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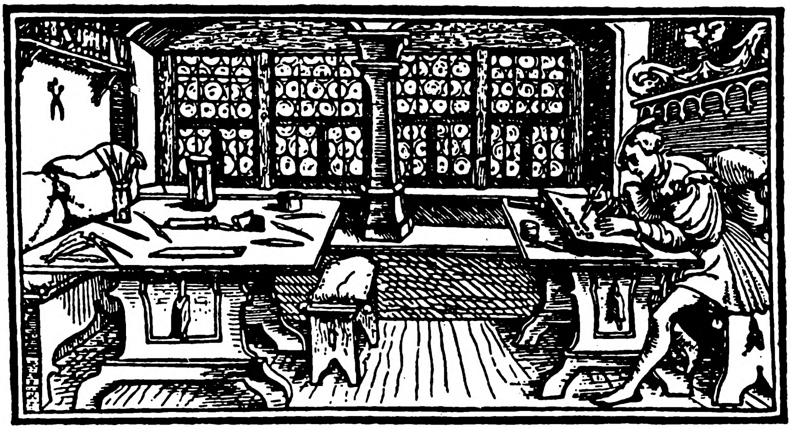

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SUMAR - CONTENTS - SOMMAIRE

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ON THE D.D. STANCU METHOD OF PARAMETERS

DUMITRU ACU

Dedicated to Professor D.D. Stancu on his 70^{th} birthday

Abstract. In the paper one uses the parameters' method of D.D. Stancu for the construction of the quadrature formulae with a high algebraic accuracy degree and for the deduction of the Hildebrand's V-method.

1. Introduction.

39 years ago prof. D.D. Stancu [7] used for constructing a n-point quadrature formula of Gaussian-type called by us [2] "the D.D. Stancu method of parameters". Further D.D. Stancu has applied this method to constructing of the quadrature formulae with the high algebraic degree of exactness (see [8], [9]).

We have used the D.D. Stancu method of parameters to the extension of Hildebrand's V-method ([1]).

In the section 2 of the paper we consider "the D.D. Stancu method of parameters" applied in order to construct quadrature formulae with simple knots and high algebraic degree of exactness, presenting some different proofs given by D.D. Stancu.

In the section 3, we study the Hildebrand's V- method.

2. The D.D. Stancu method of parameters.

Let $w : (a,b) \to (0,\infty)$ be a weight function so that its moments are $c_j = \int_a^b x^j w(x)dx$, $j = 0,1,2,\ldots$. The interval (a,b) is finite or infinite.

We suppose to construct the quadrature formulae with simple knots $x_1 \ldots x_n$ all placed in $[a,b]$, with algebraic degree of exactness $n+k-1$, k natural number, $1 \le k \le n$,

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by the form

$$\int_a^b w(x)f(x)dx = \sum_{i=1}^{n} A_i f(x_i) + R(f) \ . \tag{1}$$

For this aim we use "the D.D. Stancu method of parameters".

Corresponding to the method of parameters, we consider the parameters $\alpha_1, \dots, \alpha_k$ placed in the interval (a, b) and different from $x_1, \dots, x_n$. We use the notation:

$$u(x) = \prod_{i=1}^{n}(x - x_i) \quad , \quad v(x) = \prod_{j=1}^{k}(x - \alpha_j) \ ,$$

$$u_i(x) = \frac{u(x)}{x - x_i} \ , \ i = \overline{1,n} \quad , \quad v_j(x) = \frac{v(x)}{x - \alpha_j} \ , \ j = \overline{1,k} \ .$$

Now we write the Lagrange's interpolation formula for the function f and the knots x_i, $i = \overline{1,n}$, and $\alpha_j, j = \overline{1,k}$. We have

$$f(x) = L_{n+k-1}(x_1, \dots, x_n; \alpha_1, \dots, \alpha_n; f|x) + r(\alpha, x) \tag{2}$$

where

$$L_{n+k-1}(x_1, \dots, x_n; \alpha_1, \dots, \alpha_k; f|x) = \tag{3}$$

$$= \sum_{i=1}^{n} \frac{u_i(x)}{u_i(x_i)} \cdot \frac{v(x)}{v(x_i)} f(x_i) + \sum_{j=1}^{k} \frac{u(x)}{u(\alpha_j)} \cdot \frac{v_j(x)}{v_j(\alpha_j)} f(\alpha_j)$$

is the Lagrange's interpolation polynomial and

$$r(f, x) = u(x)v(x)[x, x_1, \dots, x_n, \alpha_1, \dots, \alpha_k; f] \tag{4}$$

is the remainder of the interpolation formula. In (4) the symbol of the square brackets represents divided difference of f on the knots $x_1, \dots, x_n$, $\alpha_1, \dots, \alpha_k$.

Multiplying (2) with $w(x)$ and then, integrating from a to b, we obtain

$$\int_a^b w(x)f(x)dx = \sum_{i=1}^{n} A_i f(x_i) + \sum_{j=1}^{k} B_j f(x_j) + R_{n+k-1}(f) \tag{5}$$

with

$$A_i = \int_a^b w(x) \frac{u_i(x)}{u_i(x_i)} \cdot \frac{v(x)}{v(x_i)} dx \quad , \quad i = \overline{1,n} \ , \tag{6}$$

$$B_j = \int_a^b w(x) \frac{u(x)}{u(\alpha_j)} \cdot \frac{v_j(x)}{v_j(\alpha_j)} dx \quad , \quad j = \overline{1,k} \tag{7}$$

and

$$R_{n+k-1}(f) = \int_a^b w(x)u(x)v(x)[x, x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_k; f]dx \ . \tag{8}$$

Now we select the knots $x_1, \ldots, x_n$ such that the quadrature formula (5) is of type (1). Then we have

$$B_j = 0 \quad , \quad j = \overline{1,k} \ . \tag{9}$$

Taking into account (7), the conditions (9) are equivalent with

$$\int_a^b w(x)u(x)v_j(x)dx = 0 \quad , \quad j = \overline{1,k} \ .$$

From here we obtain that the conditions (9) are equivalent with

$$\int_a^b w(x)u(x)x^j dx \quad , \quad j = \overline{1,k-1} \ . \tag{10}$$

Theorem 1. *If the knots of the quadrature formula (1) are selected such that $u(x)$ satisfies the relations (10), then coefficients A_i, $i = \overline{1,n}$, are independent of the parameters $\alpha_1, \ldots, \alpha_k$ and they can be calculated with the formula*

$$A_i = \int_a^b w(x)\frac{u_i(x)}{u_i(x_i)}dx \quad , \quad i = \overline{1,n} \ . \tag{11}$$

Proof 1^{st}. From (6) we have

$$A_i = \int_a^b w(x)\frac{u(x)}{u_i(x_i)(x-x_i)}\left[1 + \frac{v(x)}{v(x_i)} - 1\right]dx =$$

$$= \int_a^b w(x)\frac{u(x)}{u_i(x_i)(x-x_i)}dx + \int_a^b w(x)\frac{u(x)}{u_i(x_i)v(x_i)}\frac{v(x)-v(x_i)}{x-x_i}dx \ .$$

Because $(v(x) - v(x_i))/(x - x_i)$ is a polynomial of the $k-1$ degree, using the conditions (10), it follows that the last integral is equal to 0. Hence

$$A_i = \int_a^b w(x)\frac{u_i(x)}{u_i(x_i)}dx \quad , \quad i = \overline{1,n} \ ,$$

from where we conclude that the coefficients A_i, $i = \overline{1,n}$, are independent of the parameters $\alpha_1, \ldots, \alpha_k$.

Proof 2^{nd}. First we prove that the coefficients A_i, $i = \overline{1,n}$, don't depend on the parameters α_j, $j = \overline{1,k}$. In this order we calculate the partial derivatives $\dfrac{\partial A_i}{\partial \alpha_j}$, $i = \overline{1,n}$,

$j = \overline{1,k}$. Using the expressions (6) we have:

$$\frac{\partial A_i}{\partial \alpha_j} = \int_a^b w(x)\frac{u_i(x)}{u_i(x_i)} \cdot \frac{v_j(x)}{v_j(x_i)} \cdot \frac{\partial}{\partial \alpha_j}\left(\frac{x-\alpha_i}{x_i-\alpha_j}\right)dx =$$

$$= \int_a^b w(x)\frac{u_i(x)}{u_i(x_i)} \cdot \frac{v_j(x)}{v_j(x_i)} \cdot \frac{x-x_i}{(x_i-\alpha_j)^2}dx =$$

$$= \int_a^b w(x)\frac{u(x)}{u_i(x_i)} \cdot \frac{v_j(x)}{v_j(x_i)(x-\alpha_j)^2}dx = 0 \quad , \quad j=\overline{1,k} \ , \ i=\overline{1,n} \ ,$$

hence it follows that the coefficients A_i are independent of the parameters α_j, $j = \overline{1,k}$.

Now, we consider

$$I_i(x_1,\dots,x_n,\alpha_1,\dots,\alpha_k) = \int_a^b w(x)u_i(x)v(x)dx \quad , \quad i=\overline{1,n} \ . \tag{12}$$

For fixed i we take $\alpha_j = x_i$ in (12) and we obtain

$$I_i(x_1,\dots,x_n,\alpha_1,\dots,\alpha_k)\Big|_{\alpha_j=x_i} = \int_a^b w(x)u_i(x)v_j(x)(x-x_i)dx =$$

$$= \int_a^b w(x)u(x)v_j(x)dx = 0 \quad , \quad j=\overline{1,k} \ .$$

From here we conclude that $I_i(x_1,\dots,x_n,\alpha_1,\dots,\alpha_k)$ is divisible by $v(x_i)$, that is

$$I_i(x_1,\dots,x_n,\alpha_1,\dots,\alpha_k) = K_i(x_1,\dots,x_n,\alpha_1,\dots,\alpha_k)v(x_i), \tag{13}$$

$i = \overline{1,n}$.

From (6), (12) and (13) it follows:

$$A_i = \frac{K_i(x_1,\dots,x_n,\alpha_1,\dots,\alpha_k)}{u_i(x_i)} \quad , \quad i=\overline{1,n}$$

whence we obtain

$$\frac{\partial K_i(x_1,\dots,x_n,\alpha_1,\dots,\alpha_k)}{\partial \alpha_j} = 0 \quad , \quad j=\overline{1,k} \ ,$$

which proves us that the functions K_i, $i = \overline{1,n}$, don't depend on the parameters α_j, $j = \overline{1,k}$.

From (12) and (13) we have:

$$\frac{\partial^k I_i}{\partial\alpha_1\partial\alpha_2\dots\partial\alpha_k} = (-1)^k \int_a^b w(x)u_i(x)dx \quad , \quad i=\overline{1,n}$$

and

$$\frac{\partial^k I_i}{\partial \alpha_1 \partial \alpha_2 \ldots \partial \alpha_k} = (-1)^k K_i(x_1, \ldots, x_n) \quad , \quad i = \overline{1, n}$$

where we find

$$K_i(x_1, \ldots, x_n) = \int_a^b w(x) u_i(x) dx \quad , \quad i = \overline{1, n} \ .$$

Therefore

$$I_i = v(x_i) \int_a^b w(x) u_i(x) dx \quad , \quad i = \overline{1, n}$$

and using (6) we obtain

$$A_i = \int_a^b w(x) \frac{u_i(x)}{u_i(x_i)} dx \quad , \quad i = \overline{1, n}$$

Q.E.D.

Theorem 2. *If the relations (10) are true, then for $k = n-1$ and $k = n$, the coefficients A_i can be calculated with the formula*

$$A_i = \int_a^b w(x) \left(\frac{u_i(x)}{u_i(x_i)} \right)^2 dx \quad , \quad i = \overline{1, n} \ . \qquad (14)$$

Proof. Using (11) we have:

$$\begin{aligned} A_i &= \int_a^b w(x) \frac{u_i(x)}{u_i(x_i)} dx = \\ &= \int_a^b w(x) \frac{u_i(x)}{u_i(x_i)} dx \left[\frac{u_i(x)}{u_i(x_i)} - \frac{u_i(x)}{u_i(x_i)} + 1 \right] dx = \\ &= \int_a^b w(x) \left[\frac{u_i(x)}{u_i(x_i)} \right]^2 dx - \int_a^b w(x) \frac{u(x)}{(u_i(x_i))^2} \frac{u_i(x) - u_i(x_i)}{x - x_i} dx \end{aligned}$$

$i = \overline{1, n}$.

Using (10) it follows that the last integral is null and we obtain (14).

Now we conclude that it holds:

Theorem 3. *The quadrature formula (1) has the algebraic degree of exactness $n + k - 1$, $1 \leq k \leq n$, if and only if the polynomial $u(x)$ of the knots $x_1, \ldots, x_n$ satisfies the relations (10).*

Remark 1. *For $k = n$ from Theorem 3 we obtain the classical result relating to Gaussian quadrature formulas.*

3. The Hildebrand's V-method.

Using Hildebrand's V-method (see [4]) we can give another form for the tions (10).

Theorem 4. *The conditions (10) are true if and only if there exists a polynomi of degree n, with n distinct and real zeros on the interval (a, b), such that to exist of polynomials $(V_j)_{j=0}^{k}$ defined as it follows:*

$$V_0(x) = w(x)u(x)$$

$$V_j(x) = \int_0^x V_{j-1}(t)dt \quad , \quad j = \overline{1,k}$$

with

$$V_j(a) = 0 \quad and \quad V_j(b) = 0 \quad , j = \overline{1,k} \; .$$

Proof. Let's suppose the conditions (10) to be valid. Then we consider polynomials $(V_j)_{j=0}^{k}$ defined by (15). Taking into account (10) we have:

$$V_j(a) = 0 \quad , \quad j = \overline{1,k}$$

and

$$V_1(b) = \int_a^b w(x)u(x)dx = 0$$

$$V_2(b) = \int_a^b V_1(x)dx = xV_1(x)\Big|_a^b - \int_a^b w(x)u(x)xdx = 0$$

etc.

Viceversa, if we chose the polynomial $u(x)$ such that to exist the set of polynomi $(V_j)_{j=0}^{k}$, given by (15) and (16), then $u(x)$ verifies the conditions (10).

Really, from $V_1(b)$ it results

$$\int_a^b w(x)u(x)dx = 0$$

From $V_2(a) = 0$ and $V_2(b) = 0$, using the integration by parts, we find:

$$0 = V_2(b) = \int_a^b V_1(x)dx = V_1(x)\Big|_a^b - \int_a^b w(x)u(x)xdx \; ,$$

where

$$\int_a^b w(x)u(x)xdx = 0 \quad etc. \quad Q.E.D.$$

If the quadrature formula (1) has the algebraic degree of exactness $n+k-1$, $1 \leq k \leq n$, then it is an interpolating quadrature formula on the nodes x_i, $i=\overline{1,n}$, and its remainder admits the representation (see [3]):

$$R_{n+k-1}(f) = \int_a^b w(x)u(x)[x, x_1, x_2, \dots, x_n; f]dx \ . \tag{17}$$

From (17), with the help of the polynomials V_j, $j=\overline{0,k}$, integrating by parts, we can write

$$R_{n+k-1}(f) = \int_a^b \frac{d^k}{dx^k}[x, x_1, x_2, \dots, x_n; f]V_k(x)dx \ . \tag{18}$$

which is Hildebrand's representation (see [4]) for the remainder of the formula (1).

The form (18) for the remainder of the quadrature formula (1) has used by F. Locher ([5], [6]) for the constructing of the V-optimal quadrature formulas (see [1], [2]).

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ON MODIFIED BETA OPERATORS

OCTAVIAN AGRATINI

Dedicated to Professor D.D. Stancu on his 70^{th} birthday

Abstract. In the present paper, we deal with an integral operator of beta-type $L_{m,y}$ depending on a positive real parameter y. We give an estimation of the order of approximation by using the first order modulus of continuity. Also, we prove an asymptotic formula of Voronovskaja type and we show that the operator preserves the Lipschitz constants.

1. Introduction

In time, several integral operators which are associated with beta-type probability distributions have been discussed and interesting properties have been proved. In this respect, we mention the papers [1], [2], [5], [6], [7].

Let us denote by $L_B[0,\infty)$ the linear space of real bounded functions defined on $[0,\infty)$ and Lebesgue measurable. In [8] D.D. Stancu introduced a new beta second-kind approximating operator defined on $L_B[0,\infty)$ as:

$$(L_m f)(x) = L_m(f(t),x) = \frac{1}{B(mx, m+1)} \int_0^\infty f(t) \frac{t^{mx-1}}{(1+t)^{mx+m+1}} dt, \quad x > 0, \tag{1}$$

and $(L_m f)(0) = f(0)$, $B(\cdot,\cdot)$ being the beta function.

This is an integral linear positive operator of Feller type which reproduces the linear

functions. Starting from L_m defined by (1), we introduce and investigate a sequence of linear positive operators depending on a parameter $y > 0$. These modified beta operators are defined as follows:

$$(L_{m,y} f)(x) = \frac{1}{B(my, m+1)} \int_0^\infty f(t+x) \frac{t^{my-1}}{(1+t)^{my+m+1}} dt, \quad x \geq 0, \tag{2}$$

where $f \in L_B[0,\infty)$.

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It is clear that for any $x > 0$ we have:

$$(L_{m,x}f)(0) = (L_m f)(x).$$

This type of construction was used in the papers [3] and [4] for an integra modification of Szász operators.

The next section provides the main results of this paper. All the proofs and th necessary supporting results are provided in section three.

2. Main results

Theorem 1. *Let $a > 0$ and $f \in C[0,\infty)$ such that $L_{m,y}(|f|,x) < \infty$. For any $m > 1$ and $y \in [0,a]$ we have the following inequality:*

$$|(L_{m,y}f)(x) - f(x+y)| \leq \left(1 + \sqrt{a(a+1)}\right) \omega\left(f, \frac{1}{\sqrt{m-1}}\right), \tag{3}$$

where $\omega(f,\cdot)$ represents the modulus of continuity of the function f.

Corollary 1. *Let $0 < a < \infty$. If $f \in L_B[0,\infty) \cap C[0,2a]$, then, for any fixed $y \in [0,a]$, we have:*

$$\lim_{m\to\infty} (L_{m,y}f)(x) = f(x+y), \tag{4}$$

uniformly in $[0,a]$.

Theorem 2. *Let $a > 0$ and $f \in L_B[0,\infty)$ differentiable in some neighborhood of a point $x + y \in [0,a]$ such that at this point f'' exists. Then we have:*

$$\lim_{m\to\infty} m(f(x+y) - (L_{m,y}f)(x)) = -\frac{y(y+1)}{2} f''(x+y). \tag{5}$$

Theorem 3. *Let $a > 0$ and $f \in L_B[0,\infty)$. Then $f \in \mathrm{Lip}_{[0,a]}(A,\mu)$ if and only if $L_{m,y}f \in \mathrm{Lip}_{[0,a]}(A,\mu)$, where $A > 0$ and $\mu \in (0,1]$.*

It is appropriate to remark here that $g \in \mathrm{Lip}_{[0,a]}(A,\mu)$ if for any x and y belonging to $[0,a]$ we have:

$$|g(x) - g(y)| \leq A|x-y|^{\mu}. \tag{6}$$

3. Proofs

Before we proceed with the proofs, we recall some useful relations:

$$(L_{m,y}e_0)(x) = (L_m e_0)(y) = 1, \tag{7}$$

$$(L_{m,y}e_1)(x) = x + y, \tag{8}$$

$$L_m((t-x)^2; x) = \frac{x(x+1)}{m-1}, \tag{9}$$

where $e_k(t) = t,\ t \geq 0,\ k = 0, 1.$

Proof of Theorem 1. We have

$$|(L_{m,y}f)(x) - f(x+y)| \leq$$
$$\leq \frac{1}{B(my, m+1)} \int_0^\infty |f(t+x) - f(y+x)| \frac{t^{my-1}}{(1+t)^{my+m+1}} dt.$$

It is verified that for any $\delta > 0$

$$|f(t+x) - f(y+x)| \leq \left(1 + \frac{1}{\delta}|t-y|\right) \omega(f,\delta).$$

From the above relations, by making use of the Cauchy inequality and of the relations (7), (8), (9) we can write successively:

$$|(L_{m,y}f)(x) - f(x+y)| \leq$$
$$\leq \left(1 + \frac{1}{\delta}\frac{1}{B(my, m+1)} \int_0^\infty |t-y| \frac{t^{my-1}}{(1+t)^{my+m+1}} dt\right) \omega(f,\delta) \leq$$
$$\leq \left(1 + \frac{1}{\delta} L_m^{1/2}((t-y)^2; y)\right) \omega(f,\delta) = \left(1 + \frac{1}{\delta}\sqrt{\frac{y(y+1)}{m-1}}\right) \omega(f,\delta).$$

But $y \leq a$ and if we take $\delta = 1/\sqrt{m-1}$, we arrive at inequality (3). □

Proof of Theorem 2. Because f has a finite second order derivative at a point $x+y \in [0,a]$ then f can be expanded by Taylor's formula:

$$f(t) = f(x+y) + (t-x-y)f'(x+y) + \frac{(t-x-y)^2}{2} f''(x+y) + (t-x-y)^2 r_{2,y}(t),$$

where $r_{2,y}$ is a real valued function having the property: $r_{2,y}(t) \to 0$ as $t \to x+y$. Using (7) and (8), we get:

$$(L_{m,y}f)(x) - f(x+y) = \frac{y(y+1)}{2(m-1)} f''(x+y) + R_{2,y}(x),$$

where $R_{2,y}$ is given by:

$$R_{2,y}(x) = \frac{1}{B(my, m+1)} \int_0^\infty (t-y)^2 r_{2,y}(t+x) \frac{t^{my-1}}{(1+t)^{my+m+1}} dt.$$

Taking into account that $r_{2,y}(t+x)$ tends to zero when t tends to y, it follows that for every $\varepsilon > 0$ there exists an $\delta > 0$ so that for every t for which $|t-y| < \delta$, we have $|r_{2,y}(t+x)| < \varepsilon$. Since $r_{2,y}$ is bounded on $[0,a]$, for every t for which $|t-y| \geq \delta$, we deduce:

$$|r_{2,y}(t+x)| \leq M \leq M\delta^{-2}(t-y)^2.$$

Consequently, the inequality

$$|r_{2,y}(t+x)| \leq \varepsilon + M\delta^{-2}(t-y)^2$$

holds. By choosing $\varepsilon = \frac{1}{m^2}$, after few calculations, we obtain

$$\lim_{m\to\infty} mR_{2,y}(x) = 0$$

which leads us to the desired result. Further, the convergence from (5) is uniform if f'' is continuous on $[0,a]$. □

Proof of Theorem 3. Let $f \in L_B[0,\infty) \cap \mathrm{Lip}_{[0,a]}(A,\mu)$ and $x_1, x_2 \in [0,a]$ such that $x_1 + y \leq a$, $x_2 + y \leq a$. Considering (2), (6) and (7) it results

$$|(L_{m,y}f)(x_1) - (L_{m,y}f)(x_2)| \leq$$
$$\leq \frac{1}{B(my,m+1)} \int_0^\infty |f(t+x_1) - f(t+x_2)| \frac{t^{my-1}}{(1+t)^{my+m+1}} dt \leq A|x_1 - x_2|^\mu.$$

Thus, $L_{m,y}f$ preserves Lipschitz constants.

Now, we assume $L_{m,y}f \in \mathrm{Lip}_{[0,a]}(A,\mu)$. For any integer $m > 1$ and $x_i + y \in [0,a]$, $(i = 0,1)$, we can write:

$$|f(x_1+y) - f(x_2+y)| - |(L_{m,y}f)(x_1) - f(x_1+y)| - |f(x_2+y) -$$
$$- (L_{m,y}f)(x_2)| \leq |(L_{m,y}f)(x_1) - (L_{m,y}f)(x_2)| \leq A|x_1 - x_2|^\mu.$$

With the help of relation (4), we obtain easily that $f \in \mathrm{Lip}_{[0,a]}(A,\mu)$. This completes the proof of Theorem 3. □

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APPROXIMATION OF FRACTALS WITH NEURAL NETWORKS

PETER ANDRAS

Dedicated to Professor D.D. Stancu on his 70th birthday

Abstract. The fractals could be find in many cases of chaotic behavior of natural processes. The approximation of fractal forms plays an important role in chaotic control or chaotic classification. The artificial neural networks can be used for the approximation of the fractal shapes. A new type of neural network is introduced in this paper, which has a generalized tree-like structure, and can approximate with good results the fractal shapes. This network functions relatively fast with many neurons.

1. Introduction

The fractals are very nature-like geometrical forms, which are used in many fields for nature-like modeling. They could be find in many cases of chaotic behavior of natural processes.

An interesting property of the fractal forms is that they are some limits of basically iterative transformations of some simple shapes. The approximations of fractal forms play an important role in chaotic control or chaotic classification. Because of the formally simple but technically very difficult description of the fractal shapes it is hard to generate good approximations of them with classical tools.

The artificial neural networks can be used for the approximation of the fractal shapes. The reduced Coulomb potential networks ([5]), the self-organizing maps ([6]), or the modified Coulomb potential networks ([5]) can approximate the fractal shapes to some degree. One common problem of them is that the good approximation by them needs too many data and requires too complex networks that work very slow.

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A new type of neural network is introduced in this paper, which has a generalized tree-like structure, and can approximate with good results the fractal shapes. This network functions relatively fast with many neurons.

2. The Mandelbrot Set

The Mandelbrot set ([4]) is one of the most studied fractal shape. It was discovered in the 80s by the American mathematician B.B. Mandelbrot, who founded the field of fractal geometry.

The Mandelbrot set can be defined as the set of those complex numbers (z), for which the following series has a non-infinity limit or at least is bounded:

$$z_0 = z, z_{n+1} = z_n^2 + z_0.$$

So if $z_n \in A$, $\forall\ n$ where A is a bounded set, then z is in the Mandelbrot set. The Mandelbrot set is a typical fractal set, which is very chaotic and is self- similar. It is possible to show theoretically that the Mandelbrot set is included in the square $[-2, 2] \times [-2, 2]$, and if for a z, the (z_n) series contains a member, which has a modulus greater than 4, then that z is out of the Mandelbrot set ([4]).

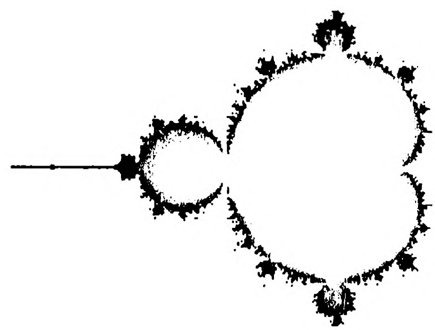

Figure 1

One method to get a picture of this set is to calculate for each z from the bidimensional interval $[-2, 2] \times [-2, 2]$ the trajectory of the (z_n) series for $n = 1, 30$, and to consider the point z member of the Mandelbrot set, if there is no z_k with modulus greater than 4 in its series. By this method it is possible to get approximations of the Mandelbrot set. An image of the Mandelbrot set is presented in Figure 1. The Mandelbrot set it is used in this paper as test set for the proposed neural network.

3. The Generalized Tree-structured Neural Networks

The description of the basic type of the tree-structured neural networks can be found in [1] or [2].

The proposed network is formed by a tree of neurons, a final output neuron. The tree has a root neuron where the input data enters the network. The data is processed through the network to the output neuron. The final output of the network is 1 if the point represented by the input data is member of the goal set (or approximated set), and 0 if it is not member of the goal set.

The neurons represent hyperspheres with center c and radius r. Each neuron has a type which could be Member or Non-Member. Each neuron is connected to the output neuron, and has an incoming connection from one other neuron. A neuron could have multiple output connections to many other neurons in the tree. The neurons with the same type represent non-overlapping hyperspheres. The neural tree looks like in the Figure 2..

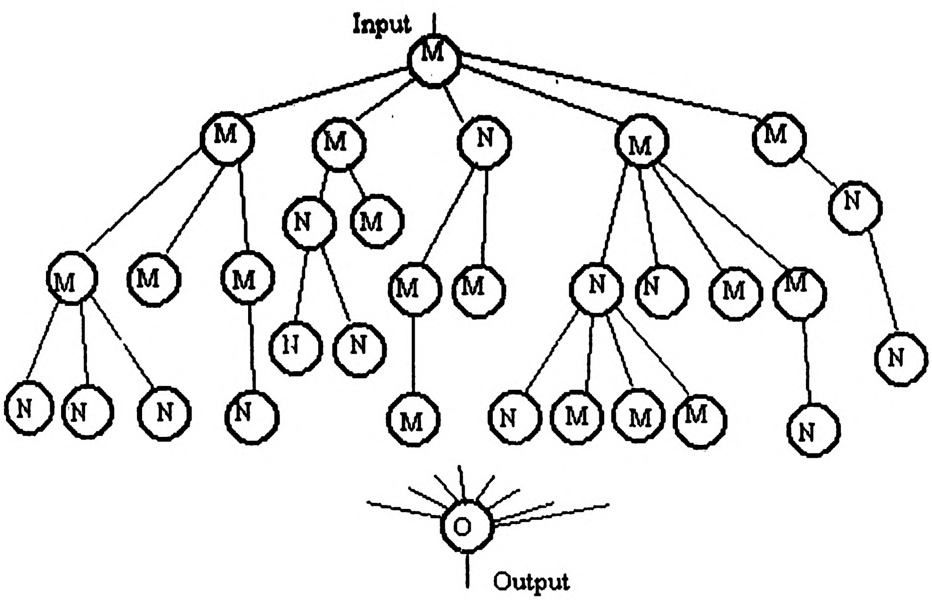

Figure 2

The network is built up through the training. At the initial moment there is only the root neuron and the output neuron. New neurons are added to the network during the training.

The functioning algorithm of a neuron is the following:

Algorithm 1. *(The Functioning Algorithm of the Neurons)*

incoming data is x. Let us consider that neuron is representing the hyper-

the type of the neuron is T, the output connections to the other neurons *of type T are in the list A, and the output connections to other neurons of type non-T are in the list B.*

Step 1. *If $x \in S(c,r)$ then go to Step 2. else go to Step 3.*

Step 2. *If the list B is not empty, then let b be the number of the neuron in the list B for which $d(x,c_b) \leq d(x,c_k)$ for each neuron k from the list B, the input data (x) is transmitted to neuron b, otherwise go to Step 4.*

Step 3. *If the list A is not empty, then let a be the number of the neuron in the list A for which $d(x,c_a) \leq d(x,c_k)$ for each neuron k from the list A, the input data (x) is transmitted to neuron a, otherwise go to Step 5.*

Step 4. *Transmit to the output neuron the type T stop signal. Step 5. Transmit to the output neuron the type not-T stop signal.*

The training algorithm of the network is the following:

Algorithm 2. *(The Training Algorithm of the Generalized Tree-structured Neural Network) The incoming data is x.*

Step 1. *The incoming data enters the root neuron.*

Step 2. *The incoming data is processed by the neurons of the network still one of the neurons send the stop signal to the output neuron.*

Step 3. *The output neurons output is 1 if the received stop signal is a Member - stop signal and 0 otherwise.*

Step 4. *If the resulted output is incorrect then a new neuron is added to the network. The new neuron has as its center the point x, its type is set to be the type of point x (Member if x is in the goal set, non-Member otherwise), and the radius of the new neuron is so, that the hypersphere of the new neuron is tangent to hypersphere of the last activated neuron of the network tree.*

Observation: *The hypersphere of the new neuron could be internally or externally tangent to the hypersphere of the last activated neuron. If there is created a neuron with the same type as of the last activated neurons, then the hypersphere of the new neuron will externally tangent, otherwise it will be internally tangent to the hypersphere of the last activated neuron.*

Now we can state the functioning algorithm of the network, which is practically the same as the first three steps of the training algorithm.

Algorithm 3. *(The Functioning Algorithm of the Generalized Tree-structured Neural Network)*

The incoming data is x.

Step 1. *The incoming data enters the root neuron.*

Step 2. *The incoming data is processed by the neurons of the network still one of the neurons send the stop signal to the output neuron.*

Step 3. *The output neurons output is 1 if the received stop signal is a Member - stop signal and 0 otherwise.*

The network is trained with probabillistic data, which means that the input data is selected randomly from the space region which contains the goal set. The training is supervised training, so for each training point is necessary to know if it is in the goal set or not. The result after the training is a network which represents a mixture of hyperspheres belonging to the types Member and non-Member.

The advantages of this network compared to other neural networks (self- organizing maps, Coulomb potential networks) are that it has a tree structure, which permits a fast functioning, it represents multilevel combination of different parts belonging to different classes, and it reduces considerably the number of the neurons needed for the representation of the classes.

Because of its multilevel combination property (this is realized by the internal structuring of the interior of the hyperspheres represented by the neurons, using other neurons which represents other hyperspheres belonging the other class, and which are inside of the neurons hypersphere) these networks could be used for very detailed delimitation of very complex shapes, which is the case of the fractals too.

The proposed network can be more generalized in order to represent not only two sets (goal set and outer space of the goal set), but many different sets.

4. The Performance of the Generalized Tree-structured Neural Networks

The basic approximation properties of the tree-structured network are shown in [1]. There it is proven that the tree-structured neural networks have the universal approximation property, and they realize practically what is predicted theoretically based on their theoretical approximation property.

Here will be shown that the proposed generalized tree-structured neural network increases its performance, measured as percentage of the correctly classified points, through the training procedure.

Let us consider first that G is the goal set and D is a set which contains G and which contains the training points. Let us note by S_i the hypersphere represented by the neuron nr. i., and let consider the sets $F_i = S_i - G$, and $E_i = S_i - F_i$. So F_i is that part of S_i which is out of G, and E_i is that part of S_i which is inside of G. Let N_k be the union of $S_i, i = 1, k$, taking in account the type of the neurons too, so

$$N_k = \left(\bigcup_{i \in A} S_i\right) \setminus \left(\bigcup_{i \in B} S_i\right),$$

where A is the set of the indexes of the neurons with type Member, and B is the set of the indexes of the neurons with type non- Member. Let us note with MP_k the percentage of covering of G by N_k, and MCP_k the percentage of non-covering of $D - G$ by N_k (this is one minus the covering percentage of $D - G$ by N_k). So we have that

$$MP_k = \frac{\sigma\ (N_k \cap G)}{\sigma\ (G)}$$
$$MCP_k = \frac{\sigma\ ((D - N_k) \cap (D - G))}{\sigma\ (D - G)}.$$

Let us imagine the network as graph, and in addition let us note the connections to the neurons with the same type by e, and to the neurons of the other type by i. Then we have the network as in the Figure 3.

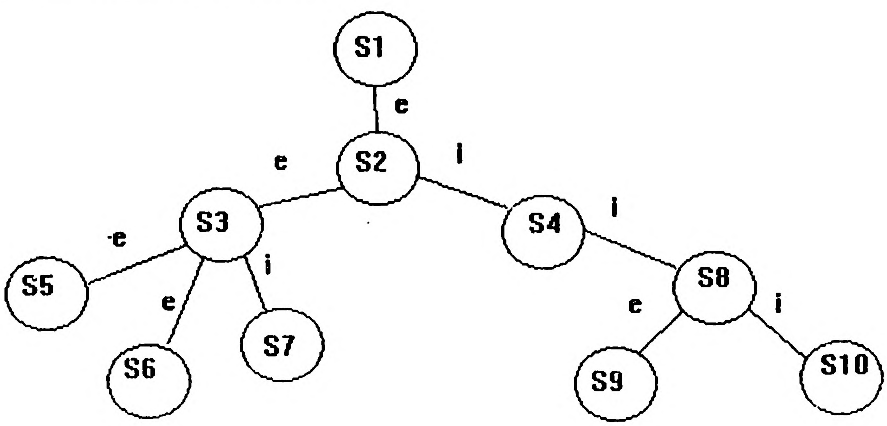

Figure 3

Let us consider now a subtree of the neural tree where all of the internal links are e-links, and all of the external links of the group are i-links. For convenience let us consider that the neurons are numbered consecutively in this subtree, so we are supposing that they were added consecutively to the network (this supposition does not make any important modification in the structure or in the functioning of the network). This case is presented in Figure 4.

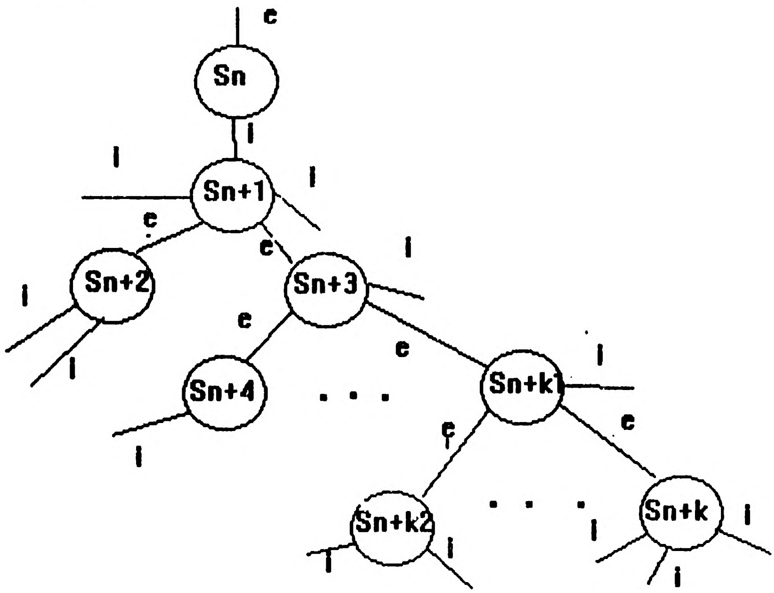

Figure 4

Then we have that

$$MP_n = \frac{\sigma(N_n \cap G)}{\sigma(G)},$$
$$MP_{n-1} = \frac{\sigma(N_{n-1} \cap G)}{\sigma(G)},$$

and

$$N_n = N_{n-1} \cup S_n.$$

Furthermore we have that

$$N_{n+k} = N_n - (\bigcup_{i \geq 1} S_{n+i}) = N_{n-1} \cup (S_n - (\bigcup_{i \geq 1} S_{n+i}),$$

so it results

$$N_{n-1} \subseteq N_{n+k},$$

which means that

$$MP_{n-1} \leq MP_{n+k}.$$

On the other hand we have that

$$MCP_n = \frac{\sigma((D - N_n) \cap (D - G)}{\sigma(D - G)},$$

$$D - N_{n+k} = D - (N_k - (\bigcup_{i\geq 1} S_{n+i})).$$

So it results that

$$D - N_n \subset D - N_{n+k},$$

which means that

$$MCP_n < MCP_{n+k}.$$

So we have shown that the value of MP is not decreased by this group, and the value of MCP is increased by it.

The similar case is when all of the internal links of the group are i-links and all of the external links are e-links. In this case we have that MP is increased by the added group and MCP is not decreased. The performance of the network is resulted from MP and MCP through the following formula:

$$h = p_1 MP + p_2 MCP,$$

where p_1 and p_2 are the percentages of the goal set and of the complementary of the goal set. So in both cases results that the performance of the network will be increased, because of MP or MCP is increased. It is easy to realize that the all neural tree is formed by such groups, so its performance is increased by adding these groups. So it results that the performance of the network will increase through the training procedure. This result is in concordance with the results in [1].

5. Application

As was mentioned before, the Mandelbrot set is used here to test the performance of the proposed neural network. For comparison, in [3] can be found the results of the self-organizing maps and of the reduced Coulomb potential networks for the same test, with similar training data.

Ten thousand points were used for the training, which were randomly selected with uniform distribution. The result was a network with 451 neurons. The performance of the network was the following:

- all correct classification: 96.9%
- correct classification of the member points: 97.6%

The graphical approximation of the Mandelbrot set, realized by this network is presented in the Figure 5.

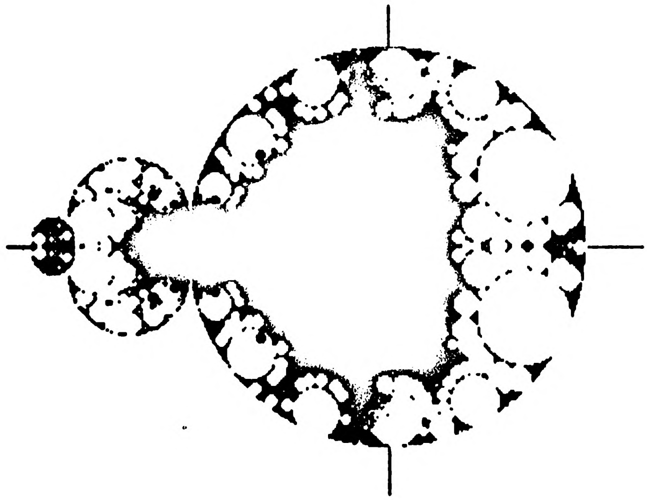

Figure 5

6. Conclusions

The proposed generalized tree-structured neural networks are applicable for the approximation and representation of very complex shapes as the fractals. Their approximation property can be usefully applied in chaotic control or chaotic classification, which have applicability in chemical process control, physiological signal classification, or atmospheric physics. The proposed networks could be generalized more, to represent even multi - class structures, where are not only one goal set, but many.

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ON A BLENDING OPERATOR OF BERNSTEIN-STANCU TYPE

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Dedicated to Professor D.D. Stancu on his 70^{th} birthday

Abstract. Considering the generalized univariate operator Bernstein-Stancu type (1.1) we construct the blending operator (2.1) and we establish some properties of this operator.

1. Introduction

D.D.Stancu has constructed in [10] the positive linear operator

$$B_m^{<\alpha,p,q>} : C[0,1] \to C[0,1],$$

defined for any $f \in C[0,1]$ and any $x \in [0,1]$ by

$$B_m^{<\alpha,p,q>}(f)(x) = \sum_{k=0}^{m} w_{m,k}(x,\alpha)\, f\,(\frac{k+p}{m+q}) \tag{1.1}$$

where $\alpha = \alpha(m) \geq 0$, $0 \leq p \leq q$ and

$$w_{m,k}(x,\alpha) = \binom{m}{k} \cdot \frac{x^{[k,-\alpha]}(1-x)^{[m-k,-\alpha]}}{1^{[m,-\alpha]}}. \tag{1.2}$$

In (1.2), $x^{[k,-\alpha]}$ is the factorial power of x with the exponent k and the increment -α, i.e.

$$x^{[k,-\alpha]} = x(x+\alpha)...(x+(k-1)\alpha). \tag{1.3}$$

Clearly, if $\alpha = 0, p \neq 0$, or $q \neq 0, p = q = 0$ the operator (1.1) reduces to others operators of Bernstein-Stancu type [8], [9].

Let us denote $I = [0,1]$, $I^2 = [0,1] \times [0,1], \mathbf{R}^{\mathbf{I}^2} = \{f \mid f : I^2 \to \mathbf{R}\}$.

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If $f,g : I \to \mathbf{R}$, $\alpha = \alpha(m) \geq 0, 0 \leq p \leq q, \beta = \beta(n) \geq 0, 0 \leq s \leq t$ we denote by $B_m^{<\alpha,p,q>}$, $B_n^{<\beta,s,t>}$ the operators defined for any $f \in \mathbf{R}^I, g \in \mathbf{R}^I$ by (1.1) and respectively by

$$B_n^{<\beta,\, s,\, t>}(g)(y) = \sum_{l=0}^{n} w_{n,\, l}(y, \beta)\, g\left(\frac{l+s}{n+t}\right),\ y \in I. \tag{1.4}$$

We suppose that $f \in \mathbf{R}^{I^2}$ and we denote by ${}_xB_m^{<\alpha,\, p,\, q>}$, ${}_yB_n^{<\beta,\, s,\, t>}$ the parametric extensions of $B_m^{<\alpha,\, p,\, q>}, B_n^{<\beta,\, s,\, t>}$ defined respectively by

$${}_xB_m^{<\alpha,\, p,\, q>}(f)\ (x,y) = \sum_{k=0}^{m} w_{m,k}(x,\alpha)\ f\ (\frac{k+p}{m+q}, y) \tag{1.5}$$

$${}_yB_n^{<\beta,\, s,t>}(f)\ (x,y) = \sum_{l=0}^{n} w_{n,\,l}(y,\beta)\, f\ (x, \frac{l+s}{n+t}). \tag{1.6}$$

Using these extensions we shall construct a sequence $\{U_{m,n}\}_{(m,n)\in \mathbf{N}^2}$ so that $U_{m,\,n}(f) \to f$, uniformly on I^2 for any function f , B-continuous on I^2.

We recall that f is namely B-continuous at $(x,y) \in I^2$ if and only if

$$\lim_{(x',\ y')\to(x\ ,y)} \Delta_{x',y'}[f : x, y] = 0 \tag{1.7}$$

where

$$\Delta_{x',y'}[f; x, y] = f(x',y') - f(x,y') - f(x',y) + f(x,y) \tag{1.8}$$

If (1.7) holds for any point of I^2, we say that f is B-continuous on I^2 and we denote by $C_b(I^2)$ the set of B-continuous functions on I^2.

An important result for the approximation of B-continuous functions is the following Korovkin type criterion.

Theorem 1.1. *[1] Let $\{L_{m,n}\}_{m,n\in N^2}$ be a sequence of positive linear operators transforming functions of $\mathbf{R}^{I^2}$ into functions of $\mathbf{R}^{I^2}$ so that for all $(x,y) \in I^2$ one has*

(i) $L_{m,n}(e)(x,y) = 1$

(ii) $L_{m,n}(\varphi)(x,y) = x + u_{m,n}(x,y)$

(iii) $L_{m,n}(\phi)(x,y) = y + v_{m,n}(x,y)$

(iv) $L_{m,n}(\varphi^2 + \phi^2)(x,y) = x^2 + y^2 + w_{m,n}(x,y)$

where $u_{m,n}(x,y), v_{m,n}(x,y), w_{m,n}(x,y)$ converge to zero uniform on I^2 as m, n tend to infinity. If $f(\cdot,*) \in C_b(I^2)$ and $(x,y) \in I^2$, we put

$$U_{m,n}(f)(x,y) = L_{m,n}(f(\cdot,y) + f(x,*) - f(\cdot,*); x, y)$$

Under these conditions, for every $f \in C_b(I^2)$ the sequence $\{U_{m,n}(f)\}$ converges to f uniformly on I^2.

To estimate the error in the approximation of $f \in R^{I^2}$ by $U_{m,n}(f)$, the mixed modulus of continuity ω_{mixed} is used. This modulus is the function $\omega_{mixed} : \mathbf{R}_+ \times \mathbf{R}_+$, defined by

$$\omega_{mixed}(\delta_1, \delta_2) = \sup\{\Delta_{x',y'}[f; x, y]|\ |x-y| < \delta_1, |y'-y| < \delta_2\} \tag{1.9}$$

for any $\delta_1, \delta_2 \in R_+ = (0, +\infty)$.

The most important properties of ω_{mixed} are contained in

Theorem 1.2. *[2]The mixed modulus of continuity has the properties:*

(i) $\omega_{mixed}(\delta_1, \delta_2) \leq \omega_{mixed}(\delta_1^*, \delta_2^*)$, *for any* $\delta_1, \delta_2, \delta_1^*, \delta_2^* \in \mathbf{R}_+$ *with* $\delta_1 < \delta_1^*$, $\delta_2 < \delta_2^*$.

(ii) $\omega_{mixed}(\lambda_1\delta_1, \lambda_2\delta_2) \leq (1+\lambda_1)(1+\lambda_2)\omega_{mixed}(\delta_1, \delta_2)$,*for any* $\delta_1, \delta_2, \lambda_1, \lambda_2 \in \mathbf{R}_+$.

2. Main results.

Lemma 2.1. *Let be $f \in R^{I^2}$. The parametric extensions (1.4) and (1.5) of the generalized Bernstein-Stancu operator (1.1) comute. Their product is the linear positive operator $L_{m,n} : \mathbf{R}^{\mathbf{I}^2} \to \mathbf{R}^{\mathbf{I}^2}$, defined for any $f \in \mathbf{R}^{I^2}$ by*

$$L_{m,n}(f)(x,y) = \sum_{k=0}^{m}\sum_{l=0}^{n} w_{m,k}(x,\alpha) w_{n,l}(y,\beta) f\left(\frac{k+p}{m+q}, \frac{l+s}{n+t}\right). \tag{2.1}$$

Proof. The conclusions of the lemma 2.1 are verified by simply computation. □

Lemma 2.2. *The operator $B_{m,n}^{<\alpha,\beta,p,q,s,t>} : C_b(I^2) \to C_b(I^2)$,*

$$B_{m,n}^{<\alpha,\beta,p,q,s,t>}(f)(x,y) = L_{m,n}[f(\cdot, y) + f(x, *) - f(\cdot, *); (x,y)] \tag{2.2}$$

where $f(\cdot, *) \in C_b(I^2)$, is a well-defined linear operator on $C_b(I^2)$.

Proof. If $(x,y) \in I^2$ is fixed, the B-continuity of f implies that of the function $F(\cdot, *) = f(\cdot, y) + f(x, *) - f(\cdot, *)$. This is a consequence of the fact that for all $(u,v), (s,t) \in I^2$ one has $\Delta_{u,v}[F; s, t] = -\Delta_{u,v}[f; s, t]$ independently of $(x,y) \in I^2$. Hence $B_{m,n}^{<\alpha,\beta,p,q,s,t>}$ is a well-defined linear operator on $C_b(I^2)$. □

Remark 2.1. By the linearity of $L_{m,n}$ it results that

$$B_{m,n}^{<\alpha,\beta,p,q,s,t>} =_x B_m^{<\alpha,\ p,\ q>} \bigoplus_y B_n^{<\beta,\ p,\ q>} =_x B_m^{<\alpha,\ p,\ q>} + B_n^{<\beta,\ p,\ q>} - L_{m,n},$$

i.e. $B_{m,n}^{<\alpha,\beta,p,q,s,t>}$ is the boolean sum of the parametric extensions ${}_xB_m^{<\alpha,\ p,\ q>}$ *and* ${}_yB_n^{<\beta,\ p,\ q>}$.

Remark 2.2. The operator $B_{m,n}^{<\alpha,\beta,p,q,s,t>}$ associates to the function $f \in C_b(I^2)$, the pseudo-polynomial $B_{m,n}^{<\alpha,\beta,p,q,s,t>}(f)$, defined by

$$B_{m,n}^{<\alpha,\beta,p,q,s,t>}(f)(x,y) = \sum_{k=0}^{m}\sum_{l=0}^{n} w_{m,k}(x,\alpha)w_{n,l}(y,\beta)[f(\frac{k+p}{m+q},y) + f(x,\frac{l+s}{n+t}) - f(\frac{k+p}{m+q},\frac{l+s}{n+t})]. \tag{2.3}$$

If $p = q = s = t = 0$, one obtains the Bernstein-Stancu blending operator $B_{m,n}^{<\alpha,\beta>}$ studied in the paper [1] and also in our paper [3],[4].

If $p = q = s = t = 0$ and $\alpha = \beta = 0$, one obtains the classical Bernstein operator $B_{m,n}$, which was considered in [7].

Lemma 2.3. *If* $0 \leq \alpha(m) \to 0(m \to \infty), 0 \leq \beta(n) \to 0(n \to \infty)$, *the operator* $L_{m,n}$ *has the following properties:*

(i) $L_{m,n}(e)(x,y) = 1$

(ii) $L_{m,n}(\varphi)(x,y) = x + u_{m,n}(x,y)$

(iii) $L_{m,n}(\phi)(x,y) = y + v_{m,n}(x,y)$

(iv) $L_{m,n}(\varphi^2 + \phi^2)(x,y) = x^2 + y^2 + w_{m,n}(x,y)$

where $u_{m,n}(x,y), v_{m,n}(x,y), w_{m,n}(x,y)$ *converge to zero uniformly on* I^2 *as* m,n *tend to infinity.*

Proof. By direct computation, we obtain

$$B_{m,n}^{<\alpha,\beta,p,q,s,t>}(e)(x,y) = 1, \qquad (\forall)(x,y) \in I^2$$

$$B_{m,n}^{<\alpha,\beta,p,q,s,t>}(\varphi)(x,y) = x + \frac{p-qx}{m+q}, \qquad (\forall)(x,y) \in I^2$$

$$B_{m,n}^{<\alpha,\beta,p,q,s,t>}(\phi)(x,y) = y + \frac{s-ty}{n+t}, \qquad (\forall)(x,y) \in I^2$$

$$B_{m,n}^{<\alpha,\beta,p,q,s,t>}(\varphi^2+\phi^2)(x,y) = x^2 + y^2 + \frac{m^2}{(m+q)^2}\frac{x(1-x)}{1+\alpha}\left(\frac{1}{m}+\alpha\right) -$$

$$-\frac{q(2m+q)}{(m+q)^2}x^2 + \frac{p(p+2mx)}{(m+q)^2} + \frac{n^2}{(n+t)^2}\frac{y(1-y)}{1+\beta}\left(\frac{1}{n}+\beta\right) -$$

$$-\frac{t(2n+t)}{(n+t)^2}y^2 + \frac{s(s+2ny)}{(n+t)^2}$$

and from these expressions it follows that the assertions are valid. □

Theorem 2.1. *If* $0 \le \alpha(m) \to 0 (m \to \infty), 0 \le \beta(n) \to 0 (n \to \infty)$, *then* $\{B_{m,n}^{<\alpha,\beta,p,q,s,t>}(f)\}_{(m,n)\in\mathbb{N}^2}$ *converges to* f, *uniformly on* I^2, *for any* $f \in C_b(I^2)$.

Proof. From the hypothesis, it results that $L_{m,n}$ has the properties (i)-(iv) from the Lemma 2.3. Then we obtain the conclusion of the statement as a consequence of the theorem 1.1. □

Remark 2.3. If $p = q = s = t = 0$,the theorem 2.1. is reduced to a result established in [1] and also in [4].

If $p = q = s = t = \alpha = \beta = 0$, the theorem 2.1. is reduced to the main result of the paper [7].

Theorem 2.2. *If* $f \in C_b(I^2)$, *one has*

$$\sup_{(x,y)\in I^2} q \mid f(x,y) - B_{m,n}^{<\alpha,\beta,p,q,s,t>}(f)(x,y) \le$$

$$\le \left(\frac{3}{2}\right)^2 \omega_{mixed}\left(\frac{2}{m+q}\sqrt{p(p+2m)+\frac{m(1+m\alpha)}{4(1+\alpha)}}, \frac{2}{n+t}\sqrt{s(s+2n)+\frac{n(1+n\beta)}{4(1+\beta)}}\right) \tag{2.4}$$

*Proof.*Taking into account of the definition of ω_{mixed} one has

$$(2.5)\quad \mid f(x,y) - B_{m,n}^{<\alpha,\beta,p,q,s,t>}(f)(x,y) \le \sum_{k=0}^{m}\sum_{k=0}^{n}$$

$$\omega_{mixed}\left(\mid x - \frac{k+p}{m+q}\mid, \mid y - \frac{l+s}{n+t}\mid\right) w_{m,k}(x,\alpha)w_{n,l}(y,\beta).$$

Let be $\delta_1 > 0, \delta_2 > 0$;applying the theorem 2.1 (ii) with $\lambda_1 = \frac{|x-\frac{k+p}{m+q}|}{\delta_1}, \lambda_2 = \frac{|y-\frac{l+s}{n+t}|}{\delta_2}$,it results

$$(2.6)\qquad | f(x,y) - B_{m,n}^{<\alpha,\beta,p,q,s,t>}(f)(x,y) \leq$$
$$\leq \omega_{mixed}(\delta_1,\delta_2)\cdot\left(1+\frac{1}{\delta_1}\sum_{k=0}^{m} | x-\frac{k+p}{m+q} | w_{m,k}(x,\alpha)\right)\cdot$$
$$\left(1+\frac{1}{\delta_2}\sum_{k=0}^{n} | y-\frac{l+s}{n+t} | w_{n,l}(y,\beta)\right)\cdot$$

From the inequality of Schwarz, one has

$$(2.7)\qquad \sum_{k=0}^{m} | x-\frac{k+p}{m+q} | w_{m,k}(x,\alpha) \leq \sqrt{\sum_{k=0}^{m}\left(x-\frac{k+p}{m+q}\right)^2 w_{m,k}(x,\alpha)}$$
$$\sum_{k=0}^{n} | y-\frac{l+s}{n+t} | w_{n,l}(y,\beta) \leq \sqrt{\sum_{l=0}^{n}\left(y-\frac{l+s}{n+t}\right)^2 w_{n,l}(y,\beta)}$$

After some transformations, from (2.6) and (2.7) it results

$$(2.8)\qquad | f(x,y) - B_{m,n}^{<\alpha,\beta,p,q,s,t>}(f)(x,y) \leq$$
$$\leq \omega_{mixed}(\delta_1,\delta_2)\cdot\left(1+\frac{1}{\delta_1}\cdot\frac{1}{m+q}\sqrt{p(p+2m)+\frac{m(1+m\alpha)}{4(1+\alpha)}},\right)\cdot$$
$$\cdot\left(1+\frac{1}{\delta_2}\cdot\frac{1}{n+t}\sqrt{s(s+2n)+\frac{n(1+n\beta)}{4(1+\beta)}}\right)$$

Choosing

$$\delta_1 = \frac{2}{m+q}\sqrt{p(p+2m)+\frac{m(1+m\alpha)}{4(1+\alpha)}},$$
$$\delta_2 = \sqrt{s(s+2n)+\frac{n(1+n\beta)}{4(1+\beta)}}$$

in (2.8) and taking the supremum, we obtain (2.4).

As a consequence of the theorem 2.2, one has

Theorem 2.3. *[3] If $f \in C_b(I^2)$,then*

$$(2.9)\qquad \sup_{(x,y)\in I^2} | f(x,y) - B_{m,n}^{<\alpha\beta>}(f)(x,y) | \leq \left(\frac{3}{2}\right)^2 \omega_{mixed}\left(\sqrt{\frac{1+m\alpha}{m(1+\alpha)}},\sqrt{\frac{1+n\beta}{n(1+\beta)}}\right)$$

Proof. In the theorem 2.2 we put $p = q = s = t = 0$. □

As a consequence of theorem 2.3., one obtains

Theorem 2.4. *[2]If $f \in C_b(I^2)$, then (2.10)* $\sup_{(x,y)\in I^2} | f(x,y) - B_{m,n}(f)(x,y) | \leq \left(\frac{3}{2}\right)^2 \omega_{mixed}\left(\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)$.

Proof. In the theorem 2.3. we put $\alpha = \beta = 0$. □

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SHAPE PRESERVING PROPERTIES OF SOME SPLINE-TYPE OPERATOR

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Dedicated to Professor D.D. Stancu on his 70^{th} birthday

Abstract. In the present note there are investigated properties of conservation of monotony and convexity of a function which is the argument of a linear and positive spline-type operator.

1.

This paper is an investigation of some shape preserving properties of the spline positive linear operator introduced and studied by Stancu [7, 8]. The considered positive linear operator is a slight generalization of the Schoenberg operator [6] by using some nodes depending on two non-negative parameters. The Schoenberg spline-type linear positive operator was investigated in detail by Marsden and Schoenberg [5] and by Marsden [3, 5]. The bivariate Schoenberg spline-type operator were considered in [1, 2]. We also remark the shape properties of the Schoenberg operator were pointed out in [3].

Let us consider a partition Δ_n (with $n \geq 1$) of the interval $[0,1]$ defined by

$$\Delta_n \,:\, 0 = x_0 < x_1 \leq x_2 \leq \cdots \leq x_{n-1} < x_n = 1,$$

and the corresponding extended partition $\Delta_{n,k}$ (with $k \geq 1$) given by

$$\Delta_{n,k} \,:\, x_{-k} = \cdots = x_0 = 0 < x_1 \leq x_2 \leq \cdots \leq x_{n-1} < 1 = x_n = \cdots = x_{n+k}.$$

The knots of these partitions satisfy the condition $x_{i-k} < x_i$, for $k < i < n$, i.e. at the most $k-1$ order of the multiplicities of knots situated in the interval $(0,1)$ is allowed.

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Using the extended partition $\Delta_{n,k}$ and the $(k+1)$-th divided differences, the B-spline functions of degree k

$$M_{i,k}(x) = \left[x_i, \ldots, x_{i+k+1}; (k+1)(t-x)_+^k\right], \qquad i = \overline{-k, n-1}, \tag{1}$$

are defined. The normalized B-splines of degree k are given by

$$N_{i,k}(x) = \frac{x_{i+k+1} - x_i}{k+1} M_{i,k}(x) = (x_{i+k+1} - x_i)\left[x_i, \ldots, x_{i+k+1}; (t-x)_+^k\right]. \tag{2}$$

Considering two real parameters α and β satisfying the condition $0 \leq \alpha \leq \beta$, the following nodes

$$\xi_{i,k}^{\alpha,\beta} = \frac{x_{i+1} + \cdots + x_{i+k} + \alpha}{k+\beta}, \quad i = \overline{-k, n-1}$$

are given. These nodes satisfy the following relations

$$0 \leq \xi_{-k,k}^{\alpha,\beta} < \xi_{-k+1,k}^{\alpha,\beta} < \cdots < \xi_{n-1,k}^{\alpha,\beta} \leq 1, \tag{1}$$

and

$$\xi_{i,k}^{\alpha,\beta} - \xi_{i-1,k}^{\alpha,\beta} = \frac{x_{i+k} - x_i}{k+\beta}, \quad i = \overline{-k, n-1}. \tag{2}$$

The generalized Schoenberg-type linear positive spline operator $S_{n,k}^{\alpha,\beta}$ introduced by Stancu is defined, for any function $f : [0,1] \to \mathbb{R}$, by the following formula

$$\left(S_{n,k}^{\alpha,\beta} f\right)(x) = S_{n,k}^{\alpha,\beta}(f; x) = \sum_{i=-k}^{n-1} N_{i,k}(x) f\left(\xi_{i,k}^{\alpha,\beta}\right), \quad x \in [0,1]. \tag{3}$$

2.

The first two derivatives of the spline function $\left(S_{n,k}^{\alpha,\beta} f\right)(x)$ will be calculated. Taking into account the formulas (1) and (2) we have

$$N'_{i,k}(x) = \begin{cases} -M_{-k+1,k-1}(x), & \text{for } i = -k, \\ M_{i,k-1}(x) - M_{i+1,k-1}(x), & \text{for } i = \overline{-k+1, n-2}, \\ M_{n-1,k-1}(x), & \text{for } i = n-1, \end{cases} \tag{4}$$

when $k > 1$.

If we use the formula (4), for the first derivative we have successively

$$\left(S_{n,k}^{\alpha,\beta} f\right)'(x) = \sum_{i=-k}^{n-1} N_{i,k}'(x) f\left(\xi_{i,k}^{\alpha,\beta}\right) \tag{3}$$

$$= M_{n-1,k-1}(x) f\left(\xi_{n-1,k}^{\alpha,\beta}\right) - M_{-k+1,k-1}(x) f\left(\xi_{-k,k}^{\alpha,\beta}\right) \tag{4}$$

$$+ \sum_{i=-k+1}^{n-2} \left[M_{i,k-1}(x) - M_{i+1,k-1}(x)\right] f\left(\xi_{i,k}^{\alpha,\beta}\right) \tag{5}$$

$$= M_{n-1,k-1}(x) f\left(\xi_{n-1,k}^{\alpha,\beta}\right) - M_{-k+1,k-1}(x) f\left(\xi_{-k,k}^{\alpha,\beta}\right) \tag{6}$$

$$+ \sum_{i=-k+1}^{n-2} M_{i,k-1}(x) f\left(\xi_{i,k}^{\alpha,\beta}\right) - \sum_{i=-k+2}^{n-1} M_{i,k-1}(x) f\left(\xi_{i-1,k}^{\alpha,\beta}\right) \tag{7}$$

$$= \sum_{i=-k+1}^{n-1} M_{i,k-1}(x) f\left(\xi_{i,k}^{\alpha,\beta}\right) - \sum_{i=-k+1}^{n-1} M_{i,k-1}(x) f\left(\xi_{i-1,k}^{\alpha,\beta}\right) \tag{8}$$

$$= \sum_{i=-k+1}^{n-1} M_{i,k-1}(x) \left[f\left(\xi_{i,k}^{\alpha,\beta}\right) - f\left(\xi_{i-1,k}^{\alpha,\beta}\right)\right] \tag{9}$$

$$= \frac{k}{k+\beta} \sum_{i=-k+1}^{n-1} M_{i,k-1}(x) \frac{x_{i+k} - x_i}{k} \left[\xi_{i-1,k}^{\alpha,\beta}, \xi_{i,k}^{\alpha,\beta}; f\right]. \tag{10}$$

In this way we have that

$$\left(S_{n,k}^{\alpha,\beta} f\right)'(x) = \sum_{i=-k+1}^{n-1} \left[f\left(\xi_{i,k}^{\alpha,\beta}\right) - f\left(\xi_{i-1,k}^{\alpha,\beta}\right)\right] M_{i,k-1}(x) \tag{11}$$

$$= \frac{k}{k+\beta} \sum_{i=-k+1}^{n-1} N_{i,k-1}(x) \left[\xi_{i-1,k}^{\alpha,\beta}, \xi_{i,k}^{\alpha,\beta}; f\right]. \tag{5}$$

In a similar manner, for the second derivative we have

$$\left(S_{n,k}^{\alpha,\beta} f\right)''(x) = \sum_{i=-k}^{n-1} N_{i,k}''(x) f\left(\xi_{i,k}^{\alpha,\beta}\right) = \frac{k}{k+\beta} \sum_{i=-k+1}^{n-1} N_{i,k-1}'(x) \left[\xi_{i-1,k}^{\alpha,\beta}, \xi_{i,k}^{\alpha,\beta}; f\right] \tag{12}$$

$$= \frac{k}{k+\beta} \sum_{i=-k+2}^{n-1} M_{i,k-2}(x) \left(\left[\xi_{i-1,k}^{\alpha,\beta}, \xi_{i,k}^{\alpha,\beta}; f\right] - \left[\xi_{i-2,k}^{\alpha,\beta}, \xi_{i-1,k}^{\alpha,\beta}; f\right]\right), \tag{13}$$

and therefore

$$\left(S_{n,k}^{\alpha,\beta} f\right)''(x) = \sum_{i=-k+2}^{n-1} M_{i,k-2}(x) \left(\xi_{i,k}^{\alpha,\beta} - \xi_{i-2,k}^{\alpha,\beta}\right) \left[\xi_{i-2,k}^{\alpha,\beta}, \xi_{i-1,k}^{\alpha,\beta}, \xi_{i,k}^{\alpha,\beta}; f\right], \tag{6}$$

when $k > 2$.

It should be observed that if $x_{i-k+1} < x_i$, for $k \leq i < n$, then $\xi^{\alpha,\beta}_{i,k-1} - \xi^{\alpha,\beta}_{i-1,k-1} = \frac{x_{i+k-1}-x_i}{k+\beta-1} > 0$, and

$$M_{i,k-2}(x) = \frac{k-1}{x_{i+k-1}-x_i} N_{i,k-2}(x) = \frac{k-1}{k+\beta-1} \cdot \frac{N_{i,k-2}(x)}{\xi^{\alpha,\beta}_{i,k-1} - \xi^{\alpha,\beta}_{i-1,k-1}} .$$

Thus the following formula

$$\left(S^{\alpha,\beta}_{n,k} f\right)''(x) = \frac{k(k-1)}{(k+\beta)(k+\beta-1)} \tag{14}$$

$$\times \sum_{i=-k+2}^{n-1} \left[\xi^{\alpha,\beta}_{i-2,k}, \xi^{\alpha,\beta}_{i-1,k}, \xi^{\alpha,\beta}_{i,k}; f\right] \frac{\xi^{\alpha,\beta}_{i,k} - \xi^{\alpha,\beta}_{i-2,k}}{\xi^{\alpha,\beta}_{i,k-1} - \xi^{\alpha,\beta}_{i-1,k-1}} N_{i,k-2}(x) \tag{7}$$

is obtained.

3.

If we use the formulas giving the first two derivatives of the spline function $\left(S^{\alpha,\beta}_{n,k} f\right)(x)$ we shall obtain the preserving properties of the linear positive operator $S^{\alpha,\beta}_{n,k}$.

Indeed, if f is an increasing (resp. decreasing) function on the interval $[0,1]$ then all divided differences in formula (5) are positive (resp. negative). Such that the first derivative of the spline function $S^{\alpha,\beta}_{n,k}(f)$ is a positive (resp. negative) function , i.e. the operator $S^{\alpha,\beta}_{n,k}$ preserves the monotonicity of the function f.

Analogously, if f is a convex (resp. concave) function on the interval $[0,1]$, then all divided differences of the second order in formula (6) are positive (resp. negative). Taking into account that $\xi^{\alpha,\beta}_{i,k} - \xi^{\alpha,\beta}_{i-2,k} > 0$ we have that the second derivative of the spline function $S^{\alpha,\beta}_{n,k}(f)$ is a positive (resp. negative) function, i.e. the operator $S^{\alpha,\beta}_{n,k}$ preserves the convexity of the function f.

We remark that the convexity of the order two is not preserved. Taking the same example as in [3], $f(x) = x^2$, which is a convex function of the order two, but the corresponding spline function $S^{\alpha,\beta}_{n,k}(f)$ $(k > 3)$ is not a convex function of the order two.

Firstly, if $f(x) = x^2$ it can be readily shown that the following formula for the third derivative of the spline function $S_{n,k}^{\alpha,\beta}(f)$

$$\left(S_{n,k}^{\alpha,\beta} f\right)'''(x) = \frac{k(k-1)}{(k+\beta)(k+\beta-1)} \tag{15}$$

$$\times \sum_{i=-k+3}^{n-1} \left(\frac{\xi_{i,k}^{\alpha,\beta} - \xi_{i-2,k}^{\alpha,\beta}}{\xi_{i,k-1}^{\alpha,\beta} - \xi_{i-1,k-1}^{\alpha,\beta}} - \frac{\xi_{i-1,k}^{\alpha,\beta} - \xi_{i-3,k}^{\alpha,\beta}}{\xi_{i-1,k-1}^{\alpha,\beta} - \xi_{i-2,k-1}^{\alpha,\beta}} \right) M_{i,k-3}(x) \tag{16}$$

holds. Taking the equally spaced knots $x_i = \frac{i}{n}$, $i = \overline{0,n}$, we have

$$\left(S_{n,k}^{\alpha,\beta} f\right)'''(0) = -\frac{2nk(k-1)(k-2)}{(k+\beta)^2},$$

and therefore $\left(S_{n,k}^{\alpha,\beta} f\right)'''(0) < 0$. This implies that the spline function $\left(S_{n,k}^{\alpha,\beta} f\right)(x)$ is not a convex function on $[0,1]$.

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HEREDITARY PREDICATES

N. BOTH AND I. PURDEA

Dedicated to Professor D.D. Stancu on his 70^{th} anniversary

Abstract. A general definition of hereditary predicates (predicative formulas) is given, as well as their relation to monotony and to elementary predicates.

The notion of heredity surpasses the frame of mathematics. As an example of hereditary property is the notion of (linear, algebraic, relational) independence (see [1]) in the sense that if a set is independent then each of (nonempty!) subset is independent.

In this article we get a generalization of hereditary properties and some particular cases of heredity, as the logical monotony (see [3], [2]).

1. Heredity and co-heredity

Let P be the set of bivalent propositions, $\Rightarrow$, the logical implication and (A, r) a relational structure, where $r = (A, A, R)$ is a binary relation.

The unary predicate $\mathcal{P} : A \to P$ is called $(r, \Rightarrow)$-*compatible* if

(comp.) $xry \;\Rightarrow\; (\mathcal{P}(x) \;\Rightarrow\; \mathcal{P}(y))$ (see [5]).

Particularly, if (A, r) is $(\mathcal{P}(M), \subseteq)$ and $\mathcal{P}$ is an unary predicate on the Power set $\mathcal{P}(M)$, then $\mathcal{P}$ is called *hereditary* predicate if

(er). $X \subseteq Y \;\Rightarrow\; (\mathcal{P}(Y) \;\Rightarrow\; \mathcal{P}(X))$.

Analogously, $\mathcal{P}$ is *co-hereditary* if

(co). $X \subseteq Y \;\Rightarrow\; (\mathcal{P}(X) \;\Rightarrow\; \mathcal{P}(Y))$.

The fact that $\mathcal{P}$ has one of the properties (co or er) will be denoted by "co $\mathcal{P}$", "er $\mathcal{P}$" respectively.

Example 1. In the vector-space $(K, V, +, \cdot)$ with two predicates: $\mathcal{P}, Q : \mathcal{P}(V) \to P$,

$$\mathcal{P}(X) = \text{"}X \text{ is independent"}$$

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$$\mathcal{Q}(X) = \text{"}X \text{ generates } V\text{"},$$

we have er $\mathcal{P}$ and co $\mathcal{Q}$.

Remark 1. Observe the "duality" between the predicates $\mathcal{P}$ and $\mathcal{Q}$ above.

Theorem 1. *(i). co* $\mathcal{P} \Leftrightarrow$ *er* $\overline{\mathcal{P}}$

(ii). co $\mathcal{P}$ *and co* $\mathcal{Q} \Rightarrow$ *co* $(\mathcal{P} \wedge \mathcal{Q})$

(iii). co $\mathcal{P}$ *and co* $\mathcal{Q} \Rightarrow$ *co* $(\mathcal{P} \vee \mathcal{Q})$

(j). er $\mathcal{P} \Leftrightarrow$ *co* $\overline{\mathcal{P}}$

(jj). er $\mathcal{P}$ *and er* $\mathcal{Q} \Rightarrow$ *er* $(\mathcal{P} \wedge \mathcal{Q})$

(jjj). er $\mathcal{P}$ *and er* $\mathcal{Q} \Rightarrow$ *er* $(\mathcal{P} \vee \mathcal{Q})$

Proof. For $X, Y \in \mathcal{P}(M)$, denote p_X, p_Y, q_X, q_Y the propositions $\mathcal{P}(X), \mathcal{P}(Y), \mathcal{Q}(X)$ and $\mathcal{Q}(Y)$ respectively. We accept as a general assumption: $X \subseteq Y$, and so, the corresponding properties may be formulated by:

$$\text{er } \mathcal{P} : P_Y \supset P_X, \quad \text{er } \mathcal{Q} : P_Y \supset P_X$$

$$\text{co } \mathcal{P} : P_X \supset P_Y, \quad \text{co } \mathcal{Q} : P_X \supset P_Y,$$

where $\supset$ is the propositional implication (as binary operation).

Now, we may prove the Theorem 1 starting from the truth-table below, having $2^4 = 16$ lines and 18 columns. The columns (7), (8), (9), (10), (11), (16), (17) and (18) represent the properties: co $\mathcal{P}$, co $\mathcal{Q}$, er $\mathcal{P}$, er $\mathcal{Q}$, co $\overline{\mathcal{P}}$, er $\overline{\mathcal{P}}$, co $(\mathcal{P} \wedge \mathcal{Q})$ and co $(\mathcal{P} \vee \mathcal{Q})$. Moreover, for the second part of Theorem 1, on finds out that er $(\mathcal{P} \wedge \mathcal{Q})$ and er $(\mathcal{P} \vee \mathcal{Q})$ is given by (13) $\supset$ (12) and (15) $\supset$ (14).

Recall the significance of the columns:

$(1) = p_X$ $\quad (7) = p_X \supset p_Y$ $\quad (13) = p_Y \wedge q_Y$

$(2) = p_Y$ $\quad (8) = q_X \supset q_Y$ $\quad (14) = p_X \vee q_X$

$(3) = q_X$ $\quad (9) = p_Y \supset p_X$ $\quad (15) = p_Y \vee q_Y$

$(4) = q_Y$ $\quad (10) = q_Y \supset q_X$ $\quad (16) = (6) \supset (5)$

$(5) = \overline{p}_X$ $\quad (11) = (5) \supset (6)$ $\quad (17) = (12) \supset (13)$

$(6) = \overline{p}_Y$ $\quad (12) = p_X \wedge q_X$ $\quad (18) = (14) \supset (15)$.

Lin. \ Col.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	0	0	0	0	1	1	1	1	1	1	1	0	0	0	0	1	1	1
2	0	0	0	1	1	1	1	1	1	0	1	0	0	0	1	1	1	1
3	0	0	1	0	1	1	1	0	1	1	1	0	0	1	0	1	1	0
4	0	0	1	1	1	1	1	1	1	1	1	0	0	1	1	1	1	1
5	0	1	0	0	1	0	1	1	0	1	0	0	0	0	1	1	1	1
6	0	1	0	1	1	0	1	1	0	0	0	0	1	0	1	1	1	1
7	0	1	1	0	1	0	1	0	0	1	0	0	0	1	1	1	1	1
8	0	1	1	1	1	0	1	1	0	1	0	0	1	1	1	1	1	1
9	1	0	0	0	0	1	0	1	1	1	1	0	0	1	0	0	1	0
10	1	0	0	1	0	1	0	1	1	0	1	0	0	1	1	0	1	1
11	1	0	1	0	0	1	0	0	1	1	1	1	0	1	0	0	0	0
12	1	0	1	1	0	1	0	1	1	1	1	1	0	1	1	0	0	1
13	1	1	0	0	0	0	1	1	1	1	1	0	0	1	1	1	1	1
14	1	1	0	1	0	0	1	1	1	0	1	0	1	1	1	1	1	1
15	1	1	1	0	0	0	1	0	1	1	1	1	0	1	1	1	0	1
16	1	1	1	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1

Finally, the proof is obtained by comparing of columns in the truth-table above, namely:

(7) and (16) for (i),

(7), (8) and (17) for (ii),

(7), (8) and (18) for (iii).

For (j), (jj), (jjj) - analogously. □

Remark 2. The converse of (ii) or (iii) is not true.

Remark 3. A "weak-converse" of (ii), namely

(iv). co($\mathcal{P} \wedge \mathcal{Q}$) $\Rightarrow$ co $\mathcal{P}$ or co $\mathcal{Q}$ holds.

Corollaries. *1. er $\mathcal{P}$ and co $\mathcal{Q}$ $\Rightarrow$ co ($\mathcal{P} \supset \mathcal{Q}$).*

2. co $\mathcal{P}$ and er $\mathcal{Q}$ $\Rightarrow$ er ($\mathcal{P} \supset \mathcal{Q}$).

2. Monotony and heredity

Let $\phi(\mathcal{P}_1, \ldots, \mathcal{P}_m; \mathcal{Q}_1, \ldots, \mathcal{Q}_n)$ be a predicative formula over $\mathcal{P}(M)$, containing the monadic predicates $\mathcal{P}_i, \mathcal{Q}_j$ only ($i = \overline{1,m}$ and $j = \overline{1,n}$).

Denote

$$S(\phi, \mathcal{P}_k, \varphi) = \phi(\mathcal{P}_1, \ldots, \mathcal{P}_{k-1}, \varphi, \mathcal{P}_{k+1}, \ldots, \mathcal{P}_m; \mathcal{Q}_1, \ldots, \mathcal{Q}_n).$$

The predicative formula ϕ is called *increasing* (or *decreasing*) *monotone* in $\mathcal{P}_k$ (see [2] and [3]) if it has the property:

(mon). $\varphi \supset \psi \Rightarrow S(\phi, \mathcal{P}_k, \varphi) \supset S(\phi, \mathcal{P}, \psi)$

or

(mon*). $\varphi \supset \psi \Rightarrow S(\phi, \mathcal{P}_k, \psi) \supset S(\phi, \mathcal{P}, \varphi)$.

Theorem 2. *If the formula ϕ is increasing in $\mathcal{P}_i$, $i = \overline{1,m}$ and decreasing in $\mathcal{Q}_j$, $j = \overline{1,n}$ then:*

a. co $\mathcal{P}$, $i = \overline{1,m}$ and er $\mathcal{Q}_j$, $j = \overline{1,n}$ $\Rightarrow$ co ϕ.

b. er $\mathcal{P}$, $i = \overline{1,m}$ and co $\mathcal{Q}_j$, $j = \overline{1,n}$ $\Rightarrow$ er ϕ.

Proof. (by induction on the construction of ϕ)

(α). If ϕ is an elementary formula then $\phi = \mathcal{P}_k$ or $= \mathcal{Q}_h$ and the proof is trivial.

(β). If ϕ has one of the forms:

$$\overline{\psi}, \quad \psi_1 \wedge \psi_2, \quad \psi_1 \vee \psi_2, \quad \psi_1 \supset \psi_2,$$

and suppose that for ψ, ψ_1, ψ_2 the Theorem 2 holds, then from Theorem 1 and his Corollaries, the proof results. □

3. Elementary predicates

Given the n-ary predicate on M, $p : M^n \to P$ (that is $p \in \Pi_M$, see [4]) and define the *universal extension* of p, $\mathcal{P}_p : \mathcal{P}(M) \to P$, $\mathcal{P}(X) = \forall\, x_i \in X : p(x_1, \ldots, x_n)$; also define the *existential extension* of p, $\mathcal{P}_p^* : \mathcal{P}(M) \to P$, $\mathcal{P}_p^*(X) = \exists\, x_i \in X : p(x_1, \ldots, x_n)$.

Call the predicate $\mathcal{P} : \mathcal{P}(M) \to P$, *n-elementary* (or simply *elementary*) if there exists $p : M^n \to P$ so that $\mathcal{P} = \mathcal{P}_p$. Analogously, $\mathcal{P}$ is called *n-coelementary* (or simply *coelementary*) if there exists an n-ary predicate p on M so that $\mathcal{P} = \mathcal{P}_p^*$. Write "el $\mathcal{P}$" (or "col $\mathcal{P}$") if $\mathcal{P}$ is elementary (or coelementary).

Theorem 3. *a) el* $\mathcal{P} \Rightarrow$ *er* $\mathcal{P}$.

b) col $\mathcal{P} \Rightarrow$ *co* $\mathcal{P}$.

Proof. Let $\mathcal{P}$ be a predicate on $\mathcal{P}(M)$ and $X, Y \subseteq M$, so that $x \subseteq Y$.

a). If el $\mathcal{P}$ then there is $p \in \Pi_M$ so that $\mathcal{P} = \mathcal{P}_p$. Therefore $\mathcal{P}(Y) = \mathcal{P}_p(Y) = \forall\, y_i \in Y : p(y_1, \ldots, y_n) \Rightarrow \forall\, y_i \in X : p(y_1, \ldots, y_n) = \mathcal{P}_p(X) = \mathcal{P}(X)$, that is er $\mathcal{P}$.

b). If col $\mathcal{P}$ then there is $p \in \Pi_M$ so that $\mathcal{P} = \mathcal{P}^*_p$. Therefore $\mathcal{P}(X) = \mathcal{P}^*_p(X) = \exists\, x_i \in X : p(x_1, \ldots, x_n) \Rightarrow \exists\, x_i \in Y : p(x_1, \ldots, x_n) = \mathcal{P}^*_p(Y) = \mathcal{P}(Y)$, that is co $\mathcal{P}$. □

Example 2. Let p be a ternary predicate on $(A, \cdot)$, $p(x_1, x_2, x_3) =$ "$(x_1x_2)x_3 = x_1(x_2x_3)$", and $\mathcal{P} = \mathcal{P}_p$. As 3-elementary, $\mathcal{P}$ is also hereditary (Theorem 3). Particularly it follows that each subgroupoid of a semigroup is semigroup too.

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AN EXTENSION OF THE SPLINE FUNCTIONS OF FOURIER TYPE

HANNELORE BRECKNER

Dedicated to Professor D.D. Stancu on his 70^{th} anniversary

Abstract. The aim of this paper is the investigation of a class of interpolation spline functions, which are more general then Fourier-type spline functions. One computes the analytical expression of this functions and a numerical example for there computation is provided.

1. Introduction

Let a and b be real numbers such that $a < b$ and let q be a positive integer. We denote by $\mathcal{H}^q[a,b]$ the real linear space of all functions $f : [a,b] \to \mathbb{R}$ whose derivatives of order q exist a.e. on $[a,b]$ and are square Lebesgue integrable. Endowed with the scalar product $\langle \cdot, \cdot \rangle_q : \mathcal{H}^q[a,b] \times \mathcal{H}^q[a,b] \to \mathbb{R}$ defined by

$$\langle f, g \rangle_q = \sum_{i=0}^{q} \int_a^b f^{(i)}(t) g^{(i)}(t) dt,$$

this space becomes a real Hilbert space (see [3], p. 155).

It is well known (see [3], p. 226) that a spline function of Fourier type represents the minimizer of the functional

$$f \in \mathcal{H}^q[-1,1] \mapsto \int_{-1}^{1} \left(f^{(q)}(t) \right)^2 dt \in \mathbb{R} \tag{1}$$

on the set consisting of all functions $f \in \mathcal{H}^q[-1,1]$ that satisfy interpolation conditions of the form

$$\int_{-1}^{1} \lambda_i(t) f(t) dt = y_i \qquad (i \in \{1, \ldots, n\}), \tag{2}$$

where n is a positive integer satisfying $n \geq q$, $\lambda_1, \ldots, \lambda_n$ are the first n consecutive Legendre polynomials with respect to the interval $[-1,1]$, and $y_1, \ldots, y_n$ are real numbers.

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The purpose of this paper is to introduce a kind of spline functions that solve a minimization problem which is more general than the problem mentioned above. For this end the special interval $[-1, 1]$ is replaced by the arbitrary interval $[a, b]$, the functional (1) by

$$f \in \mathcal{H}^q[a,b] \mapsto \int_a^b \left(f^{(q)}(t)\right)^2 dt \in \mathbb{R}, \tag{3}$$

and the Legendre polynomials $\lambda_1, \ldots, \lambda_n$ by n arbitrary polynomials $p_1, \ldots, p_n$ of degrees $q_1, \ldots, q_n$ respectively, where $q_1, \ldots, q_n$ are distinct positive integers satisfying $\{0, \ldots, q-1\} \subseteq \{q_1, \ldots, q_n\}$.

In the paper we prove the existence and uniqueness of such spline functions. Since the polynomials $p_1, \ldots, p_n$ that occur in our interpolation conditions are not orthogonal, the techniques used in the case of spline functions of Fourier type cannot be applied in our case. By investigating minutely the properties of the functions that we use, we arrive to results that have not been explicitly pointed out in the case when the Legendre polynomials were used. We also find an analytical expression for these spline functions. Their practical determination will be illustrated by a numerical example.

2. Some Preliminary Results

Let $\mathcal{L}^2[a,b]$ be the real Hilbert space consisting of all equivalence classes of functions $f : [a,b] \to \mathbb{R}$ which are square Lebesgue integrable. Let $\langle \cdot, \cdot \rangle_0$ be the scalar product and let $\|\cdot\|_0$ be the norm of this space. We denote also by f the equivalence class which contains the function f. We define $D : \mathcal{H}^q[a,b] \to \mathcal{L}^2[a,b]$ by

$$D(f) = f^{(q)}.$$

This mapping is linear and continuous. The linear subspace $D(\mathcal{H}^q[a,b])$ of $\mathcal{L}^2[a,b]$ will be denoted by H. In what follows it will be considered as a linear space endowed with the scalar product $\langle \cdot, \cdot \rangle_0$ and the norm $\|\cdot\|_0$. We also denote by D the mapping $D : \mathcal{H}^q[a,b] \to H$.

Proposition 2.1. *The space $(H, \|\cdot\|_0)$ is a Hilbert space.*

Proof. Let (f_n) be a Cauchy sequence in the space $(H, \|\cdot\|_0)$. We will show successively that for each $k \in \{1, \ldots, q-1\}$ there exists a sequence (F_n) in the space $\mathcal{H}^q[a,b]$ such that the following assertions hold:

$$(F_n^{(q-k)}) \text{ is a Cauchy sequence in } \mathcal{H}^k[a,b]; \tag{1}$$

$$D(F_n) = f_n \quad \text{for each } n \in \mathbb{N}. \tag{2}$$

Let $k = 1$ and consider an arbitrary $\epsilon > 0$. Since (f_n) is a Cauchy sequence, there exists an $n_0 \in \mathbb{N}$ such that

$$\int_a^b \Big(f_n(t) - f_m(t)\Big)^2 dt < \epsilon^2 \tag{3}$$

for all $m, n \in \mathbb{N}$ with $m \geq n_0, n \geq n_0$.

Fix an arbitrary $n \in \mathbb{N}$. Since f_n belongs to H, there exists a function $G_n \in \mathcal{H}^q[a,b]$ such that $D(G_n) = f_n$, i.e. $G_n^{(q)}(t) = f_n(t)$ a.e. $t \in [a,b]$. Because $G_n^{(q-1)}$ is an absolutely continuous function (see [1], Theorem (18.17), p. 286), it has the form

$$G_n^{(q-1)}(t) = \int_a^t f_n(s)ds + G_n^{(q-1)}(a) \quad \text{for each } t \in [a,b].$$

On the other hand, there exists a function $F_n \in \mathcal{H}^q[a,b]$ which satisfies

$$F_n^{(q-1)}(t) = G_n^{(q-1)}(t) - G_n^{(q-k-1)}(a) \quad \text{for each } t \in [a,b].$$

By derivation we obtain $F_n^{(q)}(t) = G_n^{(q)}(t)$ a.e. $t \in [a,b]$, which implies $D(F_n) = f_n$. So we have found a function $F_n \in \mathcal{H}^q[a,b]$ satisfying

$$F_n^{(q-1)}(t) = \int_a^t f_n(s)ds \quad \text{for each } t \in [a,b] \tag{4}$$

and $D(F_n) = f_n$.

The relations (3) and (4) imply

$$\Big(F_n^{(q-1)}(t) - F_m^{(q-1)}(t)\Big)^2 \leq (b-a)\int_a^b \Big(f_n(t) - f_m(t)\Big)^2 dt < (b-a)\epsilon^2$$

for all $m, n \in \mathbb{N}$ with $m \geq n_0$, $n \geq n_0$ and each $t \in [a,b]$. Hence it results that

$$\int_a^b \Big(F_n^{(q-1)}(t) - F_m^{(q-1)}(t)\Big)^2 dt \leq (b-a)^2\epsilon^2 \tag{5}$$

for all $m, n \in \mathbb{N}$ with $m \geq n_0$, $n \geq n_0$. Taking into account that

$$\|F_n^{(q-1)} - F_m^{(q-1)}\|_1^2 = \int_a^b \Big(F_n^{(q-1)}(t) - F_m^{(q-1)}(t)\Big)^2 dt + \int_a^b \Big(f_n(t) - f_m(t)\Big)^2 dt,$$

we deduce from (3) and (5) that $(F_n^{(q-1)})$ is a Cauchy sequence in $\mathcal{H}^1[a,b]$. Therefore there exists a sequence (F_n) in $\mathcal{H}^q[a,b]$ for which (1) and (2) hold.

Now we consider a number $k \in \{1,\ldots,q-1\}$ satisfying (1) and (2). We prove that these assertions are also true for $k+1$. By the hypothesis concerning k there exists a sequence (G_n) in the space $\mathcal{H}^q[a,b]$ such that $(G_n^{(q-k)})$ is a Cauchy sequence in the space $\mathcal{H}^k[a,b]$ and $D(G_n) = f_n$ for each $n \in \mathbb{N}$.

Consider an arbitrary $n \in \mathbb{N}$. Since $k \geq 1$ and $G_n \in \mathcal{H}^q[a,b]$, it follows that $G_n^{(q-k-1)}$ is differentiable and that its derivative $G_n^{(q-k)}$ is continuous. Therefore it can be written as

$$G_n^{(q-k-1)}(t) = \int_a^t G_n^{(q-k)}(s)ds + G_n^{(q-k-1)}(a) \tag{6}$$

for each $t \in [a,b]$. On the other hand, we can find a function $F_n \in \mathcal{H}^q[a,b]$ which satisfies

$$F_n^{(q-k-1)}(t) = G_n^{(q-k-1)}(t) - G_n^{(q-k-1)}(a) \quad \text{for each } t \in [a,b]. \tag{7}$$

By derivation we obtain $G_n^{(q-k+j)}(t) = G_n^{(q-k+j)}(t)$ for each $j \in \{0,\ldots,k-1\}$ and each $t \in [a,b]$, and also $F_n^{(q)}(t) = G_n^{(q)}(t)$ a.e. $t \in [a,b]$. Thus $D(F_n) = f_n$.

Take an arbitrary $\epsilon > 0$. Since $(G_n^{(q-k)})$ is a Cauchy sequence in the space $\mathcal{H}^k[a,b]$, there exists a number $n_0 \in \mathbb{N}$ such that the inequalities

$$\|G_n^{(q-k)} - G_m^{(q-k)}\|_k < \epsilon \tag{8}$$

and

$$\int_a^b \left(G_n^{(q-k)}(t) - G_m^{(q-k)}(t)\right)^2 dt \leq \epsilon^2 \tag{9}$$

hold for all $m,n \in \mathbb{N}$ with $m \geq n_0$, $n \geq n_0$. From (6), (7) and (9) it follows that

$$\left(F_n^{(q-k-1)}(t) - F_m^{(q-k-1)}(t)\right)^2 \leq (b-a)\int_a^b \left(G_n^{(q-k)}(t) - G_m^{(q-k)}(t)\right)^2 dt < (b-a)\epsilon^2$$

for $m,n \in \mathbb{N}$ with $m \geq n_0$, $n \geq n_0$ and each $t \in [a,b]$. This implies

$$\int_a^b \left(F_n^{(q-k-1)}(t) - F_m^{(q-k-1)}(t)\right)^2 dt < (b-a)^2\epsilon^2 \tag{10}$$

for $m,n \in \mathbb{N}$ with $m \geq n_0$, $n \geq n_0$. Taking into account that

$$\|F_n^{(q-k-1)} - F_m^{(q-k-1)}\|_{k+1}^2 = \int_a^b \left(F_n^{(q-k-1)}(t) - F_m^{(q-k-1)}(t)\right)^2 dt + \|G_n^{(q-k)} - G_m^{(q-k)}\|_k^2,$$

it follows from (8) and (10) that $(F_n^{(q-k-1)})$ is a Cauchy sequence in the space $\mathcal{H}^{k+1}[a,b]$. So there exists a sequence of functions in the space $\mathcal{H}^q[a,b]$ for which the assertions (1) and (2) hold.

Finally, it follows that for $k = q$ there exists a Cauchy sequence (F_n) in the space $\mathcal{H}^q[a,b]$ such that $D(F_n) = f_n$ for each $n \in \mathbb{N}$. Since $\mathcal{H}^q[a,b]$ is a Hilbert space, there exists a function $F \in \mathcal{H}^q[a,b]$ such that (F_n) converges to F. In virtue of the continuity of the mapping D the sequence $(D(F_n))$ converges to $D(F)$, which belongs to H. Therefore the sequence (f_n) is convergent. Consequently, $(H, \|\cdot\|_0)$ is a Hilbert space. □

In conclusion D is a linear, continuous and surjective mapping between two Hilbert spaces.

3. Main Result

Let $q \in \mathbb{N}$ be fixed and choose $n \in \mathbb{N}$ distinct positive integers $q_1, \dots, q_n$ such that $\{0, \dots, q-1\} \subseteq \{q_1, \dots, q_n\}$. For each $i \in \{1, \dots, n\}$ we consider a nonzero polynomial p_i of degree q_i and define the functional $\Phi_i : \mathcal{H}^q[a,b] \to \mathbb{R}$ by

$$\Phi_i(f) = \int_a^b p_i(t) f(t) dt.$$

In view of the Riesz Representation Theorem there exists for each $i \in \{1, \dots, n\}$ a unique $u_i \in \mathcal{H}^q[a,b]$ such that

$$\Phi_i(f) = \langle f, u_i \rangle_q \quad \text{for each } f \in \mathcal{H}^q[a,b].$$

We claim that the functions $u_1, \dots, u_n$ are linearly independent. Indeed, let $\alpha_1, \dots, \alpha_n$ be real numbers such that

$$\alpha_1 u_1 + \cdots + \alpha_n u_n = 0.$$

Then we have

$$\int_a^b (\alpha_1 p_1(t) + \cdots + \alpha_n p_n(t)) f(t) dt = 0$$

for each $f \in \mathcal{H}^q[a,b]$. In particular, for $f = \sum_{i=1}^{n} \alpha_i p_i$ we conclude that $f \equiv 0$. But the polynomials $p_1, \dots, p_n$ have distinct degrees and therefore they are linearly independent.

Hence we must have $\alpha_i = 0$ for each $i \in \{1, \ldots, n\}$. In other words, the functions $u_1, \ldots, u_n$ are linearly independent as claimed.

Let the mapping $\Phi : \mathcal{H}^q[a, b] \to \mathbb{R}^n$ be defined by

$$\Phi(f) = (\Phi_1(f), \ldots, \Phi_n(f)).$$

Obviously, Φ is linear and continuous. Let $\Phi^* : \mathbb{R}^n \to \mathcal{H}^q[a, b]$ be its adjoint mapping. This is also linear and continuous, and has the following expression:

$$\Phi^*(x) = \sum_{i=1}^{n} x_i u_i \quad \text{for each } x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Proposition 3.2. *The mapping $\Phi : \mathcal{H}^q[a, b] \to \mathbb{R}^n$ is surjective.*

Proof. Starting from the linearly independent polynomials $p_1, \ldots, p_n$ we construct n orthonormal polynomials $r_1, \ldots, r_n$ by the orthonormalisation process associated to the scalar product $\langle \cdot, \cdot \rangle_0$. From this construction it follows that each polynomial p_i ($i \in \{1, \ldots, n\}$) can be written in a unique way as

$$p_i = \sum_{j=1}^{i} \alpha_j^i r_j, \tag{1}$$

where α_j^i ($j \in \{1, \ldots, i\}$) are real numbers with $\alpha_i^i \neq 0$.

Let $(b_1, \ldots, b_n) \in \mathbb{R}^n$. We will find n real numbers $c_1, \ldots, c_n$ such that

$$\Phi(c_1 r_1 + \cdots + c_n r_n) = (b_1, \ldots, b_n). \tag{2}$$

Indeed, in view of (1) and the orthonormality of the polynomials $r_1, \ldots, r_n$ we have

$$\Phi_i(c_1 r_1 + \cdots + c_n r_n) = \sum_{j=1}^{n} c_j \int_a^b p_i(t) r_j(t) dt = c_1 \alpha_1^i + \cdots + c_i \alpha_i^i$$

for each $i \in \{1, \ldots, n\}$. Introducing these values into the equality (2) we obtain the following system:

$$\begin{cases} c_1 \alpha_1^1 & = & b_1 \\ & & \\ c_1 \alpha_1^i + \cdots + c_i \alpha_i^i & = & b_i \\ & & \\ c_1 \alpha_1^n \cdots + c_i \alpha_i^i + \cdots + c_n \alpha_n^n & = & b_n. \end{cases}$$

This system has a unique solution, because its determinant is $\alpha_1^1 \cdot \cdots \cdot \alpha_n^n \neq 0$. So there exist n real numbers $c_1, \ldots, c_n$ satisfying (2). Hence Φ is a surjective mapping. □

Notations. If X and Y are linear spaces and $T : X \to Y$ is a mapping, then we set

$$N(T) = \{x \in X \mid T(x) = 0\},$$

$$I(T) = \{y \in Y \mid \exists x \in X : T(x) = y\}.$$

For any $y \in \mathbb{R}^n$ the set of all functions f belonging to the space $\mathcal{H}^q[a,b]$ that satisfy the interpolation conditions

$$\int_a^b p_i(t) f(t) dt = y_i \quad (i \in \{1, \dots, n\}),$$

will be denoted by A_y, i.e.

$$A_y = \{f \in \mathcal{H}^q[a,b] \mid \Phi(f) = y\}.$$

Proposition 3.3. *The following relation holds:*

$$N(D) \cap N(\Phi) = \{0\}.$$

Proof. If $f \in N(D)$, i.e. $D(f) = 0$, we can write $f^{(q)}(t) = 0$ for a.e. $t \in [a,b]$. By a well-known theorem (see [1], Theorem (18.15), p. 284) we can see that

$$f^{(q-1)}(t) = f^{(q-1)}(a) \text{ for each } t \in [a,b],$$

i.e. $f \in \mathcal{P}_{q-1}$, where $\mathcal{P}_{q-1}$ is the set of all polynomials of degree $q-1$ at most. Therefore $N(D) = \mathcal{P}_{q-1}$.

Take an $f \in N(D) \cap N(\Phi)$. Then we have $f \in \mathcal{P}_{q-1}$ and $\Phi_i(f) = 0$ for each $i \in \{1, \dots, n\}$, which means that

$$\int_a^b p_i(t) f(t) dt = 0 \quad \text{for each } i \in \{1, \dots, n\}. \tag{3}$$

Since $p_1, \dots, p_n$ are polynomials of degree $q_1, \dots, q_n$ which satisfy $\{0, \dots, q-1\} \subseteq \{q_1, \dots, q_n\}$ and since $f \in \mathcal{P}_{q-1}$, there exist n real numbers $\alpha_1, \dots, \alpha_n$ such that

$$f(t) = \sum_{i=1}^n \alpha_i p_i(t) \quad \text{for each } t \in [a,b].$$

Multiplying each side of this equality with $f(t)$ and then integrating over the interval $[a,b]$, we obtain

$$\int_a^b \left(f(t)\right)^2 dt = \sum_{i=1}^n \alpha_i \int_a^b p_i(t) f(t) dt$$

In view of (3) it follows that

$$\int_a^b \left(f(t)\right)^2 dt = 0.$$

Therefore f must be a.e. equal to zero. So $N(D) \cap N(\Phi) = \{0\}$. □

Using the results stated in this section and in section 2 we will solve now the minimization problem formulated at the beginning of this paper.

Theorem 3.4. *For any $y \in \mathbb{R}^n$ there exists a unique $\sigma \in A_y$ such that*

$$\|D(\sigma)\|_0 = \min\{\|D(f)\|_0 \mid f \in A_y\}. \tag{4}$$

Proof. Finding a function $\sigma \in A_y$ for which (4) holds is equivalent to determinate a class $f_0 \in D(A_y)$ such that

$$\|f_0\|_0 = \min\{\|g\|_0 \mid g \in D(A_y)\}.$$

In other words we have to find the best approximation to the zero class belonging to H by using the elements from $D(A_y)$. For this end we prove that $D(A_y)$ is closed. $D(A_y)$ can be obtained from $D(A_0)$ by a translation, hence it suffices to prove that $D(A_0)$ is a closed set.

Let us define $D_1 : N(D)^\perp \to H$ by $D_1 = D\mid_{N(D)^\perp}$. Obviously D_1 is linear, continuous and satisfies $N(D_1) \subseteq N(D)$. But we also have $N(D_1) \subseteq N(D)^\perp$, which implies

$$N(D_1) \subseteq N(D) \cap N(D)^\perp = \{0\}.$$

So $N(D_1) = \{0\}$ and therefore D_1 is injective.

Since $N(D)$ is a complete subspace ($N(D) = \mathcal{P}_{q-1}$ is a linear subspace of dimension q), we can write

$$\mathcal{H}^q[a,b] = N(D) \oplus N(D)^\perp,$$

and so

$$D(\mathcal{H}^q[a,b]) = D(N(D)) + D(N(D)^\perp),$$

which means

$$H = \{0\} + D_1(N(D)^\perp) = D_1(N(D)^\perp).$$

Consequently, D_1 is surjective. Therefore D_1 is a linear, continuous and bijective mapping. From a well-known theorem concerning the continuity of the inverse mapping (see [3], Theoreme 4.2.7, p. 183) it follows that D_1 is a closed mapping.

In order to show that $D(N(\Phi))$ is closed, we prove that

$$D(N(\Phi)) = D\Big((N(D) + N(\Phi)) \cap N(D)^{\perp}\Big).$$

Obviously, we have

$$D\Big((N(D) + N(\Phi)) \cap N(D)^{\perp}\Big) \subseteq D(N(D) + N(\Phi)) \subseteq D(N(\Phi)).$$

So it remains to show that

$$D(N(\Phi)) \subseteq D\Big((N(D) + N(\Phi)) \cap N(D)^{\perp}\Big).$$

Let us choose an arbitrary $f \in N(\Phi)$. Since

$$\mathcal{H}^q[a,b] = N(D) \oplus N(D)^{\perp},$$

there exists a function $g \in N(D)$ such that $f - g \in N(D)^{\perp}$. Consequently, we have

$$f - g \in (N(D) + N(\Phi)) \cap N(D)^{\perp},$$

which implies that

$$D(f) \in D\Big((N(D) + N(\Phi)) \cap N(D)^{\perp}\Big).$$

Hence

$$D(N(\Phi)) \subseteq D\Big((N(D) + N(\Phi)) \cap N(D)^{\perp}\Big).$$

Since $N(D) = \mathcal{P}_{q-1}$ has the dimension $q - 1$ and $N(\Phi)$ is closed, it follows that $N(D) + N(\Phi)$ is closed. $N(D)^{\perp}$ is also closed and

$$D\Big((N(D) + N(\Phi)) \cap N(D)^{\perp}\Big) = D_1\Big((N(D) + N(\Phi)) \cap N(D)^{\perp}\Big),$$

where D_1 is a closed mapping. Thus $D(N(\Phi)) = D(A_0)$ is closed and hence $D(A_y)$ is also a closed linear subspace of the space H. Using a well-known theorem (see [2], Satz 16.2, p. 143), we conclude that there exists a unique best approximation $f_0 \in D(A_y)$ of the zero class (belonging to H) using the elements of $D(A_y)$.

The relation $f_0 \in D(A_y)$ implies the existence of a function $\sigma \in A_y$ for which $D(\sigma) = f_0$. In order to show the uniqueness of this function we consider two functions $\sigma_1, \sigma_2 \in A_y$ such that $D(\sigma_1) = D(\sigma_2) = f_0$. Hence $\sigma_1 - \sigma_2 \in N(D)$ and $\sigma_1 - \sigma_2 \in N(\Phi)$. From Proposition 3.2 it follows that $\sigma_1 = \sigma_2$. So there exists a unique (excepting a set of measure zero) function $\sigma \in A_y$ for which (4) holds. □

Let $y \in \mathbb{R}^n$. The function $\sigma \in A_y$ whose existence and uniqueness are assured by Theorem 3.3 is called *interpolating spline function with respect to* (D, Φ, y).

In particular, if we consider the interval $[-1, 1]$ and if the polynomials $p_1, \ldots, p_n$ are the first n consecutive Legendre polynomials with respect to the interval $[-1, 1]$, then the interpolating spline function with respect to (D, Φ, y) is exactly the spline function of Fourier type.

Next we will use the following result which can be found in [3], Theoreme 4.3.9, p. 193.

Proposition 3.5. *If T is a linear and continuous mapping between two Hilbert spaces, and T^* is its adjoint, then the following relations hold:*

[(i)]

1. $I(T) \perp = N(T^*)$, $\quad I(T^*)^\perp = N(T)$.
2. $\overline{I(T)} = N(T^*)^\perp$, $\quad \overline{I(T^*)} = N(T)^\perp$.
3. $I(T)$ *is closed if and only if* $I(T^*)$ *is closed.*

Proposition 3.6. *The following relations hold:*

$$I(D^*) = N(D)^\perp, \tag{5}$$

$$I(\Phi^*) = N(\Phi)^\perp. \tag{6}$$

Proof. Since D and Φ are surjective, it follows that $I(D) = H$ and $I(\Phi) = \mathbb{R}^n$. Therefore $I(D)$ and $I(\Phi^*)$ are also closed sets. Thus (5) and (6) hold. □

Theorem 3.7. *Let $y \in \mathbb{R}^n$, and let $\sigma \in A_y$. Then σ is the interpolating spline function with respect to (D, Φ, y) if and only if there exist n real numbers $r_1, \ldots, r_n$ such that*

$$D^*(D(\sigma)) = \sum_{i=1}^{n} r_i u_i. \tag{7}$$

Proof. Necessity. Since σ is the interpolating spline function with respect to (D, Φ, y), $D(\sigma)$ is the best approximation of $0 \in H$ by using elements of $D(A_y)$. By means of the characterization theorem of the best approximation in a Hilbert space (see [2], Satz 16.3, p. 144) we have $\langle D(\sigma), g\rangle_0 = 0$ for each $g \in D(A_0)$. Therefore $D(\sigma) \in D(A_0)^\perp$ and $\langle D^*(D(\sigma)), f\rangle_q = 0$ for each $f \in A_0$. But we have $A_0 = N(\Phi)$; thus $D^*(D(\sigma)) \in N(\Phi)^\perp$.

Taking (6) into consideration, it follows that $D^*(D(\sigma)) \in I(\Phi^*)$. On the other hand, the adjoint mapping $\Phi^* : \mathbb{R}^n \to \mathcal{H}^q[a,b]$ of Φ has the form

$$\Phi^*(x) = \sum_{i=1}^{n} x_i u_i \quad \text{for each } x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Since the functions $u_1, \dots, u_n$ are linearly independent, we have $I(\Phi^*) = \mathrm{sp}\{u_1, \dots, u_n\}$. Therefore $D^*(D(\sigma)) \in \mathrm{sp}\{u_1, \dots, u_n\}$, i.e. there exist n real numbers $r_1, \dots, r_n$ for which (7) holds.

Sufficiency. From (7) it follows that $D^*(D(\sigma)) \in \mathrm{sp}\{u_1, \dots, u_n\} = I(\Phi^*)$. According to (6) we have $I(\Phi^*) = N(\Phi)^\perp$, and hence $D^*(D(\sigma)) \in I(\Phi)^\perp$. This means that $\langle D^*(D(\sigma)), f\rangle_q = 0$ for each $f \in N(\Phi)$, i.e. $\langle D(\sigma), g\rangle_0 = 0$ for each $g \in D(N(\Phi))$. By the characterization of the best approximations in the subspace $D(A_y)$ we conclude that σ is the interpolating spline function with respect to (D, Φ, y). □

Now we are able to determine the analytical expression of the interpolating spline function with respect to (D, Φ, y). From Theorem 3.6 it follows that there exist n real numbers $r_1, \dots, r_n$ such that

$$D^*(D(\sigma)) = \sum_{i=1}^{n} r_i u_i.$$

On the other hand, (5) implies that $D^*(D(\sigma)) \in N(D)^\perp$. Consequently, we have

$$\langle \sum_{i=1}^{n} r_i u_i, p\rangle_q = 0 \quad \text{for each } p \in N(D) = \mathcal{P}_{q-1}.$$

Therefore the following relation holds

$$\sum_{i=1}^{n} r_i \int_a^b p_i(t)p(t)dt = 0 \quad \text{for each } p \in \mathcal{P}_{q-1}. \tag{8}$$

Since $\{1, t, \dots, t^{q-1}\}$ is a base of the linear space $\mathcal{P}_{q-1}$ we can see that (8) holds if and only if

$$\sum_{i=1}^{n} r_i \int_a^b p_i(t)t^j dt = 0 \quad \text{for each } j \in \{0, \dots, q-1\}. \tag{9}$$

Let us choose an arbitrary $f \in \mathcal{H}^q[a,b]$. By Taylor's formula we have

$$f(t) = \sum_{j=0}^{q-1} \frac{(t-a)^j}{j!} f^{(j)}(a) + \int_a^b \frac{(t-\tau)_+^{q-1}}{(q-1)!} f^{(q)}(\tau)d\tau \tag{10}$$

for each $t \in [a,b]$. Using (9) and (10) we obtain

$$\langle D(f), D(\sigma)\rangle_0 = \langle f, D^*(D(\sigma))\rangle_q = \sum_{i=1}^{n} r_i \langle f, u_i\rangle_q =$$

$$= \sum_{i=1}^{n} r_i \int_a^b p_i(t) f(t)dt = \int_a^b f^{(q)}(\tau) G(\tau) d\tau = \langle D(f), G\rangle_0,$$

where

$$G(\tau) = (-1)^q \sum_{i=1}^{n} r_i \int_a^{\tau} p_i(t) \frac{(\tau - t)_+^{q-1}}{(q-1)!} dt \quad \text{for each } \tau \in [a,b].$$

Since the equality $\langle D(f), D(\sigma)\rangle_0 = \langle D(f), G\rangle_0$ was stated for an arbitrary $f \in \mathcal{H}^q[a,b]$, we must have

$$\sigma^{(q)}(\tau) = G(\tau) \quad \text{for a.e. } \tau \in [a,b].$$

G is a polynomial and so it belongs to H. Since the interpolating spline function with respect to (D, Φ, y) is unique (excepting a set of measure zero) it results that

$$\sigma^{(q)}(\tau) = (-1)^q \sum_{i=1}^{n} r_i \int_a^{\tau} p_i(t) \frac{(\tau - t)_+^{q-1}}{(q-1)!} dt \quad \text{for each } \tau \in [a,b]. \tag{11}$$

By q times integration we get

$$\sigma(\tau) = \sum_{j=0}^{q-1} c_j \tau^j + (-1)^q \sum_{i=1}^{n} r_i \int_a^{\tau} p_i(t) \frac{(\tau - t)_+^{2q-1}}{(2q-1)!} dt \tag{12}$$

for each $\tau \in [a,b]$, where $c_0, \dots, c_{q-1}$ are real constants.

Conversely, if we have a function σ defined by (12), where the real constants $c_0, \dots, c_{q-1}, r_1, \dots, r_n$ satisfy (9) and $\Phi(\sigma) = y$, then σ is the interpolating spline function with respect to (D, Φ, y). The $n + q$ constants can be uniquely determinated by solving the following system:

$$\begin{cases} \displaystyle\int_a^b p_i(t)\sigma(t)dt &= y_i \quad (i \in \{1, \dots, n\}) \\ \displaystyle\sum_{i=1}^{n} r_i \int_a^b p_i(t) t^j dt &= 0 \quad (j \in \{0, \dots, q-1\}). \end{cases}$$

So the analytical expression of the interpolating spline function with respect to (D, Φ, y) is given by (12) and the constants can be determinated by solving the above system.

If we start from (11) we find another characterization of the spline function with respect to (D, Φ, y). By q times differentiation of each side of the equality (11) we get

$$\sigma^{(2q)}(\tau) = \sum_{i=1}^{n} r_i p_i(\tau) \tag{13}$$

for each $\tau \in [a, b]$ and

$$\sigma^{(j)}(a) = 0 \quad \text{for each } j \in \{q, \ldots, 2q-1\}. \tag{14}$$

For each $j \in \{q, \ldots, 2q-1\}$ we use Taylor's formula for $\sigma^{(j)}$ and have

$$\sigma^{(j)}(b) = \sum_{k=0}^{2q-1-j} \frac{(b-a)^k}{k!}\sigma^{(k+j)}(a) + \int_a^b \frac{(b-\tau)_+^{2q-1-j}}{(2q-1-j)!}\sigma^{(2q)}(\tau)d\tau.$$

From (9), (13) and (14) we obtain

$$\sigma^{(j)}(b) = \int_a^b \frac{(b-\tau)^{2q-1-j}}{(2q-1-j)!}\sigma^{(2q)}(\tau)d\tau = \sum_{i=1}^{n} r_i \int_a^b \frac{(b-\tau)^{2q-1-j}}{(2q-1-j)!}p_i(\tau)d\tau = 0$$

for each $j \in \{q, \ldots, 2q-1\}$. So the interpolating spline function σ with respect to (D, Φ, y) satisfies (13), (14) and also

$$\sigma^{(j)}(b) = 0 \quad \text{for each } j \in \{q, \ldots, 2q-1\}. \tag{15}$$

Conversely, if $\sigma \in \mathcal{H}^q[a, b]$ satisfies $\Phi(\sigma) = y$, (13), (14) and (15), then σ is the interpolating spline function with respect to (D, Φ, y). This assertion results because (13) and (14) imply (11), while (14) and (15) imply (9).

4. A Numerical Example

Let $q = 1$ and $n = 2$, let $\Phi = (\Phi_1, \Phi_2) : H^1[0,1] \to \mathbb{R}^2$ be defined by

$$\Phi_1(f) = \int_0^1 f(t)dt, \qquad \Phi_2(f) = \int_0^1 t^2 f(t)dt,$$

and let $y = (1, 11) \in \mathbb{R}^2$. We search for the analytical expression of the interpolating spline function with respect to (D, Φ, y).

We start by using the formula (12). In our particular case we have

$$\sigma(\tau) = c_0 - r_1 \int_0^\tau (\tau - t)dt - r_2 \int_0^\tau (\tau - t)t^2 dt$$

for each $\tau \in [0, 1]$. The real constants c_0, r_1, r_2 are determinated by the conditions

$$\Phi_1(\sigma) = 1, \quad \Phi_2(\sigma) = 11, \quad r_1 \int_0^1 1dt + r_2 \int_0^1 t^2 dt = 0.$$

The last equality is relation (9) applied in our particular case. The expression of σ is

$$\sigma(\tau) = -105\tau^4 + 210\tau^2 - 48 \quad \text{for each } \tau \in [0, 1].$$

Remark. There exists a relationship between the minimization problem solved in the present paper and the probability theory. If q and n are positive integers such that $n \geq q$ and if $y_1, \ldots, y_n$ are real numbers, then the functional (1.3) has a unique minimizer in the set consisting of all functions $f \subset \mathcal{H}^q[a, b]$ that satisfy the interpolation conditions

$$\begin{cases} \int_a^b f(t)dt &= 1 \\ \int_a^b t^i f(t)dt &= y_i \ (i \in \{1, \ldots, n\}). \end{cases}$$

If this function is nonnegative, then it is the probability density of a random variable whose moments of order $i \in \{1, \ldots, n\}$ are equal to y_i.

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CONNECTION BETWEEN THE NUMBER OF MULTIGRID ITERATIONS AND THE DISCRETIZED ERROR IN SOLVING A PROBLEM OF CONVECTION IN POROUS MEDIUM

IOANA CHIOREAN

Dedicated to Professor D.D. Stancu on his 70^{th} anniversary

Abstract. The connection between the number of multigrid iterations and the discretizing process error in obtaining the numerical solution of a problem of convection in porous medium is studied.

1. Introduction

Free convection in an enclosure filled with a fluid-saturated porous medium has occupied the central state in many fundamental heat transfer analysis. The study was made according with the shape of enclosure, the position of the heat sources, the boundary conditions, etc. For instance, Prasad and Kulacki [8] study the problem in a rectangular cavity with a heated vertical wall, the other one kept cold, and the horizontal ones adiabatic. In [7], Prasad reconsider the problem, but with external heat sources. Riley and Rees [9] consider a more general situation, when the walls of the cavity make a variable angle . Pop and Ingham [6] study the phenomena in a sphere, solving a 3D problem. Several other interesting results are given in Nield and Bejan [5]. The problem under consideration is that of a 2D steady laminar convection in a porous layer bounded by an inclined square enclosure with four rigid walls of constant temperature. Heat is generated by uniform distributed energy sources within the cavity. The porous layer is isotropic, homogeneous and saturated with an incompressible fluid. The heat generation creates a temperature gradient across the layer, and thereby provides a driving mechanism for natural convection within the cavity. The fluid motion obeys the equations Darcy-Oberbeck-Boussinesq.

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2. Basic equations

Taking into account the nomenclature from [1], for the case of volumetric heating given here, the governing equations are:

$$\nabla \cdot \mathbf{V}' = 0, \tag{1}$$

$$\mathbf{V}' = \frac{K}{\mu}(\rho' \mathbf{g} - \nabla p'), \tag{2}$$

$$(\rho c)_p \frac{\partial T'}{\partial t'} + (\rho c)_f (\mathbf{V}' \cdot \nabla) T' = k \nabla^2 T' + S', \tag{3}$$

$$\rho' = \rho_0' [1 - \beta (T' - T_0')] \tag{4}$$

Transforming these equations by using the dimensionless variables and the Rayleigh number, the following parabolic in T system of partial differential equations is obtained:

$$\begin{cases} \dfrac{\partial T}{\partial t} + \dfrac{\partial \psi}{\partial y}\dfrac{\partial T}{\partial x} - \dfrac{\partial \psi}{\partial x}\dfrac{\partial T}{\partial y} = \nabla^2 T + 1, \\ \nabla^2 \psi = Ra \left(\sin\phi \dfrac{\partial T}{\partial y} - \cos\phi \dfrac{\partial T}{\partial x} \right) \end{cases} \tag{5}$$

The unknowns are the temperature function T and the stream function ψ. We solve this system being situated in an enclosure with unit square section ($L = 1$), with the initial values $T_0 = \psi_0 = 0$ and the boundary conditions

$$T = \psi = 0 \text{ for } x = 0 \text{ and } 1, y = 0 \text{ and } 1. \tag{6}$$

3. The discrete problem for the steady state problem

In order to obtain the solution for system (5) with the boundary conditions (6), we studied, first, the steady state problem, by attaching to the continuous problem a discrete one, obtained by approximating the space derivatives with the centered finite differences, according with Roache [10]. The discrete solution for temperature and stream function was obtained using a multigrid algorithm with a Gauss-Seidel smoother (see Hackbusch [3]), working on equidistant grids Ω_l, defined in the following manner:

$$\Omega_l = \{(ih_l, jh_l) | 0 \leq i, j \leq N_l, h_l = 1/N_l, N_l = 2^{l+1}\}$$

Note 1. Index l stays for the level of the grid. Denoting $T_{i,j} = T(ih_l, jh_l), \psi_{i,j} = \psi(ih_l, jh_l)$ for every $0 \leq i, j \leq N_l$, the discrete system corresponding to (5) is

$$\begin{cases} \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2 * h_l} \frac{T_{i+1,j} - T_{i-1,j}}{2 * h_l} - \frac{\psi_{i+1,j} - \psi_{i-1,j}}{2 * h_l} \frac{T_{i,j+1} - T_{i,j-1}}{2 * h_l} - \\ - \frac{T_{i,j+1} + T_{i,j-1} + T_{i+1,j} + T_{i-1,j} - 4T_{i,j}}{h_l^2} = 1 + \frac{T_{i,j}}{dt} \\ \frac{\psi_{i,j+1} + \psi_{i,j-1} + \psi_{i+1,j} + \psi_{i-1,j} - 4\psi_{i,j}}{h_l^2} = Ra \left(\sin\phi \frac{T_{i,j+1} - T_{i,j-1}}{2 * h_l} - \right. \\ \left. - \cos\phi \frac{T_{i+1,j} - T_{i-1,j}}{2 * h_l} \right) \end{cases} \quad (7)$$

The considerations upon the numbers of iterations of multigrid method

As we stated above, the multigrid method with Gauss-Seidel smoother is used in solving the discretized system, according to the following general scheme :

1. Establish the work grid of points;
2. Fix the initial values for the temperature function;
3. Compute the stream function values in the inner points of the grid based on the temperature values and on the boundary values;
4. Compute the temperature function values in the inner points of the grid based on the stream values and on the boundary values;
5. Repeat Steps 3 and 4 until the convergence criterion is fulfil or a maximum number of iterations is exceeded.

Note 1. The initial values for temperature and stream functions are supposed to be zeros.

Note 2. In obtaining the convergence criterium, we used the scaled residual norm.

According with Hackbusch [3], the following property takes place:

Property. The discretization error is proportional with power $k$of the discretized step.

Taking into account this result, we following result holds:

Theorem. *If $ee_i/ee_{i-1} \leq a * 10^{-1}$, with $i = 1, \ldots, N(l)$, then the minimal needed number of multigrid iterations needed in obtaining the solution is*

$$N(l) \geq \lceil \log_2 \frac{ah_l^k}{ree_0} / \log_2 \frac{a}{10} \rceil$$

, where ee_i denotes the approximation error at the step $i+1$, ee_0 is the initial error and a is a real constant which verifies $0 < a \leq 5$.

Proof. To make the writing of the proof easier, we will denote in what follows $N(l)$ by n. According with the hypothesis, by mathematical induction, we can proof that

$$ee_n \leq ee_0 a^n 10^{-n}. \tag{8}$$

So, for $i = 1$, we get

$$ee_1/ee_0 \leq a10^{-1} \Leftrightarrow ee_1 \leq ee_0 a 10^{-1},$$

it means (8) is fulfilled. Let's suppose the relation true for $i = n - 1$:

$$ee_{n-1} \leq ee_0 a^{n-1} 10^{-(n-1)}$$

We have to prove it for $i = n$. According with the induction hypothesis

$$ee_n \leq ee_{n-1} a 10^{-1} \leq ee_0 a^{n-1} 10^{-(n-1} a 10^{-1} = ee_0 a^n 10^{-n},$$

so (8) is fulfilled. Further, based on the previous property and on (8), the following relation takes place:

$$ee_n \leq ee_0 a^n 10^{-n} \leq a h_l^k \tag{9}$$

where, without loss of generality, we used the same constant a form the theorem. Taking into account that $ee_0 > 0$ and $a/10 < 1$, we have:

$$(ee_0 a^n 10^{-n} \leq a h_l^k) \Leftrightarrow (a^{n-1} 10^{-n} \leq h_l^k / ee_0) \Leftrightarrow ((n-1)\log_2 a - n \log_2 10 \leq$$

$$\leq \log_2(h_l^k/ee_0)) \Leftrightarrow (n \log_2 \frac{a}{10} \leq \log_2(a h_l^k / ee_0)) \Leftrightarrow$$

$$\Leftrightarrow (n \geq \log_2(a h_l^k/ee_0)/\log_2 \frac{a}{10}).$$

Noting that n is an integer, the theorem is proved. □

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MULTIGRID ITERATIONS AND THE DISCRETIZED ERROR

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COEFFICIENTS CHOICE FOR A NEW DIRECT INTEGRATION OPERATOR

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Dedicated to Professor D.D. Stancu on his 70^{th} anniversary

Abstract. For a new direct integration operator, introduced by the authors in [3] and [4], the best choice of operator coefficient values is presented. The coefficients β, γ and δ are defined as the limits these coefficients approach when $\Delta t \to 0$. The coefficients θ, θ' and θ'', associated with β, γ and δ, are derived from the condition that they minimize the local error. Global error analysis, time step choice, and operator order are presented. The two body problem is considered as a numerical example showing the accuracy features of the new operator, which offers better results in comparison with Runge-Kutta 4^{th} order method.

1. Introduction

The operator applies to the direct integration of equations of the form

$$M\ddot{U} + g(\dot{U}) + f(U) = P(t) \tag{1}$$

describing the non-linear dynamical response of a system.

In Eq.1: U denotes the degree of freedom vector, and $\dot{U}, \ddot{U}$ the derivatives of U with respect to time t; M is the mass matrix; f and g are the non-linear stiffness and damping functions, respectively; $P(t)$ is the excitation function.

In particular, for a linear response $f(U) = KU$ and $g(\dot{U}) = C\dot{U}$, in which K and C are the stiffness and damping matrix, respectively. For a single-degree-of-freedom system, Eq.1 reads

$$m\ddot{u} + g(\dot{u}) + f(u) = p(t). \tag{2}$$

Direct integration operators for Eq.1 are presented in [1], [2] and [5]. The new operator analyzed in this paper has been introduced by the authors, in [3] and [4].

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2. Operator formulae for a single-degree-of-freedom (SDOF) system

We suppose that function u fulfills on the interval $I = [t_0, t_1]$, $t_1 = t_0 + \Delta t$, one of the two following conditions:

(1) $\ddot{u}$ is differentiable in $I - \{t_0\}$ or,

(2) $\ddot{u}$ is continuous in t_0 and there exists $\dddot{u}$-finite valuated or not in $I - \{t_0\}$.

Let denote function values in t_0 abd t_1 by the subscripts 0 and 1, respectively. E.G.: $u_0 = u(t_0)$, $\dot{u}_0 = \dot{u}(t_0)$, $u_1 = u(t_1)$, etc. Then, for every positive integers p, p' and p'', the remainder in the Taylor series of $u, \dot{u}$ and $\ddot{u}$ can be taken in Schlömilch-Roche form, i.e.:

$$u_1 = u_0 + \dot{u}_0(\Delta t) + \ddot{u}_0(1/2)(\Delta t)^2 + \dddot{u}\ (t_0 + \theta\Delta t)\beta(\Delta t)^3 \tag{3}$$

$$\dot{u}_1 = \dot{u}_0 + \ddot{u}_0(\Delta t) + \dddot{u}\ (t_0 + \theta'\Delta t)\gamma(\Delta t)^2 \tag{4}$$

$$\ddot{u}_1 = \ddot{u}_0 + \dddot{u}\ (t_0 + \theta''\Delta t)\delta(\Delta t) \tag{5}$$

Coefficients β, γ and δ are given by:

$$\beta = \frac{(1-\theta)^{3-p}}{2p}, \quad \gamma = \frac{(1-\theta')^{2-p'}}{p'}, \quad \delta = \frac{(1-\theta'')^{1-p''}}{p''} \tag{6}$$

in which θ, θ' and $\theta'' \in (0,1)$ and depend on p, p' and p'', respectively, and on t_1 - see [8]. If we denote by a bar the truncated Taylor series, i.e.:

$$\overline{u}_1 = u_0 + \dot{u}_0(\Delta t) + \ddot{u}_0(\Delta t)^2/2$$

$$\overline{\dot{u}}_1 = \dot{u}_0 + \ddot{u}_0(\Delta t),$$

Eqs.2a-2c become

$$u_1 = \overline{u}_1 + \dddot{u}\ (t_0 + \theta\Delta t)\beta(\Delta t)^3 \tag{7}$$

$$\dot{u}_1 = \overline{\dot{u}}_1 + \dddot{u}\ (t_0 + \theta'\Delta t)\gamma(\Delta t)^2 \tag{8}$$

$$\ddot{u}_1 = \ddot{u}_0 + \dddot{u}\ (t_0 + \theta''\Delta t)\delta(\Delta t) \tag{9}$$

Eqs.4a-4c are general formulae for deriving direct integration operators. Several operators are obtained by introducing assumptions on the variation of $\dddot{u}$ over the interval $[t_0, t_1]$. For instance, Newmark operator assume that $\dddot{u}$ is constant on the interval $[t_0, t_1]$ - see [4].

The new operator

The new operator is based on from the following assumption:

A1. | $\dddot{u}\,(t)$ varies linearly on interval $I = [t_0, t_1]$

Assumption A1 leads to

$$\dddot{u}\,(t_0 + \theta\Delta t) = \dddot{u}_0 + \theta\Delta\,\dddot{u}_1 \tag{10}$$

where

$$\Delta\,\dddot{u}_1 = \dddot{u}_1 - \dddot{u}_0 \tag{11}$$

The formulae of the new operator are

$$u_1 = \overline{u}_1 + \beta(\Delta t)^3\,\dddot{u}_0 + \theta\beta(\Delta t)^3\Delta\,\dddot{u}_1 \tag{12}$$

$$\dot{u}_1 = \overline{\dot{u}}_1 + \gamma(\Delta t)^2\,\dddot{u}_0 + \theta'\gamma(\Delta t)^2\Delta\,\dddot{u}_1 \tag{13}$$

$$\ddot{u}_1 = \ddot{u}_0 + \delta(\Delta t)\,\dddot{u}_0 + \theta''\delta(\Delta t)\Delta\,\dddot{u}_1 \tag{14}$$

$$\dddot{u}_1 = \dddot{u}_0 + \Delta\,\dddot{u}_1 \tag{15}$$

3. Coefficient choice

3.1. Coefficients β, γ and δ. The operator coefficients β, γ and δ will be chosen as the limit values these coefficients approach when $\Delta t \to 0$.

Eqs.7 can be put in the following form:

$$\frac{u_1 - \overline{u}_1}{(\Delta t)^3} = \beta\,\dddot{u}_0 + \theta\beta\Delta\,\dddot{u}_1 \tag{16}$$

$$\frac{\dot{u}_1 - \overline{\dot{u}}_1}{(\Delta t)^2} = \gamma\,\dddot{u}_0 + \theta'\gamma\Delta\,\dddot{u}_1 \tag{17}$$

$$\frac{\ddot{u}_1 - \ddot{u}_0}{\Delta t} = \delta\,\dddot{u}_0 + \theta''\delta\Delta\,\dddot{u}_1 \tag{18}$$

When $\Delta t \to 0$, the left-hand sides of Eq.8 approach $(1/6)\,\dddot{u}_0$, $(1/2)\,\dddot{u}_0$ and $\dddot{u}_0$, respectively. In the right-hand sides, if we suppose $\dddot{u}$ continuous on I, $\Delta\,\dddot{u}_1 \to 0$.

So, for any $\dddot{u}_0 \neq 0$, the limit values of β, γ and δ will be the following:

$$\beta = \frac{1}{6}, \quad \gamma = \frac{1}{2}, \quad \delta = 1 \tag{19}$$

3.2. Coefficients θ, θ' and θ''. Form Eq.3, using the values of β, γ and δ defined Eq.9, we obtain

$$\theta = 1 - (p/3)^{\frac{1}{3-p}} \quad \text{for} \quad p \neq 3,\ p \in \mathbb{N} \tag{(}$$

$$\theta' = 1 - (p'/2)^{\frac{1}{2-p'}} \quad \text{for} \quad p' \neq 2,\ p' \in \mathbb{N} \tag{(}$$

$$\theta'' = 1 - (p'')^{\frac{1}{1-p''}} \quad \text{for} \quad p'' \neq 1,\ p \in \mathbb{N} \tag{(2}$$

There exist an infinity of solutions for each of Eqs.10a-10c.

The integers p, p' and p'' will be chosen so that coefficients θ, θ' and θ'' minimi the local error of operator formulae - Eqs.7a-7c.

We suppose that derivatives $u^{(4)}$ and $u^{(5)}$ exist, and $u^{(5)}$ id bounded on I.

We denote by the superscript T the true value of u_1, expressed by Taylor expai sion with remainder in Lagrange form:

$$u_1^T = \overline{u}_1 + (1/6)\ \dddot{u}_0\ (\Delta t)^3 + (1/24) u^{(4)}(\xi)(\Delta t)^4$$

where $t_0 < \xi < t_0 + \Delta t$. The approximate solution given by Eq.7a can be expressed as

$$u_1 = \overline{u}_1 + (1/6)(\Delta t)^3\ \dddot{u}_0 + (\theta/6) u^{(4)}(\eta)(\Delta t)^{(4)}$$

where $t_0 < \eta < t_0 + \delta t$. The local error will be given by

$$Er(u_1) = u_1^T - u_1 = (1/6)[(1/4)u^{(4)}(\xi) - \theta u^{(4)}(\eta)](\Delta t)^4$$

Form this equation, $Er(u_1)$ can be bounded by some constant times $(\Delta t)^4$.

But, if we pick

$$\theta = \frac{1}{4} \tag{23}$$

the local error can be expressed as follows:

$$Er(u_1) = \frac{1}{24}[u^{(4)}(\xi) - u^{(4)}(\eta)](\Delta t)^5 = \frac{1}{24} u^{(5)}(\zeta)(\Delta t)^5.$$

Then, denoting by M_5 the superior bound of $u^{(5)}$ on I,

$$|u^{(5)}(t)| \leq M_5, \quad \text{for} \quad t \in I, \tag{24}$$

it follows that

$$|Er(u_1)| \leq \frac{1}{24} M_5 (\Delta t)^5. \tag{25}$$

The value of θ in Eq.11 corresponds to the choice of $p = 4$ in Eq.10a.

In an analogous way, it can be shown that picking

$$\theta' = \frac{1}{3} \quad \text{and} \quad \theta'' = \frac{1}{2}, \tag{26}$$

the local errors in $\dot{u}$ and $\ddot{u}$ are minimized and expressed as follows:

$$|Er(\dot{u}_1)| \leq \frac{1}{6} M_5 (\Delta t)^4 \quad \text{and} \quad |Er(\ddot{u}_1)| \leq \frac{1}{2} M_5 (\Delta t)^3 \tag{27}$$

where M_5 is defined by Eq.12b.

The values in Eq.13 correspond to the choice of $p' = 3$ and $p'' = 2$, in Eq.10b and 10c, respectively.

Definition 1. The operator coefficients defined by the Eqs.9, 11 and 13, will be called the limit coefficients.

In that follows, Eqs.7a-7c will be considered with β, γ, δ and $\theta, \theta', \theta''$, taken as the limit coefficients.

Note 1. The smaller is Δt, the closer to the exact formulae of $u_1, \dot{u}_1$ and $\ddot{u}_1$, are operator formulae 7a-7c.

Note 2. The limit coefficients are also consistent with the assumption A1, in the sense that formulae 7a-7c are exact for an assumed variation of a fourth degree polynomial for u and, a third and second degree polynomials for $\dot{u}$ and $\ddot{u}$, respectively.

4. Global error and time step choice

If we denote by TT the length of the response interval, the global error $GEr(u_1)$ can be bounded as follows:

$$|GEr(u_1)| \leq (1/24)\overline{M}_5 TT (\Delta t)^4 \tag{28}$$

where

$$|u^{(5)}(t)| \leq \overline{M}_5 \quad \text{for} \quad t \in [t_0, TT] \tag{29}$$

The time step has to be small enough in order to meet assumption A2 and use the limit coefficients. An estimation of the time step needed to keep the global error under a prescribed value, can be obtained from Eq.15a. If the global error in displacements to be less than ε, it is sufficient to pick

$$\Delta t < \left(\frac{24}{\overline{M}_5 TT} \varepsilon \right)^{1/4} .$$ (1

5. Recommended time step

The bound given by Eq.15a is overestimated. For instance, let consider t equation $\ddot{u} + 4\pi^2 u = 0$, with initial conditions $u(0) = 1$, $\dot{u}(0) = 0$, whose exact soluti is $u = \cos(2\pi t)$. For the response calculated with $\Delta t = 0.1 \ldots 0.002$ and $TT = 50$, t ratio of the estimated bound in Eq.15a to the maximum absolute error on $[0, TT]$ is abo 20.1, almost independent of Δt. E.g.: for $\Delta t = 0.02$ the estimated bound is $3.264E -$ while the maximum absolute error on $[0, TT]$ is $1.622E - 4$; the actual error at $t = T$ is $1.476E - 8$.

Consequently, the bound in Eq.16 is less than the actual value Δt required f the prescribed ε. Nevertheless, it offers the possibility to choose the time step, and mal comparison with the values required by other operators.

Example. With $\varepsilon = 10^{-6}$ and $TT = 50$, the bound for Δt equals $2.646E - 3$. F the response calculated with $\Delta t = 2.65E - 3$, the maximum absolute error on $[0, TT]$ $5.005E - 08$, at $t = 49.75$. This is about $\varepsilon/20$. The $\varepsilon = 10^{-6}$ precision is reached with time step of about $5.6E - 3$, which is about 2.1 times the estimated Δt.

Thus, for this example, the recommended time step is about two times the valu given by Eq.16.

These results show the effectiveness and the use of bound given by Eq.16.

6. Operator order and accuracy considerations

According to order definition for Nyström methods [7, p.261], the operator orde is 3. But, as the local error in u and $\dot{u}$ are bounded by some constants times h^5 and h^4 respectively, according to a relaxed order definition [7, p.275] the operator order shoul be 4.

The accuracy features are strongly enforced by the fact that the operator coe ficients are the limit coefficients - see Note 1 in # 3.

This explains the better results obtained by the proposed operator in compariso with Runge-Kutta fourth order method - see Table 1, and Fig.1 and 2.

7. Numerical example

The two body problem, expressed by the following equations [7] is considered:

$$\ddot{x} + x/r^3 = 0, \quad \ddot{y} + y/r^3 = 0.$$

The initial conditions are:

$$x(0) = 1 - e, \quad \dot{x}(0) = 0,$$

$$y(0) = 0, \quad \dot{y}(0) = [(1-e)/(1+e)]^{1/2}$$

where $r^2 = x^2 + y^2$, and e is the orbit eccentricity. The analytical solution is given by

$$x = \cos(u) - e, \quad y = \sqrt{1-e^2}\sin(u)$$

where u is given by $y - e\sin u = t$. The response is periodical with the period $T = 2\pi$.

The case of $e = 0.1$, with time steps 0.01, 0.1 and 0.5, was considered. The step range is approximately $T/600 \ldots T/12$. The integration time was $TT = 20$, which is over three times the motion period.

The solutions are computed via the new operator and via Runge-Kutta method of order fourth, using a constant time step. The computed solutions are compared with the analytical solution. The results are presented in Table 1, where the comparison criterion is the error in displacement response at $t \approx 6\pi$ (3 periods), and in Fig.3 and 4.

Table 1. Two body problem, $e = 0.1$ - ERROR in Displacements at $t \approx 6\pi$[1]

Time Step	Time	X-Displacement		Y-Displacement	
Δt	t	Proposed Operator	Runge-Kutta 4^{th} order	Proposed Operator	Runge-Kutta 4^{th} order
0.01	18.85	1.079E-12	3.810E-11	-3.354E-09	-5.738E-09
0.10	18.90	1.228E-06	1,261E-05	-3.356E-05	-1.581E-04
0.50	19.00	1.867E-03	-2.025E-02	1.655E-01	-3.750E-01

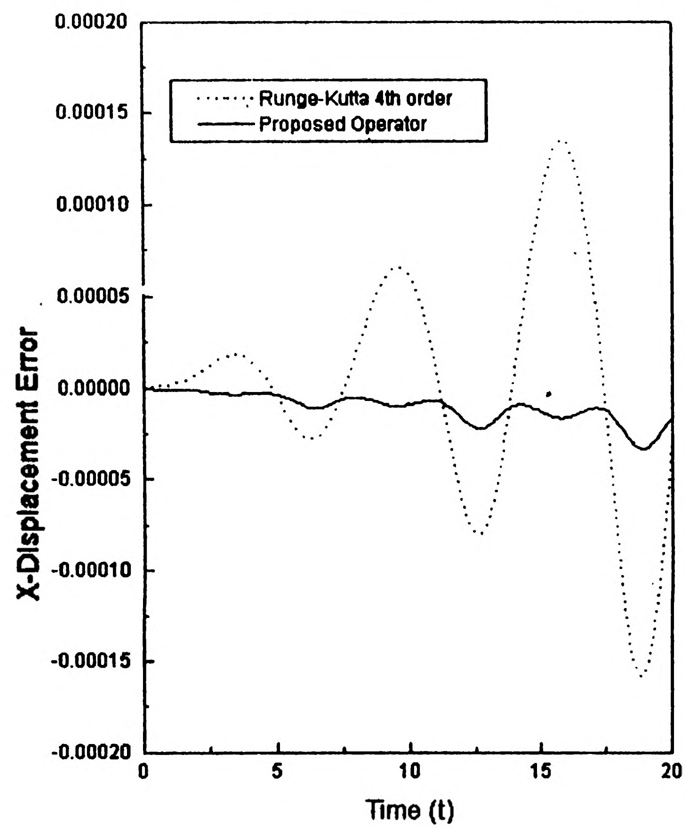

Fig.1 Two Body Problem, e=0.1, Δt=0.1 - Error in X-Displacements

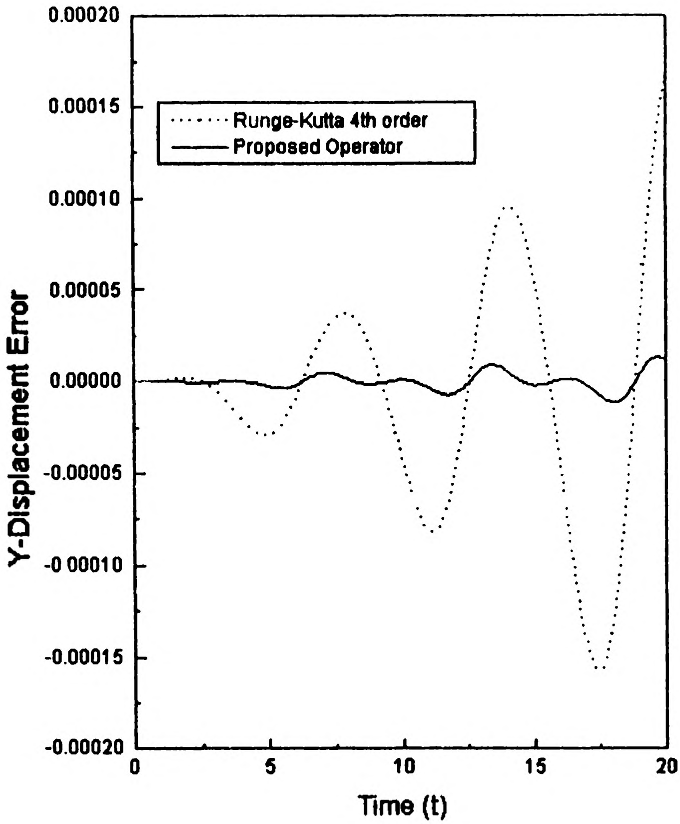

Fig.2 Two Body Problem, e=0.1, Δt=0.1 - Error in Y-Displacements

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SHEPARD OPERATORS OF LAGRANGE-TYPE

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Abstract. The interpolation operator, introduced by D. Shepard in 1964 has the degree of exactness zero (it reproduces only the constant function). Later, there have been defined Shepard operators with a degree of exactness (dex, for short) dex > 0, but these ones require information not only on the function f, but also require the values of some derivatives of f. In the present paper there aare constructed Shepard interpolation operators with a degree of exactness dex > 0, using only the values of the function f on interploation nodes.

0. In 1964, D. Shepard [5] has introduced the so called "Shepard interpolation procedure" for arbitrary located data (x_i, y_i, f_i), $i = 1, \dots, N$, on a plane domain, i.e. $(x_i, y_i) \in D$ $(D \subset \mathbf{R}^2)$ and f_i is the value of a function f $(f : D \to \mathbf{R})$ at the point (x_i, y_i).

The Shepard interpolation operator, say S_0, is defined by

$$(S_0 f)(x,y) = \sum_{i=1}^{N} A_i(x,y) f(x_i, y_i),$$

where

$$A_i(x,y) = \prod_{\substack{j=1 \\ j \neq i}}^{N} d_j^{\mu}(x,y) \Big/ \sum_{k=1}^{N} \prod_{\substack{j=1 \\ j \neq k}}^{N} d_j^{\mu}(x,y)$$

with $d_j(x,y) = ((x - x_j)^2 + (x - y_j)^2)^{1/2}$ and $\mu \in \mathbf{R}_+$.

An important characteristic of an approximation operator (procedure) is its degree of exactness (abbreviated here by "dex").

It is known that $\mathrm{dex}(S_0) = 0$ (S_0 reproduces only the constant function).

In order to increase the degree of exactness, it was defined the Shepard-type operator S_1 [5]:

$$(S_1 f)(x,y) = \sum_{i=1}^{N} A_i(x,y)[f(x_i, y_i) + (x - x_i) f^{(1,0)}(x_i, y_i) + (y - y_i) f^{(0,1)}(x_i, y_i)],$$

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that:

1. interpolates the values of f and of its first order partial derivatives $f^{(1,0)}$ and $f^{(0,1)}$ at the points (x_i, y_i), $i = 1, \dots, N$ and
2. $\text{dex}(S_1) = 1$.

Using, for example, the bivariate Taylor interpolation operator T_m:

$$(T_m f)(x,y) = \sum_{p+q \le m} \frac{(x-x_i)^p}{p!} \frac{(y-y_i)^q}{q!} f^{(p,q)}(x_i, y_i)$$

it was defined the Shepard-type operator S_m [3]:

$$S_m f = \sum_{i=1}^{N} A_i T_m f,$$

with the properties:

1) if $\mu > m$ then $(S_m f)^{(p,q)}(x_i, y_i) = f^{(p,q)}(x_i, y_i)$, $i = 1, \dots, N$, for all $p, q = 0, 1, \dots, m$; $p + q \le m$.

2) $\text{dex}(S_m) = m$.

Remark 1. Using the Taylor interpolation, it can be constructed Shepard-type operators of any degree of exactness. But it implies the values of the partial derivatives of f of higher order.

The goal of this note is to construct Shepard-type operators of higher degree of exactness using only the values of the function f at the interpolation nodes $P_i := (x_i, y_i)$, $i = 1, \dots, N$, i.e. Lagrange-type information about f.

1. Let L_i^n be the bivariate n-degree operator (associated to the point P_i), that interpolates the function f respectively at the set points

$$\{P_i, P_{i+1}, \dots, P_{i+m-1}\}, \quad i = 1, \dots, N, \; m < N, \tag{1}$$

with $P_{N+i} := P_i$, $i = 1, \dots, m-1$, and $m := (n+1)(n+2)/2$ - the number of the coefficients of a bivariate polynomial of the degree n.

Remark 2. For given N, it can be considered only operators L_i^n with n such that $(n+1)(n+2)/2 < N$, i.e. for $n \in \{1, \dots, \nu\}$, where
$\nu = \text{integer}\left[\left(\sqrt{8N+1} - 3\right)/2\right]$.

The existence and the uniqueness of the polynomials L_i^n are assured by [1, pg. 275, Theorem 17.1]:

Let $P_i := (x_i, y_i)$, $i = 1, \dots, (n+1)(n+2)/2$ be different points in plane that do not lie on the algebraic curve of nth degree $\left(\sum_{i+j \le n} a_{ij} x^i y^j = 0 \right)$. Then, for every function f defined at the points P_i, $i = 1, \dots, (n+1)(n+2)/2$ there exist a unique polynomial Q_n of nth degree that satisfies $Q_n(x_i, y_i) = f(x_i, y_i)$, $i = 1, \dots, (n+1)(n+2)/2$.

Hence, if the points P_k, $k = i, \dots, i+m-1$ of the set (1) do not lie on an algebraic curve of n th degree, for each $i = 1, \dots, N$, then L_i^n exists and it is unique for all $i = 1, \dots, N$.

Suppose that the existence and the uniqueness conditions of the operators L_i^n, $i = 1, \dots, N$ are satisfied.

We have

$$(L_i^n f)(x, y) = \sum_{k=i}^{i+m-1} l_k(x, y) f(x_k, y_k), \quad i = 1, \dots, N, \tag{2}$$

where l_k are the corresponding cardinal polynomials:

$$l_k(x_j, y_j) = \delta_{kj}, \quad k, j = i, \dots, i+m-1. \tag{3}$$

The basically properties of L_i^n are:

$$(L_i^n f)(x_k, y_k) = f(x_k, y_k), \quad k = i, \dots, i+m-1 \tag{4}$$

and

$$\operatorname{dex}(L_i^n) = n, \quad i = 1, \dots, N. \tag{5}$$

Definition 1. The operator S_n^L given by

$$(S_n^L f)(x, y) = \sum_{i=1}^{N} A_i(x, y)(L_i^n f)(x, y) \tag{6}$$

is called a Shepard operator of Lagrange-type.

Theorem 1. Let $f : D \to \mathbf{R}$ $(D \subset \mathbf{R}^2)$ be a given function, $P_i := (x_i, y_i) \in D$, $i = 1, \dots, N$ and $m := (n+1)(n+2)/2 < N$. If the points $P_i, \dots, P_{i+m-1}$ do not lie on an algebraic curve of nth degree, for all $i = 1, \dots, N$ $(P_{N+k} := P_k,\ k = 1, \dots, m-1)$, then there exist the operator S_n^L with the properties:

$$(S_n^L f)(x_j, y_j) = f(x_j, y_j), \quad j = 1, \dots, N \tag{7}$$

and

$$\operatorname{dex}(S_n^L) = n. \tag{8}$$

Proof. Taking into account that

$$A_i(x_j, y_j) = \delta_{ij}, \quad i, j = 1, \ldots, N,$$

from (6), we have:

$$(S_n^L f)(x_j, y_j) = (L_i^n f)(x_j, y_j), \quad j = 1, \ldots, N.$$

Now, using (4), the interpolation condition (7) follows.

To prove the property (8), it can be used the linearity of the operator S_n^L. This way, we nave to check that

$$S_n^L e_{pq} = e_{pq}, \quad p, q = 0, 1, \ldots, N \text{ with } p + q \leq n,$$

where $e_{pq}(x, y) = x^p y^q$

From (6), one obtains:

$$(S_n^L e_{pq})(x, y) = \sum_{i=1}^{N} A_i(x, y)(L_i^n e_{pq})(x, y).$$

But,

$$L_i^n e_{pq} = e_{pq} \text{ for } p + q \leq n$$

($\operatorname{dex}(L_i^n) = n$, for all $i = 1, \ldots, n$).

It follows that

$$(S_n^L e_{pq})(x, y) = e_{pq}(x, y) \sum_{i=1}^{N} A_i(x, y)$$

and the relation

$$\sum_{i=1}^{N} A_i(x, y) = 1$$

completes the proof. □

Next, one considers two particular cases, for $n = 1$ and $n = 2$.

2. For $n = 1$, the Shepard operator from (6), becomes

$$(S_1^L f)(x, y) = \sum_{i=1}^{N} A_i(x, y)(L_i^1 f)(x, y).$$

where

$$(L_i^1 f)(x,y) = l_i(x,y)f(x_i,y_i) + l_{i+1}(x,y)f(x_{i+1},y_{i+1}) + l_{i+2}(x,y)f(x_{i+2},y_{i+2})$$

for $i = 1,\dots,N$ ($P_{N+1} := P_1,\ P_{N+2} := P_2$). From (3) one obtains

$$l_i(x,y) = \frac{(y_{i+1}-y_{i+2})x + (x_{i+2}-x_{i+1})y + x_{i+1}y_{i+2} - x_{i+2}y_{i+1}}{(x_i - x_{i+1})(y_{i+1}-y_{i+2}) - (x_{i+1}-x_{i+2})(y_i - y_{i+1})},$$

$$l_{i+1}(x,y) = \frac{(y_{i+2}-y_i)x + (x_i - x_{i+2})y + x_{i+2}y_i - x_i y_{i+2}}{(x_{i+1}-x_{i+2})(y_{i+2}-y_i) - (x_{i+2}-x_i)(y_{i+1}-y_{i+2})},$$

$$l_{i+2}(x,y) = \frac{(y_i - y_{i+1})x + (x_{i+1}-x_i)y + x_i y_{i+1} - x_{i+1}y_i}{(x_{i+2}-x_i)(y_i - y_{i+1}) - (x_i - x_{i+1})(y_{i+2}-y_i)}.$$

Remark 3. The existence and uniqueness condition of L_i^1 is that the points P_i, P_{i+1} and P_{i+2} do not lie on a line ($Ax + By + C = 0$), or to be the vertices of a non-degenerate triangle Δ_i, for all $i = 1,\dots,N$.

Remark 4. $S_1^L f$ and $S_0 f$ (the Shepard function) use the same information about f ($f(x_i,y_i),\ i = 1,\dots,N$), but $\mathrm{dex}(S_1^L) = 1$ while $\mathrm{dex}(S_0) = 0$.

3. In the case of $n = 2$, we have

$$(S_2^L f)(x,y) = \sum_{i=1}^{N} A_i(x,y)(L_i^2 f)(x,y)$$

where $L_i^2 f$ are two-degree polynomials that interpolate f at $P_i,\dots,P_{i+5}$, respectively.

In accordance with the theorem 1, the interpolation nodes $P_i,\dots,P_{i+5}$ must not lie respectively on an algebraic curve of second degree $g_i,\ i = 1,\dots,N$. If, for some $j,\ 1 < j \le N$ this condition is not satisfied, the index order of $P_i,\ i = 1,\dots,N$ can be changed.

Remark 5. A two-degree of exactness Shepard operator of Lagrange-type, can be obtained using more information about f. Roughly speaking, the only problem is to exist some two-degree Lagrange-type polynomials which interpolate f at $P_k,\ k = 1,\dots,N$.

A sufficient condition for the existence and the uniqueness of such polynomials, say $\tilde{L}_i^2 f$, is that the six interpolation nodes to be the vertices P_i, P_{i+1}, P_{i+2} of the triangle Δ_i and the midpoints, say Q_i, Q_{i+1}, Q_{i+2}, of the sides of Δ_i [9,10], for $i = 1,\dots,N$.

Using the polynomials $\tilde{L}_i^2 f,\ i = 1,\dots,N$, one can be defined the Shepard operator $\tilde{S}_2$:

$$(\tilde{S}_2 f)(x,y) = \sum_{i=1}^{N} A_i(x,y)(\tilde{L}_i^2 f)(x,y)$$

which interpolates f at P_i, $i = 1, \ldots, N$ and $\text{dex}(\tilde{S}_2) = 2$.

Of course, $S_2 f$ also uses the values of the function f at the midpoints Q_i, $i = 1, \ldots, N+2$. But, the advantage of $\tilde{S}_2 f$ is that the existence and the uniqueness conditions of the polynomials $\tilde{L}_i^2 f$ is more simple to verify.

Appendix

We give some concrete examples and the corresponding graphs for different values of the parameter μ. For the function $f(x,y) = x\exp(-x^2 - y^2)$ on the rectangular domain D=[-2,2]x[-2,2], for N=10 and interpolation nodes given bellow:

(-1.9204, 1.9204), (-1.9204, -1.6910), (-0.9812, -1.6910),
(-0.9812, 1.3152), (0.0876, 1.5156), (0.2881, -0.0208),
(-0.1127, -0.8893), (0.9561, -1 1565), (0.7557, 0.7807),
(1.5574, -3490),

and for the the function $g(x,y) = -(x^2 + y^2)$ on $[-1,1] \times [-1,1]$ with nodes (-1, -1), (-0.5, -0.5), (0.5, -0.5), (0, 0), (0.5, 0.5), (-0.5, 0.5), (-1, 1), (1, 1), (1, -1), we will study the behavior of simple Shepard operator and Shepard-Lagrange operator (n=1) for μ=1,2 and respectively 4.

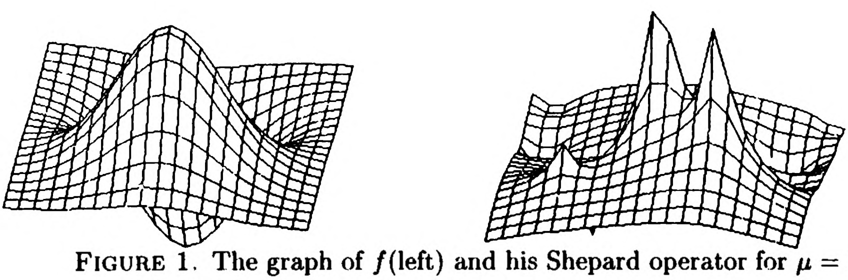

FIGURE 1. The graph of f(left) and his Shepard operator for $\mu = 1$

FIGURE 2. Shepard operators (for f) for $\mu = 2$(left) and $\mu = 4$

FIGURE 3. $S_1^L f$ for $\mu = 1$(left) and $\mu = 2$

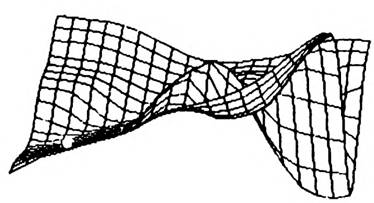

FIGURE 4. $S_1^L f$ for $\mu = 4$

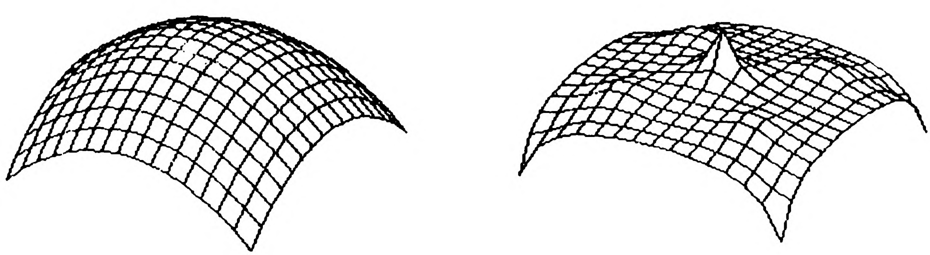

FIGURE 5. The graph of g(left) and his Shepard operator for $\mu = 1$

FIGURE 6. Shepard operators (for g) for $\mu = 2$(left) and $\mu = 4$

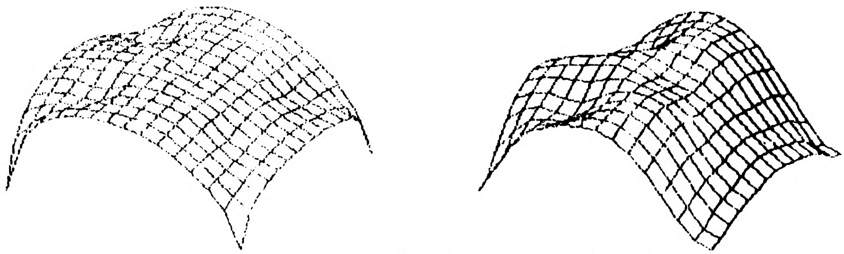

FIGURE 7. $S_1^L g$ for $\mu = 1$(left) and $\mu = 2$.

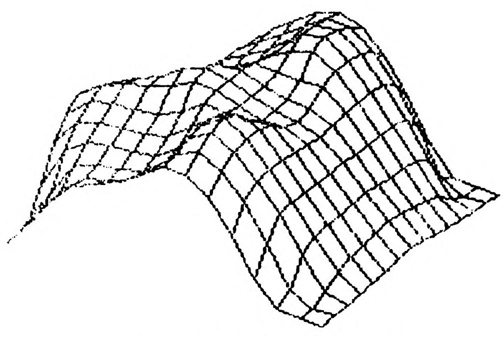

FIGURE 8. $S_1^L f$ for $\mu = 4$

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A NOTE ON THE DEGREE OF SIMULTANEOUS APPROXIMATION BY HIGHER ORDER CONVEXITY PRESERVING POLYNOMIAL OPERATORS

IOAN GAVREA AND HEINZ H. GONSKA AND DANIELA P. KACSÓ

Dedicated to Professor Dr. D.D. Stancu on the occasion of his 70th birthday

Abstract. In this paper that it is shown for certain operators preserving the shape of a function quantitative estimates for simultaneous approximation can be given by reducing the problem to the approximation of continuous functions by positive linear operators.

1. Introduction

The present note is a supplement to an earlier paper on "Pointwise estimates for higher order convexity preserving polynomial approximation" (see [3]) written by Jia-ding Cao and the second author of this note. We will show and recall here (again) that for certain operators preserving the shape of a function quantitative estimates for simultaneous approximation can be given by reducing the problem to the approximation of continuous functions by positive linear operators. Our paper can thus be viewed as being written in the spirit of three contributions by Sendov & Popov [8], Knoop & Pottinger [7], and the second author [5]. The latter paper mentioned contains the idea to involve positive linear operator estimates in terms of ω_2 into inequalities for simultaneous approximation. This type of estimate for the approximation of arbitrary continuous functions was first given by Jia-ding Cao in 1964 (see [2]); a more refined version of it can be found in [6]. In fact, this is the type of estimate which we will be using below

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2. Main Results

In this section we will give a quantitative estimate for certain operators constructed with the aid of Beatson-type kernels. We present first the following result given by Beatson [1]:

Lemma 1. *Let $f, g : C(I)$, $I = [-1, 1]$, and let the convolution operator G given by*

$$G(f,x) = (f * g)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\cos\nu)[\cos(\theta-\nu)]d\nu, \quad x = \cos\theta.$$

Let j be a non-negative integer. Then the cone of j-convex functions is invariant under the operator $G(f) = f * g$ if and only if g is j-convex.

Furthermore we will consider the operators $H_{n,s,j} : C[-1,1] \to \prod_{sn-s+j}$, where $j \in \mathbb{N}$, $s \geq j+2$ and $n \geq 1$, introduced by Cao and Gonska in [3] in the following way:

Let

$$K_{sn-s}(\arccos t) = C_{n,s}\left(\frac{\sin\left(n \arccos \frac{t}{2}\right)}{\sin\left(\arccos\frac{t}{2}\right)}\right)^{2s} = \frac{1}{2} + \sum_{k=1}^{sn-s} \rho_{k,sn-s}\cos(\arccos t),$$

where $C_{n,s}$ is a normalizing constant.

For $j \in \mathbb{N}$ they considered a j-th antiderivative of $K_{sn-s}(\arccos t)$, namely

$$F_{n,s,j}(z) := \frac{1}{(j-1)!}\int_{-1}^{z}(z-t)^{j-1}K_{sn-s}(\arccos t)dt.$$

Normalizing yields the kernel

$$\overline{F}_{n,s,j}(\nu) = \frac{\pi F_{n,s,j}(\cos\nu)}{\int_{-\pi}^{\pi} F_{n,s,j}(\cos t)dt},$$

which can also be written as

$$\overline{F}_{n,s,j}(\nu) = \frac{1}{2} + \sum_{i=1}^{sn-s+j}\lambda_{i,n,s,j}\cos i\nu.$$

The operator $H_{n,s,j}$ is then given by

$$(H_{n,s,j}f)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\cos\nu)\overline{F}_{n,s,j}(\theta-\nu)d\nu =$$

$$= \frac{1}{\int_{-\pi}^{\pi} F_{n,s,j}(\cos\nu)d\nu}\int_{-\pi}^{\pi} f(\cos\nu)F_{n,s,j}[\cos(\theta-\nu)]d\nu.$$

Using Lemma 3.8 in [4] we can now write: For $0 \leq i \leq ns - s + j$ (degree of H) we have:

$$H_{n,s,j}(T_i(t), x) = \lambda_{i,n,s,j}T_i(x), \tag{1}$$

where T_i is the i-th Chebyshev polynomial. From the latter relation it follows that $H_{n,s,j}$ transforms the polynomials of degree i into polynomials of degree i, for $0 \le i \le ns - s + j$ (i.e., $H_{n,s,j}\prod_i \subset \prod_i$).

Also from (1) we get that the coefficient of x^i in $H_{n,s,j}(e_i, x)$ is $\lambda_{i,n,s,j}$.

In [3, Theorem 4.3] and [4, Lemma 3.9] the authors proved the following results:

(i) The operators $H_{n,s,j}$ are positive. Furthermore, if the function f is i-convex, then $H_{n,s,j}(f,\cdot)$ is also i-convex, where $i \in \{0,1,\dots,j\}$.

(ii) $1 - \lambda_{i,n,s,j} = \mathcal{O}(n^{-2})$.

Theorem 1. *Let $i \in \{0,1,\dots,j\}$ be fixed. Then there exists a constant C independent of f, n, x such that the following estimate holds:*

$$\|H^{(i)}_{n,s,j}f - f^{(i)}\| \le C\left[\frac{\|f^{(i)}\|}{n^2} + \frac{1}{n}\omega_1\left(f^{(i)};\frac{1}{n}\right) + \omega_2\left(f^{(i)};\frac{1}{n}\right)\right]$$

for every $f \in C^i(I)$.

Proof. For $i \in \{0,1,\dots,j\}$ fixed, we consider the operator $H^*_{n,s,j} : C(I) \to \prod_{sn-s+j}$ defined by

$$(H^*_{n,s,j}f)(x) = \frac{1}{\lambda_{i,n,s,j}}(H^{(i)}_{n,s,j}F_i)(x),$$

where $F_i(x) = \int_0^x \int_0^{t_1} \cdots \int_0^{t_{i-1}} f(t_i)dt_i dt_{i-1}\dots dt_1$. Since $H_{n,s,j}$ preserves the convexity of order i, it follows that $H^*_{n,s,j}$ is positive.

An easy computation yields

$$H^*_{n,s,j}(e_0;x) = 1, \tag{2}$$

$$H^*_{n,s,j}(e_1;x) = \frac{\lambda_{i+1,n,s,j}}{\lambda_{i,n,s,j}}x,$$

$$H^*_{n,s,j}(e_2;x) = \frac{\lambda_{i+1,n,s,j}}{\lambda_{i,n,s,j}}x^2,$$

$$H^*_{n,s,j}((e_1-x)^2;x) = \frac{\lambda_{i+2,n,s,j} - 2\lambda_{i+1,n,s,j} + \lambda_{i,n,s,j}}{\lambda_{i,n,s,j}}x^2.$$

We will need now the following special case of a result given by Gonska and Kovacheva in [6]:

Let $K = [a,b]$, $K' = [c,d]$ and $K' \subset K$. If $L : C(K) \to B(K')$ is a positive linear operator, satisfying $Le_0 = e_0$, then for $f \in C(K)$, $x \in K'$ and each $0 < h \le \frac{1}{2}(b-a)$, the following holds:

$$|L(f;x) - f(x)| \le \frac{2}{h}|L(e_1 - x;x)|\omega_1(f;h) +$$

$$+\left[\frac{3}{2}+\frac{3}{2h}|L(e_1-x;x)|+\frac{3}{4h^2}L((e_1-x)^2;x)\right]\omega_2(f;h).$$

In our case the latter inequality becomes (writing $\lambda_i=\lambda_{i,n,s,j}$)

$$|H^*_{n,x,j}(g;x)-g(x)|\le\frac{2}{h}|x|\cdot\left|\frac{\lambda_{i+1}-\lambda_i}{\lambda_i}\right|\omega_1(g;h)+$$

$$+\left[\frac{3}{2}+\frac{3}{2h}|x|\cdot\left|\frac{\lambda_{i+1}-\lambda_i}{\lambda_i}\right|+\frac{3}{4h^2}\cdot x^2\cdot\frac{\lambda_{i+2}-2\lambda_{i+1}+\lambda_i}{\lambda_i}\right]\omega_2(g;h),$$

for every $g\in C(I)$ and every $0<h\le 1$.

For every $f\in C^i(I)$ we take in the latter inequality $g=f^{(i)}$. We obtain for $h=\frac{1}{n}$:

$$\left|\frac{H^{(i)}_{n,s,j}(f,x)}{\lambda_i}-f^{(i)}(x)\right|\le C\left[\frac{1}{n}\omega_1\left(f^{(i)};\frac{1}{n}\right)+\omega_2\left(f^{(i)};\frac{1}{n}\right)\right],$$

which implies

$$|H^{(i)}_{n,s,j}(f;x)-f^{(i)}(x)+f^{(i)}(x)(1-\lambda_i)|\le C\left[\frac{1}{n}\omega_1\left(f^{(i)};\frac{1}{n}\right)+\omega_2\left(f^{(i)};\frac{1}{n}\right)\right].$$

Now it follows immediately that

$$\|H^{(i)}_{n,s,j}(f-f^{(i)}\|\le C\left(\frac{\|f^{(i)}\|}{n^2}+\frac{1}{n}\omega_1\left(f^{(i)};\frac{1}{n}\right)+\omega_2\left(f^{(i)};\frac{1}{n}\right)\right).$$

We mention that in the latter inequality we used the fact that $1-\lambda_{i,n,s,j}\le\frac{C}{n^2}$.

□

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DESIGN OF WILSON-FOWLER SPLINES

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Dedicated to Professor Dr. D.D. Stancu on the occasion of his 70^{th} birthday

Abstract. Wilson-Fowler splines are characterized by Bézier polygons. By this characterization a natural extension to the space is given. We also present a method to design Wilson-Fowler splines and an algorithm of solving the corresponding interpolation problem with specific boundary slopes.

1. Introduction

The Wilson-Fowler spline was introduced by Wilson and Fowler [Fowler & Wilson '66] (see also [Fritsch '86]) as a planar, curvature continuous spline curve interpolating a set of planar points $\{\mathbf{P}_i\}_{i=0}^n$. For each segment joining $\mathbf{P}_i$ and $\mathbf{P}_{i+1}$ of this curve we introduce a local (u, v)-coordinate system $\{\mathbf{u}, \mathbf{v}\}$, with the variable $u \in [0, L_i]$ running along the chord connecting $\mathbf{P}_i$ and $\mathbf{P}_{i+1}, L_i = |\mathbf{P}_{i+1} - \mathbf{P}_i|$, and $v = c_i(u)$ representing the deviation from the chord as a cubic polynomial:

$$c_i(u) = \tan A_i u(u - L_i)^2/L_i^2 + \tan B_i u^2(u - L_i)/L_i^2. \tag{1.1}$$

Here A_i and B_i are the angles tangent vector at $\mathbf{P}_i$ and $\mathbf{P}_{i+1}$, respectively, makes with the i-th chord $\mathbf{P}_{i+1} - \mathbf{P}_i$. The segments are joined so that the curve they describe is GC^2 (For the definition, see [Hoschek & Lasser '93, Farin '85])

The Wilson-Fowler spline is a cubic spline. So it is natural for us to consider its Bézier representation (see [Nielson '74]). If we denote the Bézier points for the segment joining $\mathbf{P}_i$ and $\mathbf{P}_{i+1}$ as $\{\mathbf{b}_{3i}, \mathbf{b}_{3i+1}, \mathbf{b}_{3i+2}, \mathbf{b}_{3i+3}\}$, then we know by definition that $\mathbf{b}_{3i} = \mathbf{P}_i, \mathbf{b}_{3i+3} = \mathbf{P}_{i+1}$. Moreover, we have the following characterization which will be proved in Section 2.

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Theorem 1. *Let* $\{\mathbf{b}_{3i}, \mathbf{b}_{3i+1}, \mathbf{b}_{3i+2}, \mathbf{b}_{3i+3}\}$ *be a set of points in the plane. Then the cubic Bézier polynomial using it as the Bézier polygon satisfies (1.1) in the above mentioned local coordinate system if and only if for* $j = 1, 2$, *one of the following conditions holds:*

i) $\mathbf{b}_{3i+j}$ *lies on the line passing through* $\mathbf{b}_{3i}$ *and* $\mathbf{b}_{3i+3}$, *i.e.,*

$$(\mathbf{b}_{3i+j} - \mathbf{b}_{3i}) \times (\mathbf{b}_{3i+3} - \mathbf{b}_{3i}) = 0; \tag{1.2}$$

ii) $\mathbf{b}_{3i+j}$ *lies on the line which is perpendicular to the chord connecting* $\mathbf{b}_{3i}$ *and* $\mathbf{b}_{3i+3}$, *and through the point* $\mathbf{b}_{3i} + \frac{j}{3}(\mathbf{b}_{3i+3} - \mathbf{b}_{3i})$, *i.e.,*

$$(\mathbf{b}_{3i+j} - (\mathbf{b}_{3i} + \frac{j}{3}(\mathbf{b}_{3i+3} - \mathbf{b}_{3i}))) \cdot (\mathbf{b}_{3i+3} - \mathbf{b}_{3i}) = \mathbf{0}. \tag{1.3}$$

Remark. *If (1.2) is satisfied for both* $j = 1, 2$, *we know that* $v = c_i(u) \equiv 0$, *no matter how* $\mathbf{b}_{3i+1}$ *and* $\mathbf{b}_{3i+2}$ *are located. In this case, (1.1) is not suitable for the representation of the curve segment. So in Section 5 we shall restrict ourselves to the case when (1.3) is satisfied for* $j = 1, 2$.

This characterization enables us to perform the following tasks: firstly, we can extend the planar Wilson-Fowler splines to the space, see Section 3; secondly, we can easily design a Wilson-Fowler spline, see Section 4; thirdly, we can solve the interpolation problem with specific boundary slopes by finding zeros of a corresponding univariate algebraic polynomial, which we shall discuss in Section 5.

2. Proof of Theorem 1

In this section, we prove Theorem 1.

Proof of Theorem 1. The cubic Bézier curve using $\{\mathbf{b}_{3i}, \mathbf{b}_{3i+1}, \mathbf{b}_{3i+2}, \mathbf{b}_{3i+3}\}$ as the Bézier polygon is defined for $u \in [0, L_i]$ as

$$\mathbf{X}(u) = \sum_{j=0}^{3} \mathbf{b}_{3i+j} B_j^3(\frac{u}{L_i}), \tag{2.1}$$

where $L_i = |\mathbf{b}_{3i+3} - \mathbf{b}_{3i}|$ and $\{B_j^3(t)\}_{j=0}^3$ is the Bernstein basis.

Given $u \in [0, L_i]$, the vector $\mathbf{X}(u) - \mathbf{X}(0)$ has the following form in the (u, v)-coordinate system:

$$\mathbf{X}(u) - \mathbf{X}(0) = (u, (\mathbf{X}(u) - \mathbf{X}(0)) \cdot \mathbf{v}). \tag{2.2}$$

Thus, $\mathbf{X}$ satisfies (1.1) if and only if

$$(\mathbf{X}(u) - \mathbf{X}(0)) \cdot \mathbf{v} = c_i(u) = \tan A_i u(u - L_i)^2/L_i^2 + \tan B_i u^2(u - L_i)/L_i^2. \qquad (2.3)$$

We observe that (2.3) is satisfied for $u = 0, L_i$. Also, both sides of (2.3) are polynomials of degree three. So (2.3) is satisfied for $u \in [0, L_i]$ if and only if

$$\frac{d}{du}\{(\mathbf{X}(u) - \mathbf{X}(0)) \cdot \mathbf{v}\} = \frac{d}{du}c_i(u) \qquad (2.4)$$

for $u = 0, L_i$. But the latter is equivalent to the following equations:

$$\frac{3}{L_i}(\mathbf{b}_{3i+1} - \mathbf{b}_{3i}) \cdot \mathbf{v} = \tan A_i, \qquad (2.5)$$

$$\frac{3}{L_i}(\mathbf{b}_{3i+3} - \mathbf{b}_{3i+2}) \cdot \mathbf{v} = \tan B_i. \qquad (2.6)$$

Let us note that

$$(\mathbf{b}_{3i+1} - \mathbf{b}_{3i}) \cdot \mathbf{v} = \tan A_i(\mathbf{b}_{3i+1} - \mathbf{b}_{3i}) \cdot \frac{(\mathbf{b}_{3i+1} - \mathbf{b}_{3i})}{L_i},$$

and

$$(\mathbf{b}_{3i+3} - \mathbf{b}_{3i+2}) \cdot \mathbf{v} = \tan B_i(\mathbf{b}_{3i+3} - \mathbf{b}_{3i+2}) \cdot \frac{(\mathbf{b}_{3i+3} - \mathbf{b}_{3i+2})}{L_i}.$$

Thus, (2.5) is equivalent to (1.3) with $j = 1$ or $\tan A_i = 0$, and (2.6) is equivalent to (1.3) with $j = 2$ or $\tan B_i = 0$. Let us mention that (1.2) is equivalent to $\tan A_i = 0$ for $j = 1$, and to $\tan B_i = 0$ for $j = 2$. Therefore, $\mathbf{X}$ satisfies (1.1) if and only if for $j = 1, 2$, one of the statements (1.2) and (1.3) holds.

The proof of Theorem 1 is complete.

3. Wilson-Fowler splines in the space

Early in 1976, Thomas [Thomas '76] expressed the belief that the Wilson-Fowler method is a usable and competitive approach to planar curve fitting of the type which exhibits the mechanical spline syndrome. He defined a class of Wilson-Fowler splines in the space. This method has a number of practical merits. But the structure of these splines is not so easy to understand.

Concerning the characterization of planar Wilson-Fowler splines presented in Theorem 1, we define the Wilson-Fowler splines in the space as follows.

Definition 1. *Let $n \in \mathbf{N}$, $\{\mathbf{b}_j\}_{j=0}^{3n}$ be a set of points in the space $\mathbf{R}^3$. Define a spline curve of third degree with its i-th ($0 \le i \le n-1$) component spline to be the Bézier polynomial curve whose Bézier points are $\mathbf{b}_{3i}, \mathbf{b}_{3i+1}, \mathbf{b}_{3i+2}, \mathbf{b}_{3i+3}$. We say that* *this spline curve is a Wilson-Fowler spline in the space if it is GC^2 continuous and for $0 \le i \le n-1, 1 \le j \le 2$, one of the following conditions holds:*

i) $\mathbf{b}_{3i+j}$ *lies on the line containing* $\mathbf{b}_{3i}$ *and* $\mathbf{b}_{3i+3}$, *i.e.,*

$$(\mathbf{b}_{3i+j} - \mathbf{b}_{3i}) \times (\mathbf{b}_{3i+3} - \mathbf{b}_{3i}) = 0; \tag{3.1}$$

ii) $\mathbf{b}_{3i+j}$ *lies in the plane which is perpendicular to the line containing* $\mathbf{b}_{3i}$ *and* $\mathbf{b}_{3i+3}$, *and through the point* $\mathbf{b}_{3i} + \frac{j}{3}(\mathbf{b}_{3i+3} - \mathbf{b}_{3i})$, *i.e.,*

$$(\mathbf{b}_{3i+j} - (\mathbf{b}_{3i} + \frac{j}{3}(\mathbf{b}_{3i+3} - \mathbf{b}_{3i}))) \cdot (\mathbf{b}_{3i+3} - \mathbf{b}_{3i}) = 0. \tag{3.2}$$

In our opinion, the above definition is natural since we have another representation for these Wilson-Fowler splines which is similar to (1.1). For convenience, in what follows, we shall denote $< \mathbf{a}, \mathbf{b} > \in (-\pi, \pi]$ as the angle the vector $\mathbf{b}$ makes with the vector $\mathbf{a}$.

Suppose that a Wilson-Fowler spline is given as in Definition 1. Let $0 \le i \le n-1$. If (3.1) is satisfied for $j = 1$ or 2, then the i-th component Bézier curve, denoted as $\mathbf{X}_i$, is planar, hence we have the representation (1.1). So let us consider the case when (3.2) is satisfied for $j = 1, 2$, and $\mathbf{X}_i$ is not planar, which implies that for $j = 1, 2$,

$$\mathbf{b}_{3i+j} \neq \mathbf{b}_{3i} + \frac{j}{3}(\mathbf{b}_{3i+3} - \mathbf{b}_{3i}). \tag{3.3}$$

We define a local system $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ as

$$\mathbf{u} = \frac{\mathbf{b}_{3i+3} - \mathbf{b}_{3i}}{|\mathbf{b}_{3i+3} - \mathbf{b}_{3i}|}, \tag{3.4}$$

$$\mathbf{v} = \frac{\mathbf{b}_{3i+2} - (\mathbf{b}_{3i} + \frac{2}{3}(\mathbf{b}_{3i+3} - \mathbf{b}_{3i}))}{|\mathbf{b}_{3i+2} \quad (b_{3i} + \frac{2}{3}(\mathbf{b}_{3i+3} - \mathbf{b}_{3i}))|}, \tag{3.5}$$

$$\mathbf{w} = \mathbf{u} \times \mathbf{v}. \tag{3.6}$$

By (3.2) and (3.3), $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is a local coordinate system in which $\mathbf{b}_{3i}$ is the origin.

The parametric Bézier curve $\mathbf{X}_i(u)$ with $u \in [0, L_i]$, $L_i = |\mathbf{b}_{3i+3} - \mathbf{b}_{3i}|$ is defined by

$$\mathbf{X}_i(u) = \sum_{j=0}^{3} \mathbf{b}_{3i+j} B_j^3(\frac{u}{L_i}). \tag{3.7}$$

In the local coordinate system $(\mathbf{u}, \mathbf{v}, \mathbf{w})$, $\mathbf{X}_i(u)$ can be represented as $(u, c_i(u), d_i(u$ $[0, L_i]$. Then

$$c_i(u) = (\mathbf{X}_i(u) - \mathbf{b}_{3i}) \cdot \mathbf{v} = \sum_{j=1}^{2} (\mathbf{b}_{3i+j} - \mathbf{b}_{3i}) \cdot \mathbf{v} B_j^3(\frac{u}{L_i}), \tag{3.8}$$

and

$$d_i(u) = (\mathbf{X}_i(u) - \mathbf{b}_{3i}) \cdot \mathbf{w} = (\mathbf{b}_{3i+1} - \mathbf{b}_{3i}) \cdot \mathbf{w} B_1^3(\frac{u}{L_i}). \tag{3.9}$$

Write

$$\begin{aligned}
A_i &= < \mathbf{b}_{3i+3} - \mathbf{b}_{3i}, \mathbf{b}_{3i+1} - \mathbf{b}_{3i} - ((\mathbf{b}_{3i+1} - \mathbf{b}_{3i}) \ \mathbf{w})\mathbf{w} >, \\
B_i &= < \mathbf{b}_{3i+3} - \mathbf{b}_{3i}, \mathbf{b}_{3i+3} - \mathbf{b}_{3i+2} >, \\
C_i &= < \mathbf{v}, \mathbf{b}_{3i+1} - \mathbf{b}_{3i} - \frac{L_i}{3}\mathbf{u} > .
\end{aligned}$$

It can be easily seen that

$$\begin{aligned}
(\mathbf{b}_{3i+1} - \mathbf{b}_{3i}) \cdot \mathbf{v} &= \frac{L_i}{3} \tan A_i, \\
(\mathbf{b}_{3i+2} - \mathbf{b}_{3i}) \cdot \mathbf{v} &= -\frac{L_i}{3} \tan B_i, \\
(\mathbf{b}_{3i+1} - \mathbf{b}_{3i}) \cdot \mathbf{w} &= \frac{L_i}{3} \tan A_i \tan C_i.
\end{aligned}$$

Combining these formulas with (3.8) and (3.9), we obtain

$$c_i(u) = \tan A_i u(u - L_i)^2/L_i^2 + \tan B_i u^2(u - L_i)/L_i^2, \tag{3.10}$$

$$d_i(u) = \tan A_i \tan C_i u(u - L_i)^2/L_i^2. \tag{3.11}$$

Thus we have set up a representation for the Wilson-Fowler splines in the space similar to (1.1) under a local coordinate system. This representation combining (1.1) is equivalent to the definition of the Wilson-Fowler splines in the space

4. Design of Wilson-Fowler splines in the space

In this section, we consider the design of Wilson-Fowler splines in the space.

Let $\mathbf{X}(u)$ be a cubic spline curve whose i-th ($0 \leq i < n$) component spline $\mathbf{X}_i(u)$ has Bézier points $\mathbf{b}_{3i}, \mathbf{b}_{3i+1}, \mathbf{b}_{3i+2}, \mathbf{b}_{3i+3}$ and has the parametric representation

$$\mathbf{X}_i(u) = \sum_{j=0}^{3} \mathbf{b}_{3i+j} B_j^3\left(\frac{u - u_{i-1}}{L_i}\right), \quad u_{i-1} \leq u \leq u_i, \tag{4.1}$$

where

$$\begin{aligned} u_0 &\doteq 0; \\ u_i &= u_{i-1} + L_{i-1}; \\ L_i &= |\mathbf{b}_{3i+3} - \mathbf{b}_{3i}|. \end{aligned}$$

From the general design of GC^2 spline curve (see [Hoschek & Lasser '93, Nielson '84]) and Definition 1, we can design a Wilson-Fowler spline in the space as well as in the plane as follows.

Let $1 \leq i \leq n-1$. If $< \mathbf{b}_{3i+3} - \mathbf{b}_{3i}, \mathbf{b}_{3i} - \mathbf{b}_{3i-1} > \notin (-\frac{\pi}{2}, \frac{\pi}{2})$, then $\mathbf{X}$ cannot be a Wilson-Fowler spline.

If $A_i := < \mathbf{b}_{3i+3} - \mathbf{b}_{3i}, \mathbf{b}_{3i} - \mathbf{b}_{3i-1} > \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$, then

$$\mathbf{b}_{3i+1} = \mathbf{b}_{3i} + \frac{L_i}{3\cos A_i} \frac{\mathbf{b}_{3i} - \mathbf{b}_{3i-1}}{|\mathbf{b}_{3i} - \mathbf{b}_{3i-1}|}, \tag{4.2}$$

and in case $B_{i-1} := < \mathbf{b}_{3i} - \mathbf{b}_{3i-3}, \mathbf{b}_{3i} - \mathbf{b}_{3i-1} > \neq 0, \pi$,

$$\mathbf{b}_{3i+1} = \mathbf{b}_{3i} + k_i(\mathbf{b}_{3i} - \mathbf{b}_{3i-1}) \tag{4.3}$$

with $k_i = \frac{L_i \cos B_{i-1}}{L_{i-1} \cos A_i} > 0$.

If $A_i = 0$, then

$$\mathbf{b}_{3i+1} = \mathbf{b}_{3i} + k_i(\mathbf{b}_{3i} - \mathbf{b}_{3i-1}) \tag{4.4}$$

with $k_i > 0$.

The design of $\mathbf{b}_{3i+2}$ follows from Hoschek and Lasser [Hoschek & Lasser '93, pp. 223-224]. One possible choice of $\mathbf{b}_{3i+2}$ is the intersection of the line

$$\mathbf{b}_{3i} - k_i^2(\mathbf{b}_{3i-1} - \mathbf{b}_{3i-2}) + \mu(\mathbf{b}_{3i} - \mathbf{b}_{3i-1}), \quad \mu \in \mathbf{R} \tag{4.5}$$

with the plane perpendicular to the vector $\mathbf{b}_{3i+3} - \mathbf{b}_{3i}$ and through the point $\mathbf{b}_{3i} + \frac{2}{3}L_i(\mathbf{b}_{3i+3} - \mathbf{b}_{3i})$. The other possible choice is the intersection (if it exists) of the line (4.5) with the line

$$\mathbf{b}_{3i} + \mu(\mathbf{b}_{3i+3} - \mathbf{b}_{3i}), \quad \mu \in \mathbf{R}. \tag{4.6}$$

Thus, if $A_i \neq 0$, there are one or two possible choices of $\mathbf{b}_{3i+2}$. The case $A_i = 0$ can be easily discussed with possibly infinitely many choices of $\mathbf{b}_{3i+2}$. The above discussion will be used in the next section.

5. Interpolation problem with specific boundary slopes

In this section we show how to find a Wilson-Fowler spline with specific boundary slopes. We restrict our discussion to the planar case here, and moreover, to the following type.

Definition 2. *Let* $\mathbf{X}$ *be a Wilson-Fowler spline in the space given as in Definition 1. We say that* $\mathbf{X}$ *is an equally-spaced Wilson-Fowler spline if for* $0 \leq i \leq n-1, 1 \leq j \leq 2$, *(3.2) is satisfied.*

Assume that $\{\mathbf{P}_i\}_{i=0}^n$ is a set of points in the plane $\mathbf{R}^2$ and $\mathbf{T}_0$ and $\mathbf{T}_n$ are two unit vectors in $\mathbf{R}^2$ with $< \mathbf{P}_1 - \mathbf{P}_0, \mathbf{T}_0 > \neq 0, \frac{\pi}{2}, \pi, < \mathbf{P}_n - \mathbf{P}_{n-1}, \mathbf{T}_n > \neq 0, \frac{\pi}{2}, \pi$. The purpose of the interpolation problem is to find a Wilson-Fowler spline so that it interpolates the points $\mathbf{P}_i$ and have $\mathbf{T}_0, \mathbf{T}_n$ as its unit tangent vectors at $\mathbf{P}_0$ and $\mathbf{P}_n$. Denote $L_i = |\mathbf{P}_{i+1} - \mathbf{P}_i|, \gamma_i = < \mathbf{e}_1, \mathbf{P}_{i+1} - \mathbf{P}_i >$, $TA_0 = \tan < \mathbf{P}_1 - \mathbf{P}_0, \mathbf{T}_0 >$ and $TB_{n-1} = \tan < \mathbf{P}_n - \mathbf{P}_{n-1}, \mathbf{T}_n >$ since $\mathbf{T}_0$ or $\mathbf{T}_n$ cannot be perpendicular to $\mathbf{P}_1 - \mathbf{P}_0$ or $\mathbf{P}_n - \mathbf{P}_{n-1}$.

Suppose that we have an equally-spaced Wilson-Fowler spline satisfying the above conditions. For $0 \leq i \leq n-1$, in the local (u, v)-coordinate system, let

$$\mathbf{b}_{3i+1} = (\frac{L_i}{3}, r_i), \tag{5.1}$$

$$\mathbf{b}_{3i+2} = (\frac{2L_i}{3}, R_i), \tag{5.2}$$

$$A_i = < \mathbf{b}_{3i+3} - \mathbf{b}_{3i}, \mathbf{b}_{3i+1} - \mathbf{b}_{3i} >$$

$$B_i = < \mathbf{b}_{3i+3} - \mathbf{b}_{3i}, \mathbf{b}_{3i+3} - \mathbf{b}_{3i+2} >$$

Then

$$A_i = B_{i-1} - (\gamma_i - \gamma_{i-1}),$$

which implies

$$r_i = \frac{L_i}{3} \tan A_i = \frac{S_i(R_{i-1})}{T_i(R_{i-1})}, \tag{5.3}$$

where

$$T_i(x) = \begin{cases} 1 - \frac{3x}{L_{i-1}} \tan(\gamma_i - \gamma_{i-1}), & \text{if } \gamma_i - \gamma_{i-1} \neq \pm\frac{\pi}{2}, \\ x, & \text{if } \gamma_i - \gamma_{i-1} = \pm\frac{\pi}{2}; \end{cases}$$

$$S_i(x) = \begin{cases} -\frac{L_i}{L_{i-1}} x - \frac{L_i}{3} \tan(\gamma_i - \gamma_{i-1}), & \text{if } \gamma_i - \gamma_{i-1} \neq \pm\frac{\pi}{2}, \\ \frac{L_i L_{i-1}}{9}, & \text{if } \gamma_i - \gamma_{i-1} = \pm\frac{\pi}{2}, \end{cases} \tag{5.4}$$

and in terms of design parameters for GC^2 spline curve (see, [Hoschek & Lasser '93]),

$$\begin{aligned} \omega_{i,11} &= \frac{|\mathbf{b}_{3i+1} - \mathbf{b}_{3i}|}{|\mathbf{b}_{3i} - \mathbf{b}_{3i-1}|} \frac{L_{i-1}}{L_i} \\ &= \frac{\cos B_{i-1}}{\cos A_i} = \frac{\cos B_{i-1}}{\cos(B_{i-1} - (\gamma_i - \gamma_{i-1}))}. \end{aligned} \tag{5.5}$$

From (4.5), we have

$$\begin{aligned} |\mathbf{c}_i - \mathbf{b}_{3i}| &= (\frac{L_i}{L_{i-1}} \omega_{i,11})^2 |\mathbf{b}_{3i-1} - \mathbf{b}_{3i-2}| \\ &= (\frac{L_i}{L_{i-1}} \omega_{i,11})^2 \frac{L_{i-1}}{-3 \sin a_{i-1}} \\ &= -L_i \frac{L_i}{3L_{i-1}} \frac{1}{\sin a_{i-1}} (\frac{\cos B_{i-1}}{\cos(B_{i-1} - (\gamma_i - \gamma_{i-1}))})^2 \end{aligned} \tag{5.6}$$

and

$$\tan a_{i-1} = -\frac{L_{i-1}}{3(R_{i-1} - r_{i-1})}, \tag{5.7}$$

with $a_{i-1} = \mp\frac{\pi}{2}$ when $R_{i-1} = r_{i-1}$.

We also observe that

$$< \mathbf{b}_{3i+3} - \mathbf{c}_i, \mathbf{b}_{3i} - \mathbf{c}_i > = \frac{\pi}{2} - B_{i-1} + a_{i-1},$$

which is also valid for $a_{i-1} = \mp\frac{\pi}{2}$ and implies

$$\frac{|\mathbf{d}_i - \mathbf{b}_{3i}|}{|\frac{\pi}{2} - B_{i-1} + a_{i-1}|} \quad \frac{|\mathbf{c} - \mathbf{b}_{3i}|}{}$$

Hence, in case $\gamma_i - \gamma_{i-1} \neq \pm\frac{\pi}{2}$,

$$
\begin{aligned}
R_i &= \tan A_i\left(\frac{2}{3}L_i - |\mathbf{d}_i - \mathbf{b}_{3i}|\right) \\
&= \frac{-\frac{3R_{i-1}}{L_{i-1}} - \tan(\gamma_i - \gamma_{i-1})}{1 - \frac{3R_{i-1}}{L_{i-1}}\tan(\gamma_i - \gamma_{i-1})} \\
&\quad \left\{\frac{2}{3}L_i + \frac{\sin(\frac{\pi}{2} - B_{i-1} + a_{i-1})}{\sin A_i} L_i \frac{L_i}{3L_{i-1}} \frac{1}{\sin a_{i-1}} \left(\frac{\cos B_{i-1}}{\cos(B_{i-1} - (\gamma_i - \gamma_{i-1}))}\right)^2\right\} \\
&= \frac{-\frac{3R_{i-1}}{L_{i-1}} - \tan(\gamma_i - \gamma_{i-1})}{1 - \frac{3R_{i-1}}{L_{i-1}}\tan(\gamma_i - \gamma_{i-1})} \frac{L_i}{3}\{2 + \frac{3L_i}{L_{i-1}} \frac{2R_{i-1} - r_{i-1}}{L_{i-1}(1 + (\frac{3R_{i-1}}{L_{i-1}})^2)} \\
&\quad [(\frac{3R_{i-1}}{L_{i-1}} + \tan(\gamma_i - \gamma_{i-1}))(1 - \frac{3R_{i-1}}{L_{i-1}}\tan(\gamma_i - \gamma_{i-1}))^2\cos(\gamma_i - \gamma_{i-1})]^{-1} \\
&\quad [(1 - \frac{3R_{i-1}}{L_{i-1}}\tan(\gamma_i - \gamma_{i-1}))^2 + (\frac{3R_{i-1}}{L_{i-1}} + \tan(\gamma_i - \gamma_{i-1}))^2]\} \\
&= -\frac{L_i}{3}\{L_{i-1}(1 + (\frac{3R_{i-1}}{L_{i-1}})^2)(1 - \frac{3R_{i-1}}{L_{i-1}}\tan(\gamma_i - \gamma_{i-1}))^3\cos(\gamma_i - \gamma_{i-1})\}^{-1} \\
&\quad \{\frac{3L_i}{L_{i-1}}(2R_{i-1} - r_{i-1})[1 + (\frac{3R_{i-1}}{L_{i-1}}\tan(\gamma_i - \gamma_{i-1}))^2 + (\frac{3R_{i-1}}{L_{i-1}})^2 + (\tan(\gamma_i - \gamma_{i-1}))^2] \\
&\quad + 2L_{i-1}(1 + (\frac{3R_{i-1}}{L_{i-1}})^2(\frac{3R_{i-1}}{L_{i-1}} + \tan(\gamma_i - \gamma_{i-1})) \\
&\quad (1 - \frac{3R_{i-1}}{L_{i-1}}\tan(\gamma_i - \gamma_{i-1}))^2\cos(\gamma_i - \gamma_{i-1})\} \\
&:= \frac{Q_i(R_{i-1}, r_{i-1})}{P_i(R_{i-1})}.
\end{aligned} \tag{5.8}
$$

Here

$$
P_i(x) = L_{i-1}(1 + (\frac{3x}{L_{i-1}})^2)(1 - \frac{3x}{L_{i-1}}\tan(\gamma_i - \gamma_{i-1}))^3\cos(\gamma_i - \gamma_{i-1}), \tag{5.9}
$$

$$
\begin{aligned}
Q_i(x, y) &= -\frac{L_i}{3}\{\frac{3L_i}{L_{i-1}}(2x - y)[1 + (\frac{3x}{L_{i-1}}\tan(\gamma_i - \gamma_{i-1}))^2 + (\frac{3x}{L_{i-1}})^2 \\
&\quad + (\tan(\gamma_i - \gamma_{i-1}))^2] + 2L_{i-1}(1 + (\frac{3x}{L_{i-1}})^2(\frac{3x}{L_{i-1}} + \tan(\gamma_i - \gamma_{i-1})) \\
&\quad (1 - \frac{3x}{L_{i-1}}\tan(\gamma_i - \gamma_{i-1}))^2\cos(\gamma_i
\end{aligned} \tag{5.10}
$$

are univariate and bivariate polynomials, respectiv...

We notice that (5.8) is also valid for R...

If $\gamma_i - \gamma_{i-1} = \pm\frac{\pi}{2}$, say, $-\frac{\pi}{2}$ then with

$$P_i(x) = 3x(1+(\frac{3x}{L_{i-1}})^2), \tag{5.11}$$

$$Q_i(x,y) = -\frac{L_i}{3}(1+(\frac{3x}{L_{i-1}})^2)\{\frac{3L_i}{L_{i-1}}(2x-y)-2L_{i-1}\}. \tag{5.12}$$

In order that $\omega_{i,11} > 0$, from (5.5) we must require

$$\cos(\gamma_i - \gamma_{i-1}) + \sin(\gamma_i - \gamma_{i-1}) \tan B_{i-1} > 0$$

which is equivalent to

$$R_{i-1} > \frac{L_{i-1}}{3\tan(\gamma_i - \gamma_{i-1})}, \quad \text{if } \gamma_i - \gamma_{i-1} \in (-\pi, 0), \tag{5.13}$$

and

$$R_{i-1} < \frac{L_{i-1}}{3\tan(\gamma_i - \gamma_{i-1})}, \quad \text{if } \gamma_i - \gamma_{i-1} \in [0, \pi]. \tag{5.14}$$

The boundary conditions can also be described as

$$r_0 = \frac{L_0}{3} \tan < \mathbf{P}_1 - \mathbf{P}_0, \mathbf{T}_0 >, \tag{5.15}$$

and

$$R_{n-1} = -\frac{L_{n-1}}{3} \tan < \mathbf{P}_n - \mathbf{P}_{n-1}, \mathbf{T}_n > . \tag{5.16}$$

Combining the above observations, we have

Theorem 2. *Let $\{\mathbf{P}_i\}_{i=0}^n$ be a set of points in the plane $\mathbf{R}^2$ and $\mathbf{T}_0$, $\mathbf{T}_n$ be two unit vectors in $\mathbf{R}^2$ with $< \mathbf{P}_1 - \mathbf{P}_0, \mathbf{T}_0 > \neq 0, \frac{\pi}{2}, \pi, < \mathbf{P}_n - \mathbf{P}_{n-1}, \mathbf{T}_n > \neq 0, \frac{\pi}{2}, \pi$. Then there exists an equally-spaced Wilson-Fowler spline $\mathbf{X}(u)$ whose segments are of the form (4.1) and such that*

$$\mathbf{X}(u_i) = \mathbf{P}_i, \quad 0 \le i \le n, \tag{5.17}$$

$$\frac{\mathbf{X}'(0)}{|\mathbf{X}'(0)|} = \mathbf{T}_0, \tag{5.18}$$

$$\frac{\mathbf{X}'(u_n)}{|\mathbf{X}'(u_n)|} = \mathbf{T}_n, \tag{5.19}$$

if and only if there is a finite sequence of numbers $\{r_i, R_i\}_{i=0}^{n-1}$ such that for $1 \le i \le n-1$, (5.3), (5.4), (5.8), (5.9)-(5.10) (or (5.11)-(5.12) when $\gamma_i - \gamma_{i-1} = \pm\frac{\pi}{2}$), (5.13)-(5.14) and (5.15)-(5.16) are satisfied.

Thus the interpolation problem for equally-spaced Wilson-Fowler splines is reduced to the problem of finding a finite sequence of numbers ith specific properties. Let us state the following algorithm for the solution of this problem:

1. For $1 \leq i \leq n-1$, compute the polynomials S_i, T_i, P_i, Q_i;

2. Let $r_0 = \frac{L_0}{3} \tan < \mathbf{P}_1 - \mathbf{P}_0, \mathbf{T}_0 >$ and for $1 \leq i \leq n-1$, compute the rational polynomials by induction:

$$r_i(R_0) = \frac{S_i(R_{i-1})}{T_i(R_{i-1})} := \frac{f_i(R_0)}{g_i(R_0)}, \tag{5.20}$$

$$R_i(R_0) = \frac{Q_i(R_{i-1}, r_{i-1})}{P_i(R_{i-1})} := \frac{p_i(R_0)}{q_i(R_0)}; \tag{5.21}$$

3. Let $F(R_0)$ be the polynomial given by

$$F(R_0) := p_{n-1}(R_0) + \frac{L_{n-1}}{3} \tan < \mathbf{P}_n - \mathbf{P}_{n-1}, \mathbf{T}_n > q_{n-1}(R_0); \tag{5.22}$$

compute all its zeros $\{z_1, z_2, \cdots, z_l\}$ with $l \geq 0$;

4. For $1 \leq j \leq l$, compute $\{R_i(z_j)\}_{i=1}^{n-1}$ by (5.21);

5. Find the set E of all $j \in \{1, 2, \cdots, l\}$ such that for $1 \leq i \leq n-1$, $R_{i-1}(z_j)$ satisfies (5.13) and (5.14);

6. For $j \in E$, compute $\{r_i(z_j)\}_{i=1}^{n-1}$ by (5.20);

7. If $l = 0$ or $E = \phi$, then the interpolation problem has no solution;

If $l \geq 1$ and $E \neq \phi$, for any $j \in E$, we have a solution of equally-spaced Wilson-Fowler spline whose Bézier points are given by

$$\begin{aligned}
\mathbf{b}_{3i} &= \mathbf{P}_i, \quad 0 \leq i \leq n, \\
\mathbf{b}_{3i+1} &= \mathbf{P}_i + \frac{L_i}{3}(\mathbf{P}_{i+1} - \mathbf{P}_i) + r_i(z_j)\mathbf{v}_i, \\
\mathbf{b}_{3i+2} &= \mathbf{P}_i + \frac{2L_i}{3}(\mathbf{P}_{i+1} - \mathbf{P}_i) + R_i(z_j)\mathbf{v}_i,
\end{aligned}$$

where $\mathbf{v}_i$ is the unit vector such that $(\frac{\mathbf{P}_{i+1} - \mathbf{P}_i}{L_i}, \mathbf{v}_i)$ is a local coordinate system.

From the above algorithm we can see that the essential part of solving the interpolation problem for equally-spaced Wilson-Fowler splines is to find the zeros of a related algebraic polynomial.

The above algorithm can be extended to the general Wilson-Fowler splines as well as to Wilson-Fowler splines in the space.

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ON THE SPLINE APPROXIMATING METHODS FOR SECOND ORDER VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Dedicated to Professor D.D.Stancu on his 70^{th} anniversary.

Abstract. In this paper one proposes a new method of approximation of the solutions of the nonlinear second order Volterra integro-differential equations by means of spline functions. One studies the error estimation and the convergence of the proposed method.

1. Introduction

Consider the nonlinear second order Volterra type integro-differential equation of the form:

$$\begin{aligned} y''(x) &= f(x, y(x), \int_0^x K(x,t,y(t))dt), \ 0 \le x \le a \\ y(0) &= y_0, \ y'(0) = y'_0 \end{aligned} \tag{1}$$

where f and K are given function, y_0, $y'_0 \in \mathbb{R}$ and y is the unknown function.

There are many papers in the literature which consider Volterra integro-differential equation problems, the most being of the first order. For a comprehensive literature we refer to the monograph [6] and also to the papers [1], [3].

Because a lot of phenomena and processes are modeled using the second order Volterra integro-differential equations, their numerical treatment is desired.

Recently some authors ([3], [7], [10]) have proposed different methods to approximate, the solution of the problem (1) especially using the spline functions as approximations.

In this paper, extending from the ordinary differential equation we propose a new method, using polynomial spline functions to approximate the solution of the second order Volterra integro - differential equation with the initial conditions. The existence,

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uniqueness and the convergence of the constructed approximate spline solution are investigated.

2. Description of the approximating method

Following [11] we shall write problem (1) in the form

$$\begin{aligned} y'' &= f(x, y(x), z(x)),\ z(x) = \int_0^x K(x,t,y(t))dt,\ 0 \le x \le a \\ y(0) &= y_0,\ y'(0) = y_0' \end{aligned} \tag{2}$$

Throughout in this paper we suppose that the problem (2) has a unique solution $y : [0,a] \to \mathbb{R}$ and that it is smooth enough.

The function $f : [0,a] \times \mathbb{R}^2 \to \mathbb{R}$ is assumed to be sufficiently smooth function and both it, and its first derivative satisfy the following Lipschitz condition:

$$|f^{(i)}(x, u_1, v_1) - f^{(i)}(x, u_2, v_2)| \le L_i[|u_1 - u_2| + |v_1 - v_2|] \tag{3}$$
$$\forall (x, u_1, v_1), (x, u_2, v_2) \in [0,a] \times \mathbb{R}^2,\ i = 0, 1,\ f^{(0)} = f$$

Also, we assume that the kernel $K : [0,a] \times [0,a] \times \mathbb{R} \to \mathbb{R}$ is a smooth function and satisfies the following Lipschitz condition:

$$|K(x,t,y_1) - K(x,t,y_2)| \le L_3|y_1 - y_2| \tag{4}$$
$$\forall (x,t,y_1), (x,t,y_2) \in [0,a] \times [0,a] \times \mathbb{R}$$

Let Δ be the uniform partition of the interval $[0,a]$ defined by:

$$\Delta : 0 = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_N = a,\ x_k := kh,\ h := \frac{a}{N}$$

Assume that $f \in C^m([0,a] \times \mathbb{R}^2)$ and that the modulus of continuity of the function $y^{(i)}$ is $\omega(y^{(i)}, h) = \omega(h) = O(h^\alpha)$, for $i = 0, 1$ with $0 < \alpha < 1$.

Let m be an integer number with $2 \le m < N$.

Following the idea from [10] we shall construct a polynomial spline functions of degree m and class of continuity $C^1[0,a]$, on the partition Δ to approximate the exact solution of (2).

For $x \in [x_k, x_{k+1}]$, $k = 0, 1, \ldots, N-1$ we define the component of the spline function s_Δ in the following way:

$$s_\Delta(x) = s_k(x) := s_{k-1}(x_k) + \frac{s'_{k-1}(x_k)}{1!}(x - x_k) + \sum_{j=0}^{m-2} \frac{a_k^{(j)}}{(j+2)!}(x - x_k)^{j+2} \tag{5}$$

$$a_k^{(j)} := f^{(j)}(x_k, s_{k-1}(x_k), \int_0^{x_k} K(x_k, t, s_{k-1}(t))dt), j = \overline{0, m-2}, k = \overline{0, N-1}$$

where $s_{-1}(0) := y_0$ and $s'_{-1} = y'_0$ and $s_{k-1}^{(i)}(x_k), i = 0, 1$ are the left hand limits of the derivatives $s_{k-1}^{(i)}(x)$ as $x \to x_k$ of the segment $s_\Delta(x)$ defined on $[x_{k-1}, x_k]$. On the usual technique it can be shown that such a spline function $s_\Delta \in \mathbf{C}^1\ [0, a]$ exists and it is unique by the above construction.

3. Error estimation and convergence

To estimate the error, for any $x \in [x_k, x_{k+1}]$ the exact solution y of the problem (2) can be written by using the following Taylor's expansion:

$$y(x) = \sum_{i=0}^{m} \frac{y^{(i)}(x_k)}{i!}(x - x_k)^i + \frac{y^{(m+1)}(\xi_k)}{(m+1)!}(x - x_k)^{m+1} \tag{6}$$

where $\xi_k \in]x_k, x_{k+1}[$. In what follows we shall denote

$$y_k^{(i)} := y^{(i)}(x_k) \text{ for } i = 0, 1, \ldots, m+1 \text{ and } k = 1, \ldots, N-1$$

For any $x \in [0, a]$ we define the error by the usual way:

$$e(x) := |y(x) - s_\Delta(x)|,\ e'(x) := |y'(x) - s'(x)|, \tag{7}$$

and

$$e_k := |y_k - s_\Delta(x_k)|,\ e'_k := |y'_k - s'_\Delta(x_k)|$$

In what follows we need the following Lemmas.

Lemma 1. [4, p.25]. *Let α and β be nonnegative real numbers, $\beta \neq 1$ and $\{A_i\}_{i=0}^k$ be a sequence satisfying the conditions :*

$$A_0 \geq 0,\ A_{i+1} \leq \alpha + \beta A_i,\ i = 0, 1, \ldots, k.$$

Then the following inequality

$$A_{k+1} \leq \beta^{k+1} A_0 + \alpha \frac{\beta^{k+1} - 1}{\beta - 1} \tag{8}$$

holds .

Lemma 2. [10]. *Let $e^{(i)}(x)$, $i = 0, 1$ be defined as in (7). Then for $p = 0, 1$ the following inequalities hold :*

$$e^{(p)}(x) \leq \sum_{i=0}^{1-p} \frac{e_k^{(i+p)}}{i!} h^i + h\, b\, e_k + \frac{\omega(h) h^{m+2-p}}{(m+2-p)!} \tag{9}$$

where b is a constant independent of h.

Definition. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two real matrices of the some order. We say that $A \leq B$ if:*

(1) *both a_{ij} and b_{ij} are nonnegative*

(2) $a_{ij} \leq b_{ij}$, $\forall\, i, j$

In view of this definition and if we use the matrix notations:

$$E(x) := (e(x), e'(x))^T, \; E_k := (e_k, e'_k)$$

from the Lemma 2 we can write

$$E(x) \leq (I + hA)E_k + h^{m+1}\omega(h)B \tag{10}$$

where

$$A = \begin{pmatrix} b & 1 \\ b & 0 \end{pmatrix}, \; B = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{4} \end{pmatrix}, \; I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If for the matrix $T = (t_{ij})$ we define the norm by

$$||T|| := \max_i \sum_j |t_{ij}|$$

then on the basis of (10) we have:

$$||E(x)|| \leq (1 + h||A||)||E_k|| + ||B||h^{m+1}\omega(h)$$

This inequality holds for any $x \in [0, a]$. Setting $x = x_{k+1}$ it follows:

$$||E_{(k+1)}|| \leq (1 + h||A||)||E_k|| + ||B||h^{m+1}\omega(h)$$

and because $||E_0|| = 0$, using Lemma 1 we have:

$$||E(x)|| \leq ||B||\omega(h) \frac{(i + h||A||)^{k+1} - 1}{1 + h||A|| - 1} \leq \frac{||B||}{||A||}(e^{||A||a} - 1)h^m\omega(h)$$

Now it follows straightforward

$$e^{(i)}(x) \leq b_1 \omega(h) h^m, \text{for } i = 0, 1 \tag{11}$$

where $b_1 := \frac{\|B\|}{\|A\|}(e^{\|A\|a} - 1)$ is a constant independent of h.

Thus, we proved the following result:

Theorem 1. *Let y be the exact solution of the second order Volterra integro-differential equation problem* (1) *and s_Δ be the spline approximation of y, given by* (5). *If $f \in \mathbf{C}^m([0,a] \times \mathbb{R}^2)$, then the following estimations hold:*

$$\begin{aligned} |y(x) - s_\Delta(x)| &\leq b_2\omega(h)h^m \\ |y'(x) - s'_\Delta(x) &\leq b_3\omega(h)h^m \end{aligned}$$

where $x \in [0,a]$ and b_2 and b_3 are constant independent of h .

The method of approximating the solution of problem (1) by a spline function presented here has some advantages order the standard known methods for second order Volterra integro-differential equations, producing continuous, differentiable, accurate and global approximation to the exact solution and its first derivative. The step size h can be changed without additional complications and the methods need no starting values.

Note that in this paper it was assumed that the value $a_k^{(j)}$ are calculated exactly. In practical application it should be suggested to choose a suitable quadrature formula.

4. Numerical example

Consider the following Volterra integro-differential problem:

$$\begin{aligned} y''(x) &= y^2(x) + \int_0^x y(t)dt - e^{2e} + 1,\ x \in [0,1] \\ y(0) &= y'(0) = 1 \end{aligned}$$

The exact solution is $y(x) = e^x$.

Taking the step size $h = 0,1$, a quadratic $(m = 2)$ and cubic $(m = 3)$ spline approximation s evaluated at the point $x = 0,3$ gives the following performances:

	y	s	Error	y'	s'	Error
m=2	1,349846	1,347694	$0,22 \times 10^{-2}$	1,34[illegible]	1,332932	$0,17 \times 10^{-2}$
m=3	1,349846	1,349817	$0,41 \times 10^{-4}$	1,34[illegible]846	1,349810	$0,48 \times 10^{-4}$

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ERROR ESTIMATES OF SOME NUMERICAL DIFFERENTIATION FORMULAS

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Dedicated to Professor D.D.Stancu on his 70th anniversary.

Abstract. Estimates for the error of some numerical differentiation formulas of interpolatory type in the case of Hermite nodes are given.

1. Preliminaries

Denote by H_n, $n \geq 1$, the Hermite polynomials, namely

$$H_n(x) = (-1)^n e^{x^2} \left(e^{-x^2}\right)^{(n)}, \quad x \in R$$

and let x_n^k, $n \geq 1$, $1 \leq k \leq n$ be the roots of H_n so that

$$x_n^1 < x_n^2 < \cdots < x_n^n.$$

Define M as the triangular node matrix whose n-th row contains the roots of H_{2n} together with 0, i.e.

$$M = \{t_n^k : \ n \geq 1, \ -n \leq k \leq n\}$$

where

$$t_n^k = x_{2n}^{n+k}, \ t_n^{-k} = -t_n^k = -x_{2n}^{n-k+1}, \text{ if } 1 \leq k \leq n \text{ and } t_n^0 = 0.$$

Denote by $C^1(R)$ the set of all functions $f : R \to R$ which have a continuous derivative and consider the numerical differentiation formulas

$$f'(0) = D_n f + R_n f, \quad n \geq 1, \ f \in C^1(R) \tag{1}$$

where the functionals $D_n : C^1(R) \to R$ are defined by

$$D_n f = \sum_{k=-n}^{n} a_n^k f(t_n^k) \tag{2}$$

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and $R_n f = 0$ for all polynomials which have the degree at most $2n$. The last condition shows that the numerical differentiation formulas (1) are of interpolatory type.

Putting

$$W_n(x) = \prod_{k=-n}^{n} (x - t_n^k) \text{ and } l_n^k(x) = \frac{W_n(x)}{(x - t_n^k)W'(t_n^k)}$$

$n \geq 1$, $-n \leq k \leq n$, $x \in R$, by the equalities $R_n l_n^k = 0$ we get the following expressions for the coefficients a_n^k:

$$a_n^k = \begin{cases} \frac{-W_n(0)}{t_n^k W_n'(t_n^k)}, & \text{if } 1 \leq k \leq n \\ -a_n^k, & \text{if } -n \leq k \leq -1 \\ 0, \text{if } k = 0 \end{cases} \quad (3)$$

2. Estimations for the coefficients a_n^k

Let $L_n^{(\alpha)}$, $\alpha > -1$, $n \geq 1$ be the Laguerre polynomials, namely

$$L_n^{(\alpha)}(x) = \frac{1}{n!} e^x x^{-\alpha} (e^{-x} x^{n+\alpha})^{(n)}, \quad x > 0.$$

By the classical relation

$$H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{(\frac{-1}{2})}(x^2), \quad n \geq 1,$$

we deduce that

$$W_n(x) = 2^{-2n} x H_{2n}(x) = (-1)^n n! x L_n^{(\frac{-1}{2})}(x^2)$$

and

$$W_n'(x) = (-1)^n n! \left[L_n^{(\frac{-1}{2})}(x^2) + 2x^2 \left(L_n^{(\frac{-1}{2})} \right)' (x^2) \right]. \quad (4)$$

The following relations

$$L_n^{(\alpha)}(0) \sim n^\alpha \quad (5)$$

$$\text{If } L_n^{(\alpha)}(y_n^k) = 0, \ 1 \leq K \leq n, \text{ then } y_n^k \sim k^2/n \quad (6)$$

$$\text{If } L_n^{(\alpha)}(y_n^k) = 0, \ 1 \leq k \leq n, \text{ then } |L_n^{(\alpha)}(y_n^k)| \sim k^{-\alpha-3/2} n^{\alpha+1} \quad (7)$$

hold too for each $\alpha > -1$, where $a_n \sim b_n$, $b_n \neq 0$, means that there exist two real numbers $A > 0$ and $B > 0$, which don't depend on n so that $A \leq |a_n/b_n| \leq B$ for all $n \geq 1$.

Remark, too, that $y_n^k = (t_n^k)^2$, $1 \leq k \leq n$.

Using (3) and (4) we obtain:

$$a_n^k = -\frac{L_n^{\left(\frac{-1}{2}\right)}(0)}{2y_n^k\sqrt{y_n^k}\left(L_n^{\left(\frac{-1}{2}\right)}\right)'(y_n^k)}, \quad 1 \le k \le n. \tag{8}$$

It follows by (5), (6), (7) and (8) the following estimation:

$$|a_n^k| \sim \sqrt{n}/k^2. \tag{9}$$

3. Estimation for the error of formulas (1)

Let $E_n(f)$ be the degree of approximation of a function f in $C^1(R)$ by polynomials of degree at most n, i.e.

$$E_n(f) = \inf\{\sup\{|f(x) - P(x)| : x \in R\} : P \in P_n\}$$

and put

$$e_n(f) = \inf\{|f(0) - P(0)| : P \in P_n\}.$$

For each polynomials P in P_{2n} we have $D_nP = P'(0)$ so that, using (2), we get:

$$|R_nf| = |f'(0) - D_nf| = |f'(0) - P'(0) + D_n(P - f)| \le$$

$$\le |f'(0) - P'(0)| + \sum_{k=1}^{n} |a_n^k|[|f(t_n^k) - P(t_n^k)| + |f(-t_n^k) - P(-t_n^k)|].$$

It is easily seen, taking into account of (9), that:

$$|R_nf| < e_{2n-1}(f') + M\sqrt{n}E_{2n}(f),$$

where $M > 0$ don't depends on n.

Finally, we obtain:

$$\frac{|R_nf| - e_{2n-1}(f')}{E_{2n}(f)} = O\left(\sqrt{n}\right), \text{ if } E_{2n}(f) \ne 0. \tag{10}$$

4. Remark

If $\omega(f;\cdot)$ denotes the modulus of continuity of a continuous function $f : [a,b] \to R$, one proves that

$$|D_nf - f'(0)| < M_1\omega\left(f_n'; 1/\sqrt{n}\right),$$

where $M_1 > 0$ don't depends on n and $f_n : [-t_n^n, t_n^n] \to R$ is the function defined by $f_n(x) = f(x)$ for all $x \in [-t_n^n, t_n^n]$.

If f' is a uniform continuous function, one can show that

$$\lim_{n\to\infty} D_n f = f'(0),$$

which proves the convergence of the numerical differentiation formulas (1) correspo to f.

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ON THE USE OF UMBRAL CALCULUS TO CONSTRUCT APPROXIMATION OPERATORS OF BINOMIAL TYPE

ANDREI VERNESCU

Dedicated to Professor D.D.Stancu, on his 70 -th birthday

Abstract. In this expository paper it is shown that the approximation operators of Bernstein type or Stancu type can be obtained by the methods of umbral calculus.

1. The Romanian School of Numerical Analysis and Theory of Approximation, developed around Professor D.D.STANCU, has given important contributions in the field of linear approximation operators. One of the methods of constructing approximation operators is based on the umbral calculus. This calculus was developed by G.C.Rota and his collaborators (S. Roman, R. Mullin, D. Kahaner, A. Odlyzko) in some papers (see [3], [5] - [10] and the treatise [4]) and it was introduced in our country by a group of mathematicians which worked around Professor D.D. STANCU.

2. In order to construct linear approximation operators using the umbral calculus we consider a *sequence of polynomials* $(p_n)_{n\geq 0}$, where p_n is exactly of degree n, for all n. A polynomial sequence is said to be *of binomial type* if

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y) \qquad n = 0, 1, 2, \dots \tag{1}$$

We present here some basic facts about umbral calculus which we will need. Let us denote by $\mathcal{A}$ the algebra of all polynomials in one variable, with real coefficients and with Π_n the linear space of all polynomials of degree at most n. We will consider all operators $\mathcal{A} \to \mathcal{A}$ to be linear. The shift - operator $E^a\colon \mathcal{A} \to \mathcal{A}$ is defined by $(E^a_p)(x) = p(x+a)$ for all $p \in \mathcal{A}$. An operator $T\colon \mathcal{A} \to \mathcal{A}$ is called a *shift-invariant operator* if $E^aT = TE^a$, for all $a \in \mathbf{R}$. We denote $e_n(t) = t^n$, $n = 0, 1, 2, \dots$. A *delta operator* $Q\colon \mathcal{A} \to \mathcal{A}$ is defined as a shift-invariant operator for which Qe_1 is a non-zero constant. Delta operators possess

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many of the properties of the derivative operator D: $Qe_0 = 0$, $Qp \in \Pi_n$ (if $p \in \Pi_n$). A polynomial sequence $(p_n)_{n\geq 0}$ is called the sequence of *basic polynomials* for a delta operator Q if:

(i) $p_0(x) = 1$;

(ii) $p_n(0) = 0$, $n \geq 1$;

(iii) $Qp_n = np_{n-1}$.

Every delta operator Q has an unique sequence of basic polynomials associated with it. If $K: \mathcal{A} \to \mathcal{A}$ is the operator defined by $(Kp)(x) = xp(x)$ and $T: \mathcal{A} \to \mathcal{A}$ is an operator, then the *Pincherele-derivative of* T is $T' = TK - KT$. If Q is a delta operator, then Q' is shift-invariant and moreover, $(Q')^{-1}$exists. One of the most important results is the following

Theorem. *(R. Mullin, G.C.Rota, [3])*

a) If $(p_n)_n$ is a basic sequence for some delta operator Q, then it is of binomial type.

b) If $(p_n)_n$ is a sequence of polynomials of binomial type, then it is a basic sequence for the same delta operator.

c) $Q: \mathcal{A} \to \mathcal{A}$ is a delta operator if and only if $Q = DP$ for same shift-invariant operator P, where P^{-1} exists and $Dp = p'$.

d) Let $(p_n)_n$ be the sequence of basic polynomials for the delta operator $Q = DP$. Then

(i) $p_n = Q'p^{-n-1}e_n$;

(ii) $p_n = P^{-n}e_n - (P^{-n})'e_{n-1}$;

(iii) $p_n = KP^{-n}e_{n-1}$;

(iv) $p_n = K(Q')^{-1}p_{n-1}$.

3. **The approximation operator constructed by C. Manole ([2].)** Let $(p_n)_{n\geq 1}$ be a certain polynomial sequence of binomial type with $p_1(1) \neq 0$, $p_n(x) \geq 0$ for all $x \in [0, 1]$. The approximation operator of C.Manole is defined by:

$$(L_n f)(x) = \frac{1}{p_n(1)} \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(1-x) f\left(\frac{k}{n}\right) \tag{2}$$

Obviously, L_n is a linear operator of interpolatory type. Let us introduce the notations

$$\begin{cases} S_n(x, y, n) = \sum\limits_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y) \left(\frac{k}{n}\right)^m; \ m \geq 1 \\ S_0(x, y, n) = p_n(x+y) \end{cases} \tag{3}$$

Let Q be the delta operator corresponding to the binomial sequence $(p_n)_{n\geq 0}$. By using the umbral calculus, C. Manole has established in [2] two lemmas and the following

Theorem. *If Q is the delta operator associated to the binomial sequence $(p_n)_{n\geq 0}$, then*

$$S_0(x,y,n) = p_n(x+y)$$

$$S_1(x,y,n) = \frac{x}{x+y}p_n(x+y)$$

$$S_2(x,y,n) = \frac{x}{x+y}p_n(x+y) - \frac{(n-1)xy}{n}\left(Q'^{-2}p_{n-2}\right)(x+y).$$

Corollary. *Let $L_n : C[0,1] \to C[0,1]$, $n = 1,2,\ldots$ be the sequence of linear operators defined by (2). If (p_n) is the binomial sequence with respect to the delta operator Q, then*

$$(L_n e_0)(x) = 1$$

$$(L_n e_1)(x) = x$$

$$(L_n e_2)(x) = x^2 + \frac{x(1-x)}{n} + x(1-x)a_n^{(2)}$$

where the coefficients $a_n^{(k)}$ are given by the formula

$$a_n^{(k)} = \frac{(n-1)(n-2)\cdot\ldots\cdot(n-k+1)}{n^{k-1}}\left[1 - \frac{(Q'^{-k}p_{n-k})(1)}{p_n(1)}\right]$$

By means of umbral calculus it is possible to arrive at once on some classical approximation operators.

a) In the particular case of the sequence $p_k(x) = e_k(x) = x^k$ one finds the operator of **Bernstein**, defined by the formula

$$(L_n f)(x) = (B_n f)(x) = \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^k f\left(\frac{k}{n}\right)$$

b) In the particular case of the sequence $p_k(x) = p_k^{<\alpha>}(x) \overset{\text{def}}{=} x^{[k,-\alpha]} = x(x+$ $+\alpha)(x+2\alpha)\cdot\ldots\cdot(x+\overline{k-1}\alpha)$, where $\alpha \geq 0$ (which is of binomial type accordingly the Vandermonde's identity), we arrive at the operator of Stancu

$$(L_n f)(x) = (S_n f)(x) = \frac{1}{p_n^{<\alpha>}(1)}\sum_{k=0}^{n}\binom{n}{k}p_k^{<\alpha>}(x)p_{n-k}^{<\alpha>}(1-x)f\left(\frac{k}{n}\right)$$

This approximation operator was introduced by Professor D.D. Stancu in 1968 in the paper [12].

Of course, in the particular case $\alpha = 0$ we find the operator of Bernstein. c) Let

$Q = f(D)$ be the delta operator associated to (p_n), where $f(t)$ is a formal power series with $f(0) = 0$, $f'(0) \neq 0$. Noting with $u(t)$ the local inverse of $f(t)$ (i.e. $u(f(t)) = t$) it is know that

$$e^{xu(t)} = \sum_{k=0}^{\infty} p_k(x)\frac{t^k}{k!}.$$

Let $\mathcal{D}$ be a linear space of functions $f\colon [0,\infty) \to \mathbf{R}$ with the property that the series $\sum_{k=0}^{\infty} p_k(n)\frac{t^k}{k!}f\left(\frac{k}{n}\right)$, (where $x = tu'(t)$ and $n = 1,2,\dots$) is convergent for $x \in (-A, A)$. Let $J = [0, a]$, $0 < a < A$, and $L_n : f \mapsto L_n f$, $(f \in \mathcal{D})$, defined by

$$(L_n f)(x) = e^{-nu(t)} \sum_{k=0}^{\infty} p_k(n)\frac{t^k}{k!} f\left(\frac{k}{n}\right)$$

By certain particularizations we obtain some known operators: of Favard-Szasz, Bernstein and Baskakov.

4. The approximation operator constructed by L.Lupaş and A.Lupaş. ([1])

First the following notation and definition are presented: If $(p_n)_n$, $p_n(n) \neq 0$, is a basic sequence for the delta operator Q let us denote

$$w_n(Q) = 1 - \frac{n(n-1)}{p_n(n)}\left(Q'^{-2}p_{n-2}\right)(n)$$

Definition. A linear operator $Q\colon \mathcal{A} \to \mathcal{A}$ belongs to the class W if

(i) Q is a delta operator with the basic sequence (p_n)

(ii) $p'_n(0) \geq 0$; $n = 1, 2, \dots$

(iii) $\lim\limits_{n\to\infty} w_n(Q) = 0$

In [1] some particular examples are examinated.

I. The differentiation operator D, for which $w_n(D) = \frac{1}{n}$.

II. Backward difference operator ∇, for which $w_n(\nabla) = \frac{2}{n+1}$.

III. The Touchard operator T, for which $\frac{1}{n} \leq w_n(T) \leq \frac{3}{n}$.

IV The Laguerre operator L, for which $-\frac{3}{n} \leq w_n(T) \leq \frac{3}{n}$.

Let Q be a delta operator and (p_n) his basic sequence. In [1] $L_n^Q : C(I) \to C(I)$ is defined by

$$(L_n^Q f)(x) = \frac{1}{p_n(n)} \sum_{k=0}^{n} \binom{n}{k} p_k(nx) p_{n-k}(n - nx) f\left(\frac{k}{n}\right) \tag{4}$$

Using theorem 1 it is shown that

$$L_n^Q e_0 = e_0$$

$$L_n^Q e_1 = e_1$$

$$(L_n^Q e_2)(x) = e_2(x) + x(1-x)w_n(Q).$$

So: If $Q \in W$, then $\lim_{n\to\infty} w_n(Q) = 0$ and then $L_n e_k \overset{u}{\underset{n\to\infty}{\longleftarrow}} e_k$.

Using the Bohman-Korovkin theorem, there are proved in [1] the following theorems.

Theorem 1. *If $Q \in W$ and $f \in C(I)$, then $\lim_{n\to\infty} ||f - L_n^Q f|| = 0$ (where $||f|| = \max_{t\in I} |f(t)|$).*

Theorem 2. *Let $(L_n^Q)_{n\geq 1}$, $Q \in W$, be the sequence of linear positive operators defined above. Then:*

(i) If $h \in C(I)$ is convex on I, then $h(x) \leq (L_n^Q h)(x)$, $x \in I$.

(ii) If $h \in C^{(2)} I$, $m_f = \min_{x\in I} f''(x)$, $M_f = \max_{x\in I} f''(x)$, then for $x \in I$

$$\frac{1}{2} m_f \cdot w_n(Q) x(1-x) \leq (L_n^Q f)(x) - f(x) \leq \frac{1}{2} M_f \cdot w_n(Q) x(x-1).$$

Theorem 3. *Let $Q \in W$, $f \in C[0,1]$ and $\omega(f,\delta)$ be the modulus of continuity of the function f. Then*

$$\left|f(x) - \left(L_n^Q f\right)(x)\right| \leq 2\omega\left(f; \sqrt{x(1-x)\omega_n(Q)}\right)$$

$$||f - L_n^Q f|| \leq \frac{5}{4}\omega\left(f, \sqrt{\omega_n(Q)}\right)$$

Ending this paper we mention that the Bernstein operator L_n^D and the operator L_n^∇ can be obtained from some class of operators introduced by Professor D.D. Stancu in his papers [13] – [15].

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