CP) with A depending on t only and it is shown that if A is an m-dissipative operator or if it is linear and the infinitesimal generator of a strongly continuous semi-group, then the set of solutions of problem (CP) is connected. The method introduced in [56] may be used also to the case of a quasi-linear inclusion. More exactly we have the following result

Theorem 11. *We suppose that the assumptions (X_1), (A), (U_1), (F_5), (F_6), (F_7) and (S_1) are fulfilled and consider problem (CP). Then $\mathcal{S}(x_0)$ is closed and connected.*

Proof. In addition to the Lebesgue measure on I we consider the measure defined by $d\mu = \exp\left(-2c_1c_{20}\int_0^t k(s)\,ds\right)\,dt$, where $c_{20} = \exp(p)$, $p = c_2(\|x_0\| + \|m\|_1)$. The two measures are equivalent, [7, p. 157]. Also consider the space $\mathcal{L}_1(I,\mu,X)$ of (classes of) Bochner integrable functions in respect to the measure μ and to the norm $\|f\|_{1*} = \int_0^T \|f(t)\|\,d\mu(t)$. Let j be the identity map $j : \mathcal{L}^1(I,X) \to \mathcal{L}^1(I,\mu,X)$. We see that the norms $\|\cdot\|_1$ and $\|\cdot\|_{1*}$ are equivalent. Thus j establishes an homeomorphism between the spaces $\mathcal{L}^1(I,X)$ and $\mathcal{L}^1(I,\mu,X)$.

For $z \in C(I,X)$ from lemma 2.2 in [38] we have that $t \mapsto F(t,z(t))$ is measurable, and by the assumptions we have that it has closed values and it is integrably bounded. It follows that $S^1_{F_z} \neq \emptyset$, and by [24, theorem 3.2], it results that $S^1_{F_z}$ is a bounded set in $\mathcal{L}_1(I,X)$. By theorem [24, theorem 3.1] we know that $S^1_{F_z}$ is a decomposable set. Hence $S^1_{F_x} \in \mathcal{D}$ and $S^1_{F_z}$ is a bounded set in $\mathcal{L}_1(I,X)$.

Take $f \in \mathcal{L}_1(I,X)$. Based on theorem 8 the equation (5) with $g = f$ has a unique mild solution on I. We define the map $d : \mathcal{L}_1(I,X) \to C(I,X)$ such that to a member $f \in \mathcal{L}_1(I,X)$ corresponds the mild solution of the above mentioned quasi-linear equations, namely $d(f) = x$ iff $x(t) = \mathcal{U}_x(t,0)x_0 + \int_0^t \mathcal{U}_x(t,s)f(s)\,ds$, $t \in I$. From the corollary 1 it follows that d is a Lipschitz map, so it is continuous. But d is also one to one.

Let us take the following set-valued map $\Phi : \mathcal{L}_1(I,X) \to \mathrm{C}(\mathcal{L}_1(I,X))$ defined by $\Phi(f) = S^1_{F(\cdot,d(f))}$. It follows that $\Phi(f) \in \mathcal{D}$ and $\Phi(f)$ is bounded in $\mathcal{L}_1(I,X)$, for each $f \in \mathcal{L}_1(I,X)$.

There holds the equality $\mathcal{S}(x_0) = \{d(f) \mid f \in \Phi(f)\}$. If $x \in \mathcal{S}(x_0)$ there exists $f \in S^1_{F_x}$ such that $x(t) = \mathcal{U}_x(t,0)x_0 + \int_0^t \mathcal{U}_x(t,s)f(s)\,ds$, $t \in I$. Then $x = d(f)$. Hence $f \in S^1_{F(\cdot,d(f))}$, that is $f \in \Phi(f)$. Vice versa, we suppose that $x = d(f)$ with $f \in \Phi(f)$. Then $x(t) = \mathcal{U}_x(t,0)x_0 + \int_0^t \mathcal{U}_x(t,s)f(s)\,ds$, $t \in I$ and $f \in S^1_{F(\cdot,d(f))} = S^1_{F_x}$. Thus $x \in \mathcal{S}(x_0)$.

We define the multifunction $\tilde{\Phi} : \mathcal{L}^1(I,\mu,X) \to \mathrm{P}(\mathcal{L}^1(I,\mu,X))$ by $\tilde{\Phi} = j\Phi j^{-1}$. Obviously, $\tilde{\Phi}(f) \in \mathcal{D}$, $\tilde{\Phi}(f)$ is bounded for each $f \in \mathcal{L}^1(I,\mu,X)$, and the sets of fixed points of the two multifunctions Φ and $\tilde{\Phi}$ are equal.

We intend to show that the set of the fixed points of the multifunction $\tilde{\Phi}$ is an absolute retract, [9, p. 85]. Since the map j is an homeomorphism, the set of the fixed points of the multifunction Φ is an absolute retract, [9, p. 86]. Its image by d still remains an absolute retract and coincides with $\mathcal{S}(x_0)$. Hence $\mathcal{S}(x_0)$ being a retract, it is connected, [27, p. 27], and closed.

Take $f, h \in \mathcal{L}_1(I,X)$, $x = d(f)$, $y = d(h)$ and $\alpha_1 = \int_0^T \exp\left[-2c_1c_{20}\int_0^t k(s)\,ds\right] dt$. For each $v \in S^1_{F_x}$ and $\varepsilon > 0$ we choose $u \in S^1_{F_y}$ such that $\|v(t) - u(t)\| \le \mathrm{d}(v(t), F(t,y(t))) + \varepsilon\alpha_1^{-1} \le \mathrm{D}(F(t,x(t)), F(t,y(t))) + \varepsilon\alpha_1^{-1} \le k(t)\|x(t) - y(t)\| + \varepsilon\alpha_1^{-1}$. Then

$$\begin{aligned}
\|u - v\|_{1*} &= \int_0^T \exp\left[-2c_1c_{20}\int_0^t k(s)\,ds\right] \|u(t) - v(t)\|\,dt \\
&\le \int_0^T \exp\left[-2c_1c_{20}\int_0^t k(s)\,ds\right] k(t)\|x(t) - y(t)\|\,dt + \varepsilon\alpha\alpha^{-1},
\end{aligned}$$

and by theorem 8 we have

$$\leq \int_0^T \exp\left[-2c_1c_{20}\int_0^t k(s)\,ds\right] k(t)c_1c_{20}\int_0^t \|f(s)-h(s)\|\,ds\,dt+\varepsilon$$

$$\leq -\frac{1}{2}\int_0^T \left[\exp\left(-2c_1c_{20}\int_0^t k(s)\,ds\right)\right]' \int_0^t \|f(s)-h(s)\|\,ds\,dt+\varepsilon$$

$$= -\frac{1}{2}\exp\left[-2c_1c_{20}\int_0^t k(s)\,ds\right]\int_0^t \|f(s)-h(s)\|\,ds\Big|_0^T$$

$$+\frac{1}{2}\int_0^T \exp\left[-2c_1c_{20}\int_0^t k(s)\,ds\right]\|f(t)-h(t)\|\,dt+\varepsilon$$

$$\leq \frac{1}{2}\|f-h\|_{1*}+\varepsilon.$$

Since ε is arbitrary it follows that $d(v,\tilde{\Phi}(h)) \leq \frac{1}{2}\|f-h\|_{1*}$ and $d(\tilde{\Phi}(f),\tilde{\Phi}(h)) \leq \frac{1}{2}\|f-h\|_{1*}$. If we change f by h and vice versa, then we have

$$D(\tilde{\Phi}(f),\tilde{\Phi}(h)) \leq \frac{1}{2}\|f-h\|_{1*}.$$

Based on [10, theorem 1], the set of the fixed points of the multifunction $\tilde{\Phi}$ is an absolute retract. Then the set of the fixed points of the multifunction Φ is an absolute retract. Then $\mathcal{S}(x_0)$ is an absolute retract, too. Thus the theorem is proved. □

6. Relaxation result

The *relaxation* theorems (also called Filippov-Wazewski theorems and appeared in [19], [58]) concern with the case when the set of solutions of a differential inclusion (whose right-hand side is not convex) is dense in the set of solutions of a differential inclusion whose right-hand side is, usually, the convex hull of the right-hand side of the first inclusion.

Such theorem may be considered as an existence one for the first inclusion since if it is proved that the convexified inclusion has a solution and the set of solutions of the first inclusion is dense in the set of solutions of the convexified inclusion, then the first inclusion has a solution, too.

The importance of the relaxation theorems in the qualitative theory of the differential inclusions and control theory is emphasized in [3, pp. 123-124]. Recent results on this topic may be found, e.g. in [22], [59], [55] and [30].

Let us consider two Cauchy problems, namely (CP) and the relaxed one

$$\frac{dx(t)}{dt} \in A(t, x(t))x(t) + \overline{co}F(t, x(t)), \quad \text{a.e. } t \in I, \quad \text{and } x(0) = a. \qquad (\mathrm{CP_c})$$

Theorem 12 ([39]). *Suppose there are satisfied the following assumptions: (X_1), (A), (U_1), (F_5), (F_6), (F_7) and (S_1). Then*

$$\overline{\mathcal{S}_{S^1_{\overline{co}F}}(a)} = \overline{\mathcal{S}_{S^1_F}(a)}.$$

Remarks. (a) Theorem 12 is a generalization of the theorems 2.1 and 2.5 in [22]. The generalization concerns the fact that we get the similar results for the corresponding quasi-linear case as well as the fact that in theorem 2.5 in [22] it is supposed the integrable function in the assumption (F_6) is equal to the function k in (F_5), which is not necessary. This last observation was remarked also in [55].

(b) In [55, theorem 3.6] it is proved (for the case of the linear inclusions and under the supplementary assumption that F has weak compact values) the following equality

$$\mathcal{S}_{S^1_{\overline{co}F}}(a) = \overline{\mathcal{S}_{S^1_F}(a)}.$$

7. Periodic solutions

The last part of this paper exhibits two theorems on a boundary value problem and as a particular case it results sufficient conditions for the existence of a periodic solution.

Consider the following boundary value problem for quasi-linear inclusion

$$\frac{dx(t)}{dt} \in A(t, x(t))x(t) + F(t, x(t)), \quad \text{a.e. } t \in I \quad \text{and } Lx = 0, \qquad (\mathrm{BP})$$

where L is a linear and continuous operator from $C(I, X)$ in X.

In order to get existence results we reduce the boundary value problem to a fixed point problem. This reduction may be performed by the general method presented in [35], [36].

We need some assumptions

(L) L is a continuous bounded linear operator from the Banach space $C(I, X)$ onto X. Let's take $D = \ker L$. Hence $D \in \mathrm{CCo}(C(I, X))$.

(L_1) For every $v \in D$ we consider the linear mapping $L_{1v} : AC(I,X) \to \mathcal{L}^1(I,X)$ and it is the same with L_1 in [36, p. 18], if A does not depend on w. Otherwise, L_{1v} is the linear and onto mapping defined by $L_{1v}x(\cdot) = \frac{dx(\cdot)}{d\cdot} - A(\cdot, v)x(\cdot)$.

(S_v) For each $v \in D$ S_v is the unique pseudo-inverse of the restriction of L to $\ker L_{1v}$ and it is denoted by S if A does not depend on w. Since we have a set of mappings S_v it is naturally to impose a condition dependence on v. Hence we suppose there are c and $q \in \mathbb{R}_+$ such that $\|S_v\| \leq c$, $\|S_v - S_u\| \leq q\|u - v\|$, $u, v \in D$.

(P) For each $v \in D$ we define the linear and continuous projector P_{1v} on $C(I,X)$ by $P_{1v}(x) = \mathcal{U}_v(\cdot, 0)x(0)$. For each $v \in D$ let P_{3v} be the linear and continuous projector from $\ker L_{1v}$ to $\ker L_{1v}$ defined by $P_{3v}(\mathcal{U}_v(\cdot,0)x_c) = \mathcal{U}_v(\cdot,0)x_{c_1}$, x_c and x_{c_1} being two fixed elements in X such that $\operatorname{Im} P_{3v} = \ker(L|_{\ker L_{1v}})$.

(C) Suppose that the following compatibility condition is satisfied $\forall v \in D$, $(1_X - L_{3v}S_v)(L\int_0^t \mathcal{U}_v(t,s)f(s)\,ds) = 0$, $t \in I$, $f \in S^1_{F_x}$, where $L_{3v} = L|_{\ker L_{1v}}$.

Under the assumptions (X_1), (A), (F_1) by a *mild solution* of the boundary value problem (BP) we mean a function $x \in C(I,X)$ which satisfies

$$x(t) = \mathcal{U}_x(t,0)x(0) + \int_0^t \mathcal{U}_x(t,s)f(s)\,ds, \quad Lx = 0, \qquad t \in I, \quad f \in S^1_{F_x}.$$

Remark. From [36] it follows that the set of mild solutions of problem (BP) is contained in the set of the fixed points of the mapping $\psi : D \to \mathrm{P}(D)$, $\psi(v) = C_v(v)$ defined by $C_v(x) = \{y \in D \mid y(t) = P_{3v}(P_{1v}(x)) - S_v L\int_0^t \mathcal{U}_v(t,s)f(s)\,ds + \int_0^t \mathcal{U}_v(t,s)f(s)\,ds,\ t \in I,\ f \in S^1_{F_x}\}$. Taking into account [35] it follows that we may suppose the first term in the formula of y as zero, that is $P_{3v}(P_{1v}(x)) = 0$, for each $x \in D$.

The t-*section* of $\psi(D)$ is $C(t) = \{y(t) \mid y \in C_v(v),\ v \in D\}$. Similarly to ($M_1$) we need the assumption

(M_2) If A depends on w, suppose that for each $t \in I$, $C(t)$ is relatively compact in X.

For an arbitrary positive μ let us denote $c_3 = (c||L||+1)\,[c_1||k_\mu||_1 + c_2T||m||_1] + qc_1||L||\,||m||_1$.

Theorem 13 ([37]). *If there are satisfied* (X_2), (A), (U_1), (F_2), (F_5), (F_6), (F_3) *or* (F_4), (L), (L_1), (S), (P), (C) *and, moreover,* $0 < c_3 < 1$, *then there exists a mild solution of problem (BP) in* D.

Theorem 14 ([37]). *Suppose that there are satisfied the following assumptions*

(i) (X_2), (A), (F_2), (F_{5-6}), (U_1), (U_3), (L), (L_1), (S_v), (P), (C);

(i) (M_2) *or* (U_2);

(iii) (F_3) *or* (F_4),

then there exists a mild solution of problem (BP) *in* D.

Remarks. (a) If operator A in (BP) does not depend on w, then in [46] it is proved a stronger result.

(b) If in (BP) we take $Lx = x(0) - x(T)$, then, based on theorems 13 and 14, we get sufficient conditions for existence of periodic solutions.

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FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, BABEŞ-BOLYAI UNIVERSITY, 3400 CLUJ-NAPOCA, ROMANIA
E-mail address: mmarian@math.ubbcluj.ro

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ON A CLASS OF VOLTERRA INTEGRAL EQUATIONS WITH DEVIATING ARGUMENT

VIORICA MUREŞAN

Abstract. Existence and data dependence results for some Volterra integral equations with linear deviating of the argument are given.

1. Introduction

Differential-functional equations with linear deviating of the argument have been studied in many papers ([1]-[10], [18], [19],...).

In [9], by using the Picard operators' technique and a suitable Bielecki norm, we have given existence and uniqueness theorems for some Volterra integral equations which contain a linear deviating of the argument.

In this paper we study the existence and the data dependence for the solutions of the following Volterra integral equation with linear deviating of the argument:

$$x(t) = x(0) + \int_0^t f(s, x(\lambda s))ds, \quad t \in [0, b],\ 0 < \lambda < 1.$$

We use the weakly Picard operators' technique, a fixed point theorem given by Rus in [12] and some data dependence results given by Rus and Mureşan in [17].

2. A fixed point theorem

Let (X, d) be a metric space and $A : X \to X$ an operator. We denote by F_A the fixed point set of A, that is

$$F_A := \{x \in X |\ A(x) = x\}.$$

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We have:

Theorem 2.1. (Rus [12]) *Let (X,d) be a complete metric space and $A : X \to X$ a continuous operator. We suppose that there exists $\alpha \in [0,1[$ such that*

$$d(A^2(x), A(x)) \leq \alpha d(x, A(x)), \text{ for all } x \in X.$$

Then:

a) $F_A \neq \emptyset$;

b) $A^n(x) \to x^*(x)$ *as* $n \to \infty$, *for all* $x \in X$, *and* $x^*(x) \in F_A$.

3. Volterra integral equations with deviating argument

We consider the following Volterra integral equation with deviating argument:

$$x(t) = x(0) + \int_0^t f(s, x(\lambda s))ds, \quad t \in [0,b],\ 0 < \lambda < 1, \tag{3.1}$$

where $f \in C([0,b] \times \mathbf{R})$.

We have

Theorem 3.1. *We suppose that there exists $L > 0$ such that*

$$|f(s,u) - f(s,v)| \leq L|u - v|, \text{ for all } s \in [0,b] \text{ and all } u, v \in \mathbf{R}.$$

Then the equation (3.1) has solutions in $C[0,b]$.

Proof. Let $(C[0,b], ||\cdot||_B)$ be, where

$$||x||_B = \max_{t\in[0,b]}(|x(t)|e^{-\tau t}), \quad \tau > 0.$$

We consider the operator

$$A : (C[0,b], ||\cdot||_B) \to (C[0,b], ||\cdot||_B),$$

defined by

$$A(x)(t) := x(0) + \int_0^t f(s, x(\lambda s))ds, \quad t \in [0,b],\ 0 < \lambda < 1, \tag{3.2}$$

which is a continuous operator.

This operator is not a contraction.

We have

$$A^2 : (C[0,b], \|\cdot\|_B) \to (C[0,b], \|\cdot\|_B),$$

$$A^2(x)(t) := x(0) + \int_0^t f\left(s, x(0) + \int_0^{\lambda s} f(u, x(\lambda u))du\right) ds.$$

It follows that

$$|A^2(x)(t) - A(x)(t)| \leq L \int_0^t \left| x(0) - x(\lambda s) + \int_0^{\lambda s} f(u, x(\lambda u))du \right| ds.$$

By denoting $\lambda s = v$, we obtain

$$|A^2(x)(t) - A(x)(t)| \leq \frac{L}{\lambda} \int_0^{\lambda t} \left| x(0) - x(v) + \int_0^{v} f(u, x(\lambda u))du \right| dv =$$

$$= \frac{L}{\lambda} \int_0^{\lambda t} |A(x)(v) - x(v)| e^{-\tau v} e^{\tau v} dv \leq$$

$$\leq \frac{L}{\lambda\tau} \|A(x) - x\|_B (e^{\tau\lambda t} - 1) \leq \frac{L}{\lambda\tau} \|A(x) - x\|_B e^{\tau t}.$$

Therefore,

$$|A^2(x)(t) - A(x)(t)| e^{-\tau t} \leq \frac{L}{\lambda\tau} \|A(x) - x\|_B, \text{ for all } t \in [0,b].$$

So, we have that

$$\|A^2(x) - A(x)\|_B \leq \frac{L}{\lambda\tau} \|A(x) - x\|_B, \text{ for all } x \in C[0,b].$$

We can choose τ so that $\frac{L}{\lambda\tau} < 1$. Let $\tau = \frac{L}{\lambda} + 1$ be.

We denote

$$\frac{\frac{L}{\lambda}}{\frac{L}{\lambda} + 1} = \alpha.$$

Thus

$$\|A^{n+1}(x) - A^n(x)\|_B \leq \alpha^n \|A(x) - x\|_B$$

and

$$\|A^{n+p}(x) - A^n(x)\|_B \leq \frac{\alpha^n}{1-\alpha} \|A(x) - x\|_B, \text{ for all } n \in \mathbf{N} \text{ and all } p \in \mathbf{N},\ p \geq 2.$$

So $(A^n(x))_{n\in\mathbb{N}^*}$ is a Cauchy sequence, for all $x \in C[0,b]$. Because $(C[0,b],d)$, where $d(x,y) = ||x-y||_B$, is a complete metric space, we have that $(A^n(x))_{n\in\mathbb{N}^*}$ is a convergent sequence, for all $x \in C[0,b]$.

We denote $A^\infty(x) = \lim\limits_{n\to\infty} A^n(x)$. From $A^{n+1}(x) = A(A^n(x))$ and the continuity of the operator A we have that $A^\infty(x) \in F_A$, that is $F_A \neq \emptyset$.

So, the equation (3.1) has solutions in $C[0,b]$. □

4. An example of weakly Picard operator

We have

Definition 4.1. (Rus [16]) Let (X,d) be a metric space. An operator $A : X \to X$ is a weakly Picard operator if the sequence $(A^n(x))_{n\in\mathbb{N}^*}$ converges for all $x \in X$ and its limit, denoted by $A^\infty(x)$, is a fixed point of A.

For more details about the Picard operators and the weakly Picard operators see [13]-[16].

Let $(C[0,b], ||\cdot||_C)$ be, where $||x||_C = \max\limits_{t\in[0,b]} |x(t)|$.

We consider the following operator:

$$A : (C[0,b], ||\cdot||_C) \to (C[0,b], ||\cdot||_C),$$

defined by

$$A(x)(t) := x(0) + \int_0^t f(s, x(\lambda s))ds, \quad t \in [0,b],\ 0 < \lambda < 1, \tag{4.1}$$

where f is as in the Theorem 3.1.

We have

Theorem 4.1. *The operator A defined by (4.1) is a weakly Picard operator.*

Proof. We consider $(C[0,b], ||\cdot||_B)$, where

$$||x||_B = \max_{t\in[0,b]} \left(|x(t)|e^{-\left(\frac{L}{\lambda}+1\right)t}\right).$$

From the proof of the Theorem 3.1 we have that the operator

$$A : (C[0,b], ||\cdot||_B) \to (C[0,b], ||\cdot||_B),$$

$$A(x)(t) := x(0) + \int_0^t f(s, x(\lambda s))ds, \quad t \in [0,b],\ 0 < \lambda < 1,$$

is a weakly Picard operator.

But $\|\cdot\|_C$ on $C[0,b]$ is metric equivalent with $\|\cdot\|_B$ on $C[0,b]$. Therefore, the operator A defined by (4.1) is a weakly Picard operator. □

Remark 4.1. The operator

$$A : (C[0,b], \|\cdot\|_C) \to (C[0,b], \|\cdot\|_C),$$

defined by

$$A(x)(t) := \int_0^t f(s, x(\lambda s))ds, \quad t \in [0,b],\ 0 < \lambda < 1,$$

is a Picard operator (F_A has a unique fixed point).

So the integral equation

$$x(t) = \int_0^t f(s, x(\lambda s))ds, \quad t \in [0,b],\ 0 < \lambda < 1,$$

has a unique solution in $C[0,b]$ (Theorem 3.1.1, [9]).

5. Data dependence of the solutions set

Let (X, d) be a metric space. We use the following notations:

$$P(X) = \{Y \subseteq X \mid Y \neq \emptyset\},$$

$$P_{b,cl}(X) = \{Y \in P(X) \mid Y \text{ is bounded and closed}\}$$

and

$$O_A(x) = \{x, A(x), A^2(x), \dots, A^n(x), \dots\} \text{ (the orbit of } x \in X).$$

Then we have

$$\delta(Y) = \sup\{d(a,b) \mid a, b \in Y\}, \text{ the diameter of } Y \in P(X)$$

and

$$H : P_{b,cl}(X) \times P_{b,cl}(X) \to \mathbf{R}_+,$$
$$H(Y,Z) = \max\left(\sup_{a\in Y}\inf_{b\in Z} d(a,b), \sup_{b\in Z}\inf_{a\in Y} d(a,b)\right),$$

the Hausdorff-Pompeiu distance on $P_{b,cl}(X)$ set.

Let $A, B : (X,d) \to (X,d)$ two operators for which there exists $\eta > 0$ such that $d(A(x), B(x)) < \eta$, for all $x \in X$. The data dependence problem of the solutions set is to estimate the "distance" between the two fixed point sets F_A and F_B of these operators.

In order to study the data dependence of the solutions set of the equation (3.1), we need the following result:

Theorem 5.1. (Th.2.4, [17]) *Let (X, d) be a complete metric space and $A, B : X \to X$ two orbitally continuous operators. We suppose that:*

(i) there exists $\alpha \in [0, 1[$ such that $d(A^2(x), A(x)) \leq \alpha d(x, A(x))$, for all $x \in X$

and

$d(B^2(x), B(x)) \leq \alpha d(x, B(x))$, for all $x \in X$;

(ii) there exists $\eta > 0$ such that $d(A(x), B(x)) \leq \eta$, for all $x \in X$.

Then

$$H(F_A, F_B) \leq \frac{\eta}{1-\alpha}.$$

Now we consider the following Volterra integral equations with deviating argument:

$$x(t) = x(0) + \int_0^t f(s, x(\lambda s))ds, \quad t \in [0, b],\ 0 < \lambda < 1, \tag{5.1}$$

$$x(t) = x(0) + \int_0^t g(s, x(\lambda s))ds, \quad t \in [0, b],\ 0 < \lambda < 1, \tag{5.2}$$

in which λ is the same and $f, g \in C([0, b] \times \mathbf{R})$.

We have

Theorem 5.2. *We suppose that*

(i) there exists $L > 0$ such that

$$|f(s, u) - f(s, v)| \leq L|u - v|, \text{ for all } s \in [0, b] \text{ and all } u, v \in \mathbf{R},$$

and

$$|g(s, u) - g(s, v)| \leq L|u - v|, \text{ for all } s \in [0, b] \text{ and all } u, v \in \mathbf{R};$$

(ii) there exists $\eta_1 > 0$ such that

$$|f(s, u) - g(s, u)| \leq \eta_1, \text{ for all } s \in [0, b] \text{ and all } u \in \mathbf{R};$$

(iii) $Lb < 1$.

Then

(a) $F_A \neq \emptyset$ *and* $F_B \neq \emptyset$;

(b) $H_{\|\cdot\|_C}(F_A, F_B) \leq \frac{\eta_1 b}{1 - Lb}$, *where by* $H_{\|\cdot\|_C}$ *we denote the Hausdorff-Pompeiu metric with respect to* $\|\cdot\|_C$ *on* $C[0, b]$.

Proof. (a) By using the results of the Theorem 3.1 we have that $F_A \neq \emptyset$ and $F_B \neq \emptyset$.

(b) We consider the operators

$$A, B : (C[0, b], \|\cdot\|_C) \to (C[0, b], \|\cdot\|_C),$$

defined by

$$A(x)(t) := x(0) + \int_0^t f(s, x(\lambda s))ds, \quad t \in [0, b],\ 0 < \lambda < 1,$$

$$B(x)(t) := x(0) + \int_0^t g(s, x(\lambda s))ds, \quad t \in [0, b],\ 0 < \lambda < 1,$$

in which λ is the same.

Then

$$|A^2(x)(t) - A(x)(t)| \leq \frac{L}{\lambda} \int_0^{\lambda t} |A(x)(v) - x(v)|dv \leq$$
$$\leq Lb\|A(x) - x\|_C, \text{ for all } t \in [0, b].$$

Therefore,

$$\|A^2(x) - A(x)\|_C \leq Lb\|A(x) - x\|_C, \text{ for all } x \in C[0, b].$$

Similarly,

$$\|B^2(x) - B(x)\|_C \leq Lb\|B(x) - x\|_C, \text{ for all } x \in C[0, b].$$

From (ii) we obtain that

$$\|A(x) - B(x)\|_C \leq \eta_1 b, \text{ for all } x \in C[0, b].$$

By applying the Theorem 5.1 we have that

$$H_{\|\cdot\|_C}(F_A, F_B) \leq \frac{\eta_1 b}{1 - Lb}.$$

□

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DEPARTMENT OF MATHEMATICS, TECHNICAL UNIVERSITY OF CLUJ-NAPOCA, 15, C. DAICOVICIU STREET, 3400 CLUJ-NAPOCA, ROMANIA

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ON HARDY-OPIAL TYPE INTEGRAL INEQUALITIES

B.G. PACHPATTE

Abstract. The aim of the present paper is to establish some new integral inequalities of the Hardy-Opial type involving functions and their derivatives. The analysis used in the proofs is elementary and our results provide new estimates on these types of inequalities.

1. Introduction

This paper is concerned with the integral inequalities of the following type

$$\int_a^b s|u|^p dx \leq \int_a^b r|u'|^p dx, \tag{1}$$

$$\int_a^b s|u|^p|u'| dx \leq \int_a^b r|u'|^{p+1} dx, \tag{2}$$

where s and r will usually positive continuous functions on the open interval (a, b), p is a suitable constant, and the inequalities will hold for $u \in C^1(a, b)$ which satisfy certain other conditions. The inequalities of the forms (1) and (2) are called Hardy and Opial type inequalities, see [3, p.706]. A great many papers have been written dealing with integral inequalities of the type (1) and (2), probably so by the challenge of research in various branches of mathematics, where such inequalities are often the basis for proving various theorems or approximating various functions. Excellent surveys of the work on such inequalities together with many references are contained in the books by Beckenbach and Bellman [2], Hardy, Littlewood and Polya [5], Mitrinovic [6] and the papers by Beesack [3], Shum [12] and the present author [9-11]. In this paper we establish a number of new integral inequalities involving functions and their derivatives which claim their origin to the Hardy and Opial type inequalities given in

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(1) and (2). Our proofs are elementary and the inequalities developed here provide new estimates on these types of inequalities.

2. Statement of results

In this section we state our main results to be proved in this paper. In what follows, we denote by R, the set of real numbers and let $J = [a, b]$, $a < b$ for $a, b \in \mathrm{R}$.

Our first theorem deals with the inequalities in which the constants appearing do not depend on the size of the domain of definition of the function.

Theorem 1. *Let u be a real-valued continuously differentiable function defined on J such that $u(a) = u(b) = 0$.*

(a_1) *If α, p, q be nonnegative real numbers such that $q \geq 1$ and $A = (p+q)/(\alpha+1)$, then*

$$\int_a^b |t|^{\alpha}|u(t)|^{p+q}dt \leq A^q \int_a^b |t|^{\alpha+q}|u(t)|^p|u'(t)|^q dt, \tag{3}$$

$$\int_a^b |t|^{\alpha}|u(t)|^{p+q}dt \leq A^{p+q} \int_a^b |t|^{\alpha+p+q}|u'(t)|^{p+q}dt. \tag{4}$$

(a_2) *If α, p, q, r be nonnegative real numbers such that $q + e \geq 1$ and $B = (p+q+r)/(\alpha+1)$, then*

$$\int_a^b |t|^{\alpha+r}|u(t)|^{p+q}|u'(t)|^r dt \leq B^q \int_a^b |t|^{\alpha+q+r}|u(t)|^p|u'(t)|^{q+r}dt, \tag{5}$$

$$\int_a^b |t|^{\alpha+r}|u(t)|^{p+q}|u'(t)|^r dt \leq B^{p+q} \int_a^b |t|^{\alpha+p+q+r}|u'(t)|^{p+q+r}dt. \tag{6}$$

Remark 1. It is interesting to note that the inequalities obtained in (3) and (6) are similar to that of the Opial's type inequality given in (2), see also [7-10]. The inequality (3) yields the lower bound on the integral of the form arising on the left side of (2). The inequality obtained in (4) is analogous to the Hardy's type inequality given in (1) and the inequality established in (5) is different from both the inequalities in (1) and (2).

In the following theorems we establish the inequalities in which the constants appearing depend on the size of the domain of definition of the function.

Theorem 2. *Let u be a real-valued continuously differentiable function defined on J such that $u(a) = u(b) = 0$.*

(b_1) *If $p \geq 0$, $g \geq 1$ be real numbers and $K = (p+q)(b-a)/2$, then*

$$\int_a^b |u(t)|^{p+q} dt \leq K^q \int_a^b |u(t)|^p |u'(t)|^q dt, \tag{7}$$

$$\int_a^b |u(t)|^{p+q} dt \leq K^{p+q} \int_a^b |u'(t)|^{p+q} dt. \tag{8}$$

(b_2) *If p, q, r be nonnegative real numbers such that $q + r \geq 1$ and $L = (p+q+r)(b-a)/2$, then*

$$\int_a^b |u(t)|^{p+q} |u'(t)^r dt \leq L^q \int_a^b |u(t)|^p |u'(t)|^{q+r} dt, \tag{9}$$

$$\int_a^b |u(t)|^{p+q} |u'(t)|^r dt \leq L^{p+q} \int_a^b |u'(t)|^{p+q+r} dt. \tag{10}$$

Remark 2. We note that the inequalities obtained in (7) and (10) are similar to that of the Opial's type inequality given in (2) which in turn yields lower and upper bounds on the integral of the form involved on the left side of (2). The inequality (8) is analogous to the Hardy's type inequality given in (1), while the inequality obtained in (9) is different from those of the inequalities in (1) and (2).

Theorem 3. *Let u be a real-valued twice continuously differentiable function defined on J such that $u(a) = u(b) = 0$.*

(c_1) *If p_0, p_1, p_2, p_3 be nonnegative real numbers such that*

$$p_3 > 1, \quad p_3 \geq p_1, \quad p_1 + p_2 + p_3 - (p_3 - p_1)/(p_3 - 1) \geq 0,$$

and

$$M = [(p_1 + p_2 + p_3 - 1)/(p_0 + 1)]^{p_3} [(p_0 + p_1 + p_2 + p_3)(b-a)/2]^{p_3 - p_1},$$

then

$$\int_a^b |u(t)^{p_0}|u'(t)|^{p_1+p_2+p_3}dt \leq M\int_a^b |u(t)|^{p_0+p_1}|u'(t)|^{p_2}|u''(t)|^{p_3}dt, \tag{11}$$

(c_2) *If p_0, p_1, p_2, p_3, p_4 be nonnegative real numbers such that*

$$p_3 > 1, \quad p_0p_3 - p_1p_4 \geq 0, \quad p_3 + p_4 \geq p_1 + (p_1p_4)/p_3,$$

$$\sum_{i=1}^{4} p_i + (p_1p_4)/p_3 - [(p_3 + p_4 - (p_1p_4)/p_3)]/(p_3 + p_4 - 1)] \geq 0,$$

and

$$N = \left[\left(\sum_{i=1}^{4} p_i + (p_1p_4)/p_3 - 1\right)/(p_0 - (p_1p_4)/p_3 + 1\right]^{p_3} \times \left[\left(\sum_{i=0}^{4} p_i\right)(b-a)/2\right]^{(p_3-p_1)},$$

then

$$\int_a^b |u(t)|^{p_0}|u'(t)|^{p_1+p_2+p_3}|u''(t)|^{p_4}dt \leq N\int_a^b |u(t)|^{p_0+p_1}|u'(t)|^{p_2}|u''(t)|^{p_3+p_4}dt. \tag{12}$$

Remark 3. It is easy to observe that the inequality obtained in (11) is analogous to the Opial's type inequality given in (2), which yields a new upper bound on the integral of the form arising on the left side of (2). The inequality (12) is different from those of the inequalities given in (1) and (2).

3. Proof of Theorem 1

(a_1) By rewriting the integral on the left side of (3) and making use of the integration by parts, the fact that $u(a) = u(b) = 0$ and the Hölder's inequality with indices $q, q/(q-1)$ we observe that

$$\int_a^b |t|^\alpha |u(t)|^{p+q}dt = \frac{1}{\alpha+1}\int_a^b \frac{d}{dt}(|t|^{\alpha+1} sgn\, t)|u(t)|^{p+q}dt = \tag{13}$$

$$= -A\int_a^b |t|^{\alpha+1} sgn\, t|u(t)|^{p+q-1}u'(t) sgn\, u(t)dt \leq A\int_a^b |t|^{\alpha+1}|u(t)|^{p+q-1}|u'(t)|dt =$$

$$= A\int_a^b [|t|^{\alpha+1-\alpha(q-1)/q}|u(t)|^{p/q}|u'(t)|] \times [|t|^{\alpha(q-1)/q}|u(t)|^{p+q-1-(p/q)}]dt \leq$$

$$\leq A\left[\int_a^b |t|^{\alpha+q}|u(t)|^p|u'(t)|^q dt\right]^{1/q} \times \left[\int_a^b |t|^\alpha |u(t)|^{p+q}dt\right]^{(q-1)/q}.$$

If $\int_a^b |t|^\alpha |u(t)|^{p+q} dt = 0$, then (3) is trivially true, otherwise dividing both sides of (13) by $\left[\int_a^b |t|^\alpha |u(t)|^{p+q} dt\right]^{(q-1)/q}$ and then taking the qth power on both sides of the resulting inequality we get the required inequality in (3).

Rewriting the integral on the right side of (3) and using the Hölder's inequality with indices $(p+q)/p, (p+q)/q$ we observe that

$$\int_a^b |t|^\alpha |u(t)|^{p+q} dt \leq A^q \int_a^b [|t|^{(\alpha p)/(p+q)} |u(t)|^p] \times [|t|^{\alpha+q-(\alpha p)/(p+q)} |u'(t)|^q] dt \leq \tag{14}$$

$$\leq A^q \left[\int_a^b |t|^\alpha |u(t)|^{p+q} dt\right]^{p/(p+q)} \times \left[\int_a^b |t|^{\alpha+p+q} |u'(t)|^{p+q} dt\right]^{q/(p+q)}.$$

Now by following the arguments as in the last part of the proof of inequality (3) with suitable modifications, we get the required inequality in (4).

(a_2) By rewriting the integral on the left side of (5) and using the Hölder's inequality with indices $(q+r)/r, (q+r)/q$ and the inequality (4), we observe that

$$\int_a^b |t|^{\alpha+r} |u(t)|^{p+q} |u'(t)^r dt = \tag{15}$$

$$= \int_a^b [|t|^{\alpha-(\alpha q)/(q+r)} |u(t)|^{(pr)/(q+r)} (|t||u'(t)|)^r] \times [|t|^{(\alpha q)/(q+r)} |u(t)|^{p+q-(pr)/(q+r)}] dt \leq$$

$$\leq \left[\int_a^b |t|^{\alpha+q+r} |u(t)|^p |u'(t)|^{q+r} dt\right]^{r/(r+q)} \times \left[\int_a^b |t|^\alpha |u(t)|^{p+q+r} dt\right]^{q/(q+r)} \leq$$

$$\leq \left[\int_a^b |t|^{\alpha+q+r} |u(t)|^p |u'(t)|^{q+r} dt\right]^{r/(q+r)} \times \left[B^{q+r} \int_a^b |t|^{\alpha+q+r} |u(t)|^p |u'(t)|^{q+r} dt\right]^{q/(q+r)} =$$

$$= B^q \int_a^b |t|^{\alpha+q+r} |u(t)|^p |u'(t)|^{q+r} dt.$$

This completes the proof of inequality (5).

Rewriting the integral on the right side of (5) and using the Hölder's inequality with indices $(p+q)/p, (p+q)/q$ we observe that

$$\int_a^b |t|^{\alpha+r} |u(t)|^{p+q} |u'(t)|^r dt \leq \tag{16}$$

$$\leq B^q \int_a^b [|t|^{(\alpha+r)(p/(p+q))} |u(t)|^p |u'(t)|^{(rp)/(p+q)} \times [|t|^{\alpha+q+r-(\alpha+r)(p/(p+q))} |u'(t)|^{q+r-(rp)/(p+q)}] dt \leq$$

$$\leq B^q \left[\int_a^b |t|^{\alpha+r}|u(t)|^{p+q}|u'(t)|^r dt\right]^{p/(p+q)} \times \left[\int_a^b |t|^{\alpha+p+q+r}|u'(t)|^{p+q+r} dt\right]^{q/(p+q)} .$$

Now by following the arguments as in the last part of the proof of inequality (3) with suitable modification, we get the required inequality in (6). The proof is complete.

4. Proof of Theorem 2

(b_1) From the hypothesis of Theorem 2 we have

$$u^{p+q}(t) = (p+q)\int_a^b u^{p+q-1}(s)u'(s)ds = -(p+q)\int_t^b u^{p+q-1}(s)u'(s)ds. \tag{17}$$

From (17) we observe that

$$|u(t)|^{p+q} \leq [(p+q)/2 \int_a^b |u(s)|^{p+q-1}|u'(s)|ds. \tag{18}$$

Integrating both sides of (18) from a to b, and rewriting the right side of the resulting inequality and then using the Hölder's inequality with indices $q, q/(q-1)$ we have

$$\int_a^b |u(t)|^{p+q}dt \leq K\int_a^b |u(t)|^{p+q-1}|u'(t)|dt = \tag{19}$$

$$= K\int_a^b [|u(t)|^{p/q}|u'(t)|][|u(t)|^{p+q-1-(p/q)}]dt \leq$$

$$\leq K\left[\int_a^b |u(t)|^p|u'(t)|^q dt\right]^{1/q}\left[\int_a^b |u(t)|^{p+q}dt\right]^{(q-1)/a} .$$

If $\int_a^b |u(t)|^{p+q} = 0$, then (7) is trivially true, otherwise dividing both sides of (19) by $\left[\int_a^b |u(t)|^{p+q}dt\right]^{(q-1)/q}$ and then taking the qth power on both sides of the resulting inequality, we get the required inequality in (7).

By using the Hölder's inequality with indices $(p+q)/p, (p+q)/q$ on the right side of (7) and following the arguments as in the last part of the proof of inequality (7) with suitable changes, we get the required inequality in (8).

(b_2) Rewriting the integral on the left side of (9) and using the Hölder's inequality with indices $(q+r)/r$, $(q+r)/q$ and the inequality (7), we observe that

$$\int_a^b |u(t)|^{p+q}|u'(t)|^r dt = \int_a^b [|u(t)|^{(pr)/(q+r)}|u'(t)|^r][|u(t)|^{p+q-(pr)/(q+r)}]dt \leq \quad (20)$$

$$\leq \int_a^b |u(t)|^p|u'(t)|^{q+r} dt]^{r/(q+r)} \times \left[\int_a^b |u(t)|^{p+q+r} dt\right]^{q/(q+r)} \leq$$

$$\leq \left[\int_a^b |u(t)|^p|u'(t)|^{q+r} dt\right]^{r/(q+r)} \times \left[K^{q+r}\int_a^b |u(t)|^p|u'(t)|^{q+r} dt\right]^{q/(q+r)} =$$

$$= K^q \int_a^b |u(t)|^p|u'(t)|^{q+r} dt.$$

This is the required inequality in (9).

The details of the proof of inequality (10) are very close to that of the proof of inequality (6) given above with suitable changes and hence we omit it here. The proof is complete.

5. **Proof of Theorem 3**

(c_1) Be rewriting the integral on the left side of (11) and making use of the integration by parts, the fact that $u(a) = u(b) = 0$, the Hölder's inequality with indices $p_3, p_3/(p_3-1)$ and the inequality (9), we observe that

$$\int_a^b |u(t)|^{p_0}|u'(t)|^{p_1+p_2+p_3} dt = \quad (21)$$

$$= \frac{1}{p_0+1}\int_a^b \left[\frac{d}{dt}(|u(t)|^{p_0+1} sgn\, u(t)\right] \times \left[|u'(t)|^{p_1+p_2+p_3-1} sgn\, u'(t)\right] dt =$$

$$= -\left(\frac{p_1+p_2+p_3-1}{p_0+1}\right)\int_a^b |u(t)|^{p_0+1} sgn\, u(t)|u'(t)|^{p_1+p_2+p_3-1} \times u''(t)(sgn\, u'(t))^2 dt \leq$$

$$\leq \left(\frac{p_1+p_2+p_3-1}{p_0+1}\right)\int_a^b |u(t)|^{p_0+1}|u'(t)|^{p_1+p_2+p_3-2}|u''(t)| dt =$$

$$= \left(\frac{p_1+p_2+p_3-1}{p_0+1}\right)\int_a^b [|u(t)|^{(p_0+p_1)/p_3}|u'(t)|^{p_2/p_3}|u''(t)|] \times$$

$$\times [|u(t)|^{p_0+1-(p_0+p_1)/p_3}|u'(t)|^{p_1+p_2+p_3-2-(p_2/p_3)}] dt \leq$$

$$\leq \left(\frac{p_1+p_2+p_3-1}{p_0+1}\right)\left[\int_a^b |u(t)|^{p_0+p_1}|u'(t)|^{p_2}|u''(t)|^{p_3} dt\right]^{1/p_3} \times$$

$$\times\left[\int_a^b |u(t)|^{p_0+(p_3-p_1)/(p_3-1)} \times |u'(t)|^{p_1+p_2+p_3-(p_3-p_1)/(p_3-1)} dt\right]^{(p_3-1)/p_3} \leq$$

$$\leq \left(\frac{p_1+p_2+p_3-1}{p_0+1}\right)\left[\int_a^b |u(t)|^{p_0+p_1}|u'(t)|^{p_2}|u''(t)|^{p_3} dt\right]^{1/p_3} \times$$

$$\times[(p_0+p_1+p_2+p_3)(b-a)/2]^{(p_3-p_1)/p_3} \times \left[\int_a^b |u(t)|^{p_0}|u'(t)|^{p_1+p_2+p_3} dt\right]^{(p_3-1)/p_3}.$$

Now by following the arguments as in the last part of the proof of inequality (7) with suitable changes, we get the required inequality in (11).

(c_2) Rewriting the integral on the left side of (12) and using the Hölder's inequality with indices $(p_3+p_4)/p_4, (p_3+p_4)/p_3$ and the inequality (11), we observe that

$$\int_a^b |u(t)|^{p_0}|u'(t)|^{p_1+p_2+p_3}|u''(t)|^{p_4} dt = \tag{22}$$

$$= \int_a^b [|u(t)|^{(p_0+p_1)(p_4/(p_3+p_4))}|u'(t)|^{(p_2p_4)/(p_3+p_4)}|u''(t)|^{p_4}]\times$$

$$\times[|u(t)|^{p_0-(p_0+p_1)(p_4/(p_3+p_4))} \times |u'(t)|^{p_1+p_2+p_3-(p_2p_4)/(p_3+p_4)}] dt \leq$$

$$\leq \left[\int_a^b |u(t)|^{p_0+p_1}|u'(t)|^{p_2}|u''(t)|^{p_3+p_4} dt\right]^{p_4/(p_3+p_4)} \times$$

$$\times\left[\int_a^b |u(t)|^{p_0-(p_1p_4)/p_3} \times |u'(t)|^{p_1+(p_1p_4)/p_3)+p_2+(p_3+p_4)} dt\right]^{p_3/(p_3+p_4)} \leq$$

$$\leq \left[\int_a^b |u(t)|^{p_0+p_1}|u'(t)|^{p_2}|u''(t)|^{p_3+p_4} dt\right]^{p_4/(p_3+p_4)} \times$$

$$\times N\left[\int_a^b |u(t)|^{p_0+p_1}|u'(t)|^{p_2}|u''(t)|^{p_3+p_4} dt\right]^{p_3/(p_3+p_4)} =$$

$$= N\int_a^b |u(t)|^{p_0+p_1}|u'(t)|^{p_2}|u''(t)|^{p_3+p_4} dt.$$

This is the required inequality in (12). The proof is complete.

Remark 4. The multidimensional integral inequalities of the type (1) and (2) and their variants are established by many authors in the literature by using different techniques. In particular, in [4] Dubinskii has established the multidimensional inequalities analogues to the inequalities (7), (9), (11) and (12), see also [1], by using the divergence theorem, Young's inequality and imbedding theorems. Here we note that our proofs of Theorems 1-3 are quite elementary and the constants involved in these inequalities provide precise information.

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DEPARTMENT OF MATHEMATICS, MARATHWADA UNIVERSITY, AURANGABAD 431 004, (MAHARASHTRA) INDIA

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ABOUT AN INTEGRAL OPERATOR PRESERVING THE UNIVALENCE

VIRGIL PESCAR

Abstract. In this work an integral operator is studied and the author determines conditions for the univalence of this integral operator.

1. Introduction

Let A be the class of the functions f which are analytic in the unit disc $U = \{z \in C; |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

We denote by S the class of the function $f \in A$ which are univalent in U.

Many authors studied the problem of integral operators which preserve the class S. In this sense an important result is due to J. Pfaltzgraff [4].

Theorem A ([4]). *If f(z) is univalent in U, α a complex number and $|\alpha| \leq \frac{1}{4}$, then the function*

$$G_\alpha(z) = \int_o^z [f'(\xi)]^\alpha d\xi \tag{1}$$

is univalent in U.

Theorem B ([3]). *If the function $g \in S$ and α is a complex number, $|\alpha| \leq \frac{1}{4n}$, then the function defined by*

$$G_{\alpha,n}(z) = \int_o^z [g'(u^n)]^\alpha du \tag{2}$$

is univalent in U for all positive integer n.

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2. Preliminaries

For proving our main result we will need the following theorem and lemma.

Theorem C ([1]). *If the function f is regular in the unit disc U, $f(z) = z + a_2 z^2 + \ldots$ and*

$$\left(1 - |z|^2\right) \left| \frac{z f''(z)}{f'(z)} \right| \leq 1 \tag{3}$$

for all $z \in U$, then the function f is univalent in U.

Lema Schwarz ([2]). *If the function g is regular in U, $g(0) = 0$ and $|g(z)| \leq 1$ for all $z \in U$, then the following inequalities hold*

$$|g(z)| \leq |z| \tag{4}$$

for all $z \in U$, and $|g'(0)| \leq 1$, the equalities (in inequality (4) for $z \neq 0$) hold only in the case $g(z) = \epsilon z$, where $|\epsilon| = 1$.

3. Main result

Theorem 1. *Let γ be a complex number and the function $g \in A$, $g(z) = z + a_2 z^2 + \ldots$. If*

$$\left| \frac{g''(z)}{g'(z)} \right| \leq \frac{1}{n} \tag{5}$$

for all $z \in U$ and

$$|\gamma| \leq \frac{1}{\left(\frac{n}{n+2}\right)^{\frac{n}{2}} \frac{2}{n+2}} \tag{6}$$

then the function

$$G_{\gamma,n}(z) = \int_0^z \left[g'(u^n)\right]^\gamma du \tag{7}$$

is univalent in U for all $n \in N^* - \{1\}$.

Proof. Let us consider the function

$$f(z) = \int_0^z \left[g'(u^n)\right]^\gamma du. \tag{8}$$

The function

$$h(z) = \frac{1}{\gamma} \frac{f''(z)}{f'(z)}, \tag{9}$$

where the constant γ satisfies the inequality (6) is regular in U.

From (9) and (8) it follows that

$$h(z) = \frac{\gamma}{|\gamma|} \left[\frac{nz^{n-1} g''(z^n)}{g'(z^n)} \right]. \tag{10}$$

Using (10) and (5) we have

$$|h(z)| \leq 1, \tag{11}$$

for all $z \in U$. From (10) we obtain $h(0) = 0$ and applying Schwarz-Lemma we have

$$\frac{1}{|\gamma|} \left| \frac{f''(z)}{f'(z)} \right| \leq |z|^{n-1} \leq |z| \tag{12}$$

for all $z \in U$, and hence, we obtain

$$\left(1 - |z|^2\right) \left| \frac{zf''(z)}{f'(z)} \right| \leq |\gamma| \left(1 - |z|^2\right) |z|^n. \tag{13}$$

Let us consider the function Q:[0, 1] $\to R$, $Q(x) = \left(1 - x^2\right) x^n$; $x = |z|, z \in U$, which has a maximum at a point $x = \sqrt{\frac{n}{n+2}}$, and hence

$$Q(x) < \left(\frac{n}{n+2}\right)^{\frac{n}{2}} \frac{2}{n+2} \tag{14}$$

for all $x \in (0, 1)$. Using this result and (13) we have

$$\left(1 - |z|^2\right) \left| \frac{zf''(z)}{f'(z)} \right| \leq |\gamma| \left(\frac{n}{n+2}\right)^{\frac{n}{2}} \frac{2}{n+2}. \tag{15}$$

From (15) and (6) we obtain

$$\left(1 - |z|^2\right) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \tag{16}$$

for all $z \in U$. From (16) and (8) and Theorem C it follows that $G_{\gamma,n}$, is in the class S. □

Remark. For $n = 2$, we obtain $|\gamma| \leq 4$ and the function $G_{\gamma,2}$ is in the class S.

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"TRANSILVANIA" UNIVERSITY OF BRAŞOV, FACULTY OF SCIENCE, DEPARTMENT OF MATHEMATICS, 2200 BRAŞOV, ROMANIA

STUDIA UNIV. "BABEŞ-BOLYAI", MATHEMATICA, Volume XLIV, Number 1, March 1999

A GENERALIZATION OF BECKER'S UNIVALENCE CRITERION

IRINEL RADOMIR

Abstract. In the paper there is presented a sufficient univalence conditions for functions of a complex variable f, verifying the conditions $f(0) = 0$, $f'(0) = 1$. Our condition is a generalization of Becker's univalence criterion.

1. Introduction

Let A be the class of functions f, which are analytic in the unit disk $U = \{z \in C\ , |z| < 1\ \}$, with $f(0) = 0$ and $f'(0) = 1$.

Theorem 1.1. ([2]). *Let $f \in A$. If for all $z \in U$*

$$(1 - |z|^2)|\frac{zf''(z)}{f'(z)}| \leq 1 \tag{1}$$

then the function f is univalent in U.

In order to prove the main results we shall need the theory of Loewner chains.

2. Preliminaries

We denote by U_r the disk of $z-$plane, $U_r = \{\ z \in C\ :\ \ |z| < r\ \}$, where $r \in (0,1]$ and $I = [0, \infty)$.

Definition. *A function $L(z,t) : U \times I \to C$ is called a Loewner chain if*

$$L(z,t) = e^t z + a_2(t) z^2 + \dots \qquad |z| < 1,$$

is analytic and univalent in U for each $t \in I$ and if $L(z,s) \prec L(z,t)$, $0 \leq s < t < \infty$, where by $\prec$ we denote the relation of subordination.

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Theorem 2.1. (3). *Let r be a real number, $r \in (0,1]$. Let $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots$, $a_1(t) \neq 0$ be analytic in U_r for all $t \in I$, locally absolutely continuous in I and locally uniform with respect to U_r. For almost all $t \in I$ suppose*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t} \quad \text{for all } z \in U_r ,$$

where $p(z,t)$ is analytic in U such that $Re\, p(z,t) > 0$ for $z \in U$, $t \in I$.

If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z,t)/a_1(t)\}$ forms a normal family in U_r, then $L(z,t)$ has, for each $t \in I$, an analytic and univalent extension to the whole disk U.

3. Main results

Theorem 3.1. *Let $f(z) = z + a_2z^2 + \dots$ and $g(z) = z + b_2z^2 + \dots$ be analytic functions in U. If for all $z \in U$*

$$\left| \frac{zf''(z)}{f'(z)} - |z|^2\frac{zg''(z)}{f'(z)} \right| \leq 1 , \text{ and} \tag{2}$$

$$\left| \frac{z(\, f(z) - g(z)\,)''}{f'(z)} \right| \leq 1 , \tag{3}$$

then the function $g(z) + z(\, f(z) - g(z)\,)'$ is univalent in U.

Proof. We consider the function $L : U \times I \to C$ defined from

$$L(z,t) = (e^t z)f'(e^t z) - \int_0^{e^{-t}z} ug''(u)du \tag{4}$$

Because the functions f and g are analytic in U it results that the function $L(z,t)$ is analytic in U for all $t \in I$. From (4) we obtain

$$L(z,t) = e^t z + a_2(t)z^2 + \dots$$

In order to prove that $\{\, L(z,t)/e^t \,\}$ forms a normal family in U, it is sufficient to observe that there exist positive numbers k_1, k_2 such that

$$|\, f'(z) \,| \leq k_1 \quad \text{and} \quad \left| \int_0^z ug''(u)du \right| \leq k_2 ,$$

for all $z \in U_r$, $r \in (0,1]$. Therefore we have $| L(z,t)/e^t | \leq k_1 + k_2$ for all $z \in U_r$ and $t \in I$.

We consider the function $p: U_r \times I \to C$ defined by

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z} / \frac{\partial L(z,t)}{\partial t} \tag{5}$$

In order to prove that the function $p(z,t)$ has an analytic extension with positive real part in U, for all $t \in I$ it is sufficient to show that the function

$$w(z,t) = \frac{p(z,t)-1}{p(z,t)+1} \qquad z \in U_r , \tag{6}$$

can be continued analytically in U and that

$$| w(z,t) | < 1 \qquad (\forall) z \in U, \;\; t \geq 0.$$

From (4), (5) and (6) we obtain

$$w(z,t) = e^{-t} z \frac{f''(e^{-t}z)}{f'(e^{-t}z)} - e^{-3t} z \frac{g''(e^{-t}z)}{f'(e^{-t}z)} \tag{7}$$

From (3) it results that $f'(z) \neq 0$ for all $z \in U$ and hence the function $w(z,t)$ is analytic in U for all $t \in I$. We have

$$w(z,0) = \frac{z(\, f(z) - g(z)\,)''}{f'(z)}$$

and from (3) it results that $|w(z,0)| \leq 1$ for all $z \in U$. Also we have $w(0,t) = 0 < 1$. If $z \in U$, $z \neq 0$ and $t > 0$, we observe taht the function $w(z,t)$ is analytic in $\bar{U}$,because $|e^{-t}z| \leq e^{-t} < 1$ for all $z \in \bar{U}$. Using the maximum principle, for all $z \in U$ and $t > 0$, we have

$$| w(z,t) | < \max_{|\zeta|=1} |w(\zeta,t)| = |w(e^{i\theta},t)| \, , \tag{8}$$

where $\theta = \theta(t)$ is a real number. Let us denote $u = e^{-t}e^{i\theta}$. Then $|u| = e^{-t}$ and from (7) we obtain

$$w(e^{i\theta},t) = \frac{uf''(u)}{f'(u)} - |u|^2 \frac{ug''(u)}{f'(u)} \tag{9}$$

Since $|u| < 1$, from (2), (8) and (9) it results that $|w(z,t)| < 1$ for all $z \in U$, $t \geq 0$. It follows that $L(z,t)$ is a Loewner chain and hence the function $L(z,0) = g(z) + z \cdot (f(z) - g(z))'$ is univalent in U.

Remark. If $g(z) = f(z)$, from Theorem 3.1 we obtain Theorem 1.1.

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DEPARTMENT OF MATHEMATICS, "TRANSILVANIA" UNIVERSITY, BRAŞOV, ROMA-, NIA

STUDIA UNIV. "BABEŞ-BOLYAI", MATHEMATICA, Volume XLIV, Number 1, March 1999

ON CERTAIN CLASSES OF GENERALIZED CONVEX FUNCTIONS WITH APPLICATIONS, I.

J. SÁNDOR

Abstract. The aim of this series of papers is to introduce certain new concepts of generalized convex functions with applications. The first part contains results related to the η-invex functions first introduced by the author in 1988. Here are studied also η-cvazi-invexity and generalized η-pseudo-invexity with connections to well known classes of functions as subadditive or Jensen-convex functions. The second part treats the so-called A-convex functions, due to the author. Finally the part III, on Λ-invex functions, leads to a generalization of the Banach-Steinhaus theorem of condenzation of singularities.

Invex functions

A. A differentiable function $f : \mathbf{R}^n \to \mathbf{R}$ is called **invex**, if there exists a vector function $\eta(x,u) \in \mathbf{R}^n$ such that

$$f(x) - f(u) \geq (\eta(x,u))^t \nabla f(u). \tag{1}$$

This concept has been introduced by Hanson [1], who proved that, if in place of the usual convexity conditions the functions involved in a nonlinear optimization problem, all satisfy condition (1) for the same function η, then there hold true weak duality results, and that the sufficiency of the Kuhn-Tucker conditions are true. Hanson's paper was the source of inspiration for many later researches. Craven [3] has introduced the name of "invex" functions, and obtained duality theorems for fractional programming. Mond and Hanson [2] have extended the concept of invexity to polyhedral cones, Craven and Glover [5] proved that the class of invex functions

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is equivalent with the class of functions having all stationary points as global minimum points. Martin [7] has defined the Kuhn-Tucker invexity, while Jeyakumar [6] introduced weak and strong invexity.

Examples. 1) Let $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x^3$. This function is not invex, as the critical point $x = 0$ is not a global minimum point. But, as it is well known, f is cvazi-convex.

2) Let $f : \mathbf{R}^2 \to \mathbf{R}$, $f(x, y) = x^3 - 10y^3 + x - y$. It is easy to see that f is invex, but it is not cvazi-convex.

In the same way, f is called **cvazi-invex**, if

$$f(y) - f(x) \leq 0 \;\Rightarrow\; (\eta(x, y))^t \nabla f(x) \leq 0 \tag{2}$$

and **pseudo-invex**, if

$$(\eta(x, y))^t \nabla f(x) \geq 0 \;\Rightarrow\; f(y) - f(x) \geq 0. \tag{3}$$

Clearly, in example 1) f is cvazi-invex; and that by a theorem of Craven and Glover [5] we have that there is no difference between the class of invex and pseudo-invex functions.

Definition 1. Let $S \subset \mathbf{R}^n$, open, and $f : S \to \mathbf{R}$ a differentiable function. If relation (1) is valid for all $x, u \in S$, we will say that f is **invex related** to η on S.

The following propositions give certain connextions between functions invex related to η and other classes of functions.

Proposition 1. Let $S \subset \mathbf{R}^n$ be open, convex and f invex related to η on S. Let us assume that the following condition is true:

$$f(x) < f(y) \;\Rightarrow\; (x - y)^t \nabla f(y) \gtrless (\eta(x, y))^t \nabla f(y) \text{ for all } x, y \in S. \tag{4}$$

Then f is pseudo-convex (strictly pseudo-convex).

Proof. Let $x, y \in S$ and $f(x) < f(y)$. Then, in view of (1) we can write

$$(x - y)^t \nabla f(y) = [(x - y) - \eta(x, y)]^t \nabla f(y) + (\eta(x, y))^t \nabla f(y) \leq$$

$$\leq (x - y - \eta(x, y))^t \nabla f(y) + f(x) - f(y) < [(x - y) - \eta(x, y)]^t \nabla f(y) < 0.$$

If $x, y \in S$ and $f(x) \leq f(y)$, then from

$$(x - y)^t \nabla f(y) = (x - y - \eta(x, y))^t \nabla f(y) + (\eta(x, y))^t \nabla f(y) \leq$$

$$\leq (x-y-\eta(x,y))^t\nabla f(y)+f(x)-f(y)\leq (x-y-\eta(x,y))^t\nabla f(y)<0,$$

so we have strict pseudo-convexity if there is strict inequality in (4).

Proposition 2. *Let f be invex related to η on the open, convex set S. If the implication*

$$(x-y)^t\nabla f(y)>0 \ \Rightarrow\ (\eta(x,y))^t\nabla f(y)\geq (x-y)^t\nabla f(y)\quad (x,y\in S) \tag{5}$$

is true, then f is cvazi-convex (and thus, cvazi-invex, too).

Proof. We can write successively

$$f(x)-f(y)\geq (\eta(x,y))^t\nabla f(y)=(\eta(x,y)-(x-y))^t\nabla f(y)+(x-y)^t\nabla f(y)>$$

$$>(\eta(x,y)-(x-y))^t\nabla f(y)>0,$$

thus $(x-y)^t\nabla f(y)>0 \ \Rightarrow\ f(x)-f(y)>0$, implying the cvazi-convexity of f, in case of differentiable function f.

The following proposition gives a simple method of construction of new invex functions.

Proposition 3. *Let $g:\mathbf{R}\to\mathbf{R}$ be a differentiable, increasing and convex function. Let $f:\mathbf{R}^n\to\mathbf{R}$ be invex related to η. Then $g\circ f:\mathbf{R}^n\to\mathbf{R}$ is invex related to η, too.*

Proof. It is known that $g(x+t)\geq g(x)+g'(x)t$ for all $x,t\in\mathbf{R}$. Thus we have

$$g(f(x))\geq g[f(y)+(\eta(x,y))^t\nabla f(y)]\geq g[f(y)]+g'(f(y))\nabla[\eta(x,y)f(y)]=$$

$$=g[f(y)]+(\eta(x,y))^t\nabla(g\circ f)(y),$$

which means that $g\circ f$ is invex related to η.

Definition 2. ([8]) Let $f:\mathbf{R}^n\to\mathbf{R}$, and $\eta:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$ be a given function. We say that f is **η-invex** (incave) if the following inequality is true:

$$f(y+\lambda\eta(x,y))\ \substack{\leq\\ \geq}\ \lambda f(x)+(1-\lambda)f(y),\ \text{ for all } x,y\in\mathbf{R}^n,\ \text{ all } \lambda\in[0,1]. \tag{6}$$

Proposition 4. *If f is differentiable and η-invex, then f is invex related to η (on $\mathbf{R}^n$).*

Proof. Relation (6) can be rewritten in the form

$$f(y+\lambda\eta(x,y))-f(y)\leq\lambda[f(x)-f(y)].$$

Let $\lambda > 0$, so by division with λ and by taking $\lambda \to 0+$, in case of differentiable f, one easily obtains

$$(\eta(x,y))^t \nabla f(y) \leq f(x) - f(y)$$

what means that f is invex related to η.

Clearly, the converse of this property is not generally true, but there exist conditions when this converse is valid, too.

We obtain first certain connections to the class of subadditive functions.

Proposition 5. *Let $f : \mathbf{R}_+ \to \mathbf{R}$ be an η-incave function, and let us suppose that $f(0) \geq 0$ and $\eta(0,x) = -x$ for all $x \in \mathbf{R}$. Then the function f is subadditive.*

Proof. From the η-incavity of f (see Definition 2) and $f(0) \geq 0$ we have

$$f(v + \lambda\eta(0,v)) \geq \lambda f(0) + (1-\lambda) f(v) \geq (1-\lambda) f(v).$$

Let $v := x + y$ and $\lambda := \dfrac{y}{x+y} \in [0,1]$ in this relation. From the equality $x = x + y + \dfrac{x}{x+y}\eta(0, x+y)$ we immediately get $f(x) \geq \dfrac{x}{x+y} f(x+y)$. Replacing x by y we get $f(y) \geq \dfrac{x}{x+y} f(x+y)$, so by addition it results $f(x) + f(y) \geq f(x+y)$, i.e. the subadditivity of f.

Proposition 6. Let $f : (0,\infty) \to \mathbf{R}$ be a subadditive function which is η-invex, and satisfies the condition

$$f(y) \leq f\left(x + y + \frac{x}{y}\eta(x, x+y)\right) \quad \text{for all } x, y \in (0,\infty). \tag{7}$$

Let $g : (0,\infty) \to \mathbf{R}$ be defined by $g(x) = \dfrac{f(x)}{x}$. Then the function g is a decreasing function.

Proof. In $f(v + \lambda\eta(u,v)) \leq \lambda f(u) + (1-\lambda) f(v)$ let us put (with $x < y$, $x, y \in (0,\infty)$)

$$\lambda := \frac{x}{y}, \quad u := x, \quad v := x + y.$$

We can obtain the relation

$$f\left(x + y + \frac{x}{y}\eta(x, x+y)\right) \leq \frac{x}{y} f(x) + \left(1 - \frac{x}{y}\right) f(x+y) \leq$$

$$\leq \frac{x}{y} f(x) + \left(1 - \frac{x}{y}\right)[f(x) + f(y)] = f(x) + f(y) - \frac{x}{y} f(y).$$

From condition (7) we can deduce the inequality

$$f(y) \leq f(x) + f(y) - \frac{x}{y} f(y), \text{ or } \frac{f(y)}{y} \leq \frac{f(x)}{x} \text{ for } x < y.$$

Thus $g(y) \leq g(x)$ for $x < y$.

Remark. For $\eta(a, b) = a - b$ (when f is convex), relation (7) becomes $f(y) \leq f(y)$, which is always true.

B. In Definition 2 we have introduced the notion of η-invexity on the entire space $\mathbf{R}^n$. In many circumsctances, it will be important to consider such functions on a subset $S \subset \mathbf{R}^n$. Then the necessity of generalization of convex sets arises.

Definition 3. ([8]) Let $S \subset \mathbf{R}^n$ and $\eta : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ be given. We say that the set S is η-invex, if the implication

$$x, y \in S, \quad \lambda \in [0, 1] \Rightarrow y + \lambda \eta(x, y) \in S \tag{8}$$

is true.

Remarks. Clearly, all subset S is η-invex to $\eta \equiv 0$. The definition essentially says that there exists a curve in S beginning from y. It is not required that x is one of the final points of this curve for all x, y.

Examples. 1) Let $\eta_1 : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ given by $\eta_1(x, y) = x - y$ for $x, y \geq 0$, $x - y$ for $x \leq 0, y \leq 0$; $-7-y$ for $x \geq 0, y \leq 0$; $2-y$ for $x \leq 0, y \geq 0$. Let $S_1 = [-7, -2] \cup [2, 10]$. Then S_1 is η_1-invex; as an easy verification applies.

2) Let $S_1 \subset \mathbf{R}$ be η_1-invex and $S_2 \subset \mathbf{R}$ be η_2-invex, and define $S = S_1 \times S_2 \subset \mathbf{R}^n$. Let $\eta : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$\eta(x, y) = (\eta_1(x), \eta_2(y)),$$

where $x = (x_1, y_2)$, $y = (x_2, y_2)$.

Then S is η-invex. This follows immediately.

Definition 4. Let $S \subset \mathbf{R}^n$ be η-invex set, where η is given. We say that f is **η-invex** on S ($f : S \to \mathbf{R}$) if relation (6) is valid for all $x, y \in S$.

If S is open, and f is differentiable, we can state the following proposition, similar to Proposition 4.

Proposition 7. *If $S \subset \mathbf{R}^n$ is open, invex, and $f : S \to \mathbf{R}$ is differentiable, and η-invex, then f is invex related to η.*

Remark. The converse of this proposition is not true. Let $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x^2$. Since all critical points are global minima, Craven and Glover's theorem implies that f is invex related to $\eta(x,y) = \dfrac{x^2 - y^2}{2y}$ $(y \neq 0)$; 0 $(y = 0)$.

On the other hand, inequality (6) is transformed into (we omit the simple algebraic manipulations) $\lambda^2(x^4 - 2x^2y^2 + y^4) \leq 0$, which is valid only if $\lambda = 0$ or $x = y = 0$. Thus f is not η-invex.

Proposition 8. *Let η be given, and let $S \subset \mathbf{R}^n$ be η-invex. Let $X \supset S$ be an open set and $f : X \to \mathbf{R}$ invex related to η, and differentiable. Let us suppose that η satisfies the following conditions*

$$\begin{aligned} &\eta(x, x + \lambda\eta(y,x)) = -\lambda\eta(y,x), \\ &\eta(y, x + \lambda\eta(y,x)) = (1-\lambda)\eta(y,x) \quad (x, y \in \mathbf{R}^n;\ \lambda \in [0,1]). \end{aligned} \tag{9}$$

Then f is η-invex on S.

Proof. Let $y, x \in S$. Let $0 < \lambda < 1$ be given and consider $z = x + \lambda\eta(y,x)$. Clearly $z \in S$. From the invexity of f related to η we have

$$f(y) - f(z) \geq (\eta(y,z))^t \nabla f(x). \tag{10}$$

In the same manner,

$$f(x) - f(z) \geq (\eta(x,z))^t \nabla f(z). \tag{11}$$

From (10) and (11) we can derive

$$\lambda f(y) + (1-\lambda) f(x) - f(z) \geq [\lambda(\eta(y,z))^t + (1-\lambda)(\eta(x,z))^t]\nabla f(z).$$

But from the given condition (9) we have

$$\lambda(\eta(y,z))^t + (1-\lambda)(\eta(x,z))^t = [\lambda(1-\lambda) - \lambda(1-\lambda)](\eta(y,x))^t = 0$$

and the theorem is proved.

Remark. Condition (9) is not trivial. Let $S = [-7,-2] \cup [2,10]$, given in example 1) from A. Then the η-function given there verifies (9). Thus $\eta(x,y) \neq x - y$.

Definition 5. Let η be given and $S \subset \mathbf{R}^n$ an η-invex set. We say that f is **η-cvazi-invex**, if

$$f(y + \lambda\eta(x,y)) \leq \max\{f(x), f(y)\}, \quad x, y \in S, \ \lambda \in [0,1] \tag{12}$$

is true.

Theorem 1. *A function $f : S \to \mathbf{R}$ is η-cvazi-invex iff all level sets $S(f,\alpha)$ of f are η-invex sets.*

Proof. Let f be η-cvazi-invex on S, and let $x, y \in S(f,\alpha) = \{z : \ f(z) \leq \alpha\}$, $\alpha \in \mathbf{R}$. We have

$$f(y + \lambda\eta(x,y)) \leq \max\{f(x), f(y)\} = f(y), \text{ if } f(x) \leq f(y).$$

Supposing $y \in S(f,\alpha)$, we get $f(y) \leq \alpha$, so $f(x) \leq \alpha$ (thus $x \in S(f,\alpha)$), it results that $f(y+\lambda\eta(x,y)) \leq \alpha$, yielding $y+\lambda\eta(x,y) \in S(f,\alpha)$. Thus the sets $S(f,\alpha)$ are η-invex.

Let us now assume that these level sets are η-invex for all $\alpha \in \mathbf{R}$, and put $\alpha = \max\{f(x), f(y)\}$. Then $x \in S(f,\alpha)$, $y \in S(f,\alpha)$. By invexity of $S(f,\alpha)$ we have $x + \lambda\eta(g,x) \in S(f,\alpha)$, thus $f[x + \lambda\eta(y,x)] \leq \alpha = \max\{f(x), f(y)\}$, which means the η-cvazi-invexity of f.

Theorem 2. *Let $S \subset \mathbf{R}^n$ be η-invex, and let $f : S \to \mathbf{R}$ be η-cvazi-invex function. Let us suppose that the function η has the following property:*

$$x \neq y \ \Rightarrow \ \eta(x,y) \neq 0. \tag{13}$$

Then all strict-local minimum point of f is a strict global minim point of f.

Proof. Let y be a strict local minim point, which is not global, then there exists $x^* \in S$ with $f(x^*) < f(y)$. But f being η-cvazi-invex, we have $f(y+\lambda\eta(x^*,y)) \leq f(y)$, where $y + \lambda\eta(x^*,y) \in S$. This gives a contradiction with the assumption on y.

We now introduce a new class of functions, namely the class of η-pseudo-invex functions.

Definition 6. ([9]) Let $S \subset \mathbf{R}^n$ be an η-invex set. We say that the function $f : S \to \mathbf{R}$ si **η-pseudo-invex** if for all $x, y \in S$ with $f(y) < f(x)$ there exists $c > 0$ and $\alpha \in (0,1]$

such that for all $a \in (0, \alpha)$ we have the inequality

$$f(x + a\eta(y, x)) \leq f(x) - ac. \tag{14}$$

Proposition 9. *If f is η-invex, then it is η-pseudo-invex, too.*

Proof. Indeed, one has

$$f(x + \lambda\eta(y, x)) \leq \lambda f(y) + (1 - \lambda)f(x) = f(x) - \lambda[f(x) - f(y)].$$

Let now $c := f(x) - f(y) > 0$ and put $\alpha := 1$. Then inequality (14) is valid.

We can consider functions $f : M \to \mathbf{R}$ with $M \subset \mathbf{R}^n$ a nonvoid set, and $S \subset M$. We say that $x^0 \in S$ is a **local-minim** point of f **relatively** to S if there exists a vecinity $V \in \mathcal{V}(x^0)$ such that for all $x \in S \cap V$ we have $f(x^0) \leq f(x)$.

Theorem 3. Let $f : M \to \mathbf{R}$, ($S \subset M$ defined as above) an η-pseudo-invex function on the invex set S. If $x^0 \in S$ is a local-minim point of f relatively to S, then x^0 is a global-minim point of f relatively to S.

Proof. There exists $V \in \mathcal{V}(x^0)$ such that for all $x \in S \cap V$ we have $f(x^0) \leq f(x)$. Let $B(x^0, r)$ be a ball inscribed in V, with $r > 0$. Thus for all $x \in S \cap B(x^0, r)$ we have

$$f(x^0) \leq f(x). \tag{15}$$

Let us suppose now, on the contrary, that there exists $y \in S$ with $f(y) < f(x^0)$. The function f being η-pseudo-invex, there exists $c > 0$ and $\alpha \in (0, 1]$ such that for all $a \in (0, \alpha)$ we have

$$f(x^0 + a\eta(y, x^0)) < f(x^0) \tag{16}$$

where $y \neq x^0$. Let a_0 be selected such that

$$0 < a_0 < \frac{r}{||\eta(y, x^0)||}$$

(which is possible since $\eta(y, x^0) \neq 0$ for $y \neq x^0$). Put $z := x^0 + a_0\eta(y, x^0)$. From (11) we get $f(z) < f(x^0)$. On the other hand, we have

$$||z - x^0|| = a_0||\eta(y, x^0)|| < r,$$

thus $z \in B(x^0, r)$. Clearly $z \in S$ (which is η-invex), so via (15) we obtain that $f(x_0) < f(z)$, a contradiction to $f(z) < f(x^0)$. This contradiction finishes the proof of the theorem.

C. We now introduce the **Jensen-invex** sets and functions.

Definition 7. The set $U \subset \mathbf{R}^n$ will be called **η-Jensen-invex** (where, as usual, $\eta : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is given) if for all $x, y \in U$ we have $y + \frac{1}{2}\eta(x,y) \in U$.

If U is η-Jensen-invex set, then the function $f : U \to \mathbf{R}$ will be called **η-Jensen-invex function** (or η-J-invex) if

$$f\left(y + \frac{1}{2}\eta(x,y)\right) \le \frac{f(x)+f(y)}{2} \text{ for all } x, y \in U. \tag{17}$$

Remark. The η-J-invex sets (or functions) could also be named $\frac{1}{2} - \eta$-invex, as we have selected from $\lambda \in [0,1]$ the set $\lambda \in \left\{\frac{1}{2}\right\}$. The J-convex functions are J-invex for $\eta(x,y) = x - y$.

Proposition 10. *Let $f : \mathbf{R}^n \to \mathbf{R}$ be invex related to η and let us suppose that η satisfies the functional equation*

$$\eta\left(x, \frac{x+y}{2}\right) + \eta\left(y, \frac{x+y}{2}\right) = 0 \quad (x, y \in \mathbf{R}^n). \tag{18}$$

Then f is J-convex function.

Proof. $f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) = \left[f(x) - f\left(\frac{x+y}{2}\right)\right] + \left[f(y) - f\left(\frac{x+y}{2}\right)\right] \ge$

$$\ge \left(\eta\left(x, \frac{x+y}{2}\right)\right)^t \nabla f\left(\frac{x+y}{2}\right) + \left(\eta\left(y, \frac{x+y}{2}\right)\right)^t \nabla f\left(\frac{x+y}{2}\right) = 0$$

on base of (1) and (18). Thus we can deduce that

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2},$$

i.e. the J-convexity of f.

The following proposition can be proved in the same manner, and we omit its proof.

Proposition 11. *Let f be as in Proposition 10, and let us assume that η satisfies the functional equation*

$$\eta\left(x+\frac{1}{2}\eta(x,y)\right)+\eta\left(y+\frac{1}{2}\eta(x,y)\right)=0 \quad (x,y\in\mathbf{R}^n). \tag{19}$$

Then the function f is η-J-invex.

The J-invexity and the continuity of functions of a variable is contained in the following:

Proposition 12. *Let $f:\mathbf{R}\to\mathbf{R}$ be η-J-invex, where $\eta:\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ satisfies the following conditions:*

$$\eta(x,x_n)\nearrow 0 \text{ if } x_n\nearrow x,$$

and

$$\eta(x,x_n)\searrow 0 \text{ if } x_n\searrow x \quad (n\to\infty).$$

Let us suppose that there exist two sequences (x_n), (y_n), where $x_n\nearrow x_0$, $y_n\searrow x_0$ $(n\to\infty)$, $x_0\in\mathbf{R}$ such that

$$\lim_{n\to\infty} f\left[x_n+\frac{1}{2}\eta(x_n,y_n)\right]=f(x_0).$$

If there exist the (lateral) limits $f(x_0-)$ and $f(x_0+)$, then f is continuous at x_0.

Proof. In the inequality

$$f\left(x+\frac{1}{2}\eta(x,y)\right)\le\frac{f(x)+f(y)}{2}$$

put $x:=x_n$, $y:=y_n$. From the given conditions we obtain

$$f(x_0)\le\frac{f(x_0-)+f(x_0+)}{2}. \tag{20}$$

Let now $x:=x_0$, $y:=x_n$ in relation (17). From $x_0+\frac{1}{2}\eta(x_0,x_n)\nearrow x_0$ we get

$$f(x_0-)\le\frac{f(x_0)+f(x_0-)}{2},$$

or

$$f(x_0-)\le f(x_0). \tag{21}$$

Let now $y := y_n$, $x := x_0$ in (17). As above, we can deduce:

$$f(x_0+) \leq f(x_0). \tag{22}$$

From (20), (21), (22) we can deduce $f(x_0) = f(x_0-) = f(x_0+)$, yielding the continuity of f at x_0.

D. Lastly, we deal with **almost-invex** functions.

Definition 8. Let $f : S \subset \mathbf{R}^n \to \mathbf{R}$, where S is an η-invex set.

We say that the function f is **η-almost invex** if

$$f(y + \lambda\eta(x, y)) \leq \lambda f[y + \eta(x, y)] + (1 - \lambda)f(y) \tag{23}$$

holds true for all $x, y \in S$.

Remark. The name "almost invex" follows from the observation that (23) may be written also as

$$f(x + \lambda\eta(x, y)) \leq \lambda f(x) + (1 - \lambda)f(y) + \lambda g(x, y)$$

where $g(x, y) = f(y + \eta(x, y)) - f(x)$.

In the case of $g(x, y) \equiv 0$ we obtain the classical η-invex functions.

The convex functions are almost invex, as we shall see.

Definition 9. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a differentiable function. We say that ∇f is **η-increasing** function, if

$$(\eta(x, y))^t(\nabla f(x) - \nabla f(y)) \geq 0, \ \forall \ x, y \in \mathbf{R}^n. \tag{24}$$

Proposition 13. *If the function $f : \mathbf{R}^n \to \mathbf{R}$ is invex related to η, and if η is an antisymmetric function, then ∇f is η-increasing.*

Proof. We have $f(x) - f(y) \geq (\eta(x, y))^t\nabla f(y)$ and $f(y) - f(x) \geq (\eta(y, x))^t\nabla f(x)$. From $\eta(y, x) = -\eta(x, y)$, and by addition we easily get $0 \geq (\eta(x, y))^t[\nabla f(y) - \nabla f(x)]$, so (24) follows.

We now prove:

Theorem 4. *Let $f : \mathbf{R}^n \to \mathbf{R}$ be differentiable, and ∇f an η-increasing function. Then f is η-almost-invex function.*

Proof. Let us introduce the function $\phi : [0,1] \to \mathbf{R}$ by $\phi(x) = f(y + \lambda\eta(x,y))$. If $0 \leq \lambda_1 < \lambda_2 \leq 1$, put $u_1 = y + \lambda_1\eta(x,y)$ and $u_2 = y + \lambda_2\eta(x,y)$. Thus $(u_2 - u_1) = (\lambda_2 - \lambda_1)\eta(x,y)$. From the η-monotonicity of ∇f we can write $0 \leq (u_2 - u_1)^t(\nabla f(u_2) - \nabla f(u_1)) = (\lambda_2 - \lambda_1)(\eta(x,y))^t[\nabla f(u_2) - \nabla f(u_1)]$. On the other hand we have $\phi'(\lambda_1) = h^t\nabla f(u_1) \leq h^t\nabla f(u_2)$, where $h = \eta(x,y)$. Thus the application ϕ' is increasing, so this function of a single variable is convex. Therefore we have $\phi(\lambda \cdot 1 + (1-\lambda)0) \leq \lambda\phi(1) + (1-\lambda)\phi(0) = \lambda f(y + \eta(x,y)) + (1-\lambda)f(y)$, i.e. the η-almost-invexity of f.

Remark. The application $\phi(\lambda) = f(y + \lambda\eta(x,y))$ introduced above has the properties $\phi(0) = f(y)$, $\phi(1) = f(y+\eta(x,y))$. Thus if $g : [0,1] \to \mathbf{R}$ is convex, then f is η-almost-invex. If the application ϕ is cvaziconvex, i.e. $g(x) \leq \max\{g(0), g(1)\}$, $\forall\ x \in [0,1]$, we can obtain the notion of **almost-cvazi-invexity**. Thus $f : S \to \mathbf{R}$ will be **η-almost-cvazi-invex** if

$$f(y + \lambda\eta(x,y)) \leq \max\{f(y), f(y + \eta(x,y))\}.$$

However, we do not study this class of functions here.

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"BABEŞ-BOLYAI" UNIVERSITY, 3400 CLUJ-NAPOCA, ROMANIA

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ALMOST OPTIMAL NUMERICAL METHOD

SOMOGYI ILDIKO

Abstract. This paper investigates an algorithm presented by Smolyak (1963), who studied tensor product problems.

1. Introduction

The essence of these algorithms is that it is enough to know how to solve the tensor product problem for $d = 1$ efficiently. The algorithms for arbitrary d are fully determined in terms of the algorithms for generally, arbitrary linear functionals.

The choice of function values is especially interesting, since for arbitrary linear functionals we know how to solve multivariate problems.

The algorithms are linear. They depend linearly on the information. This property makes their implementation easier. In fact, the weights of the algorithm for $d \geq 2$ are given by linear combinations of the corresponding tensor product weights of the one dimensional algorithms. Information used by the algorithms is called hyperbolic cross information and had been successfully applied for a number of problems.

2. Formulation of the problem

In this section a tensor product problem will be define for a class of functions of d variables.

For $d = 1, 2, \dots$ consider

$$S_d : X_d \to Y_d$$

where X_d is a separable Banach space of functions $f : D^d \to \mathbf{R}$, $D \subset \mathbf{R}$, Y_d is either a separable Hilbert space of functions, or $\mathbf{R}$, and S_d is a continuous linear operator.

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We assume that Y_d is a tensor product,

$$Y_d = Y_1 \otimes Y_1 \otimes \ldots \otimes Y_1, \tag{1}$$

and X_1 is a Hilbert space

$$X_d = X_1 \otimes X_1 \otimes \ldots \otimes X_1$$

$$S_d = S_1 \otimes S_1 \otimes \ldots \otimes S_1.$$

The tensor product $f = f_1 \otimes \ldots \otimes f_d = \bigotimes_{k=1}^{d} f_k$ for numbers f_k is just the product $\prod_{k=1}^{d} f_k$. When f_k are scalar functions, f is a function of d variables, $f(t_1, \ldots, t_d) = \prod_{k=1}^{d} f_k(t_k)$.

The element $S_d(f)$ is approximated by $A(f) = \phi(N(f))$, where the information about f,

$$N(f) = [L_1(f), \ldots, L_n(f)], \tag{2}$$

consists of n values of continuous linear functionals L_i, and $\phi : \mathbf{R}^n \to G_d$ is a linear mapping. This results from linearity of A,

$$A(f) = \sum_{i=1}^{n} y_i L_i(f), \qquad \text{for some } y_i \in Y_d. \tag{3}$$

The error of the algorithm A is given as

$$e(A) = \sup \{ ||S_d(f) - A(f)||_{Y_d} \ : \ ||f||_{X_d} \leq 1 \}. \tag{4}$$

Due to linearity of S_d and A, we have

$$e(A) = ||S_d - A||.$$

The cost of A does not depend on the setting and it is defined as follows. We assume that the cost of computing $L_i(f)$ equals $c(d)$ for any $f \in X_d$ and any L_i. Also assume that basic arithmetic operations on reals and multiplication and addition in Y_d have a unit cost. Assuming that the elements y_i can be precomputed, the cost of the algorithm A, $cost(A)$, is bounded by

$$cost(A) \leq n(c(d) + 2) - 1.$$

The precomputation of the elements y_i is usually easy since they depend only on the corresponding elements for $d = 1$.

3. Smolyak's algorithm

As it was mentioned in the introduction, the essence of these algorithms is that they give a general construction that leads to almost optimal approximations for any dimension $d > 1$ from optimal approximation for the univariate case $d = 1$.

Assume, therefore, that for $d = 1$, we know linear algorithms (operators) U^i, $i \geq 1$, which approximate the problem $\{X_1, Y_1, S_1\}$ such that $||S_1 - U^i|| \to 0$ as $i \to \infty$. Introducing the notation

$$\Delta_0 = U_0 = 0, \qquad \Delta_i = U_i - U_{i-1}, \tag{5}$$

for $d > 1$ we approximate the tensor product problem $\{X_d, Y_d, S_d\}$ by the algorithm

$$A(q,d) = \sum_{0 \leq i_1 + i_2 + \cdots + i_d \leq d} \Delta_{i_1} \otimes \ldots \otimes \Delta_{i_d}. \tag{6}$$

Hence $f(t_1, t_2, ...t_d) = f_1(t_1) f_2(t_2) \ldots f_d(t_d)$ then

$$(A(q,d)f)(t_1, t_2, \ldots t_d) = \sum_{0 \leq i_1 + i_2 + \cdots + i_d \leq d} (\Delta_{i_1} f_1)(t_1)(\Delta_{i_2} f_2)(t_2) \ldots (\Delta_{i_d} f_d)(t_d)$$

where q is a nonnegativ integer, and $q \geq d$, because when $q < d$ one of the indices is zero, say $i_j = 0$, and $\Delta_{i_j} = 0$ implies that $A(q,d) = 0$.

We use the notation $|i| = i_1 + \cdots i_d$ for $i \in N^d$ and $i \geq j$ if $i_k \geq j_k$ for all k. By $Q(q,d)$ we mean

$$Q(q,d) = \{i = (i_1, i_2, \ldots i_d) : 1 \leq i, |i| \leq q\}$$

with $1 = (1, 1 \ldots 1)$ and $|Q(q,d)| = \binom{q}{d}$.

We have

$$\begin{aligned} A(q,d) &= \sum_{i \in Q(q,d)} \bigotimes_{k=1}^{d} \Delta_{i_k} = \sum_{i \in Q(q-1,d-1)} (\bigotimes_{k=1}^{d} \Delta_{i_k}) \otimes \sum_{i_d=1}^{g-|i|} \Delta_{i_d} \\ &= \sum_{i \in Q(q-1,d-1)} (\bigotimes_{k=1}^{d-1} \Delta_{i_k}) \otimes U_{q-|i|} \end{aligned} \tag{7}$$

since $\sum_{i=1}^{m} \Delta_i = U_m$ for any $m \geq 1$.

Observe that

$$\bigotimes_{k=1}^{d}(U_{i_k} - U_{i_{k-1}}) = \sum_{\alpha\in\{0,1\}^d} (-1)^{|\alpha|} \bigotimes_{k=1}^{d} U_{i_k-\alpha_k}$$

$\bigotimes_{k=1}^{d} U_{j_k}$ appears in $A(q,d)$ for all indices i for which $i_k = j_k + \alpha_k$ with $\alpha \in \{0,1\}^d$ and $\mid \alpha \mid \leq q- \mid j \mid$. The sign of $\bigotimes_{k=1}^{d} U_{j_k}$ in this case is $(-1)^{|\alpha|}$.

Let

$$b(i,d) = \sum_{\alpha\in\{0,1\}^d, |\alpha|\leq i} (-1)^{|\alpha|}.$$

This yield

$$A(q,d) = \sum_{j\in Q(q,d)} b(q- \mid j \mid, d) \bigotimes_{k=1}^{d} U_{j_k}.$$

We now compute $b(i,d)$. Since $|\alpha| = j$ corresponds to $\binom{d}{j}$ terms, we have

$$b(i,d) = \sum_{j=0}^{\min\{i,d\}} \binom{d}{j}(-1)^j = (-1)^i \binom{d-1}{i}.$$

In particular, $b(i,d) = 0$ for $i \geq d$. This yields the explicit form of $A(q,d)$:

Lema 1.

$$A(q,d) = \sum_{q-d+1\leq|i|\leq q} (-1)^{q-|i|} \binom{d-1}{q- \mid i \mid} (U_{i_1} \otimes \ldots \otimes U_{i_d}) \tag{8}$$

In particular, for

$$U_i(f) = \sum_{j=1}^{m_i} a_{i,j} L_{i,j}(f)$$

with $a_{i,j} \in G_1$ and continuous functionals $L_{i,j}$ we have

$$A(q,d)f = \sum_{q-d+1\leq|i|\leq q} (-1)^{q-|i|} \binom{d-1}{q- \mid i \mid} \sum_{j\leq m_i} L_{i,j}(f) g_{i,j},$$

where $L_{i,j} = \bigotimes_{k=1}^{d} L_{i_k,j_k}$, $g_{i,j} = \bigotimes_{k=1}^{d} a_{i_k,j_k}$ and $m_i = (m_{i_1}, \ldots, m_{i_d})$.

Furthermore we consider the case in which for $d = 1$ we have one of the spaces

$$F_1^r = C^r([-1,1]), \qquad r \in N$$

with the norm

$$||f|| = max(||f||_\infty, \dots, ||f^{(r)}||_\infty).$$

For $d > 1$ consider the tensor product

$$F_d^r = \{f : [-1,1]^d \to \mathbf{R} / D^\alpha f \text{ continuous if } \alpha_i \leq r \ \ \forall \ \ i\}$$

with the norm

$$||f|| = max\{||D^\alpha f||_\infty / \alpha \in N_0^d, \alpha_i \leq r\}.$$

Let

$$I_d(f) = \int_{[-1,1]^d} f(x)dx, \qquad \text{with } f \in F_d^r. \tag{9}$$

We wish to find good approximation to the functional I_d on the basis of good approximation in the univariate case, using the algorithm of Smolyak.

In the multivariate case $d \geq 1$, define

$$U_{i_1} \otimes \dots \otimes U_{i_d} = \sum_{j_1=1}^{m_{i_1}} \dots \sum_{j_d=1}^{m_{i_d}} f(x_{j_1}^{i_1}, \dots, x_{j_d}^{i_d})(a_{j_1}^{i_1}, \dots, a_{j_d}^{i_d})$$

where we assume that a sequence of quadrature formulas

$$U_i(f) = \sum_{j=1}^{m_i} f(x_j^i) a_j^i$$

is given with $m_i \in N$.

On the basis of Lemma 1 with given quadrature formulas U^i we can write the approximation formula $A(q,d)$ for general d.

$A(q,d)$ is a linear functional, and for $f \in F_d^r$, $A(q,d)(f)$ depends only through function values at a finite number of points.

Let $X^i = \{x_1^i, \dots, x_{m_i}^i\} \subset [-1,1]$ denote the set of points that correspond to U^i. Then $U_{i_1} \otimes \dots \otimes U_{i_d}$ is based on the grid $X^{i_1} \times \dots \times X^{i_d}$, and therefore $A(q,d)(f)$ depends on the values of f at the union

$$H(q,d) = \bigcup_{q-d+1 \leq |i| \leq q} (X^{i_1} \times \dots \times X^{i_d}) \in [-1,1]^d.$$

If $X_i \subset X_{i+1}$, than $H(q,d) \subset H(q+1,d)$ and $H(q,d) = \bigcup_{|i|=q} (X^{i_1} \times \ldots \times X^{i_d})$. Therefore this kind of sets seems to be the most economical choice.

In the general case we assume that the algorithm

$$U_i(f) = \sum_{j=1}^{m_i} a_{i,j} L_{i,j}(f)$$

use nested information $N_i = [L_{i,1}, L_{i,2}, \ldots, L_{i,m_i}]$. That is,

$$\{L_{i,1}, L_{i,2}, \ldots, L_{i,m_i}\} \subset \{L_{i+1,1}, L_{i+1,2}, \ldots, L_{i+1,m_{i+1}}\}, \ \forall i = 1, 2, \ldots \tag{10}$$

Since X_1 is now a Hilbert space, $L_{i,j} =< f, f_{i,j} >$ for some element $f_{i,j}$ of X_1. Hence, there exists a sequence $\{f_i\}$ in F_1 such that

$$N_i(f) = \{< f, f_1 >, < f, f_2 >, \ldots, < f, f_m >\} \ \imath = 1, 2, \ldots$$

Assume that the algorithms U_i are optimal, i.e. they minimize the error among all algorithms that use the information N_i. U_i is optimal if

$$L_i = S_1 \mathcal{P}_{\mathrm{i}}, \tag{11}$$

where $\mathcal{P}$ is the orthogonal projection on the linear subspace $span\{f_j, j = 1, 2, \ldots, m_i\} = (kerN_i)^{\perp}$. Then (11) implies optimality of the algorithm $A(q,d)$ for any d. If we note $N_{q,d}(f) = [L_{i,j}(f) : 1 \leq i, q - d + 1 \leq | i | \leq q, j \leq m_i]$ the information used by the algorithm $A(q,d)$, then for nested information N_i and optimal U_i of (11), $A(q,d) = S_d \mathcal{P}(q,d)$ where $\mathcal{P}(q,d)$ is the orthogonal projection on the linear subspace $(ker(N(q,d))^{\perp}$. Thus, in particular, $A(q,d)$ minimizes the error among all algorithms that use the same information $N_{q,d}$.

4. The Clenshaw-Curtis method

For any cubature formula Q we have the error bound

$$|I_d(f) - Q(f)| \leq ||I_d - Q|| \cdot ||f||$$

In the univariate case $d = 1$

$$\lim_{n \to \infty} n^r \cdot \inf_{Q_n}(||I_1 - Q_n||) = \beta_r \tag{12}$$

where $\beta_r > 0$ are known constants for any $\forall r \in N$, (Strauβ, 1979), and Q_n are formulas which use n function value.

Novak and Ritter suggest to use the Clenshaw-Curtis method, with a suitable choice of the sequence m_i, where m_i denotes the number of function value used by U_i, and assume that $m_i < m_{i+1}$. In light of (12) they are interested in formulas U_i with

$$\lim_{i\to\infty} \sup(m_i^r \|I_1 - U_i\|) < \infty, \qquad \forall i \in N. \tag{13}$$

and the property is true, for interpolatory formulas U_i, with positive weights.

To obtain nested sets of points, they choose

$$m_i = 2^{i-1} + 1, \quad i > 1 \text{ and } m_1 = 1. \tag{14}$$

Let

$$x_j^i = -\cos\frac{\pi(j-1)}{m_i - 1}, \qquad j = 1, 2, \dots, m_i$$

and $x_1^1 = 0$, then $U_1(j) = 2j(0)$.

The weights of the Clenshaw-Curtis formula

$$U_i(f) = \sum_{j=1}^{m_i} f(x_j^i) a_j^i$$

are caracterized by the demand that U_i is exact for all polinomials of degree less than m_i, and for $i > 1$ they are given by

$$a_j^i = a_{m_i+1-j}^i = \frac{2}{m_i - 1}\left(1 - \frac{\cos(\pi(j-1)}{m_i(m_i - 2} - 2\sum_{k=1}^{m_i-3/2} \frac{1}{4k^2 - 1} \cdot \cos\frac{2\pi k(j-1)}{m_i - 1}\right)$$

for $j = 2, \dots, m_i$ and $a_1^i = a_{m_i}^i = \frac{1}{m_i(m_i-2)}$.

For delimitation of the error, they start from the estimate in the univariate case

$$\|I_1 - U_i\| \leq \gamma_r \cdot 2^{-r \cdot i}.$$

From (6) we get

$$A(q+1,d+1) = \sum_{|i|\le q}(\Delta^{i_1}\otimes\ldots\otimes\Delta^{i_d}\otimes\sum_{k=1}^{q+1-|i|}\Delta^{i_k}) = \sum_{|i|\le q}(\Delta^{i_1}\otimes\ldots\otimes\Delta^{i_d}\otimes U_{q+1-|i|}).$$

Then for the error we can obtain the following estimate:

$$I_{d+1} - A(q+1,d+1) = (I_d - A(q,d))\otimes I_1 + \sum_{|i|\le q}\Delta^{i_1}\otimes\ldots\otimes\Delta^{i_d}\otimes(I_1 - U_{q+1-|i|}).$$

Furthermore

$$||\Delta^{i_k} \le ||I_1 - U_{i_k}|| + ||I_1 - U_{i_k-1}|| \le \gamma_r\cdot 2^{-ri_k}(1+2^r).$$

We get

$$\sum_{|i|\le q}||\Delta^{i_1}||\cdot\ldots\cdot||\Delta^{i_d}||\cdot||I_1 - U^{q+1-|i|}|| \le \binom{q}{d}\cdot\gamma_r^{d+1}\cdot(1+2^r)^d\cdot 2^{-r(q+1)}.$$

Inductively the following theorem can be obtained.

Theorem 1. *Let* $\theta_r = \max\{2^{r+1}, \gamma_r\cdot(1+2^r)\}$. *The error of the cubature formula* $A(q,d)$ *satisfies the following estimates:*

$$||I_d - A(q,d)|| \le \gamma_r\theta_r^{d-1}\binom{q}{d-1}\cdot 2^{-r\cdot q}.$$

Corollary 1. *Let* $n = n(q,d)$ *denote the number of knots used by* $A(q,d)$. *Then*

$$||I_d - A(q,d)|| = \mathcal{O}(n^{-r}\cdot(\log n)^{(d-1)(r-1)}).$$

This corrolary gives the error of $A(q,d)$ related to the number of knots from $H(q,d)$ and also gives the best error bound for Smolyak's algorithm which holds for arbitrary tensor product problems. On the other hand this method yields error of order $n^r(logn)^{(d-1)(r-1)}$ for all classes F_d^r, hence this methods are almost optimal up to logarithmic factors on a whole scale of spaces of nonperiodic functions.

Property (15) is the essential requirement for the U_i in the univariate case. Relation which also holds for the Gauss formulas. These formulas yield methods

$A(q,d)$ with a higher degree of exactness. Still Novak and Ritter prefer the Clenshaw-Curtis formulas because in this case the number of knots from $H(q,d)$ is reduced. Weights of different signs at common points are partially cancelled.

To determine the polynomial exactness they start from the fact that the Clenshaw-Curtis formula U_i is exact on $V^i = P_{m_i}$, where m_i is odd.

Theorem 2. *The cubature formula $A(q,d)$ is exact on*

$$\sum_{|i|=q} (V^{i_1} \otimes \ldots \otimes V^{i_d}).$$

The theorem can be proved by induction over d.

Remark. Theorem 2 holds for general tensor product problems if the space

$$V^i = \{f \in F_1^r / I_1(f) = U^i(f)\}$$

of exactness for the univariate problem is nested, $V^i \subset V^{i+1}$.

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"Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, 3400 Cluj-Napoca, Romania.

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CONNECTING ALGORITHMICAL PROBLEMS IN SEMIGROUPS WITH THE THEORIES OF LANGUAGES AND AUTOMATA

K.D. TARKALANOV

Abstract. This paper is a survey with new generalizations and their short proofs of some of our results. Its purpose is connecting as pointed out in the title.

A beginning (and end) of a word over an alphabet is called correct [4] if its length is not smaller than the half of the word. A finite determined semigroup is of the class $K_{\frac{1}{2}}$ [4] under the following conditions: if two determining words have a common section which is a correct beginning (a correct end) of one of them, then this section is a beginning (an end) of the other as well. (We give a generalization in [10]). The word $E = R_1R_2 \ldots R_k$ over the alphabet of a semigroup of $K_{\frac{1}{2}}$ is normal [4] if each multiplier is s subword of a determining word, the first is a correct beginning, the last being a correct end and if each R_i is a correct end of R_{i+1} is a correct beginning.

The word problem is solvable in a semigroup of the class $K_{\frac{1}{4}}$ [4].

The inequality problem which is formulated by us in [9] is in fact the problem of the deduction in a semi-true system. The one-direction substitutions represents a transformation to smaller words. By a process of a symmetrization a semigroup is obtained and the one-direction inequalities introduce an order in it. This problem is more general than the word problem. It is solvable [10] in a class of partially ordered semigroups which are given by determining inequalities. This class of ours contains the class $K_{\frac{1}{2}}$. In the basis of our generalization way suggested in [9], we are not interested in the common sections of the right determining words in the one-direction transformations from left to right.

A vocabulary (a set of words over an alphabet) is called strongly regular [7] if none of its words enters into another and no real beginning of any word is an end of any of them. The invariant deconding automata without overtaking [7] decode (the coding is an overletter one) the coding word only after its full absorption at the entrance. Until then they iet out the empty word at the exit. This sort of automata exists if and only if [7] the coding words constitute a strongly regular vocabulary.

We prove that in a (partially ordered) semigroup with a strongly regular vocabulary of the determining words the inequality/word problem is solvable [11].

In an analogical way to the relation from theorem 2.1.5 [8] (as well as theorem 1 [5]) we define [12] a congruence $\approx$ in a semigroup Π in the following way: we shall say that the element $[x]$ is in correlation to the element $[y]$ in Π if and only if for any arbitrary elements $[w]$ and $[z]$ of Π the elements $[w][x][z]$ and $[w][y][z]$ simultaneleously belong or do not belong to a given subset of elements of Π. We prove [12] the theorem: *If the full prototype $\varphi^{-1}(M)$ of the subset M of the semigroup Π with a finite number of generators Σ at the natural homomorfism φ of the free semigroup Σ^* (generated by them) is a regular language in the latter, then the congruence $\approx$ for M has a finite index in Π and M consists of full classes of elements which are equivalent to it. Then the natural homomorphic image $\Pi/\approx$ is a finite semigroup. (The natural homomorphism depicts each letter from Σ in that element from Π which contains it.)*

This theorem affords a common way of obtaining finite homomorphic images of semigroups. This can be fulfilled for the semigroups of the class $K_{\frac{1}{2}}$ and with a strongly regular vocabulary of the determining words: thus, we shall say that an element of a semigroup Π_1 of the class $K_{\frac{1}{2}}$ is normal if it consists of (a finite number - according to the theorems for the solvability of the word/inequality problem) normal words. For the subsemigroup N_1 of the normal elements $\varphi^{-1}(N_1)$ is a regular language in Σ^* [12]. The proof is effected by constructing a concrete right linear grammar [8]. We shall say that an element of a semigroup Π_2 from the other class is vocabular if all of its words (a finite number [11]) are products of determining words. For the subsemigroup N_2 from the vocabular elements $\varphi^{-1}(N_2)$ is a *regular language.*

Actually, for the strongly regular vocabulary $V = \{v_1, v_2, \dots, v_m\}$ of the determining words of Π_2 we construct the right linear grammar

$$\Gamma_2 = \{\Sigma \cup \{\mu_i |\ i = \overline{1,m}\} \cup \{\sigma\}, \Sigma, Q_w, \sigma\}$$

(the designations are clear [8]). Here Q_2 are the following rules for deduction:

$$\sigma : v_i\mu_i; \quad i = \overline{1,m};$$

$$\mu_i \to v_j\mu_j; \quad j = \overline{1,m};$$

$$\mu_i \to \Lambda.$$

It has been proved that the language generated by it coincide with the subsemigroup $\varphi^{-1}(N_2)$ of all vocabulary words in Σ^*.

The already indicated theorem from [12] gives us the opportunity by a unified method to prove

Theorem 1. *Each semigroup from the two indicated classes possesses a finite homographic image with a nontrivial subsemigroup.*

The part of this theorem for the class with strongly regular vocabularies of the determining words is not published so far.

A word from a semigroup Π_1 is called symmetric if it begins and ends in a normal word. The symmetric elements form a subsemigroups H_1 in Π_1 [12]. The word $\gamma = \gamma_1 v_{j_1} \gamma_2 v_{j_2} \dots \gamma_t v_{j_t} \gamma_{t+1}$ in a semigroup v_{j_s}, $s = \overline{1,t}$) is called right special if γ_{t+1} does not end in a nonempty proper beginning of a determining word. The right special elements form a subsemigroup H_2 in Π_2 [11]. The method applied for separating subsemigroups in the semigroups from the two classes aims each one of them to possess the following property: if a given element of the separated subsemigroup is a product of several of its elements then each of its words is a product of a word from the first multiplier multiplied by a word from the second one and so on - until the last one. This property is not fulfilled in the general case.

A definer regular algebra over an arbitrary semigroup has been introduced in [6] analogically to Kleene's algebra of regular events over a free semigroup. The general method of separating subsemigroup with the indicated property is suggested by the author and his two results from [12] and [11] can be unified in one (not published so far).

Theorem 2. *If $\mathcal{R}$ and $\mathcal{S}$ are regular expressions in a separated subsemigroup H of a semigroup of one of the two classes with a system of generators Σ then $\mathcal{R} = \mathcal{S} \Leftrightarrow \varphi^{-1}(\mathcal{R}) = \varphi^{-1}(\mathcal{S})$.*

In this way the problem of the equality $\mathcal{R} = \mathcal{S}$ is reduced to the solvable [8], [1] identity problem in the algebra of the regular events over Σ^*.

After everything said about the strongly regular vocabularies it is justified and in co-ordination with it to continue the research of the decoding automata without overtaking. We prove [13] that a homomorphic image [2] of an invariant decondin automaton without overtaking is the same automaton as the above. The existence of a nontrivial image is proved with the help of our theorem [14] for constructing factor-automata. Much later and in the particular case of automata without exits factor-automata are been studied by another author in [3]. With the help of the correspondences from the definition for an automaton homomorphism [2] we define an image of coding:

Let (H_1, H_2, H_3) be a homomorphism of the automaton $U = (S, B, A, \lambda, \beta)$ in the automaton $U' = (S', B', A', \lambda', \beta')$ and the correspondence $H_3 : A = \{a_1, a_2, \dots, a_n\} \to A' = \{a'_1, a'_2, \dots, a'_m\}$ of the exit alphabet A in the exit alphabet A' be reversible. Let U' be an invariant decondin automaton without overtaking for the conding $K_{V'}^{A'}$. Then $n \le m$ and let us admit that $H_3(a_i) = a'_i$, $i = \overline{1, n}$ (after permutations in A or in A' which does not reduce the generality). We chose only one prototype v_i at the correspondence H_2 for each coding owrd v'_i $(i = \overline{1, n})$ of the coding $K_{V'}^{A'}$, where the words v_i from the semigroup B^* have lengths, which are respectively equal to these of the words v'_i. In this way we obtain many codings K_V^A of the alphabet A with vocabularies $V = \{v_1, v_2, \dots, v_n\}$. Analogically we prove an unpublished theorem which answers the interesting reverse question:

Theorem 3. *If U' is an invariant deconding automaton without overtaking for the coding $K_{V'}^{A'}$ then the automaton U decodes invariantly without overtaking each one of the possible codings K_V^A.*

We prove also in [13] that the consecutive connection of two invariant decoding automata without overtaking is the same automaton as the above. As a consequence we point out that after a consecutive coding with the help of strongly regular vocabularies a coding is obtained again with the help of such a vocabulary.

A considerable range of the problems considered is well-grounded from: methodological and philosophical point of view in [15].

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KRASSIMIR D. TARKALANOV, PLOVDIV UNIVERSITY, DEPT. OF MATHEMATICS AND INFORMATICS, 24 TZAR ASSON STR., PLOVDIV 4000, BULGARIA

STUDIA UNIV. "BABEŞ-BOLYAI", MATHEMATICA, Volume XLIV, Number 1, March 1999

PERIHELION ADVANCE AND MANEFF'S FIELD

VASILE URECHE

Abstract. We compute the advance of the perihelion of a planetary orbit as predicted by the Maneff's gravitational law and we compare the result with the results of the general relativity theory, as well as with the observational data for Mercury and for the binary pulsar PSR 1913+16. The effects resulting from the adoption of the Maneff's potential are analysed both in the classical and the relativistic case. For the relatistic analysis we propose a new form of the metric associated to Maneff's gravitational potential.

The results show that in the classical case the advance of the perihelion (periastron) predicted in the Maneff's model is exactly half of the observed one, while putting the prediction of this model in accord with the prediction of general relativity requires a modification of the perturbating factor in Maneff's potential with a factor of 2.

The computation made in the relativistic case with the Maneff potential give a result which is not in concordance with the observational data, because in this case the advance of the perihelion is a superposition of the value due to the relativistic effect and that resulting from the modification of the potential in the Maneff case.

1. Introduction

G. Maneff considered a post-Newtonian nonrelativistic law of gravitation, assuming that the gravitational interaction between two masses m_1 and m_2 is given

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by the "force function" ([5]):

$$U = \frac{Gm_1m_2}{r}\left[1 + \frac{3G(m_1+m_2)}{2c^2r}\right], \tag{1}$$

where r is the distance between m_1 and m_2, G is the Newtonian gravitational constant, and c is the speed of light.

Recently, several *theoretical* approaches considered that the study of the consequences of adopting Maneff's potential, would be an ideal method to investigate the susceptibility to generalization of the mathematical models and techniques developed in Celestial Mechanics ([6],[1]) and Stellar Astrophysics ([9], [10]).

In this paper, we analyze the way in which the Maneff's gravitational interaction responds to one of the most important *observational* facts that have become a milestone in the evolution of the theory of gravitation: the advance of the planetary perihelion.

This phenomenon was discovered by Le Verrier in 1859 as a discrepancy between the observations and the theoretical predictions for the shift of Mercury's perihelion. Present-day measurements indicate that Mercury exhibits an excess motion in the perihelion shift of about 43" per century. The attempts to explain this phenomenon have to consider that either a hidden planet or some sort of diffuse material should orbit in the neighborhood of the Sun - or Newton's theory of gravity should suffer some adjustments. All the models involving hidden mass within Newton's theory of gravitation have constantly failed, while the excellent correlation between the observations and the theoretical predictions of Einstein's General Relativity became one of the great successes of this theory.

This paper analyses the problem of perihelion advance in a potential-independent fashion, i.e. we infer the expression for the perihelion advance as a functional of the potential expression. In section 2, we develop the Binet-like differential equation for the orbit of a body moving in a central spherical symmetric field. The form of the potential $\Phi(r)$ is not specified, so the equation explicitly depends on Φ. The potential is then particularized to Maneff's expression and the perihelion advance is computed in this case. Section 3, after introducing a general relativistic metric to be associated with a spherical symmetric potential function $\Phi(r)$, proceeds in a similar fashion to derive the perihelion advance in the relativistic framework. The problem of periastron advance for the binary pulsar is considered in Section 4. In section 5 we summarize the results, compare the observational values for the Mercury's orbit and for binary pulsar PSR 1913+16 with those theoretically predicted and drop the conclusions.

2. The classical framework

2.1. **The Binet-like equation.** We shall consider a massive body of mass M and a test particle of mass $m \ll M$ in the gravitational field of M. The effects of this field on m can be descried by the following potential Φ, which is attached to U from (1) ($m_1 = M$, $m_2 = m$):

$$\Phi(r) = -\frac{GM}{r} - \frac{3G^2M^2}{2c^2r^2} \tag{2}$$

The spherical symmetry implies that the orbit is planar, so we restrict our considerations to the two-dimensional problem, i.e. finding the equation of the orbit in the form $r = r(\theta)$ We shall start the derivation of the differential equation of the trajectory from the laws of conservation for energy and angular momentum:

$$\begin{cases} \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 + 2\Phi(r) = h \\ r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = C \end{cases} \tag{3}$$

After the change of unknown function to $u = 1/r$ we obtain from (3) the Binet-like equation:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = -u - \frac{1}{C^2}\frac{\mathrm{d}\Phi}{\mathrm{d}u} \tag{4}$$

2.2. **Solution for Newtonian potential.** If the potential is Newtonian we find the well-known conic solution:

$$u = \frac{GM}{C^2}\left[1 + e\cos(\theta - \omega)\right]. \tag{5}$$

For the adequate values of h and C this orbit will be an ellipse.

2.3. Solution for Maneff case. For Maneff potential the equation (4) becomes:

$$\frac{d^2u}{d\theta^2} = -u\left(1 - 3\frac{G^2M^2}{C^2c^2}\right) - \frac{GM}{C^2} \tag{6}$$

One observes that this equation has an exact analytic solution. If we use the notation:

$$\alpha = 3\frac{G^2M^2}{C^2c^2}$$

with $\alpha < 1$ for the realistic astrophysical situations (noncollisional orbits), the solution of (6) is:

$$u = \frac{GM}{C^2(1-\alpha)}\left[1 + e\cos(\sqrt{1-\alpha}\,\theta - \omega)\right]. \tag{7}$$

For $0 < e < 1$ and $\alpha \ll 1$ this represents approximately an allipse. If we try to identify the perihelion advance in (7) by putting it in the form:

$$u = \frac{GM}{C^2(1-\alpha)}\left[1 + e\cos(\theta - \omega - \delta(\theta))\right]. \tag{8}$$

and if we take into account the fact that usually $\alpha \ll 1$, we get:

$$\delta(\theta) = \frac{1}{2}\alpha\theta = \frac{3}{2}\frac{G^2M^2}{C^2c^2}\theta. \tag{9}$$

The perihelion advance predicted by Maneff's field is proportional with the value obtained by Einstein's relativity (see below eq. (17)), i.e. Einstein's expression for perihelion advance is twice as big as Maneff's. This problem can easily be solved by scaling the "perturbative" term in Maneff's formula. Thus, if we took the potential of the form:

$$\Phi(r) = -\frac{GM}{r} - 3\frac{G^2M^2}{c^2r^2} \tag{10}$$

we would obtain the exact relativistic formula for the perihelion advance within the classical framework.

3. The relativistic framework

3.1. **The potential-dependant metric.** Let $x^0 = ct$ be the temporal coordinate and $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$ the spherical (Schwarzschild) coordinates. Then, we shall associate to the potential Φ the following general relativistic metric ([4])

$$\mathrm{d}s^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2\,\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 + \frac{2\Phi}{c^2}} - r^2\,\mathrm{d}\theta^2 - r^2 \sin^2\theta\,\mathrm{d}\varphi^2, \tag{11}$$

where $\mathrm{d}s$ is the elementary interval.

One should note that the metric given by eq. (11) does not satisfy Einstein's field equations ([8], [2]) unless the potential Φ is Newtonian, i.e. it has an expression of the form:

$$\Phi(r) = -\frac{a}{r}.$$

Therefore, any attempt to extend this approach beyond the problem of the motion in a central field (e.g. modeling massive relativistic bodies as it is required in astrophysical or cosmological applications) should start from defining a proper adjustment to the field equations. Fortunately, it is not the case for the matter of perihelion advance, since the equation of the orbit will be straightly inferred from the equations of the geodesics.

3.2. **The Binet-like equation.** Once the relativistic metric of the field is specified the derivation of the Binet-like equation for the trajectory proceeds by computing Christoffel's symbols and then writing the equations of the geodesics. One should refer to Tolman ([8]) for the details of this derivation for the general Schwartzschild metric:

$$\mathrm{d}s^2 = e^{\nu(r)}\,\mathrm{d}t^2 - e^{\lambda(r)}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2 \sin^2\theta\,\mathrm{d}\varphi^2, \tag{12}$$

noting that our metric (3) is a particular case of (12).

The system of 10 geodesic equations for the metric (12), finally reduces to the following two equations:

$$\begin{cases} e^{\lambda}\left(\frac{\mathrm{d}r}{\mathrm{d}_s}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}_s}\right)^2 - e^{-\nu}K^2 + 1 = 0 \\ r^2\frac{\mathrm{d}\theta}{\mathrm{d}_s} = H, \end{cases} \tag{13}$$

where K is a dimensionless constant and Hc is the relativistic equivalent of C constant in the classical approach.

For the metric (3), eqs. (13) become:

$$\begin{cases} \frac{1}{1+\frac{2\Phi}{c^2}}\left(\frac{\mathrm{d}r}{\mathrm{d}_s}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}_s}\right)^2 - \frac{K^2}{1+\frac{2\Phi}{c^2}} + 1 = 0 \\ r^2\frac{\mathrm{d}\theta}{\mathrm{d}_s} = H, \end{cases} \tag{14}$$

Taking the new unknown function $u = 1/r$ we obtain the Binet-like equation:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} = -u\left(1 + \frac{2\Phi}{c^2}\right) - \frac{u^2}{c^2}\frac{\mathrm{d}\Phi}{\mathrm{d}u} - \frac{1}{c^2H^2}\frac{\mathrm{d}\Phi}{\mathrm{d}u} \tag{15}$$

3.3. **Newtonian potential.** In the case of Newtonian potential, eq. (15) becomes:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} = -u + \frac{GM}{c^2H^2} + \frac{3GM}{c^2}u^2. \tag{16}$$

This equation cannot be integrated in terms of elementary functions. The approximate approach in solving equation (16) takes into account the fact the the bounded (noncollisional) orbit is quasi-elliptical, so the solution can be put in the following form ([8]):

$$u = \frac{GM}{C^2}\left[1 + e\cos(\theta - \omega - \delta(\theta)\right]$$

where $\delta(\theta) \ll \theta$.

In the first-order analysis, one obtains for $\delta(\theta)$ the expression:

$$\delta(\theta) = 3\frac{G^2M^2}{C^2c^2}\theta. \tag{17}$$

Then, the Einstein's formula for the perihelion advance for one period will be:

$$\Delta\omega = 6\pi\frac{G^2M^2}{C^2c^2} = \frac{24\pi^3a^2}{c^2P^2(1-e^2)} \tag{18}$$

It is well-known that equation (18) accounts for the observed orbits with an excellent accuracy.

Taking into account the Kepler's third law, the expression (18) can be put in the form ([3]):

$$\Delta\omega = \frac{6\pi GM}{c^2 a(1-e^2)} \tag{19}$$

or in the terms of Schwarzschild radius:

$$R_S = \frac{2GM}{c^2} \tag{20}$$

we have

$$\Delta\omega = \frac{3\pi}{1-e^2}\frac{R_S}{a}. \tag{21}$$

3.4. **Maneff potential.** For Maneff's potential, eq. (15) becomes:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} = -u\left(1 - 3\frac{G^2M^2}{H^2c^4}\right) + \frac{GM}{H^2c^2} + \frac{3GM}{c^2}u^2 + \frac{6G^2M^2}{c^4}u^3. \tag{22}$$

In this case, the computation of the *small* perihelion advance gives the following result:

$$\delta(\theta) = \frac{9}{2}\frac{G^2M^2}{C^2c^2}\theta. \tag{23}$$

One observes that the value of the perihelion advance is a sum of the values given by eqs. (9) and (17). The effect of changing the potential in the relativistic framework is a simple superposition of the relativistic effect and the effect of changing the potential.

The prediction of eq. (23) is not in agreement with the observational data and seems not to justify the intricacies which a relativistic Maneff approach implies (such as adjusting Einstein's field equations).

4. Periastron advance of binary pulsar

The obtained reults can also be applied for the binary systems having the two components of comparable masses M_1 and M_2. In this case the product GM

must to be changed in $\mu = G(M_1 + M_2)$. Then, using the Kepler's third law, from the equation (18) we obtain the rate of the periastron advance ([11])

$$\dot{\omega} = 3\left(\frac{2\pi}{P}\right)^{5/3} \frac{G^{2/3}}{c^2(1-e^2)}(M_1 + M_2)^{2/3}. \tag{24}$$

The equation (21) shows that in the binary pulsars, where the semimajor axis a is small, the periastron advance is large. For the binary pulsar PSR 1913+16, taking $M_1 = M_p = 1.4M_\odot$, $M_2 = M_c = 1.4M_\odot$, $P = 27\,907$ sec, the observed rate of periastron advance will be obtained, namely $\dot{\omega} = 4.23$ deg yr^-1.

If we shall use the equation (9), only half of the observed value will be obtained. This means that the Maneff gravitational field explains the periastron advance in the binary pulsar PSR 1913+16 only *qualitatively* but not *quantitatively*, as this was considered in a recent paper ([7]). We observe that, if the equation (7c) from the cited paper will be used (noncollisional orbits) the same expression (9) from our paper will be obtained.

In conclusion, if we take the Maneff gravitational field as an alternative post-Newtonian nonrelativistic law of gravitation, the "perturbative" term in the Maneff potential (2) must to be scaled by the factor 2 as in the expression (10). In this way the Maneff gravitational field will explain the perihelion advance of the planetary orbits as well as the periastron advance for the binary pulsars.

5. Conclusions

Analyzing the Binet-like equations (6), (16), (22) and the perihelion advance formulae (9), (17), (23), we come to the following conclusions:

- The theoretical results of Einstein's relativity are in perfect agreement with the observational evidence. No corrections are *necessary* for this theory. It explains the planetary perihelion advance, as well as the periastron advance of binary pulsars.
- In the classical framework, Maneff's field *can* explain the phenomenon of perihelion advance qualitatively. A scaling of the second term in Maneff's formula would lead to the exact relativistic result for the angular advance

formula. The main strength of Maneff's formalism is, in our opinion, its simplicity that makes relativistic effects such as the perihelion advance amenable to the analysis of Celestial Mechanics.

- Within the relativistic framework, Maneff's potential predicts a result which is in disagreement with the observed data. It seems that there is no need to change Einstein's equations in a manner that would affect Schwartzschild solution.
- One should note, however, that the derivation of the formula for the perihelion advance was carried out within the first-order analysis. This is perfectly justified at the scale of the Solar System as well as for the binary pulsars, where the components are close to the mass points.

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"BABEŞ-BOLYAI" UNIVERSITY OF CLUJ-NAPOCA, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, STR. M. KOGĂLNICEANU 1, 3400 CLUJ-NAPOCA, ROMÂNIA

BOOK REVIEWS

S. Dragomir, L. Ornea, **Conformal Kähler Geometry**, Birkhäuser, Boston-Basel-Berlin, 1998, 327pp.+xi, ISBN 0-1817-4020-7, ISBN 3-7643-4020-7

The research contribution of these two young mathematicians in modern differential geometry are well known. The present book represents a succsessful synthesis of the recent investigations concerning the study of the locally conformal Kähler (shortly l.c.K) manifolds. If (M^{2n}, J, g) is a Hermitian manifold of complex dimension n having its complex structure J and its Hermitian metric g, then g is l.c.K. if it is conformal to some local Kählerian metric in the neighborhood of each point of M^{2n}. A manifold endowed with a l.c.K. metric is called a l.c.K. manifold. The geometry of l.c.K. manifolds has developed intensivily since the 1970s, buth there are early contributions by P. Libermann (going back to 1954.) The recent treatment of the subject was initiated in 11976. The book contains seventeen chapters dealing with the following aspects. In the first eleven chapters there are given the main achievements in the theory of l.c.K. manifolds. The last six chapters present the theory of submanifolds in l.c.K. manifolds, as developed by J.L. Cabrerizo & M.F. Andres, S. Ianuş & K. Matsumoto & L. Ornea, F. Narita a.o. The book also contains two appendices concerning Boothby-Wang fibrations, respectively Riemannian submersions.

A relevant and suggestive bibliography containing 302 references related to the subject is included.

The importance and the actuality of the subject, as well as the very rigurous and didactic presentation of the content, make out of this book a valuable contribution to present mathematics. The book is intended first of all for mathematicians, but it can be interesting also for a wide circle of readers, including mechanicists and physicists.

D. ANDRICA

D. H. H y e r s, G. I s a c and T. M. R a s s i a s, **Topics in Nonlinear Analysis and Applications,** World Scientific, Singapore, 1997, ISBN 981-02-02534-2

In this book the authors consider several aspects of Nonlinear Analysis. It covers subjects such as: Complementarity problems, Metrics on convex cones, Zero-epi operators, Variational principles, Fixed point principles, Maximal element principles.

The authors present the material in an integrated and a self-contained way. Most of the results in this book are from the last thirty years.

I recommend this very interesting book to all those who wish to undertake study and research in the Nonlinear analysis and its applications.

I. A. RUS

K r z y s z t o f J a r o s z, **Function Spaces,** Proceedings of the Third Conference on Function Spaces, May 19-23, 1998, Southern Illinois University at Edwardsville, Editor, Contemporary Mathematics vol. 232, American Mathematical Society 1999

These are the Proceedings of the Third Conference on Function Spaces organized by Professor Krzysztof Jarosz at Southern Illinois University at Edwardsville (SIUE), from May 19 to May 23, 1998. The

Proceedings of the previous two conferences, organized also by Professor Jarosz at the same University in Spring 1990 and in Spring 1994, were published by Marcel Dekker as volumes 136 and 172 of the series Lecture Notes in Pure and Applied Mathematics in 1992 and 1995, respectively.

The Third Conference was attended by over 100 participants (the list of participants included in the volume counts 114 mathematicians) from over than 25 countries. The aim of the Conference was to bring together mathematicians interested in various aspects of function spaces or in related areas, as spaces and algebras of analytic functions (of one or several variables) and operators acting on them, L^p-spaces, spaces of continuous functions, spaces of Banach-valued functions, Banach and C^*-algebras, geometry of Banach spaces. Some lectures were of expository nature, presenting in an accessible way to non-experts in the field known (to the experts) results and open problems, establishing links between various areas of investigation and opening possibilities for future joint work. Some of these lectures are included in the present volume. The papers presenting new results are also written in a manner that make them understandable by a broader audience.

The volume contains 36 papers dealing with topics as: norm attaining operators on $L_1(\mu)$ (M. D. Acosta), separating maps on spaces of continuous functions (J. Arauzo, K. Jarosz), extending linear isometries (S. J. Dilworth), fixed point property in $L_1[0,1]$ (P. N. Dowling), smoothness properties of sequence spaces (R. Gonzalo, J. A. Jaramillo) isometries in Orlicz spaces (B. Randriantonina), Banach spaces isometric to their squares (N. J. Kalton), isometries of non-commutative L^p-spaces (K. Watanabe), Weil-spectrum (P. Aiena), normal functions of several complex variables (J. T. Anderson, J. A. Cima), composition and Toeplitz operators on Hardy spaces (P. Avramidou, F. Jafari, K. Stroethoff), a survey on compact-like operators on H^∞ (M. D. Contreras, S. Diaz-Madrigal), a survey on closed ideals in familiar function algebras (P. Gorkin, R. Mostini), conditions for a linear *functional to be multiplicative* (K. Jarosz), Witt groups on C^*-algebras (C. Badea), the quaternionic Riemann theorem (S. Bernstein), convolution by means of bilinear maps (O. Blasco), Sobolev spaces of holomorphic functions (S. Krantz, M. M. Peloso), a.o.

For the Third Conference on Function Spaces a WEB page containing the abstracts, the schedule and the pictures of the participants, was created by the organizers. It is still available at the address: `http://www.siue.edu/MATH/conference.htm` .

Undoubtely that the present volume, containig contibutions written by eminent specialists all around the world, will attract a large audience, including people intersted in functional analytic methods in analysis or in their applications to other fields as partial differential equations, optimization, variatonal analysis and optimal control.

The volume is printed in excellent typographical conditions by the American Mathematical Society.

S. Cobzaş