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LA $\mathcal{T}$-TOPOLOGIE D'UN GROUPE ABÉLIEN

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REZUMAT. - $\mathcal{T}$-topologia unui grup abelian. Este introdusă o nouă topologie functorială ideală şi discretă dar non-completabilă, intim legată de grupuri abeliene mixte. Sunt demonstrate proprietăţile de bază.

RÉSUMÉ. Une nouvelle topologie fonctorielle idéale discrète mais non complétable intimement lieé aux groupes abéliens mixtes est introduite. Les propriétés de base sont démontrées.

LEMME 1. *L'ensemble* $\mathcal{T} = \{U \leq A \mid A/U$ *groupe de torsion*$\}$ *ordonné par l'inclusion est un filtre dans le treillis des sous-groupes de A.*

Démonstration. Si $U \in \mathcal{T}$ et $U \leq V$, en utilisant l'épimorphisme canonique $p_{UV} : A/U \twoheadrightarrow A/V$ on déduit immédiatement que $V \in \mathcal{T}$. Par calcul élémentaire on déduit de $U, V \in \mathcal{T}$ que $U \cap V \in \mathcal{T}$. □

La topologie linéaire déterminée par ce filtre va être nommée la $\mathcal{T}$-topologie de A. On va aussi utiliser $\mathcal{F} = \{U \in \mathcal{T} \mid U$ sans torsion$\}$.

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Remarque 1. *On a $U \in \mathcal{T} \Leftrightarrow U + T(A)$ essentiel dans A.*

Démonstration. Si $U \in \mathcal{T}$ on a $U + T(A) \in \mathcal{T}$ (lemme) donc $A/(U + T(A))$ est un groupe de torsion. L'inclusion $S(A) \leq U + T(A)$ étant évidente, $U + T(A)$ est essentiel dans A (cf. [1], vol.1, ex.10, p.87). La condition est aussi suffisante car $A/(U + T(A))$ groupe de torsion implique A/U groupe de torsion (calcul élémentaire).□

Cette équivalence généralise l'équivalence connue: $T(A)$ essentiel dans $A \Leftrightarrow A$ groupe de torsion (cf. [1], ex. 11, p.87).

PROPRIÉTÉS immediates 1. *La $\mathcal{T}$-topologie d'un groupe A est plus fine que toutes les topologies les plus connues:* ***Z****-adique, p-adique, Prüfer et la topologie de l'indice fini.*

La $\mathcal{T}$-topologie d'un groupe A est discrète ssi A est un groupe de torsion (en effet: discrète ssi $0 \in \mathcal{T}$).

La $\mathcal{T}$-topologie d'un groupe A est grossière ssi $A = 0$.

En effet, si $A \neq 0$, ϕ, 0 et A sont deja trois ouvers.□

Dans la suite l'exposition suivra un parallélisme avec [2] pour des raisons évidentes.

D'ailleurs cette dernière propriété peut être aussi déduite de [2] (3.5): A

doit être un π'_d-groupe π_t-divisible; dans notre cas $\pi'_d = \phi$ donc A est un groupe de torsion avec $A[p] = 0$ pour chaque p prèmier. Donc $A = 0$.

PROPRIÉTÉS immediates 2. *Si A est un groupe mixte ou sans torsion, $\mathcal{T}$ ne contient pas de sous-groupes de torsion.*

Si A est sans torsion $\mathcal{T} = \mathcal{F}$.

Si A est mixte $\mathcal{F}$ est un système fondamental de voisinages de 0 *dans la $\mathcal{T}$-topologie.*

Démonstration. Si U et A/U sont des groupes de torsion, A le serrait aussi. Pour la dernière affirmation on prouve que: $U \in \mathcal{T}$ ssi il existe un $F \leq U$, $F \in \mathcal{F}$. En effet si A est un groupe mixte et $U \in \mathcal{T}$ nous avons $U \neq T(U)$. Si U est sans torsion, on a rien a démontrer. Si U est mixte, soit F un sous-groupe sans torsion maximal (nommée par la suite une *st-composante*) de U. Alors F est sans torsion et U/F est groupe de torsion (on a même F net dans U). A/U étant aussi groupe de torsion, A/F en est aussi, donc $F \in \mathcal{F}$. Réciproquement, tout est clair. □

Remarque 2. Si F est net dans U et A/U est groupe de torsion, généralement il n'est pas vrai que F est net dans A. Les st-composantes d'un groupe mixte ne forment pas un système fondamental de voisinages de 0 dans la $\mathcal{T}$-topologie.

Dans le cas contraire tout sous-groupe sans torsion $U \in \mathcal{F}$ contiendrait un sous-groupe sans torsion maximal dans A.

PROPRIÉTÉS immediates 3. *La $\mathcal{T}$-topologie d'un groupe A est toujours Hausdorff.*

Cela peut se démontrer de plusieurs manières:

(i) On peut démontrer aisement que l'intersection de toutes les st-composantes est 0. Alors $\cap\{U \leq A | A/U$ de torsion$\} = 0$ a aussi lieu.

(ii) Pour vérifier que $\cap\{U \leq A | U \in \mathcal{T}\} = 0$, on revient à: pour chaque $a \in A$, $a \neq 0$ il y a un $U \in \mathcal{T}$ tel que $a \notin U$. En effet, pour chaque $a \in A$, $a \neq 0$, le lemme de Zorn est applicable dans l'ensemble des sous-groupes de A qui ne contiennet pas a. L'ensemble contient donc un sous-groupe maximal pour lequel A/M est cocyclique ([1], (25.2)). Mais alors $M \in \mathcal{T}$. □

PROPRIÉTÉS immediates 4. *La $\mathcal{T}$-topologie est fonctorielle, idéale et donc minimale.* (Voir [2]).

La classe discrète $C(\mathcal{T})$ correspondante est la classe de tous les groupes de torsion (discrète et idéale).□

Remarque 3. $C(\mathcal{T})$ n'étant pas fermée pour des puissances arbitraires, $C(\mathcal{T})$

ne provient pas d'une famille topologique de radicaux (cf. [2] (2.11)). $C(\mathcal{T})$ est fermée aux extensions (c'est donc une classe Sèrre).

PROPRIÉTÉS immediates 5. *La $\mathcal{T}$-topologie d'un groupe contient les sous-groupes essentiels.*

En effet B essentiel dans A ssi $S(A) \leq B$ et A/B est groupe de torsion. Si A est un groupe sans torsion, on a $\mathcal{T} = \mathcal{F} = \{$sous-groupes essentiels$\}$, mais si A est un groupe mixte, aucune st-composante n'est pas un sous-groupe essentiel.□

PROPRIÉTÉS immediates 6. *La $\mathcal{T}$-topologie de A n'admet pas de sous-groupes denses autres que A.*

En effet, ([2] (3,2)(b)) B est dense dans A ssi A/B a la topologie grossière et on utilise une propriété antérieure.□

Remarque 4. On peut aussi utiliser un exercice de [1] (ex. 10, p.34) pour donner une autre démonstration:

si $B \neq A$ serait un sous-groupe dense on aurait $B + U = A$ pour chaque $U \in \mathcal{T}$. Pour $U = nA$ avec n naturel, on voit que B serait aussi dense dans la topologie Z-adique, donc A/B serait divisible. A/B n'est pas un groupe de torsion parce que alors $B \in \mathcal{T}$ et on aurait $B + B = B = A$. Donc $r_0(A/B) \neq 0$ et alors il existe un sous-groupe $C/B \cong Q$ (le théorème de structure pour les groupes divisibles) qui est aussi un facteur direct. On prend par example $E/B \leq C/B$ avec $E/B \cong \mathbf{Z}$ et alors $A/(E+D)) \cong (A/B)/(E/B \oplus D/C) \cong \mathbf{Q}/\mathbf{Z}$ si $A/B = C/B \oplus D/B$. Donc $E + D \in \mathcal{T}$ et $B + (E + D) = E + D = A$, contradiction.□

PROPOSITION 1. *Si pour une topologie fonctorielle T, la classe discrète $\mathcal{C}(T)$ est une classe Sèrre alors un sous-groupe B de A est T-concordant ssi $A/B \in \mathcal{C}(T)$.*

Démonstration. Un sous-groupe est T-concordant si sa topologie de sous-espace de $T(A) = A[\mathcal{U}_A]$ coincide avec la topologie fonctorielle de B. Donc B est T-concordant ssi $\mathcal{U}_B = B \cap \mathcal{U}_A$.

Premièrement, si B est T-concordant, de $\mathcal{U}_B \subseteq B \cap \mathcal{U}_A$ on déduit que pour chaque $U \leq B$, $B/U \in \mathcal{C}$ implique $A/U \in \mathcal{C}$. $\mathcal{C}$ étant fermée aux sous-groupes, on a $0 \in \mathcal{C}$ donc on peut prendre plus haut $U = B$. Donc $A/B \in \mathcal{C}$.

Réciproquement, $\mathcal{C}$ étant fermée aux sous-groupes, on déduit $B \cap \mathcal{U}_A \subseteq \mathcal{U}_B$ (en effet pour $U \leq A$ on a $B \cap U \leq B$ et $B/(B \cap U) \cong (B + U)/U \leq A/U \in \mathcal{C}$; donc $B/(B \cap U) \in \mathcal{C}$). Si $A/B \in \mathcal{C}$ et $\mathcal{C}$ est une classe Sèrre, B/U et $A/B \in \mathcal{C}$ $\Rightarrow A/U \in \mathcal{C}$ donc $\mathcal{U}_B \subseteq B \cap \mathcal{U}_A$ et B est T-concordant.□

Cas particulier. B est $\mathcal{T}$-concordant ssi A/B est un groupe de torsion.

PROPOSITION 2. *Tous les groupes quotients sont $\mathcal{T}$-concordants.*

En effet, $C(\mathcal{T})$ étant une classe idéale de groupes on applique [2] (3.2)(b).□

Remarque 5. On peut démontrer ce resultat aussi par voie directe:

$\mathcal{T}_{A/B} = \{C/B \leq A/B \mid (A/B)/(C/B) \cong A/C$ groupe de torsion$\} = \{C/B \leq A/B \mid p_B^{-1}(C/B) = C, A/C$ groupe de torsion$\}$ = la topologie quotient de A/B. (Ici $p_B: A \rightarrow A/B$ est la projection canonique).

Remarque 6. Le foncteur induit par la $\mathcal{T}$-topologie commute avec les sommes directes ([2], (3.21)).

PROPOSITION 3. *La $\mathcal{T}$-topologie n'est pas complétable.*

En effet, la classe $C(\mathcal{T})$ contient des groupes divisibles e.g. $\boldsymbol{Z}(p^{\infty})$) donc le résultat se déduit de [2] (5.11). De la même façon on déduit que le complété de $\boldsymbol{Z}$ dans la $\mathcal{T}$-topologie est même que le complété dans la topologie $\boldsymbol{Z}$-adique: $\prod J_p$ pour tous les nombres prèmiers p (J_p est le groupe des entiers p-adiques): [2] (4.16).□

PROPOSITION 4. *Tout sous-groupe de A est fermé dans la $\mathcal{T}$-topologie de A.*

Démonstration. Si T est une topologie fonctorielle, $C(T)$ sa classe discrète et $\mathcal{D}$ le filtre de définition (i.e. $U \in \mathcal{D} \Leftrightarrow A/U \in C(T)$) pour un sous-groupe B de A, on a l'adhérence $\bar{B} = \cap\{B + U \mid U \in \mathcal{D}\}$, donc B est fermé ssi $B = \cap\{B$

$+ U|U \in \mathcal{D}\}$. On remarque alors aisement que pour deux topologies fonctorielles T_1 et T_2 avec $C(T_1) \subseteq C(T_2)$ (ou $\mathcal{D}_1 \subseteq \mathcal{D}_2$; T_2 plus fine que T_1) si B est T_1-fermé il est aussi T_2-fermé ($B \subseteq \cap\{B + U|U \in \mathcal{D}_2\} \subseteq \cap\{B + U|U \in \mathcal{D}_1\} = B$ sont toutes egalités). Mais tout sous-groupe est fermé dans la topologie de Prüfer (Fuchs, vol. 1, p.31) et alors on obtient le résultat énoncé.□

Rappelons de [2] (3.12): un sous-groupe B est T-pur dans A si chaque $D \in C(T)$ est injectif relativement à la suite exacte

$$0 \to B \to A \to A/B \to 0.$$

La définition revient à la χ_*-purité (C.P.Walker;e.g. [1], p.131) donc on a: B est T-pur dans A ssi pour chaque $U \leq B$ pour lequel B/U est groupe de torsion, B/U est facteur direct dans A/U. Raisonnant comme plus haut, pour T_2 plus fine que T_1 si B est T_2-pur, il est aussi T_1-pur. Si on prend T_1 la topologie de Prüfer, T_2 la $\mathcal{T}$-topologie et on remarque que pour $\mathcal{T}_1$ = {groupes cocycliques} les sous-groupes χ_{1*}-purs sont exactement les sous-groupes purs, on voit que $\mathcal{T}$-pur doit être une notion intermédiaire entre pur et facteur direct.

Remarque 7. Si B est un sous-groupe $\mathcal{T}$-pur de torsion alors B est un facteur direct (en effet, dans ce cas $0 \in \mathcal{D}_B$ et alors $B/0$ est un facteur direct dans $A/0$).

PROPOSITION 5. *Si* Ext($A/B,T$) = 0 *a lieu pour chaque groupe de torsion* T*, alors* B *est* $\mathcal{T}$*-pur dans* A.

En effet, B/U est un groupe de torsion, $A/B \cong (A/U)/(B/U)$, donc A/U est une extension de B/U par A/B et alors B/U est un facteur direct de A/U. On sait (cf. [1], p.189) que Ext($A/B,T$) = 0 a lieu pour chaque groupe de torsion T ssi A/B est libre, donc cette condition implique deja B facteur direct de A.□

Conclusion: les groupes A dans lesquels chaque sous-groupe $\mathcal{T}$-pur est un facteur direct sont ceux pour lesquels on a

$$0 \to T \to G \to A/B \to 0 \Rightarrow \exists U \leq B: G \cong A/U.$$

Problèmme 1. Caractériser les groupes complets dans la $\mathcal{T}$-topologie.

Problèmme 2. Quels sont les groupes pour lesquels la $\mathcal{T}$-topologie et la $\boldsymbol{Z}$-topologie coincident?

Vu que la $\mathcal{T}$-topologie est plus fine que la topologie $\boldsymbol{Z}$-adique, les groupes recherchés sont exactement ceux qui on tous les groupes quotients groupes de torsion bornés.

Problèmme 3. Caractériser les sous-groupes $\mathcal{T}$-purs dans un groupe sans torsion ou un groupe mixte.

Problèmme 4. Étudier les liens de la $\mathcal{T}$-topologie les catégories Walk et Warf.

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ON CERTAIN CLASS OF ANALYTIC FUNCTIONS

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Dedicated to Professor V. Ureche on his 60^{th} anniversary

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REZUMAT. - Asupra unei anumite clase de funcţii analitice. În lucrare este studiată o clasă de funcţii analitice în discul unitate U, notată $T_\lambda(j,\alpha,\beta)$, unde $0 \le \lambda \le 1$, $0 \le \alpha < 1$, $0 < \beta \le 1$ and $j \in \mathbf{N}_0 = \{0,1,...\}$. Sunt obţinute teoreme de deformare (delimitări ale modulelor funcţiilor şi derivatelor lor) şi estimări ale coeficienţilor dezvoltării în serie Taylor ale funcţiilor din aceste clase.

ABSTRACT. There are many classes of analytic functions in the unit disc U. We shall consider the special class $T_\lambda(j,\alpha,\beta)$ ($0 \le \lambda \le 1$, $0 \le \alpha < 1$, $0 < \beta \le 1$, $j \in \mathbf{N}_0 = \{0,1,...\}$) of analytic functions in the unit disc U. And the purpose of this paper is to show some distorsion theorems for the class $T_\lambda(j,\alpha,\beta)$. Also we show some coefficient estimates for the classes $T_\lambda(j,\alpha,\beta)$, $T_0(j,\alpha,\beta)$, and $T_\lambda(j,\alpha,1)$.

1. Introduction. Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

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which are analytic and univalent in the unit disc $U = \{z: |z| < 1\}$. We use Ω to denote the class of bounded analytic functions $w(z)$ in U satisfing the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$. For a function $f(z)$ in S, we define

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = Df(z) = z f'(z), \tag{1.3}$$

and

$$D^j f(z) = D(D^{j-1} f(z)) \quad (j \in \mathbf{N} = \{1,2,...\}). \tag{1.4}$$

The differential operator D^j was introduced by Salagean [9]. With the help of the differential operator D^j, we say that a function $f(z)$ belonging to S is in the class $S_j(\alpha)$ if and only if

$$\mathrm{Re}\{\frac{D^{j+1} f(z)}{D^j f(z)}\} > \alpha \quad (j \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}) \tag{1.5}$$

for some α $(0 \leq \alpha < 1)$, and for all $z \in U$. The class $S_j(\alpha)$ was defined by Salagean [9].

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0). \tag{1.6}$$

Further, we define the class $T(j,\alpha)$ by

$$T(j,\alpha) = S_j(\alpha) \cap T. \tag{1.7}$$

The class $T(j,\alpha)$ was studied by Hur and Oh [3] and Salagean [10] and [11]. We note that $T(0,\alpha) = T^*(\alpha)$ and $T(1,\alpha) = C(\alpha)$ were studied by Silverman [12]. For this class $T(j,\alpha)$ Salagean [10] and Hur and Oh [3] gave the following lemma.

LEMMA 1. *Let the function $f(z)$ be defined by* (1.6). *Then $f(z)$ is in the class $T(j,\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} n^j(n-\alpha)\, a_n \leq 1-\alpha. \tag{1.8}$$

The result is sharp.

The next lemma may be found in [3].

LEMMA 2. *Let the function $f(z)$ defined by* (1.6) *be in the class $T(j,\alpha)$. Then we have*

$$|z| - \frac{1-\alpha}{2^j(2-\alpha)}|z|^2 \leq |f(z)| \leq |z| + \frac{1-\alpha}{2^j(2-\alpha)}|z|^2 \tag{1.9}$$

and

$$1 - \frac{1-\alpha}{2^{j-1}(2-\alpha)}|z| \leq |f'(z)| \leq 1 + \frac{1-\alpha}{2^{j-1}(2-\alpha)}|z|. \tag{1.10}$$

The result is sharp.

Let $T_\lambda(j,\alpha,\beta)$ denote the class of functions of the form (1.1) which satisfy the condition

$$\left| \frac{\dfrac{f(z)}{g(z)} - 1}{\lambda \dfrac{f(z)}{g(z)} + 1} \right| < \beta \quad (0 \le \lambda \le 1, 0 < \beta \le 1, z \in U) \tag{1.11}$$

where

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad (b_n \ge 0) \tag{1.12}$$

is in the class $T(j,\alpha)$ ($j \in \mathbf{N}_0$; $0 \le \alpha < 1$).

We note that:

(i) $T_\lambda(0,\alpha,\beta) = \tilde{S}_\lambda(\alpha,\beta)$ (Owa [5,7]):

(ii) $T_\lambda(0,\alpha,1) = S_u(0,\alpha)$, $0 \le u = \lambda \le 1$, (Altintas [1]).

2. Distortion Theorems.

THEOREM 1. *Let the function $f(z)$ defined by* (1.1) *be in the class $T_\lambda(j,\alpha,\beta)$. Then we have*

$$|f(z)| \ge \frac{(1-\beta|z|)(2^j(2-\alpha)-(1-\alpha)|z|)|z|}{2^j(1+\lambda\beta|z|)(2-\alpha)} \tag{2.1}$$

and

$$|f(z)| \le \frac{(1+\beta|z|)(2^j(2-\alpha)|z|)|z|}{2^j(1-\lambda\beta|z|)(2-\alpha)} \tag{2.2}$$

for $z \in U$. These estimates are sharp.

Proof. We employ the same technique used by Goel and Sohi [2] and

Owa [5,6,7,8]. Since $f(z) \in T_\lambda(j,\alpha,\beta)$, after a simple computation we have

$$\frac{f(z)}{g(z)} = \frac{1-\beta w(z)}{1+\lambda\beta w(z)}, \quad w \in \Omega. \tag{2.3}$$

By using Schwarz's lemma [4], we have $|w(z)| \leq |z|$. Hence

$$\frac{1-\beta|z|}{1+\lambda\beta|z|} \leq \left|\frac{f(z)}{g(z)}\right| \leq \frac{1+\beta|z|}{1-\lambda\beta|z|}. \tag{2.4}$$

Consequently, we have the theorem with the aid of Lemma 2. By taking

$$\frac{f(z)}{g(z)} = \frac{1-\beta z}{1+\lambda\beta z} \tag{2.5}$$

and

$$g(z) = z - \frac{1-\alpha}{2^j(2-\alpha)}z^2 \tag{2.6}$$

we can see the estimates are sharp. This completes the proof of Theorem 1.

COROLLARY 1. *Under the hypotheses of Theorem* 1, *$f(z)$ is included in the disc with center at the origin and radius r_1 given by*

$$r_1 = \frac{(1+\beta)(2^j(2-\alpha)+(1-\alpha))}{2^j(1-\lambda\beta)(2-\alpha)}. \tag{2.7}$$

THEOREM 2. *Let the function $f(z)$ defined by* (1.1) *be in the class $T_\lambda(j,\alpha,\beta)$. Then we have*

$$|f'(z)| \leq \frac{(1+\beta|z|)(2^{j-1}(2-\alpha)+(1-\alpha)|z|)}{2^{j-1}(1-\lambda\beta|z|)(2-\alpha)}$$

$$+\frac{(1+\lambda)\beta(2^j(2-\alpha)+(1-\alpha)|z|)|z|}{2^j(1-\lambda\beta|z|)^2(1-|z|^2)(2-\alpha)} \tag{2.8}$$

for $z \in U$.

Proof. Since $f(z) \in T_\lambda(j,\alpha,\beta)$, by using (2.3), we obtain

$$f'(z) = \frac{1-\beta w(z)}{1+\lambda\beta w(z)} g'(z) - \frac{(1+\lambda)\beta w'(z)}{\{1+\lambda\beta w(z)\}^2} g(z), \ w \in \Omega. \tag{2.9}$$

Further, we have $|w'(z)| \leq \frac{1}{1-|z|^2}$ by means of Caratheodory's theorem [4]. Hence we obtain the theorem with the aid of Lemma 2.

Remark 1. We have not able to obtain the sharp estimate for $|f'(z)|$ for $f(z) \in T_\lambda(j,\alpha,\beta)$.

3. Coefficient Estimates

THEOREM 3. *Let the function $f(z)$ defined by* (1.1) *be in the class $T_\lambda(j,\alpha,\beta)$. Then we have*

$$|a_2| \leq \frac{1-\alpha}{2^j(2-\alpha)} + \beta(1+\lambda) \tag{3.1}$$

and

$$|a_3| \leq \frac{1-\alpha}{3^j(3-\alpha)} + \frac{1-\alpha}{2^j(2-\alpha)}\beta(1+\lambda) + \beta(1+\lambda). \tag{3.2}$$

The estimate for $|a_2|$ is sharp.

Proof. Let

$$w(z) = \sum_{n=1}^{\infty} c_n z^n \in \Omega. \tag{3.3}$$

Then we obtain [4]

$$|c_1| \leq 1. \tag{3.4}$$

and

$$|c_2| \leq 1 - |c_1|^2. \tag{3.5}$$

Since $f(z) \in T_\lambda(j,\alpha,\beta)$, by using (2.3), we have

$$f(z)\,(1+\lambda\beta w(z)) = g(z)\,(1-\beta w(z)), \quad w \in \Omega. \tag{3.6}$$

Then, on substituting the power series (1.1), (1.12), and (3.3), for the functions $f(z)$, $g(z)$, and $w(z)$, respectively, in (3.6) we get

$$\left(z+\sum_{n=2}^{\infty} a_n z^n\right)\left(1+\lambda\beta\sum_{n=1}^{\infty} c_n z^n\right) = \left(z-\sum_{n=2}^{\infty} b_n z^n\right)\left(1-\beta\sum_{n=1}^{\infty} c_n z^n\right). \tag{3.7}$$

Equating coefficients of z^2 and z^3 on both sides of (3.7), we obtain

$$a_2 = -\beta(1+\lambda)c_1 - b_2 \tag{3.8}$$

and

$$a_3 = -\beta(1+\lambda)c_2 + \lambda\beta^2(1+\lambda)c_1^2 + \beta(1+\lambda)b_2c_1 - b_3. \tag{3.9}$$

Since $g(z) \in T(j,\alpha)$, by using Lemma 1, we have

$$b_2 \leq \frac{1-\alpha}{2^j(2-\alpha)}, \tag{3.10}$$

and

$$b_3 \leq \frac{1-\alpha}{3^j(3-\alpha)}. \tag{3.11}$$

Hence we have the theorem. Further we can see that estimate for $|a_2|$ is sharp for $\frac{f(z)}{g(z)}$, defined by (2.5), where $g(z)$ is given by (2.6).

THEOREM 4. *Let the function $f(z)$ defined by* (1.1) *be in the class*

$T_\lambda(j,\alpha,\beta)$. *Then we have*

$$|a_4| \leq \frac{1-\alpha}{4^j(4-\alpha)} + \frac{1-\alpha}{3^j(3-\alpha)}\beta(1+\lambda) + \frac{1-\alpha}{2^j(2-\alpha)}\beta(1+\lambda)(1+\lambda\beta)$$
$$+\beta(1+\lambda) + 2\lambda\beta^2(1+\lambda). \tag{3.12}$$

Proof. Equating the coefficients of z^4 on both sides of (3.7), we have

$$a_4 = -b_4 - \beta(1+\lambda)c_3 - \beta(\lambda a_2 - b_2)c_2 - \beta(\lambda a_3 - b_3)c_1. \tag{3.13}$$

Since

$$c_n = \frac{1}{2\pi i}\int_{|z|=r}\frac{w(z)}{z^{n+1}}dz \quad (0<r<1;\ n\in\mathbb{N}) \tag{3.14}$$

for the coefficients c_n of analytic function $w(z)$ in the unit disc U, we obtain

$$|c_n| \leq \frac{1}{r},\ 0<r<1,\ n\in\mathbb{N},\ n\geq 1$$

and then

$$|c_n| \leq 1,\ n\in\mathbb{N},\ n\geq 1. \tag{3.15}$$

From (3.13) and (3.15) we deduce

$$|a_4| \leq |b_4| + \beta(1+\lambda) + \beta(\lambda|a_2| + |b_2|) + \beta(\lambda|a_3| + |b_3|). \tag{3.16}$$

Hence we obtain the theorem by using Lemma 1 and Theorem 3.

Remark 2. We have not been able to obtain sharp estimates for $|a_n|$ ($n \geq 3$) for the function $f(z)$ belonging to the class $T_\lambda(j,\alpha,\beta)$.

THEOREM 5. *Let the function $f(z)$ defined by* (1.1) *be in the class* $T_0(j,\alpha,\beta)$. *Then we have*

$$|a_n| \le \beta\left(\frac{2^j(2-\alpha)+1-\alpha}{2^j(2-\alpha)}\right)+\frac{1-\alpha}{n^j(n-\alpha)} \tag{3.17}$$

for any $n \ge 2$.

Proof. Since $f(z)$ belongs to the class $T_0(j,\alpha,\beta)$, from (3.7), we have

$$-\frac{1}{\beta}\sum_{n=2}^{\infty}(a_n+b_n)z^n = \left(z-\sum_{n=2}^{\infty}b_n z^n\right)\left(\sum_{n=1}^{\infty}c_n z^n\right). \tag{3.18}$$

Equating the coefficients of z^n on both sides of (3.18), we have

$$-\frac{1}{\beta}(a_n+b_n) = c_{n-1}-\sum_{m=2}^{n-1}b_m c_{n-m} \tag{3.19}$$

By using that $b_n \ge 0$, $n \ge 2$, from (3.15) and (3.19) we obtain

$$\frac{1}{\beta}|a_n+b_n| \le 1+\sum_{m=2}^{n-1}b_m. \tag{3.20}$$

But

$$\sum_{m=2}^{n-1}2^j(2-\alpha)b_m \le \sum_{m=2}^{n-1}m^j(m-\alpha)b_m \tag{3.21}$$

and by using Lemma 1 we deduce

$$\sum_{m=2}^{\infty}b_m \le \frac{1-\alpha}{2^j(2-\alpha)}. \tag{3.22}$$

From (3.20) and (3.22) we have

$$\frac{1}{\beta}|a_n+b_n| \le 1+\frac{1-\alpha}{2^j(2-\alpha)}. \tag{3.23}$$

Hence we obtain

$$|a_n| \le |a_n+b_n|+|b_n|$$
$$\le \beta\left(\frac{2^j(2-\alpha)+1-\alpha}{2^j(2-\alpha)}\right)+\frac{1-\alpha}{n^j(n-\alpha)}, \tag{3.24}$$

because

$$b_n \le \frac{1-\alpha}{n^j(n-\alpha)} \tag{3.25}$$

for any $n \ge 2$ by Lemma 1.

Remark 3. We have not been able to obtain sharp estimates for $|a_n|$ ($n \ge 2$) for the function $f(z)$ belonging to the class $T_0(j,\alpha,\beta)$.

THEOREM 6. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $T_\lambda(j,\alpha,1)$ *and* $\mathrm{Re}(a_k) \ge 0$ ($k = 2,3,\ldots,(n-1)$), *then*

$$|a_n| \le 1 + \lambda + \frac{1-\alpha}{n^j(n-\alpha)} \tag{3.26}$$

for any $n \ge 2$.

Proof. Since $f(z) \in T_\lambda(j,\alpha,1)$, from (3.7), we have

$$\sum_{n=2}^{\infty}(a_n+b_n)z^n = -\left[(1+\lambda)z + \sum_{n=2}^{\infty}(\lambda a_n - b_n)z^n\right]\left(\sum_{n=1}^{\infty}c_n z^n\right). \tag{3.27}$$

Equating the coefficients of z^n ($n \ge 2$) on both sides of (3.27) we get

$$a_2 + b_2 = -(1+\lambda)c_1 \text{ and}$$

$$a_n + b_n = -\left[(1+\lambda)c_{n-1} + (\lambda a_2 - b_2)c_{n-2} + \ldots\right.$$

$$\left. + (\lambda a_{n-1} - b_{n-1})c_1\right], \ (n \ge 3). \tag{3.28}$$

From (3.27) and (3.28) we obtain

$$\sum_{k=2}^{n}(a_k+b_k)z^k + \sum_{k=n+1}^{\infty} d_k z^k = -\left[(1+\lambda)z + \sum_{k=2}^{n-1}(\lambda a_k - b_k)z^k\right]w(z), \tag{3.29}$$

where $\sum_{k=n+1}^{\infty} d_k z^k$ converges in U. Using $|w(z)| < 1$ for $z \in U$ and Parseval's identity [4] on both sides of (3.29), we obtain

$$\sum_{k=2}^{n} |a_k+b_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |d_k|^2 r^{2k}$$
$$\leq (1+\lambda)^2 r^2 + \sum_{k=2}^{n-1} |\lambda a_k - b_k|^2 r^{2k}. \tag{3.30}$$

Since (3.30) holds for all r in the interval $0 < r < 1$, it follows that

$$\sum_{k=2}^{n} |a_k+b_k|^2 \leq (1+\lambda)^2 + \sum_{k=2}^{n-1} |\lambda a_k - b_k|^2 \tag{3.31}$$

and from (3.31) it follows that

$$|a_n+b_n|^2 \leq (1+\lambda)^2 - (1-\lambda^2)\sum_{k=2}^{n-1} |a_k|^2 - 2(1+\lambda)\sum_{k=2}^{n-1} \mathrm{Re}(a_k) b_k. \tag{3.32}$$

Since $b_n \geq 0$ for all $n \geq 2$ and $\mathrm{Re}(a_k) \geq 0$ ($k = 2,3,\dots,(n-1)$), it follows that

$$|a_n + b_n| \leq (1+\lambda). \tag{3.33}$$

Hence, by using (3.25) and (3.33), we obtain

$$|a_n| \leq |a_n+b_n| + |b_n| \leq 1+\lambda + \frac{1-\alpha}{n^j(n-\alpha)} \quad (n \geq 2). \tag{3.34}$$

This completes the proof of Theorem 6.

COROLLARY 2. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $T_\lambda(j,0,1)$ *and* $\mathrm{Re}(a_k) \geq 0$ ($k = 2,3,\dots,(n-1)$), *then*

$$|a_n| \leq 1+\lambda + \frac{1}{n^{j+1}} \quad (n \geq 2). \tag{3.35}$$

Remark 4. Since $\frac{1-\alpha}{n^j(n-\alpha)}$ is decreasing on $n (n \geq 2)$, Theorem 6 gives

$$|a_n| \leq 1+\lambda + \frac{1-\alpha}{2^j(2-\alpha)} \quad (n \geq 2) \tag{3.36}$$

for $f(z) \in T_\lambda(j,\alpha,1)$ satisfying $\mathrm{Re}(a_k) \geq 0$ ($k = 2,3,\dots,(n-1)$).

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CLOSE-TO-CONVEX FUNCTIONS WITH POSITIVE COEFFICIENTS

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REZUMAT. - Funcţii aproape convexe cu coeficienţi pozitivi. În lucrare sunt studiate funcţiile analitice în discul unitate, care satisfac condiţia Re $f'(z) < \beta$, $1 < \beta \leq 2$, $|z| < 1$.

Abstract. Let $f(z) = z + \sum_{n=2}^{\infty} |a_n| z^n$ be analytic in the unit disk $E = \{z: |z| < 1\}$. Coefficient inequalities, distortion Theorems and radius of convexity are determined for functions satisfying Re $f'(z) < \beta$, $1 < \beta \leq 2$, $z \in E$. Further it is shown that such functions are close-to-convex.

1. Introduction. Let A denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic in the unit disk $E = \{z: |z| < 1\}$. Let S be the subclass of A consisting of univalent functions in E. Let S^* be the subclass of S, the members of which are starlike (with respect to the origin) in E. A function $f \in S$ is said to be close-to-convex of order α, denoted by $f \in C(\alpha)$, $0 \leq \alpha < 1$ if there exists a function $g \in S^*$ such that Re $zf'(z)/g(z) > \alpha$ for $z \in E$.

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$C(0) = C$ is the class of close-to-convex functions.

Denote by V the subclass of S, consisting of functions of the form $f(z) = z + \sum_{n=2}^{\infty} |a_n| z^n$. Further let $P(\alpha)$ be the subclass of V consisting of functions that satisfy $\operatorname{Re} f'(z) > \alpha$, $0 \leq \alpha < 1$, $z \in E$.

Such functions are close-to-convex of order α with respect to the identity function z. Thus $P(\alpha) \subset C(\alpha)$, $P(0) = P$ is the subclass of close-to-convex functions in V.

For $1 < \beta \leq 3/2$ and $z \in E$, let $U(\beta) = \{f \in V: \operatorname{Re}(1 + zf''(z)/f'(z) < \beta\}$ and for $1 < \beta \leq 2$, $z \in E$, let $R(\beta) = \{f \in V: \operatorname{Re} f'(z) < \beta\}$.. In [5] the authors have studied the univalent functions with positive coefficients. Sarangi and Uralegaddi [3] and H.Al-Amiri [1] have studied the functions with negative coefficients that satisfy $\operatorname{Re} f'(z) > \alpha$, $0 \leq \alpha < 1$ for $z \in E$. In [2] Ozaki has proved that if $f \in A$, satisfies $\operatorname{Re}(1 + zf''(z)/f'(z)) < 3/2$, then f is univalent. And in [4] R.Singh and S.Singh have shown that such functions are close-to-convex.

In this paper coefficient in equalities, distortion Theorems and radius of convexity are determined for the class $R(\beta)$. Also it is shown that the functions in $R(\beta)$ are close-to-convex. Further it is proved that if $f \in V$, satisfies $\operatorname{Re}(1 + zf''(z)/f'(z)) < 3/2$, then $0 < \operatorname{Re} f'(z) \leq 2$, $z \in E$.

We need the following result [5].

THEOREM A. *$f \in U(\beta)$ if and only if*

$$\sum_{n=2}^{\infty} n(n-\beta)\,|a_n| \le \beta - 1.$$

2. Coefficient inequalities.

THEOREM 1. *If $f \in V$ and $\sum_{n=2}^{\infty} n|a_n| \le 1-\alpha$ $(0 \le \alpha < 1)$ then $f \in P(\alpha)$.*

Proof. Let $\sum_{n=2}^{\infty} n|a_n| \le 1-\alpha$. It suffices to show that $|f'(z)-1| < 1-\alpha$.

We have

$$\begin{aligned} |f'(z)-1| &= \left|\sum_{n=2}^{\infty} n a_n z^{n-1}\right| \\ &\le \sum_{n=2}^{\infty} n|a_n|\,|z|^{n-1} \\ &\le \sum_{n=2}^{\infty} n|a_n|. \end{aligned}$$

The last expression is bounded above by $1-\alpha$ by hypothesis. Hence $|f'(z)-1| < 1-\alpha$ and the theorem is proved.

THEOREM 2. *$f \in R(\beta)$ if and only if $\sum_{n=2}^{\infty} n|a_n| \le \beta - 1$.*

Proof. Let $\sum_{n=2}^{\infty} n|a_n| \le \beta - 1$. It suffices to prove that

$$\left|\frac{f'(z)-1}{f'(z)-(2\beta-1)}\right| < 1,\ z \in E.$$

We have

$$\left|\frac{f'(z)-1}{f'(z)-(2\beta-1)}\right| = \left|\frac{\sum_{n=2}^{\infty} n|a_n|z^{n-1}}{2(1-\beta)+\sum_{n=2}^{\infty} n|a_n|z^{n-1}}\right|$$

$$\leq \frac{\sum_{n=2}^{\infty} n|a_n|\ |z|^{n-1}}{2(\beta-1)-\sum_{n=2}^{\infty} n|a_n|\ |z|^{n-1}}$$

$$\leq \frac{\sum_{n=2}^{\infty} n|a_n|}{2(\beta-1)-\sum_{n=2}^{\infty} n|a_n|}.$$

The last is bounded above by 1 if $\sum_{n=2}^{\infty} n|a_n| \leq 2(\beta-1) - \sum_{n=2}^{\infty} n|a_n|$ which is equivalent to

$$\sum_{n=2}^{\infty} n|a_n| \leq \beta - 1. \tag{1}$$

But (1) is true by hypothesis and the theorem is proved.

Conversely suppose $\operatorname{Re} f'(z) = \operatorname{Re}\{1 + \sum_{n=2}^{\infty} n|a_n|z^{n-1}\} < \beta$, $z \in E$. Choose values of z on the real axis. Then letting $z \to 1$ through real values we obtain $1 + \sum_{n=2}^{\infty} n|a_n| \leq \beta$. That is $\sum_{n=2}^{\infty} n|a_n| \leq \beta - 1$.

COROLLARY. *If $f \in R(\beta)$ then $|a_n| \leq (\beta-1)/n$, with equality only for the functions of the form $f_n(z) = z + ((\beta-1)/n)z^n$.*

3. Distortion Theorems. The coefficient bounds enable us to prove

THEOREM 3. *If* $f \in R(\beta)$ *then*

(i) $r - \frac{\beta-1}{2} r^2 \leq |f(z)| \leq r + \frac{\beta-1}{2} r^2, \; (|z| = r)$

with equality for $f(z) = z + \frac{\beta-1}{2} z^2$.

(ii) $1 - (\beta-1)r \leq |f'(z)| \leq 1 + (\beta-1)r \quad (|z| = r)$

with equality for $f(z) = z + \frac{\beta-1}{2} z^2$.

Proof. From Theorem 2 we have

$$2\sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} n|a_n| \leq \beta - 1.$$

$$|f(z)| < r + \sum_{n=2}^{\infty} |a_n| r^n \leq r + r^2 \sum_{n=2}^{\infty} |a_n| \leq r + \frac{\beta-1}{2} r^2. \tag{2}$$

Similarly

$$|f(z)| > r - \sum_{n=2}^{\infty} |a_n| r^n > r - r^2 \sum_{n=2}^{\infty} |a_n| > r - \frac{\beta-1}{2} r^2. \tag{3}$$

The result (i) follows from (2) and (3). Similarly the result (ii) can be proved.

4. Comparable results.

THEOREM 4. *If* $f \in R(\beta)$ *then* $f \in P(2\text{-}\beta)$.

Proof. In view of Theorem 2 and Theorem 1 we must prove $\sum_{n=2}^{\infty} n|a_n| \leq \beta - 1$ implies $\sum_{n=2}^{\infty} n|a_n| \leq 1 - (2-\beta) = \beta - 1$, which is obvious.

COROLLARY. $R(2) \subset P$.

Thus the functions in $R(\beta)$ are close-to-convex.

THEOREM 5. *If* $f \in U(\beta)$ *then* $f \in R(1/(2-\beta))$.

Proof. In view of Theorem A and Theorem 2 we must show that $\sum_{n=2}^{\infty} \frac{n(n-\beta)}{\beta-1} |a_n| \le 1$ implies $\sum_{n=2}^{\infty} n|a_n| < \frac{1}{2-\beta} - 1 = \frac{\beta-1}{2-\beta}$. It is sufficient to show hat

$$\frac{n(n-\beta)}{\beta-1} \ge \frac{(2-\beta)n}{\beta-1}$$

which is equivalent to $n-\beta \ge 2-\beta$, $n = 2, 3, \ldots$ which is obvious.

COROLLARY 1. $U(3/2) \subset P$.

From the above two corollaries we have

COROLLARY 2. *If* $\mathrm{Re}(1+zf''(z)/f'(z)) < 3/2$ *then* $0 < \mathrm{Re} f'(z) \le 2$.

5. Radius of convexity.

THEOREM 6. *If* $f \in R(\beta)$ *then* $\mathrm{Re}(1+zf''(z)/f'(z)) > 0$ *in the disk* $|z| < r = r(\beta) = \operatorname{Inf}_n \left(\frac{1}{(\beta-1)n}\right)^{1/(n-1)}$, $n = 2,3,4,\ldots$. *The result is sharp for* $f_n(z) = z + \frac{\beta-1}{n} z^n$ *for some n.*

The proof is similar to that of Theorem 4 in [3]. Observe that $r(2) = 1/2$

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FIRST AND SECOND DIFFERENTIAL SUBORDINATIONS IN SEVERAL COMPLEX VARIABLES

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REZUMAT. - Subordonări diferenţiale de ordinul întâi şi doi în mai multe variabile complexe. În această lucrare sunt prezentate rezultate privind teoria subordonărilor diferenţiale de ordinul întâi şi doi pentru aplicaţii olomorfe de mai multe variabile complexe precum şi două aplicaţii referitoare la funcţii mărginite şi respectiv funcţii convexe.

1. Introduction. In several papers S.S.Miller and P.T.Mocanu [2,3] have considered analytic functions defined in the unit disc U which satisfy certain differential conditions and they determined properties of the functions themselves. Here, we consider similar relationships for mappings of several complex variables. We shall show that if a holomorphic mapping of several complex variables satisfies certain differential inequalities, then the function itself must satisfy an associated subordination.

In Section 2 we obtain several results concerning first and second order

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differential subordinations for holomorphic mappings defined in the unit ball. Section 3 contains applications of these results to bounded functions and convex functions.

We let $\mathbb{C}^n$ denote the space of n-complex variables $z = (z_1,\dots,z_n)'$, with the euclidian inner product $<z,w> = \sum_{j=1}^{n} z_j\bar{w}_j$ and the norm $\|z\| = (<z,z>)^{1/2}$. The ball $\{z\in\mathbb{C}^n: \|z\| < r\}$ will be denoted by B_r^n. For short we write B^n for B_1^n.

Vector and matrices marked with the symbol ' and * denote the transposed and the transposed conjugate vector or matrix, respectively.

We denote by $\mathcal{L}(\mathbb{C}^n)$ the space of continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^n$, i.e. the $n \times n$ complex matrices $A = (A_{jk})$, with the standard operator norm:

$$\|A\| = \sup\{\|Az\|: \|z\| \le 1\},\ A\in\mathcal{L}(\mathbb{C}^n).$$

The class of holomorphic mappings $f(z) = (f_1(z),\dots,f_n(z))'$ from B^n into $\mathbb{C}^n$ is denoted by $\mathcal{H}(B^n)$

We denote by $Df(z)$ and $D^2f(z)$ the first and the second Fréchet derivatives of f at z.

We say that $f\in\mathcal{H}(B^n)$ is locally biholomorphic (locally univalent) if f has a local holomorphic inverse at each point in B^n, or equivalently, derivative $Df(z) = \left(\dfrac{\partial f_k(z)}{\partial z_j}\right)_{1\le j,k\le n}$ is nonsingular at each point $z\in B^n$.

If $f, g \in \mathcal{H}(B^n)$, we say that f is subordinate to g (in B^n) if there exists a Schwarz function v such that $f(z) = g(v(z))$, $z \in B^n$, and we shall write $f \prec g$ to indicate that f is subordinate to g.

Now we present a previous generalization of Jack's, Miller's and Mocanu's lemma [1] which will be the basic tool in obtaining our main results.

THEOREM 1. *Let f be holomorphic in B^n with $f(0) = 0$ and $f \not\equiv 0$.*

If $\|f(\dot{z})\| = \max\limits_{\|z\| \le \|\dot{z}\|} \|f(z)\|$, $\dot{z} \in B^n$ and if $Df(\dot{z})$ is nonsingular then there exists a real positive number m which satisfies $m \le \dfrac{\|\dot{z}\|^2}{\|f(\dot{z})\|^2}$ such that:

$$((Df(\dot{z}))^*)^{-1}(\dot{z}) = m f(\dot{z}) \tag{1}$$

and

$$\frac{\|w\|^2 - \operatorname{Re}(\dot{z}^*(Df(\dot{z}))^{-1} D^2 f(\dot{z})(w,w))}{\|Df(\dot{z})w\|^2} \ge m \tag{2}$$

for all $w \in \mathbb{C}^n \setminus \{0\}$ which satisfy $\operatorname{Re} \langle w, \dot{z} \rangle = 0$.

2. First and second order differential inequalities.

LEMMA 1. *Let q be a holomorphic and univalent mapping defined on $\overline{B}^n$. Let p be holomorphic in B^n, with $p(0) = q(0)$. Suppose that there exists a point $\dot{z} \in B^n$, $\|\dot{z}\| = r_0 < 1$ such that $Dp(\dot{z})$ is nonsingular and*

$$p(\dot{z}) \in \delta q(B^n),\ p(B^n_{r_0}) \subset q(B^n). \tag{3}$$

If $\zeta_0 = q^{-1}(p(\mathring{z}))$ *then there exists a real positive number which satisfies* $m < \|\mathring{z}\|^2$ *such that:*

$$((Dp(\mathring{z}))^*)^{-1}(\mathring{z}) = m((Dq(\mathring{\zeta}))^*)^{-1}(\mathring{\zeta}) \tag{4}$$

and

$$\|w\|^2 - \mathrm{Re}\,\mathring{z}^*(Dp(\mathring{z}))^{-1}D^2p(\mathring{z})(w,w) \geq m(\|u\|^2 - \mathrm{Re}\,\mathring{\zeta}^*(Dq(\mathring{\zeta}))^{-1}D^2q(\mathring{\zeta})(u,u)) \tag{5}$$

for each $w \in \mathbb{C}^n \setminus \{0\}$ *with* $\mathrm{Re} < w, \mathring{z} > = 0$, *where* u *is defined by* $u = (Dq(\mathring{\zeta}))^{-1}Dp(\mathring{z})w$.

Proof. Let $f: q^{-1} \circ p: \bar{B}^n_{r_0} \to \bar{B}^n$. The function $f(z) = q^{-1}(p(z))$ is holomorphic in $\bar{B}^n_{r_0}$ and satisfies $f(0) = q^{-1}(p(0)) = 0$, $\|f(\mathring{z})\| = \|q^{-1}(p(\mathring{z}))\| = \|\mathring{\zeta}\| = 1$. Since $p(B^n_{r_0}) \subset q(B^n)$ we have:

$$1 = \|f(\mathring{z})\| = \max_{\|z\| \leq r_0} \|f(z)\|.$$

Also, since p is locally univalent at $\mathring{z}$ and q^{-1} is univalent on $q(B^n)$ we have that $Df(\mathring{z})$ is nonsingular. Thus f satisfies the conditions of Theorem 1. By Theorem 1 we obtain that there exists a real positive number m such that $m \leq \|\mathring{z}\|^2$ and

$$((Df(\mathring{z}))^*)^{-1}(\mathring{z}) = mf(\mathring{z}). \tag{6}$$

Since $p(z) = q(f(z))$ we have:

$$Dp(z) = Dq(f(z))Df(z) \tag{7}$$

and

$$((Dp(\mathring{z}))^*)^{-1}(\mathring{z}) = ((Dq(f(\mathring{z})))^*)^{-1}((Df(\mathring{z}))^*)^{-1}(\mathring{z}). \tag{8}$$

By using (6) and the fact that $f(\mathring{z}) = \mathring{\zeta}$ we get

$$((Dp(\mathring{z}))^*)^{-1}(\mathring{z}) = m((Dq(\mathring{\zeta}))^*)^{-1}(\mathring{\zeta})$$

so part (i) of Lemma 1 is satisfied.

Differentiating (8) at $z = \mathring{z}$ we get

$$D^2p(\mathring{z})(w,w) = D^2q(f(\mathring{z}))(Df(\mathring{z})w, Df(\mathring{z})w) + Dq(f(\mathring{z}))D^2f(\mathring{z})(w,w)$$

for each $w \in \mathbf{C}^n$.

By multiplying the above equality to the left with

$$\mathring{z}(Dp(\mathring{z}))^{-1} = \mathring{z}^*(Df(\mathring{z}))^{-1}(Dq(f(\mathring{z})))^{-1}$$

we obtain:

$$\mathring{z}^*(Dp(\mathring{z}))^{-1}D^2p(\mathring{z})(w,w) = \mathring{z}^*(Df(\mathring{z}))^{-1}(Dq(f(\mathring{z})))^{-1}D^2q(f(\mathring{z}))(Df(\mathring{z})w, Df(\mathring{z})w) + \\ + \mathring{z}^*(Df(\mathring{z}))^{-1}D^2f(\mathring{z})(w,w).$$

Substituting (6) into the above equality, using the fact that $f(\mathring{z}) = \mathring{\zeta}$ and noting $u = Df(\mathring{z})w$ we get

$$\|w\|^2 - \mathring{z}^*(Dp(\mathring{z}))^{-1}D^2p(\mathring{z})(w,w) = \|w\|^2 - \mathring{z}^*(Df(\mathring{z}))^{-1}D^2f(\mathring{z})(w,w) - \\ - m\mathring{\zeta}^*(Dq(\mathring{\zeta}))^{-1}D^2q(\mathring{\zeta})(u,u). \tag{9}$$

Next, we shall use that part (ii) of Theorem 1. If we take the real parts in (9) we have

$$\|w\|^2 - \mathring{z}^*(Dp(\mathring{z}))^{-1}D^2p(\mathring{z})(w,w) \geq m(\|u\|^2 - \mathrm{Re}\,\mathring{\zeta}^*(Dq(\mathring{\zeta}))^{-1}D^2q(\mathring{\zeta})(u,u))$$

for each $w \in \mathbf{C}^n \setminus \{0\}$ which satisfy $\mathrm{Re} < w, \dot{z} > = 0$.

Also, we note that the condition $\mathrm{Re} < w, \dot{z} > = 0$ implies $\mathrm{Re} < u, \dot{\zeta} > = 0$ where by u we denoted $u = Df(\dot{z})w$.

Indeed:

$$\begin{aligned}\mathrm{Re} < Df(\dot{z})w, \dot{\zeta} > &= \mathrm{Re} < (Dq(\dot{\zeta}))^{-1} Dp(\dot{z})w, \dot{\zeta} > = \\ &= \mathrm{Re} < Dp(\dot{z})w, ((Dq(\dot{\zeta}))^{-1})^*(\dot{\zeta}) > = \\ &= \mathrm{Re} < Dp(\dot{z})w, m((Dp(\dot{z}))^*)^{-1}(\dot{z}) > = \\ &= m\,\mathrm{Re} < w, \dot{z} > = 0.\end{aligned}$$

LEMMA 2. *Let q be a holomorphic and univalent mapping defined $\bar{B}^n$. Let p be holomorphic in B^n, locally univalent on B^n, with $p(0) = q(0)$.*

If $p \not\prec q$ then there exists $\dot{z} \in B^n$, $\dot{\zeta} \in \bar{B}^n$ with $\|\dot{\zeta}\| = 1$ and a real positive number $m \leq 1$ such that

$$\text{(i)} \qquad p(\dot{z}) = q(\dot{\zeta}) \tag{10}$$

$$\text{(ii)} \qquad ((Dp(\dot{z}))^*)^{-1}(\dot{z}) = m((Dq(\dot{\zeta}))^*)^{-1}(\dot{\zeta}) \tag{11}$$

and

$$\text{(iii)} \qquad \begin{aligned}&\|w\|^2 - \mathrm{Re}\, \dot{z}^* (Dp(\dot{z}))^{-1} D^2 p(\dot{z})(w,w) \geq \\ &\geq m(\|u\|^2 - \mathrm{Re}\, \dot{\zeta}^* (Dq(\dot{\zeta}))^{-1} D^2 q(\dot{\zeta})(u,u)),\end{aligned}$$

for each $w \in \mathbf{C}^n \setminus \{0\}$ with $\mathrm{Re} < w, \dot{z} > = 0$, where for each w, u is defined by $u = (Dq(\dot{\zeta}))^{-1} Dp(\dot{z})w$.

Proof. Since $p(0) = q(0)$ and $p(B^n) \not\subset q(B^n)$ there exists $0 < r_0 < 1$ such that $p(B_{r_0}^n) \subset q(B^n)$ and $p(\bar{B}_{r_0}^n) \cap q(\delta B^n) \neq \varnothing$.

Hence there exists $z_0 \in \bar{B}_{r_0}$ and $\zeta_0 \in \delta B^n$ such that $p(\dot{z}) = q(\dot{\zeta})$ and $p(B_{r_0}^n) \subset q(B^n)$.

The functions p,q satisfy the conditions of Lemma 1 and hence Lemma 2 is proved.

Before obtaining the main result of this section we need to specify the class of functions for which the differential inequalities will hold.

DEFINITION. Let D be a set in $\mathbf{C}^n$ and let q be holomorphic and univalent on $\bar{B}^n$.

We define $\psi[D,q]$ to be the class of mappings $\psi: \mathbf{C}^n \times \mathbf{C}^n \times B^n \to \mathbf{C}^n$ for which $\psi(r,s,z) \notin D$ when $r = q(\zeta)$, $s = m((Dq(\zeta))^*)^{-1}(\zeta)$, $z \in B^n$, $0 \leq m \leq 1$ and $\|\zeta\| = 1$.

We now present the main differential subordination result:

THEOREM 2. *Let $D \subseteq \mathbf{C}^n$, q be a holomorphic and univalent mapping defined on $\bar{B}^n$ and let $\psi \in \Psi[D,q]$. If $p \in \mathcal{H}(B^n)$ is locally univalent on B^n with $p(0) = q(0)$ and if p satisfies*

$$\psi(p(z),((Dp(z))^*)^{-1}(z),z) \in D \tag{13}$$

when $z \in B^n$, then $p \prec q$.

Proof. If $p(B^n) \not\subset q(B^n)$ there exists points $\mathring{z} \in B^n$ and $\zeta \in bB^n$ that satisfy (i) and (ii) of Lemma 2. Using these conditions with $r = p(\mathring{z})$, $s = ((Dp(\mathring{z}))^*)^{-1}(\mathring{z})$ in Definition we obtain

$$\psi(p(\mathring{z}),((Dp(\mathring{z}))^*)^{-1}(\mathring{z}),\mathring{z}) \notin D.$$

Since this contradicts (13) we must have $p(B^n) \subset q(B^n)$ and $p \prec q$.

The definition of $\psi[D,q]$ requires that the function q behave very nicely on δB^n. If this is not the case, or the behavior of q on δB^n is unknown, it may still be possible to prove $p \prec q$ by the following result:

THEOREM 3. *Let* $0 < \rho < 1$, *let* q *be holomorphic and univalent in* B^n *and suppose that:*

$$\psi(r,s;z) \in \psi[D,q_\rho] \text{ where } q_\rho(z) = q(\rho z). \tag{14}$$

If $p \in \mathcal{H}(B^n)$ *is locally univalent on* B^n *with* $p(0) = q(0)$ *and if* p *satisfies* (13), *when* $z \in B^n$ *then* $p \prec q$.

Proof. Since q_ρ is holomorphic and univalent on $\bar{B}^n$, the class $\psi[D,q_\rho]$ is well defined from (14) and Theorem 2 we obtain $p(B^n) \subset q_\rho(B^n)$, which implies $p(B^n) \subset q(B^n)$.

3. Bounded and convex functions on B^n. If we take $q(z) = Mz$ (where $M > 0$) in Definition and Theorem 2 we obtain the following result:

COROLLARY 1. *Let D be a set in $\mathbb{C}^n$ and let be $\psi: \mathbb{C}^n \times \mathbb{C}^n \times B^n \to \mathbb{C}^n$ be such that*

$$\psi(M\zeta, \frac{m}{M}\zeta, z) \notin D \tag{15}$$

where $\zeta \in \mathbb{C}^n$ with $\|\zeta\| = 1$, $m \in \mathbb{R}_+^*$ with $m \le 1$ and $z \in B^n$.

If $p \in \mathcal{H}(B^n)$ with $p(0) = 0$ and if p satisfies

$$\psi(p(z), ((Dp(z))^*)^{-1}(z), z) \in D \tag{16}$$

where $z \in B^n$, then $\|p(z)\| < M$.

We show the applicability of this result by an example.

Example. For $D = B_M^n$, $\psi(r,s,z) = r + \lambda s$ where $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \ge 0$ and $p \in \mathcal{H}(B^n)$ with $p(0) = 0$

$$\|p(z) + \lambda((Dp(z))^*)^{-1}(z)\| < M \text{ implies } \|p(z)\| < M.$$

Our final result is an application concerning subordination to convex mappings.

COROLLARY 2. *Let $h \in \mathcal{H}(B^n)$ be a convex mapping with $h(0) = 0$ and let $\lambda \in \mathbb{C}$ with* $\operatorname{Re} \lambda \ge 0$.

Let $p \in \mathcal{H}(B^n)$ locally univalent on B^n with $p(0) = 0$.

If

$$p(z) + \lambda((Dp(z))^*)^{-1}(z) \in h(B^n) \tag{17}$$

for every $z \in B^n$ *then* $p \prec h$.

Proof. Suppose that $p \nprec h$.

By using Lemma 2 we obtain that there exists $\dot{z} \in B^n$, $\mathring{\zeta} \in \bar{B}^n$ with $\|\mathring{\zeta}\| = 1$ such that:

$$p(\dot{z}) = h(\mathring{\zeta})$$

and

$$((Dp(\dot{z}))^*)^{-1}(\dot{z}) = m((Dh(\dot{z}))^*)^{-1}(\dot{z})$$

where $m \in \mathbf{R}_+$, $m \leq 1$.

If we note

$$\psi_0 = h(\mathring{\zeta}) + \lambda((Dp(\dot{z}))^*)^{-1}(\dot{z})$$

we obtain:

$$\psi_0 = h(\mathring{\zeta}) + m\lambda((Dh(\dot{z}))^*)^{-1}(\dot{z}).$$

Straightforward calcultion gives us:

$$< \psi_0 - h(\mathring{\zeta}), ((Dh(\dot{z}))^*)^{-1}(\dot{z}) > = m\lambda \|((Dh(\dot{z}))^*)^{-1}(\dot{z})\|^2 .$$

Since $m > 0$ and $\operatorname{Re} \lambda \geq 0$ we have

$$\operatorname{Re} < \psi_0 - h(\mathring{\zeta}), (Dh(\dot{z}))^*)^{-1}(\dot{z}) \geq 0$$

and taking account that $h(B^n)$ is a convex set we obtain $\psi_0 \notin h(B^n)$ which contradicts (17).

Hence our supposition was false, which means that $p \prec h$.

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PARTIAL DIFFERENTIAL SUBORDINATIONS FOR HOLOMORPHIC MAPPINGS OF SEVERAL COMPLEX VARIABLES

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REZUMAT. - Subordonări diferenţiale pentru funcţii olomorfe de mai multe variabile complexe. În această lucrare autorii consideră clase speciale de subordonări diferenţiale precum şi inegalităţi cuprinzând derivate parţiale de ordinul întâi pentru transformări olomorfe definite pe polidiscul unitate din $\mathbb{C}^n$.

Abstract. In this paper the authors consider special classes of subordinations and inequalities involving first partial derivatives of holomorphic mappings in the unit polidisc of $\mathbb{C}^n$.

1. Introduction. In several papers [6], [7], [8] S.S. Miller and P.T. Mocanu considered the analytic functions defined on the unit disc $U = \{z \in \mathbb{C}: |z| < 1\}$, which satisfy some differential inequalities or subordinations. Using the technique of subordination were obtained severeal

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results including inclusion relations, inequalities and some sufficient conditions for univalence. K. Dobrowolska, P. Liczberski in [3] and also, S. Gong, S.S. Miller in [4] proved that if an analytic function of several complex variables defined on a complete circular domain satisfy certain partial differential inequalities or subordinations, then the function itself must satisfies an associated inequality or subordination. P. Liczberski in [5] obtained some results concerning partial differential inequalities for holomorphic mappings on the open unit Euclidian ball.

In this paper we obtain a new generalization of Jack-Miller-Mocanu Lemma and then, using this result we will obtain some properties of holomorphic mappings defined on the unit polydisc of $\mathbf{C}^n$.

2. Preliminaries. We let $\mathbf{C}^n$ denote the space of n complex variables $z = \begin{pmatrix} z_1 \\ \dots \\ z_n \end{pmatrix}$ with the norm $\|z\| = \max_{1 \le j \le n} |z_j|$. By U_r^n and $H(U_r^n)$ we shall denote the open polydisc in $\mathbf{C}^n$ i.e. the set $\{z \in \mathbf{C}^n: \|z\| < r\}$, and the family of all holomorphic mappings $f: U_r^n \to \mathbf{C}^n$, respectively. If Ω is a region in $\mathbf{C}^n$ and f is a holomorphic mapping in Ω, then we denote by $Df(z)$ the Frechet derivative of f at $z \in \Omega$. Also, if F is a holomorphic function in Ω, then by

$DF(z)$ we denote the complex vector $\begin{pmatrix} \frac{\partial F(z)}{\partial z_1} \\ \dots \\ \frac{\partial F(z)}{\partial z_n} \end{pmatrix}$ for all $z \in \Omega$ and $D^2F(z)$ the complex matrix $\left[\frac{D^2F(z)}{\partial z_i \partial z_j}\right]_{1 \le i,j \le n}$. Let $L(\mathbf{C}^n)$ the space of all linear continuous operators from $\mathbf{C}^n$ into $\mathbf{C}^n$ with the standard operator norm $\|\cdot\|$ and let I be the identity in $L(\mathbf{C}^n)$, then the restriction $D^2f(z)(z, \cdot)$ of the continuous symmetric bilinear operator $D^2f(z)$ to $z \times \mathbf{C}^n$ belongs to $L(\mathbf{C}^n)$. If $z \in \mathbf{C}^n$, then z' will represent its transpose.

If Ω is a region in $\mathbf{C}^n$ and f is holomorphic in Ω, then we say that f is biholomorphic mapping in Ω if the inverse mapping f^{-1} does exist, is holomorphic on an open set $V \subseteq \mathbf{C}^n$ and $f^{-1}(V) = \Omega$.

The main results are based on the following lemmas.

LEMMA 2.1 [6,7]. *Let g be a holomorphic function in the unit disc U with $g(0) = 0$ and suppose that at $\zeta_0 \in U$ with $|\zeta_0| = r_0$, where $0 < r_0 < 1$, g satisfies the following condition*

$$|g(\zeta_0)| = \max\{|g(\zeta)|: |\zeta| \le |\zeta_0|\}$$

then there exists a real number $m \ge 1$ such that

$$\zeta_0 g'(\zeta_0) = mg(\zeta_0)$$

and

$$\operatorname{Re}\left[1 + \frac{\zeta_0 g''(\zeta_0)}{g'(\zeta_0)}\right] \geq m.$$

LEMMA 2.2 [3,4]. *Let $g\colon \bar{U} \to \mathbf{C}$ be a function which is holomorphic and univalent in $\bar{U}$ without at most one point $\zeta \in \partial U$, which is a simple pole. Let f be a holomorphic function in U_1^n with $f(0) = g(0)$. Suppose that $f(U_1^n) \not\subset g(U)$, then there exists $\zeta_0 \in \partial U$, $r_0 \in (0,1)$, $z_0 \in \bar{U}_{r_0}^n$ and a real number $m \geq 1$ such that $f(z_0) = g(\zeta_0)$ and $[Df(z_0)]'(z_0) = m\zeta_0 g'(\zeta_0)$.*

3. Main results. Now we give the main result.

THEOREM 3.1. *Let $f \in H(U_1^n)$ with $f(0) = 0$ and $f(z) \neq 0$. Let r be a real number from the open interval $(0,1)$. If for $z_0 \in \bar{U}_r^n$ we have*

$$\|f(z_0)\| = \max\{\|f(z)\| : z \in \bar{U}_r^n\}, \tag{3.1}$$

then there exist the real numbers m, s such that $s \geq m \geq 1$ and the following relations are satisfied

(i) $$\sum_{|f_k(z_0)| = \|f(z_0)\|} t_k \frac{[Df_k(z_0)]'(z_0)}{f_k(z_0)} = m;$$

(ii) $$\sum_{|f_k(z_0)| = \|f(z_0)\|} t_k \operatorname{Re}\left\{\frac{(z_0')D^2 f_k(z_0)(z_0)}{f_k(z_0)}\right\} \geq m(m-1),$$

where $t_k \geq 0$ *for each k and* $\sum_{|f_k(z_0)| = \|f(z_0)\|} t_k = 1$;

(iii) $\|Df(z_0)(z_0)\| = s\|f(z_0)\|$.

Proof. Let us denote $b = f(z_0)$, then according the condition (3.1) we can assume that $b \neq 0$. Because $(\mathbf{C}^n, \|\cdot\|)$ is normed space, Hahn Banach Theorem guarantees that there exists Λ a continuous linear function from $\mathbf{C}^n$ into $\mathbf{C}$ such that $\Lambda(b) = \|b\|$ and $|\Lambda(u)| \leq \|u\|$, for all $u \in \mathbf{C}^n$. But, it is well known that Λ can be written under the form $\Lambda(z) = \sum_{|b_k| = \|b\|} t_k \frac{\|b\|}{b_k} z_k$, for all $z \in \mathbf{C}^n$, where $z = \begin{pmatrix} z_1 \\ \dots \\ z_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ \dots \\ b_n \end{pmatrix}$, where $t_k \geq 0$ for each k and $\sum_{|b_k| = \|b\|} t_k = 1$.

Now, if we consider the complex function g, defined in the unit disc U by the formula

$$g(\zeta) = \Lambda \circ f(\zeta z_0 \|z_0\|^{-1}),\ \zeta \in U,$$

then g satisfy $g(0) = 0$ and $|g(\zeta_0)| = \max\{g(\zeta)| : |\zeta| \leq |\zeta_0|\}$, where $\zeta_0 = \|z_0\|$. So, from Lemma 2.1 we conclude that there exists a real number m, with $m \geq 1$, such that the condition (i) and (ii) are satisfied. On the other hand, the first equality can be rewritten as follows

$$\Lambda(Df(z_0)(z_0)) = m\|f(z_0)\|,$$

and using the inequality $|\Lambda(u)| \leq \|u\|$, for each $u \in \mathbf{C}^n$, then we deduce that there exists a real number s, with $s \geq m \geq 1$ and such that the last equality is satisfied.

Let us consider M be a positive number and let D be a domain in $\mathbf{C}^{2n}$ such that $(0,0)\in D$, where $0 = \begin{pmatrix} 0 \\ \dots \\ 0 \end{pmatrix}$.

Let $M_n = \bigcup_{s\geq 1} M_n^s(M)$, where $M_n^s(M) = \{(u,v)\in\mathbf{C}^{2n}: \|u\| = M, \|v\| = sM\}$ and suppose that $M_n \subseteq D$. Also let $G_n(D,M) = \{h: D \to \mathbf{C}^n$: h continuous in D, $\|h(0,0)\| < M$, $\|h(u,v)\| \geq M$, for all $(u,v) \in M_n\}$.

Using these classes and from the result of Theorem 3.1, we obtain the following result:

THEOREM 3.2. *Let $D \subseteq \mathbf{C}^{2n}$ be a domain, let $f \in H(U_1^n)$ with $f(0) = 0$ and $f(z) \neq 0$, $z \in U_1^n$. Suppose that there exists a mapping $h \in G_n(D,M)$ such that*

$$(f(z), Df(z)(z)) \in D$$

and

$$\|h(f(z), Df(z)(z))\| < M,$$

for all $z \in U_1^n$. Then $\|f(z)\| < M$, $z \in U_1^n$.

Proof. If we suppose that there exists $z_0 \in \bar{U}_r^n$, $r \in (0,1)$ with $\|f(z_0)\| = M = \max\{\|f(z)\|: z \in \bar{U}_r^n\}$, then using Theorem 3.1 we conclude that there exists a real number s, with $s \geq 1$ and such that

$\|Df(z_0)(z_0)\| = s\|f(z_0)\|$. Hence, if we denote by $u=f(z_0)$ and $v=Df(z_0)(z_0)$, then $(u,v)\in M_n^s(M)$ and because $h\in G_n(D,M)$, we deduce that $\|h(u,v)\|\geq M$, but this is a contradiction with the hypothesis. So, $\|f(z)\| < M$, for all $z \in U_1^n$.

Remark 3.1. It is interesting that this result can be used to show that some first order partial differential equations in $\mathbf{C}^n$ have bounded solution.

COROLLARY 3.1. *Let $F \in H(U_1^n)$ with $F(0) = 0$ and $\|F(z)\| < M$, for all $z \in U_1^n$. Let $h \in G_n(D,M)$ such that the differential equation*

$$h(f(z), Df(z)(z)) = F(z), f(0) = 0,$$

has a holomorphic solution f. Then $\|f(z)\| < M$, for all $z \in U_1^n$.

DEFINITION 3.1. Let f and g be holomorphic mappings on the unit polydisc U_1^n. We say that f is subordinate to g (written $f \prec g$ or $f(z) \prec g(z)$) if there exists $w\in H(U_1^n)$ with $w(0)=0$, $\|w(z)\| < 1$, for all $z\in U_1^n$ and $f = g \circ w$.

Remark 3.2. If f is subordinate to g, then $f(0) = g(0)$ and $f(U_1^n) \subseteq g(U_1^n)$. But, if g is biholomorphic in U_1^n, then easily we show that $f \prec g$ if and only if $f(0) = g(0)$ and $f(U_1^n) \subseteq g(U_1^n)$.

Now applying the result of Theorem 3.1, we obtain the following result.

THEOREM 3.3. *Let $f \in H(U_1^n)$, g be a biholomorphic in U_ρ^n, for some $\rho > 1$, with $f(0) = g(0)$. Suppose that f is not subordinate to g, then there exist* *an* $r_0 \in (0,1)$, *the points* $z_0 \in U_1^n$, $\|z_0\| \le r_0$, $\zeta_0 \in \partial U_1^n$ *and* m, s *be real numbers such that* $s \ge m \ge 1$ *and at* $z = z_0$ *the following relations are satisfied*

(i) $$f(z_0) = g(\zeta_0),\ f(\bar{U}_{r_0}^n) \subseteq g(\bar{U}_1^n);$$

(ii) $$\sum_{|\zeta_0^k|=1} t_k \frac{[D\tilde{g}_k(w_0)]' \, Df(z_0)(z_0)}{\zeta_0^k} = m,$$

where $w_0 = g(\zeta_0)$, $g^{-1}(w_0) = \begin{pmatrix} \tilde{g}_1(w_0) \\ \dots \\ \tilde{g}_n(w_0) \end{pmatrix}$, $\zeta_0 = \begin{pmatrix} \zeta_0^1 \\ \dots \\ \zeta_0^n \end{pmatrix}$ *and* $t_k \ge 0$ *for each* k, $\sum_{|\zeta_0^k|=1} t_k = 1$;

(iii) $$s\|[Dg(\zeta_0)]^{-1}\|^{-1} \le \|Df(z_0)(z_0)\| \le s\|Dg(\zeta_0)\|.$$

Proof. Since f is not subordinate to g and $f(0)=g(0)$ then $f(U_1^n) \not\subset g(U_1^n)$, hence we can get $r_0 \in (0,1)$ and the points $z_0 \in U_1^n$, $z_0 \in \bar{U}_{r_0}^n$, $\zeta_0 \in \partial U_1^n$, with $f(z_0) = g(\zeta_0)$ and $f(\bar{U}_{r_0}^n) \subseteq g(\bar{U}_1^n)$. On the other hand, if we consider the mapping $h(z) = (g^{-1} \circ f)(z)$, $z \in U_{r_0}^n$, then $h \in H(\bar{U}_{r_0}^n)$, $h(0) = 0$ and $1 = \|h(z_0)\| = \max\{\|h(z)\| : z \in \bar{U}_{r_0}^n\}$. Using the result of Theorem 3.1 and the properties of the norm of a linear operator from $\mathbf{C}^n$, a straightforward calculation shows our result.

Now, we are able to define the concept of "admissible class" in the case of several variables. This concept is given in the following definition.

DEFINITION 3.2. Let D and Ω be domains from $\mathbf{C}^n$ and $\mathbf{C}^{2n}$, respectively.

Let g be a biholomorphic in U_ρ^n for some $\rho > 1$, $\zeta_0 \in \partial U_1^n$ and m, t_k positive numbers, for each k, with $m \geq 1$, and $\sum_{|\zeta_0^k|=1} t_k = 1$.

$$\text{Let us } H_n^m(g) = \left\{(u,v) \in \mathbf{C}^{2n} : u = g(\zeta_0),\ \sum_{|\zeta_0^k|=1} t_k \frac{[D\tilde{g}_k(w_0)]' v}{\zeta_0^k} = m\right\},$$

where $w_0 = g(\zeta_0)$, $g^{-1}(w_0) = \begin{pmatrix} \tilde{g}_1(w_0) \\ \dots \\ \tilde{g}_n(w_0) \end{pmatrix}$ and $\zeta_0 = \begin{pmatrix} \zeta_0^1 \\ \dots \\ \zeta_0^n \end{pmatrix}$. Also, let

$H_n(g) = \bigcup_{\substack{m \geq 1 \\ \zeta_0 \in \partial U_1^n}} H_n^m(g)$, and suppose that $H_n(g) \subseteq \Omega$ and $(g(0),\ 0) \in \Omega$, respectively. The admissible class $\psi_n^n(D, \Omega, g)$ consist of those continuous mappings ψ_n: $\Omega \times U_1^n \to \mathbf{C}^r$ which satisfy

$$\psi_n(g(0), 0; 0) \in D$$

and

$$\psi_n(u, v; z) \notin D,$$

for all $(u, v) \in H_n(g)$ and $z \in U_1^n$.

Using this definition and from Theorem 3.3 we obtain the following result.

THEOREM 3.4. *Let $f \in H(U_1^n)$, g be a biholomorphic in U_ρ^n for some $\rho > 1$ and $f(0) = g(0)$. Suppose that there exists $\psi_n \in \psi_n^n(D, \Omega, g)$ such that*

$$(f(z), Df(z)(z)) \in \Omega$$

and

$$\psi_n(f(z), Df(z)(z); z) \in D,$$

for all $z \in U_1^n$. *Then f is subordinate to g.*

Proof. If we suppose that f is not subordinate to g, then from Theorem 3.3, we can get points $z_0 \in U_1^n$, $\zeta_0 \in \partial U_1^n$, and the real numbers $t_k \geq 0$, for each k, $\sum_{|\zeta_0^k|=1} t_k = 1$, $m \geq 1$ such that at $z = z_0$ the conditions (i) and (ii) are satisfied. Let us $u = f(z_0)$ and $v = Df(z_0)(z_0)$, then it is clear that $(u, v) \in H_n^m(g) \subseteq H_n(g)$, hence using the Definition 3.2 we conclude that $\psi_n(u, v; z_0) \notin D$, but this is a contradiction with the hypothesis. So, f is subordinate to g.

Furthermore we suppose that D is a special domain in $\mathbf{C}^n$, such that there exists h a biholomorphic mapping in U_1^n, with $h(U_1^n) = D$. But, it is clear that this assertion is not true for all domains in $\mathbf{C}^n$.

We denote the class $\psi_n^n(D, \Omega, g)$ by $\psi_n^n(h, \Omega, g)$ and following the result of Theorem 3.4 we obtain:

COROLLARY 3.3. *Let* Ω *be a domain in* $\mathbf{C}^{2n}$, *let g,h biholomorphic mappings in* U_ρ^n *for some* $\rho > 1$ *and let* $f \in H(U_1^n)$ *with* $f(0) = g(0)$. *Suppose that there exists a holomorphic mapping* $\psi_n \in \psi_n^n(h, \Omega, g)$ *such that*

$\psi_n(g(0), 0; 0) = h(0)$.

If

$$(f(z), Df(z)(z)) \in \Omega$$

and

$$\psi_n(f(z), Df(z)(z); z) \prec h(z),\ z \in U_1^n, \tag{3.2}$$

then $f(z) \prec g(z),\ z \in U_1^n$.

Remark 3.3. The biholomorphic mapping g is said to be a dominant of the differential subordination (3.2) if $f(z) \prec g(z)$ for all $f(z)$ satisfying (3.2). If $\tilde{g}$ is a dominant of (3.2) and $\tilde{g}(z) \prec g(z)$ for all dominants g of (3.2), then $\tilde{g}$ is said to be the best dominant of (3.2).

The following result gives a sufficient condition that g to be the best dominant of the subordination (3.2).

THEOREM 3.5. *Let Ω be a domain in $\mathbf{C}^{2n}$, let g, h biholomorphic mappings in U_ρ^n, for some $\rho > 1$ and let $f \in H(U_1^n)$ such that $f(0) = g(0)$. Suppose that there exists a holomorphic mapping $\psi_n \in \psi_n^n(h, \Omega, g)$ such that $\psi_n(g(0), 0; 0) = h(0)$ and g is a solution of differential equation*

$$\psi_n(g(z), Dg(z)(z); z) = h(z),\ z \in U_1^n. \tag{3.3}$$

If

$$\psi_n(f(z), Df(z)(z); z) \prec h(z),$$

then $f(z) \prec g(z)$, $z \in U_1^n$ *and* g *is the best dominant.*

Proof. Using Corollary 3.3, we conclude that f is subordinate to g and because g is a solution of differential equation (3.3), then from Remark 3.3 easily we deduce that g is the best dominant of (3.2).

4. Examples. Finally we obtain some applications which point out the usefulness of the above results.

If M is a positive number, let g: $U_1^n \to \mathbf{C}^n$, given by $g(z) = Mz$, for all $z \in U_1^n$, then g is biholomorphic in U_1^n and is easy to show that in this case the class $H_n^m(g)$ consist of those $(u,v) \in \mathbf{C}^{2n}$, with $u = M\zeta_0$, $u_k = Me^{i\theta_k}$, for $|\zeta_0^k| = 1$, $\sum_{|\zeta_0^k|=1} t_k v_k e^{-i\theta_k} = mM$, where $\zeta_0 \in \partial U_1^n$, $t_k \geq 0$, $\theta_k \in \mathbf{R}$ for all k, $\sum_{|\zeta_0^k|=1} t_k = 1$ and $m \geq 1$. Let $H_n(0) = \bigcup_{\substack{m\geq 1\\ \zeta_0 \in \partial U_1^n}} H_n^m(g)$, with $g(z) = Mz$, $z \in U_1^n$. Let D and Ω be domains in $\mathbf{C}^n$ and $\mathbf{C}^{2n}$, respectively and suppose that $H_n(0) \subseteq \Omega$. Let us $\psi_n^n(0)$ the class of those continuous mappings ψ_n: $\Omega \times U_1^n \to \mathbf{C}^n$ such that $\psi_n(0,0;0) \in D$ and $\psi_n(u,v;z) \notin D$, for all $(u,v) \in H_n(0)$ and $z \in U_1^n$.

An immediate application of Theorem 3.4 is given in the next result:

THEOREM 4.1. *Let* $f \in (U_1^n)$ *with* $f(0) = 0$ *and suppose that there exists a mapping* $\psi_n \in \psi_n^n(0)$ *such that*

$$(f(z), Df(z)(z)) \in \Omega$$

and

$$\psi_n(f(z), Df(z)(z); z) \in D,\ z \in U_1^n.$$

Then $\|f(z)\| < M,\ z \in U_1^n$.

The following theorem consists a direct application of Theorem 4.1.

THEOREM 4.2. *Let M, N be positive real numbers, let a and b functions defined in* U_1^n *which satisfy*

$$|a(z) + mb(z)| \geq \frac{N}{M},$$

for all $m \geq 1$ *and* $z \in U_1^n$. *Let* $f \in H(U_1^n)$ *such that* $f(0) = 0$ *and*

$$\|a(z)f(z) + b(z)Df(z)(z)\| < N,\ z \in U_1^n,$$

then $\|f(z)\| < M,\ z \in U_1^n$.

COROLLARY 4.1. *Let* α *be a function defined in* U_1^n, *which satisfies* $\mathrm{Re}\left[\frac{1}{\alpha(z)}\right] \geq -\frac{1}{2},\ z \in U_1^n$. *Let* $f \in H(U_1^n)$ *with* $f(0) = 0$ *and suppose that*

$$\|f(z) + \alpha(z)Df(z)(z)\| < 1,\ z \in U_1^n,$$

then $\|f(z)\| < 1,\ z \in U_1^n$.

For $a(z) \equiv 0$ in Theorem 4.2, we deduce:

COROLLARY 4.2. *Let M, N be positive real numbers, let b*: $U_1^n \to \mathbf{C}$ *be a function which satisfies in* U_1^n *the following condition* $|b(z)| \geq \frac{N}{M}$, $z \in U_1^n$. *Let* $f \in H(U_1^n)$ *such that* $f(0) = 0$ *and*

$$\|b(z)Df(z)(z)\| < N, \ z \in U_1^n,$$

then $\|f(z)\| < M$, $z \in U_1^n$.

The final result is as follows:

THEOREM 4.3. *Let* $f \in H(U_1^n)$ *and let g be a biholomorphic convex in* U_1^n *with* $f(0) = g(0) = 0$ *and* $Dg(0) = I$. *Let B, C be holomorphic functions in* U_1^n *and* $E \in H(U_1^n)$, $E(0) = 0$, *which satisfy*

$$\operatorname{Re} B(z) \geq |C(z)-1| - \operatorname{Re}[C(z)-1] + \alpha \|E(z)\|, \ z \in U_1^n,$$

where α *is a positive real number, with* $\alpha > 4$. *Suppose that*

$$B(z)Df(z)(z) + C(z)f(z) + E(z) \prec g(z), \ z \in U_1^n, \tag{4.1}$$

then $f \prec g$.

The proof is based on the following T.J. Suffridge's result [10].

LEMMA 4.1. *Suppose that* h: $U_1^n \to \mathbf{C}^n$ *is a convex biholomorphic mapping, with* $h(0) = 0$, *then there exists a nonsingular mapping* $T \in L(\mathbf{C}^n)$ *and analytic convex univalent functions* f_j, $j \in \{1, ..., n\}$ *of one variable such that* $h(z) = T\begin{pmatrix} f_1(z_1) \\ \dots \\ f_n(z_n) \end{pmatrix}$, *for all* $z = \begin{pmatrix} z_1 \\ \dots \\ z_n \end{pmatrix} \in U_1^n$.

Proof of Theorem 4.3. Using Lemma 4.1 by easily computation we deduce that there exists analytic and convex functions g_j, $j \in \{1, ..., n\}$ of one variable such that $g_j'(0) = 1$, $j \in \{1, ..., n\}$ and

$$g(z) = \begin{pmatrix} g_1(z_1) \\ \dots \\ g_n(z_n) \end{pmatrix}, \ z = \begin{pmatrix} z_1 \\ \dots \\ z_n \end{pmatrix} \in U_1^n.$$

Since $\alpha > 4$, there exists $r_0 \in (0,1)$ such that $\alpha = \dfrac{(1+r_0)^2}{r_0}$ and $\alpha > \dfrac{(1+r)^2}{r} > 4$, for $r_0 < r < 1$.

If we set $f^r(z) = f(rz)$ and $g^r(z) = g(rz)$, for $r_0 < r < 1$, then from (4.1) we obtain that

$$B^r(z)Df^r(z)(z) + C^r(z)f^r(z) + E^r(z) \prec g^r(z), \ z \in U_1^n \tag{4.2}$$

and $r_0 < r < 1$.

If we suppose that f^r is not subordinate to g^r for some $r \in (r_0,1)$, then there exists an integer $k \in \{1, ..., n\}$ such that $f_k^r(U_1^n) \not\subset g_k^r(U)$. Using Lemma 2.2, we can get points $z_0 \in U_1^n$, $\zeta_k \in \partial U$, and a real number $m_k \geq 1$ such that $f_k^r(z_0) = g_k^r(\zeta_k)$ and $[Df_k^r(z_0)]'(z_0) = m_k \zeta_k g_k^{r\prime}(\zeta_k)$.

If we denote by $\psi_k(z) = B^r(z)[Df_k^r(z)]'(z) + C^r(z)f_k^r(z) + E_k^r(z)$, $z \in U_1^n$ and $\lambda_k = \dfrac{\psi_k(z_0) - g_k^r(\zeta_k)}{\zeta_k g_k^{r\prime}(\zeta_k)}$, then $\operatorname{Re}\lambda_k = m_k \operatorname{Re} B^r(z_0) + \operatorname{Re}\left\{[C^r(z_0) - 1]\dfrac{g_k^r(\zeta_k)}{\zeta_k g_k^{r\prime}(\zeta_k)}\right\} +$

$$+\operatorname{Re}\left[\frac{E_k^r(z_0)}{\zeta_k g_k^{r'}(\zeta)}\right] \geq m_k \operatorname{Re} B^r(z_0) + \operatorname{Re}[C^r(z_0) - 1] - |C^r(z_0) - 1| - 4|E_k^r(z_0)| \geq 0,$$

using the inequality from the hypothesis and also, from very known relations for convex univalent functions in the unit disc U:

$$\operatorname{Re}\left[\frac{z g_k'(z)}{g_k(z)}\right] > \frac{1}{2} \text{ and } |g_k'(z)| > \frac{1}{(1+|z|)^2}, \ z \in U.$$

Now, using the fact that $\zeta_k g_k^{r'}(\zeta_k)$ is the outward normal to the boundary of the convex domain $g_k^r(U)$, we obtain that $\psi_k(z_0) \notin g_k^r(U)$, but this is a contradiction with (4.2). So, we must have f^r subordinate to g^r, for all $r_0 < r < 1$, hence letting $r \to 1^-$, we deduce f subordinate to g.

If $n = 1$ in Theorem 4.3, then this result was obtained by S.S. Miller and P.T. Mocanu [8].

If $C(z) \equiv 1$ in Theorem 4.3, we obtain

COROLLARY 4.3. *Let $f \in H(U_1^n)$ and let g be a biholomorphic convex in U_1^n with $f(0) = g(0) = 0$ and $Dg(0) = I$. Let B be holomorphic function in U_1^n and $E \in H(U_1^n)$ with $E(0) = 0$ and suppose that*

$$\operatorname{Re} B(z) \geq \alpha \|E(z)\|, \ z \in U_1^n,$$

where α is a positive number, with $\alpha > 4$.

If

$$B(z)\,Df(z)\,(z) + f(z) + E(z) \prec g(z),\ z \in U_1^n,$$

then $f \prec g$.

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PARABOLIC SYSTEMS WITH DISCONTINUOUS NONLINEARITY

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REZUMAT. - Sisteme parabolice cu nonlinearitate discontinuă. Se studiază rezolvabilitatea problemei Cauchy-Dirichlet pentru sisteme parabolice cu neliniaritate discontinuă.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipshitz boundary $\partial\Omega$. We consider the Cauchy-Dirichlet problem

$$\frac{\partial u_i}{\partial t} - L_i u = f_i(\cdot,u) \quad \text{in} \quad D_T = \Omega \times (0,T), \quad i = 1,\dots,N \tag{1}$$

$$u_i(x;t)\big|_{x\in\partial\Omega} = 0, \quad u_i(x,0) = \varphi_i(x) \quad x \in \Omega, \quad i = 1,\dots,N \tag{2}$$

where L_i are second order linear differential operators with real coefficients of the form

$$L_i(u) = \sum_{j=1}^{N} \sum_{k,l=1}^{n} \frac{\partial}{\partial x_k}\left[a_{kl}^{ij} \frac{\partial u_j}{\partial x_l}\right] - \sum_{j=1}^{N} a_0^{ij} u_j \quad i = 1,\dots,N \tag{3}$$

and $f_i : \Omega \times (0,T) \times \mathbb{R}^N \to \mathbb{R}$ are given functions.

Let

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$$L^2(\Omega, \mathbf{R}^N) = \{u = (u_1,\ldots,u_N) \mid u_i \in L^2(\Omega) \quad i = 1,\ldots,N\}$$

with the scalar product resp. norm

$$(u,v)_{L^2(\Omega,\mathbf{R}^N)} = \int_\Omega \sum_{i=1}^n u_i v_i \, dx, \quad \|u\|^2_{L^2(\Omega,\mathbf{R}^N)} = \int_\Omega \sum_{i=1}^n |u_i|^2 \, dx; \tag{4}$$

$$H_0^1(\Omega,\mathbf{R}^N) = \left\{ u \in L^2(\Omega,\mathbf{R}^N) \,\middle|\, \frac{\partial u_i}{\partial x_k} \in L^2(\Omega), u_i|_{\partial\Omega} = 0 \quad i = 1,\ldots,N, \quad k = 1,\ldots,n \right\}$$

with the scalar product resp. norm

$$(u,v)_{H_0^1(\Omega,\mathbf{R}^N)} = \int_\Omega \sum_{i=1}^N \sum_{k=1}^n \frac{\partial u_i}{\partial x_k} \frac{\partial v_i}{\partial x_k} \, dx, \quad \|u\|^2_{H_0^1(\Omega,\mathbf{R}^N)} = \int_\Omega \sum_{i=1}^N \sum_{k=1}^n \left| \frac{\partial u_i}{\partial x_k} \right|^2 dx \tag{5}$$

and $H^{-1}(\Omega, \mathbf{R}^N)$ the dual space of $H_0^1(\Omega, \mathbf{R}^N)$.

Besides these we need some other spaces too.

Let X a Banach or Hilbert space. We denote by $C([0,T], X)$ the linear space of the continuous functions $u : [0,T] \to X$ with the norm

$$\|u\|_{C([0,T],X)} = \sup_{t\in[0,T]} \|u(t)\|_X$$

Analogously if X is a Hilbert space let $L^2(0,T; X)$ the set of the measurable functions $u : (0,T) \to X$ for which $\int_0^T \|u(t)\|_X^2 \, dt < +\infty$. In $L^2(0,T;X)$ we use the scalar product

$$(u,v)_{L^2(0,T;X)} = \int_0^T (u(t),v(t))_X \, dt. \tag{6}$$

If X' is the dual space of X we can similarly define the spaces $C[[0,T], X']$, $L^2(0,T; X')$.

We shall use the following notations:

$$\begin{aligned} V &= L^2(0,T;L^2(\Omega,\mathbf{R}^N)) = L^2(D_T,\mathbf{R}^N), \quad & W = L^2(0,T;H_0^1(\Omega,\mathbf{R}^N)) \\ W' &= L^2(0,T;H^{-1}(\Omega,\mathbf{R}^N)), & Z = C([0,T),\ L^2(\Omega,\mathbf{R}^N)) \end{aligned} \tag{7}$$

If the system $Lu = (L_1u,...,L_Nu)$ is elliptic and weakly closed [2] for all $t \in (0,T)$ or is strongly elliptic [13], the coefficients a_0^{ij} satisfy some "sign" conditions, then for all $f_i \in L^2(D_T)$ (here f_i does not depend on u) and for all $\varphi_i \in L^2(\Omega)$ there exists a unique weak solution $u \in W \cap Z$ of the problem (1) - (2) and an estimate of the form

$$\|u\|_W \leq C\left(\|f\|_V + \|\varphi\|_{L^2(\Omega,\mathbf{R}^N)}\right) \tag{8}$$

is true.

For Cauchy-Dirichlet problem see [1, 6, 8, 14].

If the functions f_i depend on u and satisfy Caratheodory type conditions, then many existence results were obtained using various methods for nonlinear operatos [6, 8, 10].

In the tehnical applications appear various problems for parabolic systems with initial and boundary condition which contain discontinuous nonlinearities. In the study of these problems usually the inclusions differentials are applied. We use here a simple method proposed by S. Carl [4].

In this paper we study the solvability of the Cauchy-Dirichlet problem (1)-

(2) in the case when f_i does not depend explicitly on x and t and f_i has discontinuities in $u_1,\dots,u_N$. We assume in the sequel that a_{kl}^{ij}, $a_0^{ij} \in L^\infty(D_T)$ and we build the bilinear forms $a_i : W \times W \to \mathbf{R}$

$$a_i(u,v_i) = \int_{D_T} \sum_{j=1}^{N} \left[\sum_{k,l=1}^{n} a_{kl}^{ij} \frac{\partial u_j}{\partial x_k} \frac{\partial v_i}{\partial x_l} + a_0^{ij} u_j v_i \right] dx\,dt \tag{9}$$

DEFINITION 1. We say that $u \in W$ is a weak solution of (1) - (2) if $u \in Z$, $\frac{\partial u}{\partial t} \in W'$, $\left\langle \frac{\partial u}{\partial t}, v \right\rangle \in L^1(0,T)$, $f_i(u) \in L^2(D_T)$ and

$$\int_0^T \left\langle \frac{\partial u}{\partial t}, v \right\rangle dt + \sum_{i=1}^{N} a_i(u,v_i) = (f(u),v)_{L^2(D_T,\mathbf{R}^N)} \quad \forall v \in W \tag{10}$$

$$u(x,0) = \varphi(x) \qquad \text{a.e. on } \Omega \tag{11}$$

Here $\left\langle \frac{\partial u}{\partial t}, v \right\rangle$ stays for the pairing of the functional $\frac{\partial u}{\partial t}(t) \in H^{-1}(\Omega,\mathbf{R}^N)$ with $v(\cdot,t) \in H_0^1(\Omega,\mathbf{R}^N)$.

We introduce in $L^2(D_T,\mathbf{R}^N)$ a partially ordering relation. One says that $u \le v$ if and only if $v - u \in L_+^2(D_T,\mathbf{R}^N) = \{ w \in L^2(0,T;\mathbf{R}^N) \mid w_i(x) \ge 0 \text{ a.e. on } D_T \}$. Let $W_+ = W \cap L_+^2(D_T,\mathbf{R}^N)$. If $\underline{u}, \overline{u} \in L^2(D_T,\mathbf{R}^N)$ and $\underline{u} \le \overline{u}$, we denote

$$[\underline{u},\overline{u}] = \{ u \in L^2(D_T,\mathbf{R}^N) \mid \underline{u} \le u \le \overline{u} \}.$$

DEFINITION 2. We call $u \in W$ a weak upper solution of (1) - (2) if in definition 1 condition (10) is changed into

$$\int_0^T \left\langle \frac{\partial u}{\partial t}, v \right\rangle dt + \sum_{i=1}^{N} a_i(u,v_i) \ge (f,v)_{L^2(D_T,\mathbf{R}^N)} \quad \forall v \in W_+ \tag{12}$$

Similarly we define the weak lower solutions changing the sign "$\ge$" in

(12) into "≤".

We assume that

α1) The system $(L_1 u,\ldots,L_N u)$ is strongly elliptic or weakly closed

α2) There exists a positive constant M_1 such that for all $M \geq M_1$ the Cauchy-Dirichlet problem

$$\frac{\partial u}{\partial t} - Lu + Mu = g \quad \text{in } D_T, \quad u(x,t)\,|_{x\in\partial\Omega} = 0, \quad u(x,0) = \varphi(x) \qquad (13)$$

has a unique weak solution u for all $g \in L^2(D_T, \mathbf{R}^N)$ and $\varphi \in L^2(\Omega, \mathbf{R}^N)$. For the parabolic operator $\frac{\partial}{\partial t} - L + MI$ the weak maximum and minimum principle are true in the sense that: $u \in W$, $u(x,0) = 0$ on Ω and

$$A_M(u,v) := \int_0^T \left\langle \frac{\partial u}{\partial t}, v \right\rangle dt + \sum_{i=1}^N a_i(u,v_i) + M \int_{D_T} \sum_{i=1}^N u_i v_i \, dx\, dt \geq 0 \;\; \forall v \in W_+, \qquad (14)$$

implies $u(x,t) \geq 0$ a.e. on D_T; and from $u \in W$, $u(x,0) = 0$ on Ω, and from $A_M(u,v) \leq 0$ $\forall v \in W_+$ results that $u(x,t) \leq 0$ a.e. on D_T.

Conditions α1 and α2 are obviously fulfilled if $L_i u$ contains only the function u_i

$$L_i u = \sum_{k,l=1}^n \frac{\partial}{\partial x_k}\left[a_{kl}^i \frac{\partial u_i}{\partial x_l} \right] - a_0^i u_i \qquad i = 1,\ldots,N \qquad (15)$$

and there exists $\mu > 0$ such that

$$\sum_{k,l=1}^n a_{kl}^i(x,t)\xi_k \xi_l \geq \mu \sum_{k=1}^n \xi_k^2 \text{ for a.e. } (x,t) \in D_T, \; \forall \xi \in \mathbf{R}^n, \; i = 1,\ldots,N. \quad (16)$$

For the maximum and minimum principles see [3, 5, 7, 12].

β1) There exists a positive constant M_2 such that the functions

$$F_i(\tau) = f_i(\tau) + M\tau_i \quad \tau \in \mathbb{R}^N \ \ i = 1,...,N \tag{17}$$

are monotone increasing for every $M \geq M_2$, e.g.

$$F_i(\tau^1) \leq F_i(\tau^2) \text{ if } \tau_j^1 \leq \tau_j^2 \ j = 1,...,N$$

β2) There exist a finite or countable number of surfaces $S_k \subset \mathbb{R}^N$ for which we have a representation

$$S_k = \{\tau = (\tau_1,...,\tau_N) \in \mathbb{R}^N \mid \tau_N = \psi_{Nk}(\tau'), \quad \tau' = (\tau_1,...,\tau_{N-1}) \in \mathbb{R}^{N-1}\}, \tag{18}$$

where $\psi_{Nk} \in C^1(\mathbb{R}^{N-1})$ and

$$\psi_{N,k}(\tau') > \psi_{N,k-1}(\tau') \quad \forall \tau' \in \mathbb{R}^{N-1}, \ \ \forall k$$

The functions $f_i : \mathbb{R}^N \to \mathbb{R}$ are continuous on $\mathbb{R}^N \setminus \bigcup_k S_k$, f_i has one-side limits on S_k e.g.

$$f^-(\tau) = \lim_{\substack{\xi \to \tau \\ \xi_N < \tau_N}} f(\xi_1,...,\xi_N), \quad f^+(\tau) = \lim_{\substack{\xi \to \tau \\ \xi_N > \tau_N}} f(\xi_1,...,\xi_N)$$

exist and are finite.

γ) The Cauchy-Dirichlet problem (1) - (2) has a lower solution $\underline{u}$ and an upper solution $\overline{u}$ such that $\underline{u} \leq \overline{u}$.

LEMMA 1. *We assume that the conditions* β1), β2) *and* γ) *are fulfilled and* $M \geq \max\{M_1, M_2\}$ *is a constant. Then*

1°. *For every* $u \in [\underline{u}, \overline{u}]$ *the function* $F(u) = f(u) + Mu$ *belongs to* $L^2(D_T, \mathbb{R}^N)$.

2° If $u, v \in [\underline{u}, \overline{u}]$ and $u \leq v$ then $F(u) \leq F(v)$.

3° *The set* $\{F(u) \mid u \in [\underline{u}, \overline{u}]\}$ *is bounded in* $L^2(D_T, \mathbb{R}^N)$.

For the proof see [11].

Let $M_0 = \max\{M_1, M_2\}$, $M \geq M_0$ a constant, $\varphi \in L^2(\Omega, \mathbb{R}^N)$ a fixed element and $w \in [\underline{u}, \overline{u}]$ an arbitrary function.

THEOREM 1. *If the hypotheses* α1, α2, β1, β2 *and* γ *are satisfied, then the Cauchy-Dirichlet problem*

$$\begin{aligned} &\frac{\partial u}{\partial t} - Lu + Mu = f(w) + Mw \quad \text{on } D_T \\ &u(x,t)|_{x\in\partial\Omega} = 0, \quad u(x,0) = \varphi(x) \end{aligned} \tag{19}$$

has a unique weak solution $u \in [\underline{u}, \overline{u}]$. If $\varphi \in H_0^1(\Omega, \mathbb{R}^N)$ *then* $u \in W \cap L^2(0, T, H^2(\Omega, \mathbb{R}^N))$ *and* $\frac{\partial u}{\partial t} \in V$.

Proof. By Lemma 1 $F(w) = f(w) + Mw \in L^2(D_T, \mathbb{R}^N)$. In this case the unique solvability of (19) results from α1) and α2). Let u the weak solution of (19). The function $\overline{u}$ is a weak upper solution of (1) - (2) with the same $\varphi \in L^2(\Omega, \mathbb{R}^N)$, thus we have

$$A_M(u, v) = \int_{D_T} \sum_{i=1}^{N} F_i(w)\, v_i \, dx\, dt \qquad \forall v \in W$$

$$A_M(\overline{u}, v) \geq \int_{D_T} \sum_{i=1}^{N} F_i(\overline{u})\, v_i \, dx\, dt \qquad \forall v \in W_+$$

$$u(x, o) = \overline{u}(x, 0) = \varphi(x) \qquad \text{a.e. on } \Omega$$

For the function $\overline{u} - u$ we obtain then

$$A_M(\overline{u} - u, v) \geq \int_{D_T} [F_i(\overline{u}) - F_i(w)]\, v_i \, dx\, dt \geq 0 \qquad \forall v \in W_+$$

and $(\overline{u} - u)(x,0) = 0$.

Applying the maximum principles the last two formulae give $\overline{u} - u \geq 0$ a.e. on D_T. Simillarly we obtain $\underline{u} - u \leq 0$, and then $\underline{u} \leq u \leq \overline{u}$.

If $\varphi \in H_0^1(\Omega, \mathbf{R}^N)$ then $u \in L^2(0, T, H^2(\Omega, \mathbf{R}^N))$ and $\dfrac{\partial u}{\partial t} \in V$ (see [1]).

Let $M_3 > M_0$. We consider the family of the cauchy-Dirichlet problem (19) when w describes the interval $[\underline{u}, \overline{u}]$, $M \in [M_0, M_3]$ and φ is the same function for all problems. We denote by u_{wM} the weak solution of (19).

THEOREM 2. *There exist positive constants* C_2 and C_3 *such that*

$$\|u_{wM}\|_W \leq C_2 \qquad \forall w \in [\underline{u}, \overline{u}], \ \ \forall M \in [M_0, M_3] \tag{20}$$

$$\left\| \frac{\partial u_{wM}}{\partial t} \right\|_{W'} \leq C_3 \qquad \forall w \in [\underline{u}, \overline{u}], \ \ \forall M \in [M_0, M_3] \tag{21}$$

Proof. $F(w) = f(w) + Mw \in V$ so from conditions $\alpha 1$) and $\alpha 2$) results that there exists a constant $C > 0$ such that for the solutions u_{wM} of the problem (19) we have

$$\|u_{wM}\|_W \leq C(\, \|F(w)\|_V + \|\varphi\|_{L^2(\Omega, \mathbf{R}^N)}) \qquad \forall w \in [\underline{u}, \overline{u}] \tag{22}$$

According to Lemma 1 $\{\, \|F(w)\|_V \mid w \in [\underline{u}, \overline{u}] \,\}$ is bounded, φ is the same for all M, therefore there exists $C_2 > 0$ such that (20) is true. The estimate (21) is a consequence of (20) and

$$\int_0^T \left\langle \frac{\partial u_{wM}}{\partial t}, v \right\rangle dt + \sum_{i=1}^N a_i(u_{wM}, v_i) + M(u_{wM}, v)_V = \int_{D_T} \sum_{i=1}^N F_i(w)\, v_i\, dx\, dt \quad \forall v \in W$$

LEMMA 2. *Let* $u^1, u^2, \dots, u^k, \dots$ *a bounded monotone sequence (increasing or decreasing) in* W *for which* $\{ \| \frac{\partial u_k}{\partial t} \| \mid k = 1,2,\dots\}$ *is also bounded. Then* $(u^k)_{k\geq 1}$ *is weakly convergent in* W*, strongly convergent in* $L^2(D_T, \mathbf{R}^N)$ *and* $\int_0^T \left\langle \frac{\partial u_k}{\partial t}, v \right\rangle dt \to \int_0^T \left\langle \frac{\partial u}{\partial t}, v \right\rangle dt \quad \forall v \in W \ (u = \lim u^k)$.

Proof. The monotonicity of $(u^k)_{k\geq 1}$ means here the monotonicity of the components of $u^k = (u_1^k, \dots, u_N^k)$. Then Lemma 2 results from [4].

THEOREM 3. *Let* $\underline{u}, \overline{u} \in W$ *be one lower resp. upper solution of the Cauchy-Dirichlet problem* (1) - (2). *Assume that the conditions* α1), α2), β1), β2) *and* γ) *are fulfilled and* $f^+(\tau) = f(\tau)$ (or $f^-(\tau) = f(\tau)$) *for every* $t \in \bigcup_k S_k$. *Then there exists at least one weak solution* $u \in [\underline{u}, \overline{u}]$ *of problem* (1) - (2).

Proof. We use a constructive iterative method proposed by S. Carl [4] solving an infinite sequence of problems. Let $\varphi \in L^2(\Omega, \mathbf{R}^N)$ be the given function in (2). We chose an $M \in [M_0, M_3]$ and start with the problem

$$\begin{aligned} &\frac{\partial U^1}{\partial t} - LU^1 + MU^1 = f(U^0) + MU^0 \quad \text{in } \Omega \times (0,T) \\ &U^1(x,t)|_{x\in\partial\Omega} = 0, \quad U^1(x,0) = \varphi(x) \end{aligned} \tag{23}$$

where $U^0 = \overline{u}$.

(23) has a unique weak solution U^1. Thus we have

$$A_M(U^0, v) \geq \int_{D_T} \sum_{i=1}^{N} f_i(U^0)\, v_i\, dx\, dt + M \int_{D_T} \sum_{i=1}^{N} U_i^0\, v_i\, dx\, dt \qquad \forall v \in W_+$$

$$A_M(U^1, v) = \int_{D_T} \sum_{i=1}^{N} f_i(U^0)\, v_i\, dx\, dt + M \int_{D_T} \sum_{i=1}^{N} U_i^0\, v_i\, dx\, dt \qquad \forall v \in W$$

The last two formulae give

$$A_M(U^0 - U^1, v_i) \geq 0 \quad \forall v \in W_+$$

In the same way we get

$$A_M(\underline{u} - U^1, v) \leq 0 \quad \forall v \in W_+$$

Using the maximum resp. minimum principle we obtain

$$\underline{u} \leq U^1 \leq U^0 = \overline{u} .$$

In the same manner the sequence $U^1, U^2, \dots, U^k, \dots$ is built solving the Cauchy-Dirichlet problems

$$\begin{cases} \dfrac{\partial U^{k+1}}{\partial t} - L U^{k+1} + M U^{k+1} = f(U^k) + M U^k \\ U^{k+1}(x, t)\,|_{x \in \partial\Omega} = 0, \quad U^{k+1}(x, 0) = \varphi(x) \quad x \in \Omega \end{cases} \tag{24}$$

It is obvious that

$$\underline{u} \leq U^{k+1} \leq U^k \leq \dots \leq U^1 \leq U^0 = \overline{u} .$$

By Theorem 2 the sequence $(u^k)_{k \geq 1}$ is bounded in W, and $\left\| \dfrac{\partial U^k}{\partial t} \right\|_{W'} \leq C_3$. Then from Lemma 2 results that $(u^k)_{k \geq 1}$ is strongly convergent in V and weakly convergent in W. U^k is the weak solution of (24), thus

$$\int_0^T \left\langle \frac{\partial U^{k+1}}{\partial t}, v \right\rangle dt + \sum_{i=1}^N a_i(U^{k+1}, v_i) + M \int_{D_T} \sum_{i=1}^N U_i^{k+1} v_i \, dx\, dt$$
$$= \int_{D_T} \sum_{i=1}^N f_i(U^k) v_i \, dx\, dt + M \int_{D_T} \sum_{i=1}^N U_i^k v_i \, dx\, dt \tag{25}$$

But according to Lemma 2 U_k converges strongly to an $U \in L^2(D_T, \mathbf{R}^N)$, $U^k \to U$ weakly in W, $\frac{\partial u}{\partial t} \in W'$ and $\int_0^T \left\langle \frac{\partial U^k}{\partial t}, v \right\rangle dt \to \int_0^T \left\langle \frac{\partial U}{\partial t}, v \right\rangle dt$.

Consequently after passing to limit the left side of (25) is

$$\int_0^T \left\langle \frac{\partial U}{\partial t}, v \right\rangle dt + \sum_{i=1}^N a_i(U, v_i) + M \int_{D_T} \sum_{i=1}^n U_i v_i \, dx\, dt \tag{26}$$

We shall show that the limit of the right side of (25) exists and is equal to

$$\int_{D_T} \sum_{i=1}^N f_i(U) v_i \, dx\, dt + M \int_{D_T} \sum_{i=1}^N U_i v_i \, dt$$

$U^k(x,t)$ converges decreasing to $U(x,t)$ a.e. on D_T, f_i is continuous on $\mathbf{R}^N \setminus \bigcup_j S_j$, $f_i(\tau) = f_i^+(\tau)$ on S_j, thus $f_i(U^k(x,t)) \to f_i(U(x,t))$ a.e. on D_T and from Theorem 2 results that

$$\int_{D_T} f_i(U^k(x,t)) \cdot v_i(x,t) \, dx\, dt \mid \leq \|f_i(U^k)\|_{L^2(D_T)} \cdot \|v_i\|_{L^2(D_T)} \leq C$$

where C is a conveniently chosen constant. Thus we can pass to limit in the right side of (25), too, and we obtain

$$\int_0^T \left\langle \frac{\partial U}{\partial t}, v \right\rangle dt + \sum_{i=1}^{N} a_i(U, v_i) = \int_{D_T} \sum_{i=1}^{N} f_i(U)\, v_i \, dx\, dt \qquad \forall v \in W$$

which means that U is a weak solution of problem (1) - (2).

If $f_j(\tau) = f_j^-(\tau)$ on S_k then starting with $u^0 = \underline{u}$ (lower solution of (1) - (2)) we can build a convergent sequence $u^1, u^2, \ldots, u^k, \ldots$, u_k is the solution of

$$\begin{cases} \dfrac{\partial u^{k+1}}{\partial t} - L u^{k+1} + M u^{k+1} = f(u^k) + M u^k \quad \text{in } D_T \\ u^{k+1}(x,t)\big|_{x \in \partial\Omega} = 0, \quad u^{k+1}(x,0) = \varphi(x) \end{cases} \tag{27}$$

For the solution $u = \lim u^{k+1}$ we have then

$$\underline{u} \le u \le \overline{u}$$

REMARK 1. For both cases $f_i(\tau) = f_i^+(\tau)$ and $f_i(\tau) = f_i^-(\tau)$ we may start the iteration method with any $U^0, u^0 \in [\underline{u}, \overline{u}]$. The sequences built by the method (24) resp. (27) may converge to an element different from U resp. u obtained in Theorem 3. We have the following

THEOREM 4. a) *If in Theorem* 3 $f_i(\tau) = f_i^+(\tau)$ $\tau \in \bigcup_k S_k$, *then the solution* U *of the Cauchy-Dirichlet problem* (1) - (2) *obtained in the proof of Theorem* 3 *is maximal in the sense that for all solutions* $w \in [\underline{u}, \overline{u}]$ *of problem* (1) - (2) *we have* $w \le U$.

b) *If* $f_i(\tau) = f_i^-(\tau)$ $\tau \in \bigcup_k S_k$, *then the solution* u *obtained by algorithm* (27) *is minimal, that is* $u \leq w$ *for any solution* $w \in [\underline{u}, \overline{u}]$.

For the proof see [11].

REMARK 2. using differential inclusions we may weaken the assumptions about the operator L and functions f_i [9], but in this case we can not apply the simple constructive method offered by the monotone iterative technique.

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LACUNARY INTERPOLATION BY CUBIC SPLINE

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REZUMAT. - Interpolare lacunară prin funcţii spline cubice. În această notă se studiază o problemă de interpolare prin funcţii spline cubice. Se dă o metodă de construire a soluţiei şi se evaluează eroarea aproximării.

As a generalization of polynomial Birkhoff interpolation, I.J.Schoenberg [22] had initiated the studying of lacunary interpolation by spline functions. Next, involving the values of a given function and of certain of its derivatives it was studied different particular cases of lacunary spline interpolation problems.

The goal of this note is to study such a lacunary spline problem and to give a method to construct corresponding solution. Also the approximation error is evaluated.

Let $S_p(3.\Delta_n)$ be the set of the cubic splines for the partition

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$$\Delta_n: a = x_0 < x_1 < \ldots < x_n = b,\ n \in \mathbf{N},\ n > 1, \tag{1}$$

i.e.:

$$\begin{cases} 1)\ s|_{[x_{i-1}, x_i]} \in P_3, \\ 2)\ s \in C^2[a,b] \end{cases} \quad i = \overline{1,n} \tag{2}$$

For a spline $s \in S_p(3,\Delta_n)$ we can write

$$s''(x) = M_{i-1} + \frac{M_i - M_{i-1}}{x_i - x_{i-1}}(x - x_{i-1}) \text{ for all } x \in [x_{i-1}, x_i],\ i = \overline{1,n} \tag{3}$$

where $M_k = s''(x_k)$, $k = \overline{0,n}$.

For $f \in C[a,b]$ one considers the conditions

$$\begin{cases} s(x_i) = f(x_i) \\ s'(x_i) = m_i \end{cases} \quad i = \overline{0,n}. \tag{4}$$

As a solution of the differential problem (3)-(4), one obtains [24]

$$s(x) = \frac{M_i - M_{i-1}}{6(x_i - x_{i-1})}(x - x_{i-1})^3 + \frac{M_{i-1}}{2}(x - x_{i-1})^2 + m_{i-1}(x - x_{i-1}) + f(x_{i-1}) \tag{5}$$

for $x \in [x_{i-1}, x_i]$, $i = \overline{1,n}$.

In some supplementary conditions [24] the solution s of the above differential problem can be unique determined.

Now, one considers the following lacunary spline interpolation problem: find the cubic spline $s \in S_p(3,\Delta_n)$ that interpolates the data

$$\begin{cases} f(x_0), \ldots, f(x_n) \\ f''(t_1), \ldots, f''(t_n), \quad t_i \in (x_{i-1}, x_i),\ i = \overline{1,n} \\ f''(x_0) \end{cases} \tag{6}$$

Without lost of generality, it can be considered the interval [0,1] and its uniform partition Δ_n given by the nodes

$$x_i = \frac{i}{n}, \quad i = \overline{0,n}$$

with the norm $h = 1/n$.

The points t_i can be written as

$$t_i = x_{i-1} + \alpha h, \text{ with } \alpha \in (0,1), \quad i = \overline{1,n} \tag{7}$$

THEOREM 1. *Let* $f\colon [0,1] \to \mathbf{R}$ *be a given function for which there exist* $f''(0)$ *and* $f''(t_i)$, $t_i \in (x_{i-1}, x_i)$ *for all* $i = \overline{1,n}$. *Then, there exists a unique cubic spline* $s \in S_p(3,\Delta_n)$ *such that*

$$\begin{aligned} s\left(\frac{i}{n}\right) &= f\left(\frac{i}{n}\right), \quad i = \overline{0,n} \\ s''(0) &= f''(0) \\ s''(t_i) &= f''(t_i), \quad i = \overline{1,n} \end{aligned} \tag{8}$$

and

$$s(x) = \frac{n}{6}(\lambda_i - \lambda_{i-1})\left(x - \frac{i-1}{n}\right)^3 + \frac{\lambda_{i-1}}{2}\left(x - \frac{i-1}{n}\right)^2 + \left[n\left(f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right) - \frac{\lambda_i + 2\lambda_{i-1}}{6n}\right]\left(x - \frac{i-1}{n}\right) + f\left(\frac{i-1}{n}\right), \quad x \in \left(\frac{i-1}{n}, \frac{i}{n}\right), \quad i = \overline{1,n} \tag{9}$$

with λ_i *some parameters.*

Proof. We look for the function s as a solution of the differential problem

$$s''(x) = \frac{\lambda_i - \lambda_{i-1}}{h}\left(x - \frac{i-1}{n}\right) + \lambda_{i-1}, \quad x \in \left(\frac{i-1}{n}, \frac{i}{n}\right) \tag{10}$$

$$s\left(\frac{i}{n}\right) = f\left(\frac{i}{n}\right), \quad i = \overline{1,n} \ . \tag{11}$$

We have

$$s(x) = \frac{\lambda_i - \lambda_{i-1}}{6h}\left(x - \frac{i-1}{n}\right)^3 + \frac{\lambda_{i-1}}{2}\left(x - \frac{i-1}{n}\right)^2 + C_1\left(x - \frac{i-1}{n}\right) + C_2,$$

$x \in \left(\frac{i-1}{n}, \frac{i}{n}\right)$, $i = \overline{1,n}$, where the constants C_1 and C_2 are obtained from the conditions (11). One obtains

$$C_2 = f\left(\frac{i-1}{n}\right)$$

$$C_1 = n\left[f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right] - \frac{\lambda_i + 2\lambda_{i-1}}{6h}, \ i = \overline{1,n}$$

and (9) follows.

To find the parameters λ_i, $i = \overline{0,n}$, there are used the interpolation conditions (8).

One obtains the follow ng system

$$\begin{cases} \lambda_0 = f''(0) \\ \lambda_{i-1} + n(\lambda_i - \lambda_{i-1})\left(t_i - \frac{i-1}{n}\right) = f''(t_i), \ i = \overline{1,n} \end{cases} \tag{12}$$

Using the relation

$$t_i = \frac{i-1}{n} + \alpha\frac{1}{n}, \ \alpha \in (0,1)$$

the system (12) becomes

$$\begin{cases} \lambda_0 = f''(0) \\ \alpha\lambda_i + (1-\alpha)\lambda_{i-1} = f''(t_i), \ i = \overline{1,n}, \ \alpha \in (0,1) \end{cases} \tag{13}$$

This system has a unique solution, so the theorem is completely proved.

THEOREM 2. *If* $f \in C^2[0,1]$ *and* $f \in \mathrm{Lip}_L[0,1]$, *with* L *the Lipschitz constant, then for the corresponding cubic spline interpolation* s, *we have:*

$$\text{i)}\ |s(x)-f(x)| \le \frac{2L}{n} + \frac{1}{6n^2}\left[|\lambda_i-\lambda_{i-1}| + |\lambda_i+2\lambda_{i-1}| + 3|\lambda_{i-1}|\right]$$

$$\text{ii)}\ |s'(x)-f'(x)| \le L + \frac{1}{6n}\left[3|\lambda_i-\lambda_{i-1}| + |\lambda_i+2\lambda_{i-1}| + 6|\lambda_{i-1}|\right] + \qquad (14)$$

$$+\,\omega_1\left(\frac{1}{n}\right) + \left|f'\left(\frac{i-1}{n}\right)\right|$$

$$\text{iii)}\ |s''(x)-f''(x)| \le |\lambda_i-\lambda_{i-1}| + |\lambda_{i-1}| + \omega_2\left(\frac{1}{n}\right) + \left|f''\left(\frac{i-1}{n}\right)\right|,$$

where $\omega_j\left(\frac{1}{n}\right)$ *is the modulo of continuity of* $f^{(j)}$, $j = 1,2$ *on* $\left[\frac{i-1}{n},\frac{i}{n}\right]$, $i = \overline{1,n}$.

Proof. Taking into account that

$$\left|x - \frac{i-1}{n}\right| \le \left|\frac{i}{n} - \frac{i-1}{n}\right| = \frac{1}{n},\quad x \in \left[\frac{i-1}{n},\frac{i}{n}\right],\quad i = \overline{1,n},$$

$$|f(x)-f(y)| \le L|x-y|,$$

and (9), one obtains:

$$\text{i)}\ |s(x)-f(x)| \le \frac{n}{6}|\lambda_i-\lambda_{i-1}|\cdot\frac{1}{n^3} + \frac{|\lambda_{i-1}|}{2}\cdot\frac{1}{n^2} + nL|x_i-x_{i-1}|\cdot\frac{1}{n} +$$

$$+\frac{|\lambda_i+2\lambda_{i-1}|}{6n}\cdot\frac{1}{n} + L|x_{i-1}-x| \le$$

$$\le \frac{|\lambda_i-\lambda_{i-1}|}{6n^2} + \frac{|\lambda_{i-1}|}{2n^2} + \frac{|\lambda_i+2\lambda_{i-1}|}{6n^2} + \frac{2L}{n}$$

$$\text{ii)}\ |s'(x)-f'(x)| \le \frac{n}{2}|\lambda_i-\lambda_{i-1}|\cdot\frac{1}{n^2} + |\lambda_{i-1}|\cdot\frac{1}{n} + L + \frac{|\lambda_i+2\lambda_{i-1}|}{6n} +$$

$$+\,|f'(x)| \le \frac{1}{2n}|\lambda_i-\lambda_{i-1}| + \frac{1}{6n}|\lambda_i+2\lambda_{i-1}| +$$

$$+\frac{1}{n}|\lambda_{i-1}| + L + \left|f'(x)-f'\left(\frac{i-1}{n}\right)\right| + \left|f'\left(\frac{i-1}{n}\right)\right| \le$$

$$\le L + \omega_1\left(\frac{1}{n}\right) + \frac{1}{6n}\left[3|\lambda_i - \lambda_{i-1}| + |\lambda_i + 2\lambda_{i-1}| + 6|\lambda_{i-1}|\right] + $$
$$+\left|f'\left(\frac{i-1}{n}\right)\right|$$

and

$$\text{iii)}\ |s''(x) - f''(x)| \le n|\lambda_i - \lambda_{i-1}|\left|x - \frac{i-1}{n}\right| + |\lambda_{i-1}| + \left|f''(x) - f''\left(\frac{i-1}{n}\right)\right| +$$
$$+\left|f''\left(\frac{i-1}{n}\right)\right| \le |\lambda_i - \lambda_{i-1}| + |\lambda_{i-1}| + \omega_2\left(\frac{1}{n}\right) + \left|f''\left(\frac{i-1}{n}\right)\right|$$

and the theorem is completely proved.

Remark 1. If $\alpha = 1/2\left(t_i = \frac{x_{i-1}+x_i}{2}\right)$ from (13) it follows that $\lambda_i = 2f''(t_i) - \lambda_{i-1}$ and the inequalities (14) becomes:

$$\text{i)}\ |s(x) - f(x)| \le \frac{2L}{n} + \frac{1}{6n^2}\left[\left|2f''\left(\frac{2i-1}{2n}\right) - 2\lambda_{i-1}\right| + \left|2f''\left(\frac{2i-1}{2n}\right) + \lambda_{i-1}\right| + \right.$$
$$\left. + 3|\lambda_{i-1}|\right]$$

$$\text{ii)}\ |s'(x) - f'(x)| \le L + \frac{1}{6n^2}\left[3\left|2f''\left(\frac{2i-1}{2n}\right) - 2\lambda_{i-1}\right| + \right.$$
$$\left. + \left|2f''\left(\frac{2i-1}{2n}\right) + \lambda_{i-1}\right| + 6|\lambda_{i-1}|\right] + \omega_1\left(\frac{1}{n}\right) + \left|f'\left(\frac{i-1}{n}\right)\right| \qquad (14')$$

$$\text{iii)}\ |s''(x) - f''(x)| \le \left|2f''\left(\frac{2i-1}{2n}\right) - 2\lambda_{i-1}\right| + |\lambda_{i-1}| + \omega_2\left(\frac{1}{n}\right) + \left|f''\left(\frac{i-1}{n}\right)\right|$$

Remark 2. If $\lambda_i \ge \lambda_{i-1} \ge 0,\ i = \overline{1,n}$ then

$$\text{i)}\ |s(x) - f(x)| \le \frac{2L}{n} + \frac{1}{3n^2}(\lambda_i + 2\lambda_{i-1})$$

$$\text{ii)}\ |s'(x)-f'(x)| \le L + \frac{1}{6n}(4\lambda_i+5\lambda_{i-1}) + \omega_1\left(\frac{1}{n}\right) + \left|f'\left(\frac{i-1}{n}\right)\right| \qquad (14'')$$

$$\text{iii)}\ |s''(x)-f''(x)| \le \lambda_i + \omega_2\left(\frac{1}{n}\right) + \left|f''\left(\frac{i-1}{n}\right)\right|$$

for all $x \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$, $i=\overline{1,n}$.

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DEFICIENT SPLINE APPROXIMATIONS FOR SECOND ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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REZUMAT. - Aproximări prin funcţii spline cu deficienţă pentru ecuaţii diferenţiale de tip neutral cu întârziere. În această lucrare se consideră un procedeu de rezolvare numerică a ecuaţiei diferenţiale de ordinul al doilea cu argument modificat utilizând funcţii spline polinomiale de gradul $m \geq 3$ şi clasă de continuitate C^{m-2}. Se studiază estimarea erorii procedeului de colocaţie dat, împreună cu convergenţa metodei. Un exemplu numeric ilustrează eficienţa metodei.

Abstract. A collocation procedure with polynomial spline functions of degree $m \geq 3$ and continuity class C^{m-2} is considered for numerical solution of a second order initial value problem for neutral delay differential equations. The estimation of the error as well as the convergence of the deficient cubic spline approximations is investigated.

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1. Introduction. In recent years functional differential equations have been applied in various fields of science and consequently, a large number of papers on the theory of functional differential equations has been published. The divergence of higher degree spline function methods arises from the enforcement of the continuity requirement on the spline functions. It is possible to generate convergent higher order methods by relaxing the continuity. This type of method will be considered in this paper. Here a spline approximation for the numerical solution of neutral delay differential equations has been introduced with degree $m \geq 3$ and continuity class C^{m-2}.

2. Description of the spline collocation method. Consider the following second order initial value problem for neutral delay differential equations:

$$y''(t) = f(t, y(t), y(g(t)), y'(g(t))),\ t \in [a,b]$$

$$y(t) = \varphi(t),\ y'(t) = \varphi'(t),\ t \in [a,b],\ \alpha \leq a < b \tag{2.1}$$

The function g, called the delay function, is assumed to be continuous on the interval $[\alpha,b]$, and to satisfy the inequality $\alpha \leq g(t) < t$, $t \in [a,b]$, and $\varphi \in C^{m-1}[\alpha,a]$, where $m > 2$. Assume that the functional

$$f\colon [a,b] \times C^1[a,b] \times C^1[\alpha,b] \times C[\alpha,b] \to \mathbf{R}$$

satisfies the following conditions H_1 and H_2:

H_1. For any $x \in C^1[\alpha,b]$, the mapping $t \to f(t,x(t),x(\cdot),x'(\cdot))$ is continuous on $[a,b]$.

H_2. The following Lipschitz condition holds:

$$\|f(t,x_1(t),y_1(\cdot),z_1(\cdot)) - f(t,x_2(t),y_2(\cdot),z_2(\cdot))\|$$

$$\leq L_1\left(\|x_1 - x_2\|_{[\alpha,t]} + \|y_1 - y_2\|_{[\alpha,t-\delta]} + \|z_1 - z_2\|_{[\alpha,t-\delta]}\right)$$

$$+ L_2\|z_1 - z_2\|_{[\alpha,t]}$$

with $L_1 \geq 0$, $0 \leq L_2 < 1$, $\delta > 0$, for any $t \in [a,b]$, $x_1,x_2 \in C^1[a,b]$, $y_1,y_2,z_1,z_2 \in C[\alpha,a]$. Under conditions H_1 and H_2, the problem (2.1) has a unique solution $y \in C^2[a,b] \cap C[\alpha,b]$; see [1,2].

As it is well known, jump discontinuities can occur in various higher order derivatives of the solution even if f,g,φ are analytic in their arguments. Such jump discontinuities are caused by the delay function g and propagate from the point a as the order of derivative increases. We denote the jump discontinuities by $\{\xi_i\}$, which are the roots of the equations $g(\xi_i) = \xi_{i-1}$; $\xi_0 = a$ is the jump discontinuity of φ. Since in this paper g does not depend on y, we can consider the jump discontinuities to be known for sufficiently high order derivatives and to be such that

$$\xi_0 < \xi_1 < \ldots < \xi_{k-1} < \xi_k < \ldots, \xi_M$$

We shall construct a deficient polynomial spline approximating function of degree $m \geq 3$ and deficiency 2, denoted by s: $[a,b] \to \mathbf{R}$.

Consider the interval $[\xi_j, \xi_{j+1}]$, $j = 0, \ldots, M-1$, subdivided by a uniform partition by the knots

$$\xi_j = t_0 < t_1 < \ldots < t_k < t_{k+1} < \ldots < t_N = \xi_{j+1},$$

where $t_k = t_0 + kh$ and $h = (\xi_{j+1} - \xi_j)/N$. The spline function s approximating the solution of (2.1) is defined on each subinterval $[t_k, t_{k+1}]$ by

$$s(t) = \sum_{i=0}^{m-2} \frac{s^{(i)}(t_k)}{i!}(t - t_k)^i + \frac{a_k}{(m-1)!}(t - t_k)^{m-1} + \frac{b_k}{m!}(t - t_k)^m, \tag{2.2}$$

where $s^{(i)}(t_k)$, $0 \leq i \leq m-1$, are left-hand limits of the derivatives as $t \to t_k$ of the segment of s defined on $[t_{k-1}, t_k]$, and the parameters a_k and b_k are determined from the following collocation conditions:

$$s''(t_k + h/2) =$$

$$f\big(t_k + h/2,\, s(t_k + h/2),\, s(g(t_k + h/2)),\, s'(g(t_k + h/2))\big) \tag{2.3}$$

$$s''(t_{k+1}) = f\big(t_{k+1}, s(t_{k+1}), s(g(t_{k+1})), s'(g(t_{k+1}))\big) \tag{2.4}$$

In this way, we obtain a spline function of degree $m \geq 3$ and class C^{m-2} over the entire interval $[\xi_j, \xi_{j+1}]$, with the knots $\{t_k\}_{k=0}^{N}$. It remains to show that, for h sufficiently small, the parameters a_k and b_k can be uniquely determined from

(2.3) and (2.4).

THEOREM 2.1. *If f satisfies conditions* H_1 *and* H_2, $\varphi \in C^{m-1}[\alpha,a]$, $\alpha \leq g(t) \leq t$, $t \in [\alpha,b]$, *and if h is sufficiently small, then there exists a unique spline approximating solution of problem* (2.1) *given by* (2.3)-(2.4).

Proof. If is sufficient to proof that a_k and b_k can be uniquely determined from (2.3) and (2.4). Substituting (2.2) in (2.3) and (2.4), we have

$$a_k = \frac{(m-3)!}{h^{m-3}}\Bigg\{2^{m-2} f\Bigg(t_k + \frac{h}{2}, A_k(t_k + \frac{h}{2}) + \frac{a_k}{(m-1)!}(\frac{h}{2})^{m-1} + \frac{b_k}{m!}(\frac{h}{2})^m,$$

$$A_k(g(t_k + \frac{h}{2})) + \frac{a_k}{(m-1)!}\left(g(t_k + \frac{h}{2}) - t_k\right)^{m-1} + \frac{b_k}{m!}\left(g(t_k + \frac{h}{2}) - t_k\right)^m,$$

$$A_k'(g(t_k + \frac{h}{2}) + \frac{a_k}{(m-2)!}\left(g(t_k + \frac{h}{2}) - t_k\right)^{m-2} + \frac{b_k}{(m-1)!}\left(g(t_k + \frac{h}{2}) - t_k\right)^{m-1}\Bigg)$$

$$- 2^{m-2} A_k''(t_k + \frac{h}{2})$$

$$-f\Bigg(t_{k+1}, A_k(t_{k+1}) + \frac{a_k}{(m-1)!}(h)^{m-1} + \frac{b_k}{m!}(h)^m,$$

$$A_k(g(t_{k+1})) + \frac{a_k}{(m-1)!}(g(t_{k+1}) - t_k)^{m-1} + \frac{b_k}{m!}(g(t_{k+1}) - t_k)^m,$$

$$A_k'(g(t_{k+1})) + \frac{a_k}{(m-2)!}(g(t_{k+1}) - t_k)^{m-2} + \frac{b_k}{(m-1)!}(g(t_{k+1}) - t_k)^{m-1}\Bigg) + A_k''(t_{k+1})\Bigg\}$$

$$b_k = \frac{2(m-2)!}{h^{m-2}}\Bigg\{f\Bigg(t_{k+1}, A_k(t_{k+1}) + \frac{a_k}{(m-1)!}(h)^{m-1} + \frac{b_k}{m!}(h)^m,$$

$$A_k(g(t_{k+1})) + \frac{a_k}{(m-1)!}(g(t_{k+1}) - t_k)^{m-1} + \frac{b_k}{m!}(g(t_{k+1}) - t_k)^m,$$

$$\left. A_k'(g(t_{k+1})) + \frac{a_k}{(m-2)!}(g(t_{k+1}) - t_k)^{m-2} + \frac{b_k}{(m-1)!}(g(t_{k+1}) - t_k)^{m-1} \right) - A_k''(t_{k+1})$$

$$- 2^{m-3} f\left(t_k + \frac{h}{2}, A_k(t_k + \frac{h}{2}) + \frac{a_k}{(m-1)!}(h)^{m-1} + \frac{b_k}{m!}(h)^m, \right.$$

$$A_k(g(t_k + \frac{h}{2})) + \frac{a_k}{(m-1)!}\left(g(t_k + \frac{h}{2}) - t_k\right)^{m-1} + \frac{b_k}{m!}\left(g(t_k + \frac{h}{2}) - t_k\right)^m,$$

$$\left. A_k'(g(t_k + \frac{h}{2})) + \frac{a_k}{(m-2)!}\left(g(t_k + \frac{h}{2}) - t_k\right)^{m-2} + \frac{b_k}{(m-1)!}(g(t_k + \frac{h}{2}) - t_k)^{m-1} \right) + 2^{m-3} A_k'' t_k + \frac{h}{2}) \Big\}$$

where

$$A_k(t) = \sum_{j=0}^{m-2} \frac{S^{(j)}(t_k)}{j!}(t - t_k).$$

Thus we have

$$a_k = G_1(a_k, b_k)$$

$$b_k = G_2(a_k, b_k). \tag{2.5}$$

Using assumption H_2 for $\frac{5Lh(m+5h)}{m(m-1)} < 1$ the application G: $\mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$(a_k, b_k) \to (G_1(a_k, b_k), G_2(a_k, b_k)) \tag{2.6}$$

is a contraction mapping and has a unique solution (a_k, b_k), which can be found by iteration. This completes the proof of the theorem.

In order to make a connection between the spline method and discrete multistep methods, we present the following theorem, which gives the relationship between the value of any spline function $s \in S_m$ and its second derivative at the knots.

THEOREM 2.2. [3] *For any spline function* $s \in S_m$, $s \in C^{m-2}[a,b]$, $m \geq 3$, *there exists a unique linear consistency relation between the quantities* $s(t_k)$ *and* $s''(t_k)$, $k = 0, 1, \ldots, m-2$, *namely*

$$\sum_{j=0}^{m-2} a_j^{(m)} s(t_{j+\nu}) = h^2 \sum_{j=0}^{m-3} b_j^{(m)} s''(t_{j+\nu}),\ 0 \leq \nu \leq N-m+1, \tag{2.7}$$

where

$$a_j^{(m)} := (m-1)!\left[Q_{m-1}(j+1) - 2Q_{m-1}(j) + Q_{m-1}(j-1)\right],$$

$$b_j^{(m)} := (m-1)!\ Q_{m+1}(j+1), \tag{2.8}$$

and

$$Q_k(t) := \frac{1}{(k-1)!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (t-i)_+^{k-1}.$$

THEOREM 2.3. *The spline approximating values* $s(t_k)$, $k = \overline{0,N}$ *of the above procedure are exactly the values furnished by the following multidiscrete method*

$$\sum_{j=0}^{m-1} a_j^{(m)} Y_{j+\nu} = h^2 \sum_{j=0}^{m-1} b_j^{(m)} Y''_{j+\nu},\ 0 \leq \nu \leq N-m+1, \tag{2.9}$$

where the coefficients $a_j^{(m)}$ *and* $b_j^{(m)}$ *are given by* (2.8), *if the starting values*

$$y_0 = s(t_0), y_1(t_1), \ldots, y_{m-3} = s(t_{m-2}) \tag{2.10}$$

are used.

Proof. For h small enough, only one set of values $\{y_k\}_k$ is satisfying the relation (2.9) with the starting values (2.10). But obviously the spline values $\{s_k\}_k$ are satisfying (2.9) on the basis of the consistency relation (2.7) and also the starting values (2.10). That means the spline values must coincide with the values given by the discrete multistep method (2.9).

In the sequel, we shall be concerned with estimating the error in the approximation of the solution of (2.1) by splines as well as with the convergence of the approximation s to the exact solution y as $h \to 0$.

We now define the step function $s^{(m)}$ at the knots $\{t_k\}_{k=1}^{N-1}$ by the usual arithmetic mean:

$$s^{(m)}(t_k) = \frac{1}{2}\left[s^{(m)}(t_k - \frac{1}{2}h) + s^{(m)}(t_k + \frac{1}{2}h)\right] \tag{2.11}$$

We need the following lemmas:

LEMMA 2.1. *Let* $s{:}[a,b] \to \mathbf{R}$ *be the spline approximating function and* y *be the unique solution of problem* (2.1). *If*

$$|s(t_k) - y(t_k)| < Kh^p, \quad |s(g(t_k)) - y(g(t_k))| < Kh^p,$$

$$|s'(g(t_k)) - y'(g(t_k))| < Kh^p,$$

where K *is a constant, then there exists a constant* K_1 *such that*

$$|s(t_k)-y(t_k)|<K_1h^p,\ |s''(t_k)-y''(t_k)|<K_1h^p.$$

Proof. Using the Lipschitz condition H_2, we have

$$|s''(t_k)-y''(t_k)|$$

$$=|f(t_k,s(t_k),s(g(t_k)),s'(g(t_k)))-f(t_k,y(t_k),y(g(t_k),y(g(t_k)),y'(g(t_k)))|$$

$$\leq L\{|s(t_k)-y(t_k)|+|s(g(t_k))-y(g(t_k))|+|s'(g(t_k))-y'(g(t_k))|\}$$

$$L\{Kh^p+Kh^p+Kh^p\}=3KLh^p.$$

If $K_1:=\max\{K, 3KL\}$, then we have

$$|s(t_k)-y(t_k)|<K_1h^p,\ |s''(g(t_k))-y''(g(t_k))|<K_1h^p.$$

LEMMA 2.2. [4] *Let* $y\in C^{m+1}[a,b]$, *and s be the spline function of degree m and class* $C^{(m-2)}$ *with the knots* $\{t_k\}_k$. Suppose that the following relations hold:

$$|s^{(r)}(t_k)-y^{(r)}(t_k)|=O(h^{p_r}),\ |s^{(r)}(g(t_k))-y^{(r)}(g(t_k))|=O(h^{p_r}),$$

$$0\leq r\leq m-2,\ 0\leq k\leq N,$$

$$|s^{(m)}(t)-y^{(m)}(t)|=O(h),\ t_k<t<t_{k+1},\ 0\leq k\leq N.$$

Then it follows:

$$|s(t)-y(t)|=O(h^p),\ p:=\min\{p_1,1+p_1,\ldots,(m-2)+p_{m-2}\},$$

$$p_m=1,\ \forall t\in[a,b]$$

and

$$|s^{(m)}(t) - y^{(m)}(t)| = O(h),\ t \in [a,b].$$

3. Cubic spline function approximating the solution. By Theorem 2.3 for $m = 3$, the cubic approximating spline function of degree 3 and defficiency 2 yields the same values at the knots as the discrete multistep method based on the following recurrence formula:

$$y_{k+1} - 2y_k + y_{k-1} = \frac{h^2}{6}\left[y_{k+1} + 4y_k + y_{k-1}\right] = \frac{h^2}{6}\left[f_{k+1} + 4f_k + f_{k-1}\right] \qquad (3.1)$$

where

$$f_j = f\left(t_j, y(t_j), y(g(t_j)), y'(g(t_j))\right),$$

if the starting values $y_0 = s(t_0)$ and $y_1 = s(t_0+h)$ are used. The discrete method (3.1) has degree of exactness three provided that the starting values y_0 and y_1 have third order accuracy.

As in [3], it is easy to prove that the starting values $y_0 = s(t_0)$ and $y_1 = s(t_0+h)$ have the same order of exactness as the recurrence formula (3.1); therefore we can conclude that

$$|s(t_k) - y(t_k)| = O(h^3),\ |s''(t_k) - y''(t_k)| = O(h^3).$$

The second relation follows from Lemma 2.1 for $p = 3$.

THEOREM 3.1. *If* $f \in C^2([a,b] \times C^1[a,b] \times C^1[\alpha,b] \times C[\alpha,b])$ *and s is* *the cubic spline function of degree* 3 *and defficiency* 2 *approximating the solution of* (2.1), *then there exists a constant K, independent of h, such that, for h sufficiently small and* $t \in [a,b]$,

$$|y(t) - s(t)| < Kh^3,\ |y'(t) - s'(t)| < Kh^2,\ |y'''(t) - s'''(t)| < Kh.$$

Proof. The proof is similar to the proof of Theorem 3.1 in [5].

4. Numerical Example. Consider the following neutral delay differential equation.

$$y''(t) = \cos t - \frac{1}{2}y(t) + \frac{1}{2}y(t-\pi) - y'(t-\pi),\ t \geq 0$$

$$y(t) = 1,\ -\pi \leq t \leq 0$$

The exact solution for this problem with the given initial function is:

$$y(t) = 1 - 2\cos t + 2\cos\left(\frac{\sqrt{3}}{3}t\right),\ \text{for } t \in [0,\pi].$$

Table I shows cubic approximations and Table II shows deficient spline approximations of order 3 and continuity class 1, for $h = \frac{22}{1400}$.

Table I

k	a_k	$s(t_k)$	$y(t_k)$	$e(t_k)$
0	0.369943	1.000000	1.000000	2.0161678549E-07
1	-0.208136	1.000002	1.000000	1.3896369637E-06
2	-0.015305	1.000004	1.000000	3.2549742173E-06

k	a_k	$s(t_k)$	$y(t_k)$	$e(t_k)$
3	0.231232	1.000006	1.000001	5.7729739638E-06
4	-0.385548	1.000010	1.000001	9.2825812317E-06
5	0.429349	1.000015	1.000001	1.3213615603E-05

Table II

k	a_k	b_k	$s(t_k)$	$y(t_k)$	$e(t_k)$
0	3.206419	0.013867	1.000396	1.000000	3.9586596358E-04
1	2.790963	0.012070	1.001532	1.000000	1.5321755618E-03
2	2.429338	0.010506	1.003313	1.000000	3.3129796484E-03
3	2.114568	0.009145	1.005655	1.000001	5.6547611457E-03
4	1.840582	0.007960	1.008486	1.000001	8.4848242204E-03
5	1.602096	0.006928	1.011741	1.000001	1.1739892165E-02

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RADIAL MOTION IN MANEFF'S FIELD

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Dedicated to Professor V. Ureche on his 60th anniversary

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REZUMAT. - Mişcarea radială în câmpul Maneff. Se studiază mişcarea radială în cadrul problemei celor două corpuri în câmpul gravitaţional post-newtonian nerelativist propus de G.Maneff (caracterizat de un potenţial cvasiomogen). Pe baza integralei prime a energiei, se stabilesc traiectorii de coliziune sau evadare pentru toate valorile şi pentru cele două orientări posibile ale vitezei iniţiale.

Proposed in 1924, Maneff's post-Newtonian nonrelativistic gravitational law [5-8] proved itself able to describe accurately the secular motions of both perihelia of inner planets and Moon's perigee. As showed in [4], Maneff's law

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provides the same good theoretical approximation for these phenomena as the relativity. Reconsidered recently (starting with F.N.Diacu's researches), Maneff's potential appeared much less commonplace than at first sight, showing interesting and surprising properties (see [1-3,9]). This field has implications not only in physics and (celestial) mechanics, but also in astrodynamics, cosmogony, astrophysics [10], even in atomic physics (see [1]).

In this note we shall consider the radial motion in Maneff's field, more precisely the rectilinear motion in the framework of the two-body problem with the potential function (e.g. [1,3,9])

$$U = \frac{Gm_1m_2}{r}\left(1 + \frac{3G(m_1+m_2)}{2c^2r}\right), \tag{1}$$

where m_1,m_2 = the masser, r = distance between m_1 and m_2, G = Newtonian gravitational constant, c = speed of light.

It is easy to see that, with the potential function (1), the relative motion of m_2, say, with respect to m_1 will be described by the equation

$$\ddot{\mathbf{r}} = -\frac{\mu\mathbf{r}}{r^3} - 3\left(\frac{\mu}{c}\right)^2\frac{\mathbf{r}}{r^4} \tag{2}$$

with $\mu = G(m_1+m_2)$. In polar coordinates (r,u), (2) transforms as (see [9])

$$\ddot{r} - r\dot{u}^2 + \frac{\mu}{r^2} + 3\frac{(\mu/c)^2}{r^3} = 0, \tag{3}$$

$$r\ddot{u} + 2\dot{r}\dot{u} = 0, \tag{4}$$

system to which we attach the initial conditions

$$(r,u,\dot{r},\dot{u})(t_0) = (r_0,u_0,V_0\cos\alpha,V_0\sin\alpha/r_0), \tag{5}$$

where V = velocity, α = angle between initial radius vector and initial velocity (remind that we study the motion of m_2 in a frame originated in m_1).

The force field is central, so the angular momentum is conserved and (4) provides the first integral

$$r^2\dot{u} = C, \tag{6}$$

where $C = r_0 V_0 \sin\alpha$ is the constant angular momentum. The first integral of energy can also be easily obtained by the usual technique

$$V^2 = \dot{r}^2 + (r\dot{u})^2 = \frac{2\mu}{r} + 3\frac{(\mu/c)^2}{r^2} + h, \tag{7}$$

where the constant of energy h results to have the expression

$$h = V_0^2 - 2\frac{\mu}{r_0} - 3\frac{(\mu/c)^2}{r_0^2}. \tag{8}$$

In the following we shall consider only the rectilinear motion (α = 0 or $\alpha = \pi$, so C = 0). In this case (7) leads to $V^2=\dot{r}^2$, but the integral of energy explicited by (7) and (8)

$$V^2 = V_0^2 + 2\mu\left(\frac{1}{r} - \frac{1}{r_0}\right) + 3\left(\frac{\mu}{c}\right)^2\left(\frac{1}{r^2} - \frac{1}{r_0^2}\right), \tag{9}$$

keeps the same expression as in the general case.

We shall study the motion for all values of V_0. The domains in which the motion is possible, featured by the condition $V^2 \geq 0$, will be pointed out, and the characteristics of the motion as well.

Let us first introduce the following abridging notation

$$V_1 = \frac{c}{\sqrt{3}} + \frac{\sqrt{3}\,\mu}{c r_0}, \quad V_2 = \sqrt{2\frac{\mu}{r_0} + 3\frac{(\mu/c)^2}{r_0^2}}\,. \tag{10}$$

Suppose that $V_0 > V_1$. In this case $h > c^2/3$. If $\alpha = 0$ (radial motion outwards), m_2 will follow an escape trajectory on which V decreases continuously, tending to $\sqrt{h}$ when r tends to infinity. If $\alpha = \pi$ (radial motion inwards), we have a collision trajectory with continuously increasing velocity such that $V \to \infty$ for $r \to 0$.

For $V_0 = V_1$, we have $h = c^2/3$. The possible scenarios are the same: escape path with decreasing velocity ($V \to \sqrt{h} = c/\sqrt{3}$ when $r \to \infty$) for $\alpha = 0$, and collision path with increasing velocity ($V \to \infty$ when $r \to 0$) for $\alpha = \pi$.

Let now consider $V_2 < V_0 < V_1$, which means $0 < h < c^2/3$. All is like previously: the motion directed outwards is decelerated but leads however to escape, while the motion directed inwards is accelerated and leads to collision. At limits V tends to the same values $\sqrt{h}$ and ∞, respectively.

For $V_0 = V_2$ we have $h = 0$. The scenario is identical: $\alpha = 0$ means escape trajectory with $V \to 0$ for $r \to \infty$, while $\alpha = \pi$ leads to collision (with $V \to \infty$

when $r \to 0$).

Lastly, consider $V_0 < V_2$, meaning $h < 0$. If $\alpha = 0$, then m_2 moves outwards with decreasing velocity, such that for

$$r = \frac{3\mu/c^2}{-1 + \sqrt{[1 + 3\mu/(c^2 r_0)]^2 - 3(V_0/c)^2}} \tag{11}$$

m_2 stops, then it starts inwards and collides with m_1 ($V \to \infty$ for $r \to 0$). If $\alpha = \pi$, we have a collision path with continuously increasing velocity, tending to infinity when $r \to 0$.

Notice that V_1 has no physical, but only mathematical importance (this value of V_0 annuls the discriminant of the second degree polynomial function $V = V(1/r)$ given by (9)), while V_2 has a precise physical significance (this value of V_0 annuls the constant of energy).

Concluding, in Maneff's field the radial motion has no other end but escape or collision, just like in the Newtonian field. By analogy with this last one (and by abuse of language), we shall call V_2 (for which $h = 0$) "parabolic velocity". So, the "hyperbolic/parabolic-type" ($V_0 \geq V_2$) rectilinear motion directed outwards in Maneff's field leads to escape with decreasing velocity (which tends to the corresponding value $\sqrt{h} \geq 0$ when $r \to \infty$). The "elliptic-type" ($V_0 < V_2$) rectilinear motion directed outwards cannot lead to escape; m_2 stops at a finite distance (11), then reverses the sense of motion and directs itself with increasing velocity to collision. As to the rectilinear motion directed inwards from the beginning, it ends in collision for any value of the initial velocity.

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ANIVERSĂRI

PROFESORUL DR. VASILE URECHE LA 60 DE ANI

Vasile POP*

Profesorul universitar dr. Vasile Ureche s-a născut în 7 decemrie 1934, în satul Poiana Ilvei, comuna Măgura Ilvei, localităţi situate pe mioriticele plaiuri năsăudene.

Crescut într-o minunată familie de români harnici, buni gospodari, a cunoscut de mic valoarea şi seriozitatea muncii, calitate care a cultivat-o tot timpul.

Ajuns la vârsta şcolară, în perioada 1941-1945 urmează şcoala primară în satul natal, după care se înscrie la Liceul "George Coşbuc" din Năsăud, pe care îl absolvă în anul 1952 ca şef de promoţie.

În condiţiile campaniei de distrugere a ţăranului mijlocaş, pe care se baza economia agrară a ţării, prin sistemul de cote şi impozite către stat, era extrem de greu din punct de vedere material, pentru un copil provenind dintr-o astfel de familie, să urmeze studii superioare.

De aceea profesorul Vasile Ureche şi-a permis un răgaz de doi ani (1952-1954) în care a funcţionat ca profesor suplinitor de matematică la şcoala din Leşu Ilvei şi şcoala generală din Ilva Mare.

În anul 1954 devine student la Facultatea de Matematică-Fizică, secţia Matematică de la Universitatea "Victor Babeş" din Cluj. După absolvirea facultăţii (1959) este numit pentru cinci luni, profesor de matematică la Şcoala generală din comuna Rebrişoara, judeţul Bistriţa-Năsăud, după care, la propunerea profesorului Gheorghe Chiş, devine asistent universitar la disciplina Astronomie de la Facultatea de Matematică a Universităţii "Babeş-Bolyai" din Cluj.

În perioada octombrie 1965 - aprilie 1966 beneficiază de o specializare la cele mai prestigioase observatoare astronomice din fosta URSS.

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Ca urmare a susţinerii cu succes, în anul 1968, a tezei de doctorat cu titlul "Contribuţii la interpretarea curbelor de lumină ale sistemelor binare strânse", elaborată sub îndrumarea profesorului de. doc. Gheorghe Chiş, devine lector universitar, funcţie pe care o deţine până în anul 1990.

În anul 1990 ocupă prin concurs, ca măsură reparatorie, direct postul de profesor universitar şi devine conducător de doctorat în specialitatea astronomie. Începând cu anul 1992 este şi şeful catedrei de mecanică şi astronomie.

Toţi cei care i-au fost studenţi cunosc claritatea, rigoarea şi bogăţia de informaţii a cursurilor, seminariilor şi lucrărilor practice conduse de profesorul Vasile Ureche.

Sub conducerea domniei sale a fost susţinută o teză de doctorat (lector dr. A.M.Imbroane) şi au fost sprijiniţi mai mulţi tineri în elaborarea tezelor de doctorat.

Activitatea ştiinţifică a domnului profesor Vasile Ureche, concretizează în peste 100 de lucrări publicate din care peste 30 în străinătate, este orientată pe următoarele domenii principale:

- teoria sistemelor stelare binare strânse şi interpretarea curbelor de lumină;
- teoria relativităţii, modele de structură a stelelor relativiste;
- fotometrie stelară, stele variabile şi teoria pulsaţiilor stelare.

Dintre rezultatele mai importante obţinute îmi permit să scot în evidenţă câteva:

- modelul elipsoid-elipsoid construit pentru interpretarea curbelor de lumină ale binarelor strânse;
- studiile privind teoria pulsaţiei stelelor în rotaţie şi perturbate mareic;
- studiile privind proprietăţile structurale, condiţiile de stabilitate, geometria continuului spaţiu-timp şi câmpul gravitaţional la stelele relativiste;
- modelul stelar liniar relativist;
- o nouă clasă de modele stelare relativiste de tip "stepenar";
- clasa de modele stelare barotropice al căror autor este;
- o nouă clasă de modele analitice pentru stelele pitice albe.

Aceste cercetări cu deschiderea largă în astronomie au fost comunicate la numeroase congrese şi simpozioane naţionale şi internaţionale. De altfel, prof. dr. Vasile Ureche a fost între anii 1978-1990, preşedinte al subcomisiei nr. 5 "Stele duble" din cadrul comisiei de colaborare multilaterală a academiilor de ştiinţe din ţările est europene pe tema "Fizica şi evoluţia stelelor". În această calitate, în anul 1982, a organizat conferinţa internaţională cu tema "Obiecte relativiste în sisteme binare strânse".

La toate acestea se adaugă contribuţii la 30 de contracte de cercetare ştiinţifică dintre care la 15 a fost responsabil de contract.

Fiind recunoscut ca o autoritate în domeniu, prof. dr. Vasile Ureche conduce seminarul ştiinţific cu tema "Structură şi evoluţie stelară" la care pe lângă cadre didactice, participă studenţi, doctoranzi şi cercetători de la Observatorul Astronomic.

Pentru a veni în sprijinul bunei pregătiri a studenţilor şi cercetătorilor în domeniul astronomiei, a scris singur sau în colaborare, şase cărţi dintre care menţionăm manualul universitar de Astronomie (în colaborare cu prof. dr. Pal) şi prestigiosul tratat în două volume intitulat "Universul", distins în anul 1990 cu premiul "Gheorghe Lazăr" al Academiei Române pe anul 1987.

O mărturie a recunoaşterii naţionale şi internaţionale de care se bucură profesorul dr. Vasile Ureche o constituie alegerea domniei sale ca membru al Comitetului Naţional Român de Astronomie (din anul 1990 face parte din biroul acestui comitet), membru al Uniunii Astronomice Internaţionale şi membru fondator al Societăţii Europene de Astronomie. Revista "Romanian Astronomical Journal" îl are ca membru în colectivul său de redacţie şi mai multe reviste străine îl au ca recenzent.

Mergând pe drumul luminat de iluştri săi înaintaşi George Coşbuc, Liviu Rebreanu, Grigore Moisil, care s-au născut pe aceleaşi mirifice meleaguri năsăudene, putem afirma că profesorul dr. Vasile Ureche a devenit o stea de primă mărime care dorim să strălucească mulţi ani pe firmamentul astronomiei româneşti.

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