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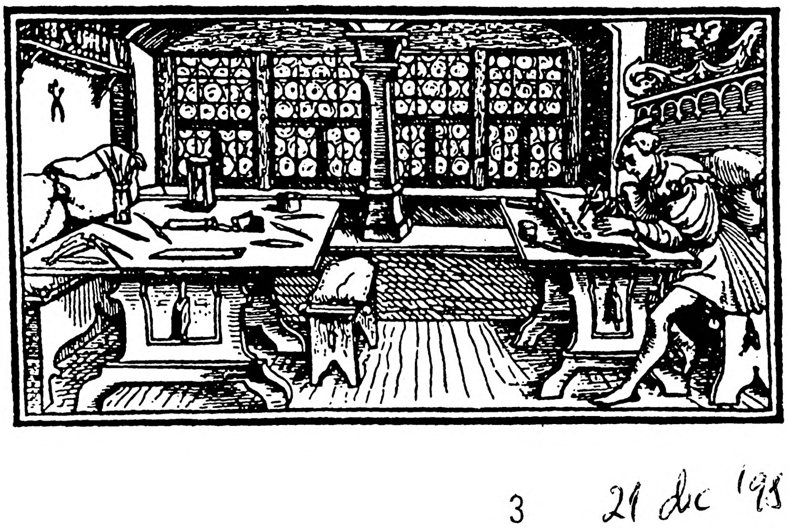

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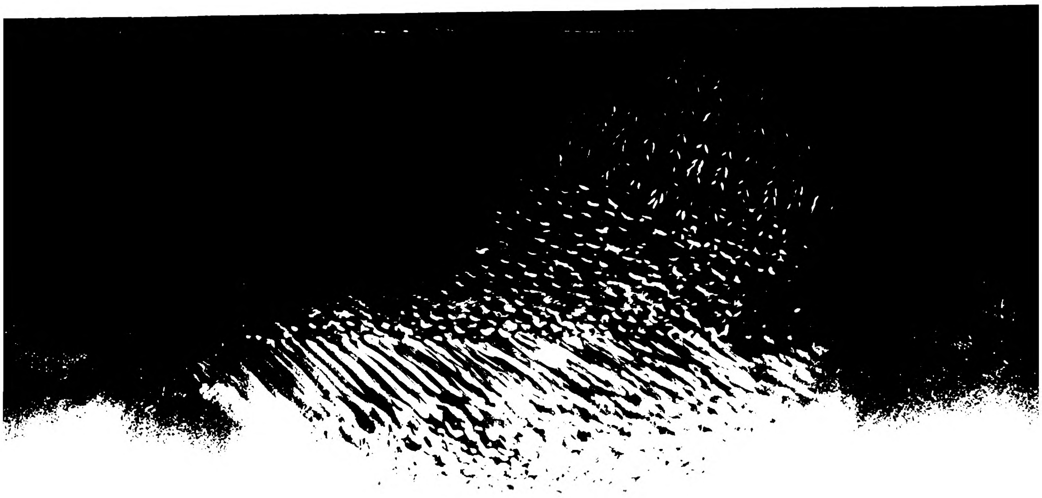

STUDIA UNIV. "BABEŞ-BOLYAI", MATHEMATICA, Volume XLI, Number 3, September 1996

SPLINE APPROXIMATION FOR FIRST ORDER FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. A collocation procedure with spline functions which are not neccessarily polynomial is considered for the numerical solution of first order Fredholm integro-differential equations. A connection with a one step method is the ingredient in the study of the convergence of the spline methods.

1. Introduction

Consider the nonlinear first-order Fredholm integro-differential equation of the form:

$$\begin{aligned} y' &= f(x, y(x), \int_0^a K(x,t,y(t))dt), \quad 0 \leq x \leq a \qquad (1.1) \\ y_{(0)} &= y_0 \end{aligned}$$

where f and K are given functions and y is the unknown function to be found.

There are a number of important problems and phenomena which are modelled using such kind of integro-differential equation, therefore their numerical treatment is desired.

While for the numerical solving of Volterra integro-differential equations a lot of methods are known, for the Fredholm integro-differential equations in the literature only a few are considered. Linz [6] considered numerical methods for the linear form of (1.1) by transforming it into a second kind of integral equation. Phillips [9] considered the non-linear form of (1.1). For a more recent paper on linear equation see Volk [10]. Garey and Gladwin [5] have adapted for (1.1) some direct numerical methods from the Volterra integro-differential equations. They investigated also the convergence of those direct methods, but most results are given only for linear problems. Gheorghe Micula and Graeme Firweather [8] have adapted for the nonlinear form of (1.1) some direct numerical spline methods.

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The estimation of error and the convergence of the spline methods were investigated on the basis of an established connection with multistep methods.

In this paper, we present a new method for the approximate solution for the nonlinear form of (1.1). We use spline functions which are not necessarily polynomial for finding the approximate solution. The method is a one step method $O(h^{m+\alpha})$ in $y^{(i)}(x)$ where $i = 0,1$ and the modulus of continuity of y' is $O(h^{\alpha})$, $0 < \alpha \leq 1$ and m is an arbitrary positive integer which equals to the number of iteration used in computing the spline functions. Condition leading to a unique solution y for equation (1.1) can be found in Anselone and Moore [1] for the linear case and in Phillips [9] for the nonlinear problem. For a deep investigation of the discrete Galerkin methods for nonlinear integral equations see Atkinson and Potra [3], [4].

2. Assumption and procedure

Following [5], we shall write problem (1.1) in the following form:

$$\begin{aligned} y'(x) &= f(x, y(x), z(x)), \;\; y(0) = y_0, \;\; 0 \leq x \leq a \\ z(x) &= \int_0^a K(x,t,y(t))dt \end{aligned} \tag{2.1}$$

Suppose that $f : [0,a] \times R^2 \to R$ is a smooth function satisfying the following Lipschitz condition in respect to the last two arguments:

$$\begin{aligned} | f(x,y_1,z_1) \;\; - \;\; f(x,y_2,z_2) | &\leq L_1 \{| y_1 - y_2 | + | z_1 - z_2 |\} \\ &\forall (x,y_1,z_1),(x,y_2,z_2) \in [0,a] \times R^2 \end{aligned} \tag{2.2}$$

Also, assume that the kernel $K : [0,a] \times [0,a] \times R \to R$ is a smooth bounded function satisfying the Lipschitz condition:

$$\begin{aligned} &| K(x,t,y_1) - K(x,t,y_2) | \leq L_2 | y_1 - y_2 | \\ &\forall (x,t,y_1),(x,t,y_2) \in [0,a] \times [0,a] \times R \end{aligned} \tag{2.3}$$

These conditions assure the existence of a unique solution y of problem (2.1). Let Δ be a uniform partition of the interval [0,a] defined by the following points:

$$\Delta : 0 = x_0 < x_1 \cdots < x_k < x_{k+1} \cdots < x_n = a$$

where $x_k = kh,\ h = \frac{a}{n} < 1$. Assume the function y' has a modulus of continuity

$$\omega(y', h) = \omega(h) = O(h^{\alpha}), \quad 0 < \alpha \le 1.$$

Choosing the required positive integer m, then for $x \in [x_k, x_{x+1}]$, $k = 0, 1, \ldots, n-1$, we define the spline functions approximating the solution $y(x)$ by $s_{\Delta}(x)$ where

$$S_{\Delta}(x) = \overset{[m]}{\underset{k}{S}}(x) = \overset{[m]}{\underset{k-1}{S}}(x_k) + \int_{x_k}^{x} f(x, \overset{[m-1]}{\underset{k}{S}}(x), \overset{[m-1]}{\underset{k}{Z}}(x))dx \tag{2.4}$$

$$\overset{[m-1]}{\underset{k}{Z}}(x) = \int_{x_k}^{x} K(x, t, \overset{[m-1]}{\underset{k}{S}}(t))dt$$

where $\overset{[m]}{\underset{-1}{S}}(x_0) = \overset{[m]}{\underset{-1}{S}}(t) = y_0$ and $\overset{[m]}{\underset{k-1}{S}}(x_k)$ is the left hand limit of $S_{k-1}^{[m]}(x)$ as $x \to x_k$ of the segment of $S_{\Delta}(x)$ defined on $[x_{k-1}, x_k]$.

In equation (2.4), we use the following m iterations $x_k \le x \le x_{x+1}$, $k = 0, 1, \ldots, n-1$ and $i = 1, 2, \ldots, m$

$$\begin{aligned}
\overset{[j]}{\underset{k}{S}}(x) &= \overset{[m]}{\underset{k-1}{S}}(x_k) + \int_{x_k}^{x} f(x, \overset{[j-1]}{\underset{k}{S}}(x), \overset{[j-1]}{\underset{k}{Z}}(x))dx \\
\overset{[j-1]}{\underset{k}{Z}}(x) &= \int_{0}^{a} K(x, t, \overset{[j-1]}{\underset{k}{S}}(t))dx \\
\overset{[0]}{\underset{k}{S}}(x) &= \overset{[m]}{\underset{k-1}{S}}(x_k) + M_k(x - x_k) \\
M_k &= f\left(x_k, \overset{[m]}{\underset{k-1}{S}}(x_k), \int_{0}^{a} K(x, t, \overset{[m]}{\underset{k-1}{S}}(t))dt\right)
\end{aligned} \tag{2.5}$$

Obviously such $S_{\Delta}(x) \in C[0, a]$ exists and is unique.

3. Error estimate and convergence

To estimate the error it is convenient to represent the exact $y(x)$ in various forms as described by the following scheme:

$$\overset{[0]}{y}(x) = y(x) = y_k + y'(\xi_k)(x - x_k), \tag{3.1}$$

where $x_k < \xi_k < x_{k+1}$ and $y_k = y_{(x_k)}$.

For $1 \le j \le m$, we write

$$\overset{[j]}{y}(x) = y(x) = y_k + \int_{x_k}^{x} f(x, \overset{[j-1]}{y}(x), \overset{[j-1]}{Z}(x))dx \tag{3.2}$$

$$\overset{[j-1]}{Z}(x) = \int_{0}^{a} K(x, t, \overset{[j-1]}{y}(t))dt. \tag{3.3}$$

Moreover, we denote the estimated error of $y(x)$ at any point $x \in [0,a]$ by:

$$e(x) = | y(x) - s_\Delta(x) |, \; e_k = | y_k - s_\Delta(x_k) | \tag{3.4}$$

Lemma 3.1 (7.p.24). *Let α and β be positive real numbers, $\{A_i\}_{i=1}^m$ be a sequence satisfying $A_1 \geq 0$ and $A_i \leq \alpha + \beta A_{i+1}$ for $i = 1,2,\ldots,m-1$ then:*

$$A_1 \leq \beta^{m-1} A_m + \alpha \sum_{i=0}^{m-2} \beta^i \tag{3.5}$$

Lemma 3.2 (7.p.25). *Let α and β be non negative real numbers, $q \neq 1$ and $\{A_i\}_{i=0}^k$ be a sequance satisfying $A_0 \geq 0$ and $A_{i+1} \leq \alpha + \beta A_i$ for $i = 0,1,\ldots,k-1$ then:*

$$A_{k+1} \leq \beta^{k+1} A_0 + \frac{\alpha[\beta^{k+1} - 1]}{\beta - 1} \tag{3.6}$$

Definition 3.1. For any $u \in [x_k, x_{k+1}]$, $k = 0,1,\ldots,n-1$ and $j = 0,1,\ldots,m$, we define the operator $T_{kj}(u)$ by,

$$T_{kj}(u) = \left| \overset{[m-j]}{y}(u) - \overset{[m-j]}{\underset{k}{S}}(u) \right|$$

whose norm is defined by:

$$\|T_{kj}\| = \max_{u \in [x_k, x_{k+1}]} \{T_{kj}(u)\}$$

Lemma 3.3. *For any $x \in [x_k, x_{k+1}]$, $k = 0,1,\ldots,n-1$ and $j = 1,2,\ldots,m$.*

$$\|T_{km}\| \leq (1 + hb_0)e_k + h\omega(h) \tag{3.7}$$

$$\|T_{k1}\| \leq \left(\sum_{i=0}^{m} b_0^i \right) e_k + b_0^{m-1} h^m \omega(h) \tag{3.8}$$

where $b_0 = L_1 + L_2 L_2 a$, is a constant independent of h.

Proof. Using (3.1), (2.5), (2.2), (2.3) and (3.3)

$$\begin{aligned} \left| \overset{[0]}{y}(x) - \overset{[0]}{\underset{k}{S}} \right| &\leq e_k + |y'(\xi_k) - M_k\, |x - x_k|| \leq \\ &\leq e_k + |y'(\xi_k) - y_k'|\, |x - x_k| + |y_k' - M_k|\, |x - x_k| \end{aligned} \tag{3.9}$$

$$\begin{aligned} |y_k' - M_k| &= \left| f(x_k, y_k, \int_0^a K(x_k, t, y(t))dt - f(x_k, \overset{[m]}{\underset{k-1}{S}}(x_k), \int_0^a K(x_k, t, \overset{[m]}{\underset{k-1}{S}}(t))dt) \right| \leq \\ &\leq L_1 e_k + L_1 L_2 \int_0^a \left| y(t) - \overset{[m]}{\underset{k-1}{S}}(t) \right| dt \end{aligned} \tag{3.10}$$

But for $t \in [x_{k-1}, x_k]$, $e(t) = \left| y(t) - \overset{[m]}{\underset{k-1}{S}}(t) \right| \to e_k$ as $t \to x_k$, then

$$|y'_k - M_k| \le L_1 e_k + L_1 L_2 e_k \int_0^a dt \le (L_1 e_k + L_1 L_2 a) e_k \tag{3.11}$$

use (3.11) in (3.9), we obtain:

$$\|T_{km}\| \le e_k + h\omega(h) + h(L_1 + L_1 L_2 a) e_k \le [1 + h(L_1 + L_1 L_2 a] e_k + h\omega(h)$$

hence

$$\|T_{km}\| \le (1 + hb_0) e_k + h\omega(h)$$

where $b_0 = L_1 + L_1 L_2 a$ is a constant independent of h.

To prove (3.8), we compute $\|T_{kj}\|$ using (3.2), (2.5), (2.2), (2.3) and (3.3), we get:

$$\left| \overset{[m-j]}{y}(x) - \overset{[m-j]}{\underset{k}{S}} \right| \le e_k + L_1 \int_{x_k}^{x} \left\{ \left| \overset{[m-j-1]}{y}(x) - \overset{[m-j]}{\underset{k}{S}}(x) \right| + \left| \overset{[m-j-1]}{Z} - \overset{[m-j]}{\underset{k}{Z}}(x) \right| \right\} dx \le$$

$$\le e_k + L_1 \int_{x_k}^{x} \left\{ \left| \overset{[m-j-1]}{y}(x) - \overset{[m-j-1]}{\underset{k}{S}}(x) \right| + L_2 \int_0^a \left| \overset{[m-j-1]}{y}(t) - \overset{[m-j-1]}{\underset{k}{S}}(t) \right| dt \right\} dx$$

Hence

$$\begin{aligned} \|T_{kj}\| \le e_k \quad & + L_1 \int_{x_k}^{x} \left\{ \max_{x \in [x_k, x_{k+1}]} \left| \overset{[m-j-1]}{y}(x) - \overset{[m-j-1]}{\underset{k}{S}}(x) \right| \right. \\ & \left. + L_2 \int_0^a \max_{x \in [x_k, x_{k+1}]} \left\{ \left| \overset{[m-j-1]}{y}(t) - \overset{[m-j-1]}{\underset{k}{S}}(t) \right| \right\} dt \right\} dx \\ & \le e_k + L_1 \int_{x_k}^{x} \left\{ \left\| T_{k(j+1)} \right\| + L_2 \left\| T_{k(j+1)} \right\| a \right\} dx \\ & \le e_k + (L_1 + L_1 L_2 a) \left\| T_{k(j+1)} \right\| \int_0^x dx \le e_k + hb_0 \left\| T_{k(j+1)} \right\| \end{aligned}$$

Using Lemma (3.1) and an inequality (3.7), we get:

$$\|T_k\| \le (b_0 h)^{m-1} \|T_{km}\| + e_k \sum_{i=0}^{m-2} (b_0 h)^i \le \left(\sum_{i=0}^{m} b_0^i \right) e_k + b_0^{m-1} h^m \omega(h)$$

□

Lemma 3.4. *Let $e(x)$ be defined as in (3.3) then there exist some constants b_1 and b_2 independent of h such that:*

$$e(x) \le (1 + hb_1) e_k + b_2 h^{m+1} \omega(h) \tag{3.12}$$

where $b_1 = \sum_{i=0}^{m} b_0^{i+1}$ and $b_2 = b_0^m$ are constant independent of h.

Proof. Using (3.2), (2.4), (2,2), (2.3), (3.3) and (3.8), we get:

$$\begin{aligned} e(x) &= \left|y(x) - \overset{[m]}{\underset{k}{S}}(x)\right| \leq e_k + L_1 \int_{x_k}^{x} \left\{ \left|\overset{[m-1]}{y}(x) - \overset{[m-1]}{\underset{k}{S}}(x)\right| + \right. \\ &+ \left. \left|\overset{[m-1]}{Z}(x) - \overset{[m-1]}{\underset{k}{Z}}(x)\right| dt \right\} dx \leq \\ &\leq e_k + L_1 \int_{x_k}^{x} \left\{ T_{k1}(x) + + L_2 \int_{x_k}^{x} \left|\overset{[m-1]}{y}(t) - \overset{[m-1]}{\underset{k}{S}}(t)\right| dt \right\} dx \leq \\ &\leq e_k + L_1 \int_{x_k}^{x} \left\{ \max_{x \in [x_k, x_{k+1}]} \{T_{k1}(x)\} + L_2 \int_0^a \max_{t \in [x_k, x_{k+1}]} \{T_{k1}(x)\} \, dt \right\} dx \leq \\ &\leq e_k + L_1 \int_{x_k}^{x} \{ \|T_{k1}\| (1 + L_2 a) \} \, dx \leq e_k + (L_1 + L_1 L_2 a) \|T_{k1}\| h \leq \\ &\leq e_k + b_0 h \|T_{k1}\| \leq e_k + h \left(\sum_{i=0}^{m} b_0^{i+1} \right) + b_0^m h^{m+1} \omega(h) \end{aligned}$$

Hence,

$$e(x) \leq (1 + hb_1)e_k + b_2 h^{m+1} \omega(h)$$

where $b_1 = \sum_{i=0}^{m} b_0^{i+1}$ and $b_2 = b_0^m$ are constants independent of h. The inequality (3.12) holds for any $x \in [0, a]$. Setting $x = x_{k+1}$, we get:

$$e_{k+1} \leq (1 + hb_1)e_k + b_2 h^{m+1} \omega(h)$$

using Lemma (3.2) and noting that $e_0 = 0$ we get:

$$\begin{aligned} e(x) &\leq \frac{b_2}{b_1} h^m \omega(h) \left[(1 + b_1 h)^{k+1} - 1 \right] \leq \frac{b_2}{b_1} h^m \omega(h) \left[\left(1 + \frac{a b_1}{h} \right)^h - 1 \right] \leq \quad (3.1\ ) \\ &\leq \frac{b_2}{b_1} \left(e^{a b_1} - 1 \right) h^m \omega(h) = b_3 h^m \omega(h) = o\left(h^{m+\alpha} \right) \end{aligned}$$

where $b_3 = \frac{b_2}{b_1} \left(e^{a b_1} - 1 \right)$ is a constant independent of h. □

We now estimate $|y'(x) - s'_\Delta(x)|$. For this purpose using (2.4), (3.1), (2.2), (2.3), (3.3) and (3.14), we obtain

$$|y'(x) - s'_\Delta(x)| \leq b_4 h \omega(h) \qquad (3.14)$$

where $b_4 = b_1 b_3 + b_2$ is a constant independent of h. Thus, we prove the following result.

Theorem 3.1. *Let $y(x)$ be the exact solution to the problem (2.1). If $s_\Delta(x)$, given by (2.4) is the approximate solution for the problem then the inequality*

$$\left|y^{(i)}(x) - s_\Delta^{(i)}(x)\right| \le b_5 h^m \omega(h)$$

holds for all $x \in [0,a]$, $i = 0,1$ where b_5 is a constant independent of h.

Numerical Examples

The method is tested using the following two examples on the interval [0,1] with stepsize $h = 0.1$ where $m = 1$ and 2. The result tabulated below are evaluated at the point 0.4.

Example 1. Consider the Fredholm integrodifferential equation

$$y' = -y + \int_0^1 y(t)dt + e^{-1} - 1, y(0) = 1, 0 \le x \le 1.$$

The exact solution is $y = e^{-x}$

		The numerical value	The absolute error
y	$m = 1$	0.655149801	1.5 x 10^{-2}
	$m = 2$	0.668546426	1.8 x 10^{-3}
y'	$m = 1$	-0.706187504	3.5 x 10^{-2}
	$m = 2$	-0.668231103	2.1 x 10^{-3}

Example 2. Consider the Fredholm integrodifferential equation

$$y' = -y + \int_0^1 y^2(t)dt + \frac{1}{2}(e^{-2} - 1), y(0) = 1, 0 \le x \le 1.$$

The exact solution is $y = e^{-x}$

		The numerical value	The absolute error
y	$m = 1$	0.681022393	1.1 x 10^{-2}
	$m = 2$	0.674430125	4.1 x 10^{-3}
y'	$m = 1$	-0.691366681	2.1 x 10^{-2}
	$m = 2$	-0.663759865	6.6 x 10^{-3}

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ON THE APPROXIMATION OF THE LAPLACE TRANSFORM BY POSITIVE LINEAR OPERATORS

ALEXANDRA CIUPA AND IOAN GAVREA

Abstract. In this paper we study a Favard-Szasz type operator, denoted by P_n, obtained by means of Appell polynomials by A. Jakimovski and D. Leviatan [2]. We prove that if the function f is of exponential order, then the Laplace transform of the function $P_n f$ approximates the Laplace transform of the function f.

I.Introduction In 1969, A. Jakimovski and D. Leviatan [2] have obtained a Favard-Szasz type operator. Let us remind it. One considers $g(z) \equiv \sum_{n=0}^{\infty} a_n z^n$ an analytic function in the disk, $|z| < R$, $(R > 1)$, and supposes $g(1) \neq 0$, $a_n \in R$ for $n = 0, 1, \ldots$. One defines the Appell polinomials $p_k(x)$, $(k \geq 0)$ by

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k \tag{1}$$

We denote by $\mathcal{E}$, the class of real function of exponential order, i.e.

$$\mathcal{E} = \{f : [0, \infty) \to R \text{ for which there are } A, B \in R \text{ such that } |f(x)| \leq Be^{Ax}, \forall x \geq 0\}.$$

A. Jakimovski and D. Leviatan have considered the operator

$$\begin{aligned} &P_n : \mathcal{E} \to C[0, \infty), \\ &(P_n f)(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), n > 0 \end{aligned} \tag{2}$$

P_n is a Favard-Szasz type operator, because for $g(z) \equiv 1$, we have $p_k(x) = \frac{x^k}{k!}$ and so from (2) we obtain the well known operator

$$(S_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right). \tag{3}$$

B. Wood [4] has proved that the operators P_n are positive if and only if $\frac{a_n}{g(1)} \geq 1$, $n = 0, 1, \ldots,$.

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II. We will call Laplace transformable function, a function f which satisfies the conditions:

1. $f(t) = 0$, for $t < 0$.
2. f is continuous over finite interval.
3. f is of exponential order.

The Laplace transform of a such function is given by

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = F(s),$$

where $\mathcal{L}\{f(t)\}$ is a shorthand notation for the Laplace integral.

Theorem 2.1. *If $f \in \mathcal{E}$, then the function $P_n f$ is Laplace transformable for $n > N$.*

Proof. Because $f \in \mathcal{E}$ and the operators P_n are positive, we have

$$|(P_n f)(x)| \le \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \left| f\left(\frac{k}{n}\right) \right| \le B \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \cdot e^{A\frac{k}{n}}.$$

If in relation (1) we replace x by nx, we obtain

$$g(u)e^{unx} = \sum_{k=0}^{\infty} p_k(nx)u^k.$$

Let $u = e^{\frac{A}{n}}$ be and suppose that $n > \frac{A}{\ln R}$. We have

$$g\left(e^{\frac{A}{n}}\right) e^{nx\exp\left(\frac{A}{n}\right)} = \sum_{k=0}^{\infty} p_k(nx) \cdot e^{k\frac{A}{n}}$$

We multiply this relation by $B\frac{e^{-nx}}{g(1)}$ and it results

$$Bg\left(e^{\frac{A}{n}}\right) \cdot \frac{e^{nx\left(\exp\left(\frac{A}{n}\right)-1\right)}}{g(1)} = B\frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \cdot e^{k\frac{A}{n}} \tag{4}$$

or

$$Bg\left(e^{\frac{A}{n}}\right) \cdot \frac{e^{x\frac{\exp\left(\frac{A}{n}\right)-1}{\frac{A}{n}}A}}{g(1)} = B\frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \cdot e^{k\frac{A}{n}}$$

The function $h(t) = \frac{e^t-1}{t}$, $t > 0$ is increasing, so

$$h\left(\frac{A}{n}\right) < h(A) \quad \text{or} \quad \frac{\exp\left(\frac{A}{n}\right) - 1}{\frac{A}{n}} < \frac{\exp(A) - 1}{A}.$$

It results that

$$e^{nx\left(\exp\left(\frac{A}{n}\right)-1\right)} = e^{Axh\left(\frac{A}{n}\right)} < e^{Axh(A)} < e^{Ax\frac{\exp(A)-1}{A}} = e^{x(\exp(A)-1)}.$$

Coming back to (4), we obtain

$$B\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty}p_k(nx)\cdot e^{A\frac{k}{n}} = B\frac{g\left(e^{\frac{A}{n}}\right)}{g(1)}\cdot e^{nx\left(\exp\left(\frac{A}{n}\right)-1\right)} \leq B\frac{g\left(e^{\frac{A}{n}}\right)}{g(1)}\cdot e^{x(\exp A-1)}.$$

It result that

$$|(P_nf)(x)| \leq \frac{B}{g(1)}g\left(e^{\frac{A}{n}}\right)\cdot e^{x(\exp A-1)}.$$

therefore the function P_nf is of exponential order, for $n > N$, where $N = \left[\frac{A}{\ln R}\right] + 1$. □

Next, we will show the Laplace transform of function P_nf approximates the Laplace transform of function f.

Theorem 2.2. *If $f \in C[0,\infty)$ and $f \in \mathcal{E}$, then there is a R such that*

$$\lim_{n\to\infty}\mathcal{L}\{P_nf\}(s) = \mathcal{L}\{f\}(s),$$

for every $s \in C$, with Re $s > a$.

Proof. . For Re $s > 0$, we have

$$\mathcal{L}\{P_nf\}(s) = \int_0^{\infty} e^{-st}(P_nf)(t)dt = \frac{1}{g(1)}\sum_{k=0}^{\infty}\int_0^{\infty} e^{-st}e^{-nt}p_k(nt)f\left(\frac{k}{n}\right)dt =$$

$$= \frac{1}{g(1)}\sum_{k=0}^{\infty}\left[\int_0^{\infty} e^{-t(s+n)}p_k(nt)dt\right]f\left(\frac{k}{n}\right) = \frac{1}{g(1)}\sum_{k=0}^{\infty}\mathcal{L}\{p_k(nt)\}(s+n)f\left(\frac{k}{n}\right).$$

We will show that the remainder in approximation formula

$$\mathcal{L}f(s) = \mathcal{L}\{P_nf\}(s) + (R_nf)(s)$$

tends to zero when $n \to \infty$. We have

$$(R_nf)(s) = \mathcal{L}\{f - P_nf\}(s) = \int_0^{\infty} e^{-st}[f(t) - (P_nf)(t)]dt,$$

therefore

$$|(R_nf)(s)| \leq \int_0^{\infty}\left|e^{-st}\right||f(t) - (P_nf)(t)|dt =$$

$$= \int_0^{A} e^{-t\,\mathrm{Re}\,s}|f(t) - (P_nf)(t)|dt + \int_A^{\infty} e^{-t\,\mathrm{Re}\,s}|f(t) - (P_nf)(t)|dt,$$

for every $A > 0$.

Because $f \in \mathcal{E}$, it results that $M_1 > 0$ and $\exists A_1 \in \mathbf{R}$ such that $|f(t)| \leq M_1 e^{A_1 t}$, for every $t \in [0, \infty)$. By making use of Theorem 2.1., it results that for $n > A_1 \frac{1}{\ln R}$ there are $A_2 \in \mathbf{R}$ and $M_2 > 0$ such that $|(P_n f)(t)| \leq M_2 e^{A_2 t}$, for every $t \in [0, \infty)$. It results that $\exists M > 0$ and $b \in \mathbf{R}$, such that

$$|f(t) - (P_n f)(t)| \leq M e^{bt}, \quad \text{for every} \quad t \in [0, \infty). \tag{5}$$

Let $\varepsilon > 0$. From (5) it results that we can choose $A > 0$ such that

$$\left| \int_A^{\infty} e^{-st} \left[f(1) - (P_n f)(t) \right] dt \right| < \frac{\varepsilon}{2}, \quad \forall n > A_1 \frac{1}{\ln R}, \quad \forall s \in C, \quad \operatorname{Re} s > b. \tag{6}$$

We have proved [1], that if $f \in C[0, a]$, then

$$|(P_n f)(t) - f(t)| \leq C(a, n) \omega \left(f; \frac{1}{\sqrt{n}} \right),$$

where $C(a, n) = \left(1 + \sqrt{a + \frac{1}{n} \frac{g''(1) + g'(1)}{g(1)}} \right)$. By making of this inequality, we obtain

$$\left| \int_0^A e^{-st} \left[f(t) - (P_n f)(t) \right] dt \right| \leq C(A, n) \omega \left(f; \frac{1}{\sqrt{n}} \right) \frac{1 - e^{-A \operatorname{Re} s}}{|s|}.$$

Because $\lim_{\delta \to 0} \omega(f; \delta) = 0$, it results that $\exists N_1(\varepsilon)$ such that $n > N_1(\varepsilon)$ and $s \in \mathbb{C}$ with $\operatorname{Re} s = \max \{0, b\} = a$ we have

$$\left| \int_0^A e^{-st} \left[f(t) - (P_n f)(t) \right] dt \right| < \frac{\varepsilon}{2}. \tag{7}$$

From (6) and (7), it results that for $n > \max \left(A_1 \frac{1}{\ln R}, N_1(\varepsilon) \right)$ we have

$$|(R_n f)(s)| < \varepsilon, \forall s \in C, \operatorname{Re} s > a.$$

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A NOTE ON RING ENDOMORPHISMS

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Abstract. If α, β, γ and δ are unitary endomorphisms of a commutative domain R, if one of these endomorphisms is a monomorphism and $\alpha + \delta = \beta + \gamma$, then these endomorphisms are equal in pairs. This generalizes a lemma of A. Vesanen which was obtained for automorphisms of finite fields.

A. Vesanen [3, Lemma 3] proved that if F is a finite field and α, β and γ are automorphisms of F such that $\alpha = \beta + \gamma - id_F$, then one of the automorphisms α, β, γ is the identity automorphism id_F.

If R is a ring with identity and α, β are endomorphisms of R such that $\alpha(1) = \beta(1) = 1$, then $\alpha - \beta \in End(R)$ if and only if $\alpha = \beta$. Indeed, if $\alpha - \beta \in End(R)$, then $\{x \in R \mid \alpha(x) = \beta(x)\}$ is an ideal of R containing 1, whence $\alpha = \beta$.

Let now α and β be two automorphisms of the complex number field C such that $\alpha(z) + \beta(z) = 2Re(z)$ for all $z \in C$. Then it can be shown by elementary calculations that one of these automorphisms is the identity automorphism and the other one is the complex conjugation.

These results admit a common generalization and the aim of this note is to prove the following:

Theorem. *Let R be a commutative domain and let α, β, γ, δ be unitary endomorphisms of R, one of them being a monomorphism. If $\alpha + \delta = \beta + \gamma$, then these endomorphisms are equal in pairs.*

Proof. Up to some point, the proof follows the original idea of Vesanen. To start with, we write $\alpha = \beta + \gamma - \delta$. Let $x, y \in R$, so that $\beta(xy) + \gamma(xy) - \delta(xy) = (\beta(x) + \gamma(x) - \delta(x))(\beta(y) + \gamma(y) - \delta(y))$. After calculations, one obtains

$$(\beta(x) - \delta(x))(\gamma(y) - \delta(y)) + (\beta(y) - \delta(y))(\gamma(x) - \delta(x)) \quad (1)$$

$y \in R$.

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We define now three subrings of R as follows: $B = \{x \in R \mid \delta(x) = \beta(x)\}$, $C = \{x \in R \mid \delta(x) = \gamma(x)\}$ and $D = \{x \in R \mid \beta(x) = \gamma(x)\}$.

If *char* $R \neq 2$, let $y = x$ in (1). This yields $(\beta(x) - \delta(x))(\gamma(x) - \delta(x)) = 0$ for all $x \in R$. Since *char* $R \neq 2$, one obtains that

$$(\beta(x) - \delta(x))(\gamma(x) - \delta(x)) = 0 \tag{2}$$

for all $x \in R$. But (2) is equivalent to $R = B \cup C$, which forces $R = B$, or $R = C$. If $R = B$, then $\beta = \delta$ and $\alpha = \gamma$, while if $R = C$, then $\gamma = \delta$ and $\alpha = \beta$.

If *char* $R = 2$, let $y = x^2$ in (1); this yields

$$(\beta(x) - \delta(x))(\gamma(x) - \delta(x))(\beta(x) - \gamma(x)) = 0 \tag{3}$$

for all $x \in R$. But (3) is equivalent with $R = B \cup C \cup D$. In order to complete the proof it suffices to show that one of the subrings B,C,D equals R.

Since a group is not union of two of its proper subgroups, one can assume that the terms of the union are irredundant.

Suppose that B,C,D are proper subrings of R. Since the additive group of R is a union of three subgroups, it follows by a result of Haber and Rosenfeld [1] that the additive factor group $R/B \cap C \cap D$ is isomorphic to the Klein four group. Now Lemma 1 of J. Lewin [2] asserts that R contains an ideal I such that $I \subseteq B \cap C \cap D$ and $|R/I|$ is finite.

If $I = 0$, then R is a finite field and B,C,D are subfields of R. But it is straightforward to prove that a field is not a union of three of its proper subfields. Therefore one of the subfields B,C,D must equal R and as a consequence two of the endomorphisms are equal.

If $I \neq 0$, let x be a nonzero element of I. Pick an element $y \in B \setminus (C \cup D)$. Then $xy \in I \subseteq B \cap C \cap D$, so in particular $xy \in C$. Thus $\gamma(xy) = \delta(xy)$ and since $x \in C$ one obtains that $\gamma(x) = \delta(x)$. But then $\gamma(x)(\gamma(y) - \delta(y)) = 0$.

Without any loss in generality, one may assume that γ is the one-to-one endomorphism, because anyway I is contained in $B \cap C \cap D$.

But then, since $x \neq 0$, we get that $\gamma(x) \neq 0$, whence $\gamma(y) = \delta(y)$. This means that $y \in C$, against the choice of y. This contradiction proves that one of the subrings B,C,D must be equal to R, that is, two of the four endomorphisms must be equal.

If $R = B$ or $R = C$, then we are done. If $R = D$, then $\beta = \gamma$, whence $\alpha = \beta + \beta - \delta = \delta$ and the proof is complete. □

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INTERPOLATIONS TO JENSEN'S INEQUALITY WITH APPLICATION TO REFINEMENTS OF THE TRIANGLE INEQUALITY FOR NORMED LINEAR SPACES

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Abstract. Some interpolations are established for Jensen's inequality for a convex function of a weighted sum of n points. These are used to derive refinements of the triangle inequality and the Cauchy-Schwarz inequality for a normed linear space.

1. Introduction

Let $(X, \|\cdot\|)$ be a normed linear space. It is well-known that for $x, y \in X$ we have a triangle inequality

$$\| x + y \| \leq \| x \| + \| y \|,$$

and for n points $x_1, \ldots, x_n \in X$

$$\| x_1 + \cdots + x_n \| \leq \| x_1 \| + \cdots + \| x_n \| .$$

In the case of weighting with positive numbers p_i $(i = 1, \ldots, n)$, this inequality becomes

$$\left\| \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right\| \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i \| x_i \|, \tag{1.1}$$

where $P_n = \sum_{i=1}^{n} p_i$.

A simple consequence of Jensen's inequality for the convex map $x \to | x |^p$ $(p \geq 1)$ is that

$$\left\| \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right\|^p \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i \| x_i \|^p . \tag{1.2}$$

The term p_i/P_n may be interpreted as the probability that an X-valued random variable Y takes the value x_i. With this interpretation, (1.2) reads

$$[E(\| Y \|)]^p \leq E(\| Y \|^p) \text{ for } p \geq 1,$$

which provides an extension of a familiar result for real-valued random variables.

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Jensen's inequality involving a weighted or unweighted sum of n points may be interpolated using partial sums of the quantities involved (see [1-5], [7]). In terms of the foregoing probabilistic interpretation, the partial sums in the weighted case may be regarded as arising from a sampling experiment without replacement.

In fact, corresponding results exist for sampling with replacement. This is addressed in the following section, where such refinements are made in three different ways. Section 2 concludes by melding three results together to give an interpolation of the triangle inequality for a normed linear space. The left-hand side of (1.2) is given a sequence of successively larger majorants ending with the right-hand side of (1.2).

The remainder of the paper explores this extended inequality in detail for the basic case $p = 1$, putting bounds in turn on the differences between successive majorants. Alternative forms are available for each in the special case of an inner-product space and give rise to refinements of the Cauchy-Schwarz inequality. Some preliminaries for use in this case are presented in Section 3.

The differences involved in Sections 6 and 8 involve a second, independent, weighting on the points x_i. This is of particular interest in that it offers the flexibility for concrete physical applications, especially if probabilistic interpretations are utilized. The results are redolent of the fundamental inequality of information theory, namely, that if $(p_i)_1^M, (q_i)_1^M$ are probability distributions with strictly positive elements, then

$$-\sum_{i=1}^{M} p_i \log p_i \leq -\sum_{i=1}^{M} p_i \log q_i,$$

with equality if $p_i \equiv q_i$ for all i.

With one exception, each section after the third addresses the difference between two consecutive majorants. The exception, Section 7, is an additional section that considers two majorants which are separated by a third. The separator involves the second weighting alluded to whereas the other two terms do not, so rather simpler arguments are available by skipping over it.

2. Some interpolations of Jensen's inequality

Theorem 2.1. *Let k be a positive integer and $f : C \subseteq X \to \mathbf{R}$ a convex (respectively concave) function on the convex set C. Suppose $x_i \in C$, $p_i \geq 0$ $(i = 1, \ldots, n)$ and*

$P_n := \sum_{i=1}^n p_i > 0$. *Then*

$$
\begin{aligned}
f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) &\le (\ge)\frac{1}{P_n^{k+1}}\sum_{i_1,\ldots,i_{k+1}} p_{i_1}\cdots p_{i_{k+1}} f\left(\frac{1}{k+1}\sum_{j=1}^{k+1} x_{i_j}\right) \\
&\le (\ge)\frac{1}{P_n^k}\sum_{i_1,\ldots,i_k} p_{i_1}\cdots p_{i_k} f\left(\frac{1}{k}\sum_{j=1}^{k} x_{i_j}\right) \\
&\le (\ge)\cdots \\
&\le (\ge)\frac{1}{P_n^2}\sum_{i_1,i_2} p_{i_1}p_{i_2} f\left(\frac{x_{i_1}+x_{i_2}}{2}\right) \\
&\le (\ge)\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i),
\end{aligned}
$$

where $\sum_{i_1,\ldots,i_\ell}$ denotes an enumeration over all possible ordered ℓ-tuples $(i_1,\ldots,i_\ell) \in \{1,2,\ldots,n\}^\ell$. This is a slightly extended version (with replacements) of a result of Pečarić and Dragomir [9], where, instead of an infinite series of inequalities, the case $1 \le k \le n$-1 only was possible. Its proof follows that given in [9].

A related result holds in which the argument of the sandwiched function f attributes a general convex weighting to the points x_{i_j}.

Theorem 2.2. *Under the conditions of Theorem 2.1, let $q_j \ge 0$ $(1 \le j \le k)$ be such that* $Q_k := \sum_{j=1}^k q_j > 0$. *Then*

$$
f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \le (\ge)\frac{1}{P_n^k}\sum_{i_1,\ldots,i_k} p_{i_1}\cdots p_{i_k} f\left(\frac{1}{Q_k}\sum_{j=1}^k q_j x_{i_j}\right) \le (\ge)\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i).
$$

This is a similar extension allowing replacements of a result proved in [2] and [8] (see also [3-4]). The following result extends likewise a result from [5] which also compares uniform and non-uniform weightings.

Theorem 2.3. *Let the assumptions of Theorem 2.2 be satisfied. Then*

$$
\frac{1}{P_n^k}\sum_{i_1,\ldots,i_k} p_{i_1}\cdots p_{i_k} f\left(\frac{1}{k}\sum_{j=1}^k x_{i_j}\right) \le (\ge)\frac{1}{P_n^k}\sum_{i_1,\ldots,i_k} p_{i_1}\cdots p_{i_k} f\left(\frac{1}{Q_k}\sum_{j=1}^k q_j x_{i_j}\right).
$$

The three theorems above may be pieced together to yield the following refinements of the triangle inequality.

Theorem 2.4. *Let $(X, \|\cdot\|)$ be a normed linear space and $f: X \to R_+$, with $f(x) = \|x\|^p$ and $p \geq 1$. Then*

$$\begin{aligned}
\left\| \frac{1}{P_I} \sum_{i \in I} p_i x_i \right\|^p &\leq \frac{1}{P_I^{k+1}} \sum_{i_1, \ldots, i_{k+1} \in I} p_{i_1} \cdots p_{i_{k+1}} \left\| \frac{1}{k+1} \sum_{j=1}^{k+1} x_{i_j} \right\|^p \\
&\leq \frac{1}{P_I^k} \sum_{i_1, \ldots, i_k \in I} p_{i_1} \cdots p_{i_k} \left\| \frac{1}{k} \sum_{j=1}^{k} x_{i_j} \right\|^p \\
&\leq \frac{1}{P_I^k} \sum_{i_1, \ldots, i_k \in I} p_{i_1} \cdots p_{i_k} \left\| \frac{1}{Q_k} \sum_{j=1}^{k} q_j x_{i_j} \right\|^p \\
&\leq \frac{1}{P_i} \sum_{i \in I} p_i \| x_i \|^p, \qquad (2.1)
\end{aligned}$$

where $x_i \in X$, $p_i \geq 0$, $q_j \geq 0$ and $P_I > 0$, $Q_k > 0$ and I is a finite set of indices.

These results give an interpolation of results obtained in [1,7,10]. Of course, direct application of Theorems 2.1 - 2.3 to the function $f(x) = \|x\|^p$, $p \geq 1$, give extensions of results obtained in [9], [2] and [4], respectively.

3. Preliminaries on inner-product spaces

The following result will be found useful in the subsequent sections.

Proposition 3.1. *Let $(H;<,>)$ be an inner-product space and $x_i \in H$ for all $i \in I$. Further, let $p_i \geq 0$ $(i \in I)$ with $P_I > 0$ and $r_{k,j} \geq 0$ $(1 \leq j \leq k)$ with $R_k := \sum_{j=1}^{k} r_{k,j} = 1$. Then*

$$\frac{1}{P_I^k} \sum_{i_1, \ldots, i_k \in I} p_{i_1} \cdots p_{i_k} \left\| \sum_{j=1}^{k} r_{k,j} x_{i_j} \right\|^2 =$$

$$= \sum_{j=1}^{k} r_{k,j}^2 \frac{1}{P_I} \sum_{i \in I} p_i \| x_i \|^2 + \left(1 - \sum_{j=1}^{k} r_{k,j}^2\right) \left\| \frac{1}{P_I} \sum_{i \in I} p_i x_i \right\|^2 \qquad (3.1)$$

and for α a real number

$$\frac{1}{P_I^{k+1}} \sum_{i_1, \ldots, i_{k+1} \in I} p_{i_1} \cdots p_{i_{k+1}} \left\| \sum_{j=1}^{k} r_{k,j} x_{i_j} + \alpha x_{i_{k+1}} \right\|^2 =$$

$$= \left(\sum_{j=1}^{k} r_{k,j}^2 + \alpha^2 \right) \frac{1}{P_I} \sum_{i \in I} p_i \| x_i \|^2 + \left(1 - \sum_{j=1}^{k} r_{k,j}^2 + 2\alpha \right) \left\| \frac{1}{P_I} \sum_{i \in I} p_i x_i \right\|^2 \qquad (3.2)$$

Proof. A simple calculation shows that

$$\frac{1}{P_I^k}\sum_{i_1,\ldots,i_k\in I} p_{i_1}\cdots p_{i_k}\sum_{j=1}^{k} r_{k,j}x_{i_j} = \frac{1}{P_I}\sum_{i\in I} p_i x_i,$$

so that the left-hand side of (3.2) may be expanded as

$$\frac{1}{P_I^{k+1}}\sum_{i_1,\ldots,i_{k+1}\in I} p_{i_1}\cdots p_{i_{k+1}}[\| \sum_{j=1}^{k} r_{k,j}x_{i_j} \|^2 + 2\Re\left\langle \sum_{j=1} r_{k,j}x_{i_j}, \alpha x_{i_{k+1}}\right\rangle + \alpha^2 \| x_{i_{k+1}} \|^2]$$

$$= \frac{1}{P_I^k}\sum_{i_1,\ldots,i_k\in I} p_{i_1}\cdots p_{i_k} \| \sum_{j=1}^{k} r_{k,j}x_{i_j} \|^2 + 2\alpha\Re\left\langle \frac{1}{P_I}\sum_{i\in I} p_i x_i, \frac{1}{P_I}\sum_{i\in I} p_i x_i\right\rangle +$$

$$+\frac{\alpha^2}{P_I}\sum_{i\in I} p_i \| x_i \|^2 + \frac{1}{P_I^k}\sum_{i_1,\ldots,i_k\in I} p_{i_1}\cdots p_{i_k} \| \sum_{j=1}^{k} r_{k,j}x_{i_j} \|^2 +$$

$$+2\alpha \| \frac{1}{P_I}\sum_{i\in I} p_i x_i \|^2 + \frac{\alpha^2}{P_I}\sum_{i\in I} p_i \| x_i \|^2 .$$

Hence, for a given value of k, (3.2) follows immediately if (3.1) holds and it suffices to establish the latter. This we may do inductively.

Equation (3.1) is immediate for $k = 1$ (when $r_{1,1} = 1$), providing a basis for induction. For the inductive step, suppose the result is true for all choices of $r_{k,j}$ for some $k \geq 1$, so that (3.2) also holds for that value of k. For $r_{k+1,j}$ given and satisfying $R_{k+1} = 1$, choose $r_{k,j} = r_{k+1,j}/\sum_{\ell=1}^{k} r_{k+1,\ell}$ $(1 \leq j \leq k)$ and $\alpha = r_{k+1,k+1}/\sum_{\ell=1}^{k} r_{k+1,\ell}$. Then we have

$$\frac{1}{P_I^{k+1}}\sum_{i_1,\ldots,i_{k+1}\in I} p_{i_1}\cdots p_{i_{k+1}} \| \sum_{j=1}^{k+1} r_{k+1,j}x_{i_j} \|^2 =$$

$$= \frac{1}{P_I^{k+1}}(\sum_{\ell=1}^{k} r_{k+1,\ell})^2 \sum_{i_1,\ldots,i_{k+1}\in I} p_{i_1}\cdots p_{i_{k+1}} \| \sum_{j=1}^{k} r_{k,j}x_{i_j} + \alpha x_{i_{k+1}} \|^2 =$$

$$= (\sum_{\ell=1}^{k} r_{k+1,\ell})^2[(\sum_{j=1}^{k} r_{k,j}^2 + \alpha^2)\frac{1}{P_I}\sum_{i\in I} p_i \| x_i \|^2 + (1 - \sum_{j=1}^{k} r_{k,j}^2 + 2\alpha) \| \frac{1}{P_I}\sum_{i\in I} p_i x_i \|^2].$$

On substitution for $r_{k,j}$ and α, the last expression simplifies to

$$\sum_{j=1}^{k+1} r_{k+1,j}^2 \frac{1}{P_I}\sum_{i\in I} p_i \| x_i \|^2 + \left(1 - \sum_{j=1}^{k+1} r_{k+1,j}^2\right)\left\|\frac{1}{P_I}\sum_{i\in I} p_i x_i\right\|^2 ,$$

that is, (3.1) with k replaced by $k+1$, so we are done. □

Remark 3.2. We may define a sequence $Y_1, \ldots, Y_{k+1}$ of X-valued independently and identically distributed random variables, each taking the value x_i with probability p_i/P_I $(i \in I)$. Then

$$E(Y_j) = \frac{1}{P_I}\sum_{i\in I} p_i x_i \quad (1 \le j \le k+1)$$

and

$$E(\| Y_j \|^2) = \frac{1}{P_I}\sum_{i\in I} p_i \| x_i \|^2 \quad (1 \le j \le k+1).$$

By analogy with the usual case of scalar-valued random variables, it is natural to define the variance $V(Y_i)$ of Y_j by

$$V(Y_j) := E(\| Y_j \|^2) - \| E(Y_j) \|^2 .$$

Then (3.2) reads

$$V\left(\sum_{j=1}^{k} r_{k,j} Y_j + \alpha Y_{k+1}\right) = \left(\sum_{j=1}^{k} r_{k,j}^2 + \alpha^2\right) V(Y_1),$$

which is a standard result in the scalar case.

Corollary 3.3. *Under the suppositions of Proposition 3.1*

$$\frac{1}{P_I^k}\sum_{i_1,\ldots,i_k\in I} p_{i_1}\cdots p_{i_k}\left\|\sum_{j=1}^{k} r_{k,j}x_{i_j} - \frac{1}{P_I}\sum_{i\in I} p_i x_i\right\|^2 =$$
$$= \sum_{j=1}^{k} r_{k,j}^2 \left[\frac{1}{P_I}\sum_{i\in I} p_i \| x_i \|^2 - \left\|\frac{1}{P_I}\sum_{i\in I} p_i x_i\right\|^2\right].$$

Proof. The left-hand side can be expanded as

$$\frac{1}{P_I^k}\sum_{i_1,\ldots,i_k\in I} p_{i_1}\cdots p_{i_k}\left\|\sum_{j=1}^{k} r_{k,j}x_{i_j}\right\|^2 -$$
$$-2\Re\left\langle \frac{1}{P_I^k}\sum_{i_1,\ldots,i_k\in I} p_{i_1}\cdots p_{i_k}\sum_{j=1}^{k} r_{k,j}x_{i_j}, \frac{1}{P_I}\sum_{i\in I} p_i x_i\right\rangle + \left\|\frac{1}{P_I}\sum_{i\in I} p_i x_i\right\|^2$$
$$= \frac{1}{P_I^k}\sum_{i_1,\ldots,i_k\in I} p_{i_1}\cdots p_{i_k}\left\|\sum_{j=1}^{k} r_{k,j}x_{i_j}\right\|^2 - \left\|\frac{1}{P_I}\sum_{i\in I} p_i x_i\right\|^2,$$

and the desired result follows from (3.1). □

4. First majorant

The extreme terms in (2.1) coincide if (and only if) the points x_i $(i \in I)$ are all equal. We now show that the difference between the members of the first inequality in Theorem 2.4 is bounded above by

$$v(x) := \max_{1 \le i \le j \le n} \{\| x_i - x_j \|\}.$$

This result is further interpolated.

Theorem 4.1. *Let $(X, \|\cdot\|)$ be a normed linear space, $x_i \in X$, $p_i > 0$ for all $i \in I$ with $P_I > 0$. Then*

$$\begin{aligned}
0 &\le \frac{1}{P_I^{k+1}} \sum_{i_1,\dots,i_{k+1} \in I} p_{i_1} \cdots p_{i_{k+1}} \left\| \frac{1}{k+1} \sum_{j=1}^{k+1} x_{i_j} \right\| - \left\| \frac{1}{P_I} \sum_{i \in I} p_i x_i \right\| \\
&\le \frac{1}{P_I^{k+1}} \sum_{i_1,\dots,i_{k+1} \in I} p_{i_1} \cdots p_{i_{k+1}} \left\| \frac{1}{k+1} \sum_{j=1}^{k+1} x_{i_j} - \frac{1}{P_I} \sum_{i \in I} p_i x_i \right\| \\
&\le \frac{1}{P_I^{k}} \sum_{i_1,\dots,i_{k} \in I} p_{i_1} \cdots p_{i_{k}} \left\| \frac{1}{k} \sum_{j=1}^{k} x_{i_j} - \frac{1}{P_I} \sum_{i \in I} p_i x_i \right\| \\
&\le \frac{1}{P_I} \sum_{j \in I} p_j \left\| x_j - \frac{1}{P_I} \sum_{i \in I} p_i x_i \right\| \le \frac{1}{P_n^2} \sum_{i,j \in I} p_i p_j \, \| x_i - x_j \| \le \nu(x),
\end{aligned} \tag{4.1}$$

for each $k \in \mathbf{N}$.

Proof. The first inequality in (4.1) restates the beginning of Theorem 2.4 for the case 1. Now by the triangle inequality

$$\left\| \frac{1}{k+1} \sum_{j=1}^{k+1} x_{i_j} \right\| - \frac{1}{P_I} \left\| \sum_{i \in I} p_i x_i \right\| \le \left\| \frac{1}{k+1} \sum_{j=1}^{k+1} x_{i_j} - \frac{1}{P_I} \sum_{i \in I} p_i x_i \right\|,$$

so on multiplication by $p_{i_1} \dots p_{i_{k+1}} (\ge 0)$ and summation over $i_1, \dots, i_{k+1}$ we get

$$\begin{aligned}
&\frac{1}{P_I^{k+1}} \sum_{i_1,\dots,i_{k+1} \in I} p_{i_1} \cdots p_{i_{k+1}} \left\| \frac{1}{k+1} \sum_{j=1}^{k+1} x_{i_j} - \frac{1}{P_I} \sum_{i \in I} p_i x_i \right\| \\
&\ge \frac{1}{P_I^{k+1}} \sum_{i_1,\dots,i_{k+1} \in I} p_{i_1} \cdots p_{i_{k+1}} \left(\left\| \frac{1}{k+1} \sum_{j=1}^{k+1} x_{i_j} \right\| - \left\| \frac{1}{P_I} \sum_{i \in I} p_i x_i \right\| \right) \\
&= \frac{1}{P_I^{k+1}} \sum_{i_1,\dots,i_{k+1} \in I} p_{i_1} \cdots p_{i_{k+1}} \left\| \frac{1}{k+1} \sum_{j=1}^{k+1} x_{i_j} \right\| - \left\| \frac{1}{P_I} \sum_{i \in I} p_i x_i \right\|,
\end{aligned}$$

which provides the second inequality.

From inequality (2.1) with $p = 1$, we have

$$\frac{1}{P_I^{k+1}} \sum_{i_1,\ldots,i_{k+1}\in I} p_{i_1}\cdots p_{i_{k+1}} \left\| \frac{1}{k+1}\sum_{j=1}^{k+1} y_{i_j} \right\| \le \frac{1}{P_I^k} \sum_{i_1,\ldots,i_k\in I} p_{i_1}\cdots p_{i_k} \left\| \frac{1}{k}\sum_{j=1}^{k} y_{i_j} \right\|$$

for all $y_i \in X$ and $i \in I$. Choosing $y_{i_j} := x_{i_j} - \frac{1}{P_I}\sum_{i\in I} p_i x_i$ and observing that

$$\frac{1}{k+1}\sum_{j=1}^{k+1} y_{i_j} = \frac{1}{k+1}\sum_{j=1}^{k+1} x_{i_j} - \frac{1}{P_I}\sum_{i\in I} p_i x_i$$

and

$$\frac{1}{k}\sum_{j=1}^{k} y_{i_j} = \frac{1}{k}\sum_{j=1}^{k} x_{i_j} - \frac{1}{P_I}\sum_{i\in I} p_i x_i,$$

we obtain the next inequalities in (4.1). The subsequent inequality may be derived recursively.

Finally,

$$\frac{1}{P_I}\sum_{j\in I} p_j \left\| x_j - \frac{1}{P_I}\sum_{i\in I} p_i x_i \right\| = \frac{1}{P_I}\sum_{j\in I} p_j \left\| \frac{1}{P_I}\sum_{i\in I} p_i (x_j - x_i) \right\|$$
$$\le \frac{1}{P_I^2}\sum_{i,j\in I} p_i p_j \parallel x_i - x_j \parallel \le \nu(x)\frac{1}{P_I^2}\sum_{i,j\in I} p_i p_j = \nu(x)$$

and the theorem is proved. □

For inner-product spaces, an alternative convenient version is available. This case is embodied in the following corollary.

Corollary 4.2. *Let $(H;<,>)$ be an inner-product space and $x_i \in H$, $p_i \ge 0$ for all $i \in I$ and $P_I > 0$. Then*

$$0 \le \frac{1}{P_I^{k+1}} \sum_{i_1,\ldots,i_{k+1}\in I} p_{i_1}\cdots p_{i_{k+1}} \left\| \frac{1}{k+1}\sum_{j=1}^{k+1} x_{i_j} \right\| - \left\| \frac{1}{P_I}\sum_{i\in I} p_i x_i \right\| \le$$
$$\le \left[\frac{1}{k+1} \left(\frac{1}{P_I}\sum_{I\in I} p_i \parallel x_i \parallel^2 - \left\| \frac{1}{P_I}\sum_{i\in I} p_i x_i \right\|^2 \right) \right]^{1/2} \tag{4.2}$$

Proof. The first inequality is that of Theorem (4.1). By the Cauchy-Buneakowski-Schwarz inequality for multiple sums, the right-hand member of the second inequality in (4.1) is

$$\frac{1}{P_I^{k+1}} \sum_{i_1,\ldots,i_{k+1}\in I} p_{i_1}\cdots p_{i_{k+1}} \left\| \frac{1}{k+1}\sum_{j=1}^{k+1} x_{i_j} - \frac{1}{P_I}\sum_{i\in I} p_i x_i \right\| \leq$$
$$\leq \left(\frac{1}{P_I^{k+1}} \sum_{i_1,\ldots,i_{k+1}\in I} p_{i_1}\cdots p_{i_{k+1}} \left\| \frac{1}{k+1}\sum_{j=1}^{k+1} x_{i_j} - \frac{1}{P_I}\sum_{i\in I} p_i x_i \right\|^2 \right)^{1/2}.$$

The result is now immediate from Corollary 3.3 with $k+1$ in place of k and $r_{k+1,j} = 1/(k+1)$. □

The above inequality can also be regarded as a refinement

$$\frac{1}{P_I}\sum_{i\in I} p_i \| x_i \|^2 - \left\| \frac{1}{P_I}\sum_{i\in I} p_i x_i \right\|^2 \geq$$
$$\geq (k+1) \left(\frac{1}{P_I^{k+1}} \sum_{i_1,\ldots,i_{k+1}\in I} p_{i_1}\cdots p_{i_{k+1}} \left\| \frac{1}{k+1}\sum_{j=1}^{k+1} x_{i_j} \right\| - \left\| \frac{1}{P_I}\sum_{i\in I} p_i x_i \right\| \right)^2 \geq 0$$

of the Cauchy-Schwarz inequality.

5. First and second majorants

Theorem 5.1. *Let $(X, \|\cdot\|)$ be a normed linear space, $x_i \in X$, $p_i \geq 0$ t for all $i \in I$ with $P_I > 0$. Then for each $k \in \mathbb{N}$*

$$\begin{aligned}
0 &\leq \frac{1}{P_I^k} \sum_{i_1,\ldots,i_k\in I} p_{i_1}\cdots p_{i_k} \left\| \frac{1}{k}\sum_{j=1}^{k} x_{i_j} \right\| - \frac{1}{P_I^{k+1}} \sum_{i_1,\ldots,i_{k+1}\in I} p_{i_1}\cdots p_{i_{k+1}} \left\| \frac{1}{k+1}\sum_{j=1}^{k+1} x_{i_j} \right\| \\
&\leq \frac{1}{k+1}\cdot\frac{1}{P_I^{k+1}} \sum_{i_1,\ldots,i_{k+1}\in I} p_{i_1}\cdots p_{i_{k+1}} \left\| \frac{x_{i_1}+\cdots+x_{i_k}}{k} - x_{i_{k+1}} \right\| \\
&\leq \frac{1}{k+1}\cdot\frac{1}{P_I^{k}} \sum_{i_1,\ldots,i_{k}\in I} p_{i_1}\cdots p_{i_{k}} \left\| \frac{x_{i_1}+\cdots+x_{i_{k-1}}}{k-1} - x_{i_{k}} \right\| \\
&\leq \cdots \\
&\leq \frac{1}{k+1}\cdot\frac{1}{P_I^2}\sum_{i,j\in I} p_i p_j \| x_i - x_j \| \leq \frac{1}{k+1}\nu(x).
\end{aligned} \tag{5.1}$$

Proof. The first inequality in (5.1) is obvious. For the second, observe that by the triangle inequality

$$\left\|\frac{1}{k}\sum_{j=1}^{k}x_{i_j}\right\| - \left\|\frac{1}{k+1}\sum_{j=1}^{k+1}x_{i_j}\right\| \le \left\|\frac{1}{k}\sum_{j=1}^{k}x_{i_j} - \frac{1}{k+1}\sum_{j=1}^{k+1}x_{i_j}\right\| =$$
$$= \frac{1}{k+1}\left\|\frac{x_{i_1}+\cdots+x_{i_k}}{k} - x_{i_{k+1}}\right\|$$

for all $x_{i_j} \in X$, $\quad j = 1,\ldots,k+1$. On multiplying by $p_{i_1}\ldots p_{i_{k+1}} (\ge 0)$ and summing over $i_1,\ldots,i_{k_1}$ in I, we get

$$\frac{1}{k+1}\cdot\frac{1}{P_I^{k+1}}\sum_{i_1,\ldots,i_{k+1}\in I}p_{i_1}\cdots p_{i_{k+1}}\left\|\frac{x_{i_1}+\cdots+x_{i_k}}{k} - x_{i_{k+1}}\right\| \ge$$
$$\ge \frac{1}{P_I^{k+1}}\sum_{i_1,\ldots,i_{k+1}\in I}p_{i_1}\cdots p_{i_{k+1}}\left(\left\|\frac{1}{k}\sum_{j=1}^{k}x_{i_j}\right\| - \left\|\frac{1}{k+1}\sum_{j=1}^{k+1}x_{i_j}\right\|\right) =$$
$$= \frac{1}{P_I^{k}}\sum_{i_1,\ldots,i_k\in I}p_{i_1}\cdots p_{i_k}\left\|\frac{1}{k}\sum_{j=1}^{k}x_{i_j}\right\| - \frac{1}{P_I^{k+1}}\sum_{i_1,\ldots,i_{k+1}\in I}p_{i_1}\cdots p_{i_{k+1}}\left\|\frac{1}{k+1}\sum_{j=1}^{k+1}x_{i_j}\right\|,$$

and the second inequality in (5.1) is proved.

This gives

$$\frac{1}{P_I^k}\sum_{i_1,\ldots,i_k\in I}p_{i_1}\cdots p_{i_k}\left\|\frac{y_{i_1}+\cdots+y_{i_k}}{k}\right\| \le \frac{1}{P_I^{k-1}}\sum_{i_1,\ldots,i_{k-1}\in I}p_{i_1}\cdots p_{i_{k-1}}\left\|\frac{y_{i_1}+\cdots+y_{i_{k-1}}}{k-1}\right\|,$$

where $y_{i_j} \in X, 1 \le j \le k$ and $k \ge 2$. By choosing

$$y_{i_1} = x_{i_1} - x_{i_{k+1}},\ldots,y_{i_k} = x_{i_k} - x_{i_{k+1}}$$

we get

$$\frac{1}{P_I^k}\sum_{i_1,\ldots,i_k\in I}p_{i_1}\cdots p_{i_k}\left\|\frac{x_{i_1}+\cdots+x_{i_k}}{k} - x_{i_{k+1}}\right\| \le$$
$$\le \frac{1}{P_I^{k-1}}\sum_{i_1,\ldots,i_{k-1}\in I}p_{i_1}\cdots p_{i_{k-1}}\left\|\frac{x_{i_1}+\cdots+x_{i_{k-1}}}{k-1} - x_{i_{k+1}}\right\|$$

and thus

$$\frac{1}{P_I^{k+1}}\sum_{i_1,\dots,i_{k+1}\in I} p_{i_1}\cdots p_{i_{k+1}}\left\|\frac{x_{i_1}+\cdots+x_{i_k}}{k}-x_{i_{k+1}}\right\| \le$$

$$\le \frac{1}{P_I^k}\sum_{i_{k+1}\in I} p_{i_{k+1}}\sum_{i_1,\dots,i_{k-1}\in I} p_{i_1}\cdots p_{i_{k-1}}\left\|\frac{x_{i_1}+\cdots+x_{i_{k-1}}}{k-1}-x_{i_{k+1}}\right\| =$$

$$= \frac{1}{P_I^k}\sum_{i_1,\dots,i_k\in I} p_{i_1}\cdots p_{i_k}\left\|\frac{x_{i_1}+\cdots+x_{i_{k-1}}}{k-1}-x_{i_k}\right\|$$

and the third inequality in (5.1) is proved. The last inequalities follow by recursion and the end of Theorem 4.1. □

Corollary 5.2. *Let $(H;<,>)$ be an inner product space and $x_i \in H$, $p_i \ge 0$ for all $i \in I$ and $P_I > 0$. Then*

$$\begin{aligned} 0 &\le \frac{1}{P_I^k}\sum_{i_1,\dots,i_k\in I} p_{i_1}\cdots p_{i_k}\left\|\frac{1}{k}\sum_{j=1}^{k} x_{i_j}\right\| - \frac{1}{P_I^{k+1}}\sum_{i_1,\dots,i_{k+1}\in I} p_{i_1}\cdots p_{i_{k+1}}\left\|\frac{1}{k+1}\sum_{j=1}^{k+1} x_{i_j}\right\| \le \\ &\le \frac{1}{k+1}\left(\frac{1}{P_I^{k+1}}\sum_{i_1,\dots,i_{k+1}\in I} p_{i_1}\cdots p_{i_{k+1}}\left\|\frac{x_{i_1}+\cdots+x_{i_k}}{k}-x_{i_{k+1}}\right\|^2\right)^{1/2} = \qquad (5.2)\\ &= \left[\frac{1}{k(k+1)}\left(\frac{1}{P_I}\sum_{i\in I} p_i \| x_i \|^2 - \left\|\frac{1}{P_I}\sum_{i\in I} p_i x_i\right\|^2\right)\right]^{1/2}. \end{aligned}$$

Proof. The first inequality is the leading inequality of Theorem 5.1. By the Cauchy-Buneakowski-Schwarz inequality for multiple sums,

$$\frac{1}{P_I^{k+1}}\sum_{i_1,\dots,i_{k+1}\in I} p_{i_1}\cdots p_{i_{k+1}}\left\|\frac{x_{i_1}+\cdots+x_{i_k}}{k}-x_{i_{k+1}}\right\| \le$$

$$\le \left(\frac{1}{P_I^{k+1}}\sum_{i_1,\dots,i_{k+1}\in I} p_{i_1}\cdots p_{i_{k+1}}\left\|\frac{x_{i_1}+\cdots+x_{i_k}}{k}-x_{i_{k+1}}\right\|^2\right)^{1/2},$$

so that the second inequality also follows from that of Theorem 5.1. Equation (3.2) with $r_{k,j} = 1/k$ and $\alpha = -1$ yields the final statement. □

Remark 5.3. The above inequality also can be regarded as a refinement

$$\frac{1}{P_I}\sum_{i\in I} p_i \| x_i \|^2 - \left\|\frac{1}{P_I}\sum_{i\in I} p_i x_i\right\|^2 \ge k(k+1)\left(\frac{1}{P_I^k}\sum_{i_1,\dots,i_k\in I} p_{i_1}\cdots p_{i_k}\left\|\frac{1}{k}\sum_{j=1}^{k} x_{i_j}\right\| - \right.$$

$$\left. - \frac{1}{P_I^{k+1}}\sum_{i_1,\dots,i_{k+1}\in I} p_{i_1}\cdots p_{i_{k+1}}\left\|\frac{1}{k+1}\sum_{j=1}^{k+1} x_{i_j}\right\|\right)^2$$

of the Cauchy-Schwarz inequality.

6. Second and third majorants

Theorem 6.1. *Let* $(X, \|,\|)$ *be a normed linear space,* $x_i \in X$, $p_i \geq 0$ *for all* $i \in I$ *with* $P_I > 0$, $k \geq 2$ *and* $q_i \geq 0$ *with* $Q_k > 0$. *Put*

$$q_1 = \max_{1 \leq j \leq k} \left\{ \left| \frac{kq_j}{Q_k} - 1 \right| \right\}, \quad q_2 = \max_{1 \leq j \leq k} \left\{ \frac{1}{Q_k} \sum_{\ell=1}^{k} | q_i - q_\ell | \right\}$$

and let $q = \min\{q_1, q_2\}$. *Then*

$$\begin{aligned} 0 &\leq \frac{1}{P_I^k} \sum_{i_1,\ldots,i_k \in I} p_{i_1} \cdots p_{i_k} \left\| \frac{1}{Q_k} \sum_{j=1}^{k} q_j x_{i_j} \right\| - \left\| \frac{1}{k} \sum_{j=1}^{k} x_{i_j} \right\| \leq \\ &\leq \frac{1}{P_I^k} \sum_{i_1,\ldots,i_k \in I} p_{i_1} \cdots p_{i_k} \left\| \frac{1}{Q_k} \sum_{j=1}^{k} q_j x_{i_j} - \frac{1}{k} \sum_{j=1}^{k} x_{i_j} \right\| \leq \\ &\leq \frac{q}{P_I} \sum_{i \in I} p_i \| x_i \| . \end{aligned}$$

Proof. The first inequality is immediate from Theorem 2.4. By the triangle inequality

$$\left\| \frac{1}{Q_k} \sum_{j=1}^{k} q_j x_{i_j} \right\| - \left\| \frac{1}{k} \sum_{j=1}^{k} x_{i_j} \right\| \leq \left\| \frac{1}{Q_k} \sum_{j=1}^{k} q_j x_{i_j} - \frac{1}{k} \sum_{j=1}^{k} x_{i_j} \right\| = \frac{1}{kQ_k} \left\| \sum_{\ell=1}^{k} \sum_{j=1}^{k} (q_j - q_\ell) x_{i_j} \right\| .$$

Note that

$$\frac{1}{kQ_k} \left\| \sum_{\ell=1}^{k} \sum_{j=1}^{k} (q_j - q_\ell) x_{i_j} \right\| \leq \frac{1}{k} \sum_{j=1}^{k} \left\| \frac{(kq_j - Q_k)}{Q_k} x_{i_j} \right\| \leq_1 \frac{1}{k} \sum_{j=1}^{k} \| x_{i_j} \|$$

and

$$\frac{1}{kQ_k} \left\| \sum_{\ell=1}^{k} \sum_{j=1}^{k} (q_j - q_\ell) x_{i_j} \right\| \leq \frac{1}{kQ_k} \sum_{j=1}^{k} \sum_{\ell=1}^{k} | q_j - q_\ell | \| x_{i_j} \| \leq_2 \frac{1}{k} \sum_{j=1}^{k} \| x_{i_j} \| .$$

Thus

$$\left\| \frac{1}{Q_k} \sum_{j=1}^{k} q_j x_{i_j} \right\| - \left\| \frac{1}{k} \sum_{j=1}^{k} x_{i_j} \right\| \leq \left\| \frac{1}{Q_k} \sum_{j=1}^{k} q_j x_{i_j} - \frac{1}{k} \sum_{j=1}^{k} x_{i_j} \right\| \leq \frac{q}{k} \sum_{j=1}^{k} \| x_{i_j} \|$$

and so

$$\begin{aligned}
\frac{q}{P_I}\sum_{i\in I} p_i \| x_i \| &= \frac{q}{P_I^k}\sum_{i_1,\dots,i_k\in I} p_{i_1}\cdots p_{i_k}\left(\frac{\| x_{i_1} \| + \cdots + \| x_{i_k} \|}{k}\right) \geq \\
&\geq \frac{1}{P_I^k}\sum_{i_1,\dots,i_k\in I} p_{i_1}\cdots p_{i_k}\left\|\frac{1}{Q_k}\sum_{j=1}^{k} q_j x_{i_j} - \frac{1}{k}\sum_{j=1}^{k} x_{i_j}\right\| \geq \\
&\geq \frac{1}{P_I^k}\sum_{i_1,\dots,i_k\in I} p_{i_1}\cdots p_{i_k}\left(\left\|\frac{1}{Q_k}\sum_{j=1}^{k} q_j x_{i_j}\right\| - \left\|\frac{1}{k}\sum_{j=1}^{k} x_{i_j}\right\|\right)
\end{aligned}$$

and the theorem is proved. □

7. Second and fourth majorants

Theorem 7.1. *Under the assumptions of Theorem 2 we have*

$$\begin{aligned}
0 &\leq \frac{1}{P_I}\sum_{i\in I} p_i \| x_i \| - \frac{1}{P_I^k}\sum_{i_1,\dots,i_k\in I} p_{i_1}\cdots p_{i_k}\left\|\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right\| \leq \\
&\leq \frac{1}{P_I^{k+1}}\sum_{i_1,\dots,i_{k+1}\in I} p_{i_1}\cdots p_{i_{k+1}}\left\|\frac{x_{i_1} + \cdots + x_{i_k}}{k} - x_{i_{k+1}}\right\| \leq \\
&\leq \frac{1}{P_I^k}\sum_{i_1,\dots,i_k\in I} p_{i_1}\cdots p_{i_k}\left\|\frac{x_{i_1} + \cdots + x_{i_{k-1}}}{k-1} - x_{i_k}\right\| \leq \\
&\leq \cdots \\
&\leq \frac{1}{P_I^2}\sum_{i,j\in I} p_i p_j \| x_i - x_j \| \leq \\
&\leq \nu(x).
\end{aligned} \tag{7.1}$$

Proof. The first inequality follows from Theorem 2.4. From the third inequality on, the result has been given in Theorem 5.1. Also

$$\| x_{i_{k+1}} \| - \left\|\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right\| \leq \left\| x_{i_{k+1}} - \frac{x_{i_1} + \cdots + x_{i_k}}{k}\right\|.$$

If we multiply by $p_{i_1}\dots p_{i_{k+1}}$ and sum over $i_1,\dots,i_{k+1}$ in I we get

$$\begin{aligned}
&\frac{1}{P_I^{k+1}}\sum_{i_1,\dots,i_{k+1}\in I} p_{i_1}\cdots p_{i_{k+1}}\left\|\frac{x_{i_1} + \cdots + x_{i_k}}{k} - x_{i_{k+1}}\right\| \geq \\
&\geq \frac{1}{P_I^{k+1}}\sum_{i_1,\dots,i_{k+1}\in I} p_{i_1}\cdots p_{i_{k+1}}\left(\| x_{i_{k+1}} \| - \left\|\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right\|\right) \geq \\
&\geq \frac{1}{P_I}\sum_{i\in I} p_i \| x_i \| - \frac{1}{P_I^k}\sum_{i_1,\dots,i_k\in I} p_{i_1}\cdots p_{i_k}\left\|\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right\|,
\end{aligned}$$

which proves the second inequality in (7.1).

The following corollary for inner-product spaces is derived easily from (3.1).

Corollary 7.2. *Under the assumptions of Corollary 5.1*

$$\begin{aligned} 0 &\le \frac{1}{P_I}\sum_{i\in I} p_i \| x_i \| - \frac{1}{P_I^k}\sum_{i_1,\dots,i_k\in I} p_{i_1}\dots p_{i_k}\left\|\frac{x_{i_1}+\dots+x_{i_k}}{k}\right\| \le \\ &\le \left(\frac{1}{P_I^k}\sum_{i_1,\dots,i_k\in I} p_{i_1}\dots p_{i_k}\left\|\frac{x_{i_1}+\dots+x_{i_k}}{k}\right\|^2 - \left\|\frac{1}{P_I}\sum_{i\in I} p_i x_i\right\|^2\right)^{1/2} = \\ &= \left[\frac{1}{k}\left(\frac{1}{P_I}\sum_{i\in I} p_i \| x_i \|^2 - \left\|\frac{1}{P_I}\sum_{i\in I} p_i x_i\right\|^2\right)\right]^{1/2}. \end{aligned}$$

Remark 7.3. As before, the preceding result provides a refinement of the Cauchy-Schwarz inequality.

8. Third and fourth majorants

Theorem 8.1. *Under the assumptions of Theorem 6.1, we have*

$$\begin{aligned} 0 &\le \frac{1}{P_I}\sum_{i\in I} p_i \| x_i \| - \frac{1}{P_I^k}\sum_{i_1,\dots,i_k\in I} p_{i_1}\dots p_{i_k}\left\|\frac{1}{Q_k}\sum_{j=1}^k q_j x_{i_j}\right\| \le \\ &\le \frac{1}{P_I^{k+1}}\sum_{i_1,\dots,i_{k+1}\in I} p_{i_1}\dots p_{i_{k+1}}\left\|x_{i_{k+1}} - \frac{1}{Q_k}\sum_{j=1}^k q_j x_{i_j}\right\| \le \\ &\le \frac{1}{P_I^2}\sum_{i,j\in I} p_i p_j \| x_i - x_j \| \le \\ &\le \nu(x). \end{aligned} \tag{8.1}$$

Proof. The first inequality derives from Theorem 2.4 and the second is obvious from

$$\| x_{i_{k+1}} \| - \left\|\frac{q_1 x_{i_1}+\dots+q_k x_{i_k}}{Q_k}\right\| \le \left\|x_{i_{k+1}} - \frac{q_1 x_{i_1}+\dots+q_k x_{i_k}}{Q_k}\right\|$$

by a similar argument to that in the above theorem.

For the next, observe that

$$\left\|\frac{q_1 x_{i_1}+\dots+q_k x_{i_k}}{Q_k} - x_{i_{k+1}}\right\| = \left\|\frac{1}{Q_k}\sum_{j=1}^k q_i(x_{i_j} - x_{i_{k+1}})\right\| \le \frac{1}{Q_k}\sum_{j=1}^k q_j \| x_{i_j} - x_{i_{k+1}} \|.$$

If we multiply this by $p_{i_1}\dots p_{i_{k+1}}$ and sum over $i_1,\dots,i_{k+1}$ in I we deduce the third inequality in (8.1). We omit the details. □

The case of inner-product spaces is more interesting as it provides another refinement of the Cauchy-Buneakowski-Schwarz inequality. We derive the following result easily from (3.1).

Corollary 8.2. *Suppose that* $(H;<,>)$ *is an inner-product space. Under the above assumptions we have*

$$\begin{aligned} 0 &\le \frac{1}{P_I}\sum_{i\in I} p_i \| x_i \| - \frac{1}{P_I^k}\sum_{i_1,\ldots,i_k\in I} p_{i_1}\cdots p_{i_k}\left\| \frac{1}{Q_k}\sum_{j=1}^k q_j x_{i_j}\right\| \le \\ &\le \left[\frac{1}{P_I}\sum_{i\in I} p_i \| x_i \|^2 - \frac{1}{P_I^k}\sum_{i_1,\ldots,i_k\in I} p_{i_1}\cdots p_{i_k}\left\| \frac{1}{Q_k}\sum_{j=1}^k q_j x_{i_j}\right\|^2\right]^{1/2} = \\ &= \left(\left[1-\sum_{j=1}^k \left(q_j\frac{j}{Q_k}\right)^2\right]\left[\frac{1}{P_I}\sum_{i\in I} p_i \| x_i \|^2 - \left\|\frac{1}{P_I}\sum_{i\in I} p_i x_i\right\|^2\right]\right)^{1/2} \end{aligned}$$

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QUARTIC INTERPOLATORY SPLINES

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Abstract. The continuity conditions for quartic interpolatory splines with knots different from the points of interpolation are given. Boundary conditions completing such system are discussed and the existence theorem and algorithm for computing parameters of the spline are discussed.

1. Quartic splines

Let us have the increasing set of knots on the real axis

$$x_0 < x_1 < \dots < x_n < x_{n+1}. \qquad (\Delta x)$$

We shall denote $S_{41}(x)$ or simply $S(x)$ a quartic spline with the defect one on the knotset (Δx) - a functional with the following properties:

1. $S_{41}(x) \in C^3[x_0, x_{n+1}]$,

2. $S_{41}(x)$ *is a polynomial of the fourth degree on each interval* $[x_i, x_{i+1}], i = 0(1)n$.

Let us denote $S(\Delta x)$ the linear space of such splines with $\dim S(\Delta x) = n+5$. It is known (see [1]) that there are some difficulties connected with quartic splines interpolating the given function values at knots x_i, concerning the problems of existence of such splines in some cases and in the lack of localizing properties (causing then undamped error propagation) and of symmetry in boundary conditions (three free parameters).

Some part of such difficulties can be overwhelmed using separated meshes of knots of the spline x_i and points of the interpolation t_j, as was used also at quadratic splines (see [3], [4]).

In the following we will use the separated knot set

$$x_0 \le t_0 < x_1 < t_1 < \dots < t_{n-1} < x_n < t_n \le x_{n+1} \qquad (\Delta x \Delta t)$$

with stepsizes $h_j = x_{j+1} - x_j, \tau_j = t_{j+1} - t_j$. The case $t_i = (x_i + x_{i+1})/2,\ x_i = x_0 + ih$ (*equidistant set*) is the most frequently used.

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In the following we shall describe the algorithm for computing of appropriate local parameters of the spline $S_{41}(x)$ which corresponds to the conditions

3. $S_{41}(t_i) = g_i, i = 0(1)n$ (*conditions of interpolation*).

We have still four free parameters for the quartic spline on $(\Delta x \Delta t)$. We shall use some *boundary conditions* symmetrically on both sides of the interval $[x_0, x_{n+1}]$.

2. Basic relations between parameters - continuity conditions

We can write the Taylor formula with the remainder in the integral form for $S(x)$:

$$S(x) = S(x_0) + S'(x_0)(x - x_0) + \frac{1}{2}S''(x_0) + \frac{1}{2}\int_{x_0}^{x_{n+1}} S'''(t)(x-t)_+^2 dt \qquad (1)$$

where $z_+^j = z^j$ for $z \geq 0, z_+^j = 0$ for $z < 0$ ("*cut powers*").

Let us denote $T_i = S'''(x_i), g_i = S(t_i)$ the two of the local parameters of the spline. For the third derivative of $S(x)$ we can write

$$S'''(t) = T_k + (t - x_k)(T_{k+1} - T_k)/h_k \quad \text{for} \quad t \in [x_k, x_{k+1}], \quad k = 0(1)n \qquad (2)$$

2.1. Internal knots. . Using calculus of divided differences (see [2]), the third difference of both sides of (1) can be written as

$$2[t_{j-2}, t_{j-1}, t_j, t_{j+1}]S(x) = \int_{x_0}^{x_{n+1}} S'''(t)\left\{[t_{j-2}, t_{j-1}, t_j, t_{j+1}](x-t)_+^2\right\} dt. \qquad (3)$$

In our case of noncoinciding points t_i we can write the difference on the left-hand side of (3) as

$$[t_{j-2}, t_{j-1}, t_j, t_{j+1}]S(x) = \sum_{i=j-2}^{j+1} c_{ji} S(t_i) \qquad (4)$$

with coefficients $c_{ji} = 1/\prod_k (t_i - t_k)$, $k = j-2, ..., j+1$, $k \neq i$, $j = 2(1)n-1$, depending on the geometry of the knotset only. Explicitly,

$$\begin{aligned} 1/c_{j,j-2} &= -\tau_{j-2}(\tau_{j-2} + \tau_{j-1})(\tau_{j-2} + \tau_{j-1} + \tau_j), \\ 1/c_{j,j-1} &= \tau_{j-2}\tau_{j-1}(\tau_{j-1} + \tau_j), \\ 1/c_{jj} &= -\tau_{j-1}\tau_j(\tau_{j-2} + \tau_{j-1}) \\ 1/c_{j,j+1} &= \tau_j(\tau_{j-1} + \tau_j)(\tau_{j-2} + \tau_{j-1} + \tau_j) \end{aligned} \qquad (5)$$

on the general knotset.

On equidistant knotset we have simply

$$c_{j,j+1} = -c_{j,j-2} = 1/(6h^3), \quad c_{j,j-1} = -c_{j,j} = 1/(2h^3).$$

The same difference on the right-hand side of (3) is applied to the function $(x-t)_+^2$ only (in x-variable):

$$[t_{j-2}, t_{j-1}, t_j, t_{j+1}](x-t)_+^2 = \sum_{i=j-2}^{j+1} c_{ji}(t_i - t)_+^2. \tag{6}$$

This difference is equal zero for $x \notin [t_{j-2}, t_{j+1}]$. Using (2) and (4), (6), we can write (3) as

$$2[t_{j-2}, t_{j-1}, t_j, t_{j+1}]S(x) = \sum_{k=j-2}^{j+1} \int_{x_k}^{x_{k+1}} \left\{ [T_k + (1-x_k)(T_{k+1} - T_k)/h_k] \sum_{i=j-2}^{j+1} c_j i(t_i - t)_+^2 \right\} dt \tag{7}$$

For more detailed study of (7) let us further denote

$$\begin{aligned} P_{ik} &= \int_{x_k}^{x_{k+1}} (t_i - t)_+^2 dt = \frac{1}{3}\left[(t_i - x_k)_+^3 - (t_i - x_{k+1})_+^3\right] > 0 \quad \text{for } k \le i, \\ P_{ik}^3 &= \int_{x_k}^{x_{k+1}} (t_i - t)_+^3 dt = \frac{1}{4}\left[(t_i - x_k)_+^4 - (t_i - x_{k+1})_+^4\right] > 0 \quad \text{for } k \le i, \\ Q_{ik} &= \int_{x_k}^{x_{k+1}} (t - x_k)(t_i - t)_+^2 dt = -\frac{1}{3} h_k (t_i - x_{k+1})_+^3 + \frac{1}{3} P_{ik}^3 \le 0 \quad \text{for } k \le i. \end{aligned} \tag{8}$$

The values of this coefficients on equidistant mesh ($h_0 = h_n = h/2, h_i = h$ for $j = 1(1)n-1$) and $k, j \ge 1$ are given in the Table 1:

i	k	$k+1$	$k+2$	$k+3$
$24P_{ik}/h^3$	1	26	98	218
$192Q_{ik}/h^4$	1	72	328	776
$64P_{ik}^3/h^4$	1	80	544	1776

We can write now the right-hand side of (7) as

$$\sum_{k=j-2}^{j+1} \left\{ T_k \sum_{i=j-2}^{j+1} c_{ji} P_{ik} + [(T_{k+1} - T_k)/h_k] \sum_{i=j-2}^{j+1} c_{ji} Q_{ik} \right\} = \sum_{i=j-2}^{j+2} a_i^j T_j$$

with the coefficients a_i^j defined by the relations

$$
\begin{aligned}
a_{j-2}^j &= \sum_{i=j-2}^{j+1} c_{ji}(P_{i,j-2} - Q_{i,j-2}/h_{j-2}), \\
a_k^j &= \sum_{i=j-2}^{j+1} c_{ji}(P_{ik} - Q_{ik}/h_k + Q_{i,k-1}/h_{k-1}), \quad k = j-1, j, j+1 \qquad (9) \\
a_{j+2}^j &= c_{j,j+1} Q_{j+1,j+1}/h_{j+1};
\end{aligned}
$$

in the equidistant case we have

$$
a_{j-2}^j = a_{j+2}^j = 1/1152, \; a_{j-1}^j = a_{j+1}^j = 76/1152, a_j^j = 230/1152. \qquad (10)
$$

The equation (3) can be written now as

$$
2 \sum_{i=j-2}^{j+1} c_{ji} S(t_i) = a_{j-2}^j T_{j-2} + a_{j-1}^j T_{j-1} + a_j^j T_j + a_{j+1}^j T_{j+1} + a_{j+2}^j T_{j+2} \qquad (11)
$$

with coefficients c_{ji} given in (5) and a_i^j defined in (9). When we write the relations (11) for $j = 2(1)n-1$, we obtain altogether n-2 linear equations for calculation of the parameters $T_i, i = 0(1)n+1$. To complete the system of equation, we have to prescribe four another - usually boundary - conditions.

Remarks

1. The diagonal dominance condition in equations (11), sufficient for solvability of such systems,

$$
a_j^j > a_{j-2}^j + a_{j-1}^j + a_{j+1}^j + a_{j+2}^j
$$

could be written as

$$
\sum_{i=j-2}^{j+1} c_{ji} P_{ij} > \sum_{i=j-2}^{j+1} c_{ji}(P_{i,j-2} + P_{i,j-1} + P_{i,j+1} + P_{i,j+2} - 2Q_{i,j-1}/h_{j-1} + 2Q_{ij}/h_j); \qquad (12)
$$

we can see from (10) that it is fulfilled on the equidistant mesh.

2. Let us mention, that it is possible to work with the more general set of points interpolation $\{t_i, i = 0(1)n+4\}$. The system of equations (11) with $j = 2(1)n+3$ is then complete and existence of such interpolating spline depends on the solvability of such system. According to the general result of Curry-Schoenberg (see [2]) such problem has unique solution in case that t_i belongs to the support of the corresponding B-spline.

3. We can give some intuitive interpretation to the relations (11) using the values of coefficients a_i^j from (9)-(10) and the fact that

$$6\,[t_{j-2},t_{j-1},t_j,t_{j+1}]\,S(x)=S'''(z_j),\,z_j\in(t_{j-2},t_{j+1}):$$

the weighted mean of the values $T_{j-2},...,T_{j+1}$ *is equal to some value* $S'''(z_j)$.

2.2. **Multiple knots on the left boundary.** To complete the system (11) of n-2 equations for n+2 parameters $T_i, i=0(1)n+1$ on the set $(\Delta x\Delta t)$, four boundary conditions can be prescribed. In case of prescribed values of S', S'' at boundary points $x=t_0, x=t_n$, we can complete our system by the relations analogical to (3) with $j=1,2;\ n,\ n+1$, where the divided differences with multiple knots have to be used (see [2]). For any function $f(x)\in C^2[x_0,x_{n+1}]$ defined on the mesh $(\Delta x\Delta t)$ we can find that

$$\begin{aligned}&(\tau_0+\tau_1)\,[t_0,t_0,t_1,t_2]\,f=(2\tau_0+\tau_1)\left\{\tau_0^2(\tau_0+\tau_1)\right\}^{-1}f_0+(\tau_0)^{-1}f_0'-\\&-(\tau_0+\tau_1)(\tau_0^2\tau_1)^{-1}f_1+\left\{\tau_1(\tau_0+\tau_1)\right\}^{-1}f_2,\\&\tau_0^2\,[t_0,t_0,t_0,t_1]\,f=(f_1-f_0)/\tau_0-f_0'-\tau_0f_0''/2,\end{aligned}\tag{13}$$

where $f_i=f(t_i),\ f_0'=f'(t_0),\ f_0''=f''(t_0)$.
For $t\in[t_0,t_1]$ and function $f(x)=(x-t)_+^2$ in x-variable we obtain

$$\tau_0^2\,[t_0,t_0,t_0,t_1]\,(x-t)_+^2=(t_1-t)_+^2/\tau_0;$$

for $t\in[t_0,t_2]$ we find that

$$(\tau_0+\tau_1)\,[t_0,t_0,t_1,t_2]\,(x-t)_+^2=\left[\tau_1(\tau_0+\tau_1)\right]^{-1}(t_2-t)_+^2-\left[(\tau_0+\tau_1)/(\tau_0^2\tau_1)\right](t_1-t)_+^2.\tag{14}$$

The equations corresponding to (11) for $j=0,1$ are then

$$6\,[t_0,t_0,t_0,t_1]\,S(x)=\int_{t_0}^{t_1}S'''(t)\left\{[t_0,t_0,t_0,t_1]\,(x-t)_+^2\right\}dt=a_0^0T_0+a_1^0T_1+a_2^0T_2\tag{15}$$

with

$$a_0^0=(P_{10}-Q_{10}/h_0)/\tau_0^3;\ a_1^0=(P_{11}+Q_{10}/h_0-Q_{11}/h_1)/\tau_0^3,\ a_2^0=Q_{11}/(h_1\tau_0^3)$$

and

$$2\,[t_0,t_0,t_1,t_2]\,S(x) = \int_{t_0}^{t_2} S'''(t)\,\{[t_0,t_0,t_1,t_2](x-t)_+^2\}\,dt = a_0^1T_0 + a_1^1T_1 + a_2^1T_2 + a_3^1T_3, \tag{16}$$

where

$$\begin{aligned}
a_0^1 &= \alpha_0(P_{20} - Q_{20}/h_0) - \beta_0(P_{10} - Q_{10}/h_0), \quad \alpha_0 = [\tau_1(\tau_0+\tau_1)^2]^{-1},\\
a_1^1 &= \alpha_0(P_{21} - Q_{21}/h_1 + Q_{20}/h_0) - \beta_0(P_{11} - Q_{11}/h_1 + Q_{10}/h_0),\\
a_2^1 &= \alpha_0(P_{22} - Q_{22}/h_2 + Q_{21}/h_1) - \beta_0(Q_{11}/h_1),\\
a_3^1 &= \alpha_0 Q_{22}/h_2, \quad \beta_0 = (\tau_0^2\tau_1)^{-1}.
\end{aligned}$$

For equidistant case the values of coefficients in (15)-(16) not occurring in Table 1 are given in Table 2:

k	1	2	3	j	0	1	2	3	4
$24P_{k0}/h^3$	7	37	91	$192a_j^0$	34	29	1		
$192Q_{k0}/h^4$	11	67	171	$768a_j^1$	26	154	71	1	
$192P_{k0}^3/h^3$	45	525	671	$1152a_j^2$	2	75	230	76	1

2.3. Multiple knots on the right boundary. Quite similar, using some symmetry of our problem (but unfortunately with some asymmetry given in the notation) and some algebraic manipulations with the values of divided differences, we can prove that generally

$$\begin{aligned}
&\tau_{n-1}^2[t_{n-1},t_n,t_n,t_n]f = \tau_{n-1}f_n''/2 - f_nn' + (f_n - f_{n-1})/\tau_{n-1},\\
&(\tau_{n-2}+\tau_{n-1})[t_{n-2},t_{n-1},t_n,t_n]f = f_nn'/\tau_{n-1} - \{\tau_{n-2}(\tau_{n-2}+\tau_{n-1})\}^{-1}f_{n-2} + \\
&+\{(\tau_{n-2}+\tau_{n-1})/(\tau_{n-1}^2\tau_{n-2})\}f_{n-1} - (2\tau_{n-1}+\tau_{n-1})\{\tau_{n-1}^2(\tau_{n-2}+\tau_{n-1})\}^{-1}f_n.
\end{aligned} \tag{17}$$

In our special case we have

$$\begin{aligned}
&\text{for } t\in[t_{n-1},t_n]: \quad [t_{n-1},t_n,t_n,t_n](x-t)_+^2 = (\tau_{n-1}^3)^{-1}(t-t_{n-1})_+^2,\\
&\text{for } t\in[t_{n-2},t_n]: \quad [t_{n-2},t_{n-1},t_n,t_n](x-t)_+^2 = \\
&\qquad [\tau_{n-2}(\tau_{n-1}+\tau_{n-2})^2]^{-1}(t-t_{n-2})_+^2 - (\tau_{n-1}^2\tau_{n-2})^{-1}(t-t_{n-1})_+^2.
\end{aligned} \tag{18}$$

Choosing in a symmetrical way the representation of $S'''(t)$,

$$S'''(t) = T_j + (x_j - t)(T_{j-1} - T_j)/h_{j-1} \quad \text{for} \quad t\in[x_{j-1},x_j],\ j = n, n+1$$

we obtain after integration over interval $[t_{n-1}, t_n]$ and using (18), (3)

$$\begin{aligned} 2[t_{n-1}, t_n, t_n, t_n]S(x) &= \int_{t_{n-1}}^{t_n} S'''(t)[t_{n-1}, t_n, t_n, t_n](x-t)_+^2 dt = \\ &= a_{n-1}^{n+1} T_{n-1} + a_n^{n+1} T_n + a_{n+1}^{n+1} T_{n+1} \end{aligned} \tag{19}$$

with coefficients

$$\begin{aligned} a_{n-1}^{n+1} &= \bar{Q}_{n-1,n}/\tau_{n-1}^3 h_{n-1},\ a_{n+1}^{n+1} = (\bar{P}_{nn} - \bar{Q}_{nn}/h_n)/\tau_{n-1}^3, \\ a_n^{n+1} &= (\bar{P}_{n-1,n} - \bar{Q}_{n-1,n}/h_{n-1} + \bar{Q}_{nn}/h_n)/\tau_{n-1}^3. \end{aligned} \tag{20}$$

The integration over interval $[t_{n-2}, t_n]$ using (18) results in

$$\begin{aligned} 2[t_{n-2}, t_{n-1}, t_n, t_n]S(x) &= \int_{t_{n-2}}^{t_n} S'''(t)\left\{[t_{n-2}, t_{n-1}, t_n, t_n](x-t)_+^2\right\} dt = \\ &= a_{n-2}^n T_{n-2} + a_{n-1}^n T_{n-1} + a_n^n T_n + a_{n+1}^n T_{n+1} \end{aligned} \tag{21}$$

with the coefficients

$$\begin{aligned} a_{n-2}^n &= \alpha_n \bar{Q}_{n-2,n-1}/h_{n-2}, \quad \text{where} \quad \tau_n = [\tau_{n-2}(\tau_{n-1} + \tau_{n-2})^2]^{-1}, \\ a_{n-1}^n &= \alpha_n(\bar{P}_{n-2,n-1} - \bar{Q}_{n-2,n-1}/h_{n-2} + \bar{Q}_{n-1,n-1}/h_{n-1}) - \beta_n \bar{Q}_{n-1,n}/h_{n-1}, \\ a_n^n &= \alpha_n(\bar{P}_{n-1,n-1} + \bar{Q}_{n,n-1}/h_n - \bar{Q}_{n-1,n-1}/h_{n-1}) - \alpha_n(\bar{P}_{n-1,n} + \bar{Q}_{nn}/h_n - \bar{Q}_{n-1,n}/h_{n-1}), \\ a_{n+1}^n &= \alpha_n(\bar{P}_{n,n-1} - \bar{Q}_{n,n-1}/h_n) - \beta_n(\bar{P}_{nn} - \bar{Q}_{nn}/h_n), \quad \beta_n = (\tau_{n-1}^2 \tau_{n-2})^{-1}. \end{aligned} \tag{22}$$

We have used in (20), (22) the notation

$$\begin{aligned} \bar{P}_{ik} &= \frac{1}{3}\left[(x_{i+1} - t_{k-1})_+^3 - (x_i - t_{k-1})_+^3\right] \quad \text{for} \quad i = n,\ k = n-1, n; \\ &\quad i = n-1,\ k = n-1, n;\ i = n-2, k = n-1; \\ \bar{P}_{ik}^3 &= \frac{1}{4}\left[(x_{i+1} - t_{k-1})_+^4 - (x_i - t_{k-1})_+^4\right] \quad \text{for} \quad i = n-1,\ k = n-1, n; \\ &\quad i = n,\ k = n-1, n;\ i = n-2,\ k = n-1; \\ \bar{Q}_{ik} &= \frac{1}{3}\left[(x_{i+1} - t_k)_+ (x_{i+1} - t_{k-1})_+^3 - h_i(x_i - t_{k-1})_+^3\right] + \frac{1}{3}\bar{P}_{ik}^3. \end{aligned} \tag{23}$$

For equidistant case the values of these coefficients are equal to the symmetric coefficients on the left boundary given in Table 2.

2.4. Complete system of continuity conditions. The relations (16), (11) and (19), (21) form now together with (11) the complete system of $n+2$ linear equations for the parameters $T_i, i = 0(1)n+1$ with right-hand sides (see (13), (17))

$$\begin{aligned} f_j &= 2[t_{j-2}, t_{j-1}, t_j, t_{j+1}]S(x),\ j = 2(1)n-1 \\ f_0 &= 2[t_0, t_0, t_0, t_1]S(x),\ f_1 = 2[t_0, t_0, t_1, t_1]S(x), \\ f_n &= 2[t_{n-2}, t_{n-1}, t_n, t_n]S(x),\ f_{n+1} = 2[t_{n-1}, t_n, t_n, t_n]S(x), \end{aligned}$$

of the general form

$$\begin{bmatrix} a_0^0 & a_1^0 & a_2^0 & & & & & & \cdot \\ a_0^1 & a_1^1 & a_2^1 & a_3^1 & & & & & \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 & a_4^2 & & & & \\ \dots & \dots & \dots & \dots & & & & & \\ & a_{j-2}^j & a_{j-1}^j & a_j^j & a_{j+1}^j & a_{j+2}^j & & & \\ & & \dots & \dots & \dots & \dots & & & \\ & & & & a_{n-2}^n & a_{n-1}^n & a_n^n & a_{n+1}^n \\ & & & & & a_{n-1}^{n+1} & a_n^{n+1} & a_{n+1}^{n+1} \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ \dots \\ T_j \\ \dots \\ T_n \\ T_{n+1} \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \dots \\ f_j \\ \dots \\ f_n \\ f_{n+1} \end{bmatrix} \tag{24}$$

The matrix of the system (24) is fivediagonal - the diagonal dominance condition (12) can be easily completed for the first and last pair of the equations from (20), (22).

In case of equidistant mesh $(\Delta x \Delta t)$ with $h_i = \tau_i = h, h_0 = h_n = h/2$ the complete system of continuity conditions can be written as

$$\frac{1}{192}\begin{bmatrix} 34 & 29 & 1 & & & & & & & \\ 26 & 154 & 75 & 1 & & & & & & \\ 2 & 75 & 230 & 76 & 1 & & & & & \\ & 1 & 76 & 230 & 76 & 1 & & & & \\ & \dots & \dots & \dots & \dots & \dots & \dots & & & \\ & & & 1 & 76 & 230 & 76 & 1 & & \\ & & & & & 1 & 76 & 230 & 75 & 1 \\ & & & & & & 1 & 75 & 154 & 26 \\ & & & & & & & 1 & 29 & 34 \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \\ \dots \\ T_{n-2} \\ T_{n-1} \\ T_n \\ T_{n+1} \end{bmatrix} = \frac{2}{h^3} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \dots \\ f_{n-2} \\ f_{n-1} \\ f_n \\ f_{n+1} \end{bmatrix} \tag{25}$$

with diagonally dominant matrix,

$$\begin{aligned}
\bar{f}_0 &= S_1 - S_0 - hS_0' - \frac{1}{2}h^2 S_0'',\ \bar{f}_{n+1} = S_n - S_{n-1} - hS_n' + \frac{1}{2}h^2 S_n'', \\
\bar{f}_1 &= 3S_0 - 4S_1 + S_2 + 2hS_0',\ \bar{f}_n = -3S_n + 4S_{n-1} - S_{n-2} + 2hS_n', \\
\bar{f}_j &= S_{j+1} - 3S_j + 3S_{j-1} - S_{j-2},\ j = 2(1)n-2.
\end{aligned}$$

We have thus the unique solution of this system for any values on the right-hand side of (25).

We summarize our discussion from 2.1-2.4 in the following theorem.

Theorem 1. *Let $S(x)$ be a quartic spline on the knotset $(\Delta x \Delta t)$. Then between its parameters $T_i = S'''(x_i)$ and the third divided differences of the function values (and the values of appropriate derivatives on the boundary) the relations (24) hold (" continuity conditions").*

3. Boundary conditions

The matrix of the system (24) is quite determined by knotset $(\Delta x \Delta t)$ and its geometry. The prescribed values $g_i = S(t_i)$ are used for computation of the components of the right-hand side values $f_i, i = 2(1)n-1$. In the first and last two components f_i the values of S', S'' and $t = t_0$, $t = t_{n+1}$ appear as additional parameters of the problem. To determine the interpolating spline uniquely, we have to prescribe additional conditions - the values of boundary derivatives or some equivalent conditions.

3.1. **The first and second derivatives.** Given the values $S'(t_0)$, $S''(t_0)$, $S'(t_n)$, $S''(t_n)$ and function values $S(t_i) = g_i$, $i = 0(1)n$ on the knotset $(\Delta x \Delta t)$, we can immediately calculate all components on the right-hand side of the system (24) or (25). When the condition of diagonal dominance in the matrix of the system is fulfilled (as it is in the case of equidistant mesh), we have then unique solution T_i of the system for any given data.

3.2. **The conditions of periodicity.** When the spline we search has to be a periodic function, we can extend our mesh $(\Delta x \Delta t)$ in the periodic way on the left and right boundary. The simplest way is now to consider the case of the mesh

$$t_{-1} < x_0 < t_0 = a < x_1 < \dots < x_n < t_n = b < x_{n+1} < t_{n+1}$$

Writing now the relation (11) for $j = 1(1)n$ with

$$S(t_0) = S(t_n), \quad T_0 = T_n, \quad T_1 = T_{n+1}, \quad T_{-1} = T_{n-1}, \tag{26}$$

we obtain the system of n linear equations for the parameters T_j, $j = 1(1)n$ with cyclic fivediagonal matrix; the components f_i are determined now by function values only. In case of equidistant mesh the coefficients a_i^j are just the numbers given in (25).

3.3. The first and third derivatives prescribed. When the values $S'(t_i)$ and $S'''(t_i) = T_j$ at the boundaries $x = t_0, t = t_n$ are prescribed, we can handle the problem in the following way:

a) we omit the first and the last equation in (24) ($j = 0, n+1$);

b) for $j = 1, n$ we substitute in these equations for the known values T_0, T_{n+1}, $S'(t_0)$, $S'(t_n)$ on the left and right side using (13), (17).

We obtain the system of n linear equations for parameters T_j, $j = 1(1)n$ in this way.

3.4. The second and third derivatives prescribed. In case of prescribed values S'', S''' at boundaries we can use the relation (13) to eliminate the expressions

$$S_0' - (S_1 - S_0)/\tau_0 \quad \text{resp.} \quad S_n' - (S_n - S_{n-1})/\tau_{n-1}$$

from the boundary pair of equations (24) to obtain equivalent relations without the first derivatives

$$\begin{aligned}
&-S_0'' + 2(\tau_0 + \tau_1)^{-1}\left[(S_2 - S_1)/\tau_1 - (S_1 - S_0)/\tau_0\right] = \sum_{j=0}^{3}\left[\tau_0 a_j^0 + (\tau_0 + \tau_1)a_j^1\right] T_j, \\
&S_n'' - 2(\tau_{n-1} + \tau_{n-2})\left[(S_n - S_{n-1})/\tau_{n-1} - (S_{n-1} - S_{n-2})/\tau_{n-2}\right] = \\
&= \sum_{j=n-2}^{n+1}\left[\tau_{n-1} a_j^{n+1} + (\tau_{n-1} + \tau_{n-2})a_j^n\right] T_j.
\end{aligned} \tag{27}$$

With known T_0, T_{n+1} these relations complete the system (11) to the fivediagonal system of equations for T_j, $j = 1(1)n$.

3.5. The second (third) and the fourth derivatives prescribed. For the fourth one-sided derivatives at the boundaries there is

$$Q_0 = S^{(4)}(x_0 + 0) = (T_1 - T_0)/h_0, \; Q_{n+1} = S^{(4)}(x_{n+1} - 0) = (T_{n+1} - T_n)/h_n. \tag{28}$$

In case of prescribed values M_0, Q_0, M_{n+1}, Q_{n+1} we can write in (24) the first and last equation ($j = 0$, $n+1$) as

$$-T_0 + T_1 = h_0 Q_0, \qquad -T_n + T_{n+1} = h_n Q_{n+1}. \tag{29}$$

we use further relations (27) for $j = 1,n$ to complete in such a way the rearranged system (24) to $n+2$ equations for the parameters T_j, $j = 0(1)n+1$.

When the boundary values T_0, Q_0 are prescribed, we can calculate T_1 from (29) and then T_j, $j = 2(1)n + 1$ recursively from (11).

3.6. General type of boundary conditions. It is possible to consider more type of boundary conditions

$$\begin{aligned}
&a_0^0 T_0 + a_1^0 T_1 + a_2^0 T_2 = f_0,\\
&a_0^1 T_0 + a_1^1 T_1 + a_2^1 T_2 + a_3^1 T_3 = f_1,\\
&a_{n-2}^n T_{n-2} + a_{n-1}^n T_{n-1} + a_n^n T_n + a_{n+1}^n T_{n+1} = f_n,\\
&a_{n-1}^{n+1} T_{n-1} + a_n^{n+1} T_n + a_{n+1}^{n+1} T_{n+1} = f_{n+1}
\end{aligned} \tag{30}$$

with given coefficients a_i^j, f_j or various conditions which can be finally expressed in this form.

4. Local parameters of the quartic spline

For the given knot set $(\Delta x \Delta t)$ with prescribed conditions of interpolation $S(t_i) = g_i$, $i = 0(1)n$, the continuity conditions (11) completed by boundary conditions analyzed in 3.1-3.6 form together the complete system of linear equations for computing the parameters $T_i = S'''(x_i), i = 0(1)n + 1$. To the complete determination of the interpolating spline we have now to compute the remaining local parameters of the quartic spline under search.

4.1. Some local representations of the quartic spline. .

a) Let us denote

$$s_i = S(x_i),\ m_i = S'(x_i),\ M_i = S''(x_i),\ T_i = S'''(x_i),\ Q_i = S^{(4)}(x_i) \tag{31}$$

the Taylor coefficients of $S(x)$ at $x = x_i$. Then the representation

$$S(x) = s_i + m_i(x - x_i) + \frac{1}{2}M_i(x - x_i)^2 + \frac{1}{6}T_i(x - x_i)^3 + \frac{1}{24}Q_i(x - x_i)^4, x \in [x_i, x_{i+1}] \tag{32}$$

uses all local parameters concentrated in $x = x_i$ (with function values s_i unknown).

b) For the Taylor expansion $S(x)$ at $x = t_i$,

$$S(x) = \sum_{j=0}^{4} S^{(j)}(t_i)(x - x_i)^j/j! \quad \text{for} \quad x \in [x_i, x_{i+1}],$$

the needed values of derivatives at $x = t_i$ can be expressed as

$$\begin{aligned}
S^{(4)}(t_i) &= Q_i = (T_{i+1} - T_i)/h_i, \\
S^{(3)}(t_i) &= T_i + d_iQ_i \quad \text{with} \quad d_i = t_i - x_i, \\
S''(t_i) &= M_i + d_iT_i + \frac{1}{2}d_i^2Q_i = M_i + d_i\left[(2h_i - d_i)T_i + d_iT_{i+1}\right]/(2h_i), \\
S'(t_i) &= m_i + d_iM_i + \frac{1}{2}d_i^2T_i + \frac{1}{6}d_i^3Q_i \\
S(t_i) &= g_i.
\end{aligned} \tag{33}$$

c) To reduce the number of needed local parameters we can use the Taylor representation at $x = x_i$ with variable $q = (x - x_i)/h_i$

$$S(x) = \sum_{j=0}^{4} b_j^i q^j \quad \text{for} \quad x \in [x_i, x_{i+1}] \tag{34}$$

with

$$\begin{aligned}
b_1^i &= h_im_i, \\
b_2^i &= h_i^2(m_{i+1} - m_i)/2 - h_i^3(T_{i+1} + 2T_i)/12 = (h_i/2)\left[m_{i+1} - m_i - h_i^2(T_{i+1} + 2T_i)/6\right] \\
b_3^i &= h_i^3T_i/6, \\
b_4^i &= h_i^3(T_{i+1} - T_i)/24, \quad b_0^i = g_i - \sum_{j=1}^{4} b_j^i d_i^j,
\end{aligned}$$

where now $d_i = (t_i - x_i)/h_i$. We see now that it is enough to store the parameters m_i, T_i only.

d) It is possible also to use the local representation

$$\begin{aligned} S(x) &= g_i + h_i m_i (q - d_i) + h_i^2 M_i (q^2 - d_i^2)/2 + \\ &+ h_i \left[m_{i+1} - m_i - h_i (M_{i+1} + 2M_i)/3\right] (q^3 - d_i^3) + \\ &+ h_i \left[h_i (M_{i+1} + M_i)/4 - (m_{i+1} - m_i)/2\right] (q^4 - d_i^4) \quad \text{for} \quad x \in [x_i, x_{i+1}], \end{aligned} \tag{35}$$

with $q = (x - x_i)h_i$, $d_i = (t_i - x_i)/h_i$, where only parameters g_i, m_i, M_i occur.

4.2. Computing local parameters m_i, M_i. Suppose that we have computed the parameters $T_i, i = 0(1)n+1$ for the spline $S(x)$ with some boundary conditions from 3.1-3.6. The remaining local parameters we can compute as follows:

$$\begin{aligned} Q_i &= (T_{i+1} - T_i)/h_i, \quad i = 0(1)n, \\ M_{i+1} &= M_i + h_i T_i + \frac{1}{2} h_i^2 Q_i = M_i + \frac{1}{2} h_i (T_i + T_{i+1}). \end{aligned} \tag{36}$$

The last relation can be used in case of boundary conditions with M_0 or M_{n+1} given. Similarly we can use the recurrence

$$m_{i+1} = m_i + h_i M_i + \frac{1}{2} h_i^2 T_i + \frac{1}{6} h_i^3 Q_i = m_i + h_i \left[M_i + \frac{1}{6} h_i (2T_i + T_{i+1})\right] \tag{37}$$

for calculation of all m_i in case of given m_0 or m_{m+1} and known M_i, T_i. Some another relation following from the Taylor expansions of g_j, g_{j-1} at $x = x_j$,

$$\begin{aligned} (t_j - t_{j-1}) m_j &= g_j - g_{j-1} + \frac{1}{2} M_j \left[(t_{j-1} - x_j)^2 - (t_j - x_j)^2\right] - \\ &- \frac{1}{6} T_j \left[(t_j - x_j)^3 - (t_{j-1} - x_j)^3\right] - \frac{1}{24} \left[(t_j - x_j)^4 Q_j - (t_{j-1} - x_j)^4 Q_{j-1}\right] \end{aligned} \tag{38}$$

can be used to calculate the values $m_j, j = 1(1)n$.

In case of equidistant mesh with $h_0 = h_n = h/2$, $h_i = h$ the relations

$$\begin{aligned} m_1 &= (g_1 - g_0)/h - h^2 (T_2 + 13T_1 + 2T_0)/384, \\ m_j &= (g_j - g_{j-1})/h - h^2 (T_{j+1} + 14T_j + T_{j-1})/384, \\ m_n &= (g_n - g_{n-1})/h - h^2 (2T_{n+1} + 13T_n + T_{n-1})/384 \end{aligned} \tag{39}$$

can be used directly for work with the representation (34). The continuity conditions for the cubic spline $S_3(x) = S'(x)$ can be also used for computing needed parameters - e.g.

$$\begin{aligned} (M_{j-1} + 4M_j + M_{j+1})/6 &= (m_{j+1} - m_{j-1})/(2h), \; j = 1(1)n - 1 \\ (T_{j-1} + 4T_j + T_{j+1})/6 &= (m_{j-1} - 2m_j + m_{j+1})/h^2 \end{aligned} \tag{40}$$

in the equidistant case; the first one gives the recursion in case of known m_0 or m_{n+1}, the second one form the tridiagonal system of equation for computing m_j from known values of T_i.

5. Algorithm of computation, existence

Given the set of knots $(\Delta x \Delta t)$, values $g_i, i = 0(1)n$ and some of boundary conditions discussed in Section 3 for interpolating spline, we can proceed as follows:

a)formulate the boundary conditions as additional equations to the system of continuity conditions (11) in terms of parameters T_i, as discussed in Section 3;

b)calculate the coefficients of the matrix and right-hand side of resulting system of equations (24);

c)solve this system of equations (using special algorithms for systems with pentadiagonal or cyclic pentadiagonal matrix);

d)choose appropriate local representation for $S(x)$ and calculate corresponding local parameters as discussed in Section 4.

We can summarize obtained results in the following theorem.

Theorem 2. *Let us have given the mesh $(\Delta x \Delta t)$ of knots and points of interpolation with prescribed values g_i at points $t_i, i = 0(1)n$. Under the condition (12) and boundary conditions discussed in section 3 there exists the unique quartic spline interpolating the values g_i under given boundary conditions.*

6. Examples

Example 1.

For the test function $g(x) = 1/(1 + x^2)$ we have used the knotset

x_i	-6	-3		-1		1		3	6
t_i	-6		-2		0		2		6

and boundary conditions

a) $S'(-6) = S'(6) = 0, S''(-6) = S''(6) = 0$

b) $S'(-6) = 0.5, S'(6) = -0.5, S''(-6) = S''(6) = 0$

for computing parameters of the corresponding quartic spline. The resulting splines are displayed in Fig. 1 a), b).

The same data (without the second boundary derivatives) were used for corresponding quadratic splines, denoted by c), d) in Fig. 1 (the knots of the spline are marked on the x-axis, the points of interpolation are market by full dots).

Example 2.

For the function $g(x) = \frac{1}{16}e^x \sin 2x$ with the knotset

x_i	-1	0	1.5	2.5	3.1	3.8	4.2	4.7	5
t_i	-1	1	2	3	3.5	4	4.5	5	
g_i	-0.021	0.154	-0.345	-0.351	1.360	3.376	2,319	-5.046	

the quartic splines corresponding to the boundary conditions

a) $S'(-1) = 0,$ $S'(5) = -21$ (rounded exact values of g', g'')

$S''(-1) = 0,$ $S''(5) = -20$

b) $S'(-1) = -1,$ $S'(5) = -1$

$S''(-1) = 0,$ $S''(5) = -20$

are given in Fig. 2 together with quadratic splines determined by the same conditions of interpolation and boundary conditions

c) $S_2 2'(-1) = -1, S_2 2'(5) = -1.$

The local influence of the changes in prescribed function values is demonstrated in Fig. 3, where the curves correspond to the splines determined by the above data with consecutive changes

d) $S(1) = 1$ instead of $S(1) = 0.154$

e) $S(3) =$ -1$S(3) =$ -0.351

f) $S(5) =$ -3$S(5) =$ -0.046.

Example 3. For the monotone data

x_i	0	0.5	1.5	2.2	3	3.8	4.5	6	7.5	8.5	10
t_i	0	1	2	2.5	3.5	4	5	7	8	10	
g_i	-1	-1	-1	-0.4	0.4	1	1.2	1.6	2.2	3	

and boundary conditions

a) $S'(0) = S''(0) = 0, S'(10) = S''(10) = 0$

b) $S'(0) = S''(0) = 0, S'(10) = 0.5, S''(10) = 1$

the corresponding splines are shown in Fig. 4, together with quadratic spline with boundary condition

c) $S'(0) = 0, S'(10) = 0.5$.

We can see that both quartic and quadratic spline don't generally preserve monotonicity of the data in the left constant part. The quadratic spline preserves the monotonicity in the interval [4,7], but the quartic spline reaches here unwanted "undulation".

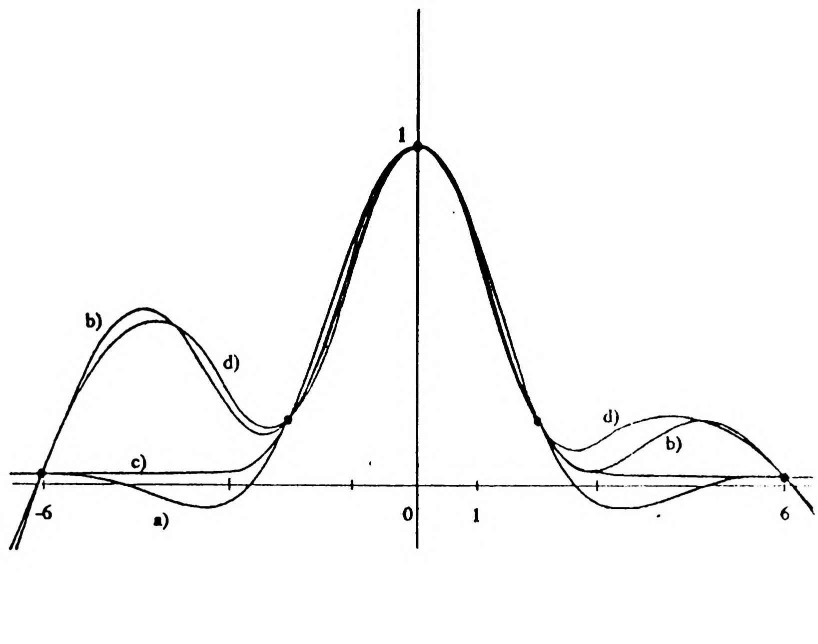

Figure 1

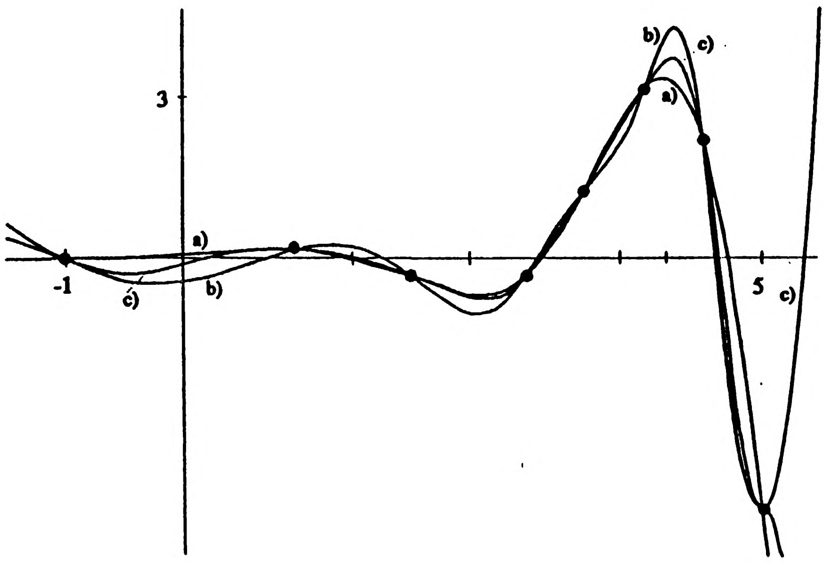

Figure 2

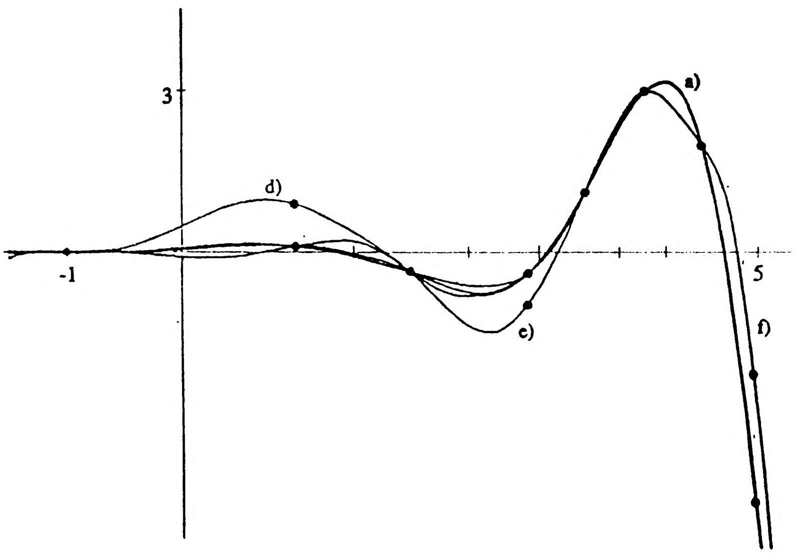

Figure 3

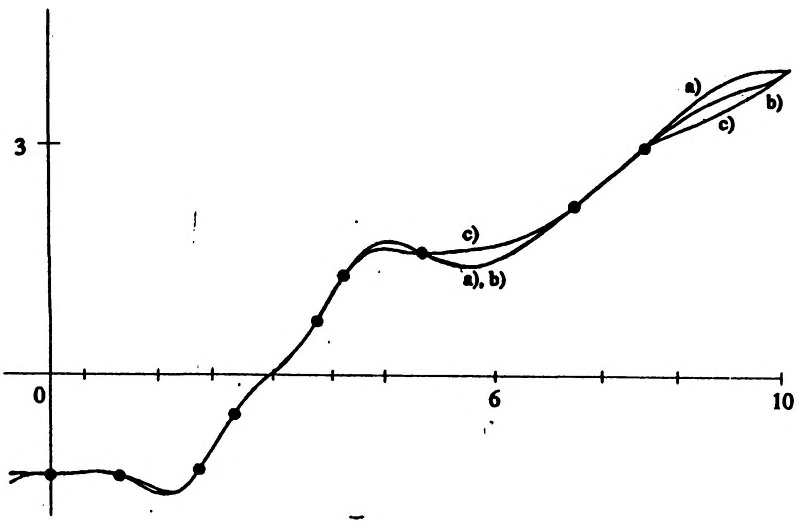

Figure 4

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ON SOME SUFFICIENT CONDITIONS OF ALMOST STARLIKENESS OF ORDER 1/2 IN $\mathbb{C}^n$

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Abstract. In this paper the author obtains some sufficient conditions of almost starlikeness of order 1/2 for holomorphic mappings defined on the ball in $\mathbb{C}^n$.

1. Introduction.

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, ..., z_n)'$ with the Euclidian inner product $< z, w >= \sum_{j=1}^{n} z_j \bar{w}_j$ and the norm $\| z \|=< z, z >^{1/2}$, for all $z \in \mathbb{C}^n$. The open Euclidian ball $\{z \in \mathbb{C}^n : \| z \| < r\}$ is denoted by B_r and the open unit Euclidian ball is abbreviated by $B_1 = B$. The origin $(0, 0, ..., 0)'$ is always denoted by 0. As usual, by $L(\mathbb{C}^n, \mathbb{C}^n)$ we denote the space of all continuous linear operator from $\mathbb{C}^n$ into $\mathbb{C}^n$ with the standard operator norm. The letter I will always represent the identity operator in $L(\mathbb{C}^n, \mathbb{C}^n)$. The class of holomorphic mappings from a domain $G \subseteq \mathbb{C}^n$ into $\mathbb{C}^n$ is denoted by $H(G)$. A mapping $f \in H(G)$ is said to be locally biholomorphic in G if its Fréchet derivative $Df(z) = \left(\frac{\partial f_j(z)}{\partial z_k}\right)_{1 \le j,k \le n}$ as an element of $L(\mathbb{C}^n, \mathbb{C}^n)$ is nonsingular at each point $z \in G$. A mapping $f \in H(G)$ is called biholomorphic on G, if the inverse mapping f^{-1} does exist, is holomorphic on a domain Ω and $f^{-1}(\Omega) = G$.

If $D^2 f(z)$ means the Fréchet derivative of the second order of $f \in H(G)$ at the point z, then of course $D^2 f(z)$ is a continuous bilinear operator from $\mathbb{C}^n \times \mathbb{C}^n$ into $\mathbb{C}^n$ and its restriction $D^2 f(z)(u, \cdot)$ to $u \times \mathbb{C}^n$ belongs to $L(\mathbb{C}^n, \mathbb{C}^n)$. The symbol " ′ " means the transpose of elements and matrix defined on $\mathbb{C}^n$.

For our purpose, we shall use the following definitions and results.

Definition 1.1. A holomorphic mapping $f : B \to \mathbb{C}^n$ is starlike if f is biholomorphic on B, $f(0) = 0$, and $(1 - t)f(B) \subseteq f(B)$, for all $t \in [0, 1]$.

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Lemma 1.1. *([5]) Let* $B \to \mathbb{C}^n$ *be a locally biholomorphic mapping on* B *with* $f(0) = 0$. *Then* f *is starlike iff*

$$\mathrm{Re} < [Df(z)]^{-1} f(z), z > > 0,$$

for all $z \in B \setminus \{0\}$.

Definition 1.2. Let $f : B \to \mathbb{C}^n$ be a locally biholomorphic mapping on B with $f(0) = 0$ and $Df(0) = I$. We say that f is almost starlike of order 1/2 on B if

$$\mathrm{Re} < [Df(z)]^{-1} f(z), z > > \frac{1}{2} \| z \|^2,$$

for all $z \in B \setminus \{0\}$.

It is clear that if f is almost starlike of order 1/2 on B, then according to Lemma 1.1, f is starlike, hence univalent on B.

Let $\mathcal{M} = \{f \in H(B) : f(0) = 0, \mathrm{Re} < f(z), z > > 0, z \in B \setminus \{0\}\}$.

Lemma 1.2. *([1]). Let* $p \in H(B)$ *with* $p(0) = 0$ *and* $Dp(0) = aI$, *where* a *is a complex number, with Re* $a > 0$. *Suppose that* $p \notin \mathcal{M}$, *then there exist* $z_0 \in B \setminus \{0\}$ *and a real number* m *such that the following relations hold:*

(i) $\mathrm{Re} < p(z_0), z_0 >= 0$ *and* $\mathrm{Re} < p(z), z > > 0$ *for all* $0 <\| z \|<\| z_0 \|$,

(ii) $\left[\overline{Dp(z_0)}\right]' z_0 + p(z_0) = m z_0$

(iii) $\mathrm{Re} < D^2 p(z_0)(v, v), z_0 > +2\mathrm{Re} < Dp(z_0)v, v >\geq m \| v \|^2$,

for all $v \in \mathbb{C}^n \setminus \{0\}$, *with* $\mathrm{Re} < z_0, v >= 0$, *where* $m \leq -\frac{1}{2\mathrm{Re}\, a} \mid a \quad is \mid^2$ *and*

$s = \frac{1}{\|z_0\|^2} Im < p(z_0), z_0 >$.

2. Main results

Theorem 2.1. *Let* f *be a locally biholomorphic mapping on* B, *with* $f(0) = 0$ *and* *I. Suppose that*

$$\mathrm{Re} < [Df(z)]^{-1} D^2 f(z)(x, x), z ><\| x \|^2 + \frac{1}{2} \left| \left\langle x, \frac{z}{\| z \|} \right\rangle \right|^2,$$

for all $z \in B \setminus \{0\}$ *and* $x \in \mathbb{C}^n \setminus \{0\}$ *with* $\mathrm{Re} \left\langle x, \frac{z}{\|z\|^2} \right\rangle = \frac{1}{2}$, *then* f *is almost* *order 1/2 on* B.

Proof. Let $p(z) = [Df(z)]^{-1}f(z)$, then p is holomorphic on B, $p(0) = 0$ and $Dp(0) = I$. If we show that $\mathrm{Re} < p(z), z > \; > \frac{1}{2} \| z \|^2$, for all $z \in B \setminus \{0\}$, then f will be almost starlike of order $1/2$ on B.

Let $q(z) = p(z) - \frac{1}{2}z, z \in B$, then q is holomorphic on B, $q(0) = 0$ and $Dq(0) = \frac{1}{2}I$. It is enough to show that $\mathrm{Re} < q(z), z >> 0$, for all $z \in B \setminus \{0\}$.

Suppose that $\mathrm{Re} < q(z), z > \not> 0$ in all points of $B \setminus \{0\}$, then according to Lemma 1.2, there exists $z_0 \in B \setminus \{0\}$ and $m \in \mathbb{R}$ such that

$$\mathrm{Re} < q(z_0), z_0 >= 0,$$

$$[\overline{Dq(z_0)}]' z_0 + q(z_0) = m z_0,$$

where $m \leq -\left[\frac{1}{4} + s^2\right]$ and $s = \frac{1}{\|z_0\|^2}\mathrm{Im} < q(z_0), z_0 >$.

Straightforward calculation yields:

$$[Df(z)]^{-1}D^2f(z)(p(z), \cdot) = I - Dp(z), z \in B \setminus \{0\},$$

hence at $z = z_0$, we have

$$[Df(z_0)]^{-1}D^2f(z_0)(p(z_0), p(z_0)) = p(z_0) - Dp(z_0)p(z_0).$$

Multiplying with $z = z_0$ in the both sides of above equality we obtain:

$$< [Df(z_0)]^{-1}D^2f(z_0)(p(z_0), p(z_0)), z_0 >=< p(z_0), z_0 > - < [\overline{Dp(z_0)}]' z_0, p(z_0) >$$

On the other hand, it is clear that

$$[\overline{Dp(z_0)}]' z_0 = (m+1)z_0 - p(z_0),$$

so,

$$< [Df(z_0)]^{-1}D^2f(z_0)(p(z_0), p(z_0)), z_0 >=\| p(z_0) \|^2 - m < p(z_0), z_0 > .$$

Since $m \leq -\left(\frac{1}{4} + s^2\right)$ and $s = \frac{1}{\|z_0\|^2}\mathrm{Im} < p(z_0), z_0 >$, we obtain:

$$\mathrm{Re} < [Df(z_0)]^{-1}D^2f(z_0)(p(z_0), p(z_0)), z_0 >=\| p(z_0) \|^2 - \frac{m}{2} \| z_0 \|^2 \geq$$

$$\geq \| p(z_0) \|^2 + \frac{1}{2}\left|\left\langle p(z_0), \frac{z_0}{\| z_0 \|}\right\rangle\right|^2.$$

Let $x = p(z_0)$, then $x \neq 0$ and $\mathrm{Re} < x, z_0 >= \frac{1}{2} \| z_0 \|^2$. So, the above inequality is a contradiction with the hypothesis.

Hence, $\mathrm{Re} < p(z), z >> \frac{1}{2} \| z \|^2$, for all $z \in B \setminus \{0\}$. □

For $n=1$ in Theorem 2.1 we obtain the following result:

Corollary 2.1. *([3]). Let f be a holomorphic function on the unit disc U, with* $f(0)=f'(0)-1=0$ *and suppose that*

$$\left|\frac{f''(z)}{f'(z)}\right|\leq\frac{3}{2}, z\in U,$$

then $\left|\frac{zf'(z)}{f(z)}-1\right|<1$, *for all* $z\in U$.

Proof. It is clear that, if $\left|\frac{f''(z)}{f'(z)}\right|\leq\frac{3}{2}, z\in U$, then

$$\operatorname{Re}\left[\frac{\bar{z}f''(z)}{f'(z)}x^2\right]<\frac{3}{2}\mid x\mid^2, \tag{1}$$

for all $z\in U\setminus\{0\}$ and $x\in C\setminus\{0\}$ with $\operatorname{Re}[x\bar{z}]=\frac{1}{2}\mid z\mid^2$, so, applying the result of Theorem 2.1, we obtain that $\operatorname{Re}\frac{f(z)}{zf'(z)}>\frac{1}{2}$, for all $z\in U$, which is equivalent with $\left|\frac{zf'(z)}{f(z)}-1\right|<1$, for all $z\in U$. □

With the same arguments as in the proof of Theorem 2.1 we obtain:

Theorem 2.2. *Let f be a locally biholomorphic mapping on B, with* $f(0)=0$ *and* $Df(0)=I$. *Suppose that*

$$|<[Df(z)]^{-1}D^2f(z)(x,x),z>|<\|x\|^2+\frac{1}{2}\left|\left\langle x,\frac{z}{\|z\|}\right\rangle\right|^2,$$

for all $z\in B\setminus\{0\}$ *and* $x\in C^n\setminus\{0\}$ *with* $\operatorname{Re}\left\langle x,\frac{z}{\|z\|^2}\right\rangle=\frac{1}{2}$, *then f is almost starlike of order 1/2 on B.*

Theorem 2.3. *Let f be a locally biholomorphic mapping on B with* $f(0)=0$ *and* $Df(0)=I$. *Suppose that*

$$|\operatorname{Im}<[Df(z)]^{-1}D^2f(z)(x,x),z>|<\left[\operatorname{Im}\left\langle x,\frac{z}{\|z\|}\right\rangle\right]^2,$$

for all $z\in B\setminus\{0\}$ *and* $x\in C^n\setminus\{0\}$ *with* $\operatorname{Re}\left\langle x,\frac{z}{\|z\|^2}\right\rangle=\frac{1}{2}$, *then f is almost sta[illegible] order 1/2 on B.*

Proof. Let $p(z)=[Df(z)]^{-1}f(z), z\in B$, then it is enough to show that

$$\operatorname{Re}<p(z),z>\;>\frac{1}{2}\|z\|^2,$$

for all $z \in B \setminus \{0\}$. If this assertion does not hold, then as in the proof of Theorem 2.1, there exist $z_0 \in B \setminus \{0\}$ and $m \in \mathbf{R}$ such that

$$\mathrm{Re} < p(z_0), z_0 >= \frac{1}{2} \parallel z_0 \parallel^2$$

$$[\overline{Dp(z_0))}]' z_0 + p(z_0) = (m+1) z_0,$$

where $m \leq -\left(\frac{1}{4} + s^2\right)$ and $s = \frac{1}{\|z_0\|^2} \mathrm{Im} < p(z_0), z_0 >$.

Straightforward calculation yields

$$| \mathrm{Im} < [Df(z_0)]^{-1} D^2 f(z_0)(x,x), z_0 >| = -m \mid s \mid\parallel z_0 \parallel^2 \geq \left[\mathrm{Im} \left\langle x, \frac{z_0}{\parallel z_0 \parallel} \right\rangle \right]^2,$$

where $x = p(z_0)$. Since this inequality is a contradiction with the hypothesis, we conclude that $\mathrm{Re} < p(z), z >> \frac{1}{2} \parallel z \parallel^2$, for all $z \in B \setminus \{0\}$. □

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SOME INTEGRAL OPERATORS AND HARDY CLASSES

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Abstract. In the paper one obtains results concerning the Hardy class of the integral operators (3).

1. Introduction

Let A denote the of functions $f(z) = z + a_2z^2 + \dots$ that are analytic in the unit disk U, and S denote the subset of A consisting of univalent functions. During the last several years many authors have used various methods to study different types of integral operators $I(f)$ mapping subsets of S into S. In [4] the authors develop a more general type of integral operator which maps subsets of A into S:

$$I_{\phi,\varphi}(f) = \left[\frac{\beta+\gamma}{z^\gamma\phi(z)}\int_0^z f^\alpha(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}}, \quad \text{where} \quad \alpha,\beta,\gamma,\delta \tag{1}$$

In [6] P.T.Mocanu defines the second order integral operator

$$F(z) = \frac{1}{z^\gamma\phi(z)}\int_0^z \left(t^{\gamma-\beta-1}\frac{\phi(t)}{\varphi(t)}\int_0^t f(s)s^{\beta-1}\varphi(s)ds\right)dt. \tag{2}$$

On the other hand, in [3] O.Fekete determines the Hardy class for the second order integral operator (2). In this paper we obtain some results for the Hardy class of integral operators:

$$I_g(f) = \left[\frac{\beta+\gamma}{z^\gamma}\int_0^z \left[\frac{f(t)}{t}\right]^\alpha \left[\frac{g(t)}{t}\right]^\delta t^{\alpha+\delta-1}dt\right]^{\frac{1}{\beta}}, \tag{3}$$

studied by many authors [5] and the more general operator (1).

We define the n-order integral operator for the integral operators (1) and (3), and we determine the Hardy class.

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2. Preliminaries

For $f \in A$ and $z = r\,e^{i\theta}$ we denote

$$M(r,f) = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta\right)^{\frac{1}{p}} & \text{for} \quad 0 < p < \infty \\ \sup\limits_{0\le\theta\le2\pi} \left|f(re^{i\theta})\right| & \text{for} \quad p = \infty \end{cases}$$

A function is said to be of Hardy class H^p $(0 < p \leq \infty)$ if $M(r,f)$ remains bounded as $r \to 1^-$. H^∞ is the class of bounded analytic functions in the unit disk.

We shall need the following lemmas:

Lemma 1. *If $f' \in H^p, 0 < p < 1$, then $f \in H^{\frac{p}{1-p}}$. If $f' \in H^p, p \geq 1$, then $f \in H^\infty$.*

Lemma 2. *If $f \in H^p$ and $g \in H^2$ then $fg \in H^{\frac{pq}{p+q}}$.*

Lemma 3. *If $f \in H^p$ and $F(z) = \int_0^z f(t)dt$ then $F \in H^{\frac{p}{1-p}}$, for $0 < p < 1$ and $F \in H^\infty$ for $p \geq 1$.*

Lemma 4. *If $f \in H^p, p \geq 1, g'/g \in H^q, q \geq 1$ and L is defined by:*

$$L(f)(z) = \frac{1}{z^C g(z)} \int_0^z f(t)t^{C-1}g(t)dt, f, g \in A, z \in U$$

then

(i) $L(f) \in H^{\frac{pq}{p+q-pq}}$ and $L'(f) \in H^{\frac{pq}{p+q}}$ for $p < \frac{q}{q-1}$;
(ii) $L(f) \in H^\infty$ and $L'(f) \in H^{\frac{pq}{p+q}}$ for $p \geq \frac{q}{q-1}$.

Lemma 5. *If $f \in H^p, p > 1, g'/g \in H^q, q > 1$ and L_g is defined by*

$$L_g(f)(z) = \frac{1}{g(z)} \int_0^z f(t)g'(t)dt, f, g \in A, z \in U$$

then

(i) $L_g(f) \in H^{\frac{pq}{p+q-pq}}$ and $L_g g'(f) \in H^{\frac{pq}{p+q}}$ for $p < \frac{q}{q-1}$;
(ii) $L_g(f) \in H^\infty$ and $L_g g'(f) \in H^{\frac{p}{p+q}}$ for $p \geq \frac{q}{q-1}$.

Lemma 1, Lemma 2 and Lemma 3 are well known (see [2]). Lemma 4 and Lemma 5 were proved in [3].

3. Results for the integral operator I_g

Let be $\alpha, \beta, \gamma, \delta \in R^*_+, f, g \in A$ and I_g defined by (3) we have the following results:

Theorem 1. *$I_g = J_{\frac{1}{\beta}} \circ K \circ I \circ J_{\alpha,g}$ where*

$$\begin{aligned} J_{\alpha,g}(f) &= z\left[\frac{f(z)}{z}\right]^{\alpha}\left[\frac{g(z)}{z}\right]^{\delta}; \\ I(f) &= \int_0^z f(t)t^{\alpha-\delta-2}dt; \\ K(f) &= \frac{\beta+\gamma}{z^{\beta+\gamma-1}}f(z); \\ J_{\frac{1}{\beta}}(f) &= z\left[\frac{f(z)}{z}\right]^{\frac{1}{\beta}}. \end{aligned}$$

Proof. A simple computation yields the results of the theorem. □

Theorem 2. *If $f \in H^p$ and $g \in H^q$ then*

$$J_{\alpha,g}(f) \in H^{\frac{pq}{\alpha q+\delta p}}.$$

Proof. If $f \in H^p$ then $F(z) = z\left[\frac{f(z)}{z}\right]^{\alpha} \in H^{\frac{p}{\alpha}}$ and if $g \in H^q$ then $G(z) = \left[\frac{g(z)}{z}\right]^{\delta} \in H^{\frac{q}{\delta}}$. From Lemma 2, we have $F \cdot G \in H^{\frac{pq}{\alpha q+\delta p}}$ and $J_{\beta,g}(f) \in H^{\frac{pq}{\alpha q+\delta p}}$. □

Theorem 3. *If $f \in H^p, p < 1$, then $I(f) \in H^{\frac{p}{1-p}}$ and if $f \in H^p, p \geq 1$, then $I(f) \in H^{\infty}$.*

Proof. $I(f) = \int_0^z f(t)t^{\alpha+\delta-2}dt$ and $I'(f) = f(z) \cdot z^{\alpha+\delta-2}$.
Hence $I'(f) \in H^p, p < 1$ and $I(f) \in H^{\frac{p}{1-p}}$ or $I'(f) \in H^p, p \geq 1$ and $I(f) \in H^{\infty}$ (Lemma 1). □

Theorem 4. *If $f \in H^p$ then $J_{\frac{1}{\beta}}(f) \in H^{\beta p}$.*

Proof. $J_{\frac{1}{\beta}}(f) = z\left[\frac{f(z)}{z}\right]^{\frac{1}{\beta}}$ obtaining $J_{\frac{1}{\beta}}(f) \in H^{\beta p}$. □

Theorem 5. *If $f \in H^p, g \in H^q, p, q, \alpha, \beta, \delta \in R^*_+$ then*

(i) if $\frac{pq}{\delta p+\alpha q} < 1$ then $I_g(f) \in H^{\frac{\beta pq}{\delta p+\alpha q-pq}}$;

(ii) if $\frac{pq}{\delta p+\alpha q} \geq 1$ then $I_g(f) \in H^{\infty}$.

Proof. (i) If $f \in H^p, g \in H^q$ from Theorem 1 and Lemma 2 we have $J_{\beta,g}(f) \in H^{\lambda}$, $\lambda = \frac{\frac{p}{\alpha}\cdot\frac{q}{\delta}}{\frac{p}{\alpha}+\frac{q}{\delta}} = \frac{pq}{\delta p+\alpha q}$. Hence $I\left(J_{\beta,g}(f)\right) \in H^{\frac{\lambda}{1-\lambda}}$, and $I\left(J_{\beta,g}(f)\right) \in H^{\frac{pq}{\delta p+\alpha q-pq}}$, Using the definition of K we have $K((J_{\beta,g}(f))) \in H^{\frac{pq}{\delta p+\alpha q-pq}}$, and from Theorem 4 $J_{\frac{1}{\beta}}(K(I(J_{\beta,g}(f)))) \in H^{\frac{\beta pq}{\delta p+\alpha q-pq}}$ and $I_g(f) \in H^{\frac{\beta pq}{\delta p+\alpha q-pq}}$;

(ii) If $\frac{pq}{\delta p+\alpha q} \geq 1$ then $I(J_{\beta,g}(f)) \in H^\infty$ (from Lemma 1). Hence $K(I(J_{\beta,g}(f))) \in H^\infty$ and from Theorem 4 $I_g(f) \in H^\infty$. □

4. Results for the integral operator $I_{\phi,\varphi}$

The integral operator $I_{\phi,\varphi}$ defined by (1) was introduced by S.S. Miller, P.T. Mocanu and M.O. Reade in 1978 [5] and more generally in 1991 [4].

Theorem 6. *If the integral operator $I_{\phi,\varphi}$ is defined by (1) then $I_{\phi,\varphi} = J_{\frac{1}{\beta}} \circ I \circ J_\alpha$ where $J_\alpha(f) = z\left[\frac{f(z)}{z}\right]^\alpha$ and $I(f) = \frac{1+\eta}{z^\eta \phi(z)}\int_0^z f(t)\varphi(t)dt$, $\eta = \beta+\gamma-1, \delta = \beta+\gamma-\alpha$.*

Proof. A simple computation yields the results of the theorem. □

Theorem 7. *If $f \in H^p, p \geq \alpha, g'/g \in H^q, q > 1, \alpha > 0$, and $\phi(z) = \varphi(z) = g(z)$ then:*

(i) $I_{\phi,\varphi}(f) \in H^{\frac{\beta pq}{p+\alpha q-pq}}$; for $\frac{p}{\alpha} < \frac{q}{q-1}$;

(ii) $I_{\phi,\varphi}(f) \in H^\infty$ for $\frac{p}{\alpha} \geq \frac{q}{q-1}$ where

$$I_{\phi,\varphi}(f) = \left[\frac{\beta+\gamma}{z^\gamma g(z)}\int_0^z f^\alpha(t)g(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}}.$$

Proof. (i) If $f \in H^p$ then $J_\alpha(f) \in H^{\frac{p}{\alpha}}$ and from Lemma 4 we have $I(J_\alpha(f)) \in H^{\frac{\frac{p}{\alpha}q}{\frac{p}{\alpha}+q-\frac{p}{\alpha}q}}$ and $I(J_\alpha(f)) \in H^{\frac{pq}{p+\alpha q-pq}}$.

Hence $J_{\frac{1}{\beta}}(I(J_\alpha(f))) \in H^{\frac{\beta pq}{p+\alpha q-pq}}$. for $\frac{p}{\alpha} < \frac{q}{q-1}$;

(ii) If $f \in H^p$ then $J_\alpha(f) \in H^{\frac{p}{\alpha}}$ and from Lemma 4 $I(J_\alpha(f)) \in H^\infty, \frac{p}{\alpha} \geq \frac{q}{q-1}$ hence $J_{\frac{1}{\beta}}(I(J_\alpha(f))) \in H^\infty$. □

Theorem 8. *If $f \in H^p, p \geq \alpha, g'/g \in H^q, q > 1$ and $\phi(z) = g(z), \varphi(z) = g'(z)$ then*

(i) $I_{\phi,\varphi}(f) \in H^{\frac{\beta pq}{p+\alpha q-pq}}$ for $\frac{p}{\alpha} \geq \frac{q}{q-1}$;

(ii) $I_{\phi,\varphi}(f) \in H^\infty$ for $\frac{p}{\alpha} \geq \frac{q}{q-1}$.

Proof. Using Lemma 5 and the methods of Theorem 7 the results of Theorem 8 follow. □

Theorem 9. *If $f \in H^p, \varphi \in H^q, \frac{1}{\phi} \in H^r$ then*

(i) if $pq < p+\alpha q$ then $I_{\phi,\varphi}(f) \in H^{\frac{pqr}{\beta(pq+pr+\alpha qr-pqr)}}$;

(ii) if $pq \geq p+\alpha q$ then $I_{\phi,\varphi}(f) \in H^r$.

Proof. A straightforward computation shows that $I_{\phi,\varphi}$, as given by (1), can be written as $I_{\phi,\varphi} = J_{\frac{1}{\beta}} \circ A \circ B \circ J_\alpha$ where

$$J_\alpha(f) = z\left[\frac{f(z)}{z}\right]^\alpha;$$
$$B(f) = \frac{1+n^n}{z}\int_0^z f(t)\varphi(t)t^{n-1}dt, \eta = \beta+\gamma-1;$$
$$A(f) = \frac{1}{\phi(z)}f(z).$$

(i) Since $f \in H^p$ we have $J_\alpha(f) \in H^{\frac{p}{\alpha}}$. From Lemma 2 we have $f^\alpha(z)\varphi(z) \in H^\lambda$, $\lambda = \frac{\frac{p}{\alpha}q}{\frac{p}{\alpha}+q} = \frac{pq}{p+\alpha q}$ and from Lemma 3 we have $B(J_\alpha(f)) \in H^\mu$, where $\mu = \frac{\frac{pq}{p+\alpha q}}{1-\frac{pq}{p+\alpha q}} = \frac{pq}{p+\alpha q-pq}$ for $pq < p+\alpha q$.

From Lemma 2 we obtain $A(B(J_\alpha(f))) \in H^\nu$ where

$$\nu = \frac{r\frac{pq}{p+\alpha q-pq}}{r+\frac{pq}{p+\alpha q-pq}} = \frac{pqr}{pq+pr+\alpha qr-pqr};$$
$$J_{\frac{1}{\beta}}(A(B(J_\alpha(f)))) \in H^{\frac{pqr}{\beta(pq+pr+\alpha qr-pqr)}}.$$

Hence $I_{\phi,\varphi}(f) \in H^{\frac{pqr}{\beta(pq+pr+\alpha qr-pqr)}}$.

(ii) If $pq \geq p+\alpha q$ then $B(J_\alpha(f)) \in H^\infty$ hence $F(z) = B(J_\alpha(f(z)))$ is bounded in the unit disk

$$M_p(r, I_{\phi,\varphi}) = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{\phi(re^{i\theta})}F(re^{i\theta})\right|^p d\theta \leq K\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{\phi(re^{i\theta})}\right|^p d\theta,$$

K - constant. Hence $I_{\phi,\varphi} \in H^r$.

□

5. The n-order integral operators I_g and $I_{\phi,\varphi}$

Let be f, g_i analytic functions, $\alpha_i, \beta_i, \gamma_i, \delta_i \in R_+^*, i \in \{1, 2, \ldots, n\}$ and

$$I_{g_i}(f) = \left[\frac{\beta_i+\gamma_i}{z^{\gamma_i}}\int_0^z\left[\frac{f(t)}{t}\right]^{\alpha_i}\left[\frac{g_i(t)}{t}\right]^{\delta_i} t^{\alpha_i-\delta_i-1}dt\right]^{\frac{1}{\beta_i}}.$$

We define the n-order integral operator $I_g^n = I_{g_n} \circ I_{g_{n-1}} \circ \cdots \circ I_{g_1}, n \in \mathrm{N}$

Theorem 10. *If $f \in H^p, g_i \in H^{q_i}, \alpha_i, \beta_i, \gamma_i, \delta_i \in R_+^*$ then*

(i) $I_g^n(f) \in H^{\lambda_n}$ where $\lambda_n = \frac{\beta_n q_n \lambda_{n-1}}{\alpha_n q_n + \lambda_{n-1}(\delta_n - q_n)}, \lambda_0 = p, n \in \mathrm{N}^*$ if $\alpha_i q_i + \lambda_{i-1}(\delta_i - q_i) > 0, i \in 1, 2, \ldots, n$

$I_g^n(f) \in H^\infty$ for $\alpha_n q_n + \lambda_{n-1}(\delta_n - q_n) \leq 0$ and $\alpha_i q_i + \lambda_{i-1}(\delta_i - q_i) > 0$.

Proof. (i) If $f \in H^p, g_i \in H^{q_i}, \alpha_i, \beta_i, \gamma_i, \delta_i \in R_+^*$ from Theorem 5 we have $I_g g'(f) = I_{g_i}(f) \in H^{\lambda_1}, \lambda_1 = \frac{\alpha_1 q_1 p}{\alpha_1 q_1 + p(\delta_1 - q_1)}$ hence $\lambda_1 = \frac{\beta_1 q_1 \lambda_0}{\alpha_1 q_1 + \lambda_0(\delta_1 - q_1)}$. We suppose that $I_g^{n-1}(f) \in H^{\lambda_{n-1}}$ and from Theorem 5 we obtain $I_g^n(f) = I_{g_n}\left(I_g^{n-1}(f)\right) \in H^{\lambda_n}$, $\lambda_n = \frac{\beta_n q_n \lambda_{n-1}}{\alpha_n q_n + \lambda_{n-1}(\delta_n - q_n)}$ if $\alpha_i q_i + \lambda_{i-1}(\delta_i - q_i) > 0$, $(\forall i) i \in 1, 2, \ldots, n$.

(ii) If $\alpha_n q_n + \lambda_{n-1}(\delta_n - q_n) \le 0$ and $\alpha_i q_i + \lambda_{i-1}(\delta_i - q_i) > 0$, $I_g^n(f) \in H^\infty, n \in \mathrm{N}$. □

Let be f, ϕ_i, φ_i analytic functions, $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathrm{R}_+^*$ and

$$I_{\phi_i,\varphi_i}(f) = \left[\frac{\beta_i + \gamma_i}{z^{\gamma_i}\phi_i(z)} \int_0^z f^{\alpha_i}(t)\varphi_i(t)t^{\delta_i - 1}dt\right]^{\frac{1}{\beta_i}}.$$

We define the n-order integral operator $I_{\phi,\varphi}^n$:

$$I_{\phi,\varphi}^n(f) = I_{\phi_n,\varphi_n} \circ I_{\phi_{n-1},\varphi_{n-1}} \circ \cdots \circ I_{\phi_1,\varphi_1}, n \in \mathrm{N}^*.$$

Theorem 11. *If $f \in H^p, \varphi_i \in H^{q_i}, \frac{1}{\phi_i} \in H^{r_i}, \alpha_i, \beta_i, \delta_i \in \mathrm{R}_+^*$ then*

(i) $I_{\phi,\varphi}^n(f) \in H^{\lambda_n}$ where λ_n is defined by the recurrent formula:

$$\begin{cases} \lambda_n = \frac{\lambda_{n-1}}{\beta_n} \cdot \frac{q_n r_n}{\alpha_n q_n r_n + \lambda_{n-1}(q_n + r_n - q_n r_n)}, & n \in \mathrm{N}^* \\ \lambda_0 = p \end{cases} \tag{4}$$

where $\alpha_i \cdot q_i \cdot r_i + \lambda_{i-1}(q_i + r_i - q_i r_i) > 0, i \in 1, 2, \ldots, n$;

(ii) $I_{\phi,\varphi}^n(f) \in H^\infty$ if $\alpha_i \cdot q_i \cdot r_i + \lambda_{i-1}(q_i + r_i - q_i r_i) > 0, i \in 1, 2, \ldots, n$ and $\alpha_n \cdot q_n \cdot r_n + \lambda_{n-1}(q_n + r_n - q_n r_n) \le 0$.

Proof. (i) Since $f \in H^p, \varphi_1 \in H^{q_1}, \frac{1}{\phi_1} \in H^{r_1}$ from Theorem 9 we have $I_{\phi_1,\varphi_1}(f) \in H^{\lambda_1}$, where $\lambda_1 = \frac{p}{\beta_1} \cdot \frac{q_1 r_1}{\alpha_1 q_1 r_1 + p(q_1 + r_1 - q_1 r_1)}$ for $\alpha_1 \cdot q_1 \cdot r_1 + p(q_1 + r_1 - q_1 r_1) > 0$. We suppose that $I_{\phi,\varphi}^{n-1} \in H^{\lambda_{n-1}}$, then $I_{\phi,\varphi}^n(f) = I_{\phi_n,\varphi_n}\left(I_{\phi,\varphi}^{n-1}(f)\right) \in H^{\lambda_n}$ where $\lambda_n = \frac{\lambda_{n-1}}{\beta_n} \cdot \frac{q_n r_n}{\alpha_n q_n r_n + \lambda_{n-1}(q_n + r_n - q_n r_n)}$ for $\alpha_i \cdot q_i \cdot r_i + \lambda_{i-1}(q_i + r_i - q_i r_i) > 0, i \in 1, 2, \ldots, n$.

(ii) $\alpha_i \cdot q_i \cdot r_i + \lambda_{i-1}(q_i + r_i - q_i r_i) > 0, i \in 1, 2, \ldots, n-1$ and $\alpha_n \cdot q_n \cdot r_n + \lambda_{n-1}(q_n + r_n - q_n r_n) \le 0$ then from Theorem 8 $I_{\phi,\varphi}^n(f) \in H^\infty$.

Remark 1. If $(\exists)k, k \in 1, 2, \ldots, n-1, \alpha_k \cdot q_k \cdot r_k + \lambda_{k-1}(q_k + r_k - q_k r_k) \le 0$ $I_{\phi,\varphi}^k(f) \in H^\infty$ and to determine the Hardy classes we have $I_{\phi,\varphi}^{k+1}(f) \in H^{\lambda_{k+1}}$ wh

$$\lambda_{k+1} = \frac{1}{\beta_{k+1}} \frac{r_{k+1} q_{k+1} \frac{k+1}{1 - q_{k+1}}}{r_{k+1} + \frac{q_{k+1}}{1 - q_{k+1}}} = \frac{1}{\beta_{k+1}} \frac{r_{k+1} q_{k+1}}{r_{k+1} + q_{k+1} - r_{k+1} q_{k+1}}$$

and returning to (4) we obtain results for $\lambda_0 = \lambda_{k+1}$.

Theorem 12. *If* $f \in H^p, \frac{g_i i'}{g_i} \in H^{q_i}, q_i > 1, \alpha_i, \beta_i \in \mathbb{R}_+^*$ *and* $\phi_i(z) = \varphi_i(z) = g_i(z)$, z U *then*

(i) $I^n_{\phi,\varphi}(f) \in H^{\lambda_n}$ *where*

$$\begin{cases} \lambda_n = \lambda_{n-1}\frac{\beta_n q_n}{\alpha_n q_n + \lambda_{n-1}(1-q_n)}, & n \in \mathbb{N}^* \\ \lambda_0 = p \end{cases} \tag{5}$$

for $\alpha_i q_i + \lambda_{i-1}(1-q_i) > 0$ *and* $i \in 1, 2, \ldots, n$;

(ii) $I^n_{\phi,\varphi}(f) \in H^\infty$ *for* $\alpha_i q_i + \lambda_{i-1}(1-q_i) > 0, i \in 1, 2, \ldots, n-1$ *and* $\alpha_n q_n + \lambda_{n-1}(1-q_n) \leq 0$.

Proof. (i) Since $f \in H^p, \frac{g_i i'}{g_i} \in H^{q_i}$, from Theorem 7 we have $I^1_{\phi,\varphi}(f) = I_{\phi_1,\varphi_1}(f) \in H^{\lambda_1}, \lambda_1 = p\frac{\beta_1 q_1}{\alpha_1 q_1 + p(1-q_1)}$ for $\alpha_1 q_1 + p(1-q_1) > 0$. We suppose that $I^{n-1}_{\phi,\varphi}(f) \in H_{\lambda_{n-1}}$, then $I^n_{\phi,\varphi}(f) = I_{\phi_n,\varphi_n}\left(I^{n-1}_{\phi,\varphi}(f)\right) \in H^{\lambda_n}$ where $\lambda_n = \lambda_{n-1}\frac{\beta_n q_n}{\alpha_n q_n + \lambda_{n-1}(1-q_n)}$ for $\alpha_n q_n + \lambda_{n-1}(1-q_n) > 0$;

(ii) From Theorem 7, if $\alpha_n q_n + \lambda_{n-1}(1-q_n) \leq 0$ we have $I^n_{\phi,\varphi}(f) \in H^\infty$. □

Remark 2. If $(\exists)k, k \in 1, 2, \ldots, n-1, \alpha_k q_k + \lambda_{k-1}(1-q_k) \leq 0$ then $I^k_{\phi,\varphi}(f) \in H^\infty$ and to determine the Hardy classes we have $I^{k+1}_{\phi,\varphi}(f) \in H^{\lambda_{k+1}}$ where $\lambda_{k+1} = q_{k+1}$ and returning to (5) we obtain results for $\lambda_0 = \lambda_{k+1}$.

Remark 3. From Theorem 8 we obtain an analogous result for $f \in H^p, \frac{g_i i'}{g_i} \in H^{q_i}, \alpha_i, \beta_i \in \mathbb{R}_+^*$ and $\phi_i(z) = g(z), \varphi_i(z) = g_i i'(z)$.

6. Some particular cases

For $n = 2, \alpha_i = 1, \beta_i = 1$, we obtain the second order integral operators studyed in [3]. In that case $I^2_{\phi,\varphi}(f)$ is

$$I^2_{\phi,\varphi}(f) = \frac{1}{z^{\gamma_2}\phi_2(z)}\int_0^z \left(t^{\gamma_1-\gamma_2-1}\frac{\varphi_2(t)}{\varphi_1(t)}\int_0^t f(s)s^{\gamma_2-1}\varphi_1(s)ds\right)dt.$$

For $\phi_2(z) = \varphi_2(z) = g_2(z), \phi_1(z) = \varphi_1(z) = g_1(z)$ and if $f \in H^p$, $\frac{g_i i'}{g_i} \in H^{q_i}$, $i = 1, 2, q_i > 1$ Theorem 11 we obtain $I^2_{\phi,\varphi}(f) \in H^{\lambda_2}$ where

$$\begin{aligned} \lambda_2 &= \lambda_1 q_2 \frac{2}{q_2 + \lambda_1(1-q_2)} = pq_1\frac{1}{q_1 + p(1-q_1)} \cdot q_2 \frac{2}{q_2 + \frac{pq_1}{q_1+p(1-q_1)}(1-q_2)} = \\ &= \frac{q_1 q_2 p}{q_1 q_2 + pq_2 + pq_1 - 2pq_1 q_2}. \end{aligned}$$

[illegible] (z) [illegible] $g_i i'(z)$ and if $f \in H^p, g_i i'/g_i \in H^{q_i}, q_i > 1$ from Remark [illegible] with [illegible] λ_2. These results were obtained in [3].

[illegible] dev[illegible] starlikeness the operator $F(z) = \left[\frac{\beta+\gamma^\gamma}{z}\int_0^z f(t)^{\beta+\gamma-2}dt\right]^{\frac{1}{\beta}}$ [illegible] $\beta, \gamma \in$ R, $\beta > 0, \beta + \gamma > 0$.

[illegible] Ref[illegible] 0 *then* $F \in H^{\frac{\beta p}{1-p}}, p < 1$.

[illegible]

$$= \left[\frac{\beta+\gamma^\gamma}{z} f(z)\right]^{\frac{1}{\gamma}}$$

$$[illegible] = \int_0^z f(t)t^{\beta+\gamma-2}dt.$$

[illegible] Theorem 4 $J_{\frac{1}{\beta}} \in$ [illegible] Re$f > 0, f \in H^p, p < 1$, and from theorem 3 we [illegible] $\lambda = \frac{\beta p}{1-p}, p < 1$. □

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A NEW GENERALIZATION OF PFALTZGRAFF'S INTEGRAL OPERATOR

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Abstract. The paper is dealing with a kind of operators which preserve a specific class of functions and construct a generalization of such of operator, invented by Pfaltzgraff.

Let A denote the class of functions f which are analytic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ with $f(0) = 0$ and $f'(0) = 1$

Let S denote the class of functions $f \in A$, f univalent in U.

Many authors studyed the problem of integral operators which preserve the class S. In this sence, the result due to Pfaltzgraff ([5]) is well-known.

Theorem A. *([5]) Let $f \in S$, $\delta \in \mathbb{C}$. If $|\delta| \leq 1/4$, then the function*

$$F(z) = \int_0^z (f'(u))^\delta du \tag{1}$$

is univalent in U.

In the papers [2], [4] were obtained other integral operators which preserve the univalence and in the same time they generalize, in different manner, the result due to Pfaltzgraff.

Theorem B. *([4]) Let $f \in S, \delta \in \mathbb{C}, n \in \mathbb{N}$. If $|\delta| \leq \frac{1}{4n}$, then the function*

$$F_{\delta,n}(z) = \int_0^z (f'(u^n))^\delta du \tag{2}$$

is univalent in U.

Theorem C. *([2]) Let $f \in S, \alpha, \in \mathbb{C}$. If $|\alpha - 1| < 1$ and $|\delta| \leq (1 - |\alpha - 1|)/4$, then the function $F_{\alpha,\delta}$,*

$$F_{\alpha,\delta}(z) = \left(\alpha \int_0^z u^{\alpha-1} (f'(u))^\delta du\right)^{1/4} \tag{3}$$

is analytic and univalent in U.

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In this paper we obtain a generalization of the Theorem A,B and C. First let us state some results which will be used in the sequel.

Lemma A. *([1]) Let $f \in S$. Then*

$$\left| -2 \mid z \mid^2 + (1- \mid z \mid^2) \frac{zf''(z)}{f'(z)} \right| \leq 4 \mid z \mid, \ (\forall) z \in U$$

Theorem D. *([3]). Let $g \in A$. Let α, β and c be complex numbers,* $\text{Re}\,\alpha > 0$, $\text{Re}(\alpha + \beta) > 0$, $\text{Re}\,\beta/\alpha > -1/2$, $\mid c \mid < 1$ *and* $\mid c(\alpha+\beta) + \beta \mid + \mid \beta \mid \leq \mid \alpha + \beta \mid$. *If*

$$\left| c \mid z \mid^{2(\alpha+\beta)} + \frac{1- \mid z \mid^{2(\alpha+\beta)}}{\alpha+\beta} \left(\frac{zg''(z)}{g'(z)} - \beta \right) \right| \leq 1, \tag{4}$$

for all $z \in U \setminus \{0\}$, then the function G_α,

$$G_\alpha(z) = \left(\alpha \int_0^z u^{\alpha-1} g'(u) du \right)^{1/\alpha} \tag{5}$$

is analytic and univalent in U.

We shall prove our result using Theorem D in the particular case $\beta = n\text{-}\alpha$, where $n \in \mathbf{N}$ For this choice, from Theorem D we get the following:

Corollary 1. *Let $g \in A$. Let α, c be complex numbers and let n be a positive integer number. If* $\mid \alpha - n \mid < n$, $\mid c \mid < 1$, $\mid cn + n - \alpha \mid + \mid n - \alpha \mid \leq n$ *an*

$$\left| c \mid z \mid^{2n} + \frac{1- \mid z \mid^{2n}}{n} \left(\frac{zg''(z)}{g'(z)} + \alpha - n \right) \right| \leq 1, \tag{4'}$$

for all $z \in U \setminus \{0\}$, then the function G_α,

$$G_\alpha(z) = \left(\alpha \int_0^z u^{\alpha-1} g'(u) du \right)^{1/\alpha}$$

is analytic and univalent in U.

Theorem 1. *Let $f \in S$, $n \in \mathbf{N}$, $\alpha, \delta \in \mathbf{C}$. If $\mid \alpha - n \mid < n$ and $\mid \delta \mid \leq \frac{n - \mid \alpha - n \mid}{4n}$, then the function $F_{\alpha,\delta,n}$,*

$$F_{\alpha,\delta,n}(z) = \left(\alpha \int_0^z u^{\alpha-1} (f'(u^n))^\delta du \right)^{1/\alpha} \tag{6}$$

is analytic and univalent in U.

Proof. Because the function f is univalent in U, we can choose the analytic branch of $(f'(u^n))^\delta$ equal to 1 at the origin and then the function g belongs to A, where

$$g(z) = \int_0^z (f'(u^n))^\delta du$$

We have

$$\frac{zg''(z)}{g'(z)} = \delta n \frac{z^n f''(z^n)}{f'(z^n)} \tag{7}$$

In view of (7), from (4') we get

$$\left| c \mid z \mid^{2n} + \frac{1 - \mid z \mid^{2n}}{n} \left(\frac{zg''(z)}{g'(z)} + \alpha - n \right) \right| =$$

$$= \left| \delta \left(-2 \mid z \mid^{2n} + (1 - \mid z \mid^{2n}) \frac{z^n f''(z^n)}{f'(z^n)} \right) + \left(c + 2\delta + 1 - \frac{\alpha}{n} \right) \mid z \mid^{2n} + \frac{\alpha - n}{n} \right| .$$

If $c = -2\delta - 1 + \alpha/n$, from $\mid \alpha - n \mid < n$ and $\mid \delta \mid \leq (n - \mid \alpha - n \mid)/(4n)$ it results that $\mid c \mid < 1$ and also $\mid cn + n - \alpha \mid + \mid n - \alpha \mid = \mid 2n\delta \mid + \mid \alpha - n \mid < n$.

Using Lemma A and in view of assertion $\mid \delta \mid \leq (n - \mid \alpha - n \mid)/(4n)$ it follows that

$$\left| c \mid z \mid^{2n} + \frac{1 - \mid z \mid^{2n}}{n} \left(\frac{zg''(z)}{g'(z)} + \alpha - n \right) \right| \leq 4 \mid \delta \mid + \frac{\mid \alpha - n \mid}{n} \leq 1$$

From Corollary 1, we conclude that the function $F_{\alpha,\delta,n}$

$$F_{\alpha,\delta,n}(z) = \left(\alpha \int_0^z u^{\alpha-1} g'(u) du \right)^{1/\alpha} = \left(\alpha \int_0^z u^{\alpha-1} (f'(u^n))^\delta du \right)^{1/\alpha}$$

is analytic and univalent in U. □

Remarks.

1. For $\alpha = 1$ from Theorem 1 we obtain Theorem B.
2. For $n = 1$ Theorem 1 becomes Theorem C.
3. From $\alpha = 1$ and $n = 1$ from Theorem 1 we refined Theorem A.

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ON CERTAIN NONLINEAR HIGHER ORDER INTEGRODIFFERENTIAL EQUATIONS

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Abstract. In this paper we study the existence, uniqueness and error estimates of solutions of certain nonlinear higher order integrodifferential equations with the given initial conditions. We shall use Banach fixed point theorem in an appropriate space of functions endowed with Bielecki type norm establish our results.

1. Introduction

In this paper we consider the nonlinear higher order integrodifferential equations of the forms:

$$\left(r(t)y^{(n-1)}(t)\right)' = F\left(t, y(t), \int_0^t \int_0^{s_1} \cdots \int_0^{S_{n-2}} \frac{1}{r(s_{n-1})} \times \right. \\ \left. \times \int_0^{s_{n-1}} f\left(s_n, y(s_n)\right) ds_n ds_{n-1} \ldots ds_1\right), \tag{1.1}$$

with the given initial conditions

$$y(0) = y_0, \ y^{(i-1)}(0) = 0, \ i = 2, 3, \ldots, n; \tag{1.2}$$

$$\left(r(t)y'(t)\right)^{(n-1)} = G\left(t, y(t), \int_0^t \frac{1}{r(s_1)} \int_0^{s_1} \times \cdots \times \right. \\ \left. \times \int_0^{s_{n-1}} g\left(s_n, y(s_n)\right) ds_n ds_{n-1} \ldots ds_1\right), \tag{1.3}$$

with the given initial conditions

$$y(0) = y_0, \left(r(0)y'(0)\right)^{i-2} = 0, i = 2, 3, \ldots, n; \tag{1.4}$$

$$\left(r(t)y^{(n)}(t)\right)^{(n)} = H\left(t, y(t), \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \frac{1}{r(s_n)} \int_0^{s_n} \int_0^{\sigma_1} \times \cdots \times \right. \\ \left. \times \int_0^{\sigma_{n-1}} h\left(\sigma_n, y(\sigma)\right) d\sigma_{n-1} \ldots d\sigma_1 \times ds_n ds_{n-1} \ldots ds_1\right), \tag{1.5}$$

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with the given initial conditions

$$\begin{aligned} y(0) = y_0, \quad y^{(i-1)}(0) &= 0, \quad i = 2, 3, \ldots, n, \\ \left(r(0)y^{(n)}(0)\right)^{(i-1)} &= 0, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{1.6}$$

where $r(t)$ is a real-valued positive continuous function defined for $t \in I = [0, \infty)$; $f, g, h : I \times R \to R$; $F, G, H : I \times R \times R \to R$ are continuous and y_0 is a given real constant and R denotes the set of real numbers.

In the past few years there has been a great deal of interest in the study of the oscillatory and asymptotic behavior of the solutions of the equations of the above forms when the integral terms involved therein are absent, see [2,5,6,11,12]. Although, a great many papers have appeared related to the special versions of such equations, it seems to us, however, that very little is known about the global existence of the solutions of such equations. Our objective here is to establish results on global existence, uniqueness and error estimates of the solutions of equations (1.1)-(1.2), (1.3)-(1.4) and (1.5)-(1.6) belonging to an appropriate space of functions. The analysis used in the proof is based on application of the Banach fixed point theorem in an appropriate space of functions endowed with Bielecki type norm, see [1,3,4,8-10]. In fact, our work in the present paper is motivated in part by the interesting results given by Morchalo [7] for special versions of equations (1.1) and (1.3) with $r(t) = 1$ and the given boundary conditions and the study of the special versions of such equations by various investigators in [2,5,6,11,12].

2. Statement of results

Let B be the space of continuous functions $\emptyset : I \to R$ such that

$$| \Phi(t) | = O(\exp(Lt)), \tag{2.1}$$

where L is a positive constant. In the space B we define the norm (see, Bielecki [1], also [3,4,8-10])

$$\| \Phi \| = \sup_{tI} [| \phi(t) \exp(-Lt)|]. \tag{2.2}$$

It is easy to see that B with norm defined in (2.2) is a Banach space. We note that the condition (2.1) implies that there exists a nonnegative constant M such that

$$| \phi | \leq M \exp(Lt), tI. \tag{2.3}$$

Using (2.3) in (2.2), we observe that

$$\| \phi \| \leq M. \tag{2.4}$$

We are now in a position to formulate our main results to be proved in this paper.

Theorem 1. *Suppose that*

(H_1) there exist nonnegative continuous functions $p_1(t)$, $q_1(t)$ defined for $t \in I$ and a constant $\alpha \geq 0$ such that

$$| F(t,y,z) - F(t,\bar{y},\bar{z}) | \leq p_1(t) (| y - \bar{y} | + | z - \bar{z} |),$$
$$| f(t,y) - f(t,\bar{y}) | \leq q_1(t) | y - \bar{y} |,$$

and

$$\int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} p_1(t_n) [\exp(Lt_n)+$$
$$+ \int_0^{t_n} \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \int_0^{s_{n-1}} q_1(s_n) \exp(Ls_n) ds_n ds_{n-1} \dots ds_1 \Big] \times$$
$$\times dt_n dt_{n-1} \dots dt_1 \leq \alpha \exp(Lt),$$

for $t \in I$;

(H_2) there exists a constant $\beta \geq 0$ such that

$$| y_0 | + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} \left| F\left(t_n, 0, \int_0^{t_n} \int_0^{s_1} \cdots \times \right.\right.$$
$$\left.\left. \times \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \int_0^{s_{n-1}} f(s_n, 0) ds_n ds_{n-1} \dots ds_1 \right) \right| \times$$
$$\times dt_n dt_{n-1} \dots dt_1 \leq \beta \exp(Lt),$$

for $t \in I$.

If $\alpha < 1$, then there exists a unique solution $y \in B$ of the equations (1.1)-(1.2). Further, for any $y_0(t) \in B$, the sequence $\{y_m(t)\}$ given successively by

$$y_m(t) = y_0 + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} F(t_n, y_{m-1}(t_n), \tag{2.5}$$
$$\int_0^{t_n} \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \int_0^{s_{n-1}} f(s_n, y_{m-1}(s_n)) \times$$
$$\times ds_n ds_{n-1} \dots ds_1) dt_n dt_{n-1} \dots dt_1,$$

for m = 1, 2, ..., converges in B to a unique solution y of the equations (1.1)-(1.2). Moreover,

$$\| y_m - y \| \leq \frac{\alpha^m}{1-\alpha} \| y_1 - y_0 \|, \tag{2.6}$$

for m = 1, 2, ..., where $0 \leq \alpha < 1$ is a constant defined as above.

Theorem 2. *Suppose that*

(H_3) There exist nonnegative continuous functions $p_2(t), q_2(t)$ defined for $t \in I$ and a constant $\alpha \geq 0$ such that

$$| G(t,y,z) - G(t,\bar{y},\bar{z}) | \leq p_2(t) (| y - \bar{y} | + | z - \bar{z} |),$$
$$| g(t,y) - g(t,\bar{y}) | \leq q_2(t) | y - \bar{y} |,$$

and

$$\int_0^t \frac{1}{r(t_1)} \int_0^{t_1} \cdots \int_0^{t_{n-1}} p_2(t_n) \left[\exp(Lt_n) + \int_0^{t_n} \frac{1}{r(s_1)} \int_0^{s_1} \cdots \times \right.$$
$$\left. \times \int_0^{s_{n-1}} q_2(s_n) \exp(Ls_n) ds_n ds_{n-1} \ldots ds_1 \right] dt_n dt_{n-1} \ldots dt_1 \leq \alpha \exp(Lt);$$

(H_4) there exists a nonnegative constant β such that

$$| y_0 | + \int_0^t \frac{1}{r(t_1)} \int_0^{t_1} \cdots \int_0^{t_{n-1}} \left| G \left(t_n, 0, \int_0^{t_n} \frac{1}{r(s_1)} \int_0^{s_1} \cdots \times \right.\right.$$
$$\left.\left. \times \int_0^{s_{n-1}} g(s_n, 0) ds_n ds_{n-1} \ldots ds_1 \right) \right| dt_n dt_n \ldots dt_1 \leq \beta \exp(Lt)$$

for $t \in I$.

If $\alpha < 1$, then there exists a unique solution $y \in B$ of the equations (1.3)-(1.4). Further, for any $y_0(t) \in B$, the sequence $\{y_m(y)\}$ given successively by

$$y_m(t) = y_0 + \int_0^t \frac{1}{r(t_1)} \int_0^{t_1} \cdots \int_0^{t_{n-1}} G \left(t_n, y_{m-1}(t_n), \int_0^{t_n} \frac{1}{r(s_1)} \times \right. \tag{2.7}$$
$$\left. \times \int_0^{s_1} \cdots \int_0^{s_{n-1}} g\left(s_n, y_{m-1}(s_n)\right) ds_n ds_{n-1} \ldots ds_1 \right) \times dt_n dt_{n-1} \ldots dt_1,$$

for m = 1, 2, ... converges in B to a unique solution y of the equations (1.3)-(1.4). Moreover

$$\| y_m - y \| \leq \alpha^m \frac{m}{1-\alpha} \| y_1 - y_0 \|, \tag{2.8}$$

for m = 1, 2, ..., where $0 \leq \alpha < 1$ is a constant as defined above.

Theorem 3. *Suppose that*

(H_5) there exists nonnegative continuous functions $p_3(t)$, $q_3(t)$ defined for $t \in I$ and a constant $\alpha \geq 0$ such that

$$| H(t,y,z) - H(t,\bar{y},\bar{z}) | \leq p_3(t)(| y-\bar{y} | + | z-\bar{z} |),$$
$$| h(t,y) - h(t,\bar{y}) | \leq q_3(t) | y-\bar{y} |,$$

and

$$\int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \frac{1}{r(t_n)} \int_0^{t_n} \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} p_3(\tau_n) [\exp(L\tau_n) + \\ + \int_0^{\tau_n} \int_0^{s_1} \cdots \int_0^{s_{n-1}} \frac{1}{r(s_n)} \int_0^{s_n} \int_0^{\sigma_1} \cdots \int_0^{\sigma_{n-1}} q_3(\sigma_n) \exp(L\sigma) \times \\ \times d\sigma_n d\sigma_{n-1} \dots d\sigma_1 ds_n ds_{n-1} \dots ds_1] \, d\tau_n d\tau_{n-1} \dots d\tau_1 \times dt_n dt_{n-1} \dots dt_1 \leq \alpha \exp(Lt),$$

for $t \in I$;

(H_6) there exists a constant $\beta \geq 0$ such that

$$| y_0 | + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \frac{1}{r(t_n)} \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} \Big| H(\tau_n, 0, \int_0^{\tau_n} \int_0^{s_1} \cdots \times \\ \times \int_0^{s_{n-1}} \frac{1}{r(s_n)} \int_0^{s_n} \int_0^{\sigma_1} \cdots \int_0^{\sigma_{n-1}} h(\sigma_n, 0) d\sigma_n d\sigma_{n-1} \dots d\sigma_1 \times \\ \times ds_n ds_{n-1} \dots ds_1) \Big| \, d\tau_n d\tau_{n-1} \dots d\tau_1 dt_n dt_{n-1} \dots dt_1 \leq \beta \exp(Lt),$$

for $t \in I$. If $\alpha < 1$, then there exists a unique solution $y \in B$ of the equations (2.5)-(2.6). Further, for $y_0(t) \in B$, the sequence $\{y_m(t)\}$ given successively by

$$\begin{aligned} y_m(t) &= y_0 + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \frac{1}{r(t_n)} \int_0^{t_n} \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} H(\tau_n, y_{m-1}(\tau_n) \times \\ &\quad \times \int_0^{\tau_n} \int_0^{s_1} \cdots \int_0^{s_{n-1}} \frac{1}{r(s_n)} \int_0^{s_n} \int_0^{\sigma_1} \cdots \int_0^{\sigma_{n-1}} h(\sigma_n, y_{m-1}(\sigma_n)) \times \\ &\quad \times d\sigma_n d\sigma_{n-1} \dots d\sigma_1 ds_n ds_{n-1} \dots ds_1) d\tau_n d\tau_{n-1} \dots d\tau_1 dt_n dt_{n-1} \dots dt_1, \end{aligned} \tag{2.9}$$

for $m = 1, 2, \dots$, converges is B to a unique solution y of the equations (1.5)-(1.6). Moreover,

$$\| y_m - y \| \leq \alpha^m \frac{m}{1-\alpha} \| y_1 - y_0 \|, \tag{2.10}$$

for $m = 1, 2, \dots$, where $0 \leq \alpha < 1$ is a constant as defined above.

3. Proofs of Theorems 1-3

Since the proofs resemble one another, we give the details of the proof of Theorem 1 only; the proofs of the Theorems 2 and 3 can be completed by following the proof of Theorem 1 with suitable modifications.

For $y \in B$ we define the operator T by

$$\begin{aligned} Ty(t) &= y_0 + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} F\left(t_n, y(t_n), \int_0^{t_n} \int_0^{s_1} \cdots \times \right. \\ &\quad \left. \times \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \int_0^{s_{n-1}} f(s_n, y(s_n))\, ds_n ds_{n-1} \ldots ds_1 \right) \times dt_n dt_{n-1} \ldots dt_1, \end{aligned} \tag{3.1}$$

for $t \in I$. Clearly, the solution of equations (1.1)-(1.2) is a fixed point of the operator equation

$$Ty(t) = (t). \tag{3.2}$$

Now we shall prove that T maps B into itself. From (3.1), (H_1), (H_2), (2.2), (2.4), it follows that

$$\begin{aligned} |\, Ty(t) \,| &\le |\, y_0 \,| + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} |\, F(t_n, y(t_n), \\ &\quad \int_0^{t_n} \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \int_0^{s_{n-1}} f(s_n, y(s_n)) ds_n ds_{n-1} \ldots ds_1) \\ &\quad - F(t_n, 0, \int_0^{t_n} \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \int_0^{s_{n-1}} f(s_n, 0) \times \\ &\quad \times ds_n ds_{n-1} \ldots ds_1) \,|\, dt_1 dt_{n-1} \ldots dt_1 \\ &\quad + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} |\, F(t_n, 0, \int_0^{t_n} \int_0^{s_1} \cdots \times \\ &\quad \times \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \int_0^{s_{n-1}} f(s_n, 0) ds_n ds_{n-1} \ldots ds_1) \,|\, \times dt_n dt_{n-1} \ldots dt_1 \\ &\le \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} p_1(t_n) [\,|\, y(t_n) \,| \times \\ &\quad \times \exp(-Lt_n) \exp(Lt_n) + \int_0^{t_n} \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \times \\ &\quad \times \int_0^{s_{n-1}} q_1(s_n) \,|\, y(s_n) \,|\, \exp(-Ls_n) \exp(Ls_n) ds_n ds_{n-1} \ldots ds_1] \times \\ &\quad \times dt_n dt_{n-1} \ldots dt_1 + \beta \exp(Lt) \end{aligned} \tag{3.3}$$

$$\leq \| y \| \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} p_1(t_n)[\exp(Lt_n)$$
$$+ \int_0^{t_n} \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \int_0^{s_{n-1}} q_1(s_n)\exp(Ls_n) \times$$
$$\times ds_n ds_{n-1} \ldots ds_1] dt_n dt_{n-1} \ldots dt_1 + \beta\exp(Lt) \leq [M\alpha + \beta]\exp(Lt).$$

This shows that T maps B into itself.

Now we verify that the operator T is a contraction map. Let $y, z \in B$, then from (3.1), (H_1), (2.2), it follows that

$$| Ty(t) - T(t) | \leq \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} | F(t_n, y(t_n), \tag{3.4}$$
$$\int_0^{t_n} \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \int_0^{s_{n-1}} f(s_n, y(s_n)) \times ds_n ds_{n-1} \ldots ds_1) -$$
$$-F(t_n, z(t_n), \int_0^{t_n} \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \times$$
$$\times \int_0^{s_{n-1}} f(s_n, z(s_n)) ds_n ds_{n-1} \ldots ds_1) | \times dt_n dt_{n-1} \ldots dt_1 \leq$$
$$\leq \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} p_1(t_n)[| y(t_n) - z(t_n) | \times$$
$$\times \exp(-Lt_n)\exp(Lt_n) + \int_0^{t_n} \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \times$$
$$\times \int_0^{s_{n-1}} q_1(s_n) | y(s_n) - z(s_n) | \exp(-Ls_n)\exp(Ls_n) \times$$
$$\times ds_n ds_{n-1} \ldots ds_1] dt_n dt_{n-1} \ldots dt_1 \leq$$
$$\leq \| y - z \| \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} p_1(t_n) \times$$
$$\times [\exp(Lt_n) + \int_0^{t_n} \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \int_0^{s_{n-1}} \times$$
$$\times q_1(s_n)\exp(Ls_n) ds_n d_{n-1} \ldots ds_1] dt_n dt_{n-1} \ldots dt_1 \leq$$
$$\leq \alpha \| y - z \| \exp(Lt).$$

From (3.4) we have

$$\| Ty - Tz \| \leq \alpha \| y - z \| . \tag{3.5}$$

Since $\alpha < 1$, it follows from Banach fixed point theorem that T has a unique fixed point in B. The fixed point of T is however a solution of the equations (1.1)-(1.2).

Let $y_0(t) \in B$ be given. Then we can determine a sequence $\{y_m(t)\}$ successively from

$$y_m(t) = Ty_{m-1}(t), m = 1, 2, ... \tag{3.6}$$

It is easy to observe that, the sequence determined from (3.6) converges to unique solution $y \in B$ of the equation (3.2). Since (3.2) and (3.6) are the operator equations of (1.1)-(1.2) and (2.5) respectively, we conclude that the sequence $\{y_m(t)\}$ given by (2.5) converges in B to the unique solution $y(t)$ of (3.2) and hence to the unique solution of (1.1)-(1.2). The error estimate follows immediately from the contraction property of the operator T and the proof of Theorem 1 is complete.

4. Further generalizations

In this section we indicate in brief the further applications of our approach to the study of more general nonlinear higher order integrodifferential equations of the forms:

$$\begin{aligned}(r(t)y^{(n-1)}(t) - a(t, y(t)))' \\ = F\left(t, y(t), \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \int_0^{s_{n-1}} f(s_n, y(s_n)) \times ds_n ds_{n-1} \dots ds_1\right),\end{aligned} \tag{4.1}$$

with the given initial conditions

$$\begin{aligned} y(0) = y_0, \qquad & \left(r(0)y^{(n-1)}(0) - a(0, y(0))\right) = 0, \\ y^{(i-1)}(0) \;\; = \;\; & 0, \;\; i = 2, 3, ..., n-1; \end{aligned} \tag{4.2}$$

$$\begin{aligned}(r(t)y'(t) - b(t, y(t)))^{(n-1)} = \\ = G\left(t, y(t), \int_0^t \frac{1}{r(s_1)} \int_0^{s_1} \cdots \int_0^{s_{n-1}} g\left(s_n, y(s_n)\right) ds_n ds_{n-1} \dots ds_1\right),\end{aligned} \tag{4.3}$$

with the given initial conditions

$$y(0) = y_0, \; (r(0)y'(0) - b(0, y(0)))^{(i-2)} = 0, i = 2, 3, ..., n; \tag{4.4}$$

$$\begin{aligned}\left(r(t)y^{(n)}(t) - c(t, y(t))\right)^{(n)} = \\ = H\left(t, y(t), \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \frac{1}{r(s_n)} \int_0^{s_n} \int_0^{\sigma_1} \cdots \int_0^{\sigma_{n-1}} \right. \\ \left. \times h(\sigma_n, y(\sigma_n)) d\sigma_n d\sigma_{n-1} \dots d\sigma_1 ds_n ds_{n-1} \dots ds_1\right),\end{aligned} \tag{4.5}$$

with the given initial conditions

$$y(0) = y_0, y^{(i-1)}(0) = 0, \ \ i = 2, 3, ..., n, \qquad (4.6)$$
$$\left(r(0)y^{(n)}(0) - c(0, y(0))\right)^{(i-1)} = 0, \ \ i = 2, 3, ..., n,$$

where r, f, g, h, F, G, H, y_0 are as defined in Section 1, $a,b,c : I \times R \to R$ are continuous functions such that $r(t)y^{(n-1)}(t) - a(t, y(t))$ is continously differentiable on I; $r(t)y'(t) - a(t, y(t))$ is (n-1)-times continuously differentiable on I and $r(t)y^{(n)}(t) - c(t, y(t))$ is n-times continuously differentiable on I. The following result similar to Theorem 2 can now be formulated regarding equations (4.3)-(4.4).

Theorem 4. *Assume that the hypotheses (H_3) and (H_4) in Theorem 2 hold. Suppose that*

(H_7) there exists a nonnegative continuous function $k(t)$ defined for $t \in I$ and a constant $\alpha_1 \geq 0$ such that

$$| b(t, y) - b(t, \bar{y}) | \leq k(t) \, | y - \bar{y} |,$$

and

$$\int_0^t \frac{1}{r(t_1)} k(t_1) \, \exp(Lt_1) dt_1 \leq \alpha_1 \exp(Lt),$$

for $t \in I$;

(H_8) there exists a constant $\beta_1 \geq 0$ such that

$$\int_0^t \frac{1}{r(t_1)} \, | b(t_1, 0) | \, dt_1 \leq \beta_1 \exp(Lt),$$

for $t \in I$. If $\alpha_0 = \alpha + \alpha_1 < 1$, then there exists a unique solution $y \in B$ of equations (4.3)-(4.4). Further, for any $y_0(t) \in B$, the sequence $\{y_m(t)\}$ given successively by

$$\begin{aligned} y_m(t) \ &= \ y_0 + \int_0^t \frac{1}{r(t_1)} b(t_1, y_{m-1}(t_1)) dt_1 + \\ &+ \int_0^t \frac{1}{r(t_1)} \int_0^{t_1} \cdots \int_0^{t_{n-1}} G\left(t_n, y_{m-1}(t_n), \int_0^{t_n} \frac{1}{r(s_1)} \times \right. \\ &\left. \times \int_0^{s_1} \cdots \int_0^{s_{n-1}} g\left(s_n, y_{m-1}(s_n)\right) ds_n ds_{n-1} ... ds_1 \right) \times dt_n dt_{n-1} ... dt_1, \end{aligned} \qquad (4.7)$$

for m = 1, 2, ..., converges in B to a unique solution y of the equations (4.3)-(4.4). Moreover,

$$\| y_m - y \| \leq \frac{\alpha_0^m}{1 - \alpha_0} \| y_1 - y_0 \|, \qquad (4.8)$$

for $m = 1, 2, \ldots$, *where* $0 \leq \alpha_0 < 1$ *is a constant defined as above.*

The details of the proof of this theorem is very close to that of the proof of Theorem 1 given in Section 3, with suitable modifications and hence we omit it here. Finally, we note that the formulation of results similar to that of Theorems 1 and 3 for the equations (4.1)-(4.2) and (4.5)-(4.6) are quite straight in view of the result given in Theorem 4 and hence we do not discuss them here.

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ON THE DUAL CONE OF A CONE OF SEQUENCES

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Abstract. In the paper there are generalized or adapted some rezults from the paper [1] to the case of a cone of sequences.

1. Introduction.

Let S_n denote the set of finite real sequences $(a_k)_{k=0}^n$. As it is known (see [2]), a subset $K_n \subseteq S_n$ is called a cone if:

$$(pa_k)_{k=0}^n \in K_n, \quad \text{for every} \quad p \geq 0, \quad (a_k)_{k=0}^n \in K_n.$$

The cone K_n is convex if:

$$(a_k + b_k)_{k=0}^n \in K_n \quad \text{for all} \quad (a_k)_{k=0}^n, \quad (b_k)_{k=0}^n \in K_n.$$

The dual cone K_n^* of K_n is defined by:

$$K_n^* = \left\{ (p_k)_{k=0}^n \in S_n : \sum_{k=0}^n p_k a_k \geq 0, \ \forall (a_k)_{k=0}^n \in K_n \right\}.$$

As it stated in [2], characterizations of the elements of a dual cone were obtained for the first time for convex functions by T. Popoviciu (see [5] for more references). They were transposed for convex sequences by J.E. Pečarić in [4]. In [6] we can find more results and references to papers in which it is determined the dual cone of some cones of sequences. Essentially the cones K_n which are considered in these papers consist in different kinds of convex or of starshaped sequences.

In this paper we want to generalize and to transpose to above context the results from [1].

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2. Some results of M.P. Drazin

The finite differences Δ^k are defined for any sequence $(a_i)_{i\geq 0}$ recurrently by:

$$\Delta^0 a_i = a_i,\ \Delta^1 a_i = a_{i+1} - a_i,\ \Delta^k a_i = \Delta^1(\Delta^{k-1} a_i),\ k \geq 2,\ i \geq 0.$$

M.P. Drazin proved in [1] the following results:

i) For any sequence $(a_i)_{i\geq 0}$ and any Y holds:

$$\sum_{i=0}^{n} \binom{n}{i} y^i a_i = (-1)^n \sum_{j=0}^{n} \binom{n}{j} (-1-y)^j \Delta^{n-j} a_j, \quad n \geq 0. \tag{1}$$

ii) If:

$$(-1)^{n-j} \Delta^{n-j} a_j \geq 0, \quad \text{for} \quad 0 \leq j \leq n,$$

with at least one inequality, then:

$$\sum_{i=0}^{n} \binom{n}{i} y^i a_i > 0, \quad \text{for} \quad y > -1. \tag{2}$$

iii) If:

$$\Delta^{n-j} a_j \geq 0, \quad \text{for} \quad 0 \leq j \leq n,$$

with at least one inequality, then:

$$(-1)^n \sum_{i=0}^{n} \binom{n}{i} y^i a_i > 0, \quad \text{for} \quad y < -1. \tag{3}$$

3. A new proof

Let us remind the following notation:

$$i^{(k)} = i(i-1)\ldots(i-k+1).$$

In [2] and [3] are given two identities which we can use for proving and generalizing (1):

$$\sum_{i=0}^{n} p_i a_i = \sum_{k=0}^{n} \left(\frac{1}{k!} \sum_{i=k}^{n} i^{(k)} p_i \right) \Delta^k a_0 \tag{4}$$

and

$$\sum_{i=0}^{n} p_i a_i = \sum_{k=0}^{n} \left(\frac{1}{(n-k)!} \sum_{i=0}^{k} (n-i)^{(n-k)} p_i \right) (-1)^{(n-k)} \Delta^{n-k} a_k. \tag{5}$$

For $p_i = \binom{n}{i} y^i$, (5) becomes:

$$\sum_{i=0}^{n} \binom{n}{i} y^i a_i = (-1)^n \sum_{k=0}^{n} (-1)^k \left(\sum_{i=0}^{k} \binom{n-i}{n-k} \binom{n}{i} y^i \right) \Delta^{n-k} a_k$$

$$= (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1-y)^k \Delta^{n-k} a_k,$$

i.e. (1).

Similarly, for $p_i = \binom{n}{i} y^{n-i}$, (4) gives:

$$\sum_{i=0}^{n} \binom{n}{i} y^{n-i} a_i = \sum_{k=0}^{n} \binom{n}{k} (1+y)^{n-k} \Delta^k a_0.$$

4. The main results

Let $r = (r_k)_{k=0}^n$, $r_k \in \{0,1\}$ be a given sequence. We define:

$$K_{n,r} = \left\{ (a_k)_{k=0}^n : (-1)^{r_k} \Delta^k a_0 \geq 0, \quad \text{for} \quad 0 \leq k \leq n \right\},$$

and

$$L_{n,r} = \left\{ (a_k)_{k=0}^n : (-1)^{r_k} \Delta^{n-k} a_k \geq 0, \quad \text{for} \quad 0 \leq k \leq n \right\}.$$

Obviously $K_{n,r}$ and $L_{n,r}$ are convex cones. Using (4) we have the following result:

Theorem 1. *If $p_k (k = 0, 1, \ldots, n)$ are real numbers such that:*

$$(-1)^{r_k} \sum_{i=k}^{n} i^{(k)} p_i \geq 0, \quad \textit{for} \quad 0 \leq k \leq n,$$

then $(p_k)_{k=0}^n \in K^*_{n,k}$.

Also, using (5) we get:

Theorem 2. *If the real numbers p_k $(k = 0, 1, \ldots, n)$ are such that:*

$$(-1)^{r_k + n - k} \sum_{i=0}^{k} (n-i)^{(n-k)} p_i \geq 0, \quad \textit{for} \quad 0 \leq k \leq n,$$

then $(p_k)_{k=0}^n \in L^*_{n,r}$.

Consequence. *Let f be a function continuous on [0,n], differentiable n times in (0,n), positive for $x = n$ and*

$$(-1)^k f^{(k)}(x) \geq 0, \quad for \quad 1 \leq k \leq n, \ \ n-k < x < n.$$

If the real numbers p_k $(k = 0, 1, \ldots, n)$ are such that:

$$\sum_{i=0}^{k} (n-i)^{(n-k)} p_i \geq 0, \quad for \quad k = 0, 1, \ldots, n \tag{6}$$

then:

$$\sum_{i=0}^{n} p_i f(i) \geq 0.$$

Proof. From (5) we have:

$$\sum_{i=0}^{n} p_i f(i) = \sum_{k=0}^{n} \frac{1}{(n-k)!} \left(\sum_{i=0}^{k} (n-i)^{(n-k)} p_i \right) (-1)^{n-k} \Delta^{n-k} f(k).$$

As it is proved in [1], for $k = 0, 1, \ldots, n-1$, there is a $x_k \in (k, n)$ such that:

$$\Delta^{n-k} f(k) = f^{(n-k)}(x_k).$$

We can consider also $x_n = n$ and from (6) we get the result. □

As an application, we can prove the following generalization of a proposition from [1]:

Theorem 3. *Let $f_1, \ldots, f_q$ be given conti ous functions on the interval $0 \leq x \leq n$, each differentiable n times in the open interval and positive for $x = n$; suppose also th :*

$$(-1)^k f_j^{(k)}(x) \geq 0, \quad j = 1, \ldots, q; \quad k = 1, \ldots, n; \quad n-k \leq x \leq n.$$

If p_k $(k = 0, 1, \ldots, n)$ are real numbers verifying (6) then:

$$\sum_{i=0}^{n} p_i \left(\prod_{j=1}^{q} f_j(i) \right) \geq 0.$$

As in [1], this result can be illustrated by the set o functions:

$$f_j(x) = (1 + a_j x)^{-b_j}, \quad j = 1, \ldots, q$$

where a_j and b_j can be any positive numbers.

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ON THE CONTINUATION PRINCIPLE FOR NONEXPANSIVE MAPS

RADU PRECUP

Abstract. In this note the continuation principle (nonlinear alternative) for nonexpansive maps on Hilbert spaces (see [5]) is extended in two directions: 1) to the case of uniformly convex Banach spaces; 2) for nonexpansive maps on a not necessarly convex set of a Hilbert space. In the proofs we use the Leray-Schauder continuation principle for condensing maps [7], [9] (we can also use Granas' continuation principle for contractions on complete metric spaces [6]).

In [5], the following nonlinear alternative for nonexpansive maps was proved by means of the Banach fixed point theorem.

Theorem A [5]. *Let H be a Hilbert space and C the closed ball $\{x \in H; |x| \leq c\}$. Then each nonexpansive map $T: C \to H$ has at least one of the following properties:*

(a) *T has a fixed point.*

(b) *There is $x \in \partial C$ and $\lambda \in]0,1[$ such that $x = \lambda T(x)$.*

In what follows we shall prove the following two generalizations of Theorem A:

Theorem 1. *Let E be an uniformly convex Banach space and U a bounded open convex set of E with $0 \in U$. Then each nonexpansive map $T : \bar{U} \to E$ has at least one of the following properties:*

(a) *T has a fixed point.*

(b) *There is $x \in \partial U$ and $\lambda \in]0,1[$ such that $x = \lambda T(x)$.*

Theorem 2. *Let H be a Hilbert space and U a bounded open set of H (not necessarily convex) with $0 \in U$. Then each nonexpansive map $T : \bar{U} \to H$ has at least one of the following properties:*

(a) *T has a fixed point.*

(b) *There is $x \in \partial U$ and $\lambda \in]0,1[$ such that $x = \lambda T(x)$.*

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Recall that a Banach space E is said to be *uniformly convex* provided that for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\| x + y \| \leq 2(1 - \delta)$ for every $x, y \in E$ satisfying $\| x \| = \| y \| = 1$ and $\| x - y \| \geq \varepsilon$.

Each uniformly convex Banach space is reflexive (see, for example, [4]), and each Hilbert space is uniformly convex as follows from the parallelogram equation $| x - y |^2 + | x + y |^2 = 2(| x |^2 + | y |^2)$. For example, the spaces $L^p(\Omega)$ with $\Omega \subset \mathbf{R}^N$ measurable are uniformly convex for $1 < p < \infty$ (see [4]).

For the proofs we need some lemmas essentially due to Browder.

Lemma 1. *Let E be an uniformly convex Banach space, D a bounded convex set of E and $T : D \to E$ a nonexpansive map. Then, for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $x_0, x_1 \in D$, $\| x_0 - T(x_0) \| \leq \delta$ and $\| x_1 - T(x_1) \| \leq \delta$, it follows $\| x - T(x) \| \leq \varepsilon$ for any x of the form $x = (1 - \lambda)x_0 + \lambda x_1$ with $\lambda \in]0, 1[$.*

For the proof see [2] or [8, Teorema 1.4.2].

Lemma 2. *Let E be an uniformly convex Banach space, D a bounded closed convex set of E and $T : D \to E$ a nonexpansive map. If $(x_n) \subset D, x_n \to x_0$ weakly and $x_n - T(x_n) \to y_0$ in norm, then $x_0 - T(x_0) = y_0$.*

For the proof see the proof of Teorema 1.4.3 a) in [8].

Proof of Theorem 1. Suppose (b) does not hold. Then, $x \neq \lambda T(x)$ for all $x \in \partial U$ and $\lambda \in [0, 1[$. For each fixed $\lambda \in]0, 1[$, the map λT is a contraction and so, it is condensing. Then, by the Leray-Schauder continuation principle for condensing maps (see [7], [9]), there exists $x_\lambda \in U$ such that $x_\lambda - \lambda T(x_\lambda) = 0$. Let us denote by x_n such an element x_λ for $\lambda = 1 - 1/n$, $n \in \mathbb{N}^*$. Then, passing if necessarily to a subsequence, we may suppose that (x_n) converges weakly to some x_0. On the other hand, from $x_n - (1 - 1/n)T(x_n) = 0$, it follows that $x_n - T(x_n) \to 0$ in norm. Then, from Lemma 2, we get $x_0 - T(x_0) = 0$. Thus, (a) holds and the proof is complete.

Remark. If in particular, $T(\bar{U}) \subset \bar{U}$, then (b) in Theorem 1, clearly, does not hold. In this case, conclusion (a) follows directly by the following theorem of Browder-Kirk: *If E is an uniformly convex Banach space, D is a bounded closed convex set of E and $T : D \to D$ is nonexpansive, then there exists $x \in D$ with $T(x) = x$.*

In the case of Hilbert spaces, we may renounce at the assumption that U is convex and also give a much simpler proof:

Proof of Theorem 2. Also suppose (b) does not hold. The sequence (x_n) obtained in the proof of Theorem 1 satisfies:

$$\langle (n-1)^{-1}x_n - (m-1)^{-1}x_m, x_n - x_m\rangle = \langle T(x_n) - T(x_m), x_n - x_m\rangle - | x_n - x_m |^2 \leq 0$$

for all $n, m > 1$. Denote $r_n = (n-1)^{-1}$ and use the equality

$$2\langle r_n x_n - r_m x_m, x_n - x_m\rangle = (r_n + r_m) | x_n - x_m |^2 + (r_n - r_m)(| x_n |^2 - | x_m |^2).$$

Then, we obtain

$$0 \leq (r_n + r_m) | x_n - x_m |^2 \leq (r_n - r_m)(| x_m |^2 - | x_n |^2).$$

Since (r_n) is a decreasing sequence, we get that $(| x_n |)$ is an increasing sequence. In addition, U being bounded, $(| x_n |)$ is also bounded and thus, convergent. Next, from

$$| x_n - x_m |^2 \leq (| x_m |^2 - | x_n |^2)(r_n - r_m)/(r_n + r_m),$$

it follows that (x_n) is convergent. It is clear that its limit is a fixed point of T and the proof is complete.

Example. Let H be a Hilbert space and let us consider the boundary value problem

$$\begin{cases} u'' = f(t, u, u') & \text{for } 0 < t < 1 \\ u(0) = u(1) = 0 \end{cases} \tag{1}$$

where $f : [0,1] \times H^2 \to H$ satisfies

(i) $f(\cdot, u, v)$ is measurable for any fixed $u, v \in H$; there exist $1 < p < \infty$ and $h \in L^\infty(0,1)$ such that $f(\cdot, 0, 0) \in L^p(0,1;H)$ and

$$| f(t, u_1, v_1) - f(t, u_2, v_2) | \leq h(t)(| u_1 - u_2 | + | v_1 - v_2 |)$$

for all $u_1, u_2, v_1, v_2 \in H$ and a.e. $t \in [0,1]$.

We look for a weak solution $u \in W_0^{1,p}(0,1;H) \cap W^{2,p}(0,1;H)$ to problem (1).

Let $G(t,s)$ be the Green function, i.e. $G(t,s) = (1-t)s$ for $s \leq t$ and $G(t,s) = (1-s)t$ for $s > t$. Also, denote by C the smallest constant in the Wirtinger-Poincaré inequality:

$$\int_0^1 | u |^p \, dt \leq C^p \int_0^1 | u' |^p \, dt, \qquad u \in W_0^{1,p}(0,1;H)$$

(see [1]).

Theorem 3. *Let* (i) *holds. Also assume*

(ii) *there is $r > 0$ such that*

$$\langle u, f(t,u,v)\rangle + |v|^2 > 0 \quad \text{for} \quad \text{a.e.} \quad t \in [0,1]$$

and whenever $|u| \geq r$ and $\langle u,v\rangle = 0$;

(iii) $(C+1)^p \int_0^1 \left\{ \int_0^1 (|G_t(t,s)| \cdot h(s))^q \, ds \right\}^{p/q} dt \leq 1.$

Then (1) *has at least one solution.*

Proof. Problem (1) is equivalent with

$$u(t) = -\int_0^1 G(t,s) f(s, u(s), u'(s)) \, ds, \qquad 0 < t < 1.$$

We shall apply Theorem 1 to $E = W_0^{1,p}(0,1;H)$ and $T : E \to E$,

$$T(u)(t) = -\int_0^1 G(t,s) f(s, u(s), u'(s)) \, ds.$$

By the uniform convexity of $L^p(0,1;H)$, it easily follows that $W_0^{1,p}(0,1;H)$ endowed with norm

$$\| u \| = \left(\int_0^1 |u'|^p \, dt \right)^{1/p},$$

is also uniformly convex.

Next we have

$$\| T(u) - T(v) \|^p = \int_0^1 \left| \int_0^1 G_t(t,s) \left(f(s,u(s),u'(s)) - f(s,v(s),v'(s)) \right) ds \right|^p dt$$

$$\leq \int_0^1 \left\{ \int_0^1 |G_t(t,s)| \cdot h(s) (|u(s) - v(s)| + |u'(s) - v'(s)|) \, ds \right\}^p dt \leq$$

$$\leq \int_0^1 (|u(s) - v(s)| + |u'(s) - v'(s)|)^p \, ds \cdot \int_0^1 \left\{ \int_0^1 (|G_t| \cdot h(s))^q \, ds \right\}^{p/q} dt$$

where $1/p + 1/q = 1$, by Hölder's inequality.

Further, since

$$\int_0^1 (|u(s) - v(s)| + |u'(s) - v'(s)|)^p \, ds \leq$$

$$\leq \left\{ \left(\int_0^1 |u(s) - v(s)|^p \, ds \right)^{1/p} + \left(\int_0^1 |u'(s) - v'(s)|^p \, ds \right)^{1/p} \right\}^p \leq$$

$$\leq (C+1)^p \| u - v \|^p,$$

we obtain

$$\| T(u) - T(v) \| \leq (C+1)B \| u - v \|$$

where $B = \left[\int_0^1 \left\{ \int_0^1 (| G_t(t,s) | h(s))^q ds \right\}^{p/q} dt \right]^{1/p}$.

Thus, by (iii), T is nonexpansive.

Finally, by a standard reasoniement, from (ii), we get a number $R > 0$ such that $\| u \| < R$ for each $u \in W_0^{1,p}(0,1;H)$ solution to $u = \lambda T(u)$ for some $\lambda \in]0,1[$. Therefore, (b) does not hold and so T has a fixed point.

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ON AN EXPONENTIAL TOTIENT FUNCTION

JÓZSEF SÁNDOR

Abstract. It is introduced and studied an arithmetical function, analogous to the Euler function, for exponential divisors.

1. Introduction.

Let $n > 1$ be a positive integer, $n = p_1^{a_1} \dots p_r^{a_r}$ be its canonical form. The number $d = p_1^{b_1} \dots p_r^{b_r}$ is called an exponential divisor or e-divisor of n if $b_i \mid a_i \quad (i = 1, 2, \dots, r)$. This notion is due to Straus and Subbarao [11]. Let $\sigma_e(n)$ be the sum of all e-divisors of n and by convention $\sigma_e(1) = 1$. Analogously, let $d_e(n)$ denote the number of exponential divisors of n, with $d_e(1) = 1$. We call n e-perfect if $\sigma_e(n) = 2n$. In [11] it is proved the non-existence of odd e-perfect numbers, with related results. For other results on e-perfect numbers as well as e-superperfect numbers (i.e. satisfying $\sigma_e(\sigma_e(n)) = 2n$; for such a terminology, see e.g. [8]), see [4]. For density problems, e-perfect numbers not divisible by 3, or e-multiperfect numbers, etc. see [3], [7], [2], [1]. For results on $d_e(n)$ we quote e.q. [5].

2. The e-totient function

First we remark that:

Lemma 1. *If d is a common exponential divisor of a and b $(a, b, d > 1)$ then a, b, d can be written as $a = p_1^{a_1} \dots p_r^{a_r}$, $b = p_1^{b_1} \dots p_r^{b_r}$, $d = p_1^{d_1} \dots p_r^{d_r}$, where $d_1 \mid (a_1, b_1), \dots, d_r \mid (a_r, b_r)$.*

This Lemma permits the introduction of a notion of exponential-comprimality:

Definition 1. We say that a and b are exponentially coprime (or e-coprime), when for each common e-divisor d of a and b in Lemma 1 we have $d_1 = 1, \dots, d_r = 1$.

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Definition 2. Let $n > 1$ and let $\varphi_e(n)$ denote the number of all $1 < a < n$ which are e-comprime with n (in notation: $(a,n)_e = 1$). The function $\varphi_e(n)$ will be called as the e-totient function. Let $\varphi_e(1) = 1$, by convention.

Theorem 1. *The function $\varphi_e(n)$ is multiplicative and for $1 < n = p_1^{a_1} \dots p_r^{a_r}$ we have*

$$\varphi_e(n) = \varphi(a_1) \dots \varphi(a_r) \tag{1}$$

where φ denotes the classical Euler totient function.

Proof. . Clearly $\varphi_e(p^k) = \varphi(k)$ since $(a,p^k)_e = 1$ only when $a = p^r$ with $(r,k) = 1$. If q^s is another prime power, then $\varphi_e(q^s) = \varphi(s)$ and $\varphi_e(p^k q^s) = \varphi(k)\varphi(s) = \varphi_e(p^k)\varphi_e(q^s)$. Indeed, $(a, p^k q^s)_e = 1$ iff $a = p^k k' q^s s'$ with $(k', k) = 1$ and $(s', s) = 1$. It is immediate that there are $\varphi(k)\varphi(s)$ such $a's$. The general case, when n is a product of r prime powers, can be proved in a completely analogous way. This shows that

$$\varphi_e(mn) = \varphi_e(m)\varphi_e(n) \quad \text{for} \quad (m,n) = 1 \tag{2}$$

where $(m,n) = 1$ means that m and n are coprime in the usual sense. □

Theorem 2. *Let $n = p_1^{a_1} \dots p_r^{a_r} > 1$. Then*

$$\sum_{d|_e n} \varphi_e(d) = a_1 \dots a_r \tag{3}$$

Proof. By definition $d \,|_e\, n$ iff $d = p_1^{d_1} \dots p_r^{d_r}$ with $d_i \mid a_i$ $(1 \le i \le r)$. Thus, by using (1), $\sum_{d|_e n} \varphi_e(d) = \sum_{d_1|a_1,\dots,d_r|a_r} \varphi(d_1)\dots\varphi(d_r) = \left(\sum_{d_1|a_1} \varphi(d_1)\right) \dots \left(\sum_{d_r|a_r} \varphi(d_r)\right) = a_1 \dots a_r$ by the well known Gauss identity $\sum_{d|k} \varphi(d) = k$. □

Theorem 3. a) *If $n \,|_e\, m$, then*

$$\varphi_e(n) \mid \varphi_e(m) \quad (n, m > 1) \tag{4}$$

b) *One has*

$$\varphi_e(n) d_e(n) \ge a_1 \dots a_r \tag{5}$$

Proof. a) If $n \,|_e\, m$, then $m = p_1^{b_1} \dots p_r^{b_r}$ and $n = p_1^{a_1} \dots p_r^{a_r}$ with $a_i \mid b_i$ $(1 \le i \le r)$. Then since it is well known that for $a_i \mid b_i$ we have $\varphi(a_i) \mid \varphi(b_i)$, we conclude with $\varphi_e(n) = \varphi(a_1)\dots\varphi(a_r) \mid \varphi(b_1)\dots\varphi(b_r)$.
b) This follows by a) since $\varphi_e(d) \le \varphi_e(n)$ and $\sum_{d|_e n} 1 = d_e(n)$, the number of exponential divisors of n. □

Theorem 4. *If* $n = p_1^{a_1} \dots p_r^{a_r} > 1$ *then*

a)

$$\varphi_e(n) d_e(n) \geq a_1 \dots a_r \tag{6}$$

b) If all a_i $(1 \leq i \leq r)$ *are odd, then*

$$\varphi_e(n) d_e(n) \geq \sigma(a_1) \dots \sigma(a_r) \tag{7}$$

where $\sigma(a)$ *denotes the sum of ordinary divisors of* a.

Proof. a) This is the same as (5), but here we obtain a new proof. By $d_e(n) = d(a_1) \dots d(a_r)$, relation (1) and the inequality $\varphi(a) d(a) \geq a$ due to R. Sivaramakrishnan [10] implies at once (6).

b) For odd a, in [9] it is proved that $\varphi(a) d(a) \geq \sigma(a)$. Inequality (7) follows on the same lines as (6). □

Finally, we prove:

Theorem 5. *The maximal order of magnitude of* $\log \varphi_e(n)$ *is*

$$\frac{\log 4}{5} \cdot \frac{\log n}{\log \log n} \tag{8}$$

Proof. We need the following theorem due to Drodova and Freiman (See [6], p.125):

Let $f(n)$ *be a multiplicative function with the proper ty* $f(p^k) = g(k)$ *where* p *is a prime, and* $g(k)$ *depends only on* k. *Suppose* $g(k) \geq 1$ *and that there exists* k_0 *with* $g(k_0) > 1$. *Assume that for a certain number* $a > 0$ *one has* $\log g(k) = o(k^{1-a})$. *Then the maximal order of magnitude of* $\log f(n)$ *is given by* $\frac{\log g(m)}{m} \cdot \frac{\log n}{\log \log n}$, *where* m *is defined by* $(\log g(k))/k$ $\{\leq (\log g(m))/m$ *for* $k \leq m$ *and* $< (\log g(m))/m$ *for* $k > m$. $\}$

In our case $g(k) = \varphi(k)$ and $\log g(k) = \log \varphi(k) < c\sqrt{k}$ ($c > 0$, constant), thus $a = 1/2$ can be chosen. We have $\varphi(k) \leq k$-1 for $k > 2$. On the other hand, the function $f(x) = (x-1)^{1/x}$ $(x > 1)$ has a derivative $f'(x) = \left(\frac{x}{x-1} - \ln(x-1)\right)/x^2$. Since $\ln 3 = 1,09 \dots < 1 + \frac{1}{3} = 1,3\dots$ and $\ln 4 = 1,38 \dots > 1 + \frac{1}{4} = 1,2\dots$; the equation $\frac{x}{x-1} - \ln(x-1) = 0$ has a single root $t_0 \in (3,4)$. Thus $f(x)$ has an absolute maximum at t_0. Since $3^{1/4} < 4^{1/5}$, clearly $(k-1)^{1/k} \leq 4^{1/5} = (\varphi(5))^{1/5}$ and $(k-1)^{1/k} < 4^{1/5}$ for $k > 5$. This shows that $m = 5$ can be selected, an this finishes the proof of the theorem. □

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ON THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF CERTAIN FOURTH ORDER NON-AUTONOMOUS DIFFERENTIAL EQUATIONS

CEMIL TUNÇ

Abstract. Our aim in this paper is to present sufficient conditions, under which all solutions of (1.1) are uniformly bounded and tend to zero an $t \to \infty$.

1. Introduction and statement of the result

We consider the equation

$$x^{(4)} + a(t)\varphi(x,\dot{x},\ddot{x},\dddot{x})\dddot{x} + b(t)f(x,\dot{x},\ddot{x}) + c(t)g(x,\dot{x}) + d(t)h(x) = p(t,x,\dot{x},\ddot{x},\dddot{x}) \quad (1.1)$$

where a, b, c,d, φ , f, g, h and p are continuous functions for the arguments displayed explicity and the dots as usual indicate differentiation with respect to t. We shall henceforth suppose that the functions a, b, c, d are positive and differentiable in $R^+ = [0,\infty)$ and that the derivatives $\frac{\partial}{\partial x}\varphi(x,y,z,u)$, $\frac{\partial}{\partial y}\varphi(x,y,z,u)$, $\frac{\partial}{\partial u}\varphi(x,y,z,u)$, $\frac{\partial}{\partial x}f(x,y,z)$, $\frac{\partial}{\partial y}f(x,y,z)$, $\frac{\partial}{\partial x}g(x,y)$, $\frac{\partial}{\partial y}g(x,y)$ and $h'(x)$ exists and are continuous for all x, y, z and u. Furthermore, it will be assumed that all functions and solutions are real. Special cases of the differential equation (1.1) have been treated in Abou-El-Ela [1] and [2], Hara [3], Tunç [4] and others. The purpose of the paper is to prove the following:

Theorem. *Futher to the fundamental assumption on a, b, c, d, φ , f, g, h and p, suppose that:*

(I) $A \geq a(t) \geq a_0 > 0$, $B \geq b(t) \geq b_0 > 0$, $C \geq c(t) \geq c_0 > 0$, $d \geq d(t) \geq d_0 > 0$, *for* $t \in R^+$.

(II) $\varphi(x,y,z,u) \geq \alpha_1 > 0$ *for all x, y, z and u;* $\alpha_2 > 0, \alpha_4 > 0$.

(III) $g(x,0) = 0$ *and* $\frac{\partial}{\partial y}g(x,y) \geq \alpha_3 > 0$ *for all x and y.*

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(IV) There is a finite constant $\delta_0 > 0$ such that

$$a_0 b_0 c_0\ \alpha_1 \alpha_2 \alpha_3 - C^2 \alpha_3 \frac{\partial}{\partial y} g(x,y) - A^2 D \alpha_1 \alpha_4 \varphi(x,y,z,0) \geq \delta_0$$

for all x, y and z.

(V) $0 \leq \frac{\partial}{\partial y} g(x,y) - \frac{g(x,y)}{y} \leq \delta_1 < \frac{2D\alpha_4\delta_0}{C a_0 c_0^2 \alpha_1 \alpha_3^2}$ *for all x and $y \neq 0$.*

(VI) $\left(\frac{1}{z}\right) \int_0^z \varphi(x,y,\zeta,0)d\zeta - \varphi(x,y,z,0) \leq \delta_2 < \frac{2\delta_0}{A a_0^2 c_0 \alpha_1^2 \alpha_3}$ *for all x, y and $z \neq 0$.*

(VII) $y\frac{\partial}{\partial x}\varphi(x,y,z,0) \leq 0, z\frac{\partial}{\partial x}\varphi(x,y,z,0) \leq 0, y\frac{\partial}{\partial y}\varphi(x,y,z,0) \leq 0$ *and* $z\frac{\partial}{\partial y}\varphi(x,y,z,0)$ *for all x, y and z.*

(VIII) $f(x,y,0) = 0, \frac{\partial}{\partial y} f(x,y,z) \leq 0, y\int_0^z \frac{\partial}{\partial x} f(x,y,\zeta)d\zeta \leq 0$ *for all x, y and z, and*

$$0 \leq \frac{f(x,y,z)}{z} - \alpha_2 \leq \frac{\varepsilon_0 c_0^3 \alpha_3^3}{BD^2\alpha_4^2}$$

for all x, y and $z \neq 0$, where δ_0 is a positive constant such that

$$\begin{aligned} \varepsilon_0 < \varepsilon \ = \ \min \Bigg[& \frac{1}{a_0\alpha_1}, \frac{D\alpha_4}{c_0\alpha_3}, \frac{\delta_0}{4a_0c_0\alpha_1\alpha_3\Delta_0}, \frac{Cc_0\alpha_3}{4D\alpha_4\Delta_0}\left(\frac{2\alpha_4 D\delta_0}{Ca_0\alpha_1 c_0^2\alpha_3^2} - \delta_1\right), \\ & \frac{Aa_0\alpha_1}{4\Delta_0}\left(\frac{2\delta_0}{Aa_0^2 c_0 \alpha_1^2 \alpha_3} - \delta_2\right)\Bigg] \end{aligned} \tag{1.2}$$

with

$$\Delta_0 = \left(\frac{a_0 b_0 c_0 \alpha_1 \alpha_2}{C} + \frac{a_0 b_0 c_0 \alpha_2 \alpha_3}{AD\alpha_4}\right).$$

(IX) $\left[\frac{\partial}{\partial x} g(x,y)\right]^2 \leq \frac{\alpha_1 \delta_0 (\varepsilon - \varepsilon_0) a_0}{16C^2}$ *for all x and y, and* $\frac{1}{y}\int_0^y \frac{\partial}{\partial x} g(x,\eta)d\eta \leq \frac{\alpha_3(\varepsilon-\varepsilon_0)c_0}{4C}$ *for all x and $y \neq 0$.*

(X) $h(0) = 0, h(x)\mathrm{sgn}x > 0 (x \neq 0), H(x) = \int_0^x h(\xi)d\xi \to \infty$ *as* $|x| \to \infty$ *and*

$$0 \leq \alpha_4 - h'(x) \leq \frac{\varepsilon\Delta_0 a_0^2 \alpha_1^2}{D}$$

for all x

(XI) $z\frac{\partial}{\partial u}\varphi(x,y,z,u) + \Delta_2 y \frac{\partial}{\partial u}\varphi(x,y,z,u) \geq 0$ *for all x, y, z and u, where*

$$\Delta_2 = \frac{\alpha_4 D}{c_0 \alpha_3} + \varepsilon \tag{1.3}$$

(XII) $\int_0^\infty \gamma_0(t)dt < \infty, d'(t) \to \infty$ *as* $t \to \infty$, *where* $\gamma_0(t) = | a'(t) | + b_+ +'(t) + | c'(t) | + d'(t) |, b_+ +'(t) = \max b'(t), 0.$

(XIII) $| p(t,x,y,z,u) | \leq p_1(t) + p_2(t)\left[H(x) + y^2 + z^2 + u^2\right]^{\frac{\delta}{2}} + \Delta\left(y^2 + z^2 + u^2\right)^{\frac{1}{2}}$, *where Δ, δ are cosnstants such that $0 \leq \delta \leq 1, \Delta \geq 0$ and $p_1(t)$, $p_2(t)$ are non-negative continuous functions satisfying*

$$\int_0^\infty p_i(t)dt < \infty (i = 1,2). \tag{1.4}$$

If Δ is sufficiently small, then every solution $x(t)$ of (1.1) is uniformly bounded and satisfies

$$x(t) \to 0, \dot{x}(t) \to 0, \ddot{x}(t) \to 0, \dddot{x}(t) \to 0 \quad as \quad t \to \infty. \tag{1.5}$$

Remark 1. When we take $a(t) = b(t) = c(t) = d(t) = 1$, and $\varphi(x, \dot{x}, \ddot{x}, \dddot{x}) = f_1(\dot{x}, \ddot{x})$, and $f(x, \dot{x}, \ddot{x}) = f_2(\ddot{x})$, and $g(x, \dot{x}) = f_3(\dot{x})$, and $h(x) = \alpha_4 x$ and finally $p(t, x, \dot{x}, \ddot{x}, \dddot{x})$ 0, the conditions (I) - (XIII) of the theorem are reduced to those of Abou-El-Ela [1].

Remark 2. When $\varphi(x, \dot{x}, \ddot{x}, \dddot{x}) = f(\ddot{x}), f(x, \dot{x}, \ddot{x}) = \phi(\dot{x}, \ddot{x})$ and $g(x, \dot{x})$ depends only on $\dot{x}$, then the conditions (I) - (XIII) become similar to those of Hara [3].

Remark 3. When $\varphi(x, \dot{x}, \ddot{x}, \dddot{x}) = \varphi(\dot{x}, \ddot{x}, \dddot{x})$, and $g(x, \dot{x})$ depends only on $\dot{x}$, then the conditions (I) - (XIII) of the theorem are reduced to Tunç [4].

2. The Function $V_0(t, x, y, z, u)$

Equation (1.1) has an equivalent system

$$\begin{aligned} \dot{x} &= y, \quad \dot{y} = z, \quad \dot{z} = u \\ \dot{u} &= -a(t)\varphi(x, y, z, u)u - b(t)f(x, y, z) - c(t)g(x, y) - d(t)h(x) + p(t, x, y, z, u). \end{aligned} \tag{2.1}$$

The main tool in the theorem is the differentiable function $V_0 = V_0(t,x,y,z,u)$ defined by:

$$\begin{aligned} 2V_0 &= 2\Delta_2 d(t) \int_0^x h(\xi)d\xi + 2c(t) \int_0^y g(x, \eta)d\eta + \\ &+ 2\Delta_1 b(t) \int_0^z f(x, y, \zeta)d\zeta + 2a(t) \int_0^z \zeta\varphi(x, y, \zeta, 0)d\zeta \\ &+ 2\Delta_2 a(t)y \int_0^z \varphi(x, y, \zeta, 0)d\zeta + [\Delta_2\alpha_2 b(t) - \Delta_1\alpha_4 d(t)]y^2 - \\ &- \Delta_2 z^2 + \Delta_1 u^2 + 2d(t)yh(x) \\ &+ 2\Delta_1 d(t)zh(x) + 2\Delta_1 c(t)zg(x, y) + 2\Delta_2 yu + 2zu + k. \end{aligned} \tag{2.2}$$

where

$$\Delta_1 = \frac{1}{a_0\alpha_1} + \varepsilon, \tag{2.3}$$

Δ_2 being the constant defined by (1.3) and k is a positive constant to be determined later in the proof.

First discuss somme important inequalities.

Let Φ_1 be the function defined by

$$\Phi_1(x,y,z,0)=\begin{cases}\left(\frac{1}{z}\right)\int_0^z \varphi(x,y,\zeta,0)d\zeta, & z\neq 0\\ \varphi(x,y,0,0), & z=0.\end{cases} \tag{2.4}$$

Then

$$\Phi_1(x,y,z,0) \geq \alpha_1>0 \quad \text{for all } x,y \text{ and } z, \tag{2.5}$$

$$\Phi_1(x,y,z,0) - \varphi(x,y,z,0)\leq \delta_2 \quad \text{for all } x,y \text{ and } z \tag{2.6}$$

from (II) and (VI).

Further we define

$$\Phi_3(x,y)=\begin{cases}\frac{g(x,y)}{y}, & y\neq 0\\ \frac{\partial}{\partial y}g(x,0), & y=0\end{cases} \tag{2.7}$$

Now

$$\Phi_3(x,y)\geq \alpha_3 \quad \text{for all } x \text{ and } y, \tag{2.8}$$

$$0\leq \frac{\partial}{\partial y}g(x,y)-\Phi_3(x,y)\leq \delta_1 \quad \text{for all } x \text{ and } y \tag{2.9}$$

by using (III) and (V). By using (1.3), (2.3), (I) and (IV) we have

$$\alpha_2 b(t)-\Delta_1 c(t)\frac{\partial}{\partial y}g(x,y)-\Delta_2 a(t)\varphi(x,y,z,0)\geq \delta_0\frac{0}{a_0c_0\alpha_1\alpha_3}-\varepsilon\Delta_0 \text{ for all } x,y,z \text{ and all } t \tag{2.10}$$

Similarly, we can easily obtain

$$\Delta_1 - \frac{1}{a(t)\Phi_1(x,y,z,0)}\geq \varepsilon \quad \text{for all } x,y,z \text{ and all } t\in R^+ \tag{2.11}$$

$$\Delta_2 - \frac{\alpha_4 D}{c(t)\Phi_3(x,y)}\geq \varepsilon \quad \text{for all } x,y \text{ and all} t\in R^+. \tag{2.12}$$

Since $\Phi_1(x,y,z,0)=\varphi(x,y,\tilde{z},0), \tilde{z}=\theta z, 0\leq\theta\leq 1$, we have

$$\alpha_2 b(t)-\Delta_1 c(t)\frac{\partial}{\partial y}g(x,y)-\Delta_2 a(t)\Phi_1(x,y,z,0)\geq \delta_0\frac{0}{a_0c_0\alpha_1\alpha_3}-\varepsilon\Delta_0 \tag{2.13}$$

for all x,y,z and all $t\in R^+$; by (2.10). The following two lemmas are essential for the actual proof of the theorem.

Lemma 1. *Subject to the assumptions (I) - (XI) of the theorem, there are positive constants D_1 and D_2 such that*

$$D_1\left[H(x)+y^2+z^2+u^2+k\right]\le V_0\le D_2\left[H(x)+y^2+z^2+u^2+k\right] \tag{2.14}$$

for all x,y,z and u

Proof. Since $f(x,y,0)=0$ and $\frac{f(x,y,z)}{z}\ge\alpha_2(z\neq 0)$, it is clear that

$$2\Delta_1 b(t)\int_0^z f(x,y,\zeta)d\zeta\ge\Delta_1\alpha_2 b(t)z^2.$$

Then we have

$$\begin{aligned}2V_0 &\ge \left[2\Delta_2 d(t)\int_0^x h(\xi)d\xi-\frac{d^2(t)h^2(x)}{c(t)\Phi_3(x,y)}\right]+\\
&+ \left[\Delta_2\alpha_2 b(t)-\Delta_1\alpha_4 d(t)-\Delta_2^2 a(t)\Phi_1(x,y,z,0)\right]y^2+\\
&+ 2c(t)\int_0^y g(x,\eta)d\eta-c(t)y^2\Phi_3(x,y)+\\
&+ \left[\Delta_1\alpha_2 b(t)-\Delta_2-\Delta_1^2 c(t)\Phi_3(x,y)\right]z^2+2a(t)\int_0^z\zeta\varphi(x,y,\zeta,0)d\zeta-\\
&- a(t)z^2\Phi_1(x,y,z,0)+\left[\Delta_1-\frac{1}{a(t)\Phi_1(x,y,z,0)}\right]u^2+\\
&+ \frac{c(t)}{\Phi_3(x,y)}\left[\frac{d(t)}{c(t)}h(x)+y\Phi_3(x,y)+\Delta z\Phi_3(x,y)\right]^2+\\
&+ \frac{a(t)}{\Phi_1(x,y,z,0)}\left[\frac{u}{a(t)}+z\Phi_1(x,y,z,0)+\Delta_2 y\Phi_1(x,y,z,0)\right]^2+k.\end{aligned}$$

By using (2.11) we find

$$\left[\Delta_1-\frac{1}{a(t)\Phi_1(x,y,z,0)}\right]u^2\ge\varepsilon u^2.$$

Thus it follows that

$$2V_0\ge V_1+V_2+V_3+\varepsilon u^2+k, \tag{2.15}$$

where

$$\begin{aligned}
V_1 &= 2\Delta_2 d(t)\int_0^x h(\xi)d\xi \frac{d^2(t)h^2(x)}{c(t)\Phi_3(x,y)},\\
V_2 &= \left[\Delta_2\alpha_2 b(t) - \Delta_1\alpha_4 d(t) - \Delta_2^2 a(t)\Phi_1(x,y,z,0)\right]y^2 + 2c(t)\int_0^y g(x,\eta)d\eta -\\
&- c(t)y^2\Phi_3(x,y),\\
V_3 &= \left[\Delta_1\alpha_2 b(t) - \Delta_2 - \Delta_1^2 c(t)\Phi_3(x,y)\right]z^2 + 2a(t)\int_0^z \zeta\varphi(x,y,\zeta,0)d\zeta -\\
&- a(t)z^2\Phi_1(x,y,z,0).
\end{aligned}$$

The function V_1 can be estimated as in [4]. In fact, the estimates there show that

$$V_1 \geq 2\varepsilon d_0 \int_0^x h(\xi)d\xi_5$$

From (1.3), (2.3), (I), (III) and (2.13) we find

$$\Delta_2\alpha_2 b(t) - \Delta_1\alpha_4 d(t) - \Delta_2^2 a(t)\Phi_1(x,y,z,0) > \frac{D\alpha_4}{c_0\alpha_3}\left(\delta_0\frac{0}{a_0c_0\alpha_1\alpha_3} - \varepsilon\Delta_0\right).$$

Since $yg(x,y) = \int_0^y g(x,\eta)d\eta + \int_0^y g_y(x,\eta)d\eta$, then

$$2c(t)\int_0^y g(x,\eta)d\eta - c(t)y^2\Phi_3(x,y) \geq \left(-\frac{\delta_1 C}{2}\right)y^2, \text{ by (2.9)}.$$

Therefore

$$V_2 \geq \left[\frac{D\alpha_4}{c_0\alpha_3}\left(\delta_0\frac{0}{a_0c_0\alpha_1\alpha_3} - \varepsilon\Delta_0\right) - \frac{C\delta_1}{2}\right]y^2 \geq \frac{C}{4}\left(\frac{2\alpha_4 D\delta_0}{Ca_0\alpha_1c_0^2\alpha_3^2} - \delta_1\right)y^2, \text{ by (1.2)}.$$

Also, using (1.3), (2.3), (I), (II), (2.9) and (2,10) we get

$$\begin{aligned}
\Delta_1\alpha_2 b(t) - \Delta_2 - \Delta_1^2 c(t)\Phi_3(x,y) &= \Delta_1\left[\alpha_2 b(t) - \Delta_1 c(t)\Phi_3(x,y) - \Delta_2 a(t)\varphi(x,y,z,0)\right]\\
&+ \Delta_2\left[\Delta_1 a(t)\varphi(x,y,z,0) - 1\right] >\\
&> \Delta_1\left[\alpha_2 b(t) - \Delta_1 c(t)\Phi_3(x,y) - \Delta_2 a(t)\varphi(x,y,z,0)\right]\\
&> \left(\frac{1}{a_0\alpha_1}\right)\left[\delta_0\frac{0}{a_0c_0\alpha_1\alpha_3} - \varepsilon\Delta_0\right]
\end{aligned}$$

Further

$$a(t)\left[\int_0^z \varphi(x,y,\zeta,0) - \Phi_1(x,y,\zeta,0)\right]\zeta d\zeta \geq -\left(\frac{A\delta_2}{2}\right)z^2, \text{ by (2.6)}.$$

Therefore

$$V_3 \geq \frac{A}{4}\left(\frac{2\delta_0}{Aa_0^2c_0\alpha_1^2\alpha_3} - \delta_2\right)z^2, \text{ by (1.2)}.$$

From the estimates of V_1, V_2 and V_3 we obtain

$$2V_0 \geq 2\varepsilon d_0 H(x) + \frac{C}{4}\left(\frac{2\alpha_4 D\delta_0}{Ca_0\alpha_1 c_0^2\alpha_3^2} - \delta_1\right)y^2 + \frac{A}{4}\left(\frac{2\delta_0}{Aa_0^2 c_0\alpha_1^2\alpha_3} - \delta_2\right)z^2 + \varepsilon u^2 + k.$$

In this case it is clear that there exists a positive constant D_1 such that

$$V_0 \geq D_1\left[H(x) + y^2 + z^2 + u^2 + k\right].$$

From the assumption of the theorem it is clear that

$$\begin{aligned}
\varphi(x,y,z,0) &< \frac{a_0 b_0 c_0 \alpha_2 \alpha_3}{A^2 D\alpha_4},\\
g(x,y) &\leq \frac{a_0 b_0 c_0 \alpha_1 \alpha_2{}^2}{C}y,\\
f(x,y,z) &\leq \left(\alpha_2 + \frac{\varepsilon c_0^3\alpha_3^3}{BD^2\alpha_4^2}\right)z \text{ and}\\
h^2(x) &\leq 2\alpha_4 H(x).
\end{aligned}$$

Therefore we can see easily that there exists a positive constant D_2 satisfying

$$V_0 \leq D_2\left[H(x) + y^2 + z^2 + u^2 + k\right].$$

Thus the proof is now complete. □

Lemma 2. *Suppose that the conditions of the theorem hold. Then there are positive constants D_4, D_5 and D_6 such that*

$$\begin{aligned}
\dot{V}_0 &\leq -D_5(y^2+z^2+u^2) + \sqrt{3}D_6(y^2+z^2+u^2)^{\frac{1}{2}}\{p_1(t)+p_2(t)\} + \\
&+ \sqrt{3}D_6 p_2(t)\left[H(x)+y^2+z^2+u^2\right] + D_4\gamma_0 V_0 \qquad (2.16)
\end{aligned}$$

Proof. An easy calculation from (2.2) and (2.1) shows that

$$\begin{aligned}
\frac{d}{dt}V_0 &= \frac{\partial V_0}{\partial u}\dot{u} + \frac{\partial V_0}{\partial z}u + \frac{\partial V_0}{\partial y}z + \frac{\partial V_0}{\partial x}y + \frac{\partial V_0}{\partial t} = -\Delta_1 a(t)u^2\varphi(x,y,z,u) - \\
&- \Delta_2 b(t)yf(x,y,z) - \Delta_2 c(t)yg(x,y) - b(t)zf(x,y,z) + u^2 + \\
&+ \Delta_1 b(t)y\int_0^z \frac{\partial}{\partial x}f(x,y,\zeta)d\zeta + \Delta_1 b(t)z\int_0^z \frac{\partial}{\partial y}f(x,y,\zeta)d\zeta + \\
&+ a(t)z\int_0^z \zeta\frac{\partial}{\partial y}\varphi(x,y,\zeta,0)d\zeta + \Delta_2 a(t)yz\int_0^z \frac{\partial}{\partial y}\varphi(x,y,\zeta,0)d\zeta + \\
&+ \Delta_2 a(t)z\int_0^z \varphi(x,y,\zeta,0)d\zeta + [\Delta_2\alpha_2 b(t) - \Delta_1\alpha_4 d(t)]yz +
\end{aligned}$$

$$\begin{aligned}
&+ \Delta_1 c(t) yz \frac{\partial}{\partial x} g(x,y) + \Delta_1 c(t) z^2 \frac{\partial}{\partial y} g(x,y) + c(t) y \int_0^y \frac{\partial}{\partial x} g(x,\eta) d\eta + \\
&+ d(t) y^2 h'(x) + \Delta_1 d(t) yz h'(x) - a(t) [\varphi(x,y,z,u) - \varphi(x,y,z,0)] zu - \\
&- \Delta_2 a(t) [\varphi(x,y,z,u) - \varphi(x,y,z,0)] yu + a(t) y \int_0^z \zeta \frac{\partial}{\partial x} \varphi(x,y,\zeta,0) d\zeta + \\
&+ \Delta_2 a(t) y^2 \int_0^z \frac{\partial}{\partial x} \varphi(x,y,\zeta,0) d\zeta + (\Delta_2 y + z + \Delta_1 u) p(t,x,y,z,u) + \frac{\partial V_0}{\partial t}
\end{aligned}$$

From (VII) and (VIII) it follows that

$$\begin{aligned}
&z \int_0^z \frac{\partial}{\partial y} f(x,y,\zeta) d\zeta \le 0, \\
&y \int_0^z \zeta \frac{\partial}{\partial x} \varphi(x,y,\zeta,0) d\zeta \le 0, \ z \int_0^z \zeta \frac{\partial}{\partial y} \varphi(x,y,\zeta,0) d\zeta \le 0, \\
&z \int_0^z y \frac{\partial}{\partial y} \varphi(x,y,\zeta,0) d\zeta \le 0 \text{ and} \\
&\int_0^z \frac{\partial}{\partial x} \varphi(x,y,\zeta,0) d\zeta \le 0.
\end{aligned}$$

Then we find

$$\frac{d}{dt} V_0 \le -(V_4 + V_5 + V_6 + V_7 + V_8 + V_9) + (\Delta_2 y + z + \Delta_1 u) p(t,x,y,z,u) + \frac{\partial V_0}{\partial t}, \quad (2.17)$$

where

$$\begin{aligned}
V_4 &= \Delta_2 c(t) y g(x,y) - \alpha_4 d(t) y^2 - \Delta_1 c(t) yz \frac{\partial}{\partial x} g(x,y) - c(t) y \int_0^y \frac{\partial}{\partial x} g(x,\eta) d\eta, \\
V_5 &= \left[\alpha_2 b(t) - \Delta_1 c(t) \frac{\partial}{\partial y} g(x,y)\right] z^2 - \Delta_2 a(t) z \int_0^z \varphi(x,y,\zeta,0) d\zeta \\
V_6 &= [\Delta_1 a(t) \varphi(x,y,z,u) - 1] u^2, \\
V_7 &= b(t) z f(x,y,z) - \alpha_2 b(t) z^2 + \Delta_2 b(t) y f(x,y,z) - \alpha_2 \Delta_2 b(t) yz, \\
V_8 &= \alpha_4 d(t) y^2 - d(t) h'(x) y^2 + \Delta_1 \alpha_4 d(t) yz - \Delta_1 d(t) h'(x) yz, \\
V_9 &= a(t) [\varphi(x,y,z,u) - \varphi(x,y,z,0)] zu + \Delta_2 a(t) [\varphi(x,y,z,u) - \varphi(x,y,z,0)] yu.
\end{aligned}$$

The functions V_7 and V_8 are the same as in [4]. The estimates for V_7, V_8 there give that

$$V_7 \ge -(\varepsilon c_0 \alpha_3) y^2, \quad (2.18)$$

$$V_8 \ge -(\varepsilon \Delta_0) z^2. \quad (2.19)$$

By using (I), (2.8) and (2.12) we obtain

$$\begin{aligned} V_4 &\geq c(t)\Phi_3(x,y)\left[\Delta_2 - \frac{\alpha_4 D}{c(t)\Phi_3(x,y)}\right]y^2 - \Delta_1 c(t)yz\frac{\partial}{\partial x}g(x,y) - c(t)y\int_0^y \frac{\partial}{\partial x}g(x,\eta)d\eta \geq \\ &\geq (\varepsilon c_0\alpha_3)y^2 - \Delta_1 c(t)yz\frac{\partial}{\partial x}g(x,y) - c(t)y\int_0^y \frac{\partial}{\partial x}g(x,\eta)d\eta. \end{aligned} \tag{2.20}$$

Combining (2.18) and (2.20) we have

$$\begin{aligned} V_4 + V_7 &\geq (\varepsilon - \varepsilon_0)c_0\alpha_3 y^2 - \Delta_1 c(t)yz\frac{\partial}{\partial x}g(x,y) - c(t)y\int_0^y \frac{\partial}{\partial x}g(x,\eta)d\eta \geq \\ &\geq (\varepsilon - \varepsilon_0)c_0\alpha_3 y^2 - \Delta_1 c(t)yz\frac{\partial}{\partial x}g(x,y) - c(t)\left[\frac{1}{y}\int_0^y \frac{\partial}{\partial x}g(x,\eta)d\eta\right]y^2 \geq \\ &\geq \frac{3}{4}(\varepsilon - \varepsilon_0)c_0\alpha_3 y^2 - \Delta_1 c(t)yz\frac{\partial}{\partial x}g(x,y) = \\ &= \frac{1}{2}(\varepsilon - \varepsilon_0)c_0\alpha_3 y^2 + \frac{1}{4}(\varepsilon - \varepsilon_0)c_0\alpha_3\left[y^2 - \frac{4c(t)\Delta_1}{(\varepsilon - \varepsilon_0)\alpha_3 c_0}yz\frac{\partial}{\partial x}g(x,y)\right] \geq \\ &\geq \frac{1}{2}(\varepsilon - \varepsilon_0)c_0\alpha_3 y^2 - \frac{c^2(t)\Delta_1^2}{(\varepsilon - \varepsilon_0)\alpha_3 c_0}\left[\frac{\partial}{\partial x}g(x,y)\right]^2 z^2 \geq \\ &\geq \frac{1}{2}(\varepsilon - \varepsilon_0)c_0\alpha_3 y^2 - \left(\delta_0\frac{0}{4a_0c_0\alpha_1\alpha_3}\right)z^2, \end{aligned}$$

by using (I), (IX) and (2.3) and (1.2).

$$V_5 = \left[\alpha_2 b(t) - \Delta_1 c(t)\frac{\partial}{\partial y}g(x,y) - \Delta_2 a(t)\Phi_1(x,y,z,0)\right]z^2 \geq \left(\delta_0\frac{0}{a_0c_0\alpha_1\alpha_3} - \varepsilon\Delta_0\right)z^2,$$

by (2.13) In case we use (I), (II) and (2.3), we find

$$V_6 = [\Delta_1 a(t)\varphi(x,y,z,u) - 1]u^2 \geq \varepsilon a_0\alpha_1 u^2$$

From (XI) for $u \neq 0$ we obtain

$$V_9 = a(t)\left[z\varphi_u(x,y,z,\theta u) + \delta_2 y\varphi_u(x,y,z,\theta u)\right]u^2 \geq 0, 0 \leq \theta \leq 1$$

but $V_9 = 0$ when $u = 0$. Hence $V_9 \geq 0$ for all x, y, z and u. On gathering all these estimates into (2.17) we deduce that

$$\begin{aligned} \dot{V}_0 &\leq -\frac{1}{2}(\varepsilon - \varepsilon_0)c_0\alpha_3 y^2 - \left(\delta_0\frac{0}{4a_0c_0\alpha_1\alpha_3}\right)z^2 - \varepsilon a_0\alpha_1 u^2 + \\ &+ (\Delta_2 y + z + \Delta_1 u)p(t,x,y,z,u) + \frac{\partial V_0}{\partial t}, \end{aligned}$$

since

$$\varepsilon < \delta_0 \frac{0}{4a_0c_0\alpha_1\alpha_3\Delta_0}$$

by (1.2). From (2.2) we have

$$\begin{aligned}\frac{\partial V_0}{\partial t} &= a'(t)\left[\int_0^z \zeta\varphi(x,y,\zeta,0)d\zeta + \Delta_2 y \in t_0^z\varphi(x,y,\zeta,0)d\zeta\right] \\ &+ b'(t)\left[\Delta_1\int_0^z f(x,y,\zeta)d\zeta + \left(\Delta_2\frac{2}{2}\right)\alpha_2 y^2\right] + c'(t)\left[\int_0^y g(x,\eta)d\eta + \Delta_1 z g(x,y)\right] \\ &+ d'(t)\left[\Delta_2\int_0^x h(\xi)d\xi - \left(\Delta_1\frac{1}{2}\right)\alpha_4 y^2 + h(x)y + \Delta_1 h(x)z\right].\end{aligned}$$

On the basis of the proof of [4, Lemma 2] it can be shown

$$\frac{\partial V_0}{\partial t} \le D_3 [| a'(t) | + b_+ +'(t) + | c'(t) | + | d'(t) |] \left[H(x) + y^2 + z^2 + u^2\right] \le D_4\gamma_0 V_0,$$

where $D_4 = D_2\frac{2}{D_1}$. Hence can find a positive D_5 such that

$$\dot{V}_0 \le -2D_5(y^2+z^2+u^2) + \Delta_2 y + z + \Delta_1 u)p(t,y,z,u) + D_4\gamma_0 V_0.$$

Let $D_4 = \max \Delta_2, 1, \Delta_1$. We have

$$\begin{aligned}\dot{V}_0 &\le -2D_5(y^2+z^2+u^2) + \sqrt{3}D_6(y^2+z^2+u^2)^{\frac{1}{2}} \mid p(t,x,y,z,u) \mid + D_4\gamma_0 V_0 \\ &\le -2D_5(y^2+z^2+u^2) + \sqrt{3}D_6(y^2+z^2+u^2)^{\frac{1}{2}} \\ &\quad \left\{p_1(t) + p_2(t)\left[H(x)+y^2+z^2+u^2\right]^{\frac{\delta}{2}} + \Delta(y^2+z^2+u^2)^{\frac{1}{2}}\right\} + D_4\gamma_0 V_0.\end{aligned}$$

In that case Δ can be fixed as follovs $\Delta = \frac{D_5}{\sqrt{3}D_6}$. With this limitation on Δ we have

$$\begin{aligned}\dot{V}_0 &\le -D_5(y^2+z^2+u^2) + \sqrt{3}D_6(y^2+z^2+u^2)^{\frac{1}{2}} \\ &\quad \left\{p_1(t) + p_2(t)\left[H(x)+y^2+z^2+u^2\right]^{\frac{\delta}{2}}\right\} + D_4\gamma_0 V_0\end{aligned}$$

Since $[H(x)+y^2+z^2+u^2]^{\frac{\delta}{2}} \le 1 + [H(x)+y^2+z^2+u^2]^{\frac{1}{2}}$, it is clear that

$$\begin{aligned}\dot{V}_0 &\le -D_5(y^2+z^2+u^2) + \sqrt{3}D_6(y^2+z^2+u^2)^{\frac{1}{2}}\{p_1(t)+p_2(t)\} \\ &+ \sqrt{3}D_6 p_2(t)\left[H(x)+y^2+z^2+u^2\right] + D_4\gamma_0 V_0.\end{aligned}$$

3. Completion of the proof

See [4].

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CORRECTION: LA $\mathcal{T}$-TOPOLOGIE D'UN GROUPE ABELIEN

GRIGORE CĂLUGĂREANU

Abstract. In the paper published in the same journal, vol.40, no.4, 1995, p.5-12 the Proposition 1 is not true.

1. Introduction

Dans Zbl.Math. 857.20038, Adolf Mader remarque que la Proposition 1 qui suit est incorrecte, la topologie p-adique étant un contreexample. Voici la Proposition incriminée:

Proposition 1. *"Si pour une topologie fonctorielle T, la classe discrète $\mathcal{C}(T)$ est une classe Sèrre alors un sous-groupe B de A est T-concordant ssi $A/B \in \mathcal{C}(T)$.*

Démonstration. Premièrement, si B est T-concordant, de $\mathcal{U}_B \subseteq B \cap \mathcal{U}_A$ on déduit que pour chaque $U \leq B, B/U \in \mathcal{C}$ implique $A/U \in \mathcal{C}$. $\mathcal{C}$ étant fermée aux sous-groupes, on a $0 \in \mathcal{C}$ donc on peut prendre plus haut $U = B$. Donc $A/B \in \mathcal{C}$.

Réciproquement,.......□

Cas particulier. B est $\mathcal{T}$-concordant ssi A/B est un groupe de torsion."

2. Correction

Un sousgroupe est *T-concordant* [1] si sa topologie de sous-espace de $T(A) = A[\mathcal{U}_A]$ coïncide avec la topologie fonctorielle de B. Donc B est T-concordant ssi $\mathcal{U}_B = B \cap \mathcal{U}_A$.

En effet, pour établir que pour un sousgroupe T-concordant B de A, $A/B \in \mathcal{C}$ a lieu, le raisonement fait est négligent: $B \in \mathcal{U}_B = B \cap \mathcal{U}_A$ implique seulement l'existence d'un sousgroupe C, tel que $B \leq C \leq A$ et $A/C \in \mathcal{C}(T)$, mais qui peut être different de B.

D'ailleurs 0 est evidemment un sousgroupe T-concordant dans n'importe quel groupe A. Si A est un groupe sans-torsion et T est la topologie p-adique (ou la $\mathcal{T}$-topologie), $A/0 \in \mathcal{C}(T)$ est clairement faux.

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Le reste de la démonstration est correcte: seulement "B est T-concordant dans A si $A/B \in C(T)$" reste vrai (avec le cas particulier correspondant).

3. Errata

Dans la Proposition 2 il faut remplacer **concordant** avec **coconcordant**.

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