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TORSION IN Γ-LATTICES

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Dedicated to Professor Ioan Purdea at his 60^{th} anniversary

Abstract. After general properties of Γ-lattices, a new notion of torsion is given and some of its connections with purity are established.

1. Introduction

Let $(\Gamma, \cdot, 1)$ be a monoid. A lattice L is called **Γ-latice** ([3]) if it is provided with a multiplication $\varphi : \Gamma \times L \to L$ (we shall denote by $\gamma a = \varphi(\gamma, a)$) which satisfies the following axioms

$\Gamma 1 : \gamma a \leq a$

$\Gamma 2 : \gamma(a \vee b) = \gamma a \vee \gamma b$

$\Gamma 3 : (\gamma\gamma')a = \gamma(\gamma' a)$

$\Gamma 4 : 1.a = a$

The source of this notion is the lattice of all the submodules of a given module M over a commutative ring R with identity on which the monoid of the principal ideals of R operates in a natural way: $\varphi(rR, A) = rA$ $(r \in R, A \leq M)$.

Remark 1.1. *This monoid naturally acts also on quotient R-modules.*

Moreover, this monoid has a special element : the zero ideal. In order to get suitable definitions for purity, divisibility and torsion and to recover some of the standard results one must consider a zero element in the monoid Γ. This is called a Γ_0-lattice if it satisfies the axiom

$\Gamma 0$: for each $a \in L$, $0.a = 0$ holds.

We say that Γ **has no zero-divisors** if $\gamma \neq 0, \delta \neq 0$ imply $\gamma \cdot \delta \neq 0$.

A subset $C \subsetneq L$ is called a **system of generators** for L if each element of L is a union of elements from C. A system of generators is called **closed** if $\Gamma \cdot C \subseteq C$.

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As in [2] we use the quotient sublattice notation $b/a = \{c \in L | a \leq c \leq b\}$. An element $c \in L$ is called **cycle** if $c/0$ is a noetherian and distributive sublattice. Clearly, using $\Gamma 1$, for any cycle c and any $\gamma \in \Gamma$, γc is a cycle too.

In a Γ_0-lattice an element $d \in L$ is called **divisible** if $\forall 0 \neq \gamma \in \Gamma : \gamma d = d$.

In a Γ-lattice L an element p is called **pure** (see [1]) if $\gamma p = p \wedge \gamma 1, \forall \gamma \in \Gamma$.

2. Elementary results

In what follows Γ will denote a (non-necessary commutative) monoid. For the proofs of the following simple results see [1].

Lemma 2.1. *In any Γ-lattice, $\gamma.0 = 0, \forall \gamma \in \Gamma$.*

Consequence 2.1. *0 is divisible in each Γ_0-lattice.*

- One can consider, for a fixed $\gamma \in \Gamma$, the upper semi-morphism (according to $\Gamma 2$) $\varphi_\gamma : L \to L, \varphi_\gamma(a) = \gamma a, \forall a \in L$. Hence

Lemma 2.2. *φ_γ is an order-preserving morphism.*

Hence

Lemma 2.3. *(i) $a \leq b \Rightarrow \gamma a \leq \gamma b$. Moreover,*

(ii) $\gamma(a \wedge b) \leq \gamma a \wedge \gamma b$.

A subset B of a Γ-lattice L is called a **Γ-stable** if $\forall \gamma \in \Gamma, \gamma B \subseteq B$.

Clearly (using $\Gamma 1$) the sublattices $a/0$ are Γ-stable and in general not every sublattice $1/a$ (or b/a) is Γ-stable.

Proposition 2.1. *A sublattice b/a is Γ-stable iff a is divisible.*

Lemma 2.4. *Each divisible element is also pure.*

Reconsidering 1.1 we consider on quotient sublattices b/a the following Γ-lattice structure:

$\forall \gamma \in \Gamma, \forall c \in b/a : \gamma * c = (\gamma c) \vee a$

enlarging in this way the notion of **Γ-sublattice** (Γ-stable sublattices).

Obviously, if a is divisible this is the natural Γ-lattice structure on b/a obtained by restriction.

3. Torsion and purity in Γ_0-lattices

In this section we give some properties of a new notion of torsion in a Γ_0-lattice L connected with purity (continuing [1]). In this context special new conditions on Γ_0-lattices seem to be necessary.

Observe that the inequality $\bigvee \{\gamma a | \gamma a \leq b\} \leq b \wedge \gamma 1$ holds for each $b \in L$ and each $\gamma \in \Gamma$.

Clearly, if $b = \gamma x$ for a suitable $x \in L$ this is an equality: indeed, both members are equal to b. Generally, if $b \notin \Gamma \cdot L$ this could be no equality.

In the sequel we shall call a Γ-lattice **dense** if for each $\gamma \in \Gamma$ and each $b \in L$ the equality $\bigvee \{\gamma a | \gamma a \leq b\} = b \wedge \gamma 1$ holds.

We use **bounded** elements in a Γ_0-lattice, i.e. elements $b \in L$ such that there is an $0 \neq \gamma \in \Gamma$ with $\gamma b = 0$. We shall denote by B the set of all the bounded elements of L.

For a Γ_0-lattice L the **torsion part** $t(L)$ is defined as the union of all the bounded (compact) elements. Then L is called a **torsion lattice** if $t(L) = 1$ resp. $t \in L$ is called a **torsion element** if $t = t(t/0)$. The lattice L is called **torsion-free** if $t(L) = 0$ resp. $u \in L$ is called a **torsion-free element** if $t(u/0) = 0$.

A closed system of generators $C \subseteq L$ is called **good** if $C \cap (t(L)/0) \subseteq B$, i.e., the generators $c \in C$ such that $c \leq t(L)$ are bounded (as concrete examples one could consider the compact elements in algebraic H-noetherian lattices or the cycles in cyclic generated lattices).

We first record in a Γ_0-lattice L the following simple properties:

(a) If $a \leq b$ and b is bounded then a is also bounded. In particular, by $\Gamma 1$, if b is bounded, γb is bounded too, for each $\gamma \in \Gamma$.

(b) Each atom is bounded or divisible.

(c) If C is a system of generators for L then any bounded element b is an union of bounded generators $\{c_i\}_{i \in I} \subseteq C$. Moreover, if for $0 \neq \gamma$ we have $\gamma b = 0$ then $\forall i \in I : \gamma c_i = 0$.

Consequently, *if the Γ_0-lattice L has no divisible atoms*

(d) The socle $s(L) \leq t(L)$.

(e) If u is a torsion-free element then $u/0$ has no atoms.

If Γ has no zero-divisors

(f) For each $0 \neq \gamma \in \Gamma$, b is bounded iff γb is bounded.

Indeed, if γb is bounded there is $0 \neq \delta \in \Gamma : \delta(\gamma b) = (\delta\gamma) b = 0$. Γ having no zero-divisors, $\delta\gamma \neq 0$ and so b is bounded. The rest is (a).

Proposition 3.1. *If Γ has no zero-divisors the "radical" property:*

$t(1/t(L)) = t(L)$,

holds in a Γ_0-lattice L with a good system of generators C.

Proof. By definitions: $1/t(L)$ is a torsion-free sublattice $\Leftrightarrow$

$\forall b \in 1/t(L), \exists 0 \neq \gamma \in \Gamma : \gamma * b = t(L) \Rightarrow b = t(L) \Leftrightarrow$

$\exists 0 \neq \gamma \in \Gamma : \gamma b \leq t(L) \leq b \Rightarrow b = t(L)$. The lattice L having a (good) system of generators C, the inequality $b \leq t(L)$ can be verified as follows: $\forall c \in C, c \leq b \Rightarrow c \leq t(L)$.

Indeed, if $c \leq b$, by 2.3 $\gamma c \leq \gamma b \leq t(L)$. C being a good system of generators γc is also a generator and it is bounded. Hence by (f) c is bounded too and $c \leq t(L)$.□

Consequence 3.1. *If Γ has no zero-divisors, L is cycle generated Γ_0-lattice and the cycles in $t(L)/0$ are bounded then L has the "radical" property.*□

Consequence 3.2. *If Γ has no zero-divisors and L is an algebraic and H-noetherian Γ_0-lattice then L has the "radical" property.*□

Another condition we need in the propositions that follows is:

for each $0 \neq \gamma \in \Gamma, \gamma a \leq \gamma b \Rightarrow a \leq b$ for elements in torsion-free Γ_0-lattices (*).

Proposition 3.2. *If for an element p in a dense Γ_0-lattice L the sublattice $1/p$ is torsion-free then p is pure. Conversely, if p is pure in a torsion-free Γ_0-lattice L with (*) then $1/p$ is also torsion-free.*

Proof. Indeed, $1/p$ is torsion-free iff $\exists 0 \neq \gamma \in \Gamma : \gamma u \leq p \leq u \Rightarrow u = p$. Using the density of L we prove the inequality $p \wedge \gamma 1 \leq \gamma p$ as follows: $\gamma a \leq p \wedge \gamma 1 \Rightarrow \gamma(a \vee p) = \gamma a \vee \gamma p \leq p \wedge \gamma 1 \leq p \Rightarrow a \vee p = p \Rightarrow a \leq p \Rightarrow \gamma a \leq \gamma p$.

Conversely, for $0 \neq \gamma \in \Gamma : \gamma u \leq p \leq u$ and $\gamma p = p \wedge \gamma 1$ we have to prove that $u = p$.

First, observe that $\gamma u \leq p, \gamma u \leq \gamma 1 \Rightarrow \gamma u \leq p \wedge \gamma 1 = \gamma p$ and $p \leq u \Rightarrow \gamma p \leq \gamma u$ so that $\gamma u = \gamma p$. One has finally to use (*).□

Proposition 3.3. *If Γ has no zero-divisors, in a dense Γ_0-lattice L with a good system of generators, $t(L)$ is pure.*

Proof. This is an immediate consequence of 3.1 and 3.2.□

Proposition 3.4. *In a dense torsion-free* Γ_0*-lattice* L *with* $(*)$*, an intersection of pure elements is pure.*

Proof. Let $\{p_i\}_{i\in I}$ be a family of pure elements of L and let $\overline{p} = \bigwedge_{i\in I} p_i$. The lattice being dense it suffices to verify that for each $0 \neq \gamma \in \Gamma$: $\gamma a \leq \overline{p} \wedge \gamma 1$ implies $\gamma a \leq \gamma \overline{p}$.

Indeed, $\gamma a \leq \overline{p} \wedge \gamma 1 = (\bigwedge_{i\in I} p_i) \wedge \gamma 1 = \bigwedge_{i\in I}(p_i \wedge \gamma 1) = \bigwedge_{i\in I}(\gamma p_i)$ implies $\gamma a \leq \gamma p_i$ for each $i \in I$. Now the condition $(*)$ implies $a \leq p_i$ for each $i \in I$ and so $a \leq \overline{p}$. Hence $\gamma a \leq \gamma \overline{p}$ by 2.3.□

Final remark. Although with a promising start, Γ-lattices require too much special conditions in order to obtain important results.

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ON THE REMAINDER TERM IN MULTIVARIATE APPROXIMATION

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Dedicated to Professor Ioan Purdea at his 60^{th} anniversary

1. Introduction An efficient procedure to construct multivariate approximation operators is to extend the known results from the univariate case. An algebraic approach of this technique has been developed in [4]. It was shown that any collection of commuting projectors generates a distributive lattice, each of whose elements provide an approximation for a given function. The maximal element of the lattice, which is the "Boolean sum" of the lattice generator operators was identified as "algebrically maximal" approximation operator and the minimal element, which is the "product" of the lattice generators as the "algebrically minimal" approximation operator of the lattice.

Next, in [1], the algebrically maximal and the algebrically minimal operators were characterized by there approximation order: the Boolean sum operator has the maximum approximation order while the product operator has the minimum approximation order among the all elements of the lattice. The proof of these extremally properties is based on the representation of the corresponding remainder operators: the remainder operator corresponding to the Boolean sum of the lattice generators is the product of the remainder operators corresponding to the generator operators and the remainder corresponding to the product of the generators is the Boolean sum of the corresponding remainder operators.

The problem which appears here is to find the remainder operator corresponding to an arbitrary element of the lattice, which is the purpose of this paper.

Also, for the characterization of an interpolation operators will be used the degree of exactness (dex).

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2. Let X be a real linear space and P_1, P_2 projectors defined on X. One denotes by P_1P_2 the product and by $P_1 \oplus P_2$ ($P_1 \oplus P_2 = P_1 + P_2 - P_1P_2$) the Boolean sum of the projectors P_1 and P_2. If $P_1P_2 = P_2P_1$ then P_1 and P_2 are commuting projectors.

Let $P_1, \ldots, P_n$ be commuting projectors defined on X. The algebraic operations of product and Boolean sum yield now projectors.

Let us remind some useful properties of the commuting projectors: if P_1, P_2, P_3 are commuting projectors then:

$$P_1P_2 \text{ and } P_1 \oplus P_2 \text{ are projectors} \tag{1}$$

$$P_1 \oplus P_2 = P_2 \oplus P_1 \tag{2}$$

$$P_1 \oplus (P_2 \oplus P_3) = (P_1 \oplus P_2) \oplus P_3 \tag{3}$$

$$P_1(P_2P_3) = (P_1P_2)P_3 \tag{4}$$

$$P_1(P_2 \oplus P_3) = (P_1P_2) \oplus (P_1P_3) \tag{5}$$

$$P_1 \oplus (P_2P_3) = (P_1 \oplus P_2)(P_1 \oplus P_3) \tag{6}$$

One denoted by $\mathcal{P}$ the set of all projectors generated from the projectors $P_1, \ldots, P_n$ by the operations of product and Boolean sum. $P_1, \ldots, P_n \in \mathcal{P}$, are said to be generator (or primaly) projectors of $\mathcal{P}$. With respect to the order relation " $\leq$ ": $P \leq Q$ iff $PQ = P$, for $P, Q \in \mathcal{P}$, $\mathcal{P}$ is a lattice, i.e. $\inf\{P, Q\} = PQ$ and $\sup\{P, Q\} = P \oplus Q$ for all $P, Q \in \mathcal{P}$. More than that, $\mathcal{P}$ is a distributive lattice (properties (5) and (6)).

3. Mutivariate approximation

Let $D \subset \mathbf{R}^n$ be a rectangular domain, say $D = X_{i=1}^{n}[a_i, b_i]$, and $\mathcal{F}_n$ a set of real functions defined on D.

One considers as generator projectors, the interpolation operators P_i, $P_i : \mathcal{F}_n \to \mathcal{G}_i$, that interpolate a function $f \in \mathcal{F}_n$ with respect to the variable x_i, for $i = 1, \ldots, n$. So, $\mathcal{G}_i$, $i = 1, \ldots, n$, are sets of functions of $n-1$ variables. One reminds that $P_1, \ldots, P_n$ are commuting projectors. Let $\mathcal{P}_n$ be the lattice generated by $P_1, \ldots, P_n$. $S = P_1 \oplus \cdots \oplus P_n$ and $P = P_1 \ldots P_n$ are the maximal respectively the minimal element of $\mathcal{P}_n$. As, P_if can be considered an approximation of f, one denotes by R_if the remainder term, where R_i is the remainder operator: $R_i = I - P_i$, with I the identity operator, for all $i = 1, \ldots, n$.

The arising problem is: *for a given $Q \in \mathcal{P}_n$ which is the corresponding remainder operator, say R_Q.*

It was already mentioned that $R_S = R_1 \dots R_n$ and $R_P = R_1 \oplus \cdots \oplus R_n$. Hence, we have the following decompositions of the identity operator:

$$I = S + R_S \tag{7}$$

and

$$I = P + R_P \tag{8}$$

The proof of these identities are based on the mathematical induction principle. So, for $n = 2$, (7) becomes

$$I = P_1 \oplus P_2 + R_1 R_2 \tag{9}$$

Taking into account that $R_i = I - P_i,\ i = 1, 2$, (9) is easy to verify.

Using the associativity property of the Boolean sum and product operations, the relation (7) follows for any $n > 2$. In the same way can be justified the identity (8).

Each decomposition of the identity operator

$$I = P + R$$

generates an approximation formula

$$f = Pf + Rf.$$

Now, let Q be an arbitrary element of $\mathcal{P}_n$. The problem is to determine the remainder operator R_Q, i.e.

$$Q + R_Q = I.$$

Theorem. *If $Q \in \mathcal{P}_n$ is of the form*

$$Q = (P_1 \dots P_{i_1}) \oplus (P_{i_1+1} \dots P_{i_2}) \oplus \cdots \oplus (P_{i_{n-1}+1} \dots P_{i_n})$$

then

$$R_Q = (R_1 \oplus \cdots \oplus R_{i_1})(R_{i_1+1} \oplus \cdots \oplus R_{i_2}) \dots (R_{i_{n-1}+1} \oplus \cdots \oplus R_{i_n}). \tag{10}$$

Proof. From (7) it follows that

$$R_Q = R_{P_1 \dots P_{i_1}} R_{P_{i_1+1} \dots P_{i_2}} \dots R_{P_{i_{n-1}+1} \dots P_{i_n}}.$$

But, from (8), we have

$$R_{P_1 \ldots P_{i_1}} = R_1 \oplus \cdots \oplus R_{i_1}$$

$$R_{P_{i_{n-1}+1} \ldots P_{i_n}} = R_{i_{n-1}+1} \oplus \cdots \oplus R_{i_n}$$

and (10) is proved.

It follows the rule: *the remainder operator R_Q corresponding to the interpolating operator Q is obtained by changing in Q each generator operator P_i by the corresponding remainder operator R_i and the product operation by Boolean sum and the Boolean sum by product.*

Some simple examples are:

(11) $$Q_1 = P_1(P_2 \oplus P_3), \quad R_{Q_1} = R_1 \oplus (R_2 R_3)$$

(12) $$Q_2 = P_1 \oplus (P_2 P_3), \quad R_{Q_2} = R_1(R_2 \oplus R_3)$$

(13) $$Q_3 = (P_1 P_2) \oplus (P_3 P_4), \quad R_{Q_3} = (R_1 \oplus R_2)(R_3 \oplus R_4)$$

(14) $$Q_4 = (P_1 \oplus P_2)(P_3 \oplus P_4), \quad R_{Q_4} = (R_1 R_2) \oplus (R_3 R_4)$$

Homogeneous approximation formulas

Let $f \in \mathcal{F}_n$ be given and $Q \in \mathcal{P}_n$. The decomposition of the identity operator $I = Q + R_Q$ generates the approximation formula for the function f:

$$f = Qf + R_Q f,$$

with $R_Q f$, the remainder term. For example, the two extremal elements of $\mathcal{P}_n$, S and P generate the so called algebrical maximal respectively algebrical minimal formulas, i.e.

$$f = Sf + R_S f$$

and

$$f = Pf + R_P f.$$

Definition 1. Let $Q \in \mathcal{P}_n$ be given. The number $r \in \mathbb{N}$, with the property that $Qf = f$ for all $f \in \mathbf{P}_r^n$ (the set of all polynomials in n variables of the total degree at most r) and there exists a polynomial $g \in \mathbf{P}_{r+1}^n$ such that $Qg \neq g$, is called the degree of exactness of the operator Q, i.e. $\mathrm{dex}(Q) = r$.

Remark 1. The conditions $Qf = f$ for all $f \in \mathbf{P}_r^n$ and there exists $g \in \mathbf{P}_{r+1}^n$ such that $Qg \neq g$ are equivalente with $Qe_{ij} = e_{ij}$ for all $i, j \in \mathbb{N}$, $i + j \leq r$ and there exists $p, q \in \mathbb{N}$ with $p + q = r + 1$ such that $Qe_{pq} \neq e_{pq}$, where $e_{ij}(x, y) = x^i y^j$.

It is known that the approximation order of the operators S and P are given by:

$$\operatorname{ord}(S) = \operatorname{ord}(P_1) + \cdots + \operatorname{ord}(P_n)$$

respectively

$$\operatorname{ord}(P) = \min\{\operatorname{ord}(P_1), \ldots, \operatorname{ord}(P_n)\}.$$

We also have:

Theorem 2. $\operatorname{dex}(S) = \operatorname{dex}(P_1) + \cdots + \operatorname{dex}(P_n)$, $\operatorname{dex}(P) = \min\{\operatorname{dex}(P_1), \ldots, \operatorname{dex}(P_n)\}$ *and* $\operatorname{dex}(P) \leq \operatorname{dex}(Q) \leq \operatorname{dex}(S)$ *for all* $Q \in \mathcal{P}_n$.

Following the Remark 1, the proof is reduced to a direct verification.

So, the Boolean sum operator S has the maximum degree of exactness while the product P has the minimum degree of exactness, among all elements of $\mathcal{P}_n$.

But, Sf approximates the function f in terms of functions of $n-1, \ldots, 1$ variables, while Pf is a scalar approximation of f.

Remark 2. $P \in \mathcal{P}_n$ is the only element of $\mathcal{P}_n$ with the property that Pf is a scalar approximation of f. For any $Q \in \mathcal{P}_n$, $Q \neq P$, Qf has at least one free variable of f. For example,

$$Q_1 f := (P_1P_2 + P_1P_3 - P_1P_2P_3)f$$

with Q_1 from (11), contains two free variables: x_3 in the term P_1P_2f and x_2 in P_1P_3f.

Starting with an approximation formula

$$f = Qf + R_Q f, \quad Q \in \mathcal{P}_n,\ Q \neq P,$$

in order to obtain a scalar approximation formula, we can use next approximation levels. If $Q_1 = P_1^1(P_2^1 \oplus P_3^1)$ (example from (11)), where the upper index marks the approximation level number, then the corresponding approximation formula is generated by the identity

$$I = (P_1^1P_2^1 + P_1^1P_3^1 - P_1^1P_2^1P_3^1) + (R_1^1 + R_2^1R_3^1 - R_1^1R_2^1R_3^1).$$

If, in a second level of approximation, it is used the operators P_3^2 and P_2^2 with $R_3^2 = I - P_3^2$ and $R_2^2 = I - P_2^2$, one obtains

$$I = (P_1^1P_2^1P_3^2 + P_1^1P_2^2P_3^1 - P_1^1P_2^1P_3^1) + (R_1^1 + R_2^1R_3^1 - R_1^1R_2^1R_3^1 + P_1^1P_2^1R_3^2 + P_1^1P_3^1R_2^2),$$

which generates a scalar approximation formula: $(P_1^1P_2^1P_3^2 + P_1^1P_2^2P_3^1 - P_1^1P_2^1P_3^1)f$ is a scalar approximation of f and $(R_1^1 + R_2^1R_3^1 - R_1^1R_2^1R_3^1 + P_1^1P_2^1R_3^2 + P_1^1P_3^1R_2^2)f$ is the corresponding remainder term.

It is obviously to see that the remainder operator R_Q for $Q \neq S$ and $Q \neq P_i$, $i = 1,\ldots,n$, is the sum of many terms. The approximation order of the interpolation operator Q must be taken with respect to each term of R_Q. In the above example the number of the terms is five: R_1^1, $R_2^1R_3^1$, $R_1^1R_2^1R_3^1$, $P_1^1P_2^1P_3^2$ and $P_1^1P_3^1R_2^2$. The degree of exactness corresponding to these terms are $\text{dex}(P_1^1)$, $\text{dex}(P_2^1) + \text{dex}(P_3^1)$, $\text{dex}(P_1^1) + \text{dex}(P_2^1) + \text{dex}(P_3^1)$, $\text{dex}(P_3^2)$ respectively $\text{dex}(P_2^2)$.

Let $Q \in \mathcal{P}_n$, $Q \neq S$ and $Q \neq P_i$, $i = 1,\ldots,n$ and R_Q the corresponding remainder operator.

Definition 2. If the degree of exactness corresponding to each term of the remainder operator R_Q is the same then Q is called a homogeneous approximation operator and

$$f = Qf + R_Qf$$

a homogeneous approximation formula.

In the considered example

$$Q = P_1^1P_2^1P_3^2 + P_1^1P_2^2P_3^1 - P_1^1P_2^1P_2^1$$

and

$$R_Q = R_1^1 + R_2^1R_3^1 - R_1^1R_2^1R_3^1 + P_1^1P_2^1R_3^2 + P_1^1P_3^1 + R_2^2,$$

Q is a homogeneous approximation operator if

$$\text{dex}(P_1^1) = \text{dex}(P_2^1) + \text{dex}(P_3^1) = \text{dex}(P_1^1) + \text{dex}(P_2^1) + \text{dex}(P_3^1) = \text{dex}(P_2^2) = \text{dex}(P_3^2)$$

that has the only solution $\text{dex}(P_1^1) = \text{dex}(P_2^1) = \text{dex}(P_3^1) = \text{dex}(P_2^2) = \text{dex}(P_3^2) = 0$. But the remainder operator R_Q can be changed in a convenable way. One of them is

$$\tilde{R}_Q = R_1^1 + R_2^1R_3^1(P_1^1 + R_1^1) - R_1^1R_2^1R_3^1 + P_1^1P_2^1R_3^2 + P_1^1P_3^1R_2^2 \quad (P_1^1 + R_1^1 = I)$$

or

$$\tilde{R}_Q = R_1^1 + P_1^1R_2^1R_3^1 + P_1^1P_2^1P_3^2 + P_1^1P_3^1R_2^2.$$

It follows that the formula

$$f = Qf + \tilde{R}_Qf \tag{15}$$

with

$$Q = P_1^1 P_2^1 P_3^2 + P_1^1 P_2^2 P_3^1 - P_1^1 P_2^1 P_3^1$$

is a scalar homogeneous interpolation formula if

$$\text{dex}(P_1^1) = \text{dex}(P_2^1) + \text{dex}(P_3^1) = \text{dex}(P_3^2) = \text{dex}(P_2^2). \tag{16}$$

For example, such a formula is obtained for: $P_1^1 := H^x_{2m+1}$, $P_2^1 := H^y_{m+1}$, $P_3^1 := L^z_m$, $P_2^2 := H^y_{2m+1}$ and $P_3^2 := H^z_{2m+1}$, where L^v_n and H^v_n are Lagrange respectively Hermite interpolation operator of the degree n, which interpolate a function f with respect to the variable v. The degree of exactness of the obtained operator is $2m + 1$.

So, starting with the formula given by the operators Q_1 and R_{Q_1} from (11), we can obtain scalar interpolation operators (scalar interpolation formulas) of any degree of exactness.

A second example, we are looking for, is given by the operator of example (12). Initial formula is

$$f = (P_1 + P_2P_3 - P_1P_2P_3)f + (R_1R_2 + R_1R_3 - R_1R_2R_3)f,$$

that is not a homogeneous one. In order to get a homogeneous formula the remainder operator can be change as follows:

$$R_1R_2+R_1R_3-R_1R_2R_3 = R_1R_2(P_3+R_3)+R_1R_3-R_1R_2R_3 = P_3R_1R_2+R_1R_3 \quad (P_3+R_3 = I).$$

This way, one obtains

$$f = (P_1 + P_2P_3 - P_1P_2P_3)f + (P_3R_1R_2 + R_1R_3)f. \tag{17}$$

Obviously, if $\text{dex}(R_2) = \text{dex}(R_3)$ then (17) is a homogeneous approximation formula. But, it is not a scalar formula, as can be seen, $P_1 + P_2P_3 - P_1P_2P_3$ is not a scalar approximation operator.

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A GENERALIZATION OF SOME OF ORE'S THEOREMS

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Dedicated to Professor Ioan Purdea at his 60^{th} anniversary

Abstract. The paper completes the results from [2] with new properties of finite π-solvable primitive groups, where π is an arbitrary set of primes. Thus we obtain a generalization for π-solvable groups of some of ORE's theorems from [5] given for solvable groups and being of special interest in the formation theory.

1. Preliminaries

All groups considered in the paper are finite. We shall denote by π an arbitrary set of primes and by π' the complement to π in the set of all primes.

Definition 1.1. a) Let G be a group, M and N two norma; subgroups of G such that $N \subseteq M$. The factor M/N is called a ***chief factor*** of G if M/N is a minimal normal subgroup of G/N.

b) A group G is said to be π-*solvable* if every chief factor of G is either a solvable π-group or a π'-group. Particularly, for π the set of all primes we obtain the notion of solvable group.

Definition 1.2. a) Let G be a group and W a subgroup of G. We define

$$core_G W = \cap\{W^g / g \in G\},$$

where $W^g = g^{-1}Wg$.

b) W is a stabilizer of G if W is a maximal subgroup of G and $core_G W = 1$.

c) A group G is *primitive* if there is a stabilizer W of G.

The following results will be used to prove the main theorems of this paper.

Theorem 1.3. ([1]) *Any solvable minimal normal subgroup of a finite group is abelian.*

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Theorem 1.4 (Schur-Zassenhaus) ([3], p.16) *Let G be a finite group and H a normal abelian subgroup of G such that $|G : H|$ and $|H|$ are relatively prime. Then:*

(a) H has a complement K in G, i.e. $HK = G$ and $H \cap K = 1$;

(b) all complements of H in G are conjugate under H.

Theorem 1.5. ([4], p.18) *If G is a group and M, M_1 are two normal subgroups of G such that $M \cap M_1 = 1$, then M and M_1 commute elementwise, i.e. $mm_1 = m_1 m$ for any $m \in M$ and $m_1 \in M_1$.*

Theorem 1.6. (Dedekind identity) ([4], p.8) *If G is a group and A, B, C are subgroups of G such that $A \subseteq C \subseteq AB$, then*

$$C = (AB) \cap C = A(B \cap C).$$

Theorem 1.7. ([2]) *Let G be a primitive group and W a stabilizer of G. Then:*

(i) for any normal subgroup $K \neq 1$ of G we have $KW = G$;

(ii) for any minimal normal subgroup M of G we have $MW = G$;

(iii) there is not a normal subgroup $K \neq 1$ of G such that $K \subseteq W$.

2. Frattini argument for π-solvable groups

In [4], p.35, 7.8. the following well-known theorem called the "Frattini argument" is given: Let G be a group, N a normal subgroup of G and P a Sylow p-subgroup of N. Then $G = NN_G(P)$.

Our later considerations need a new form of the Frattini argument which we give below.

We remind that a subgroup H of a group G is called a *Hall π-subgroup* of G if $|H|$ is a π-number and $|G : H|$ is a π'-number. We also remind the Hall-Čunihin theorem:

Theorem 2.1. (Hall-Čunihin, [4], p.660) *If G is a π-solvable group, then:*

(a) G has Hall π-subgroups and Hall π'-subgroups;

(b) all Hall π-subgroups of G are conjugate in G; all Hall π'-subgroups of G are conjugate in G.

Theorem 2.2. (The Frattini argument for π-solvable groups) *Let G be a π-solvable group, N a normal subgroup of G and P a Hall π-subgroup (or a Hall π'-subgroup) of N. Then $G = NN_G(P)$.*

Proof. Clearly $NN_G(P) \subseteq G$. Let now $g \in G$. Then $P^g \subseteq N^g = N$, hence P^g is also a Hall π-subgroup (or a Hall $\pi\prime$-subgroup) of N. But N, as a subgroup of the π-solvable group G, is a π-solvable group too. Thus, applying 2.1, P and P^g are conjugate in N. It follows that $P^g = P^n$, where $n \in \mathrm{N}$. This implies $gn^{-1} \in N_G(P)$. Then

$$g = (gn^{-1})n \in N_G(P)N = NN_G(P).$$

This proves that $G \subseteq NN_G(P)$, hence $G = NN_G(P)$. □

3. A generalization of some of ORE's theorems

Given in [5] for solvable groups, the so-called ORE's theorems are of special interest in the formation theory. Here we establish a generalization for π-solvable groups of some of ORE's theorems, where π is an arbitrary set of primes. Particularly, for π the set of all primes, we obtain ORE's theorems.

In [2] we proved the following results similar to some of ORE's:

Theorem 3.1. *Let G be a primitive π-solvable group. If G has a minimal normal subgroup which is a solvable π-group, then G has one and only one minimal normal subgroup.*

Corollary 3.2. *If G is a primitive π-solvable group, then G has at most one minimal normal subgroup which is a solvable π-group.*

Corollary 3.3. *If a primitive π-solvable group G has a minimal normal subgroup which is a solvable π-group, then G has no minimal normal subgroups which are $\pi\prime$-groups.*

Theorem 3.4. *If G is a primitive π-solvable group and N is a minimal normal subgroup of G which is a solvable π-group, then $C_G(N) = N$.*

The first result of this paper examines the converse of 3.4:

Theorem 3.5. *Let G be a π-solvable group such that:*

(i) there is a minimal subgroup M of G which is a solvable π-group and $C_G(M) = M$;

(ii) there is a minimal normal subgroup L/M of G/M such that L/M is a $\pi\prime$-group. Then G is primitive.

Proof. Suppose $M = G$. Then $G/M = 1$, hence $L/M = 1$ giving a contradiction. Thus $M \neq G$. Further, by 1.3 M is abelian.

By (ii) $|L/M|$ is a $\pi\prime$-number and by (I) $|M|$ is a π-number. It follows that $(|L/M|, |M|) = 1$. Applying now theorem 1.4, we conclude that M has a complement L_0 in L, i.e. $ML_0 = L$ and $M \cap L_0 = 1$.

Put $W = N_G(L_0)$. We shall prove that W is a stabilizer of G, i.e. W is a maximal subgroup of G and $core_G W = 1$.

Indeed, $W \neq G$, for otherwise $N_G(L_0) = G$ and hence $L_0 \lhd G$. So M and L_0 are two normal subgroups of G such that $M \cap L_0 = 1$. By 1.5 M and L_0 commute elementwise. Hence $L_0 \subseteq C_G(M) = M$. Thus $L = ML_0 = M$ and $L/M = 1$ contradicting (ii).

We note that $MW = G$ and $M \cap W = 1$. Indeed, applying 2.2 to the π-solvable group G, $L \lhd G$ and L_0 a Hall $\pi\prime$-subgroup of L (since $L_0 \simeq L_0/1 = L_0/M \cap L_0 \simeq ML_0/M = L/M$ is a $\pi\prime$-group and $|L : L_0| = |ML_0 : L_0| = |M : M \cap L_0| = |M|$ is a π-number), we obtain:

$$G = LN_G(L_0) = ML_0N_G(L_0) = MN_G(L_0) = MW.$$

To prove that $M \cap W = 1$, let us first show that $M \cap W \lhd G$. Let $g \in G = MW$, $g = m_1 w$, with $m_1 \in M$, $w \in W$ and let $m \in M \cap W$. Then

$$g^{-1}mg = (m_1 w)^{-1} m (m_1 w) = v^{-1}(m_1^{-1} m m_1) w,$$

where $m_1^{-1} m m_1 \in M \cap W$ since $M \cap W$ is normal in the abelian group M, and

$$w^{-1}(m_1^{-1} m m_1) w \in M \cap W$$

since $M \cap W$ is normal in W. Hence $g^{-1}mg \in M \cap W$. Now from $M \cap W \lhd G$, $M \cap W \subseteq M$ and M minimal normal subgroup of G it follows that $M \cap W = 1$ or $M \cap W = M$. The last condition is impossible because it implies that $M \subseteq W$ and hence the contradiction $G = MW = W$. So $M \cap W = 1$.

To prove that W is a maximal subgroup of G, we remind that $W \neq G$ and let us show that $W \leq W^* < G$ imply $W = W^*$. Suppose that $W < W^*$. Let $w^* \in W^* \setminus W \subset G = MW$. It follows that $w^* = mw$, with $m \in M$ and $w \in W$. Hence $m = w^* w^{-1} \in M \cap W^*$. But $G = MW \subseteq MW^* \subseteq G$ imply $G = MW^*$. Hence $M \cap W^* = 1$ (proof like the above $M \cap W = 1$). Thus $m = 1$ and $w^* = w \in W$, a contradiction. Then $W = W^*$.

Finally, we prove that $core_G W = 1$. Since $M \cap core_G W \lhd G$, $M \cap core_G W \subseteq M$, $M \cap core_G W \neq M$ (for otherwise $M \subseteq core_G W$ and so the contradiction $G = MW = W$)

and M being a minimal normal subgroup of G we have $M \cap core_G W = 1$. By 1.5 M and $core_G W$ commute elementwise. It follows that $core_G W \subset C_G(M) = M$ which implies $core_G W = M \cap core_G W = 1$. □

The following two theorems generalize some of ORE's theorems.

Theorem 3.6. *If G is a π-solvable group satisfying (i) and (ii) from 3.5, then ant two stabilizers W_1 and W_2 of G are conjugate in G.*

Proof. By 3.5 G is primitive. Like in the proof of theorem 3.5 we note that $M \neq G$ and M is abelian. By 3.1 M is the only minimal normal subgroup of G.

Let $W = N_G(L_0)$ be the stabilizer of G given in the proof of theorem 3.5. Hence $ML_0 = L$ and $M \cap L_0 = 1$. We also know that $MW = G$ and $M \cap W = 1$.

We shall prove that W and W_1 are conjugate in G, and that W and W_2 are conjugate in G. It follows that W_1 and W_2 are conjugate in G. It is enough to prove for W and W_1, the proof for W and W_2 being similar.

Put $L_1 = W_1 \cap L$. Let us show that $L_0 = W \cap L$. First we note that $L_0 \subseteq ML_0 = L$, $L_0 \subseteq N_G(L_0) = W$ hence $L_0 \subseteq W \cap L$. Conversely, if $x \in W \cap L = N_G(L_0) \cap L$ then $x \in N_G(L_0)$ and $x \in L = ML_0 = L_0M$ which imply $L_0^x = L_0$, where $x = l_0 m$, $l_0 \in L_0$, $m \hat{\imath} M$. So $(L_0^1)^m = L_0$ which means that $m \in N_G(L_0) = W$. Then $m \in M \cap W = 1$ hence $m = 1$ and $x = l_0 \in L_0$. This proves that $W \cap L \subseteq L_0$.

We know that L_0 is a complement of M in L. L_1 is also a complement of M in L. Indeed, $ML_1 = M(W_1 \cap L)$ and by 1.6 $M(W_1 \cap L) = (MW_1) \cap L$. So $ML_1 = (MW_1) \cap L$. But $MW_1 = G$ for otherwise we have $W_1 \subseteq MW_1 \subset G$ which implies $W_1 = MW_1$ since W_1 is maxim in G and so $M \subseteq W_1$, in contradiction with 1.7.(iii). Thus $ML_1 = G \cap L = L$. Further, $M \cap L_1 = 1$ since

$$M \cap L_1 = M \cap (W_1 \cap L) = (M \cap W_1) \cap L$$

and $M \cap W_1 = 1$ as we shall see below. First note that $M \cap W_1 \lhd G$. Indeed, if $x \in G = MW_1$, $x = m_1 w_1$ with $m_1 \in M$, $w_1 \in W_1$, and $m \in M \cap W_1$ then, using that M is abelian and that $M \cap W_1 \lhd W_1$, we have:

$$x^{-1}mx = (m_1w_1)^{-1}m(m_1w_1) = w_1^{-1}m_1^{-1}mm_1w_1 = w_1mm_1^{-1}m_1w_1 = w_1^{-1}mw_1 \in M \cap W_1.$$

Now from $M \cap W_1 \lhd G$, $M \cap W_1 \subseteq M$, $M \cap W_1 \neq M$ (for otherwise $M \subseteq W_1$, contradicting 1.7.(iii)) and M minimal norma subgroup of G we obtain $M \cap W_1 = 1$.

By 1.4.(b) L_0 and L_1 are conjugate under M, i.e. $L_0 = L_1^m$ for some $m \in M$. Further, $L_0 \subseteq W \cap W_1^m$ since $L_0 = N_G(L_0) \cap L_0 = W \cap L_0 \subseteq W$ and $L_1 0 = L_1^m = (W_1 \cap L)^m \subseteq W_1^m$. Moreover, from $L_0 \lhd N_G(L_0) = W$ and $L_0 = L_1^m = (W_1 \cap L)^m \lhd W_1^m$ it follows $L_0 \lhd WW_1^m$.

We shall prove that $W = W_1^m$, which means that W and W_1 are conjugate in G. Let us suppose that $W \neq W_1^m$. From $W \leq WW_1^m \leq G$ and $W \neq WW_1^m$ ($W = WW_1^m$ is impossible because it implies $W_1^m \subseteq W$ hence $W_1 \subseteq W^k \subset G$ and $W_1 = W^k$, where $k = m^{-1}$, since W_1 is maximal in G; but this leads to the contradiction $W = W_1^m$) since W is maximal in G it can be inferred that $WW_1^m = G$. Thus $L_0 \lhd WW_1^m = G$ so that $W = N_G(L_0) = G$, a contradiction. It follows that $W = W_1^m$. □

Theorem 3.7. *If G is a primitive π-solvable group, $V < G$ such that there is a minimal normal subgroup M of G which is a solvable π-group and $MV = G$, then V is a stabilizer of G.*

Proof. $M \cap V$ is a normal subgroup of G. Indeed, let $g \in G = MV = VM$, $g = vm$ for some $v \in V$, $m \in M$ and let $x \in M \cap V$. Since $M \cap V \lhd V$ and since by 1.3 M is abelian we have:

$$g^{-1}xg = (vm)^{-1}x(vm) = m^{-1}(v^{-1}xv)m = m^{-1}m(v^{-1}xv) = (v^{-1}xv \in M \cap V.$$

Now $M \cap V = 1$ since M is a minimal normal subgroup of G and since $M \cap V \lhd G$, $M \cap V \subseteq M$, $M \cap V \neq M$ (supposing $M \cap V = M$ it follows $M \subseteq V$ and so $G = MV = V$, in contradiction with $V < G$).

Let us verify that V is a stabilizer of G.

First, V is a maximal subgroup of G. Indeed, $V \neq G$ and we shall prove that $V \leq V^* < G$ imply $V = V^*$. Suppose $V < V^*$ and let $v^* \in V^* \setminus V \subset G = MV$. Then $v^* = mv$ for some $m\hat{\imath}M$, $v \in V$, Hence $m = v^*v^{-1} \in M \cap V^*$. We prove that $M \cap V^* = 1$. From $G = MV \subseteq MV^* \subseteq G$ it follows $MV^* = G$. Hence $M \cap V^* \lhd G$ (as the above proof for $M \cap V \lhd G$). Since M is a minimal normal subgroup of G and since $M \cap V^* \lhd G$, $M \cap V^* \subseteq M$, $M \cap V^* \neq M$ (supposing $M \cap V^* = M$ we have $M \subseteq V^*$, hence $G = MV^* = V^*$, a contradiction) it follows $M \cap V^* = 1$. Thus $m = 1$ and so $v^* = v \in V$, a contradiction.

Finally, $core_G V = 1$. Indeed, suppose $core_G V \neq 1$. By 3.1 M is the only minimal normal subgroup of G. Thus since $core_G V \lhd G$ we have $M \subseteq core_G V$. But $core_G V \subseteq V$ and so $M \subseteq V$. Then $G = MV = V$, a contradiction. □

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ON A CLASS OF MODULES WHOSE NON-ZERO ENDOMORPHISMS ARE MONOMORPHISMS

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Dedicated to Professor Ioan Purdea at his 60^{th} anniversary

Abstract. In this paper are established some results concerning a class of modules, denoted by $\mathcal{M}$, consisting of all non-zero R-modules with the property that every non-zero endomorphism of A is a monomorphism. If $A \in \mathcal{M}$, then A is indecomposable, $End_R(A)$ is a domain and $Ann_R a = Ann_R A$ for every $0 \neq a \in A$. If R is commutative and $A \in \mathcal{M}$, it is shown that $Ann_R A$ is a prime ideal of R, A is a torsion-free $R/Ann_R A$-module and if A is uniform then A is isomorphic to a submodule of $Ann_{E(R/Ann_R A)}(Ann_R A)$.

1. Introduction

In this paper we denote by R an associative ring with non-zero identity and all R-modules are left unital R-modules. The ring R will be considered as a left module over itself. By an homomorphism we understand an R-homomorphism.

Let A be an R-module. Then we denote by $E(A)$ an injective envelope of A and by $End_R(A)$ the ring of endomorphisms of A. If $0 \neq B \subseteq A$ and $0 \neq I \subseteq R$, we denote $Ann_R B = \{r \in R \mid rb = 0, \forall b \in B\}$ and $Ann_A I = \{a \in A \mid ra = 0, \forall r \in I\}$. If $0 \neq a \in A$, $Ann_R\{a\}$ is denoted by $Ann_R a$. An R-module A is said to be faithful if $Ann_R A = 0$.

A submodule B of an R-module A is said to be essential in A if $B \cap Ra \neq 0$ for every $0 \neq a \in A$ ([2], Chapter 1, Definition 2.12.1). By $B \leq A$ we shall denote that B is a submodule of the R-module A and if A is an essential extension of B, this will be denoted by $B \trianglelefteq A$. A non-zero R-module A is said to be uniform in case each of its non-zero submodules is essential in A ([1], p.294).

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Let R be a domain and let A be an R-module. Then A is called divisible if $rA = A$ for every $0 \neq r \in R$ and A is called torsion-free if $ra \neq 0$ for every $0 \neq r \in R$ and $0 \neq a \in A$ ([4], p.32 and p.34).

An R-module A is said to be quasi-injective if for every $B \leq A$ each homomorphism $f : B \to A$ extends to an endomorphism of A ([3], p.333).

Throughout this paper we denote by $\mathcal{M}$ a class of non-zero R-modules which has the following property: a non-zero R-module A belongs to $\mathcal{M}$ if and only if every non-zero endomorphism $f \in End_R(A)$ is a monomorphism.

Remarks. *a)* For example, every simple R-module is contained in the class $\mathcal{M}$.

b) If A and B are two R-modules such that $A \in \mathcal{M}$ and $B \cong A$, then $B \in \mathcal{M}$.

2. Main results

Theorem 1. *Let $A \in \mathcal{M}$. Then:*

(i) A is indecomposable ;

(ii) $End_R(A)$ is a domain ;

(iii) A is a left torsion-free $End_R(A)$-module.

Proof. *(i)* Suppose that A is not indecomposable. Then there exist non-zero R-modules B and C such that $A = B \oplus C$. Define the homomorphisms $f : A \to B$ by $f(b, c) = b$ and $g : B \to A$ by $g(b) = (b, 0)$ for every $b \in B$ and $c \in C$. It follows that $0 \neq gf \in End_R(A)$ and gf is not a monomorphism, hence $A \notin \mathcal{M}$, which represents a contradiction.

(ii). Let $f, g \in End_R(A)$ non-zero endomorphisms. Then f and g are monomorphisms. Suppose that $fg = 0$. Then $f(g(a)) = 0$ for every $a \in A$. Since f is a monomorphism, we have $g(a) = 0$ for every $a \in A$, i.e. $g = 0$. This provides a contradiction.

(iii). It is well-known that A is an $End_R(A)$-module if we define $fa = f(a)$ for every $a \in A$ and $f \in End_R(A)$. Let $0 \neq f \in End_R(A)$ and $0 \neq a \in A$. Then $fa = f(a) \neq 0$, because f is a monomorphism. Hence A is a left torsion-free $End_R(A)$-module.

Lemma 2. *Let $A \in \mathcal{M}$ be a quasi-injective R-module and let $0 \neq B \leq A$. Then $B \in \mathcal{M}$.*

Proof. Denote by $i : B \to A$ the inclusion monomorphism and let $0 \neq f \in End_R(B)$. Since A is quasi-injective, there exists $h \in End_R(A)$ such that $hi = if$. It follows that $h \neq 0$ and thus h is a monomorphism. Therefore f is a monomorphism. Hence $B \in \mathcal{M}$.

In the sequel, we shall suppose that the ring R is commutative.

Theorem 3. *Let $A \in \mathcal{M}$. Then:*

(i) $Ann_R a = Ann_R A$ for every non-zero element $a \in A$;

(ii) $Ann_R A$ is a prime ideal of R ;

(iii) A is a torsion-free $R/Ann_R A$-module ;

(iv) If A is uniform, then A is isomorphic to a submodule of the module $Ann_{E(R/Ann_R A)}(Ann_R A)$.

Proof. *(i)* Let $r \in R$ such that $r \notin Ann_R A$ and let $0 \neq a \in A$. Then exists $b \in A$ such that $rb \neq 0$. We define the endomorphism $g : A \to A$ by $g(x) = rx$ for every $x \in A$. Since $g(b) = rb \neq 0$, it follows that g is a monomorphism. Therefore $g(a) = ra \neq 0$, i.e. $r \notin Ann_R A$. Hence $Ann_R a \subseteq Ann_R A$. Obviously we have $Ann_R A \subseteq Ann_R a$. Therefore $Ann_R A = Ann_r a$.

(ii). Let $r, s \in R$ such that $rs \in Ann_R A$ and let $0 \neq a \in A$. Then we have $Ann_R A = Ann_R a$. Suppose that $s \notin Ann_R A$. It follows that $sa \neq 0$. But $rs \in Ann_R A$, hence $rsa = 0$. Therefore $r \in Ann_R(sa) = Ann_R A$. Hence $Ann_R A$ is a prime ideal of R.

(iii). Since $Ann_R A$ is a prime ideal of R, $R/Ann_R A$ is an integral domain. Denote $\bar{r} = r + Ann_R A$ for every $r \in R$. Then A becomes an $R/Ann_R A$-module if we define $\bar{r}a = ra$ for every $r \in R$ and $a \in A$. Let $0 \neq a \in A$ and $0 \neq \bar{r} \in R/Ann_R A$. Suppose that $\bar{r}a = 0$. Then $ra = 0$, hence $r \in Ann_R a = Ann_R A$, i.e. $\bar{r} = 0$. This provides a contradiction. Therefore $\bar{r}a \neq 0$. Thus A is a torsion-free $R/Ann_R A$-module.

(iv). We have $A \trianglelefteq E(A)$. Denote $p = Ann_R A = Ann_R a$ for every $a \in A$. Then $Ra \cong R/p$ for every $a \in A$. Since A is uniform, it follows that $E(A) \cong E(Ra) \cong$ $\cong E(R/p)$ ([4], Chapter 2, Proposition 2.28). Hence A is isomorphic to a submodule of $Ann_{E(R/Ann_R A)}(Ann_R A)$.

Corollary 4. *Let $A \in \mathcal{M}$ be a faithful R-module. Then:*

(i) R is an integral domain ;

(ii) A is a torsion-free R-module ;

(iii) If A is uniform, then A is isomorphic to a submodule of $E(R)$.

Proof. By Theorem 3, $Ann_R A = 0$ is a prime ideal of R, hence R is an integral domain.

Theorem 5. *Let A be a non-zero R-module. Then the following statements are equivalent:*

(i) A is uniform and $A \in \mathcal{M}$;

(ii) $A \cong B$, where $0 \neq B \trianglelefteq Ann_{E(R/p)}p$ for a prime ideal p of R.

Proof. *(i)$\Longrightarrow$(ii).* Assume (i). Let $p = Ann_R A$, which is a prime ideal of R. Now the result follows by Theorem 3.

(ii) $\Longrightarrow$ (i). Assume (ii). For every $0 \neq a \in E(R/p)$ we have $Ann_R a \subseteq p$ ([4], Lemma 2.31). Hence $Ann_R a = p$ for every $0 \neq a \in Ann_{E(R/p)}p$. Therefore we have $Ann_R B = Ann_R a = p$ for every $0 \neq a \in B$. Since $E(R/p)$ is an indecomposable injective R-module, it follows that B is uniform ([4], Chapter 2, Proposition 2.28). Let $0 \neq f \in End_R(B)$. Then there exists $0 \neq a \in B$ such that $f(a) \neq 0$. Suppose that f is not a monomorphism. Then there exists $0 \neq b \in B$ such that $f(b) = 0$. Since B is uniform, there exist $r, s \in R$ such that $0 \neq ra = sb \in Ra \cap Rb$. Hence $rf(a) = f(ra) = f(sb) = sf(b) = 0$, i.e. $r \in Ann_R f(a) = p$. Therefore $ra = 0$, which is a contradiction. Thus f is a monomorphism. It follows that $B \in \mathcal{M}$, which means that A is uniform and $A \in \mathcal{M}$.

Corollary 6. *For every prime ideal p of R, $R/p \in \mathcal{M}$.*

Corollary 7. *Let $A \in \mathcal{M}$. Then $Ra \in \mathcal{M}$ for every $0 \neq a \in A$.*

Theorem 8. *Let $A \in \mathcal{M}$ be a faithful R-module which is not injective. Then there exists $0 \neq f \in End_R(A)$ which is not an isomorphism.*

Proof. Suppose that every non-zero $f \in End_R(A)$ is an isomorphism. By Corollary 4, R is an integral domain and A is a torsion-free R-module. It follows that R is isomorphic to a subring of the ring $End_R(A)$, hence $rA = A$ for every non-zero element $r \in R$, i.e. A is divisible. Then A is injective ([4], Chapter 2, Proposition 2.7). This provides a contradiction. Thus there exists $0 \neq f \in End_R(A)$ which is not an isomorphism.

Example 9. *Let R be an integral domain. Then R is uniform and the ideal $Ann_R(E(R)) = 0$ is a prime ideal of R. If A is a non-zero submodule of $E(R)$, then $A \in \mathcal{M}$. Hence $E(R) \in \mathcal{M}$. Since $E(R)$ is an indecomposable injective R-module, every non-zero endomorphism $f \in End_R(E(R))$ is an isomorphism ([4], Chapter 3, Lemma 3.10). If A is a non-zero proper submodule of $E(R)$, then A is not injective because $E(R)$ is indecomposable. By Theorem 8, there exists $0 \neq f \in End_R(A)$ which is not an isomorphism.*

Theorem 10. *Let $A \in \mathcal{M}$ be an injective R-module and denote $p = Ann_R A$. Then $A \cong E(R/p)$ and $End_R(A)$ is a division ring.*

Proof. Let $0 \neq a \in A$. By Theorem 3, $p = Ann_R A = Ann_R a$ is a prime ideal of R. But $aR \cong R/Ann_R a = R/p$, hence $E(R/p) \cong E(aR) \leq A$. By Theorem 1, A is indecomposable, hence $A \cong E(R/p)$. Let $0 \neq f \in End_R(A)$. Then f is a monomorphism. Since A is an indecomposable injective R-module, it follows that f is an isomorphism ([4], Chapter 3, Lemma 3.10). Therefore $End_R(A)$ is a division ring.

Example 11. *Let $R = \mathbb{Z}$ be the ring of integers and $\mathbb{Q}$ the additive group of rational numbers. Then $\mathbb{Q} \in \mathcal{M}$, $\mathbb{Q} = E(\mathbb{Z})$ and $End_{\mathbb{Z}}(\mathbb{Q}) \cong \mathbb{Q}$ is a field.*

Theorem 12. *Let $A \in \mathcal{M}$ be a quasi-injective R-module and denote $p = Ann_R A$. If $A \leq B \leq Ann_{E(A)} p$, then $B \in \mathcal{M}$.*

Proof. We have $Ann_R a = p$ for every $0 \neq a \in Ann_{E(A)} p$. Let $0 \neq f \in End_R(B)$. Then there exists $0 \neq b \in B$ such that $f(b) \neq 0$. Since $A \trianglelefteq B$, there exists $r \in R$ such that $0 \neq rb \in A \cap Rb$. Therefore $r \notin p$ and $f(rb) = rf(b) \neq 0$, i.e. $f|_A \neq 0$. But f extends to a $g \in End_R(E(A))$. Since A is quasi-injective, we have $g(A) \subseteq A$, hence $f(A) \subseteq A$ ([2], p.252). Let $h \in End_R(A)$ be defined by $h(a) = f(a)$ for every $a \in A$. Since $h(b) = f(b) \neq 0$, it follows that h is a monomorphism. Suppose now that f is not a monomorphism. Then there exists $0 \neq c \in B$ such that $f(c) = 0$. Also there exists $s \in R$ such that $0 \neq sc \in A \cap Rc$. We have $h(sc) = f(sc) = sf(c) = 0$, which is a contradiction. Therefore f is a monomorphism.

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REMODELLING GIVEN BEZIER SPLINE CURVES AND SURFACES

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Dedicated to Professor Ioan Purdea at his 60th anniversary

Abstract. Rational Bézier splines offer many possibilities to control the shapes of curves and surfaces, but their relative complex equations lead at rather complicated formulas for derivatives and, consequently, smoothness conditions. In this paper we present a manner of partial or total remodelling given polynomial Bézier spline curve (part 2) and surface (part 4), preserving their class of continuity. The method consists in performing degree elevations that depend by real parameters. The curvatures of Bézier spline curve in the initial, final and joint points are also studied in part 3. The theory is illustred by some figures with initial (as witnesses) and remodeled spline curves and surfaces, respectively.

1. Introduction

1.1. Let g be a given $C^r[u_0, u_L]$ polynomial Bézier spline curve of degree m, corresponding to the control points $b_i \in \Re^3, i = \overline{0, mL}$, with the breakpoints $b_{mI}, I = \overline{1, L-1}$ and the breakvalues of the parameter $u, u_k, k = \overline{1, L}, u_0 < u_1 < \ldots < u_L$. This spline curve is represented over an interval $[u_I, u_{I+1}], I = \overline{0, L-1}$ by the following equation

$$g(u) = \sum_{k=0}^{m} B_{m,k}\left(\frac{u - u_I}{u_{I+1} - u_I}\right) b_{mI+k}, \qquad u \in [u_I, u_{I+1}], \tag{1}$$

where $B_{m,k}(t) = \binom{m}{k}(1-t)^{m-k}t^k, k = \overline{0, m}$, are the Bernstein polynomials.

The C^r conditions of g, on the jonction points $u_I, I = \overline{1, L-1}$, are

$$(\Delta_I^u)^i \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} b_{mI-i+k} = (\Delta_{I-1}^u)^i \sum_{k=o}^{i} \binom{i}{k} (-1)^{i-k} b_{mI+k} \tag{2}$$

$i = \overline{0, r}$, where $\Delta_I^u = u_{I+1} - u_I$, [2], p.92.

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1.2. Consider a polynomial Bézier spline surface G of degrees m and n relative to the parameters u and v respectively, over the two-dimensional interval $D = [u_0, u_L]\mathrm{x}[\mathrm{v}_0, \mathrm{v}_M]$, having the control points $b_{ij}, i = \overline{0, mL}, j = \overline{0, nM}$, with breakpoints $b_{mI,nJ}, I = \overline{1, L-1}, J = \overline{1, M-1}$ and u_k and v_l, $k = \overline{1, L-1}$, $l = \overline{1, M-1}$ the breakvalues of the parameters.

On $D_{IJ} = [u_I, u_{I+1}]\mathrm{x}[\mathrm{v}_J, \mathrm{v}_{J+1}]$ the surface G has the equation

$$G(u,v) = \sum_{k=0}^{m}\sum_{l=0}^{n} B_{m,k}\left(\frac{u-u_I}{u_{I+1}-u_I}\right) B_{n,l}\left(\frac{v-v_J}{v_{J+1}-v_J}\right) b_{mI+k,nJ+l} \tag{3}$$

Two patches of G corresponding to the domains $D_{I-1,J}$ and $D_{I,J}$ are r times continuously differentiable across their common curve $G(u_I, v), v \in [v_J, v_{J+1}]$ if the following conditions

$$(\Delta_I^u)^i \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} b_{mI-i+k,nJ+l} = (\Delta_{I-1}^u)^i \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} b_{mI+k,nJ+l}, \tag{4}$$

are fulfiled [2], p.272, for any $i = \overline{0, r}$, and $l = \overline{0, n}$.

Analogous, two patches of G, corresponding to the domains $D_{I,J-1}$ and $D_{I,J}$ are s times continuously differentiable, across their common curve

$$G(u, v_J), u \in [u_I, v_{I+1}]$$

if are fulfiled conditions

$$(\Delta_J^v)^j \sum_{l=0}^{j} \binom{j}{l} (-1)^{j-l} b_{mI+k,nJ-j+l} = (\Delta_{J-1}^v)^j \sum_{l=0}^{j} \binom{j}{l} (-1)^{j-l} b_{mI+k,nJ+l}, \tag{5}$$

for any $j = \overline{0, s}$ and $k = \overline{0, m}$.

The Bézier spline surface G is $C^{r,s}(D)$ if the conditions (4) and (5) are fulfiled for every $I = \overline{0, L-1}$ and $J = \overline{0, M-1}$.

2. Remodelling a Given C^r Bézier Spline Curve

Consider the polynomial Bézier spline curve g defined in part 1. We will remodel this curve by making a degree elevation corresponding to the following variable points, determined by a set of real parameters $\alpha_k^{(I)} \in (0,1), k = \overline{1, m},\ I = \overline{0, L-1}$,

$$b^*_{(m+1)I+k} = \begin{cases} b_{mI}, & \text{if } k = 0, \\ (1-\alpha_k^{(I)}) b_{mI+k-1} + \alpha_k^{(I)} b_{mI+k}, & \text{if } k = \overline{1, m}, \\ b_{m(I+1)}, & \text{if } k = m+1. \end{cases} \tag{6}$$

One observes that if

$$\alpha_k^{(I)} = \frac{m+1-k}{m+1},$$

then $b^*_{(m+1)I+k}$, $k = \overline{0, m+1}$, are the known Bézier points, which are used in the classical degree elevation, [2], p.52.

The Bézier spline curve g^* corresponding to the points (6), over the interval $[u_I, u_{I+1}]$, has the equation

$$g^*(u) = \sum_{k=0}^{m+1} B_{m+1,k}\left(\frac{u-u_I}{u_{I+1}-u_I}\right) b^*_{(m+1)I+k},\ u \in [u_I, u_{I+1}],\ I = \overline{0, L-1}. \tag{7}$$

The spline curve g^* will be $C^r[u_0, u_L]$ if, similar to (2), are fulfiled the conditions

$$(\Delta_I^u)^i \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} b^*_{(m+1)I-i+k} = (\Delta_{I-1}^u)^i \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} b^*_{(m+1)I+k} \tag{8}$$

for any $i = \overline{0, r}$ and $I = \overline{1, L-1}$.

From here, taking into account by (6) and that the conditions (2) hold, results that g^* is $C^r[u_0, u_L]$, $r \le [m/2]$ if and only if,

$$\begin{cases} \alpha_p^{(I)} = 1 - p\alpha_m^{(I-1)}, p = \overline{1, r}, I = \overline{1, L-1} \\ \\ \alpha_{m-p+1}^{(I)} = p\alpha_m^{(I)}, p = \overline{2, r}, I = \overline{0, L-2} \end{cases} \tag{9}$$

The other parameters $\alpha_p^{(0)}$, $p = \overline{1, m-r}$, $\alpha_p^{(L-1)}$, $p = \overline{r+1, m}$, $\alpha_p^{(I)}$, $p = \overline{r+1, m-r}$, $I = \overline{1, L-2}$ and $\alpha_m^{(I)}$, $I = \overline{0, L-2}$, from the open interval $(0,1)$, are arbitrary. Therefore, the Bézier spline curve $g^* \in C^r[u_0, u_L]$ has the following control points

$$b_k^* = \begin{cases} b_0, & \text{if } k = 0 \\ (1-\alpha_k^{(0)}) b_{k-1} + \alpha_k^{(0)} b_k, & \text{if } k = \overline{1, m-r} \\ \left[1 - (m+1-k)\alpha_m^{(0)}\right] b_{k-1} + & \\ \qquad (m+1-k)\alpha_m^{(0)} b_k, & \text{if } k = \overline{m-r+1, m} \\ b_m, & \text{if } k = m+1 \end{cases} \tag{10}$$

$$b^*_{(m+1)I+k} = \begin{cases} b_{mI}, & \text{if } k = 0 \\ k\alpha_m^{(I-1)} b_{mI+k-1} + \\ \quad +(1 - k\alpha_m^{(I-1)}) b_{mI+k}, & \text{if } k = \overline{1,r} \\ (1 - \alpha_k^{(I)}) b_{mI+k-1} + \alpha_k^{(I)} b_{mI+k}, & \text{if } k = \overline{r+1, m-r} \\ \left[1 - (m+1-k)\alpha_m^{(I)}\right] b_{mI+k-1} + \\ \quad +(m+1-k)\alpha_m^{(I)} b_{mI+k}, & \text{if } k = \overline{m-r+1, m} \\ b_{m(I+1)}, & \text{if } k = m+1 \end{cases} \tag{11}$$

for any $I = \overline{1, L-2}$, and

$$b^*_{(L-1)(m+1)+k} = \begin{cases} b_{(L-1)m}, & \text{if } k = 0 \\ k\alpha_m^{(L-2)} b_{(L-1)m+k-1} + \\ \quad +(1 - k\alpha_m^{(L-2)}) b_{(L-1)m+k}, & \text{if } k = \overline{1,r} \\ (1 - \alpha_k^{(L-1)}) b_{(L-1)m+k-1} + \\ \quad \alpha_k^{(L-1)} b_{(L-1)m+k}, & \text{if } k = \overline{r+1, m} \\ b_{Lm}, & \text{if } k = m+1 \end{cases} \tag{12}$$

Example. Consider the quadratic spline Bézier curve of $C^1[0,4]$ corresponding to the following points ($m = 2, L = 5$):

$$b_0(10,16), \quad b_1(7,21), \quad b_2(3,9), \quad b_3(0,0), \quad b_4(-4,0), \quad b_5(-8,0),$$
$$b_6(-6,3), \quad b_7(-4,6), \quad b_8(2,2), \quad b_9(8,-2), \quad b_{10}(10,3),$$

with the breakpoints $b_{2I}, I = \overline{1,4}$ and breakvalues of the parameter u, deduced with the chord length parametrization method: $u_0 = 0, u_1 = 1, u_2 = 7/4, u_3 = 5/2, u_4 = 13/4, u_5 = 4$. The dotted curves are Bézier spline g, as witness curves.

The parameter values corresponding to these figures are:

Fig.1: $\alpha_1^0 = 2/3$, $\alpha_2^0 = 1/20$, $\alpha_2^1 = 1/3$, $\alpha_2^2 = 1/3$, $\alpha_2^3 = 1/3$, $\alpha_2^4 = 1/10$;

Fig.2: $\alpha_1^0 = 1/10$, $\alpha_2^0 = 1/10$; $\alpha_2^1 = 1/3$, $\alpha_2^2 = 1/3$, $\alpha_2^3 = 1/100$, $\alpha_2^4 = 1/5$.

We remark that if $\alpha_2^{(I-1)}$, $I = \overline{1, L-1}$, decreases, then the Bézier spline curve one extends in the vicinity of joint point $b_{(m+1)I}$.

3. Curvature of Bézier Spline Curve g^*

Let $K(u)$ and $K^*(u)$ be the curvatures of the curves g and g^* respectively, with $r \geq 2$. Next we will deduce the dependence of $K^*(u)$ by $K(u)$ and the introduced

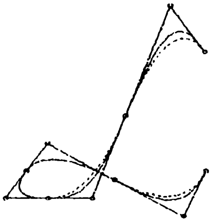

FIGURE 1

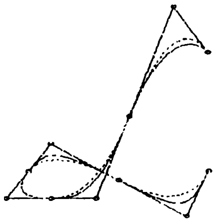

FIGURE 2

parameters, on $u = u_I, I = \overline{0,L}$. One knows that the formulas of $K(u_I)$ and $K^*(u_I)$ are

$$K(u_I) = \begin{cases} \dfrac{m-1}{m}\dfrac{\|\Delta b_{mI+1} \wedge \Delta b_{mI}\|}{\|\Delta b_{mI}\|^3}, & \text{for } I = \overline{0, L-1}, \\ \\ \dfrac{m-1}{m}\dfrac{\|\Delta b_{mL-2} \wedge \Delta b_{mL-1}\|}{\|\Delta b_{mL-1}\|^3}, & \text{for } I = L, \end{cases} \tag{13}$$

and

$$K^*(u_I) = \begin{cases} \dfrac{m}{m+1}\dfrac{\|\Delta b^*_{(m+1)I+1} \wedge \Delta b^*_{(m+1)I}\|}{\|\Delta b^*_{(m+1)I}\|^3}, & \text{for } I = \overline{0, L-1}, \\ \\ \dfrac{m}{m+1}\dfrac{\|\Delta b^*_{Im-1} \wedge \Delta b^*_{Im}\|}{\|\Delta b^*_{Im}\|^3}, & \text{for } I = L, \end{cases} \tag{14}$$

where "$\wedge$" denotes the vector product.

By direct calculus, taking into account by (10), (11), (12), and (13), the formula (14) becomes

$$\begin{aligned} K^*(u_0) &= \frac{m^2}{m^2-1}\frac{\alpha_2^{(0)}}{\left(\alpha_1^{(0)}\right)^2}K(u_0), \\ K^*(u_I) &= \frac{m^2}{m^2-1}\frac{1-2\alpha_m^{(I-1)}}{\left(1-\alpha_m^{(I-1)}\right)^2}K(u_I), I = \overline{1, L-1}, \\ K^*(u_L) &= \frac{m^2}{m^2-1}\frac{1-\alpha_{m-1}^{(L-1)}}{\left(1-\alpha_m^{(L-1)}\right)^2}K(u_L). \end{aligned} \tag{15}$$

In figure 3, the variation of $K^*(u_I)$, with respect to $\alpha_m^{(I-1)}, 0 < \alpha_m^{(I-1)} < 1/2$, $I = \overline{1, L-1}$, is shown.

Remarks: a) From (15) results that we have the possibility to control the curvature on any breakpoint $u_I, I = \overline{0, L}$.

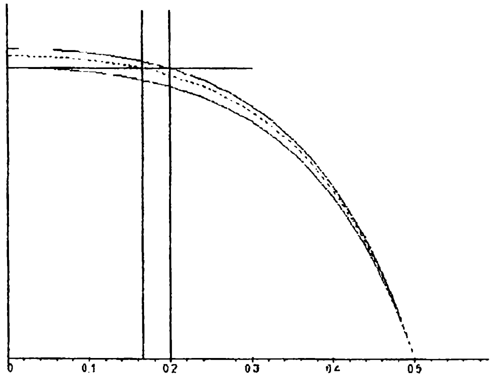

FIGURE 3

b) For any positive integer $m, 4 \leq m < \infty$, and $0 \leq \alpha_m^{(I-1)} \leq 1/2, I = \overline{1, L-1}$, we have

$$\frac{1-2\alpha_m^{(I-1)}}{\left(1-\alpha_m^{(I-1)}\right)^2} K(u_I) \leq K^*(u_I) \leq \frac{16}{15} \frac{1-2\alpha_m^{(I-1)}}{\left(1-\alpha_m^{(I-1)}\right)^2} K(u_I), I = \overline{1, L-1}.$$

c) If $0 < \alpha_m^{(I-1)} \leq \frac{1}{m+1}$, then $K^*(u_I) \geq K(u_I)$,
if $\frac{1}{m+1} \leq \alpha_m^{(I-1)} \leq \frac{1}{2}$, then $K^*(u_I) \leq K(u_I)$,
if $\alpha_m^{(I-1)} = \frac{1}{m+1}$ results $K^*(u_I) = K(u_I)$,
for any $I = \overline{1, L-1}$, and any positive integer $m, 4 \leq m < \infty$.
d) If $\alpha_1^{(0)} = \frac{m}{m+1}$, $\alpha_2^{(0)} = \frac{m-1}{m+1}$, then $K^*(u_0) = K(u_0)$,
and for $\alpha_{m-1}^{(L-1)} = \frac{2}{m+1}$, $\alpha_m^{(L-1)} = \frac{1}{m+1}$ results $K^*(u_L) = K(u_L)$.

4. Remodelling a Given $C^{r,s}$ Bézier Spline Surface

4.1. First we consider the particular Bézier spline surface which results from §1, part 1.2, for $M = 1$. Over any domain $D_{I,1} = [u_I, u_{I+1}]\text{x}[\text{v}_0, \text{v}_1]$, $\text{I} = \overline{0, \text{L}-1}$, the Bézier spline surface denoted G_1, has the equation

$$G_1(u,v) = \sum_{k=0}^{m} \sum_{l=0}^{n} B_{m,k}\left(\frac{u-u_I}{u_{I+1}-u_I}\right) B_{n,l}\left(\frac{v-v_0}{v_1-v_0}\right) b_{mI+k,l}. \qquad (16)$$

Assuming that $G_1(u,v)$ is $C^r[u_0, u_L]$ with respect to variable u (G_1 is evidently indefinite differentiable with respect to variable v) then the conditions (4), in the particular case $I = 0$, are fulfiled, for any $l = \overline{0, n}$.

We will remodel the above surface, preserving its class of smoothness, performing a degree elevation relative to the variable u, with the aid of following control points

$$b^*_{(m+1)I+k,l} = \begin{cases} b_{mI,l}, & \text{if } k = 0 \\ \left(1-\alpha^{(I)}_{k,l}\right) b_{mI+k-1,l} + \alpha^{(I)}_{k,l} b_{mI+k,l}, & \text{if } k = \overline{1,m} \\ b_{m(I+1),l}, & \text{if } k = m+1, \end{cases} \tag{17}$$

where the parameters $\alpha^{(I)}_{k,l} \in (0,1)$, $I = \overline{0,L-1}$, $l = \overline{0,n}$. We again remark that if $\alpha^{(I)}_{k,l} = \dfrac{m+1-k}{m+1}$, $l = \overline{0,n}$, $I = \overline{0,L-1}$, then (17) are the prints of the degree elevation with respect to variable u.

Bézier spline surface corresponding to these points over domain $D_{I,1}, I = \overline{0,L-1}$, has the equation

$$G^*_1(u,v) = \sum_{k=0}^{m+1}\sum_{l=0}^{n} B_{m+1,k}\left(\frac{u-u_I}{u_{I+1}-u_I}\right) B_{n,l}\left(\frac{v-v_0}{v_1-v_0}\right) b^*_{(m+1)I+k,l}. \tag{18}$$

This surface will be C^r across the curve $G^*_1(u_I,v)$, $v \in [v_0,v_1]$ if, similar to (4) for $I = 0$, the following conditions are fulfiled

$$(\Delta^u_I)^i \sum_{k=0}^{i}\binom{i}{k}(-1)^{i-k} b^*_{(m+1)I-i+k,l} = (\Delta^u_{I-1})^i \sum_{k=0}^{i}\binom{i}{k}(-1)^{i-k} b_{(m+1)I+k,l},$$

for any $i = \overline{0,r}$ and $l = \overline{0,n}$.

From here, taking into account by (17) and (4), with $I = 0$, one obtains, similar to (9) that

$$\begin{aligned} \alpha^{(I)}_{p,l} &= 1 - p\alpha^{(I-1)}_{m,l}; \quad p = \overline{1,r}, I = \overline{1,L-1}, \\ \alpha^{(I)}_{m-p+1,l} &= p\alpha^{(I)}_{m,l}; \quad p = \overline{2,r}, I = \overline{0,L-2}. \end{aligned} \tag{19}$$

The parameters $\alpha^{(0)}_{p,l}$, $p = \overline{1,m-r}$; $\alpha^{(L-1)}_{p,q}$, $p = \overline{r+1,m}$; $\alpha^{(I)}_{p,q}$, $p = \overline{r+1,m-r}$, $I = \overline{1,L-2}$, and $\alpha^{(I)}_{m,l}$, $I = \overline{0,L-2}$ take arbitrary values from the interval $(0,1)$, for any $l = \overline{0,n}$.

4.2. Now we consider the remodelling, in this manner, of a Bézier spline surface $G(u,v)$, given in part 1.2, preserving its class of $C^{r,s}(D)$. As in previous case, we make a degree elevation using the following control points, depending by two sets of parameters

$$b^*_{(m+1)I+k,(n+1)I+l} = \begin{cases} b_{m(I+\frac{k}{m+1}),n(I+\frac{l}{n+1})}, \\ \qquad \text{if } (k,l) \in \{0, m+1\}\text{x}\{0, n+1\}, \\ \left[1-\alpha_{k,l}^{(I,J)}, \alpha_{k,l}^{(I,J)}\right] \; A \; \left[1-\beta_{k,l}^{(I,J)}, \beta_{k,l}^{(I,J)}\right]^T, \\ \qquad \text{if } \begin{array}{l} k=\overline{1,m}, l=\overline{1,n}, \\ I=\overline{0,L-1}, J=\overline{0,M-1}, \end{array} \end{cases} \tag{20}$$

where A is the matrix

$$A = \begin{bmatrix} b_{mI+k-1,nJ+l-1} & b_{mI+k-1,nJ+l} \\ \\ b_{mI+k,nJ+l-1} & b_{mI+k,nJ+l} \end{bmatrix}$$

The Bézier spline surface G^* corresponding to these control points has, on the domain D_{IJ}, the equation

$$G^*(u,v) = \sum_{k=0}^{m+1} \sum_{l=0}^{n+1} B_{m+1,k}\left(\frac{u-u_I}{u_{I+1}-u_I}\right) B_{n+1,l}\left(\frac{v-v_J}{v_{J+1}-v_J}\right) b^*_{(m+1)I+k,(n+1)J+l} \tag{21}$$

By direct calculus one deduces that G^* is $C^{r,s}(D)$, in hypothesis that G is $C^{r,s}(D)$, if and only if

$$\begin{array}{l} \alpha_{k,n-j}^{(I,J-1)} = \alpha_{k,j}^{(I,J)}, I=\overline{0,L-1}, J=\overline{1,M-2}, k=\overline{0,m}, j=\overline{1,s}, \\ \beta_{m-i,l}^{(I-1,J)} = \beta_{i,l}^{(I,J)}, I=\overline{1,L-2}, J=\overline{0,M-1}, l=\overline{0,n}, i=\overline{1,r} \end{array}$$

and, similar to (19),

$$\begin{array}{l} \alpha_{m-p,l}^{(I,J)} = p\alpha_{m,l}^{(I,J)} = \alpha_m^{(I,J)}, p=\overline{2,r},\ I=\overline{0,L-2},\ J=\overline{0,M-1} \\ \alpha_{p,l}^{(I,J)} = 1-p\alpha_{m,l}^{(I-1,J)}, p=\overline{1,r},\ I=\overline{1,L-1},\ J=\overline{0,M-1} \\ \beta_{k,n-q}^{(I,J)} = q\beta_{k,n}^{(I,J)} = \beta_n^{(I,J)}, q=\overline{2,s},\ I=\overline{0,L-1},\ J=\overline{0,M-2} \\ \beta_{k,q}^{(I,J)} = 1-q\beta_{k,n}^{(I,J-1)} = \beta_n^{(I,J)}, q=\overline{1,s},\ I=\overline{0,L-1},\ J=\overline{1,M-1}. \end{array} \tag{22}$$

The other parameters, from the interval $(0,1)$,

$$\begin{array}{l} \alpha_{p,l}^{(0,J)}, p=\overline{1,m-r}; \alpha_{p,l}^{(I,J)}, p=\overline{r+1,m-r}; \\ \alpha_m^{(I,J)}, I=\overline{0,L-2}, J=\overline{0,M-1}; \alpha_{p,l}^{(L-1,J)}, p=\overline{m-r,m} \end{array}$$

and

$$\beta_{k,q}^{(I,0)}, q = \overline{1, n-s};\ \beta_{k,q}^{(I,J)}, q = \overline{s+1, n-s};$$
$$\beta_{n}^{(I,J)}, I = \overline{0, L-1}, J = \overline{0, M-2};\ \beta_{k,q}^{(I,M-1)}, q = \overline{n-s, n},$$

are arbitrary.

Example. We consider the following Bézier control points:

		l=0	l=1	l=2	l=3
	k=0	(0,0,3)	(0,1,2)	(0,3,1)	(0,4,3)
$b_{2\cdot 0+k,l}$	k=1	(1,0,1)	(1,1,4)	(1,3,2)	(1,4,4)
	k=2	(2,0,2.5)	(2,1,3)	(2,3,1)	(2,4,3)
	k=0	(2,0,2.5)	(2,1,3)	(2,3,1)	(2,4,3)
$b_{2\cdot 1+k,l}$	k=1	(3,0,4)	(3,1,2)	(3,3,0)	(3,4,2)
	k=2	(4,0,2.5)	(4,1,3)	(4,3,1)	(4,4,2.5)
	k=0	(4,0,2.5)	(4,1,3)	(4,3,1)	(4,4,2.5)
$b_{2\cdot 2+k,l}$	k=1	(5,0,1)	(5,1,4)	(5,3,2)	(5,4,3)
	k=2	(6,0,3)	(6,1,0)	(6,3,1)	(6,4,2)

For $\alpha_{k,l}^{(I)} = \frac{1}{3}$, $I = \overline{0,2}$, $k = \overline{0,2}$, $l = \overline{0,3}$ one obtains the Bézier spline surface from Figure 4, corresponding to the usual degree elevation. The parameter values are: $u \in [0,3]$ and $v \in [0,1]$; in Figure 5 the parameter values are: $\alpha_{1,l}^{0} = 1/2$, $\alpha_{2,l}^{0} = 1/500$, $\alpha_{2,l}^{1} = 1/1000$, $\alpha_{2,l}^{2} = 4/5$; $l = \overline{0,3}$.

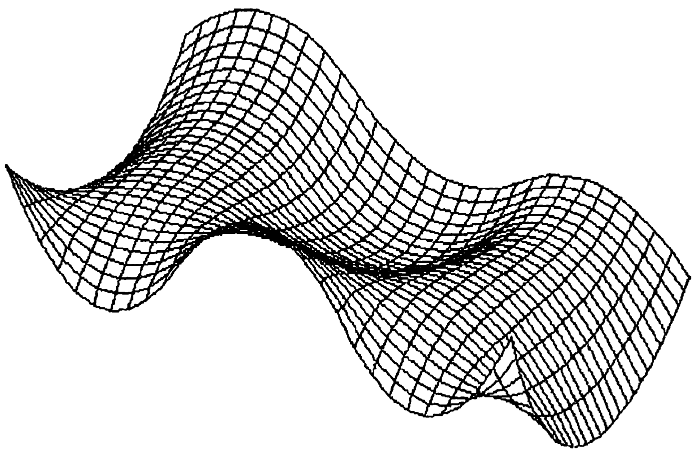

FIGURE 4. Surface 1

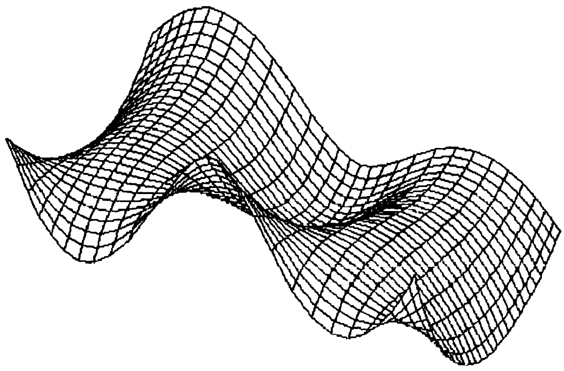

FIGURE 5. Surface 2

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DISTRIBUTIVE NONCOMMUTATIVE LATTICES OF TYPE (S)

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Dedicated to Professor Ioan Purdea at his 60^{th} anniversary

Abstract. The noncommutative lattices of type (S) were first defined in [1]. In this paper on introduces the notion of distributive noncomutative lattice of type (S) and one studies their properties as compared to those known from the classical lattices theory.

1. The triplet $(L, \wedge, \vee)$, where L is a nonvoid set, $\wedge$ and $\vee$ are two binary operations defined in L, is named noncommutative lattice of type (S), if, for all $a, b, c \in L$:

$$(A).\begin{cases} (a \wedge b) \wedge c = a \wedge (b \wedge c) \\ (a \vee b) \vee c = a \vee (b \vee c) \end{cases}$$

$$(B).\begin{cases} a \wedge (a \vee b) = a \\ a \vee (a \wedge b) = a \end{cases}$$

$$(S).\begin{cases} a \wedge (b \vee c) = a \wedge (c \vee b) \\ a \vee (b \wedge c) = a \vee (c \wedge b). \end{cases}$$

We observe that this system of laws is selfdual, so in $(L, \wedge, \vee)$ holds the duality principle.

This special class of noncommutative lattices was first defined in [1]. Then, in [1] and [2] are presented a few properties of this class of noncommutative lattices. From these properties, we mention the following:

(1.1). If $(L, \wedge, \vee)$ is a noncommutative lattice of type (S), then for all $a, b, c \in L$:

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$$(1).\begin{cases} a\wedge a=a \\ a\vee a=a \end{cases} \qquad (2).\begin{cases} a\wedge b=(a\wedge b)\vee(b\wedge a) \\ a\vee b=(a\vee b)\wedge(b\vee a) \end{cases}$$

$$(3).\begin{cases} a\wedge(b\vee a)=a \\ a\vee(b\wedge a)=a \end{cases} \qquad (4).\begin{cases} a\wedge(b\wedge c)=a\wedge(c\wedge b) \\ a\vee(b\vee c)=a\vee(c\vee b) \end{cases}$$

$$(5).\begin{cases} a\wedge b\wedge a=a\wedge b \\ a\vee b\vee a=a\vee b. \end{cases}$$

2. In this paper we introduce the notion of distributive noncommutative lattice of type (S) and we study its properties in analogy with the well known ones from the classical theory of lattices.

Let $(L,\wedge,\vee)$ a noncommutative lattice of type (S). If the identities below hold for all $a,b,c\in L$, let us accept the following notations:

$(D_r^{\wedge\vee})$. $(a\vee b)\wedge c=(a\wedge c)\vee(b\wedge c)$

$(D_r^{\vee\wedge})$. $(a\wedge b)\vee c=(a\vee c)\wedge(b\vee c)$

$(D_l^{\wedge\vee})$. $a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$

$(D_l^{\vee\wedge})$. $a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c)$

(S_r). $a\wedge c=b\wedge c$ and $a\vee c=b\vee c\Rightarrow a=b$

(S_{rl}). $a\wedge c=b\wedge c$ and $c\vee a=c\vee b\Rightarrow a=b$

(S_{lr}). $c\wedge a=c\wedge b$ and $a\vee c=b\vee c\Rightarrow a=b$

(S_l). $c\wedge a=c\wedge b$ and $c\vee a=c\vee b\Rightarrow a=b.$

The noncommutative lattice of type (S), is named distributive if, for all $a,b,c\in L$, it verifies $(D)=\{(D_r^{\wedge\vee}),(D_r^{\vee\wedge}),(D_l^{\wedge\vee}),(D_l^{\vee\wedge})\}$.

We obtain an example of distributive noncommutative lattice of type (S), if we define in the cartesian product $P(M)\times P(M)=\{(A,B)\mid A\subseteq M,\ B\subseteq M\}$, the operations "$\wedge$" and "$\vee$" thus:

$$(A_1,B_1)\wedge(A_2,B_2)=(A_1,B_1\bigcap B_2)$$

$$(A_1,B_1)\vee(A_2,B_2)=(A_1,B_1\bigcup B_2)$$

We observe that these operations are not commutative.

The noncommutative lattice of type (S) is named with simplifications if for any $a,b,c\in L$, it verifies the system of laws: $(S)=\{(S_r),(S_{rl}),(S_{lr}),(S_l)\}$.

All the theorems below refer to the noncommutative lattices of type (S). We will prove a few properties of distributive noncommutative lattices of type (S).

(2.1). $(D_r^{\wedge\vee})\bigcap(D_l^{\wedge\vee}) \Rightarrow (D_l^{\vee\wedge})$ and $(D_r^{\vee\wedge})\bigcap(D_l^{\vee\wedge}) \Rightarrow (D_l^{\wedge\vee})$

Proof. If for all $a, b, c \in L$ hold $(D_r^{\wedge\vee})$ and $(D_l^{\wedge\vee})$, then, using the properties from theorem (1.1) we obtain:

$$\begin{aligned}(a \vee b) \wedge (a \vee c) &= [a \wedge (a \vee c)] \vee [b \wedge (a \vee c)] = a \vee [b \wedge (a \vee c)] = \\ &= a \vee [(b \wedge a) \vee (b \wedge c)] = [a \vee (b \wedge a)] \vee (b \wedge c)] = a \vee (b \wedge c),\end{aligned}$$

So $(D_l^{\vee\wedge})$ is true, namely $(D_r^{\wedge\vee})\bigcap(D_l^{\wedge\vee}) \Rightarrow (D_l^{\vee\wedge})$.

The other implication from (2.1) is the dual of this first.

The following sentence is a result of the theorem (2.1).

(2.2). The noncommutative lattice of type (S) is distributive if and only if, it verifies the system of laws $\{(D_r^{\wedge\vee}), (D_r^{\vee\wedge}), (D_l^{\wedge\vee})\}$ or the system $\{(D_r^{\wedge\vee}), (D_r^{\vee\wedge}), (D_l^{\vee\wedge})\}$.

(2.3). $(D_r^{\wedge\vee})\bigcap(D_l^{\wedge\vee}) \Leftrightarrow (D_r^{\vee\wedge})\bigcap(D_l^{\vee\wedge})$

Proof. If for all $a, b, c \in L$, the rules $(D_r^{\wedge\vee})$ and $(D_l^{\wedge\vee})$ are true, then, using the definiton of noncommutative lattices of type (S) and the properties from (1.1) we obtain:

$$\begin{aligned}(a \vee c) \wedge (b \vee c) &= [a \wedge (b \vee c)] \vee [c \wedge (b \vee c)] = [a \wedge (b \vee c)] \vee c = \\ &= [(a \wedge b) \vee (a \wedge c)] \vee c = (a \wedge b) \vee [(a \wedge c) \vee c] = \\ &= (a \wedge b) \vee [c \vee (a \wedge c)] = (a \wedge b) \vee c,\end{aligned}$$

and respective

$$\begin{aligned}(a \vee b) \wedge (a \vee c) &= [a \wedge (a \vee c)] \vee [b \wedge (a \vee c)] = a \vee [b \wedge (a \vee c)] = \\ &= a \vee [(b \wedge a) \vee (b \wedge c)] = [a \vee (b \wedge a)] \vee (b \wedge c) = \\ &= a \vee (b \wedge c),\end{aligned}$$

So, the equalities $(D_r^{\vee\wedge})$ and $(D_l^{\vee\wedge})$ are true, namely $(D_r^{\wedge\vee})\bigcap(D_l^{\wedge\vee}) \Rightarrow (D_r^{\vee\wedge})\bigcap(D_l^{\vee\wedge})$. The inverse one is the dual of the first.

An immediate result of this theorem is the following sentence:

(2.4). The noncommutative lattice of type (S) is distributive if and only if, it verifies the system of laws $\{(D_r^{\wedge\vee}), (D_l^{\wedge\vee})\}$ or the system $\{(D_r^{\vee\wedge}), (D_l^{\vee\wedge})\}$.

(2.5). $(D_r^{\wedge\vee}) \Rightarrow (D_l^{\vee\wedge})$ and $(D_r^{\vee\wedge}) \Rightarrow (D_l^{\wedge\vee})$

Proof. Obviously, it will be enough to prove the first implication, because the second is the dual of the first.

If for all $a, b, c \in L$, $(D_r^{\wedge\vee})$ is true, then, using the definition of noncommutative lattices of type (S) and the theorem (1.1), we obtain:

$$\begin{aligned}(a \vee b) \wedge (a \vee c) &= [a \wedge (a \vee c)] \vee [b \wedge (a \vee c)] = \\ &= a \vee [b \wedge (a \vee c)] = \\ &= a \vee [(a \vee c) \wedge b] = \\ &= a \vee [(a \wedge b) \vee (c \wedge b] = \\ &= [a \vee (a \wedge b)] \vee (c \wedge b) = \\ &= a \vee (c \wedge b) = \\ &= a \vee (b \wedge c),\end{aligned}$$

So, the rule $(D_l^{\vee\wedge})$ is true. The second implication is the dual of the first.

An immediate result of this theorem is the following:

(2.6). The noncommutative lattice of type (S), is distributive, if and only if, it verifies the system of laws $\{(D_r^{\wedge\vee}), (D_r^{\vee\wedge})\}$.

We observe that theorem (2.1) can be considered a result of the theorem (2.5).

(2.7). $(D) \Rightarrow (S_r) \bigcap (S_{rl}) \bigcap (S_{lr})$

Proof. We suppose that, for every $a, b, x \in L$, the equalities: $a \wedge x = b \wedge x$, $a \vee x = b \vee x$ are true. Using the distributivity laws, we obtain:

$$\begin{aligned}a &= a \wedge (a \vee x) = a \wedge (b \vee x) = (a \wedge b) \vee (a \wedge x) = \\ &= (a \wedge b) \vee (b \wedge x) = (a \wedge b) \vee (x \wedge b) = (a \vee x) \wedge b = (b \vee x) \wedge b = \\ &= (b \wedge b) \vee (x \wedge b) = b \vee (x \wedge b) = b,\end{aligned}$$

so (S_r) is true, namely $(D) \Rightarrow (S_r)$.

Then, if we suppose that for $x, a, b \in L$ the equalities $a \wedge x = b \wedge x, x \vee a = x \vee b$ are true, then, using the distributivity laws, we obtain

$$\begin{aligned}a &= a \wedge (x \vee a) = a \wedge (x \vee b) = (a \wedge x) \vee (a \wedge b) = (b \wedge x) \vee (a \wedge b) = \\ &= [b \vee (a \wedge b)] \wedge [x \vee (a \wedge b)] = b \wedge [x \vee (a \wedge b)] = b \wedge [(x \vee a) \wedge (x \vee b)] = \\ &= b \wedge [(x \vee b) \wedge (x \vee b)] = b \wedge (x \vee b) = b,\end{aligned}$$

so (S_{rl}) is true, namely $(D) \Rightarrow (S_{rl})$.

The imnlication $(D) \Rightarrow (S_{lr})$ is the dual of $(D) \Rightarrow (S_{rl})$.

(2.8). $(S_l) \Rightarrow (S_r) \bigcap (S_{rl}) \bigcap (S_{lr})$

Proof. We suppose that, for $x, a, b \in L$, the equalities $a \wedge x = b \wedge x$ and $a \vee x = b \vee x$ are true.

Using the laws wich define the noncommutative lattices of type (S), and the property (5) of theorem (1.1) we obtain:

$$x \wedge a = x \wedge a \wedge x = x \wedge b \wedge x = x \wedge b$$

$$x \vee a = x \vee a \vee x = x \vee b \vee x = x \vee b,$$

Applying (S_l) we obtain that $a = b$, namely $(S_l) \Rightarrow (S_r)$.

If for $a, b, x \in L$, the equalities $a \wedge x = b \wedge x$, $x \vee a = x \vee b$ are true, then, $x \wedge a = x \wedge a \wedge x = x \wedge b \wedge x = x \wedge b$, and, by aplying (S_l) we have that $a = b$, namely $(S_l) \Rightarrow (S_{rl})$.

The implication $(S_l) \Rightarrow (S_{lr})$ is the dual of $(S_l) \Rightarrow (S_{rl})$

(2.9). $(S_{rl}) \bigcup (S_{lr}) \Rightarrow (S_r)$

Proof. If for $a, b, x \in L$, the equalities $a \wedge x = b \wedge x$ and $a \vee x = b \vee x$ are true, then, $x \vee a = x \vee a \vee x = x \vee b \vee x = x \vee b$, so, aplying (S_{rl}) we obtain $a = b$. The implication $(S_{lr}) \Rightarrow (S_r)$ is the dual of the first.

(2.10). If in the noncommutative lattice of type (S), the rule (S_l) is true, then the two binary operations are commutative, namely $(L, \wedge, \vee)$ becomes lattice.

Proof. We suppose that in $(L, \wedge, \vee)$, the rule (S_l) is true. Then, using (3) and (4) from theorem (1.1), we obtain that for every $x, a, b \in L$:

$$x \wedge (a \wedge b) = x \wedge (b \wedge a) \quad \text{and} \quad x \vee (a \wedge b) = x \vee (b \wedge a).$$

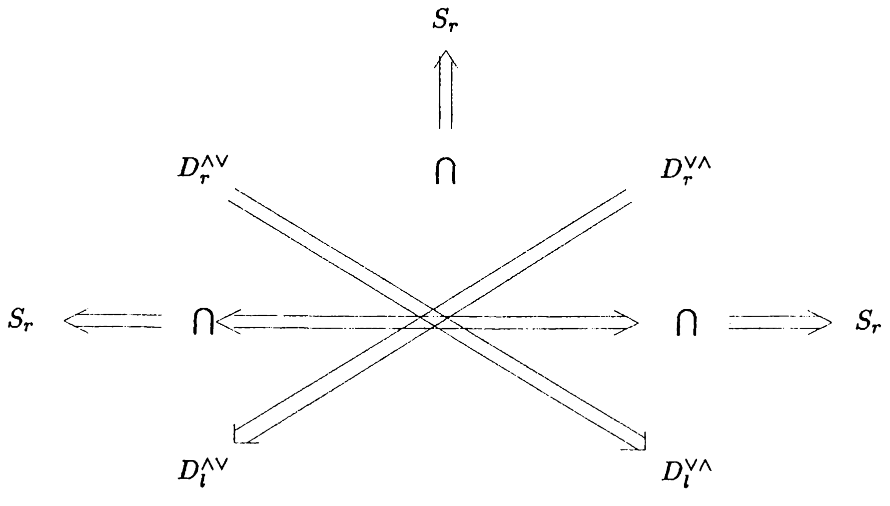
By applying (S_s), we obtain $a \wedge b = b \wedge a$ for all $a, b \in L$, namely the operation "$\wedge$" is commutative. The commutativity of "$\vee$" results analogously.

From this theorem results that in a distributive noncommutative lattice of type (S), the rules (S_s) are not necessarily true.

(2.10) also shows that any noncommutative lattice of type (S) with left simplifying is lattice.

We observe that the theorem (2.8) can be considered a result of theorem (2.10).

The main results from this paper can be represented by the following diagram:



It is known that, if $(L, \wedge\vee)$ is a lattice, then the following equivalences are true:

$(D) \Leftrightarrow (D_r^{\wedge\vee}) \Leftrightarrow (D_r^{\vee\wedge}) \Leftrightarrow (D_l^{\wedge\vee}) \Leftrightarrow (D_l^{\vee\wedge})$

$(S) \Leftrightarrow (S_d) \Leftrightarrow (S_{rl}) \Leftrightarrow (S_{lr}) \Leftrightarrow (S_s)$

$(D) \Leftrightarrow (S)$.

These equivalences are not true if $(L, \wedge\vee)$ is a noncommutative lattice of type (S), without being a lattice.

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INDUCTION OF GRADED INTERIOR ALGEBRAS

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Dedicated to Professor Ioan Purdea at his 60th anniversary

Abstract. We introduce interior $\mathcal{O}^\alpha H$-algebras graded by a finite group Γ and generalized induction for these algebras. This situation occurs in the study of source algebras of blocks of normal subgroups and our construction unifies various constructions introduced by Lluis Puig.

1. Introduction

Induction for interior G-algebras was introduced by L. Puig [P1], this being the fundamental construction linking an interior G-algebra with its source algebra. Given a subgroup H of G and an interior H-algebra H over a complete discrete valuation ring $\mathcal{O}$, the induced interior G-algebra is $\mathcal{O}G \otimes_{\mathcal{O}H} B \otimes_{\mathcal{O}H} \mathcal{O}G$, with multiplication inspired by that of the endomorphism algebra $\mathrm{End}_{\mathcal{O}G}(\mathcal{O}G \otimes_{\mathcal{O}H} M)$, where M is an $\mathcal{O}H$-module.

Later some generalizations were needed in order to deal with more involved problems. Algebras interior for a twisted group algebra were considered in [P2]; dealing with blocks of normal subgroups in [KP] imposed the construction of of G-algebra extensions; finally, noninjective induction was introduced in [P3] and [P4] in order to study bimodules inducing equivalences between interior algebras.

The aim of this note is to unify these constructions. We shall consider $\mathcal{O}$-algebras A graded by a group Γ, endowed with a grade preserving $\mathcal{O}$-algebra map $\mathcal{O}^\alpha H \to A$, where $\mathcal{O}^\alpha H$ is the twisted group algebra defined by the cocycle $\alpha \in Z^2(H, \mathcal{O}^*)$, and H has a a normal subgroup N such that $G = H/N$ is a subgroup of Γ. This degree of generality is needed; this situation occurs for instance when one considers the source algebra of a G-invariant block of $\mathcal{O}^\alpha N$. We have in mind later applications to Clifford theory, and recall that similar contexts have been considered in recent work of E. Dade.

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Key words and phrases. interior algebras,blocks.

Most of our conventions and notations will follow those of [P2], [T] and [NV], except that we use the notation of [K] for twisted group algebras. The needed definitions will be given in each section, but some standard facts from these sources will be used without comments. In Section 2 we discuss graded algebras and their exomorphisms; Γ-graded interior $\mathcal{O}^\alpha H$-algebras are introduced in Section 3. Injective induction for these algebras is defined and studied in Section 4, while in the last section we introduce the generalized induction.

2. Twisted group algebras, interior algebras and group extensions

2.1. We fix a p-modular system $(\mathcal{K},\mathcal{O},k)$, where $\mathcal{O}$ is a complete discrete valuation ring, $\mathcal{K}$ is the quotient field of $\mathcal{O}$ and $k=\mathcal{O}/J(\mathcal{O})$ is the residue field of $\mathcal{O}$. The case $k=\mathcal{O}=\mathcal{K}$ is not excluded.

2.2. Let $A=\bigoplus_{g\in G}A_g$ be a G-graded $\mathcal{O}$ algebra, where G is a finite group and the additive subgroups A_g, $g\in G$ are $\mathcal{O}$-free of finite rank.

We shall be interested in some particular cases. Recall that A is *strongly graded* if $A_gA_h=A_{gh}$ for all $g,h\in G$, and A is a *crossed product* if $A_g\cap U(A)\neq\emptyset$. In this case, denoting

$$hU(A)=\bigcup_{g\in G}(A_g\cap U(A)),$$

we have the group extension

$$\epsilon(A):\qquad 1\to U(A_1)\to hU(A)\overset{deg}{\to}G\to 1.$$

If $\epsilon(A)$ splits, then A is a *skew group algebra*. We shall also discuss twisted group algebras later.

2.3. If $B=\bigoplus_{g\in G}B_g$ is another G-graded $\mathcal{O}$-algebra, then a homomorphism $f\colon A\to B$ of $\mathcal{O}$-algebras (not necessarily unital) is called *G-graded* (*grade preserving*) if $f(A_g\subseteq B_g$ for all $g\in G$.

More generally, let $\phi\colon G\to H$ be a group homomorphism, $B=\bigoplus_{h\in H}B_h$ a H-graded $\mathcal{O}$-algebra, and denote by $\mathrm{Res}_\phi(B)$ the G-graded algebra $\mathrm{Res}_\phi(B)=\bigoplus_{g\in G}B_{\phi(g)}$. Then a homomorphism $f\colon A\to B$ of $\mathcal{O}$-algebras is called *graded* if $f(A_g)\subseteq B_{\phi(g)}$ for all $g\in G$, that is, f induces a grade preserving map, still denoted $f\colon A\to\mathrm{Res}_\phi(B)$.

If ϕ is just the inclusion $G\subseteq H$, then we shall simply denote $B_G=\bigoplus_{g\in G}B_g$. Observe that construction of B_H is functorial, that is, if $f\colon B\to B'$ is a homomorphism of H-graded algebras, then f induces in an obvious way a homomorphism of G-graded algebras $f_G : B_G \to B'_G$. In this situation, the G-graded algebra A can be trivially regarded as an H-graded algebra by defining $A_h = 0$ for $h \in H \setminus G$.

Another important situation which will occur in Section 5 is when $\phi\colon G \to H$ is surjective. Then the G-graded algebra A can be made into a H-graded algebra by defining $A_h = \bigoplus_{g\in\phi^{-1}(h)} A_g$.

Returning to the case when both A and B are G-graded, remark further that $f\colon A \to B$ induces a group homomorphism $f^*\colon U(A) \to U(B)$ by $f^*(a) = f(a-1)+1$. We also have that

$$f(a^{a^*}) = f(a)^{f^*(a^*)},$$

where $a^{a^*} = (a^*)^{-1}aa^*$. Moreover, if f is unital, and A and B are crossed products, then f induces a homomorphism of group extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U(A_1) & \longrightarrow & hU(A) & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \phi} & & \\ 1 & \longrightarrow & U(B_1) & \longrightarrow & hU(B) & \longrightarrow & H & \longrightarrow & 1 \end{array}$$

2.4. The group $hU(A)$ acts on A_1 as $\mathcal{O}$-algebra automorphisms, and on $U(A_1)$ as group automorphisms. Moreover, $hU(A_{Z(G)}) = deg^{-1}(Z(G))$ acts on A as grade-preserving automorphisms, and on $hU(A)$ as automorphisms of group extensions.

2.5. Let A and B be two G-graded $\mathcal{O}$-algebras. Then $A\otimes_{\mathcal{O}} B$ is naturally $G\times G$-graded, and if $\delta(G)$ denotes the diagonal subgroup of $G\times G$, then $(A\otimes_{\mathcal{O}} B)_{\delta(G)} = \bigoplus_{g\in G}(A_g \otimes_{\mathcal{O}} B_g)$ is again a G-graded algebra. We shall denote $\Delta(A, B^{op}) = (A\otimes_{\mathcal{O}} B)_{dG}$, this being coherent with the notation of [M]. The G-grading of B^{op} is given by $B^{op}_g = B_{g^{-1}}$, and by this convention, $\Delta(A,B) = (A\otimes_{\mathcal{O}} B^{op})_{\delta(G)}$. Moreover, if A and B are strongly graded (crossed products), then $A\otimes_{\mathcal{O}} B$ and $\Delta(A,B)$ are strongly graded (crossed products).

2.6. Definition. Let A and B be G-graded algebras. A *graded exomorphism* $\tilde{f}\colon A \to B$ is the set obtained by composing the grade-preserving homomorpfism $f\colon A \to B$ with the inner automorphisms of A and B given by conjugation with elements of of A_1 and B_1 respectively. Denote by $\widetilde{\mathrm{Hom}}_{gr}(A,B)$ the set of graded exomorphisms $\tilde{f}\colon A \to B$.

To obtain $\tilde{f}$ it suffices to compose f only with the above inner automorphisms of B. This implies that graded exomorphisms can be composed.

The exomorphism $\tilde{f}$ is called an *embedding* if $\mathrm{Ker} f = 0$ and $\mathrm{Im} f = f(1)Bf(1)$. Clearly, $\tilde{f}$ is an embedding if and only if $\tilde{f}_1 : A_1 \to B_1$ is an embedding of $\mathcal{O}$-algebras, where $f_1 : A_1 \to B_1, f_1(a) = f(a)$.

Let $\tilde{f} \in \widetilde{\mathrm{Hom}}_{gr}(A, B)$, $\tilde{g} \in \widetilde{\mathrm{Hom}}_{gr}(B, C)$ and $\tilde{h} = \tilde{g} \circ \tilde{f}$. It follows by this remark and [P2, Lemma 3.4] that: if $\tilde{g}$ is an embedding then $\tilde{f}$ is uniquely determined by $\tilde{h}$, and $\tilde{f}$ is an embedding if and only if $\tilde{h}$ is an embedding.

2.7. We end this section with by discussing an important example. Let $\alpha \colon H \times H \to \mathcal{O}^*$ a 2-cocycle (where $\mathcal{O}^* = U(\mathcal{O})$), and consider the twisted group algebra $\mathcal{O}^\alpha H = \{a\bar{x} \mid x \in H,\ a \in \mathcal{O}\}$ with multiplication $\bar{x}\bar{y} = \alpha(x, y)\overline{xy}$ for all $x, y \in H$. Clearly, $\mathcal{O}^\alpha H$ is a particular case of an H-graded crossed product, and if β is another 2-cocycle, then $\mathcal{O}^\alpha H \simeq \mathcal{O}^\beta H$ as H-graded algebras if and only if $\alpha\beta^{-1} \in B^2(H, \mathcal{O}^*)$. If N is a subgroup of H, we shall still denote $\mathcal{O}^\alpha N = \mathcal{O}^{res^H_N \alpha} N$, where $res^H_N \alpha \in Z^2(N\mathcal{O}^*)$.

We shall be interested in other gradings, too. If N is a normal subgroup of H, and $G = H/N$, then $\mathcal{O}^\alpha H$ is naturally graded by G.

We recall from [K] some properties of twisted group algebras.

(1.7.1) $\mathcal{O}^\alpha H \otimes_{\mathcal{O}} \mathcal{O}^{\alpha'} H' \simeq \mathcal{O}^{\alpha \times \alpha'}(H \times H')$ via $(\bar{h} \otimes \bar{h}') \leftrightarrow \overline{(h, h')}$.

(1.7.2) $(\mathcal{O}^\alpha H)^{op} \simeq \mathcal{O}^{\alpha^{-1}} H$ via $\bar{h} \leftrightarrow \overline{h^{-1}}$.

(1.7.3) If $\alpha, \beta \in Z^2(H, \mathcal{O}^*)$ then $\mathcal{O}^{\alpha\beta} H \simeq (\mathcal{O}^\alpha H \otimes_{\mathcal{O}} \mathcal{O}^\beta H)_{\delta(H)}$ via $\tilde{h} \leftrightarrow \bar{h} \otimes_{\mathcal{O}} \hat{h}$. (Notice that we have taken here the diagonal with respect to the H-grading.)

3. Graded interior algebras

We shall now describe our main object of study.

3.1. Definition. Let H and Γ be finite groups, $\mu \colon H \to \Gamma$ a group homomorphism, $N = \mathrm{Ker}\mu$, and $G = \mathrm{Im}\mu$. We also denote by μ the induced injective homomorphism $G \to \Gamma$. Let $\alpha \in Z^2(H, \mathcal{O}^*)$ and A a Γ-graded $\mathcal{O}$-algebra endowed with a graded homomorphism $\psi \colon \mathcal{O}^\alpha H \to A$ (that is, $\psi(\bar{h}) \in A_{\mu(h)}$ for all $h \in H$). Then (A, μ, ψ) (or simply A is called a *Γ-graded interior $\mathcal{O}^\alpha H$-algebra*, and μ, ψ are the *structural maps* of A.

3.2. Examples. a) Clearly, if N is a normal subgroup of H and $G = H/N$, then $\mathcal{O}^\alpha H$ is a G-graded interior $\mathcal{O}^\alpha H$-algebra.

b) If $e \in Z(\mathcal{O}^\alpha N)$ is a G-invariant idempotent, then $e\mathcal{O}^\alpha H$ is a G-graded interior $\mathcal{O}^\alpha H$-algebra with structural map $a\bar{h} \mapsto ea\bar{h}$, for all $a \in \mathcal{O}$, $h \in H$.

c) Let U be an $\mathcal{O}^\alpha N$-module and $M = \mathcal{O}^\alpha H \otimes_{\mathcal{O}^\alpha N} U = \mathrm{Ind}_{\mathcal{O}^\alpha N}^{\mathcal{O}^\alpha H} U$ with the usual G-grading. The $\mathcal{O}$-algebra $A = \mathrm{End}_{\mathcal{O}}(M)^{op}$ has a G-grading given by $A_g = \{f \in A \mid f(M_x) \subseteq M_{gx}$ for all $x \in G\}$. Now define $\psi\colon \mathcal{O}^\alpha H \to A$ by $\psi(\bar{h})(\bar{h}' \otimes u) = \bar{h}'\bar{h} \otimes u$. One can easily verify that A becomes a G-graded interior $\mathcal{O}^\alpha H$-algebra.

3.3. Let (A, μ, ψ) be a Γ-graded interior $\mathcal{O}^\alpha H$-algebra and $A, \mu', \psi')$ a γ-graded interior $\mathcal{O}^\beta H$-algebra. We have the group extension $N \times N \to H \times H \to G \times G$, and denote by $\delta_G(H) = \{(x, y) \in H \times H \mid xN = yN\}$ the "diagonal" of $H \times H$ w.r.t. G. Then $A \otimes_{\mathcal{O}} B$ is a $\Gamma \times \Gamma$-graded interior $\mathcal{O}^{\alpha \times \beta}(H \times H)$-algebra, and $\Delta(A, B^{op})$ is a $\delta(\Gamma)$-graded interior $\delta_G(H)$-algebra (and also a $\delta(H)$-algebra by restriction).

3.4. Observe that $A_{\mu(G)}$ is a $\mu(G)$-graded crossed product and μ, ψ induce the homomorphism

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U(\mathcal{O}^\alpha N) & \longrightarrow & hU(\mathcal{O}^\alpha H) & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mu} & & \\ 1 & \longrightarrow & U(A_1) & \longrightarrow & hU(A_{\mu(G)}) & \longrightarrow & \mu(G) & \longrightarrow & 1 \end{array}$$

of group extensions. Although H may not be a subgroup of $hU(\mathcal{O}^\alpha H)$, it still acts on A_1 by conjugation. Actually, $A_{\mu(G)}$ is determined by A_1, the group extension $N \to H \to G$, and the action of H on A_1. Indeed, the homomorphism $\mathcal{O}^\alpha H \to A_{\mu(G)}$ of G-graded algebras (identifying G with $\mu(G)$) determines a structure of a G-graded $\mathcal{O}^\alpha H$-bimodule on $A_{\mu G}$ and also a map

$$(\mathcal{O}^\alpha H \otimes_{\mathcal{O}} (\mathcal{O}^\alpha H)^{op})_{\Delta_\Gamma(\mathcal{O}^\alpha H)} A_1 \to (A \otimes_{\mathcal{O}} A^{op}) \otimes_{\Delta(A)} A_1$$

of G-graded $(\mathcal{O}^\alpha H, \mathcal{O}^\alpha H)$-bimodules (where $\Delta(A) = (A \otimes_{\mathcal{O}} A^{op})_{\delta(G)}$, see [M, Section 2]). Since the 1-component of this map is just the identity map of A_1, it follows that

$$(\mathcal{O}^\alpha H \otimes_{\mathcal{O}} (\mathcal{O}^\alpha H)^{op})_{\Delta_\Gamma(\mathcal{O}^\alpha H)} A_1 \to A \quad (\bar{x} \otimes \bar{y}) \otimes a \mapsto xay^{-1}$$

is an isomorphism of G-graded $\mathcal{O}^\alpha H$-bimodules.

3.5. Definition. A *homomorphism* $f\colon A \to A'$ of Γ-graded interior $\mathcal{O}^\alpha H$-algebras is a graded $\mathcal{O}$-algebra map satisfying $f\bar{x} \cdot a \cdot \bar{y}) = \bar{x} \cdot f(a) \cdot \bar{d}$ for all $x, y \in H$ and $a \in A$. We still denote by $\mathrm{Hom}_{gr}(A, A')$ the set of these homomorphisms.

The *exomorphism* $\tilde{f}\colon A \to A'$ is the orbit of f under the action of $U(A_1^H) \times U({A'_1}^H)$ on $\mathrm{Hom}_{gr}(A, A')$.

Since this orbit coincides with the orbit under the action of $U(A_1'^H)$, it follows that the exomorphisms of Γ-graded interior $\mathcal{O}^\alpha H$-algebras can be composed, and we denote by $\widetilde{\mathrm{Hom}}_{[}gr(A,A')$ the set of exomorphisms $\tilde{f}\colon A\to A'$.

3.6. Let ρ be a homomorphism

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N_1 & \longrightarrow & H_1 & \longrightarrow & G_1 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & N & \longrightarrow & H & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

of group extensions such that $G_1\to G$ is injective (otherwise we replace N_1 with the kernel of the composition $H_1\to H\to G$). Then $\mathrm{Res}_\rho A$ is, by definition, the Γ-graded interior $\mathcal{O}^{\alpha_1}H_1$-algebra $(A,\mu\circ\rho,\psi\circ\rho)$, where $\alpha_1=res_\rho\alpha\in Z^2(H_1,\mathcal{O}^*)$.

Moreover, a homomorphism $f:A\to B$ of Γ-graded interior $\mathcal{O}^\alpha H$ algebras induces obviously the homomorphism $\mathrm{Res}_\rho(f):\mathrm{Res}_\rho A\to\mathrm{Res}_\rho B$ of Γ-graded interior $\mathcal{O}^{\alpha_1}H_1$-algebras.

4. Injective induction for graded interior algebras

4.1. Consider the group extension $N\to H\to G$ and the subgroups K of H, $N\cap K$ of N and $K/K\cap N\simeq KN/N$ of G, and let $[H/K]$ be a complete set of representatives for the left cosets of K in H. Denote by ρ all these inclusion maps and, for $\alpha\in Z^2(H,\mathcal{O}^*)$, we also denote by α the element $\mathrm{Res}_K^H\alpha\in Z^2(K,\mathcal{O}^*)$. Let finally $\mu\colon G\to\Gamma$ an injective group homomorphism.

4.2. Let B be a Γ-graded interior $\mathcal{O}^\alpha K$-algebra with structural maps $\mu'=\mu\circ\rho:KN/N\to\Gamma$ and $\psi'\colon\mathcal{O}^\alpha K\to B$. Consider the $(\mathcal{O}^\alpha H,\mathcal{O}^\alpha H)$-bimodule $A=\mathcal{O}^\alpha H\otimes_{\mathcal{O}^\alpha K}B\otimes_{\mathcal{O}^\alpha K}\mathcal{O}^\alpha H$, and define the $\mathcal{O}$-bilinear multiplication

$$(\bar{x}\otimes b\otimes\bar{y})(\bar{x}'\otimes b'\otimes\bar{y}')=\begin{cases}0, & \text{if } yx'\in K\\ \bar{x}\otimes b\cdot\bar{y}\bar{x}'\cdot b'\otimes\bar{y}', & \text{if} yx'\in K\end{cases}$$

and the map

$$\psi\colon\mathcal{O}^\alpha H\to A,\quad \bar{x}\mapsto\sum_{y\in[H/K]}\bar{x}\bar{y}\otimes 1_B\otimes\bar{y}^{-1}.$$

4.3. Proposition. *A is a Γ-graded interior $\mathcal{O}^\alpha H$-algebra with structural maps μ and ψ.*

Proof. It can be easily verified that the multiplication is well defined and associative, and that A is an $\mathcal{O}$-algebra with unit element $1_A = \sum_{y\in[H/K]} \bar{y} \otimes 1_B \otimes \bar{y}^1$. Also, ψ is a well defined $\mathcal{O}$-algebra map.

The grading of A is defined as follows. If $g \in \Gamma$, $x, y \in H$ and $b \in B_g$, then $\bar{x} \otimes b \otimes \bar{y}$ is a homogeneous element of degree $\mu(xN)g\mu(yN) \in \Gamma$. It follows that for $g = xN \in G$ we have

$$A_g = \sum_{y\in[H/K]} \bar{x}\bar{y} \otimes_{\mathcal{O}^\alpha K} B_1 \otimes_{\mathcal{O}^\alpha K} \bar{y}^{-1}.$$

In particular, $A_1 = \sum_{y\in[H/K]} \bar{y} \otimes B_1 \otimes \bar{y}^{-1}$ is a subalgebra of A. It also follows that the structural map ψ is grade-preserving.

4.4. Definition. We shall say that the Γ-graded interior $\mathcal{O}^\alpha H$-algebra A is ***induced*** from B, and we denote $A = \mathrm{Ind}_\rho(B) = \mathrm{Ind}_{\mathcal{O}^\alpha K}^{\mathcal{O}^\alpha H}(B)$.

The construction is functorial, since if $f : B \to B'$ is a homomorphism of Γ-graded interior $\mathcal{O}^\alpha K$-algebras, then $\mathrm{Ind}_\rho(f) = id \otimes f \otimes id : \mathrm{Ind}_\rho(B) \to \mathrm{Ind}_\rho(B')$ is a homomorphism of Γ-graded interior $\mathcal{O}^\alpha H$-algebras.

4.5. Since the subalgebra A_G is a G-graded crossed product, it can be constructed from B_1 in an alternative way. Indeed, we have that

$$\begin{aligned} A_G &= (\mathcal{O}^\alpha H \otimes_{\mathcal{O}} (\mathcal{O}^\alpha H)^{op}) \otimes_{\mathcal{O}^\alpha K \otimes_{\mathcal{O}} (\mathcal{O}^\alpha K)^{op}} B_G \\ &\simeq (\mathcal{O}^\alpha H \otimes_{\mathcal{O}} (\mathcal{O}^\alpha H)^{op}) \otimes_{\mathcal{O}^\alpha K \otimes_{\mathcal{O}} (\mathcal{O}^\alpha K)^{op}} ((\mathcal{O}^\alpha K \otimes_{\mathcal{O}} (\mathcal{O}^\alpha K)^{op}) \otimes_{\Delta_\Gamma(\mathcal{O}^\alpha K)} B_1) \\ &\simeq (\mathcal{O}^\alpha H \otimes_{\mathcal{O}} (\mathcal{O}^\alpha H)^{op}) \otimes_{\Delta_\Gamma(\mathcal{O}^\alpha H)} (\Delta_\Gamma(\mathcal{O}^\alpha H) \otimes_{\Delta_\Gamma(\mathcal{O}^\alpha K)} B_1) \end{aligned}$$

where we have also denoted $\Delta_\Gamma(\mathcal{O}^\alpha K) = \Delta_{K/K\cap N}(\mathcal{O}^\alpha K)$. Then, for $g = xH \in G$,

$$A_g \simeq (\bar{x} \otimes_{\mathcal{O}} 1) \otimes_{\Delta_\Gamma(\mathcal{O}^\alpha H)} (\Delta_\Gamma(\mathcal{O}^\alpha H) \otimes_{\Delta_\Gamma(\mathcal{O}^\alpha K)} B_1),$$

and A_1 is the $\Delta_\Gamma(\mathcal{O}^\alpha H)$-module $\Delta_\Gamma(\mathcal{O}^\alpha H) \otimes_{\Delta_\Gamma(\mathcal{O}^\alpha K)} B_1$.

4.6. Proposition. *Let M be an $\mathcal{O}^\alpha(K \cap N)$-module and $B = \mathrm{End}_{\mathcal{O}}(\mathrm{Ind}_{\mathcal{O}(K\cap N)}^{\mathcal{O}^\alpha K} M)$. Then*

$$\mathrm{Ind}_{\mathcal{O}^\alpha K}^{\mathcal{O}^\alpha H}(B) \simeq \mathrm{End}_{\mathcal{O}}(\mathrm{Ind}_{\mathcal{O}^\alpha(K\cap N)}^{\mathcal{O}^\alpha H}(M))$$

as Γ-graded interior $\mathcal{O}^\alpha H$-algebras.

Proof. By construction, B is a KN/N-graded crossed product, which can be trivially regarded as a Γ-graded interior $\mathcal{O}^\alpha K$-algebra, since $KN/N \leq G \leq \Gamma$. Denote $A =$

$\mathrm{End}_{\mathcal{O}}(\mathrm{Ind}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha(K\cap N)}(M))$. Since $\mathrm{Ind}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha(K\cap N)}(M)) \simeq \mathrm{Ind}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha N}(\mathrm{Ind}^{\mathcal{O}^\alpha n}_{\mathcal{O}^\alpha(K\cap N)}(M))$, A is a G-graded crossed product by Example 2.2.c).

First, we define an $\mathcal{O}$-linear action of $\mathrm{Ind}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}(B)$ on $\mathrm{Ind}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha(K\cap N)}(M)$. Let $f \in B_g$, $v \in \mathrm{Ind}^{\mathcal{O}^\alpha K}_{\mathcal{O}^\alpha(K\cap N)}(M))$ and $x, y, z \in H$, and define

$$(\bar{x} \otimes f \otimes \bar{y})(\bar{z} \otimes v) = \begin{cases} \bar{x} \otimes f(\bar{y}^{-1}\bar{z}v), & \text{if } y^{-1}z \in K \\ 0, & \text{otherwise.} \end{cases}$$

If f is homogeneous of degree $g \in G$ and v is homogeneous of degree $h \in KN/N$, then $z \otimes v$ is homogeneous of degree $xNgy^{-1}zNh$. By [T, (6.4)], this action induces an isomorphism of interior $\mathcal{O}^\alpha H$-algebras, and by the above remarks, it is also grade-preserving.

4.7. Proposition. *Let $L \le K \le H$ and C a Γ-graded interior $\mathcal{O}^\alpha L$-algebra. Then there is an isomorphism of Γ-graded interior $\mathcal{O}^\alpha H$-algebras*

$$\mathrm{Ind}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}(\mathrm{Ind}^{\mathcal{O}^\alpha K}_{\mathcal{O}^\alpha L} C) \simeq \mathrm{Ind}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha L}(C).$$

Proof. Using [T, Proposition 16.3], one can check that the map

$$\gamma\colon \mathrm{Ind}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}(\mathrm{Ind}^{\mathcal{O}^\alpha K}_{\mathcal{O}^\alpha L} C) \to \mathrm{Ind}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha L}(C), \quad \bar{x} \otimes (\bar{y} \otimes c \otimes \bar{y}') \otimes \bar{x}' \mapsto \bar{x}\bar{y} \otimes c \otimes \bar{y}'\bar{x}'$$

is an isomorphism of of interior $\mathcal{O}^\alpha H$-algebras. By Definition 2.1 it also follows that $\gamma$is grade-preserving.

4.8. Proposition. Let $K \le H$, A a Γ-graded interior $\mathcal{O}^\alpha H$-algebra and B a Γ-graded interior $\mathcal{O}^\beta K$-algebra, where $\alpha, \beta \in Z^2(H, \mathcal{O}^*)$. Then there is an isomorphism

$$\delta\colon \Delta_\Gamma(A \otimes_{\mathcal{O}} \mathrm{Ind}^{\mathcal{O}^\beta H}_{\mathcal{O}^\beta K}(B)) \to \mathrm{Ind}^{\mathcal{O}^{\alpha\beta} H}_{\mathcal{O}^{\alpha\beta} K}(\Delta_\Gamma(\mathrm{Res}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K} A \otimes_{\mathcal{O}} B))$$

of Γ-graded interior $\mathcal{O}^{\alpha\beta} H$-algebras.

Proof. Define δ by

$$a \otimes (\hat{x} \otimes b \otimes \hat{y}) \mapsto \tilde{x} \otimes (\bar{x}^{-1} \cdot a \cdot \bar{y}^{-1} \otimes b) \otimes \tilde{y},$$

where $a \in A_g$ and $b \in B_g$. Then δ is an isomorphism of Γ-graded interior $\mathcal{O}^{\alpha\beta} H$-algebras, having inverse δ^{-1} defined by

$$\tilde{x} \otimes (a \otimes b) \otimes \tilde{y} \mapsto \bar{x} \cdot a \cdot \bar{y} \otimes \hat{x} \otimes b \otimes \hat{y}.$$

4.9. Let B be a Γ-graded interior $\mathcal{O}^\alpha K$-algebra, and consider the homomorphism of Γ-graded interior $\mathcal{O}^\alpha K$-algebras

$$d^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K} : B \to \mathrm{Res}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}\mathrm{Ind}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}B, \qquad b \mapsto 1 \otimes b \otimes 1.$$

This map determines the *canonical embedding*

$$\tilde{d}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}(B) : B \to \mathrm{Res}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}\mathrm{Ind}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}B, \qquad b \mapsto 1 \otimes b \otimes 1.$$

4.10. Proposition. *Let $\tilde{g} : B \to \mathrm{Res}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}A$ be an embedding of Γ-graded interior $\mathcal{O}^\alpha K$-algebras, and assume that $1 \in \mathrm{Tr}^H_K(g(1))$, $(g(1) \cdot g(1)^x = 0$ for all $x \in H/K$, and that $g(1)$ centralizes A_1.*

Then there is a unique exomorphism $\tilde{f} : \mathrm{Ind}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}(B) \to A$ such that $\tilde{g} = \mathrm{Res}^H_K(\tilde{f}) \circ \tilde{d}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}(B)$.

Proof. If $\tilde{f}$ exists, we may take $f(\bar{x} \otimes b \otimes \bar{y}) = \bar{x} \cdot g(b) \cdot \bar{y}$ for any $x, y \in H$, $b \in B$. Conversely, let $f : \mathrm{Ind}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}(B) \to A$ be defined by this formula; as in [T, Proposition 16.6], we obtain that f is an isomorphism of interior $\mathcal{O}^\alpha H$-algebras, and since g is grade-preserving, f is grade-preserving too. Moreover, $\tilde{f}$ does not depend on the choice of g in $\tilde{g}$, since if $b \in (B_1^K)^*$, then $\mathrm{Tr}^H_K(g(b)) \in (A_1^H)^*$ and $(\bar{x} \cdot g(b) \cdot \bar{y})^{\mathrm{Tr}^H_K(g(b))} = \bar{x} \cdot g(b) \cdot y$ for all $x, y \in H$ and $b \in B$.

5. Generalized induction

We are now going to define the induction of graded interior algebras through an arbitrary group homomorphism $\phi : H \to H'$.

5.1. Consider the commutative diagram of groups

$$\begin{array}{ccccc}
K \cap N & \longrightarrow & K & \xrightarrow{\mu} & \Lambda \\
\downarrow & & \downarrow & & \downarrow \\
N = \mathrm{Ker}\mu & \longrightarrow & H & \xrightarrow{\mu} & \Gamma \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\
N' = \mathrm{Ker}\mu' & \longrightarrow & H' & \xrightarrow{\mu'} & \Gamma'
\end{array}$$

where $K = \mathrm{Ker}(\phi\colon H \to H')$ and $\Lambda = \mathrm{Ker}(\phi\colon \Gamma \to \Gamma')$. Let $G = H/N \simeq \mu(H) \leq \Gamma$ and $G' = H'/N' \simeq \mu'(H') \leq \Gamma'$. Then $K/K \cap N \simeq KN/N$ is isomorphic to a subgroup of Λ, and $H/KN \simeq (H/N)/(KN/N)$ is isomorphic to a subgroup of Γ/Λ, so to a subgroup of Γ'.

Let further $\alpha' \in Z^2(H', \mathcal{O}^*)$ and $\alpha = res_\phi(\alpha) \in Z^2(H, \mathcal{O}^*)$. It follows that $res^H_K\alpha = 1$ and $\mathcal{O}^\alpha K = \mathcal{O}K$, and ϕ induces a homomorphism $\bar{\phi}\colon \mathcal{O}^\alpha H \to \mathcal{O}^{\alpha'} H'$ of $\mathcal{O}$-algebras with image $\bar{\phi}(\mathcal{O}^\alpha H) \simeq \mathcal{O}^{\alpha'}(H/K)$.

We can regard $\mathcal{O}^\alpha H$ and $\mathcal{O}^{\alpha'} H$ as H/KN-graded algebras in the usual way, and also as Γ'-graded algebras (where the components of degree g' not belonging to the image of H are trivial). Similarly, $\mathcal{O}^{\alpha'} H'$ can be regarded as a H'/N'-graded algebra, and also as a Γ'-graded algebra. Then $\bar{\phi}\colon \mathcal{O}^\alpha H \to \mathcal{O}^{\alpha'} H'$ is a homomorphism of Γ'-graded algebras.

5.2. Let (A, μ, ψ) be a Γ-graded interior $\mathcal{O}^\alpha H$-algebra, and as above, regard A as a Γ/Λ-graded algebra, and also as a Γ'-graded algebra. Then the structural map $\psi\colon \mathcal{O}^\alpha H \to A$ is a homomorphism of Γ'-graded algebras.

By [P4, 3.2], $(\mathcal{O} \otimes_{\mathcal{O}K} A)^K$ is an $\mathcal{O}$-algebra with multiplication $(1 \otimes a)(1 \otimes b) = 1 \otimes ab$. Also, ψ factorizes through $\mathcal{O}^{\alpha'}(H/K)$, so $(\mathcal{O} \otimes_{\mathcal{O}K} A)^K$ becomes an interior $\mathcal{O}^{\alpha'}(H/K)$-algebra.

If $g \in \Gamma$, define the grade of $a \otimes_{\mathcal{O}K} a \in \mathcal{O} \otimes_{\mathcal{O}K} A)^K$ to be $\phi(g) \in \Gamma'$. This is clearly well-defined, and $(\mathcal{O} \otimes_{\mathcal{O}K} A)^K$ is a Γ'-graded interior $\mathcal{O}^{\alpha'}(H/K)$-algebra.

5.3. Now, by definition, let

$$\mathrm{Ind}_\phi(A) = \mathrm{Ind}^{H'}_{\phi(H)}((\mathcal{O} \otimes_{\mathcal{O}K} A)^K) = \mathcal{O}^{\alpha'} H' \otimes_{\mathcal{O}^{\alpha'}(H/K)} (\mathcal{O} \otimes_{\mathcal{O}K} A)^K \otimes_{\mathcal{O}^{\alpha'}(H/K)} \mathcal{O}^{\alpha'} H'.$$

By the preceding section, $\mathrm{Ind}_\phi(A)$ is a Γ'-graded interior $\mathcal{O}^{\alpha'} H'$-algebra with multiplication

$$(\bar{x}' \otimes (1 \otimes a) \otimes \bar{s}')(\bar{y}' \otimes (1 \otimes b) \otimes \bar{t}') = \begin{cases} \bar{x} \otimes (1 \otimes a \cdot \bar{z} \cdot b) \otimes \bar{t}', & \text{if } s'y' = \phi(z) \\ 0 & \text{otherwise,} \end{cases}$$

where z is a suitable element of H. The structural maps are $\mu'\colon H' \to \Gamma'$ and $\psi'\colon \mathcal{O}^{\alpha'} H' \to \mathrm{Ind}\phi(A)$ (preserving Γ'-gradings) defined by

$$\psi'(\bar{x}') = \bar{x}' \cdot \mathrm{Tr}^{H'}_{\phi(H)}(1 \otimes (1 \otimes 1) \otimes 1) = \mathrm{Tr}^{H'}_{\phi(H)}(1 \otimes (1 \otimes 1) \otimes q) \cdot \bar{x}'.$$

5.4. For $\sigma \in \mathrm{Aut}(K)$, denote, as in [P4, 2.3], $N_A^{\sigma}(K) = \{a \in A \mid a \cdot \bar{x} = \overline{\sigma(x)} \cdot a\}$, and let

$$N_A(K) = \sum_{\sigma \in \mathrm{Aut}(K)} N_A^{\sigma}(K).$$

Then, recalling that $\mathcal{O}^{\alpha}K = \mathcal{O}K$ and $\phi(K) = 1$, it follows immediately that $N_A(K)$ inherits from A a structure of a Γ'-graded interior $\mathcal{O}^{\alpha}H$-algebra. Moreover, the map

$$d_{\phi}(A)\colon N_A(K) \to \mathrm{Res}_{\phi}(\mathrm{Ind}_{\phi}(A)), \quad a \mapsto 1 \otimes (a \otimes a) \otimes 1$$

is a homomorphism of Γ'-graded interior $\mathcal{O}^{\alpha}H$-algebra, and by [P4, 3.4.4], if $\mathcal{O} \otimes_{\mathcal{O}K} A$ is a projective $\mathcal{O}K$-module, it induces an embedding $N_A(K)/\mathrm{Ker}(d_{\phi}(A)) \to \mathrm{Res}_{\phi}(\mathrm{Ind}_{\phi}(A))$ of Γ'-graded interior $\mathcal{O}^{\alpha}H$-algebras.

Further, if $f\colon A \to B$ is a homomorphism of Γ-graded interior $\mathcal{O}^{\alpha}H$-algebras, then f induces a grade-preserving map $f\colon N_A(K) \to N_B(K)$ and $\mathrm{Ind}_{\phi}(f)\colon \mathrm{Ind}_{\phi}(A) \to \mathrm{Ind}_{\phi}(B)$ such that we have the following commutative diagram.

$$\begin{array}{ccc}
\mathrm{Ind}_{\phi}(A) & \xrightarrow{\mathrm{Ind}_{\phi}(F)} & \mathrm{Ind}_{\phi}(B) \\
{\scriptstyle d_{\phi}(A)}\uparrow & & \uparrow{\scriptstyle d_{\phi}(B)} \\
N_A(K) & \xrightarrow{\quad f \quad} & N_B(K)
\end{array}$$

It is not hard to see that Propositions 4.6, 4.7 and 4.8 can be generalized to this situation. In essence, one has to check that the maps defined in [P4, 3.7, 3.13 and 3.17] are grade-preserving. We shall only give here a common generalization of Proposition 4.6 and [P4, Proposition 3.7]

5.5. Let M be a G/N-graded $\mathcal{O}^{\alpha}H$-algebra, that is, there is an $\mathcal{O}^{\alpha}H$-module M_1 such that $M = \mathcal{O}^{\alpha}H \otimes_{\mathcal{O}^{\alpha}N} M_1$.We can regard M as a H/KN-graded $\mathcal{O}^{\alpha}H$-module, and also as a Γ'-graded $\mathcal{O}^{\alpha}H$-module. Then $\mathrm{Ind}_{\phi}(M) = \mathcal{O}^{\alpha'}H' \otimes_{\mathcal{O}^{\alpha}H} M$ is a Γ'-graded $\mathcal{O}^{\alpha'}H'$-module, where for $h' \in H'$, $g' \in |gamma'$ and $m \in M_{g'}$, the element $\bar{h}' \otimes m$ has, by definition, degree $\mu'(h')g'$.

5.6. Proposition. *If M is $\mathcal{O}$-free, then there is an isomorphism of Γ'-graded interior $\mathcal{O}^{\alpha'}H'$-algebras*

$$ind_{\phi,M}\colon \mathrm{Ind}_{\phi}(\mathrm{End}_{\mathcal{O}}(M)) \to \mathrm{End}_{\mathcal{O}}(\mathrm{Ind}_{\phi}(M)).$$

Proof. By [P4, 3.7], for $x', s' \in H'$ and $1 \otimes f \in (|CO \otimes_{OK} \mathrm{End}_{\mathcal{O}}(M))^K$, $ind_{\phi,M}(\bar{x} \otimes (1 \otimes f) \otimes \bar{s}')$ is, by definition, the $\mathcal{O}$-linear endomorphism of $\mathrm{Ind}_\phi(M)$ mapping $\bar{y}' \otimes m$ to zero or to $\bar{x}' \otimes f(\bar{z} \cdot m)$, whenever there is $z \in H$ such that $s'y' = \phi(z)$. It is straightforward to check that $ind_{\phi,M}$ is grade-preserving.

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SUFFICIENT CONDITIONS FOR STARLIKENESS

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Dedicated to Professor Ioan Purdea at his 60^{th} anniversary

Abstract. In this paper we will study a differential subordination of the form:

$$\frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec h(z)$$

where h(z) is an univalent function in the unit disc U and we will obtain sufficient conditions of starlikeness for a function $f(z) = z + a_2 z^2 + \dots$ analytic in U.

We will obtain our results by using the differential subordination method developed in [1], [2] and [3].

1. Introduction and preliminaries

Let A denote the class of analytic functions in the unit disc $U=\{z,\ |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$.

Also, let $S^* = \left\{ f \in A,\ Re\frac{zf'(z)}{f(z)} > 0,\ z \in U \right\}$ be the class of starlike functions in U.

In [7] the authors considered the class of functions $f \in A$ which satisfy the condition:

$$Re\left\{\alpha\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right\} > 0, \qquad z \in U, \tag{1}$$

for $\alpha \geq 0$ where $\frac{f(z)}{z} \neq 0,\ z \in U$.

In [4] and [7] different types of starlike functions were investigated.

In [5] condition (2) was replaced by:

$$\frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec 1 + \lambda z, \tag{2}$$

$$\frac{f(z)}{z} \neq 0,\ z \in U, \text{ where } \alpha > 0 \text{ and } \lambda > 0.$$

In this paper we will consider a more general differential subordination of the form (1),where h is an univalent function in U.

We will need the following notions and lemmas to prove our main results.

If f and F are analytic functions in U,then we say that f is subordinate to F, written $f \prec F$, or $f(z) \prec F(z)$, if there is a function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$ and if $f(z) = F(w(z))$, $z \in U$. If F is univalent then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Lemma A. *([1], [2], [3]) Let q be univalent in $\overline{U}$ with $q'(\zeta) \neq 0$, $|\zeta| = 1$, $q(0) = a$ and let $p(z) = a + p_1 z + \ldots$ be analytic in U, $p(z) \not\equiv a$. If $p \not\prec q$ then there exist $z_0 \in U$, $\zeta_0 \in \partial U$ and $m \geq 1$ such that:*

(*i*) $p(z_0) = q(\zeta_0)$

(*ii*) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$.

The function $L(z,t)$, $z \in U$, $t \geq 0$ is a subordination chain if $L(z,t) = a_1(t)z + a_2(t)z^2 + a_3(t)z^3 + \ldots$ is analytic and univalent in U for any $t \geq 0$ and if $L(z,t_1) \prec L(z,t_2)$ when $0 \leq t_1 \leq t_2$.

Lemma B. *([6]) The function $L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots$ with $a_1(t) \neq 0$ for $t \geq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$ is a subordination chain if and only if there are the constants $r \in (0,1]$ and $M > 0$ such that:*

(*i*) *$L(z,t)$ is analytic in $|z| < r$ for any $t \geq 0$, locally absolute continuous in $t \geq 0$ for every $|z| < r$ and satisfies $|L(z,t)| \leq M|a_1(t)|$ for $|z| < r$ and $t \geq 0$.*

(*ii*) *there is a function $p(z,t)$ analytic in U for any $t \geq 0$ and measurable in $[0,\infty)$ for any $z \in U$ so that $Re\, p(z,t) > 0$ for $z \in U$, $t \geq 0$ and*

$$\frac{\partial L(z,t)}{\partial t} = z\frac{\partial L(z,t)}{\partial z} p(z,t) \text{ for } |z| < r \text{ and for almost any } t \geq 0.$$

2. Main results

Theorem 1. *Let the function:*

$$h(z) = 1 + (2\alpha + 1)\mu z + \alpha\mu^2 z^2, \tag{3}$$

where

$$\alpha > 0 \qquad and \qquad 0 < \mu \le 1 + \frac{1}{2\alpha}. \tag{4}$$

If $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ is analytic in U and satisfies the condition:

$$\alpha z p'(z) + \alpha p^2(z) + (1-\alpha)p(z) \prec h(z), \tag{5}$$

then $p(z) \prec 1 + \mu z$ and this result is sharp.

Proof. If we let $q(z) = 1+\mu z$, $\mu > 0$ and $\psi(p(z), zp'(z)) = \alpha z p'(z) + \alpha p^2(z) + (1-\alpha)p(z)$, then $\psi(q(z), zq'(z)) = h(z)$.

We will show that $\psi(p(z), zp'(z)) \prec h(z)$ implies $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

If we let $L(z,t) = \psi(q(z), (1+t)zq'(z)) = 1 + (\alpha t + 2\alpha + 1)\mu z + 2\alpha\mu^2 z^2$, then it is easy to show that:

$$\frac{z\frac{\partial}{\partial z}L(z,t)}{\frac{\partial}{\partial t}L(z,t)} = \frac{1}{\alpha}[(\alpha t + 2\alpha + 1) + 2\alpha\mu z],\ |z| < 1, \qquad \text{and}$$

$$Re\frac{z\frac{\partial}{\partial z}L(z,t)}{\frac{\partial}{\partial t}L(z,t)} = Re\frac{1}{\alpha}(\alpha t + 2\alpha + 1 + 2\alpha\mu z) \ge \frac{1}{\alpha}(\alpha t + 2\alpha + 1 - 2\alpha\mu)$$

Using now the condition (5) we obtain :

$$Re\frac{z\frac{\partial}{\partial z}L(z,t)}{\frac{\partial}{\partial t}L(z,t)} \ge \frac{1}{\alpha}\left[\alpha t + 2\alpha + 1 - 2\alpha\left(1 + \frac{1}{2\alpha}\right)\right] = \frac{1}{\alpha}(\alpha t) = t \ge 0$$

Hence $Re\dfrac{z\frac{\partial}{\partial z}L(z,t)}{\frac{\partial}{\partial t}L(z,t)} \ge 0$, and by Lemma B we deduce that $L(z,t)$ is a subordination chain.

In particular, for $t = 0$ we have $L(z,0) = h(z) \prec L(z,t)$, for $t \ge 0$.

If we suppose that $p(z)$, is not subordonate to $q(z)$, then by Lemma A there exist $z_0 \in U$, $\zeta_0 \in \partial U$ such that $p(z_0) = q(\zeta_0)$ with $|\zeta_0| = 1$, and $z_0 p'(z_0) = (1+t)\zeta_0 q'(\zeta_0)$, with $t \ge 0$.

Therefore $\psi_0 = \psi(p(z_0), z_0 p'(z_0)) = \psi(q(\zeta_0), (1+t)\zeta_0 q'(\zeta_0)) = L(\zeta_0, t)$, $t \ge 0$.

Since $h(z_0) = L(z_0, 0)$ we deduce that $\psi_0 \notin h(U)$, which contradicts condition (6). Hence $p(z) \prec q(z)$ and since $\psi(q(z), zq'(z)) = h(z)$ we deduce that q is the best dominant.

Corollary 1. *If $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ is analytic in U and satisfies the condition: $\alpha z p'(z) + \alpha p^2(z) + (1-\alpha)p(z) \prec 1 + \lambda z$, where $\lambda = \mu(2\alpha + 1 - \alpha\mu)$*

$$\text{and} \qquad 0 < \mu \le \left(1 + \frac{1}{2\alpha}\right), \qquad \text{then} \qquad p(z) \prec 1 + \mu z.$$

Proof. For $|z| = 1$ from (5) we deduce $|h(z) - 1| = \mu|2\alpha + 1 + \alpha\mu z| \ge \mu(2\alpha + 1 - \alpha\mu)$.

If we put $\lambda = \mu(2\alpha + 1 - \alpha\mu)$ we obtain $1 + \lambda z \prec h(z)$ and from Theorem 1 we deduce that $p(z) \prec 1 + \mu z$.

If we put $p = \dfrac{zf'}{f}$, where $f \in A$ then Theorem 1 can be written in the following equivalent form.

Theorem 2. *Let $h(z) = 1 + (2\alpha + 1)\mu z + \alpha\mu^2 z^2$, where $\alpha > 0$, $0 < \mu \le \left(1 + \frac{1}{2\alpha}\right)$. Let $f \in A$, with $\frac{f(z)}{z} \neq 0$, satisfy the condition:*

$$\frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec h(z).$$

Then

$$\frac{zf'(z)}{f(z)} \prec 1 + \mu z$$

and $1 + \mu z$ is the best dominant.

Corollary 2. *Let $f \in A$, with $\frac{f(z)}{z} \neq 0$, satisfy the condition:*

$$\frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec 1 + \lambda z$$

where $\lambda = \mu(2\alpha + 1 - \alpha\mu)$ and $0 < \mu \le \left(1 + \frac{1}{2\alpha}\right)$.

Then

$$\frac{zf'(z)}{f(z)} \prec 1 + \mu z.$$

3. Particular cases

I. If $\alpha = 1$, then $0 < \mu \leq \frac{3}{2}$.

a). If we take $\mu = 1$ then $\lambda = 2$ and from Corollary 2 we obtain:

If $f \in A$ satisfies the condition

$$\left|\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)}+1\right)-1\right| < 2 \text{ then } \left|\frac{zf'(z)}{f(z)}-1\right| < 1.$$

This result was obtained in [5].

b) If we take $\mu = \frac{1}{2}$ then $\lambda = \frac{5}{4}$ and from Corollary 2 we obtain the following condition for starlikeness. If $f \in A$ statisfies the condition

$$\left|\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)}+1\right)-1\right| < \frac{5}{4} \text{ then } \left|\frac{zf'(z)}{f(z)}-1\right| < \frac{1}{2}.$$

c) If we take $\mu = \frac{2}{3}$ then $\lambda = \frac{9}{4}$ and from Corollary 2 we deduce:

If $f \in A$ statisfies the condition

$$\left|\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)}+1\right)-1\right| < \frac{9}{4} \text{ then } \left|\frac{zf'(z)}{f(z)}-1\right| < \frac{3}{2}.$$

II. If $\alpha = 2$, then $0 < \mu \leq \frac{5}{4}$.

a). If we take $\mu = 1$ then $\lambda = \mu(2\lambda + 1 - \lambda\mu) = 3$ and from Corollary 2 we deduce:

If $f \in A$ satisfies the condition

$$\left|\frac{zf'(z)}{f(z)}\left(2\frac{zf''(z)}{f'(z)}+1\right)-1\right| < 3 \text{ then } \left|\frac{zf'(z)}{f(z)}-1\right| < 1.$$

b) If $\mu = \frac{1}{4}$ then $\lambda = \frac{9}{8}$ and from Corollary 2 we deduce: If $f \in A$, statisfies the condition

$$\left|\frac{zf'(z)}{f(z)}\left(2\frac{zf''(z)}{f'(z)}+1\right)-1\right| < \frac{9}{8} \text{ then } \left|\frac{zf'(z)}{f(z)}-1\right| < \frac{1}{4}.$$

c) If $\mu = \frac{5}{4}$, then $\lambda = \frac{25}{8}$ and from Corollary 2 we deduce:

If $f \in A$, statisfies the condition

$$\left|\frac{zf'(z)}{f(z)}\left(2\frac{zf''(z)}{f'(z)}+1\right)-1\right| < \frac{25}{8} \text{ then } \left|\frac{zf'(z)}{f(z)}-1\right| < \frac{5}{4}.$$

d) If $\mu = \frac{1}{2}$, then $\lambda = 2$ and from Corollary 2 we deduce:

If $f \in A$, statisfies the condition

$$\left|\frac{zf'(z)}{f(z)}\left(2\frac{zf''(z)}{f'(z)}+1\right)-1\right|<2 \text{ then } \left|\frac{zf'(z)}{f(z)}-1\right|<\frac{1}{2}.$$

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ON THE CONVEXITY OF SUPPORTED SETS

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Dedicated to Professor Ioan Purdea at his 60^{th} anniversary

Abstract. It is proved that if Y is a closed subset with nonvoid interior of a real topological vector space, such that the support points are dense in its boundary, then the set Y is convex.

1. Introduction

Given a subset Y of a real topological vector space (t.v.s. for short) X, an element x_0 of $\overline{Y}$ is said to be a *support point* for Y, if there is $x^* \in X^* x^* \neq 0$, such that $x^*(x_0) = \sup x^*(Y)$ or $x^*(x_0) = \inf x^*(Y)$. As usual we denote by X^* the dual space to X. Taking $-x^*$ instead of x^* it is obvious that we can always suppose that x^* attains its supremum at x_0. In Euclidean spaces this notion was first considered by H. Minkowski [12]. Denoting by spt$Y$ the set of all support points of the set Y and by bd$Y$ its boundary then spt$Y$ $\subset$bd$Y$ (see Lemma 1.1 below). By a famous result of E. Bishop and R.R. Phelps [2], if X is a Banach space and $Y \subset X$ is closed and convex, then spt$Y$ is dense in the boundary of Y. A kind of converse of this result will be proved in Section 3 of this paper: if Y is closed with nonvoid interior and spt$Y$ is dense in bd$Y$ then Y is convex. The study of necessary conditions for some optimization problems (such as best approximation, optimal control, mathematical economics) motivates the investigation of the opposite inclusion bd$Y$ $\subset$ spt$Y$. We say that the set Y is *totally supported* provided spt$Y$ =bd$Y$.

In the following theorem we present a list of some known supported sets.

Theorem 1.1. *A subset Y of a real t.v.s. X is totally supported provided one of the following (not necessarily independent) conditions is fulfilled:*

(a) X is the Euclidean space $\mathbb{R}^n$ and Y is convex (see [13]*)*

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(b) X is a normed space and Y is closed convex with nonvoid interior.

(c) X is a t.v.s. and Y is a p-convex subset of X with nonvoid interior (see [15]*).*

Recall that a subset Y of a vector space X is called p-*convex,* for $0 < p < 1$, if $px + (1-p)y \in Y$ for all $x, y \in Y$. Remark that there exist convex sets consisting only of support points. Namely, S. Rolewicz [18] constructed a closed convex set in a nonseparable Hilbert space X, not contained in any closed hyperplane in X, such that int$Y = \emptyset$ and spt$Y = Y$. A similar example is performed in [11] for Banach spaces containing uncountable minimal systems.

The present note is concerned with the investigation of the converse problem: In what conditions on the space X and on the set Y the total supportability of the set Y guarantees its convexity ? In the following theorem we list some known results in this direction.

Theorem 1.2. *Let X be a real t.v.s. and Y a closed subset of X with nonvoid interior. Then the implication "Y is totally supported $\Rightarrow$ Y is convex " is valid whenever one of the following (again not independent) conditions holds:*

(a) X is the Euclidean space $\mathbb{R}^n$ (see [4] *for Y bounded and* [13] *in general).*

(b) X is a prehilbertian space (see [13]*).*

(c) X is a general topological vector space (see [20]*, Th.3)*

The proofs given in [5] and [20] are only sketched, so that we shall present in Section 2 detailed proofs, preparing the proofs of some more general results given in Section 3, namely the cases when the support points are only dense in the boundary of Y or when the topology of X is not a vector toplogy, but merely a vectorial group topology (see[14]).

Infinite dimensional spaces may contain convex closed sets which are not totally supported (*a fortiori* they must have empty interior). Such a set is $Y = \{x \in L^2[0,1] : |x(t)| \leq 1, \text{a.e.} t \in [0,1]\}$ in the Hilbert space $L^2[0,1]$ (see [13, pp. 531–532]). Moreover in [10, pp. 97–98], it is constructed a precompact closed convex set Y in an incomplete inner product space with spt$Y = \emptyset$, and in [3, Corollary 2] it is proved that every normed space of countable algebraic dimension contains such a set. In the last quoted work it is stated also the conjecture: A real normed space is incomplete if and only if it contains a bounded closed convex set Y with spt$Y = \emptyset$. This conjecture is false for non-normable t.v.s. : there exist metrizable complete locally convex spaces containing bounded closed convex sets having no support points (see [16]). Other supportless convex sets in separable normed spaces are constructed in [8, 9].

In Banach spaces the situation radically improves, namely, in [13, Theorem 14] and, subsequently in [2, Theorem 1], it is proved that any closed convex set Y of a real Hilbert space, or of a real Banach space, respectively, satisfies $\overline{\text{spt}Y} = \text{bd}Y$. A subset Y of a t.v.s. is called ***densely supported*** provided $\overline{\text{spt}Y} = \text{bd}Y$. In Section 3 we shall prove that every densely supported closed set with nonvoid interior is convex.

2. Convexity of totally supported sets

We start by a lemma.

Lemma 2.1. *Let X be a t.v.s. over the field $\mathbb{K}$ of real or complex numbers. If $x^* : X \to \mathbb{K}$ is a non-null continuous linear functional and Y a nonvoid open convex subset of X then the set $x^*(Y)$ is open in $\mathbb{K}$.*

*Proof.*Let $\lambda_0 \in x^*(Y)$ and $y_0 \in Y$ be such that $\lambda_0 = x^*(y_0)$. Since $x^* \neq 0$, there exists $x_0 \in X$ with $x^*(x_0) = 1$. As $0x_0 = 0$ and $Y - y_0$ is a neighborhood of $0 \in X$, there exists a number $r > 0$ such that

$$D_r x_0 \subset Y - y_0 \tag{2.1}$$

where $D_r = \{\lambda \in \mathbb{K} : |\lambda| < r\}$. If $\lambda \in D_r$ then, by (2.1), $\lambda x_0 \in Y - y_0$ and

$$\lambda_0 + \lambda = x^*(y_0) + \lambda x^*(x_0) = x^*(y_0) + \lambda x_0) \in x^*(Y)$$

showing that $\lambda_0 + D_r \subset x^*(Y)$.□

Lemma 2.2. *If Y is a subset of a t.v.s. X then spt$Y \subset$ bd$Y$.*

Proof. Suppose there exists a point $x_0 \in$spt$Y\setminus$bd$Y$. It follows the existence of a functional $x^* \in X^*$, $x^* \neq 0$, such that $c = x^*(x_0) = \sup x^*(Y)$. Since $x_0 \in$int$Y$, by Lemma 2.1, there exists $\epsilon > 0$ such that the interval $[c-\epsilon, c+\epsilon]$ is contained in $x^*(Y)$, yielding the contradiction

$$c = \sup x^*(Y) \geq \sup x^*(\text{int}Y) \geq c + \epsilon.$$

Lemma is proved.□

Lemma 2.3. *Let Y be a closed subset of a real t.v.s. X and let $x, y, z \in Y$. If there exist $\alpha, \beta, \gamma > 0$, $\alpha + \beta + \gamma = 1$, such that $\alpha x + \beta y + \gamma z \in \mathrm{spt}Y$ then $x, y, z \in \mathrm{spt}Y$.*

Proof. Supposing $x_0 = \alpha x + \beta y + \gamma z \in \mathrm{spt}Y$ it follows that there exists $x^* \in X^*$, $x^* \neq 0$, such that

$$c = x^*(x_0) = \sup x^*(Y).$$

implying

$$x^*(x - x_0) \leq 0, \qquad x^*(y - x_0) \leq 0, \qquad x^*(z - x_0) \leq 0. \tag{2.2}$$

¿From the identity

$$(\alpha + \beta + \gamma)x^*(x_0) = \alpha x^*(x) + \beta x^*(y) + \gamma x^*(z)$$

one obtains

$$\alpha x^*(x - x_0) + \beta x^*(y - x_0) + \gamma x^*(z - x_0) = 0$$

which, by (2.2), gives

$$x^*(x - x_0) = x^*(y - x_0) = x^*(z - x_0) = 0$$

showing that $x, y, z \in \mathrm{spt}Y$.□

Theorem 2.4. *Let X be a real t.v.s. and Y a closed subset of X with nonvoid interior. If Y is totally supported then Y is convex.*

Proof. Suppose the contrary, i.e. there exist $x, y \in Y$ and $\lambda \in]0,1[$ such that

$$u = \lambda x + (1 - \lambda)y \notin Y \tag{2.3}$$

Let $z \in \mathrm{int}Y$ and let $f : [0,1] \to X$ be defined by $f(t) = tu + (1-t)z$, $t \in [0,1]$. Put

$$t_0 = \sup\{t \in [0,1] : f(t) \in Y\} \tag{2.4}$$

and let $(t_n) \subset]0,1[$ be such that $f(t_n) \in Y$ and $t_n \to t_0$. It follows

$$x_0 = f(t_0) = \lim_{n \leftarrow \infty} f(t_n) \in \overline{Y} = Y$$

and $t_0 < 1$ (by the definition (2.3) of u).

Show now that $x_0 \notin \mathrm{int}Y$. For if contrary, then as Y is a neighborhood of x_0 it would exists $\epsilon > 0$ such that $t_0 + \epsilon < 1$ and $f(t_0 + \epsilon) \in Y$, in contradiction to (2.4).

Therefore

$$x_0 = t_0 u + (1 - t_0)z \in \mathrm{bd}Y = \mathrm{spt}Y. \tag{2.5}$$

By (2.5) and (2.3), x_0 can be written in the form

$$x_0 = t_0(\lambda x + (1 - \lambda)y) + (1 - t_0)z = \alpha x + \beta y + \gamma z \tag{2.6}$$

where

$$\alpha = \lambda t_0 > 0, \quad \beta = (1 - \lambda)t_0 > 0, \text{ and } \gamma = 1 - t_0 > 0.$$

Since $\alpha + \beta + \gamma = 1$, by Lemma 2.3 one obtains $z \in \mathrm{spt}Y = \mathrm{bd}Y$, in contradiction to $z \in \mathrm{int}Y$. □

Remark. The following examples show that the conditions " Y is closed" and "int$Y \neq \emptyset$" are essential for the validity of Theorem 2.4.

The set

$$Y = [0,1] \times [0,1] \setminus \{(x_1, 0) : 0 < x_1 < 1\}$$

contained in $X = \mathbb{R}^2$, is totally supported without being convex.

Also, the set $Y = \{0,1\} \subset \mathbb{R}$ verifies $\mathrm{spt}Y = \mathrm{bd}Y$ and is not convex.

3. The convexity of densely supported sets

Recall that a subset Y of a t.v.s. X is called densely supported if $\overline{\mathrm{spt}Y} = \mathrm{bd}Y$.

Theorem 3.1. *Let X be areal t.v.s. and Y a closed subset of X having nonvoid interior. If Y is densely supported then Y is convex.*

Proof. Supposing the contrary, there exist $x, y \in Y$ and $\lambda \in]0,1[$ such that

$$u = \lambda x + (1 - \lambda)y \notin Y. \tag{3.1}$$

Let $z' \in \mathrm{int}Y$ and let $V = \mathrm{int}Y$. Then V is an open neghborhood of z' and, reasoning like in the proof of Theorem 2.4, one can find a number $t_0 \in]0,1[$ such that

$$x_0' = t_0 + (1 - t_0)z' \in \mathrm{bd}Y. \tag{3.2}$$

It follows $z' = \mu(x_0' - t_0 u)$, where $\mu = (1 - t_0)^{-1} > 1$, so that

$$z' = u + \mu(x_0' - u). \tag{3.3}$$

Since spt$Y$ is dense in bd$Y$, $x_0' \in$ bd$Y$ and $W = x_0' + \mu^{-1}(V - z')$ is a neighborhood of x_0', there exists $x_0 \in W \cap$ spt$Y$, implying

$$x_0 - x_0' \in \mu^{-1}(V - z'). \tag{3.4}$$

Let

$$z = u + \mu(x_0' - u) \tag{3.5}$$

By (3.3),(3.4) and (3.2) we get

$$z - z' = \mu(x_0 - x_0') \in V - z',$$

showing that $z \in V = \text{int}Y$. By (3.5) and (3.2)

$$\alpha x + \beta y + \gamma z = x_0 \in \text{spt}Y$$

where

$$\alpha = \lambda(\mu - 1)/\mu, \qquad \beta = (1 - \lambda)(\mu - 1)/\mu \text{ and } \gamma = 1/\mu.$$

Since $\alpha, \beta, \gamma > 0$ and $\alpha+\beta+\gamma = 1$, by Lemma 2.3 one obtains $z \in \text{spt}Y \subset \text{bd}Y$, in contradiction with $z \in \text{int}Y$, □

Corollary 3.2. *Let X be a real t.v.s. and Y a closed convex subset of X with nonvoid interior. Then the following assertions are equivalent:*

(a) Y is convex;

(b) Y is totally supported;

(c) Y is densely supported.

Proof. The implication $(b) \Rightarrow (c)$ is obvious. The implication $(c) \Rightarrow (a)$ is contained in Theorem 3.1 and the implication $(a) \Rightarrow (b)$ is contained in assertion (c) of Theorem 1.1. □

Remarks

1.Archimedes (see [22]) defined the convexity of a set in $\mathbb{R}^3$ as a totally supported set, a definition which is in concordance with Corollary 3.2.

2. A careful examination of the proofs shows that Lemma 2.3 and Theorems 2.4 and 3.1 remain valid in the case when the topology of the vector space X is not a vectorial topology. Namely, it is sufficient to suppose that

(i) the addition $(x, y) \to x + y$ is continuous from $X \times X$ to X, and

(ii) for every fixed $x \in X$ the multiplication by scalars $(\lambda, x) \to \lambda x$ is a continuous function from $\mathbb{R}$ to X

(see [14] for properties of such spaces called topological vector groups).

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ON THE LEVEL SETS OF (Γ, Ω)-QUASICONVEX FUNCTIONS

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Dedicated to Professor Ioan Purdea at his 60th anniversary

Abstract. The aim of this paper is to show that the characteristic property of the real-valued quasiconvex functions to have convex level sets can be naturally extended in the class of (Γ, Ω)-quasiconvex functions, introduced by us in [5], which in particular contains the cone-quasiconvex vector-valued functions in the sense of Dinh The Luc [3].

1. Preliminaries

Quasiconvex functions play an important role in scalar and vector optimization, their characteristic property to have convex level sets being succesfully explored in order to derive optimality conditions or to study some topological properties of the efficient sets. Some fundamental properties concerning these topics can be found for instance in [2] or [3].

Our study here is based on the concept of (Γ, Ω)-quasiconvexity, introduced by us in [5] in order to describe some common properties of different classes of generalized quasiconvex functions in a unifying way.

For this aim we only need to endoved the domain of the (Γ, Ω)-quasiconvex functions with an abstract convexity induced by a set-valued mapping Γ, and to consider a binary relation Ω in the codomain. In the sequel we consider $\Gamma : E_1 \times E_1 \to 2^{E_1}$ and $\Omega : E_2 \to 2^{E_2}$, where E_1 and E_2 are two arbitrary nonempty sets.

We recall that a subset X of E_1 is said to be Γ-*convex* [5] iff

$$\Gamma(x^1, x^2) \subset X, \ \forall\, x^1, x^2 \in X.$$

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Obviously, the concept of Γ-convexity permits an unifying treatement of those notions of generalized convexity in which the line segments determined by two points are replaced by a continuous arc or by a discret subset of the domain.

On the other hand, when the codomain E_2 is not endoved with linear or topological structure, we shall need to replace the preordering induced by a pointed convex cone by an arbitrary binary relation Ω. Throughout the paper, this relation will be identified with the set-valued mapping $\Omega : E_2 \to 2^{E_2}$ defined by $\Omega y = \{y' \in E_2 \mid (y,y') \in \Omega\}$, $\forall\, y \in E_2$. We shall also associate to Ω the following relations: $\Omega^- y = \{y' \in E_2 \mid y \in \Omega y'\}$ and $\Omega^c y = E_2 \setminus (\Omega y)$, $\forall\, y \in E_2$.

Given a nonempty subset Y of E_2 we denote by $\Omega Y = \cup\{\Omega y \mid y \in Y\}$ the first order section of Y in the sense of J. Riguet [7] and by $[\Omega]Y = \cap\{\Omega y :\ y \in Y\}$ the second order section of Y, which is nowadays known as the *polar set* of Y.

By means of composite polarities, S. Dolecki and Ch. Malivert [1] have introduced the *cyrtological closure* operator $\mathrm{cl}_{\Omega^-} : 2^{E_2} \to 2^{E_2}$ defined by

$$\mathrm{cl}_{\Omega^-} Y = [\Omega][\Omega^-]Y,\ \forall\, Y \subset E_2.$$

As we shall see, the concepts of Γ-convexity and cyrtological closure are the key tools that we need to derive the main results of this work.

2. The characterization of the (Γ, Ω)-quasiconvexity by means of dominant level sets

Let us now recall [5] the definition of the (Γ, Ω)-quasiconvex functions:

Definition 2.1. Let $X \subset E_1$ be a nonempty and Γ-convex set. A function $f : X \to E_2$ is said to be (Γ, Ω)-*quasiconvex* on X if

$$f(\Gamma(x^1,x^2)) \subset \mathrm{cl}_{\Omega^-}\{f(x^1), f(x^2)\},\ \forall\, x^1,x^2 \in X.$$

This definition calls for a few comments:

i) It is easy to see that f is (Γ, Ω)-*quasiconvex* on X if and only if

$$\forall\, x^1,x^2 \in X,\ \forall\, y \in E_2,\ f(\{x^1,x^2\}) \subset \Omega y \ \Rightarrow\ f(\Gamma(x^1,x^2)) \subset \Omega y. \quad (1)$$

ii) The terminology used in the above definition is relative; in fact, this definition concerns quasiconvexity as well as quasiconcavity, since we can interchange Ω with Ω^-.

For instance, if E_1 and E_2 are linear spaces and C is a convex cone in E_2, then for Γ and Ω defined by

$$\Gamma(x^1, x^2) = \text{co}\{x^1, x^2\} = \{tx^1 + (1-t)x^2 \mid t \in [0,1]\}, \ \forall\, x^1, x^2 \in E_1$$

and

$$\Omega y = y - C, \ \forall\, y \in E_2,$$

the (Γ, Ω)-quasiconvexity coincides with the cone-quasiconvexity in the sense of Dinh The Luc [3, 4].

It is known that if the euclidean space $E_2 = \mathbb{R}^n$ is partially ordered by the positive cone $C = \mathbb{R}^n_+$ then a vector-valued function $f = (f_1, \ldots, f_n) : X \to \mathbb{R}^n$ is (Γ, Ω)-quasiconvex if and only if their scalar components $f_1, \ldots, f_n$ are quasiconvex in the usual sense. Obviously, if we replace C by $-C$ then f is (Γ, Ω)-quasiconvex if and only if $f_1, \ldots, f_n$ are quasiconcave in the usual sense.

Definition 2.2. Let $f : X \to E_2$ be a function defined on a nonempty subset X of E_1. Given $y \in E_2$, the set

$$L_f(y) = \{x \in X \mid f(x) \in \Omega y\} \tag{2}$$

is called the *level set* of f corresponding to the *level* y.

The (Γ, Ω)-quasiconvexity can be characterized by means of these level sets as follows:

Proposition 2.1. *If the function $f : X \to E_2$ is defined on a nonempty and Γ-convex set $X \subset E_1$, then the following assertions are equivalent:*

i) f is (Γ, Ω)-quasiconvex on X;

ii) $L_f(y)$ is a Γ-convex set, $\forall y \in E_2$.

Proof. The implication i) $\Rightarrow$ ii) follows directly from the above definitions.

To prove the converse implication, let $x^1, x^2 \in X$ and $y \in E_2$ be such that $f(\{x^1, x^2\}) \subset \Omega y$. Then x^1, $x^2 \in L_f(y)$ and by ii) we obtain $\Gamma(x^1, x^2) \subset L_f(y)$, i.e. $f(\Gamma(x^1, x^2)) \subset \Omega y$. Using the relation (1), we conclude that f is (Γ, Ω)-quasiconvex on X. □

In what follows we shall refine this result by taking account of the following categories of level sets:

Definition 2.3. Let $f : X \to E_2$ be a function defined on a nonempty subset X of E_1 and let $y \in E_2$. We shall say that $L_f(y)$ given by (2) is:

i) a *dominant level set*, if $y \in \Omega f(X)$;

ii) an *attainable level set*, if $y \in f(X)$.

Obviously, if Ω is reflexive then every attainable level set is a dominant one.

Remark 2.1. The above definition is motivated by the fact that real-valued quasiconvex functions have some special properties which cannot be extended in the general case of (Γ, Ω)-quasiconvex functions without strong assumptions on their codomain. Indeed, a real-valued function $f : X \to \mathbb{R}$ is quasiconvex on a convex nonempty subset X of a linear space if and only if all their attainable level sets are convex. As shown by Example 3.2 this property fails to be true even in the case of cone-quasiconvex vector functions.

The following result show that we can although characterize the (Γ, Ω)-quasiconvexity using only dominant level sets:

Corollary 2.1. *Let $X \subset E_1$ be a nonempty and Γ-convex set. A function $f : X \to E_2$ is (Γ, Ω)-quasiconvex on X if and only if their dominant level sets are Γ-convex.*

Proof. Follows immediately from Proposition 2.1, because for any point $y \in E_2 \setminus \Omega^- f(X)$ the corresponding level set $L_f(y)$ is empty. □

3. The characterization of the (Γ, Ω)-quasiconvexity by means of attainable level sets

In order to obtain some characterizations of the (Γ, Ω)-quasiconvex functions in terms of attainable level sets, the following preliminary result will be usefull:

Lemma 3.1. *Let $f : X \to E_2$, where X is a nonempty subset of E_1. If $\Omega \subset E_2 \times E_2$ is transitive, then the set-valued mapping $L_f : E_2 \to 2^{E_2}$ given by (2) is isotone, i.e.*

$$L_f(y_1) \subset L_f(y_2),\ \forall\, y_1, y_2 \in E_2,\ y_1 \in \Omega y_2.$$

Proof. Let $y_1, y_2 \in E_2$ such that $y_1 \in \Omega y_2$ and let $x \in L_f(y_1)$. Then, by definition of L_f we have $f(x) \in \Omega y_1$ and therefore $f(x) \in \Omega y_1$, since Ω is transitive. □

Remark 3.1. The assumption on the transitivity of Ω in Lemma 3.1 is essential. This is illustrated by the following example:

Example 3.1. Consider $E_1 = E_2 = \mathbb{R}^2$, $\Gamma(x^1, x^2) = \mathrm{co}\{x^1, x^2\}$, $\forall x^1, x^2 \in \mathbb{R}^2$ and let Ω be given by $\Omega y = y + C$, $\forall y \in \mathbb{R}^2$, where $C = \mathbb{R}^2 \setminus (\mathbb{R}_+^*)^2$. Obviously, C is a closed non convex cone and therefore the induced relation Ω is reflexive but it is not transitive.

If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is the identic function on $\mathbb{R}^2$, i.e. $f(x) = x$, $\forall\, x \in \mathbb{R}^2$, it is easy to see that the mapping L_f is given by

$$L_f(y) = \{x \in \mathbb{R}^2 \mid x \in \Omega y\} = y + C, \ \forall y \in \mathbb{R}^2$$

and it is not isotone. In fact, for $y^1 = (0,1), y^2 = (1,0)$ and we have $y^1 \in \Omega y^2$, but $y^0 = (2,1) \in L_f(y_1) \setminus L_f(y_2)$.

Proposition 3.1. *Let X be a nonempty and Γ-convex subset of E_1 and $f : X \to E_2$. If Ω is a complete preordering in E_2, i.e. $(\Omega \cup \Omega^-)(y) = E_2$, $\forall\, y \in E_2$, then the following assertions are equivalent:*

i) f is (Γ, Ω)-quasiconvex on X;

ii) $L_f(f(x))$ is Γ-convex, $\forall\, x \in X$.

Proof. The implication i) $\Rightarrow$ ii) is a simple consequence of Proposition 2.1.

To prove the converse implication, suppose that ii) is true and consider some arbitrary points $y \in E_2$ and $x^1, x^2 \in L_f(y)$. Since Ω is a complete relation we can suppose, without loss of generality, that $f(x^1) \in \Omega f(x^2)$. Moreover, Ω being reflexive, we have $x^1, x^2 \in L_f(f(x^2))$ and therefore $\Gamma(x^1, x^2) \subset L_f(f(x^2))$ according to assumption ii).

On the other hand, we have $f(x^2) \in \Omega y$. Since Ω is transitive, by Lemma 3.1 we can conclude that $L_f(f(x^2)) \subset L_f(y)$ and consequently $\Gamma(x^1, x^2) \subset L_f(y)$. The assertion i) follows than again from Proposition 2.1. □

Remark 3.2. Without the assumption on the completeness of Ω, the implication ii) $\Rightarrow$ i) in Proposition 3.1 fails to be true even in the class of cone-quasiconvex vector-valued functions, as shown by the following example:

Example 3.2. Consider $E_1 = \mathbb{R}$, $E_2 = \mathbb{R}^2$, $\Gamma(x^1, x^2) = \mathrm{co}\{x^1, x^2\}$, $\forall x^1, x^2 \in \mathbb{R}$ and $\Omega y = y - C$, $\forall y \in \mathbb{R}^2$, where $C = \mathbb{R}^2_+$. Remark that in this case Ω is reflexive and transitive, but it is not complete in $\mathbb{R}^2$.

It is easy to see that the function $f : X = [0,1] \to \mathbb{R}^2$, defined by

$$f(x) = \begin{cases} (x, 1-x) & \text{if} \quad x \in]0,1] \setminus \{1/2\} \\ (1/2, 1/2) & \text{if} \quad x = 0 \\ (0,1) & \text{if} \quad x = 1/2. \end{cases}$$

satisfies the condition ii) in Proposition 3.1, since for any point $x \in X$, $L_f(f(x)) = \{x\}$ is a convex set, but f is not (Γ, Ω)-quasiconvex on X, because for $y = (3/4, 3/4)$, the level set $L_f(y) = \{0\} \cup [1/4, 3/4] \setminus \{1/2\}$ is not convex.

Theorem 3.1. *Let $X \subset E_1$ be a nonempty and Γ-convex set and $f : X \to E_2$. If Ω is a complete preordering in E_2 then the following assertions are equivalent:*

i) f is (Γ, Ω)-quasiconvex on X;

ii) The set-valued mapping $L_f \circ f : X \to 2^X$ is $(\Gamma, \supset)$-quasiconvex on X.

Proof. We first notice that statement ii) can be rewritten as follows:

$$L_f(f(\Gamma(x^1, x^2))) \subset L_f(f(\{x^1, x^2\})), \ \forall x^1, x^2 \in X. \tag{3}$$

Indeed, by (1) the function $L_f \circ f$ is $(\Gamma, \supset)$-quasiconvex on X if and only if

$$\forall \, x^1, x^2 \in X, \ \forall \, Y \in 2^X, Y \supset L_f(f(\{x^1, x^2\})) \Longrightarrow Y \supset L_f(f(\Gamma(x^1, x^2))).$$

Suppose now that f is (Γ, Ω)-quasiconvex on X and consider two arbitrary points $x^1, x^2 \in X$. By the completeness of Ω we can suppose, without loss of generality, that $f(x^1) \in \Omega f(x^2)$ i.e. $x^1 \in L_f(f(x^2))$. On the other hand, since Ω is reflexive, we have also $x^2 \in f(\Gamma(x^1, x^2)) \in \Omega f(x^2)$ and hence $\Gamma(x^1, x^2) \subset L_f(f(x^2))$ i.e. $f(\Gamma(x^1, x^2)) \subset \Omega f(x^2)$. Using the transitivity of Ω we obtain

$$L_f(f(\Gamma(x^1, x^2))) \subset L_f(f(x^2)) \subset L_f(f(\{x^1, x^2\}))$$

and hence condition (3) is fulfilled.

Conversely, if $L_f \circ f$ is $(\Gamma, \supset)$-quasiconvex on X then using Lemma 3.1 we conclude that for any points $y \in E_2$ and $x^1, x^2 \in L_f(y)$ we have

$$L_f(f(x^i)) \subset L_f(y), \ \forall \, i \in \{1, 2\}.$$

On the other hand, the reflexivity of Ω implies that $\Gamma(x^1, x^2) \subset L_f(f(\Gamma(x^1, x^2)))$ and using the assumption (3) we finally infer that $\Gamma(x^1, x^2) \subset L_f(y)$, $\forall \, y \in E_2$, which means that f is (Γ, Ω)-quasiconvex on X. □

Remark 3.3. Even if the implication ii) $\Rightarrow$ i) in Theorem 3.1 is valid without the completeness assumption on Ω, this assumption cannot be dropped for the converse implication, as we can see from the following example:

Example 3.3. Consider $E_1 = \mathbb{R}, X = [0,1]$ and let Γ and Ω be given by

$$\Gamma(x^1,x^2) = [\min\{x^1,x^2\}, \max\{x^1,x^2\}],\ \forall x^1,x^2 \in \mathbb{R} \text{ and } \Omega y = y - \mathbb{R}^2_+,\ \forall y \in \mathbb{R}^2.$$

It is easy to see that the function $f : X \to \mathbb{R}^2$ defined by $f(x) = (x, 1-x)$, $\forall\, x \in X$ is (Γ,Ω)-quasiconvex on X because f has Γ-convex level sets:

$$L_f(y) = \begin{cases} \{x\} & \text{if} \quad f(x) \in \Omega y \\ \emptyset & \text{if} \quad f(x) \in \Omega^c y. \end{cases}$$

On the other hand, we can see that the function $L_f \circ f$ is not $(\Gamma,\supset)$-quasiconvex on X because for $x^1 = 0$ and $x^2 = 1$ we have

$$L_f(f(\Gamma(x^1,x^2))) = L_f(f([0,1])) = [0,1] \not\subset L_f(f(\{x^1,x^2\})) = L_f(\{0,1\}) = \{0,1\}.$$

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DATA DEPENDENCE OF THE FIXED POINTS SET OF WEAKLY PICARD OPERATORS

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Dedicated to Professor Ioan Purdea at his 60^{th} anniversary

Abstract. Data dependence in case of the weakly Picard operators is given.

1. Introduction.

Let X be a nonempty set and $f : X \to X$ an operator. We will use the notation

$F_f = \{x \in X \mid f(x) = x\}$, the fixed points set of f;

$$O_f(x;n) = \{x, f(x), f^2(x), ..., f^n(x)\}$$

$O_f(x) = \{x, f(x), f^2(x), ..., f^n(x), ...\}$, the orbit of $x \in X$;

$$P(X) = \{A \subseteq X \mid A \neq \emptyset\}.$$

For a metric space (X, d) we have

$\delta(A) = \sup\{d(a,b) \mid a, b \in A\}$, the diameter of $A \in P(X)$;

$P_{b,cl}(X) = \{A \in P(X) \mid A$ is bounded and closed $\}$;

$H : P_{b,cl}(X) \times P_{b,cl}(X) \to \mathbb{R}_+$, $H(A,B) = \max(\sup\limits_{a\in A} \inf\limits_{b\in B} d(a,b), \sup\limits_{b\in B} \inf\limits_{a\in A} d(a,b))$, the Hausdorff- Pompeiu distance on $P_{b,cl}(X)$ set.

$H : P_{cl}(X) \times P_{cl}(X) \to \mathbb{R}_+ \cup \{+\infty\}$ - the generalized Hausdorff - Pompeiu distance.

$C(X) = \{f : X \to X \mid f$ is continous operators $\}$

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function.

Definition 1. *φ is a strict comparison function if φ satisfies the following:*

i) φ is continous;

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ii) φ *is monotone increasing ;*

iii) $\varphi^n(t) \xrightarrow{n\to\infty} 0$, *for all* $t > 0$;

iv) $t - \varphi(t) \xrightarrow{t\to\infty} \infty$;

Let (X, d) be a metric space and $f : X \to X$ an operator.

Definition 2. *The operator f is called weakly Picard if the sequence $(f^n(x))_{n\geq 1}$ converges for all $x \in X$ to a fixed point of f, which will be denote by $f^\infty(x)$.*

For more details about the weakly Picard operators see [2], [3] [4].

Definition 3. *The operator f is called a strict $\varphi-$ contraction if :*

i) φ *is a strict comparison function;*

ii) $d(f(x), f(y)) \leq \varphi(d(x, y))$, *for all* $x, y \in X$.

About the strict φ - contractions we have the next

Theorem 1. *Let (X, d) be a metric space, $f : X \to X$ a strict φ - contraction and $x \in X$. Then*

i) $d(f^i(x), f^j(x)) \leq \varphi(\delta(O_f(x; n)))$, *for all* $i, j \in \{1, 2, ..., n\}$ *with* $i < j$;

ii) for each $n \in \mathbb{N}$ *exists* $p \in \mathbb{N}$, *such that* $\delta(O_f(x; n)) = d(x, f^p(x))$;

iii) $\delta(O_f(x; n)) \leq \tau_{d(x, f(x))}$ *for each* $n \in \mathbb{N}$,

where $\tau_{d(x,f(x))} = \sup\{t \mid t - \varphi(t) \leq d(x, f(x))\}$;

For more details and results see [1],[2].

The aim of this paper is to give an answer to the following

PROBLEM "Let (X, d), be a metric space and $f, g : X \to X$ two weakly Picard operators.If exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta$, for any $x \in X$, estimate the "distance" between F_f and F_g."

2. Main results.

Lemma 2. *Let (X, d) be a metric space and $f, g : X \to X$ two weakly Picard operators.Then* $H(F_f, F_g) \leq \max(\sup_{x\in F_g} d(x, f^\infty(x)), \sup_{x\in F_f} d(x, g^\infty(x)))$.

Proof. We remark that $f^\infty(x) \in F_f$ and $g^\infty(x) \in F_g$. The proof follows from the definition of H.

Theorem 3. *Let (X,d) be a complete metric space, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a strict comparison function and $f, g : X \to X$ two orbitaly continous operators. We suppose that:*

i) $d(f(x), f^2(x)) \leq \varphi(d(x, f(x))$, for any $x \in X$ and

$d(g(x), g^2(x)) \leq \varphi(d(x, g(x))$, for any $x \in X$;

ii) there exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta$, for any $x \in X$.

Then:

a) f and g are weakly Picard operators;

b) $H(F_f, F_g) \leq \tau_\eta$, where $\tau_\eta = \sup\{t \mid t - \varphi(t) \leq \eta\}$.

Proof. a) Let $x \in X$ and $i, j \in \mathbb{N}$ with $i < j$.

We have $d(f^i(x), f^j(x)) \leq \varphi(d(f^{i-1}(x), f^{j-1}(x))) \leq$

$\ldots \leq \varphi^i(d(x, f^{j-i}(x)) \leq \varphi^i(\delta(O_f(x; j-i))) \leq \varphi^i(\tau_{d(x,f(x))})$.

Finaly, if we put $i = n$, $j = n+p$, we obtain

$d(f^n(x), f^{n+p}(x)) \leq \varphi^n(\tau_{d(x,f(x))}) \xrightarrow{n\to\infty} 0$.

Hence $(f^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence and $f^\infty(x)$ will be the limit of it. Because f is orbitaly continous then $f^\infty(x) \in F_f$.

In the inequality $d(f^n(x), f^{n+p}(x)) \leq \varphi^n(\tau_{d(x,f(x))})$ if we take $\lim\limits_{p\to\infty}$ we obtain that $d(f^n(x), f^\infty(x)) \leq \varphi^n(\tau_{d(x,f(x))})$, for any $n \in \mathbb{N}$.

Similarly, for any $x \in X$, we have the convergence of $(g^n(x))_{n\in\mathbb{N}}$ and $g^\infty(x)$, the limit of this sequence, has two properties:

$g^\infty(x) \in F_g$ and $d(g^n(x), g^\infty(x)) \leq \varphi^n(\tau_{d(x,g(x))})$, for any $n \in \mathbb{N}$.

b) From the estimation $d(f^n(x), f^\infty(x)) \leq \varphi^n(\tau_{d(x,f(x))})$, which is true for any $x \in X$ and $n \in \mathbb{N}$, we obtain for $n = 0$,that $d(x, f^\infty(x)) \leq \tau_{d(x,f(x))}$ (*).

By a similar argument we have that $d(x, g^\infty(x)) \leq \tau_{d(x,g(x))}$ (**)

From (*), (**) and ii) it follows

$$d(x, f^\infty(x)) \leq \tau_\eta, \text{ for any } x \in F_g \text{ and}$$
$$d(x, g^\infty(x)) \leq \tau_\eta, \text{ for any } x \in F_f\ ,$$

we apply Lemma 2.1. ∎

As a consequence of the Theorem 2.2 we have

Theorem 4. *Let (X, d) be a complete metric space, $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ a strict comparison function and $f_n, f : X \longrightarrow X, n \in \mathbb{N}$ orbitaly continous operators.*

We suppose that:

i) $d(f(x), f^2(x)) \leq \varphi(d(x, f(x))$, *for any* $x \in X$;

ii) $d(f_n(x), f_n^2(x)) \leq \varphi(d(x, f_n(x))$, *for any* $x \in X$ *and* $n \in \mathbb{N}$;

iii) $(f_n)_{n\in\mathbb{N}}$ *converges uniformly to* f.

Then

a) f *and* f_n, $n \in \mathbb{N}$, *are weakly Picard operators;*

b) $H(F_{f_n}, F_f) \overset{n\to\infty}{\longrightarrow} 0$;

Remark 1. *,If we take* $\varphi(t) = at$, *with* $a \in [0, 1[$, *from the Theorem 2.2 we have*

Theorem 5. *Let (X, d) be a complete metric space and $f, g : X \to X$ two orbitaly continous operators. We suppose that*

i) $d(f^2(x), f(x)) \leq ad(x, f(x))$, *for any* $x \in X$ *and*

$d(g^2(x), g(x)) \leq ad(x, g(x))$, *for any* $x \in X$;

ii) there exists $\eta > 0$ *such that* $d(f(x), g(x)) \leq \eta$, *for any* $x \in X$.

Then $H(F_f, F_g) \leq \frac{\eta}{1-a}$.

3. Applications.

Let $K_1, K_2 \in C([a, b] \times [a, b] \times \mathbb{R})$. We consider the following integral equations with deviating argument:

$$x(t) = x(a) + \int_a^b K_1(t, s, x(s))ds, \ t \in [a, b] \ (1)$$

$$x(t) = x(a) + \int_a^b K_2(t, s, x(s))ds, \ t \in [a, b] \ (2)$$

By the theorem 2.3 we have

Theorem 6. *We suppose that:*

i) $K_i(a, s, u) = 0$, *for any* $x \in [a, b]$, $s, u \in \mathbb{R}$; $(i = 1, 2)$

ii) there exists $\eta > 0$ *such that*

$| K_1(t, s, u) - K_2(t, s, u) | \leq \eta$, *for all* $t, s \in [a, b]$ *and* $u \in \mathbb{R}$;

iii) there exists $L > 0$ *such that*

$| K(t, s, u) - K(t, s, v) | \leq L | u - v |$, *for all* $t, s \in [a, b]$ *and* $u, v \in \mathbb{R}$, $(i = 1, 2)$

iv) $L(b-a) < 1$;

Let S_{K_i} be the solutions set of the equations (i) in $C[a,b]$ such that

$x(a) \in [\alpha, \beta]$ ($i = 1, 2$).

Then

a) $S_{K_i} \neq \emptyset$, $i = 1, 2$;

b) $H_{\|\cdot\|}(S_{K_1}, S_{K_2}) \leq \frac{\eta(b-a)+\beta-\alpha}{1-L(b-a)}$, *where by* $H_{\|\cdot\|}$ *we denote the Hausdorff - Pompeiu metric with respect to Cebyshev norm on* $C[a,b]$.

Proof. We consider the operators $f, g : C[a,b] \to C[a,b]$ defined by

$$f(x)(t) = x(a) + \int_a^b K_1(t, s, x(s))ds \text{ and}$$

$$g(x)(t) = x(a) + \int_a^b K_2(t, s, x(s))ds.$$

It is clear that we have

$$\|f(x) - f^2(x)\| \leq L(b-a)\|x - f(x)\|, \text{ for all } x \in C[a,b] ,$$

$$\|g(x) - g^2(x)\| \leq L(b-a)\|x - g(x)\|, \text{ for all } x \in C[a,b]$$

and

$$\|f(x) - g(x)\| \leq \beta - \alpha + (b-a)\eta,$$

we apply now the theorem 2.3. ∎

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ASYMPTOTYC FORMULAE CONCERNING ARITHMETICAL FUNCTIONS DEFINED BY CROSS-CONVOLUTIONS , VII. DISTRIBUTION OF A-SEMI-k-FREE INTEGERS

LÁSZLÓ TÓTH

Dedicated to Professor Ioan Purdea at his 60^{th} anniversary

Abstract. We define the A-semi-k-free integers as a common generalization of the k-free integers (i.e. integers not divisible by the k-th power of any prime) and of the semi-k-free integers (i.e. integers not divisible unitarily by the k-th power of any prime) in terms of Narkiewicz's regular A-convolutions. We establish asymptotic formulae for the number of A-semi-k-free integers $\leq x$ with and without assuming the Riemann hypothesis if A is a cross-convolution, investigated in our previous papers.

1. Introduction

Let A be a mapping from the set $\mathbb{N}$ of positive integers to the set of subsets of $\mathbb{N}$ such that $A(n) \subseteq D(n)$ for each n, $D(n)$ denoting the set of all (positive) divisors of n. The A-convolution of arithmetical functions f and g is given by

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d)g(n/d). \tag{1}$$

W.Narkiewicz [Nar63] defined an A-convolution to be regular if

(α) the set of arithmetical functions is a commutative ring with unity with respect to ordinary addition and the A-convolution,

(β) the A-convolution of multiplicative functions is multiplicative,

(γ) the function I, defined by $I(n) = 1$ for all $n \in \mathbb{N}$, has an inverse μ_A with respect to the A-convolution and $\mu_A(p^a) \in \{-1, 0\}$ for every prime power p^a.

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For example, the Dirichlet convolution D, where $D(n) = \{d \in \mathbb{N} : d|n\}$, and the unitary convolution U, where $U(n) = \{d \in \mathbb{N} : d||n\} = \{d \in \mathbb{N} : d|n, (d, n/d) = 1\}$, are regular.

It can be proved, see [Nar63], that an A-convolution is regular if and only if

(i) $A(mn) = \{de : d \in A(m), e \in A(n)\}$ for every $m, n \in \mathbb{N}, (m, n) = 1$,

(ii) for every prime power p^a there exists a divisor $t = t_A(p^a)$ of a, called the type of p^a with respect to A, such that $A(p^{it}) = \{1, p^t, p^{2t}, ..., p^{it}\}$ for every $i \in \{0, 1, ..., a/t\}$.

The elements of the set $A(n)$ are called the A-divisors of n. For other properties of regular convolutions see also P. J. McCarthy [McC86] and V. Sita Ramaiah [Sit78].

We say that A is a *cross-convolution* if for every prime p we have either $t_A(p^a) = 1$, i.e. $A(p^a) = \{1, p, p^2, ..., p^a\} \equiv D(p^a)$ for every $a \in \mathbb{N}$ or $t_A(p^a) = a$, i.e. $A(p^a) = \{1, p^a\} \equiv U(p^a)$ for every $a \in \mathbb{N}$. Let $P_A = P$ and $Q_A = Q$ be the sets of the primes of the first and second kind of above, respectively, where $P \cup Q = \mathbb{P}$ is the set of all primes. For $P = \mathbb{P}$ and $Q = \emptyset$ we have the Dirichlet convolution D and for $P = \emptyset$ and $Q = \mathbb{P}$ we obtain the unitary convolution U.

Furthermore, let (P) and (Q) denote the multiplicative semigroups generated by $\{1\} \cup P$ and $\{1\} \cup Q$, respectively. Every $n \in \mathbb{N}$ can be written uniquely in the form $n = n_P n_Q$, where $n_P \in (P), n_Q \in (Q)$.

If A is a cross-convolution, then

$$A(n) = \{d \in \mathbb{N} : d|n, (d, n/d) \in (P)\} \tag{2}$$

and (1) can be written in the form

$$(f *_A g)(n) = \sum_{\substack{d|n \\ (d,n/d) \in (P)}} f(d)g(n/d). \tag{3}$$

Asymptotic properties of arithmetical functions defined by cross-convolutions were investigated by us in [T97-i], [T-ii], [T-iii], [T-vi].

In this paper we define the A-semi-k-free integers as a common generalization of the k-free integers and of the semi-k-free integers (i.e. integers not divisible by the k-th power of any prime and not divisible unitarily by the k-th power of any prime, respectively) in terms of regular A-convolutions.

In case of a cross-convolution A we establish an asymptotic formula for the number of A-semi-k-free integers $\leq x$ (Theorem 4), in fact we deduce a slightly more general result concerning A-semi-k-S-free integers (Theorem 3), defined with the aid of an arbitrary subset S of $\mathbb{N}$, and we improve the order of the error term given by Theorem 4, using some known estimates regarding the Möbius function, with and without assuming the Riemann hypothesis.

Our results generalize and unify the corresponding known results concerning k-free integers and semi-k-free integers, see [W63], [SurSi73a], [SurSi73b].

2. A-semi-k-free integers

Let A be a regular convolution and let $k \in \mathbb{N}, k \geq 2$. We say that $n \in \mathbb{N}$ is A-semi-k-free if there exists no prime p such that $p^k \in A(n)$. The integer 1 is A-semi-k-free for every A and for every k.

Let the canonical prime power factorization of $n \in \mathbb{N}, n > 1$ be

$$n = p_1^{a_1} p_2^{a_2} ... p_r^{a_r}. \tag{4}$$

The integer $n > 1$ is D-semi-k-free if it is k-free, i.e. $a_i < k$ for every $i \in \{1, 2, ..., r\}$. Furthermore, $n > 1$ is U-semi-k-free if it is semi-k-free, i.e. $a_i \neq k$ for every $i \in \{1, 2, ..., r\}$, this concept was introduced by D. SURYANARAYANA [Sur71].

From (i) and (ii) it follows that $n > 1$ is A-semi-k-free if $t_i \neq k, 2t_i \neq k, ..., s_i t_i = a_i \neq k$ for every $i \in \{1, 2, ..., r\}$, where $t_i = t_A(p_i^{a_i})$.

Let $q_{A,k}, q_{D,k} \equiv q_k$ and $q_{U,k} \equiv q_k^*$ denote the characteristic functions of the sets of A-semi-k-free integers, k-free integers and semi-k-free integers, respectively.

Remark 1. If A is a cross-convolution, then $q_{A,k}(n) = q_k(n_P) q_k^*(n_Q)$ for every $n \in \mathbb{N}$. Hence if A is a cross-convolution, then n is A-semi-k-free if and only if n_P is k-free and n_Q is semi-k-free.

Theorem 1. *If A is a cross-convolution and $k \in \mathbb{N}, k \geq 2$, then*

$$q_{A,k}(n) = \sum_{d^k \in A(n)} \mu(d) = \sum_{\substack{d^k e = n \\ (d,e) \in (P)}} \mu(d) \tag{5}$$

holds for every $n \in \mathbb{N}$, where μ is the Möbius function.

Proof. Taking into account the multiplicativity it is sufficient to prove (5) for $n = p^a$, a prime power. Let $F(n) = \sum_{d^k \in A(n)} \mu(d)$. If $p \in P$, then

$$F(n) = \sum_{d^k | p^a} \mu(d) = \begin{cases} \mu(1) = 1, \text{if } a < k \\ \mu(1) + \mu(p) = 0, \text{if } a \geq k \end{cases} = q_k(p^a) = q_{A,k}(p^a).$$

If $p \in Q$, then

$$F(n) = \sum_{d^k || p^a} \mu(d) = \begin{cases} \mu(1) = 1, \text{if } a \neq k \\ \mu(1) + \mu(p) = 0, \text{if } a = k \end{cases} = q_k^*(p^a) = q_{A,k}(p^a).$$

Hence $F(n) = q_{A,k}(n)$ for every $n \in \mathbb{N}$. Observe that, using (2), $d^k \in A(n)$ if and only if $d^k e = n$ and $(d, e) \in (P)$, which completes the proof. □

If S is an arbitrary subset of $\mathbb{N}$, we say that $n \in \mathbb{N}$ is semi-k-S-free if $n = 1$ or $n > 1$ and no exponent in (4) is of the form kb, where $b \in S$.

Furthermore, in case of a cross-convolution A we say that $n \in \mathbb{N}$ is A-semi-$k-S$-free if n_P is k-free and n_Q is semi-$k - S$-free.

Let $q_{A,k,S}$ denote the characteristic function of the A-semi-k-S-free integers and define the multiplicative function $\mu_{(A,S)}$ by

$$\mu_{(A,S)}(p^a) = \begin{cases} -1, & \text{if } p \in P, a = 1 \text{ or } p \in Q, a \in S, \\ 0, & \text{otherwise,} \end{cases}$$

for every prime power p^a.

The proof of the following formula is similar to the proof of Theorem 1.

Theorem 2. *If A is a cross-convolution, $k \in \mathbb{N}, k \geq 2$ and $S \subseteq \mathbb{N}$, then*

$$q_{A,k,S}(n) = \sum_{d^k \in A(n)} \mu_{(A,S)}(d) = \sum_{\substack{d^k e = n \\ (d,e) \in (P)}} \mu_{(A,S)}(d) \tag{6}$$

holds for every $n \in \mathbb{N}$.

Observe that for $S = \{1\}$ we reobtain the A-semi-k-free integers and $\mu_{(A,\{1\})} \equiv \mu$ for every A. For $S = \{1, 2, ..., r\}$ we obtain a direct generalization of the k-skew integers of rank r, cf. E. COHEN [Co61]. In case $S = \mathbb{N}$ we have the $A - k$-free integers, i.e. integers with no k-th power A-divisors > 1, introduced by us in [T-vi] and $\mu_{(A,\mathbb{N})} \equiv \mu_A$.

3. Asymptotic formulae

In what follows all the constants implied by the O-symbols are independent of x and u.

For a cross-convolution A let

$$\zeta_P(s) \equiv \sum_{\substack{n=1\\ n\in(P)}}^{\infty} \frac{1}{n^s} = \prod_{p\in P}(1-\frac{1}{p^s})^{-1}, \quad s>1.$$

First we prove the following formula.

Theorem 3. *If A is a cross-convolution, $k\in\mathbb{N}, k\geq 2$ and $S\subseteq\mathbb{N}$, then*

$$\sum_{n\leq x} q_{A,k,S}(n) = \alpha_{A,k,S}x + O(x^{1/k}),$$

where

$$\alpha_{A,k,S} = \frac{1}{\zeta_P(k)}\prod_{p\in Q}(1-(1-\frac{1}{p})\sum_{a\in S}\frac{1}{p^{ka}}).$$

Proof. Adopting the method of [Co64], using (6) and the estimate

$$(7) \qquad \sum_{\substack{n\leq x\\ (n,u)\in(P)}} 1 = \frac{\phi(u_Q)x}{u_Q} + O(x^{\varepsilon}\sigma^*_{-\varepsilon}(u)), \quad 0\leq\varepsilon<1,$$

where ϕ is Euler's function and $\sigma^*_r(u)$ denotes the sum of r-th powers of the unitary divisors of u, cf. [T97-i], Lemma 7, we have

$$\sum_{n\leq x} q_{A,k,S}(n) = \sum_{\substack{d^k e=n\leq x\\ (d,e)\in(P)}} \mu_{(A,S)}(d) = \sum_{d\leq\sqrt[k]{x}} \mu_{(A,S)}(d) \sum_{\substack{e\leq x/d^k\\ (e,d)\in(P)}} 1$$

$$= \sum_{d\leq\sqrt[k]{x}} \mu_{(A,S)}(d)\left(\frac{\phi(d_Q)x}{d_Q d^k} + O\left(\left(\frac{x}{d^k}\right)^{\varepsilon}\sigma^*_{-\varepsilon}(d)\right)\right)$$

$$= x\sum_{d\leq\sqrt[k]{x}} \frac{\mu_{(A,S)}(d)\phi(d_Q)}{d^k d_Q} + O\left(x^{\varepsilon}\sum_{d\leq\sqrt[k]{x}}\frac{\sigma^*_{-\varepsilon}(d)}{d^{k\varepsilon}}\right)$$

$$= x\sum_{n=1}^{\infty}\frac{\mu_{(A,S)}(n)\phi(n_Q)}{n^k n_Q} + O(x\sum_{d>\sqrt[k]{x}}\frac{1}{d^k}) + O\left(x^{\varepsilon}\sum_{d\leq\sqrt[k]{x}}\frac{\sigma^*_{-\varepsilon}(d)}{d^{k\varepsilon}}\right).$$

Here the series of the main term is absolutely convergent, its general term is multiplicative in n and applying Euler's product formula its sum is given by

$$\prod_{p\in P}\left(1+\sum_{i=1}^{\infty}\frac{\mu_{(A,S)}(p^i)\phi((p^i)_Q)}{p^{ik}(p^i)_Q}\right) = \prod_{p\in P}\left(1+\sum_{i=1}^{\infty}\frac{\mu(p^i)}{p^{ik}}\right)\prod_{p\in Q}\left(1+\sum_{i=1}^{\infty}\frac{\mu_{(A,S)}(p^i)\phi(p^i)}{p^{ik}p^i}\right)$$

$$= \prod_{p\in P}\left(1-\frac{1}{p^k}\right)\prod_{p\in Q}\left(1-(1-\frac{1}{p})\sum_{a\in S}\frac{1}{p^{ik}}\right) = \zeta_P(k)\prod_{p\in Q}\left(1-(1-\frac{1}{p})\sum_{a\in S}\frac{1}{p^{ik}}\right).$$

The first and the second O-terms are both $O(x^{1/k})$, using the estimates

$$\sum_{n>x} n^{-r} = O(x^{1-r}), \quad r>1 \quad \text{and} \quad \sum_{n\le \sqrt[k]{x}} \frac{\sigma^*_{-\varepsilon}(n)}{n^{k\varepsilon}} = O(x^{1/k-\varepsilon}), \quad 0<\varepsilon<1/k.k$$

□

Theorem 4. *If A is a cross-convolution and $k\in\mathbb{N}, k\ge 2$, then*

$$\sum_{n\le x} q_{A,k}(n) = \alpha_{A,k}x + O(x^{1/k}), \tag{8}$$

where

$$\alpha_{A,k} = \frac{1}{\zeta_P(k)}\prod_{p\in Q}(1-\frac{1}{p^k}+\frac{1}{p^{k+1}}). \tag{9}$$

Proof. Apply Theorem 3 for $S=\{1\}$. □

Corollary. *If A is a cross-convolution, $k\in\mathbb{N}, k\ge 2$ and $S\subseteq\mathbb{N}$, then the asymptotic densities of the A-semi-k-S-free integers and of the A-semi-k-free integers are $\alpha_{A,k,S}$ and $\alpha_{A,k}$, respectively.*

Remark 2. From this result we reobtain, among others, the asymptotic densities of the k-skew integers of rank r, see [Co61] and of the $A-k$-free integers, see [T-vi].

Now we improve the order of the error term of formula (8) using the method of [SurSi73b] based on the following estimates regarding the Möbius function.

Lemma 1. ([SurSiv73], Lemma 3.5 and Lemma 5.2) *Let $x\ge 3, u\in\mathbb{N}$ and $\varepsilon>0$. Then*

$$M_u(x) \equiv \sum_{\substack{n\le x\\(n,u)=1}} \mu(n) = O(\sigma^*_{-1+\varepsilon}(u)x\delta(x)), 0 \tag{1}$$

where

$$\delta(x) = \exp(-A(\log x)^{3/5}(\log\log x)^{-1/5}), 1 \tag{1}$$

A being a positive constant.

If the Riemann hypothesis (R.H.) is true, then

$$M_u(x) \equiv \sum_{\substack{n\le x\\(n,u)=1}} \mu(n) = O(\sigma^*_{-1/2+\varepsilon}(u)x^{1/2}\omega(x)), 2 \tag{1}$$

where

(1) $$\omega(x) = \exp(A(\log x)(\log\log x)^{-1}),3$$

A being a positive constant.

Lemma 2. *If A is a cross-convolution and $x \geq 3$, then*

(1) $$N_A(x) \equiv \sum_{n\leq x} \mu(n)\frac{\phi(n_Q)}{n_Q} = O(x\delta(x)).4$$

If the R.H. is true, then

(1) $$N_A(x) \equiv \sum_{n\leq x} \mu(n)\frac{\phi(n_Q)}{n_Q} = O(x^{1/2}\omega(x)).5$$

Proof. We have
$$\frac{\phi(n_Q)}{n_Q} = \sum_{\substack{d|n\\ d\in(Q)}} \frac{\mu(d)}{d},$$
cf. [T-iii], Lemma 4. Therefore
$$N_A(x) = \sum_{n\leq x} \mu(n)\frac{\phi(n_Q)}{n_Q} = \sum_{\substack{de=n\leq x\\ d\in(Q)}} \mu(de)\frac{\mu(d)}{d}$$
$$= \sum_{\substack{de\leq x\\ (d,e)=1\\ d\in(Q)}} \mu(d)\mu(e)\frac{\mu(d)}{d} = \sum_{\substack{d\leq x\\ d\in(Q)}} \frac{\mu^2(d)}{d} \sum_{\substack{e\leq x/d\\ (e,d)=1}} \mu(e) = \sum_{\substack{d\leq x\\ d\in(Q)}} \frac{\mu^2(d)}{d} M_d(x/d).$$
Now using (10) with $\varepsilon < 1$ we get
$$N_A(x) = \sum_{\substack{d\leq x\\ d\in(Q)}} \frac{\mu^2(d)}{d} O(\sigma^*_{-1+\varepsilon}(d)\frac{x}{d}\delta(\frac{x}{d}))$$
$$= O(x\sum_{d\leq x} \frac{\tau(d)}{d^2}\delta(\frac{x}{d})) = O(x^{1-\varepsilon}\sum_{d\leq x} \frac{\tau(d)}{d^{2-\varepsilon}}(\frac{x}{d})^{\varepsilon}\delta(\frac{x}{d})),$$
where $\tau(m)$ stands for the number of divisors of m.

Since $x^{\varepsilon}\delta(x)$ is monotonic increasing,
$$(\frac{x}{d})^{\varepsilon}\delta(\frac{x}{d}) \leq x^{\varepsilon}\delta(x),$$
and using that $\tau(m) = O(m^{\varepsilon}), \varepsilon > 0$ we obtain $N_A(x) = O(x\delta(x))$.

If the R.H. is true, then using the estimate (12) instead of (10),
$$N_A(x) = \sum_{\substack{d\leq x\\ d\in(Q)}} \frac{\mu^2(d)}{d} O(\sigma^*_{-1/2+\varepsilon}(d)(\frac{x}{d})^{1/2}\omega(\frac{x}{d})) = O(x^{1/2}\sum_{d\leq x} \frac{\tau(d)}{d^{3/2}}\omega(\frac{x}{d})).$$

Since $\omega(x)$ is monotonic increasing, we obtain that $N_A(x) = O(x^{1/2}\omega(x))$. □

Lemma 3. *If A is a cross-convolution, $x \geq 3$ and $s > 1$, then*

(1) $$\sum_{n>x} \frac{\mu(n)\phi(n_Q)}{n^s n_Q} = O(\frac{\delta(x)}{x^{s-1}}).6$$

If the R.H. is true, then

(1) $$\sum_{n>x} \frac{\mu(n)\phi(n_Q)}{n^s n_Q} = O(\frac{\omega(x)}{x^{s-1/2}}).7$$

Proof. Using (14) and (15), these results follow by partial summation, cf. [SurSi73b], proof of Lemma 2.5. □

We also need the following result, cf. [SurSi73b], eq. (2.3).

Lemma 4. *If $\varepsilon > 0$ and $0 \leq s < 1$, then*

(1) $$\sum_{n\leq x} \frac{\sigma^*_{-\varepsilon}(n)}{n^s} = O(x^{1-s}).8$$

Theorem 5. *If A is a cross-convolution, $k \in \mathbb{N}, k \geq 2$ and $x \geq 3$, then*

(1) $$\sum_{n\leq x} q_{A,k}(n) = \alpha_{A,k}x + O(x^{1/k}\delta(x)),9$$

where $\alpha_{A,k}$ and $\delta(x)$ are given by (9) and (11), respectively.

If the R.H. is true, then

(2) $$\sum_{n\leq x} q_{A,k}(n) = \alpha_{A,k}x + O(x^{1/k}\omega(x)),0$$

where $\omega(x)$ is defined by (13).

Proof. From (5) we have

$$\sum_{n\leq x} q_{A,k}(n) = \sum_{\substack{d^k e\leq x\\ (d,e)\in(P)}} \mu(d).$$

Let $z = x^{1/k}$ and $0 < \rho = \rho(x) < 1$, where $\rho(x)$ will be chosen suitably later. If $d^k e \leq x$, then both $d > \rho z$ and $e > \rho^{-k}$ cannot simultaneously hold good, therefore

$$\sum_{n\leq x} q_{A,k}(n) = \sum_{\substack{d^k e\leq x\\ d\leq \rho z\\ (d,e)\in(P)}} \mu(d) + \sum_{\substack{d^k e\leq x\\ e\leq \rho^{-k}\\ (d,e)\in(P)}} \mu(d) - \sum_{\substack{d\leq \rho z\\ e\leq \rho^{-k}\\ (d,e)\in(P)}} \mu(d) = S_1 + S_2 - S_3,$$

say. We consider each of these sums separately.

By (7) we have

$$S_1 = \sum_{d \le \rho z} \mu(d) \sum_{\substack{e \le x/d^k \\ (e,d) \in (P)}} 1 = \sum_{d \le \rho z} \mu(d) \left(\frac{\phi(d_Q)x}{d^k d_Q} + O(\tau(d)) \right)$$

$$= x \sum_{d \le \rho z} \frac{\mu(d)\phi(d_Q)}{d^k d_Q} + O(\sum_{d \le \rho z} \tau(d))$$

$$= x \sum_{n=1}^{\infty} \frac{\mu(n)\phi(n_Q)}{n^k n_Q} + O\left(x \sum_{n > \rho z} \frac{\mu(n)\phi(n_Q)}{n^k n_Q} \right) + O(\rho z \log(\rho z))$$

$$= \alpha_{A,k} x + O(x\delta(\rho z)(\rho z)^{1-k}) + O(\rho z \log(\rho z))$$

$$= \alpha_{A,k} x + O(\rho^{1-k} z \delta(\rho z)) + O(\rho z \log z)),$$

by (16) and by the well-known estimate $\sum_{n \le x} \tau(n) = O(x \log x)$. From (10) we obtain

$$S_2 = \sum_{e \le \rho^{-k}} \sum_{\substack{d \le \sqrt[k]{x/e} \\ (d,e) \in (P)}} \mu(d) = \sum_{e \le \rho^{-k}} \sum_{\substack{d \le \sqrt[k]{x/e} \\ (d,e_Q)=1}} \mu(d) = \sum_{e \le \rho^{-k}} M_{e_Q}(\sqrt[k]{x/e})$$

$$= \sum_{e \le \rho^{-k}} O\left(\sigma^*_{-1+\varepsilon}(e_Q) (\frac{x}{e})^{1/k} \delta((\frac{x}{e})^{1/k}) \right)$$

$$= \sum_{e \le \rho^{-k}} O\left(\sigma^*_{-1+\varepsilon}(e) (\frac{x}{e})^{1/k} \delta((\frac{x}{e})^{1/k}) \right).$$

Since $\delta(x)$ is monotonic decreasing and $\rho z \le (\frac{x}{e})^{1/k}$, we have $\delta((\frac{x}{e})^{1/k}) \le \delta(\rho z)$. Therefore,

$$S_2 = O\left(x^{1/k} \delta(\rho z) \sum_{e \le \rho^{-k}} \frac{\sigma^*_{-1+\varepsilon}(e)}{e^{1/k}} \right) = O(\rho^{1-k} z \delta(\rho z)),$$

from (18).

$$S_3 = \sum_{e \le \rho^{-k}} \sum_{\substack{d \le \rho z \\ (d,e) \in (P)}} \mu(d) = \sum_{e \le \rho^{-k}} \sum_{\substack{d \le \rho z \\ (d,e_Q)=1}} \mu(d) = \sum_{e \le \rho^{-k}} M_{e_Q}(\rho z)$$

$$= \sum_{e \le \rho^{-k}} O(\sigma^*_{-1+\varepsilon}(e_Q) \rho z \delta(\rho z)) = O(\rho z \delta(\rho z) \sum_{e \le \rho^{-k}} \sigma^*_{-1+\varepsilon}(e)) = O(\rho^{1-k} z \delta(\rho z)),$$

using (10) and (18). Therefore

$$\sum_{n \le x} q_{A,k}(n) = \alpha_{A,k} x + O(\rho^{1-k} z \delta(\rho z)) + O(\rho z \log z).$$

Choosing

$$\rho = \rho(x) = (\delta(x^{1/(2k)})^{1/k},$$

and following the proof of [SurSi73b], Theorem 3.1, see also [SurSiv73], we get formula (19).

If the R.H. is true, then applying (12) and (17) instead of (10) and (16), writing $x^{1/2}\omega(x) = x(x^{-1/2}\omega(x))$, where $x^{-1/2}\omega(x)$ is monotonic decreasing, and using the above arguments with $\delta(x)$ replaced by $x^{-1/2}\omega(x)$ we obtain

$$\sum_{n\leq x} q_{A,k}(n) = \alpha_{A,k}x + O(\rho^{1-k}z(\rho z)^{-1/2}\omega(\rho z)) + O(\rho z \log z).$$

Choosing $\rho = z^{-1/(2k+1)} < 1$ we have $\rho^{1/2-k}z^{1/2} = \rho z = x^{2/(2k+1)}$. Since $\omega(x)$ is monotonic increasing, we get $\omega(\rho z) \leq \omega(z) \leq \omega(x)$. We also have $\log z = O(\omega(x))$, and obtain the estimate (20). □

For $A = D$, i.e. for k-free integers and for $A = U$, i.e. for semi-k-free integers the result of Theorem 5 is due to A. WALFISZ [W63], Satz 1, p. 192 and to D. SURYANARAYANA and R. SITA RAMA CHANDRA RAO [SurSi73a], Corollary 3.2.1 ($n = 1$), [SurSi73b], Theorems 3.1 and 3.2.

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OTHER CHARACTERIZATIONS OF THE ABELIAN GROUPS WITH THE DIRECT SUMMAND INTERSECTION PROPERTY

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Dedicated to Professor Ioan Purdea at his 60^{th} anniversary

Abstract. This work gives a series of characterizations, others than the previously known ones of the abelian groups with the direct summand intersection property, for short D.S.I.P., that is of those groups in which the intersection of any two direct summand is a direct summand as well. All through this paper by group we mean abelian group in additive notation.

1. The General Case

Definition. We say that a group A has the small direct summand intersection property (for short S.D.S.I.P.) if the intersection of any family of direct summands of A is again a direct summand in A.

Obviously, if a group has S.D.S.I.P., it also has D.S.I.P.. The converse is generally false.

Let A be a group and $Sd(A) = \{X \leq A | X$ is a direct summand in $A\}$. If A has S.D.S.I.P., then for any T, $S \in Sd(A)$, $T \cap S \in Sd(A)$ and according to [16,1.4.47.], $Sd(A)$ is a complete lattice.

Definition. A subgroup G of group A is called absolute direct summand (of A), if for any subgroup $H \leq A$, $H - G$-high in A, $A = G \oplus H$.

The absolute direct summands have been studied by Fuchs in [7]; there he demonstrated the following theorem:

Theorem 1.1. *A subgroup B of A is an absolute direct summand, in A, if and only if: B is divisible or A/B is a torsion group, whose p-component is annihilated by p^k, whenever B/pB contains an element of order p^k.*

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Let $Sda(A) = \{X \leq A | X$ is an absolute direct summand in $A\}$ be the set of absolute direct summands of A. Now we proof the following:

Theorem 1.2. *If the group A has D.S.I.P., then for any $T, S \in Sda(A)$, $T \cap S \in Sda(A)$.*

Proof. We are going to show that, together with T and S and $T \cap S$ satisfies (1.1.) as well. Let by $A = T \oplus T' = S \oplus S'$. According to the hypothesis $T \cap S$ is direct summand in S, so $S = T \cap S \oplus S''$.

Case 1. If T or S are divisible, then, according to [6,20.(E).], $T \cap S$ is divisible and $T \cap S \in Sda(A)$.

Case 2. If T and S are not divisible groups, then, according to (1.1.), A/T and A/S are torsion groups. So for any $a \in A$, there is a $n > 0$ so that $na \in T$ and there is a $m > 0$ so that $ma \in S$. Then $[m,n]a \in T \cap S$ ($[m,n]$ being the smallest common multiple of m and n). So $A/(T \cap S)$ is a torsion group. Let $A/T \cap S = \bigoplus_p (A/T \cap S)_p$ be, the direct decomposition of $A/(T \cap S)$ in its own p-subgroups, according to [6,8.4.]. We suppose that there is a $x \in T \cap S$ so that $p^k x \in p(S \cap T) \subset pS \cap pT$. So, there is $x \in S$ so that $p^k x \in pS$ and $x \in T$ so that $p^k x \in pT$. Now from (1.1.) it follows that $p^k(A/S)_p = S$ and $p^k(A/T)_p = T$. Then $p^k(A/T \cap S)_p = T \cap S$ too. So $T \cap S \in Sda(A)$.

Now we are going to present some other two necessary and sufficient conditions for a subgroup B of A to be a direct summand in A, if A has certain properties.

Theorem 1.3. *Let A be a group of finite rank, with property that the neat subgroups of A coincide with its direct summands. Then the following statements are equivalent for a subgroup B of group A:*

(a) B is a direct summand in A;

(b) for any prime number p, $r_p(A) = r_p(B) + r_p(A/B)_i$;

(c) there is a subgroup $C \leq A$, $C - B$-high in A so that for any prime number p, $p(A) \subseteq p(B) + C$.

Proof. (a) $\Rightarrow$ (b). If B is a direct summand in A, then $A \cong B \oplus A/B$ and $r(A) = r(B) + r(A/B)$ (see [5,2.2.5.]). Then: $r_0(A) + \sum_p r_p(A) = r_0(B) + \sum_p r_p(B) + r_0(A/B) + \sum_p r_p(A/B)$. But $r_0(A) = r_0(B) + r_0(A/B)$ ([5,2.2.(c)]). So $\sum_p r_p(A) = \sum_p r_p(B) + \sum_p r_p(A/B)$. For any prime number p, $r_p(A) = r(A_p) = r(S(A_p)) = \dim_{Z(p)} A[p]$, and if $A = B \oplus C$, then $A[p] = B[p] \oplus C[p]$ (there is immediate checking). So $\dim_{Z(p)} A[p] = \dim_{Z(p)} B[p] + \dim_{Z(p)}(A/B)[p]$.

(b) $\Rightarrow$ (a). If $r_p(A) = r_p(B) + r_p(A/B)$, is valid for any prime number p, then by summing up after all prime numbers and considering the relation: $r_0(A) = r_0(B) + r_0(A/B)$ (which occurs for any subgroup B of A), we obtain: $r(A) = r(B) + r(A/B)$. So $A \cong B \oplus A/B$, according to the hypothesis and to [6, p.132].

(a) $\Rightarrow$ (c). If B is direct summand in A, there is $C \leq A$ so that C is B-high in A, $A = B \oplus C$, and for any prime number p, $pA = pB + pC \subset pB + C$.

(c) $\Rightarrow$ (a). If (c) occurs, then, according to [5, consequence of 2.3.1.], $A = B \oplus C$.

Applying (1.3.) to groups with D.S.I.P. we obtain:

Corollary 1.4. *For an abelian group A, which satisfies (1).3), the following statements are equivalent:*

(a) A has D.S.I.P.;

(b) for any two direct summands T and S and for any p-prime number $r_p(A) = r_p(T \cap S) + r_p(A/T \cap S)$;

(c) for any two direct summands T and S, there is a subgroup $U \leq A$, $U - T \cap S$-high in A, so that for any p-prime number, $pA \subseteq p(T \cap S) + U$.

Further on we are going to present two characterizations of the abelian groups with D.S.I.P. using the groups of extensions.

Theorem 1.5. *Being given an abelian group A, the following statements are equivalent:*

a) A has D.S.I.P.;

b) for every decomposition $A = B \oplus C$, and $\beta : B \to C$ an epimorphism, the induced map $\beta^* : Ext(B,G) \to Ext(C,G)$ is monomorphism, for any group G.

Proof. (a) $\Rightarrow$ (b) Being given A as a group with D.S.I.P., $A = \bigoplus C$ and $\beta : B \to C$ an epimorphism, then: (E) $0 \to ker\beta \to B \to C \to 0$ is an exact splitting sequence (according to [10, Proposition 1.4.]) and represents an element from $Ext(C, ker\beta)$. From [6,51.3.] we have the following exact sequence:

$$0 \to Hom(C,G) \to Hom(B,G) \to Hom(ker\beta,G) \xrightarrow{E^*} Ext(C,G) \xrightarrow{\beta^*}$$

$$\xrightarrow{\beta^*} Ext(B,G) \to Ext(ker\beta,G) \to 0.$$

Since (E) is splitting, for any $\eta : ker\beta \to G$, $E_*(\eta) = \eta E \in Ext(C,G)$ is a splitting extension according to [6, 51.1]. So $ImE^* = 0 = ker\beta^*$ and β^* is a monomorphism.

b) $\Rightarrow$ a). We consider the two exact sequences above mentioned. If β^* is monomorphism, then $ker\beta^* = 0 = ImE^*$, that is, for any $\eta : ker\beta \to G$, ηE is a splitting extension. If $G = ker\beta$ and $\eta = 1_{ker\beta}$, we get (E) a splitting extension and according to [10, Proposition 1.4.], A has D.S.I.P..

Theorem 1.6. (the dual of (1.5.)) *Let A be an abelian group.*

a) If A has D.S.I.P., then for any $B, C \in Sd(A)$ and $\alpha : B \to C$ a monomorphism, the induced map $\alpha_* : Ext(G,B) \to Ext(G,C)$ is a monomorphism, for any group G.

b) If for any $B \leq C \in Sd(A)$ and $\alpha : B \to C$ monomorphism, the induced map $\alpha_* : Ext(G,B) \to Ext(G,C)$ is a monomorphism for any group G, then A has D.S.I.P..

Proof. a) Being given $B, C \in Sd(A)$ and $\alpha : B \to C$ a monomorphism then $\alpha(B) \simeq B$ is a direct summand in C and $(E) : 0 \to B \overset{\alpha}{\to} C \to C/B \to 0$ is an exact splitting sequence. From [6,51.3.] we get the following exact sequence:

$$0 \to Hom(G,B) \to Hom(G,C) \to Hom(G,C/B) \overset{E_*}{\to}$$

$$\overset{E_*}{\to} Ext(G,B) \overset{\alpha_*}{\to} Ext(G,C) \to Ext(G,C/B) \to 0.$$

If $\eta : G \to C/B$ is some homomorphism then $E\eta = E_*(\eta) \in Ext(G,B)$ is a splitting extension, according to [6, 51.2]. So $ImE_* = ker\alpha_* = 0$ and α_* is a monomorphism. (It can be noticed that this implication is always valid; the condition that A should have D.S.I.P. hasn't been used anywehere).

b) We consider the two exact sequences from point a). If α_* is a monomorphism, then $ker\alpha_* = 0 = ImE_*$. So, for any $\eta : G \to C/B$, $E\eta$ is a splitting extension of B by G. If $G = C/B$ and $\eta = 1_{C/B}$, (E) is a splitting extension, that is B is a direct summand in C. Now, considering, H as another direct summand in A, noting $B = H \cap C$ and $\alpha : H \cap C \to C$ the inclusion map, we find that $H \cap C \in Sd(A)$, that is A has D.S.I.P..

We close this paragraph with the following result:

Theorem 1.7. *The group A has D.S.I.P., if and only if $Tor(A,C)$ has D.S.I.P., for any group C.*

Proof. A being a group with D.S.I.P., and $Tor(A,C) = Tor(T,C) \oplus Tor(T',C) = Tor(S,C) \oplus Tor(S',C)$ two direct decompositions of $Tor(A,C)$, then $Tor(A,C) \cong Tor(T \oplus T',C) \cong Tor(S \oplus S',C)$, according to [6,62.(E).].(*) But $Tor(A,C) \simeq Tor(B,C)$ has an exact place if $A \simeq B$, as the map $\varphi : (a,m,c) \mapsto (b,m,c)$ is an isomorphism between the generators of $Tor(A,C)$ and those of $Tor(B,C)$. This means that $A \cong$

$T \oplus T' \cong S \oplus S'$ and as A has D.S.I.P., $A \cong T \cap S \oplus T'' \oplus T'$. Then $Tor(A,C) \cong Tor(T \cap S \oplus T'' \oplus T', C) \cong Tor(T \cap S, C) \oplus Tor(T'', C) \oplus Tor(T', C)$ (**). Since $Tor(T \cap S, C) = Tor(T, C) \cap Tor(S, C)$, from relation (**) it follows that $Tor(A,C)$ has D.S.I.P..

Viceversa, we suppose that $Tor(A,C)$ has D.S.I.P. and let $A = T \oplus T' = S \oplus S'$ be two direct decompositions of A. Then $Tor(A,C) \cong Tor(T \oplus T', C) \cong Tor(T,C) \oplus Tor(T',C) \cong Tor(S \oplus S', C) \cong Tor(S,C) \oplus Tor(S',C) \cong Tor(T,C) \cap Tor(S,C) \oplus Tor(U,C) \cong Tor(T \cap S, C) \oplus Tor(U,C) \cong Tor((T \cap S) \oplus U, C)$, where $U \leq A$. Then $A \cong T \cap S \oplus U$, which means A has D.S.I.P..

Since $Tor(A,C) \simeq Tor(C,A)$, we also have its symmetric of (1.7.).

Corollary 1.8. *The group C has D.S.I.P. if and only if $Tor(A,C)$ has this property, for any group A.*

2. Torsion groups

The following proposition presents a series of elementary properties of p-groups with D.S.I.P..

Proposition 2.1. A being a p-group with D.S.I.P. the following statements occur:

(a) A is a simply presented group;

(b) A has a nice system;

(c) A has a nice composition series;

(d) A has the projective property relative to all the balanced-exact sequence of p-groups;

(e) A is a direct summand of a direct sum of generalized Prüfer groups;

(f) A is totally projective;

(g) A is fully transitive;

(h) For any increasing sequence of ordinals and symbols ∞, $u = (\sigma_0, \dots, \sigma_n, \dots)$, $A(u)$ and $A/A(u)$ are totally projective.

Proof. If A is a p-group with A D.S.I.P., either is indecomposable or $A = B_p \oplus C_p$, where $B_p = \bigoplus_{m_p} Z(p)$, $C_p = 0$ or $C_p = Z(p^\infty)$ (see [12, Theorem 2.]).

(a) Since $Z(p^n)$, $n \in \mathbf{N}^*$ and $Z(p^\infty)$ are simply presented groups, and a direct sum of simply presented groups is again a simply presented group (see [6, §83.]), it follows that A is a simply presented group.

(b) By [6,83.2.], every simply presented p-group has a nice system.

(c), (d), (e), (f). The statements of points (b), (c), (d), (e) and (f) are equivalent, according to [6,81.9.] and [6,82.3.].

(g) Any totally projective p-group is fully transitive (see [11] or [6,81.4.]).

(h) Every totally projective p-group A has the enunciate property, according to [6,§83.].

The following result makes another connection between the torsion p-groups with D.S.I.P. and the torsion product.

Proposition 2.2. *A being A a p-group with D.S.I.P., E a pure subgroup in A with $Z(p^\infty) \subseteq E$, then $Tor(E,G)$ is a balanced subgroup of $Tor(A,G)$, for any group G.*

Proof. According to the hypothesis and to [12, Theorem 2.], A is the direct sum between a divisible group and a bounded group. Then any subgroup of A, that is for E as well, is a nice subgroup, as the equality from [6,79.2.] is clearly demonstrated (see [6, p.75]). If E is pure in A and $Z(p^\infty) \subseteq E$, then $pE = E \cap pA = E \cap Z(p^\infty) = Z(p^\infty)$ and $pA + E = Z(p^\infty) + E = E$. So $(A/E)^1 = 0$ and according to [6,80.(G).], E is an isotype in A. Then E becomes a balanced subgroup in A. According to [6,62.(F).,62.(D).], only the case in which G is a p-group is of some interest (the other ones being quite ordinary). From [6,63.2.], and from [6, p.75] we find that $Tor(E,G)$ is a nice subgroup in $Tor(A,G)$. We demonstrate that $Tor(E,G)$ is an izotype in $Tor(A,G)$. The equality $p^\sigma Tor(E,G) = Tor(E,G) \cap p^\sigma Tor(A,G)$ becomes, according to [6,64.2.]: $Tor(p^\sigma E, p^\sigma G) = Tor(E,G) \cap Tor(p^\sigma A, p^\sigma G)$, which is quite obvious as $p^\sigma E = E \cap p^\sigma A$. So $Tor(E,G)$ is balanced in $Tor(A,G)$.

Further on we are going to determine the ring $E(A)$ and the group $AutA$ of the endomorphisms, respectively of the automorphisms of a torsion group A, with D.S.I.P.. For the beginning we have the following basic remark:

Remark 2.3. If $A = \bigoplus_{i \in I} A_i$ is a direct decomposition of the group A in fully invariant subgroups, the ring $E(A)$ of its endomorphisms is the direct product of the rings of the endomorphisms of the groups A_i, $i \in I$, that is:

$$E(A) \cong \prod_{i\in I} E(A_i).$$

Proof. If $f \in EndA$, then $\forall\, i \in I$, $f(A_i) \subseteq A_i$. Noting $f|A_i \overset{not}{=} f_i$, $i \in I$, we obtain the map $f \mapsto (f_i)_{i\in I} \in \prod_{i\in I} E(A_i)$, which is an isomorphism from $E(A)$ to $\prod_{i\in I} E(A_i)$.

Lemma 2.4. *If A is a p-group with D.S.I.P., then:*

$$E(A) \cong A,$$

or

$$E(A) \cong M^{(f)}_{m_p\times m_p}(Z(p)) \oplus \left(\prod_{m_p} Z(p)\right) \oplus R_p \cong \left(\prod_{m_p^2} Z(p)\right) \oplus \left(\prod_{m_p} Z(p)\right) \oplus R_p,$$

where:

- *$m_p \in N$ or $m_p = \infty$;*
- *$M^{(f)}_{m_p\times m_p}(Z(p))$ is the ring of the square matrices of order m_p with elements from $Z(p)$ and the columns of which have a finite number of non-null elements;*
- *$R_p = 0$ or $R_p = Q_p^*$ - the completion, in p-adic topology of Q_p - the ring of p-adic integers.*

Proof. A being a p-group with D.S.I.P., if A is indecomposable, then there is $n \in N^*$ so that $A = Z(p^n)$. In this case $E(A) = End(Z(p^n)) \cong Z(p^n) = A$ (see [6,§43]). If A is decomposable, then, according to [12, Theorem 2.], $A = B_p \oplus C_p$, where $B_p = \bigoplus_{m_p} Z(p)$, $C_p = 0$ or $C_p = Z(p^\infty)$. So $E(A) = Hom(A,A) = Hom(B_p\oplus C_p, B_p\oplus C_p) \cong Hom(B_p,B_p)\oplus Hom(B_p,C_p)\oplus Hom(C_p,B_p)\oplus Hom(C_p,C_p) = EndB_p\oplus Hom(B_p,C_p)\oplus EndC_p$ (*) (according to [6,43.1., 43.2., 43.(A).(iii)]). The group $EndB_p = End\left(\bigoplus_{m_p} Z(p)\right)$ is isomorphic to the ring of the square matrices $(\alpha_{ij})_{i,j=\overline{1,m_p}}$, of the type $m_p \times m_p$, where $\alpha_{ij} \in End(Z(p)) \cong Z(p)$, and for which the sum of elements on each column exists in the finite topology of $E(B_p)$ (see [6,106.1.]). Noting the $M^{(f)}_{m_p\times m_p}(Z(p))$ - ring of the square matrices of order $m_p \times m_p$, having elements from $Z(p)$ with its columns having a finite number of non-null elements, we find that $E(B_p) \cong M^{(f)}_{m_p\times m_p}(Z(p))$. $Hom(B_p,C_p) = 0$ or $Hom(B_p,C_p) = Hom\left(\bigoplus_{m_p} Z(p), Z(p^\infty)\right) \cong$

$\prod_{m_p} Hom(Z(p), Z(p^\infty)) \cong \prod_{m_p} Z(p^\infty)[p] \cong \prod_{m_p} Z(p)$ (see [6,§43]). Finally, $R_p = EndC_p = 0$ or $R_p = EndC_p = End(Z(p^\infty)) \cong Q_p^*$, according to [6,§43]. Replacing in the relation (*) we obtain the first isomorphism of the statement.

The second isomorphism can be stressed out in the following way:

$$EndB_p = Hom(B_p, B_p) = Hom\left(\bigoplus_{m_p} Z(p), \bigoplus_{m_p} Z(p)\right) \cong$$

$$\cong \prod_{m_p} Hom\left(Z(p), \bigoplus_{m_p} Z(p)\right) \cong \prod_{m_p}\prod_{m_p} Hom(Z(p), Z(p)) \cong \prod_{m_p^2} EndZ(p) \cong \prod_{m_p^2} Z(p).$$

Theorem 2.5. *If A is a torsion group with D.S.I.P., then:*

$$E(A) \cong \left(\prod_{p\in P_0} A_p\right) \oplus \left(\prod_{p\in P\setminus P_0} M^{(f)}_{m_p\times m_p}(Z(p))\right) \oplus \left(\prod_{p\in P\setminus P_0}\prod_{m_p}(Z(p))\right) \oplus \left(\prod_{p\in P\setminus P_0} R_p\right) \cong$$

$$\cong \left(\prod_{p\in P_0} A_p\right) \oplus \left(\prod_{p\in P\setminus P_0}\left(\prod_{m_p^2} Z(p)\right)\right) \oplus \left(\prod_{p\in P\setminus P_0}\prod_{m_p} Z(p)\right) \oplus \left(\prod_{p\in P\setminus P_0} R_p\right),$$

where:

i) P is the set of all prime numbers and $P_0 \subseteq P$;

ii) A_p is an indecomposable p-group, for any $p \in P_0$;

iii) $M^{(f)}_{m_p\times m_p}(Z(p))$ and R_p have the same meaning like (2.4.), for any $p \in P\setminus P_0$.

Proof. According to [18,3.3.], a torsion group has D.S.I.P., if and only if it takes the form: $A = \left(\bigoplus_{p\in P_0} A_p\right) \oplus \left(\bigoplus_{p\in P\setminus P_0} A_p\right)$, where A_p is an indecomposable p-group, for any $p \in P_0$, and for any $p \in P \setminus P_0$, $A_p = B_p \oplus C_p$, with $B_p = \bigoplus_{m_p} Z(p)$, $C_p = 0$ or $C_p = Z(p^\infty)$. Since the two direct summands of the decomposition of A are fully invariant (because if A_i, $i \in I$, are fully invariant subgroups of group A, then $\sum_{i\in I} A_i$ has the same property - see [6,§2.]), it follows that

$$E(A) \cong E\left(\bigoplus_{p\in P_0} A_p\right) \oplus \left(\bigoplus_{p\in P\setminus P_0} A_p\right) \cong$$

$$\cong \left(\prod_{p\in P_0} E(A_p)\right) \oplus \left(\prod_{p\in P\setminus P_0} E(B_p \oplus C_p)\right) \cong$$

$$\cong \left(\prod_{p\in P_0} A_p\right) \oplus \left(\prod_{p\in P\setminus P_0} M^{(f)}_{m_p\times m_p}(Z(p))\right) \oplus \left(\prod_{p\in P\setminus P_0}\prod_{m_p} Z(p)\right) \oplus \prod_{p\in P\setminus P_0} R_p \cong$$

$$\cong \left(\prod_{p\in P_0} A_p\right) \oplus \left(\prod_{p\in P\setminus P_0} \prod_{m_p^2} Z(p)\right) \oplus \left(\prod_{p\in P\setminus P_0} \prod_{m_p} Z(p)\right) \oplus \left(\prod_{p\in P\setminus P_0} R_p\right),$$

according to (2.4.).

Lemma 2.6. *If A is a p-group with D.S.I.P., then there is a $n_p \in N^*$ so that the group $AutA$ is isomorphic to the multiplicative group $U(Z(p^{n_p})$, of the units of the ring $(Z(p^{n_p}), +, \cdot)$, or:*

$$AutA \cong U(M^{(f)}_{m_p\times m_p}(Z(p))) \oplus \left(\prod_{m_p} Z(p-1)\right) \oplus U(R_p) \cong$$

$$\cong \left(\prod_{m_p^2} Z(p-1)\right) \oplus \left(\prod_{m_p} Z(p-1)\right) \oplus U(R_p),$$

where:

- *$U(M^{(f)}_{m_p\times m_p}(Z(p)))$ is the multiplicative group of the units of the ring $M^{(f)}_{m_p\times m_p}(Z(p))$;*
- *$U(R_p) = 0$ or $U(R_p) \cong Z(p-1) \times J_p$ (J_p being the additive group of Q_p^*).*

Proof. A being a p-group with D.S.I.P., if A is indecomposable, then there is a $n \in N^*$, such that $A = Z(p^n)$. In this case $AutA = Aut(Z(p^n))$ is isomorphic to the group $U(Z(p^n))$, presented in the statement, as an automorphism of A is an unit of $E(A)$. If $A = B_p \oplus C_p$, with $B_p = \bigoplus_{m_p} Z(p)$, $C_p = 0$ or $C_p = Z(p^\infty)$, then an automorphism of A is a inversable element of the direct product (of the direct sum) of the rings of (2.4.). Considering that $U(Z(p)) \cong Z(p-1)$ and $U(Q_p^*) \cong Z(p-1)\times J_p$ (according to [6,127.1.]), we obtain the isomorphisms of the statement.

Similar to the proof of (2.5), but using (2.6.) the following result can be demonstrated:

Theorem 2.7. *If A is a torsion group with D.S.I.P., then:*

$$AutA \cong \left(\prod_{p\in P_0} U(Z(p^{n_p}))\right) \oplus \left(\prod_{p\in P\setminus P_0} U(M^{(f)}_{m_p\times m_p}(Z(p)))\right) \oplus \left(\prod_{p\in P\setminus P_0} \left(\prod_{m_p} Z(p-1)\right)\right) \oplus$$

$$\oplus \left(\prod_{p\in P\setminus P_0} U(R_p)\right) \cong \left(\prod_{p\in P_0} U(Z(p^{n_p}))\right) \oplus \left(\prod_{p\in P\setminus P_0} \left(\prod_{m_p^2} Z(p-1)\right)\right) \oplus$$

$$\oplus \left(\prod_{p\in P\setminus P_0} \left(\prod_{m_p} Z(p-1)\right)\right) \oplus \left(\prod_{p\in P\setminus P_0} U(R_p)\right),$$

where the notations have the same meaning like in (2.6.).

Remark 2.8. Since the groups B_p, $p \in P$, are elementary p-groups of rank m_p, these are vectorial spaces over the field $Z(p)$ (of characteristic p), and $\dim B_p = m_p$, and any automorphism of B_p is a linear transformation of this space, it follows that $M^{(f)}_{m_p \times m_p}(Z(p))$ is isomorphic to the general linear group $GL(m_p, p)$.

3. Torsion-free groups

In [18,4.1.] we have demonstrated that any torsion-free divisible group has D.S.I.P.. Using this we are going to demonstrate a few interesting results.

Theorem 3.1. *Let A be a torsion-free group with the property that for any epimorphism $\beta : B \to C$ (B and C being arbitrary groups), the induced map $\beta^* : Ext(C, A) \to Ext(B, A)$ is a monomorphism. Then A has D.S.I.P..*

Proof. If $\beta : B \to C$ is an epimorphism, then $(E)\ 0 \to ker\beta \to B \to C \to 0$ is an exact sequence. From [6,51.3.], we obtain the following exact sequence: $0 \to Hom(C, A) \to Hom(B, A) \to Hom(ker\beta, A) \overset{E^*}{\to} Ext(C, A) \overset{\beta^*}{\to} Ext(B, A) \to Ext(ker\beta, A) \to 0$. Since β^* is monic, it follows that $ImE^* = 0$, that is for any $\eta : ker\beta \to A$, $E^*(\eta) = \eta E$ is splitting. Now considering $B = D$ - the divisible hull of A, $C = D/A$, $\beta = \pi_A$ - the canonic projection of D on D/A and $\eta = 1_A$, we find that $1_A E \cong E$ is a splitting extension, that is A is a direct summand in D. Then [6,20.(E).], shows that A is divisible. Now [18,4.1.] completes the proof.

Remark 3.2. It can be easily demonstrate that the converse of (3.1.) occurs for any divisible group.

Further on we are going to see what conditions the groups A and C have to satisfy so that $Hom(A, C)$ may have D.S.I.P..

Proposition 3.3. 1) A and C being two abelian groups $Hom(A, C)$ has D.S.I.P. in any of the following situations:

a) A is torsion-free and divisible;

b) C is torsion-free and divisible;

c) A is torsion-free indecomposable, C is divisible and $A \oplus C$ has D.S.I.P.;

d) A is torsion-free with D.S.I.P. and C is torsion-free of rank 1;

e) A is torsion-free of rank 1 and C is torsion-free with D.S.I.P.;

2) If A and C are torsion-free of rank 1, with $t(A) \leq t(C)$, then for any index set I, the group $H = \bigoplus_I Hom(A, C)$ has D.S.I.P.. In particular $E = \bigoplus_I EndA$ has D.S.I.P., for any torsion-free group A of rank 1.

Proof. 1) a) If A is torsion-free and divisible, then for any group C, $Hom(A, C)$ is torsion-free and divisible ([6,43.(G).]). Now we apply [18,4.1.].

b) If C is torsion-free and divisible, then for any group A, $Hom(A, C)$ is torsion-free and divisible ([6,43(D)]. We apply [18,4.1.] once again.

c) If $A \oplus C$ has D.S.I.P., then, according to [10,3.4.1.], for any $\alpha \in Hon(A, C)$, $ker\alpha$ is a direct summand in A. Since A is indecomposable, any morphism $\alpha : A \to C$ is either null or injective. Let $0 \neq \beta \in Hom(A, C)$ be a morphism for which $n\beta = 0$, for a certain $n \in N^*$. Then for any $a \in A$, $n\beta(a) = \beta(na) = 0$. So $na = 0$, as β is injective. Since A is torsion-free, it follows that $n = 0$ - contradiction with the choice of n. So $Hom(A, C)$ is torsion-free. Now we are going to demonstrate that $Hom(A, C)$ is divisible. Since the group C is divisible, it follows that for any $\alpha \in Hom(A, C)$, with any $x \in A$ and any $n \in N^*$, there is $y \in C$ so that $\alpha(x) = uy$.

We define $\gamma : A \to C$ by: for any $x \in A$, $\gamma(x) = y$, where $y \in C$ is the solution of equation $\alpha(x) = ut$. Then $\gamma \in Hom(A, C)$ and $\alpha(x) = n\gamma(x)$, for any $x \in A$. So $Hom(A, C)$ is divisible. Now [18,4.1.] completes the proof.

d) Let A be a torsion-free group with D.S.I.P. and C torsion-free of rank 1. Then $A = D \oplus E$, with D - divisible and E - reduced, completely decomposable homogeneous of finite rank ([18,5.16.]). So there is an $n \in N$ so that $E = \bigoplus_n B$, where B is reduced torsion-free of rank 1. Then there will be $Hom(A, C) = Hom(D \oplus E, C) \cong Hom(D, C) \oplus Hom(E, C) \cong Hom(D, C) \oplus \left(Hom\left(\bigoplus_n B, C\right)\right) \cong Hom(D, C) \oplus \left(\bigoplus_n Hom(B, C)\right)$, according to [6.43.1., 43.2.]. The group $Hom(D, C)$ has D.S.I.P., according to a). From [6,85.4.] we find that $Hom(B, C)$ is either 0 (if $t(B) > t(C)$) or a torsion-free group of rank 1 and of the type $t(C) : t(A)$, (if $t(A) \leq t(C)$). If $Hom(B, C) = 0$, then $Hom(A, C) \cong Hom(D, C)$ and the proof is ready for this case. If $Hom(B, C) \neq 0$, then $\bigoplus_n Hom(B, C)$ is either torsion-free divisible or reduced homogeneous completely decomposable group of finite rank. In the former case $Hom(A, C)$ will be torsion-free divisible so it has D.S.I.P., and in the latter $Hom(E, C)$ is reduced homogeneous completely decomposable group of finite rank having D.S.I.P., according to [12, Theorem 5.]. Now [18,5.12.] completes the proof.

The proof from point e) will be similar to the one from point d).

2) If A and C are the same as in the statement, then $Hom(A, C)$ is according to [6,85.4.], torsion-free group of rank 1. Now we can apply [10, Proposition 3.4.].

From (3.3.)b) and [6,§43.] (or 18,4.1.]) will have:

Corollary 3.4. *For any abelian group A and any $m \in N^*$, the group $Hom\left(A, \bigoplus_m Q\right) \cong \prod_{r_0(A)} \left[\bigoplus_m Q\right]$ has D.S.I.P..*

For any abelian group A, the group of characters of A is $CarA = Hom(A, Q/Z)$. From (3.3.a) we find that if A is torsion-free and divisible, then $CarA$ has D.S.I.P.. The next theorem will improve this result.

Theorem 3.5. *If A is a torsion-free group with D.S.I.P., then $CarA$ has the same property.*

Proof. Let A be a torsion-free group with D.S.I.P.. According to [18,5.16.], $A = \left(\bigoplus_m Q\right) \oplus \left(\bigoplus_n C\right)$, where $m, n \in N$ or $m = \infty$, and C is reduced, of rank 1. Then according to [6,43.1., 43.2.],

$$CarA = Hom(A, Q/Z) = Hom\left(\left(\bigoplus_m Q\right) \oplus \left(\bigoplus_n C\right), \bigoplus_p Z(p^\infty)\right) \cong$$

$$\cong Hom\left(\bigoplus_m Q, \bigoplus_p Z(p^\infty)\right) \oplus Hom\left(\bigoplus_n C, \bigoplus_p Z(p^\infty)\right) \cong$$

$$= \left[\prod_m \prod_p Hom(Q, Z(p^\infty))\right] \oplus \left[\prod_n \prod_p Hom(C, Z(p^\infty))\right].$$

According to [6,43.(G).], $Hom(Q, Z(p^\infty))$ is a torsion-free and divisible group. But then $\prod_n \prod_p Hom(Q, Z(p^\infty))$ is divisible and torsion-free too (see [6,20.(E).]). Since the p-basic subgroup of C is null, according to [6,47.1], $Hom(C, Z(p^\infty))$ is divisible and torsion-free; so the groups $\prod_n \prod_p Hom(C, Z(p^\infty))$ and $CarA$ have the same property. From [18,4.1.] it follows that, $CarA$ has D.S.I.P.. ∎

Now, we are going to study the ring $E(A) = EndA$ and the group $AutA$, when A is a torsion free group with D.S.I.P..

Theorem 3.6. *If A is torsion-free with D.S.I.P., then:*

$$E(A) \cong \left(\prod_m \left(\bigoplus_m Q\right]\right) \oplus \left(\bigoplus_n \left[\bigoplus_n Q\right]\right) \oplus \left(\bigoplus_n \left(\bigoplus_n EndC\right]\right),$$

where:

- *m and n are natural numbers or $m = \infty$;*
- *C is a reduced torsion-free group of rank 1.*

Proof. Let $A = D \oplus B$ be, with $D = \bigoplus_m Q$ (divisible) and $B = \bigoplus_n C$ - reduced completely decomposable homogeneous of finite rank (C being a reduced torsion-free group of rank 1), torsion-free group with D.S.I.P., according to [10,3.3.]. Then $E(A) = Hom(A,A) = Hom(D \oplus B, D \oplus B) = Hom(D,D) \oplus Hom(D,B) \oplus Hom(B,D) \oplus Hom(B,B) = EndD \oplus Hom(B,D) \oplus EndB$, according to [6.43.1, 43.2., 43.(A).(iii).]. But $EndD = Hom\left(\bigoplus_m Q, \bigoplus_m Q\right) = \prod_m \left[\bigoplus_m Q\right]$, (see [6,§43.]), $Hom(B,D) = Hom\left(\bigoplus_n C, \bigoplus_m Q\right)$! $\prod_n Hom\left(C, \bigoplus_m Q\right) \cong \bigoplus_n \left[\bigoplus_m Q\right]$, and $EndC = Hom\left(\bigoplus_n C, \bigoplus_n C\right) \cong \bigoplus_n \left[\bigoplus_n EndC\right]$.

Making a demonstration analogous to (2.7.) we obtain:

Theorem 3.7. *If A is a torsion-free group with D.S.I.P., then:*

$$AutA \cong \left(\prod_n \left[\bigoplus_m Q^*\right]\right) \oplus \left(\bigoplus_n \left[\bigoplus_m Q^*\right]\right) \oplus \left(\bigoplus_n \left[\bigoplus_n AutC\right]\right).$$

We'll close this paragraph with some other two condition necessary for a torsion-free group to have D.S.I.P..

Theorem 3.8. *Let A be a torsion-free group. In any of the following cases, A has D.S.I.P..*

(a) The group A has the following property: if A is an endomorphic image of a group B, then B contains a direct summand isomorphic to A.

(b) There is a prime number p so that the p basic subgroup B of A, is an endomorphic image of A and A/B is divisible.

Proof. (a) If A is like in the statement, then, according to [11, Theorem 1.], $A = D \oplus F$, where D is divisible and F - free. The groups D and F from the decomposition of A have D.S.I.P., because of [18.4.1.] and respectively [18.2.2.]. Now [18, 5.12.] completes the proof.

(b) Let A be a torsion-free group and $B = B_0 \oplus B_1 \oplus \cdots \oplus B_n \oplus \ldots$ its p-basic subgroup ($B_0 = \oplus Z$ and $B_n = \oplus Z(p^n)$, $n = 1, 2, \ldots$). This leads to the conclusion that $B = \oplus Z$, so it is a free group. If $f \in EndA$ and $f(A) = B$, then $A/kerf \cong B$, according to the first theorem of isomorphism. Since B is a free group, is an exactly splitting sequence: $0 \to kerf \to A \to B \to 0$ ([6,14.4.]). So $A \cong kerf \oplus B = D \oplus C \oplus B$, where D is the maximal divisible subgroup of A, and C is a reduced group. Since A/B is a divisible group it follows that $C = 0$. So $A = D \oplus B$, with D divisible and C - free. Now we are going to judge the same as at point a).

4. Mixed groups

In [18,4.4.] we have seen that a divisible group with D.S.I.P. cannot be mixed. Because of this, according to what was demonstrated in the former paragraphs and in [6,§32,§106,§113, §127, §128], we find:

Theorem 4.1. *Let A be a divisible group with D.S.I.P.*

(a) If A is a torsion group, then:

$$E(A) \cong \prod_p Q_p^*,$$

and

$$Aut(A) \cong \prod_p [Z(p-1) \times J_p].$$

(b) If A is a torsion-free group, then:

$$E(A) \cong M^{(f)}_{r_0 \times r_0}(Q) \cong \prod_{r_0} \left[\bigoplus_{r_0} Q \right],$$

and

$$Aut(A) \cong U(M^{(f)}_{r_0 \times r_0}(Q)) \cong \prod_{r_0} \left[\bigoplus_{r_0} Q^* \right] \cong \prod_{r_0} \left[\bigoplus_{r_0} \left(Z(2) \times \underset{\chi_0}{\times} Z \right) \right],$$

where the notations are the ones presented above.

Using (2.4.)-(2.8.), (3.6.)-(3.7.), (4.1.) and [18.6.4.], the problem of the determination of $EndA$ ($AutA$), for a splitting mixed group A with D.S.I.P., will be reduced to the determination of $Hom(B, Z(p^n))$, where $n \in N^*$ and B is a reduced torsion-free of rank 1 direct summand of A.

The following results present sufficient conditions for $T(A)$ and $A/T(A)$ (A being a mixed group), to have D.S.I.P., by using the ring $E(A)$.

Proposition 4.2. Let A be a mixed group with the property that any endomorphic image of A is a direct summand in A.

(a) If $T(A)$ is bounded, then A, $T(A)$ and $A/T(A)$ have D.S.I.P..

(b) If $T(A)$ is not bounded, it may have D.S.I.P., $A/T(A)$ has (always) D.S.I.P., but A does not have this property anymore.

Proof. Let A be a mixed group with the property presented in the statement. From [15,3.1., 4.2., 5.3], we find that each p-component of A is an elementary or divisible group, and $A/T(A)$ is divisible.

(a) If $T(A)$ is bounded, then each p-component is a elementary p-group and has D.S.I.P., according to [18, 3.3.]. This means that $T(A)$ is an elementary group and has D.S.I.P. ([12, Lemma 1.]). The group $A/T(A)$ has D.S.I.P., due to [18, 4.1.]. According to the hypothesis A it is splitting and as $T(A)$ and $A/T(A)$ are, in this case, fully invariant, we can apply [12, Lemma 1.].

(b) If $T(A)$ is not bounded, then it has a divisible direct summand. From [18,4.4.] we find that $T(A)$ can have D.S.I.P., if it takes the form $\bigoplus_p Z(p^\infty)$. In this case A has a mixed divisible direct summand and, according to [19, proposition 6.], doesn't have D.S.I.P..

Proposition 4.3. Let A be a mixed group. In any of the following situations $T(A)$ and $A/T(A)$ have D.S.I.P.:

(a) The kernels and the images of the endomorphisms of A are pure subgroups in A.

(b) The ring of the endomorphisms of A is regular.

Proof. (a) If A has the property given in the statement, according to [17,5. Proposition 3.], $T(A)$ is elementary and $A/T(A)$ is divisible. Now [18,3.3.] and [18,4.1.] completes the proof.

(b) We suppose that $E(A)$ is regular. If A is not reduced, then, according to [6,112.7], A is splitting, $T(A)$ is elementary, and $A/T(A)$ is divisible. So A, $T(A)$ and $A/T(A)$ has D.S.I.P.. If A is of torsion, then $A = T(A)$ is an elementary group, so it has D.S.I.P.. Finally, if A is reduced, then, again, $T(A)$ is elementary and $A/T(A)$ is divisible.

Corollary 4.4. *Any splitting group which satisfies the conditions from (4.3.)(a) has D.S.I.P..*

In the end we present other properties of the mixed groups with D.S.I.P..

Theorem 4.5. *Let A be a splitting mixed group, with D.S.I.P. with $T = T(A)$ and $\hat{T}$ - the completion of T in the Z-adic topology. Then:*

a) for any divisible group G, $Ext(G, \hat{T})$ is isomorphic to a direct summand of a direct product of groups of the form $A/p^n A$;

b) $\hat{T}$ is isomorphic to a direct summand of a direct product of groups of the form A/p^nA;

c) if C is a reduced torsion-free summand, of rank 1, from some decomposition of A, and $(Ext(Q/Z,C))_0 = 0$, then the pure-injective hull of T and the first subgroup Ulm of the cotorsion hull of T, are isomorphic to $(Ext(Q/Z,A))_0$ (so $(Ext(Q/Z,A))_0$ is a reduced algebraically compact group).

Proof. a) Ig A is a splitting mixed group with D.S.I.P., according to [12, Theorem 4], $T^1 = 0$. From [6,39.5] we find that: $0 \to T \to \hat{T} \to \hat{T}/T \to 0$ is an exact sequence. Now [6,53.7.] implies the exactness of the sequence: $0 = Hom(G,\hat{T}) \to Hom(G,\hat{T}/T) \to Pext(G,T) \to Pext(G,\hat{T}) = 0$; the last equality occurs because $\hat{T}$ is algebraically compact group ([6,39.1.]). This leads to $Hom(G,\hat{T}/T) \cong Pext(G,T) = (Ext(G,T))^1$. From [6,51.3.] we get exactness of the sequence: $0 = Hom(G,\hat{T}) \to Hom(G,\hat{T}/T) \to Ext(G,T) \to Ext(G,\hat{T}) \to Ext(G,\hat{T}/T) = 0$ (the last equality occurs because of [6,39.5.]). So $Ext(G,T)/Hom(G,\hat{T}/T) \cong Ext(G,T)/Pext(G,T) \cong Ext(G/T)/(Ext(G,T))^1 = (Ext(G,T))_0 \cong Ext(G,\hat{T})$. Since $A = T \oplus A/T$, it follows that $(Ext(G,A))_0 = (Ext(G,T))_0 \oplus (Ext(G,A/T))_0$ (see [6,37.5.]). So $Ext(G,\hat{T})$ is a direct summand in $(Ext(G,A))_0$. According to [6,30.1.], there is a direct sum of cyclic groups $X = \bigoplus_{i\in I}\langle x_i\rangle$ and an epimorphism $\eta : X \to G$ so that $ker\eta$ is a pure subgroup in X, that is, there is the following pure-exact sequence: $0 \to ker\eta \to X \to G \to 0$. From [6,57.1.] it follows that $(Ext(X,A))_0 = (Ext(G,A))_0 \oplus (Ext(ker\eta,A))_0$. But $(Ext(X,A))_0 = \left(Ext\left(\bigoplus_{i\in I}\langle x_i\rangle, A\right)\right)_0 \cong \left(\prod_{i\in I} Ext(\langle x_i\rangle, A)\right)_0 \cong \left(\prod_p A/p^nA\right)_0 \cong \prod_p A/p^nA$, according to [6,52.2,52.(D).,37.5.].

b) By [6,56.6.] it follows that $\hat{T} \cong (Ext(Q/Z,T))_0$, which is, see [6.57.1.], a direct summand in $(Ext(Q/Z,A))_0$. Since Q/Z is divisible, the statement follows from the proof of the point a) of this theorem.

c) Let $A = T \oplus D \oplus \left(\bigoplus_n C\right)$ be a splitting mixed group, with D.S.I.P., according to [18,6.4.] (so T is the torsion part of A, D is a torsion-free divisible group, and C is a torsion-free reduced group, of rank 1). From the hypothesis and from [6,37.5.,56.6.,52.(B).] we find that $(Ext(Q/Z,A))_0 = (Ext(Q/Z,T))_0 \cong \hat{T}$, which is a reduced algebraically compact group (see [6,39.1]). If $\hat{T}$ is the pure-injective hull of T, from [6,41.9.] and [12, Theorem 4.], it follows that $\tilde{T} \cong \hat{T}$.

Corollary 4.6. *Let A be a splitting mixed group with D.S.I.P. and B its reduced homogeneous completely decomposable summand of finite rank, according to [18,6.4.], E the divisible hull and G the pure-injective hull of B. If $Hom(Q/Z, E/B) \cong Hom(Q/Z, G/B)$, then $(Ext(Q/Z, A))_0$ is a reduced algebraically compact group.*

Proof. From [6,52.3.] we find that $Ext(Q/Z, B) \cong Hom(Q/Z, E/B)$. If G is the pure-injective hull of B, then $0 \to B \to G \to G/B \to 0$ is a pure-exact sequence. According to [6,53.7.]: $0 = Hom(Q/Z, G) \to Hom(Q/Z, G/B) \to Pext(Q/Z, B) \to Pext(Q/Z, G) = 0$ is an exact sequence; the two equalities are due to [6.43.(A).(iii).] and, respectively to [6,41.5.]. This means that $Pext(Q/Z, B) \cong Hom(Q/Z, G/B)$. Then, according to the hypothesis and to [6,53.3.], we get: $(Ext(Q/Z, B))_0 = Ext(Q/Z, B)/Pext(Q/Z, B)$ $Hom(Q/Z, E/B)/Hom(Q/Z, G/B) = 0$. Now, (4.5.c) completes the proof.

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