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GENERIC CHAINS AND GENERIC DIRECTED SUBSETS OF PARTIALLY ORDERED SETS

A. ABIAN* and S. BONDARI*

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REZUMAT. - Lanţuri generice şi submulţimi dirijate generice în mulţimi parţial ordonate. În lucrare sunt date câteva rezultate privind mulţimile parţial ordonate.

Abstract. It is shown that there are cases where no D-generic directed subset of a c.i.c partially ordered set with $\chi_0 < |D| < 2^{\chi_0}$ can be a chain**.

In what follows $(P,\leq)$, or simply P, stands for a nonempty partially ordered set

In order to avoid trivial cases we assume that *P has no minimum*.

We recall [1] that a subset D of P is called a *dense* subset of P if and only if

$$\text{for every } x \in P \text{ there exists } y \in D \text{ such that } y \leq x \tag{1}$$

In is shown [1] that P has either finitely many or else at least continuum many dense subsets. In other words, there is no partially ordered set with χ_0 many dense subsets.

As expected, by *chain* of P we mean a linearly (i.e., simply) ordered subset of P. Also, as usual, a subset V of P is called a *directed* subset of P if and only if

$$\text{for every } x \in V \text{ and } y \in V \text{ there exists } z \in V \text{ such that } z \leq x \text{ and } z \leq y \tag{2}$$

Thus, V is directed if and only if every two elements of V have a lower bound. This is customarily expressed by saying that V is directed if and only if every two elements of V are *compatible* [5, p.53 and 8, p.5].

* *Iowa State University, Department of Mathematics, Ames, Iowa 50011, USA*

** *Throughout this paper the symbol χ is used to denote "aleph"*

A chain C of P is called a *generic chain* of P if and only if C has a nonempty intersection with every dense subset of P. Similarly, a directed subset V of P is called a *generic directed subset* of P if and only if V has a nonempty intersection with every dense subset of P.

We recall [2] that an element m of P is called a *molecule* of P if and only if

for every $x \in P$ and $y \in P$ if $x \leq m$ and $y \leq m$ then x and y are compatible. (3)

From Corollary 1 of [2] it follows that

P has a generic directed subset if and only if P has a molecule (4)

Many interesting partially ordered sets do not have molecules and therefore by (4) they have no generic directed subsets and consequently no generic chains, as shown in the following:

Example 1. The countable (infinite) set F of all finite dyadic sequences partially ordered by extension has no molecules. Indeed, given any finite dyadic sequence, say, 011010, there always exist two incompatible finite extensions, for instance, 011010110 and 011010111 of 011010. Thus F has no generic directed subset.

For partially ordered sets which have no molecules a restricted notion of a generic chain, as well as a restricted notion of a generic directed subset, is introduced which serves as the basic item in the construction of set-theoretical models for proving the independence of, say, the Continuum Hypothesis or the Axiom of Choice from the ZF axioms [7]. This is the notion of a D-generic chain, as well as a D-generic directed subset defined as follows [cf 3 and 8, p.134].

Let D be a set (a list) of dense subsets of P. A chain C of P is called a *D-generic chain* of P if and only if C has a nonempty intersection with every dense subset belonging to D. Similarly, a directed subset V of P is called a ***D*-*generic directed subset*** of P if and only if V has a nonemty intersection with every dense subset belonging to D. Thus, a D-generic chain or a D-generic directed subset of P need not have a nonempty intersection with every dense subset of P. They are only required to have a nonempty intersection with the dense subsets of P which belong to D.

Remark 1. Based on the Axiom of Choice, one can easily show that if D is a countable set of dense subsets of P, i.e., if $D = \{D_0, D_1, D_2, \ldots, D_n, \ldots\}$ with $n \in \omega$ then P always has a D-generic chain. Indeed, since P is nonempty, none of the D_i's is empty, so let $d_0 \in D_0$. Since D_1 is dense there exists $d_1 \in D_1$ such that $d_1 \leq d_0$. Again, since D_2 is dense there exists $d_2 \in D_2$ such that $d_2 \leq d_1 \leq d_0$. Continuing in this way, one can easily establish (in ZFC) the existence of the set $H = \{d_0, d_1, d_2, \ldots, d_n, \ldots\}$ with $d_n \in D_n$ and $d_n \leq d_{n+1}$ for every $n \in \omega$. Clearly, H is a D-generic chain of P.

However, if D is an uncountable set of dense subsets of P then P need not have any D-generic directed subset. This is the case of the partial order F of Example 1. Obviously, every directed subset of F is a chain of F. Moreover, since there are continuum many (i.e., $2^{\aleph_0}$) dyadic (finite or infinite) sequences, it follows that F has $2^{\aleph_0}$ chains. Furthermore, clearly, the complement $f - C$ of every chain C of F is a dense subset of F. Consequently, F has $2^{\aleph_0}$ dense subsets. Let D be the set of all the dense subsets of F. Thus, $|D| = 2^{\aleph_0}$ and F has no D-generic directed subset, as mentioned in Example 1.

Remark 2. From Example 1 and Remark 1 it follows that if D is a set of dense subsets of a partially ordered set P such that $|D| = 2^{\aleph_0}$ then P need not have a D-generic directed subset.

In connection with Remark 2, let us observe that the condition

$$\chi_0 < |D| < 2^{\chi_0} \tag{5}$$

still does not guarantee the existence of a D-generic directed subset of P, as shown in the following:

Example 2. Let E be the set of all finite sequences whose terms are countable (finite or infinite) ordinals, i.e., the elements $\chi_1 = \omega_1$. For instance $(3, \omega, 0, \omega, 7, \omega^2, \omega^\omega, 7, \omega^\omega+\omega+3)$ is an element of E. Let E be partially ordered by extension. For every ordinal $u \in \omega_1$ let D_u be the set of all the elements of E in each of which u occurs (as a term). Clearly, every D_u is a dense subset of E. Let D be the set of all such D_u's, i.e.,

$$D = \{D_u \mid u \in \omega_1\} \tag{6}$$

From (6) it follows that $|D| = \chi_1$ and we assume that (5) also holds (i.e., we assume that the Continuum Hypothesis CH does not hold). Obviously, every directed subset of E is also a chain of E. Now, if E had a D-generic chain G, then from (6) it would follow that $\cup G$ is a countable sequence having all the uncountably many elements of ω_1 as its terms. But this is impossible. Hence E has no D-generic directed subset even though (5) is satisfied.

Let us observe that the partially ordered set E of Example 2 has many pairwise *incompatible* (i.e., without having a lower bound) elements. Indeed, $|\{(0), (1), ..., (\omega), ..., (\omega^\omega), ...\}| = \omega_1 = \chi_1$ is a subset of E of pairwise incompatible elements. This is because no two distinct elements of $\{(0), (1), ..., (\omega), ..., (\omega^\omega), ...\}$ have a common extension. Therefore, regardless of (5), it could have been expected, that E would not have a D-generic directed subset G since the elements of G must be pairwise compatible, whereas E has too many pairwise incompatible elements.

We call a partially ordered set P a c.i.c. (*countable incompatibility condition*) partially ordered set if and only if every subset of P of pairwise incompatible elements is countable [cf. 5, p.53 and 8, p.133].

Remark 3. From Example 2 and Remark 2 it follows that if P is not a c.i.c partially ordered set then P need not have a D-generic directed subset even if $\chi_0 < |D| < 2^{\chi_0}$.

In connection with Remark 3 the following natural question arises: Let M be a model for ZFC+-CH is the following statement true or false in M?

Let P be a c.i.c partially ordered set and D be a set of dense subsets of P such that $\chi_0 < |D| < 2^{\chi_0}$. Then P has a D-generic directed subset. (7)

It has been shown [5, p.52 and 8, p.135] that there are models M in which (7) is true and there are models M in which (7) is false. In other words, (7) is consistent with and independent of ZFC+-CH axioms.

Statement (7) is the so-called Martin's [5, p.54] denoted by MA. The significance of MA is partly due to the that under MA it can be shown that infinite cardinals k with $\chi_0 < k < 2^{\chi_0}$ acquire some properties similar to $\chi_0 = \omega$.

For instance, under MA it can be shown that for every cardinal k with $\chi_0 < k < 2^{\chi_0}$ it is the case that (i) the union of k many subsets of reals each of Lebesque measure zero is of measure zero. (ii) the union of k many subsets of reals of Baire first category is of first category. (iii) $2^k = 2^{\chi_0}$. (iv) the intersection of k many dense open sets in any compact Hausdorff space with a countable base is dense.

Again, MA is widely used in Analysis, Topology and Algebra [4 and 6] for asserting the consistency and independence of various statements in ZFC+-CH. This is usually achieved by invoking MA in connection with a suitably chosen c.i.c partially ordered set P and asserting the existence of a suitably chosen D-generic directed subset G of P with $\chi_0 < |D| < 2^{\chi_0}$. Very often the generic directed subset G turns out to be a chain. However, as shown below, there are cases where it is impossible to have a generic directed subset G which is a chain.

In ZF+-CH, we prove:

THEOREM. *There exists a c.i.c partially ordered set P and a set **D** of dense subsets of P with $\chi_0 < |\boldsymbol{D}| < 2^{\aleph_0}$ such that no **D**-generic directed subset of P is a chain.*

Proof. Let S be the set of all finite subsets of χ_1 and let P be the partially ordered set $(S, \supseteq)$. Clearly every two elements of S have a lower bound (namely, their set-theoretical union) in S. Thus S has no (nonempty) subset of pairwise incompatible elements. Hence S is c.i.c. Next, for every $u \in \chi_1$, let D_u be the set of all elements of S each of which contains u as an element, i.e.,

$$D_u = \{X \mid X \text{ is a finite subset of } \chi_1 \text{ and } u \in X\} \tag{8}$$

Clearly, every D_u is a dense subset of S. Let

$$\boldsymbol{D} = \{D_u \mid u \in \chi_1\} \tag{9}$$

Obviously, $\boldsymbol{D}$ satisfies (5). Let us assume to the contrary that S has a $\boldsymbol{D}$-generic chain G. From (8) and (9) it follows that $|G| = \chi_1$. Let H be a subset of G of cardinality χ_0. Since $|G| = \chi_1$ and $|H| = \chi_0$ and G is a chain, H must have a lower bound B which must be an element of G. Clearly, $B \supseteq \cup H$ and $|\cup H| = \chi_0$ which contradicts the fact that every element of S and therefore every element of G is finite. Thus, our assumption is false and the Theorem is proved.

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A LATTICE OF Π-SCHUNCK CLASSES

Rodica COVACI*

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REZUMAT. - O latice de Π-clase Schunck. În lucrare se demonstrează că clasa $\mathcal{P}$ a tuturor π-claselor Schunck cu proprietatea P ([3]), ordonată prin incluziune, formează în raport cu operaţiile de compunere ([4]) şi de intersecţie o latice completă cu 0 şi 1.

Abstract. The paper proves that the class $\mathcal{P}$ of all π-Schunck classes with the P property ([3]), ordered by inclusion, forms respect to the operations of composition ([4]) and intersection a complete lattice with 0 and 1.

1. Preliminaries. All groups considered in the paper are finite. We shall denote by π a set of primes, by π' the complement to π in the set of all primes and by $O_{\pi'}(G)$ the largest normal π'-subgroup of a group G. A group G is said to be *π-solvable* if every chief factor of G is either a solvable π-group or a π'-group. Particularly, for π the set of all primes, one obtain the notion of solvable group. The following notions are well known in the formation theory (see [4]):

DEFINITION 1.1. a) A class X of groups is called a *homomorph* if X is closed under homomorphisms.

b) A homomorph X is a *Schunck class* if X is *primitively closed*, i.e. if any group G, all of whose primitive factor groups are in X, is itself in X.

* *"Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, 3400 Cluj-Napoca, Romania*

c) Let X be a class of groups and G a group. A subgroup H of G is called an *X-covering subgroup* of G if: (i) $H \in X$; (ii) $H \leq K \leq G$, $K_0 \trianglelefteq K$ and $K/K_0 \in X$ imply $K = HK_0$.

Further, we have respecting to a fixed set of primes π:

DEFINITION 1.2. a) Let X be a class of groups. The class X is *π-closed* if

$$G/O_{\pi'}(G) \in X \Rightarrow G \in X.$$

b) A π-closed homomorph, respectively a π-closed Schunck class will be called a *π-homomorph*, resp. a *π-Schunck class*.

In [2] is given a lattice structure on the class D of all π-Schunck classes X with the D property, i.e. with the property that in any π-solvable group, the X-maximal subgroups coincide with the X-covering subgroups.

On the other side, in [3] are introduced the π-Schunck classes with the P property.

DEFINITION 1.3. A class X of groups is said to have *the P property* if for any π-solvable group G we have:

N minimal normal subgroup of G and N π'-group $\Rightarrow$ $G/N \in X$.

Example 1.4. a) The class I of all unit groups is a π-Schunck class with the P property.

b) The class W_π of all π-solvable groups is a π-Schunck class with the P property.

The properties of the π-Schunck classes X with the P property given in [3] are connected by the X-covering subgroups of π-solvable groups. This and the results from [1] and [2] lead to the ideea of a lattice structure on the class of all π-Schunck classes with the P property.

2. The lattice of π-Schunck classes with the P property. Let us denote with $\wp$ the class of all π-Schunck classes with the P property.

DEFINITION 2.1. If X and Y are two classes of groups, we define the *composition* of X and Y: the class $\langle X,Y\rangle$ of all π-solvable groups G such that $G = \langle S,T\rangle$, where S is an X-covering subgroup of G and T is an Y-covering subgroup of G.

In preparation for the main theorem we give some lemmas.

LEMMA 2.2. *X and Y are two classes of groups, then $X \subseteq \langle X,Y\rangle$ and $Y \subseteq \langle X,Y\rangle$.*

LEMMA 2.3. ([2]) *If X and Y are π-homomorphs, then $\langle X,Y\rangle$ is a π-homomorph.*

LEMMA 2.4. ([2]) *If X and Y are π-Schunck classes, then $\langle X,Y\rangle$ is a π-Schunck class.*

LEMMA 2.5. *If $X \in \wp$ and $Y \in \wp$, then $\langle X,Y\rangle \in \wp$.*

Proof. By 2.4., $\langle X,Y\rangle$ is a π-Schunck class. We prove now that $\langle X,Y\rangle$ has the P property. Let G be a π-solvable group and N a minimal normal subgroup of G such that N is a π'-group. Since X has the P property, we have $G/N \in X$ and so G/N is its own X-covering subgroup. Similarly, since Y has the P property, we have $G/N \in Y$ and so G/N is its own Y-covering subgroup. We obtain $G/N = \langle G/N, G/N \rangle$, where G/N is its own X-covering subgroup and G/N is also its own Y-covering subgroup. Furthermore, since G is a π-solvable group, G/N is a π-solvable group. It follows that $G/N \in \langle X,Y\rangle$. ■

THEOREM 2.6. *The class $\wp$ of all π-Schunck classes with the P property, ordered by inclusion, forms respect to the operations of composition and intersection a complete lattice with* 0 *and* 1.

Proof. The result follows immediately from lemmas 2.2, 2.3., 2.4. and 2.5. In the lattice $\wp$, the class I of all unit groups is the 0 element and the class W_π of all π-solvable groups is the 1 element. ∎

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SOME INEQUALITIES AND IDENTITIES FOR MEANS

J. PEČARIĆ* and I. RAŞA**

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REZUMAT. - Câteva inegalităţi şi identităţi pentru medii. Lucrarea conţine rafinări ale unor rezultate cunoscute referitoare la diverse tipuri de medii.

1. Let $0 < a_n \le \dots \le a_1$ and $p_i > 0$, $i = 1, \dots, n$.

Denote $P_n = \sum_{i=1}^{n} p_i$, $F(a_1, \dots, a_n; p) =$

$$= \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i\right) \Big/ \left(\prod_{i=1}^{n} a_i^{p_i}\right)^{1/P_n}, \quad G(a_1, \dots, a_n; p) = \frac{1}{P_n}\sum_{i=1}^{n} p_i a_i - \left(\prod_{i=1}^{n} a_i^{p_i}\right)^{1/P_n}$$

For each $k = 1, \dots, n-1$ and $t \in [a_{k+1}, a_k]$, let $f(t) = F(t, \dots, t, a_{k+1}, \dots, a_n; p)$ and $g(t) = G(t, \dots, t, a_{k+1}, \dots, a_n; p)$.

THEOREM 1. *The functions f and g defined above are nondecreasing on* $[a_n, a_1]$.

Proof. Both functions are continuous on $[a_n, a_1]$.

Moreover, on (a_{k+1}, a_k) we have

$$\frac{d}{dt}\log f(t) = \frac{f'(t)}{f(t)} = P_k \frac{\sum_{i=k+1}^{n} p_i(t - a_i)}{P_n t\left(P_k t + \sum_{i=k+1}^{n} p_i a_i\right)} \ge 0.$$

$$g'(t) = \frac{P_k}{P_n}\left(1 - \prod_{i=k+1}^{n}\left(\frac{a_i}{t}\right)^{p_i/P_n}\right) \ge 0.$$

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* *University of Zagreb, Faculty of Textile Technology, 41000 Zagreb, Croatia*

** *Technical University, Department of Mathematics, 3400 Cluj-Napoca, Romania*

Hence f and g are nondecreasing on $[a_n, a_1]$.

COROLLARY 2. $F(a_1, \ldots, a_n;p) \geq F(a_2, a_2, a_3, \ldots, a_n;p) \geq \ldots$
$\geq F(a_k, \ldots, a_k, a_{k+1}, \ldots, a_n;p) \geq F(a_{k+1}, \ldots, a_{k+1}, a_{k+2}, \ldots, a_n;p) \geq \ldots \geq 1.$
$G(a_1, \ldots, a_n;p) \geq G(a_2, a_2, a_3, \ldots, a_n;p) \geq \ldots \geq G(a_k, \ldots, a_k, a_{k+1}, \ldots, a_n;p) \geq$
$\geq G(a_{k+1}, \ldots, a_{k+1}, a_{k+2}, \ldots, a_n;p) \geq \ldots \geq 0.$

These results are refinements of the well-known inequality between the aritmetic and the geometric means. The above proofs of this inequality are different from those given in [1].

2. Let $0 < a < b$, $A = \frac{a+b}{2}$, $G = \sqrt{ab}$, $H = \frac{2ab}{a+b}$, $L = \frac{b-a}{\log b - \log a}$, $I = \frac{1}{e}\left(b^b/a^a\right)^{\frac{1}{b-a}}$. These means are well-known. Let us consider also the mean $S = \left(a^a b^b\right)^{\frac{1}{a+b}}$; it appears in [2] (denoted by $B_2^{(1)}$) and in [3] (see also the references given in [2] and [3]).

It is easy to prove that

$$\log \frac{S}{I} = 1 - \frac{H}{L} \tag{1}$$

On the other hand, it is known (see [3]) that

$$\log \frac{I}{G} = \frac{A}{L} - 1 \tag{2}$$

Many identities and inequalities connecting the above mentioned means are given in [3].

For example,

$$\log \frac{A}{G} = \sum_{k=1}^{\infty} \frac{1}{2k} \left(\frac{b-a}{b+a}\right)^{2k} \tag{3}$$

$$\log \frac{I}{G} = \sum_{k=1}^{\infty} \frac{1}{2k+1} \left(\frac{b-a}{b+a}\right)^{2k} \tag{4}$$

Using (1) it is easy to derive

$$\log \frac{S}{I} = \sum_{k=1}^{\infty} \frac{2}{(2k-1)(2k+1)} \left(\frac{b-a}{b+a}\right)^{2k} \tag{5}$$

From (3) and (4) it follows

$$\log \frac{A}{I} = \sum_{k=1}^{\infty} \frac{1}{2k(2k+1)} \left(\frac{b-a}{b+a}\right)^{2k} \tag{6}$$

Now (5) and (6) imply

$$\log \frac{S}{A} = \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} \left(\frac{b-a}{b+a}\right)^{2k} \tag{7}$$

The relation (7) is an improvement of the inequality $S > A$ given in [2] and [4].

By using (3), (6) and (7) we obtain

$$SG < A^2 < SI \tag{8}$$

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CERTAIN SUBCLASSES OF PRESTARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

M.K. AOUF* and G.S. SĂLĂGEAN**

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REZUMAT. - Subclase de funcţii prestelate cu coeficienţi negativi. În această lucrare sunt obţinute rezultate asupra produselor Hadamard de funcţii aparţinând la două clase de funcţii prestelate cu coeficienţi negativi. De asemenea, se pun în evidenţă operatorii integrali corespunzători acestor clase.

Abstract. The object of the present paper is to obtain several interesting results for the modified Hadamard products of functions belonging to the classes $R[\alpha,\beta]$ and $C[\alpha,\beta]$ consisting of prestarlike functions with negative coefficients. Also we obtain integral operators for these classes.

1. Introduction. Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the unit disc $U = \{z: |z| < 1\}$. A function $f(z)$ of S is said to be starlike of order α if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U) \tag{1.2}$$

for some $\alpha(0 \le \alpha < 1)$. We denote the class of all starlike functions of order α by $S^*(\alpha)$. Further a function $f(z)$ of S is said to be convex of order α if and only if

* *University of Mansoura, Faculty of Science, Department of Mathematics, Mansoura, Egypt*

** *"Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, 3400 Cluj-Napoca, Romania*

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \quad (z\in U) \tag{1.3}$$

for some α ($0 \le \alpha < 1$). And we denote the class of all functions of order α by $K(\alpha)$. We note that

$$f(z)\in K(\alpha) \text{ if and only if } zf'(z)\in S^*(\alpha). \tag{1.4}$$

The classes $S^*(\alpha)$ and $K(\alpha)$ were first introduced by Robertson [4], and later were studied by Schild [5], MacGregor [1] and Pinchuk [3].

Now, the function

$$S_\alpha(z)=\frac{z}{(1-z)^{2(1-\alpha)}} \tag{1.5}$$

is the well-known extremal function for $S^*(\alpha)$. Setting

$$C(\alpha,n)=\frac{\prod_{k=2}^{n}(k-2\alpha)}{(n-1)!} \quad (n\ge 2), \tag{1.6}$$

$S_\alpha(z)$ can be written in the form

$$S_\alpha(z)=z+\sum_{n=2}^{\infty}C(\alpha,n)z^n. \tag{1.7}$$

Then we can see that $C(\alpha,n)$ is decreasing function in α and satisfies

$$\lim_{n\to\infty}C(\alpha,n)=\begin{cases}\infty & \left(\alpha<\frac{1}{2}\right)\\ 1 & \left(\alpha=\frac{1}{2}\right)\\ 0 & \left(\alpha>\frac{1}{2}\right).\end{cases} \tag{1.8}$$

Let $f*g(z)$ denote the Hadamard product of two functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z)=z+\sum_{n=2}^{\infty}b_nz^n, \tag{1.9}$$

then

$$f*g(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n. \tag{1.10}$$

Let $R(\alpha,\beta)$ be the subclass of S consisting of functions $f(z)$ such that $f * S_\alpha(z) \in S^*(\beta)$ for $0 \leq \alpha < 1$ and $0 \leq \beta < 1$. Further let $C(\alpha,\beta)$ be the subclass of S consisting of functions $f(z)$ satisfying $zf'(z) \in R(\alpha,\beta)$ for $0 \leq \alpha < 1$ and $0 \leq \beta < 1$. $R(\alpha,\beta)$ is called the class of functions α-prestalike of order β and was introduced by Sheil-Small, Silverman and Silvia [7].

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0). \tag{1.11}$$

Further we denote $R[\alpha,\beta]$ and $C[\alpha,\beta]$ the classes obtained by taking intersections, respectively, of the classes $R(\alpha,\beta)$ and $C(\alpha,\beta)$ with T, that is

$$R[\alpha,\beta] = R(\alpha,\beta) \cap T \tag{1.12}$$

and

$$C[\alpha,\beta] = C(\alpha,\beta) \cap T. \tag{1.13}$$

The class $R[\alpha,\beta]$ was recently studied by Silverman and Silvia [8] and Uralegaddi and Sarangi [9] and the class $C[\alpha,\beta]$ was studied recently by Owa and Uralegaddi [2].

In order to show our results, we shall need the following lemmas.

LEMMA 1 [8]. *Let the function $f(z)$ be defined by* (1.11). *Then $f(z)$ is in the class $R[\alpha,\beta]$ if and only if*

$$\sum_{n=2}^{\infty} (n-\beta)C(\alpha,n)\, a_n \leq 1-\beta. \tag{1.14}$$

The result is sharp.

LEMMA 2 [2]. *Let the function $f(z)$ be defined by* (1.11). *Then $f(z)$ is in the class $C[\alpha,\beta]$ if and only if*

$$\sum_{n=2}^{\infty} n(n-\beta)C(\alpha,n)\, a_n \leq 1-\beta. \tag{1.15}$$

The result is sharp.

2. Integral Operators

THEOREM 1. *Let the function $f(z)$ defined by* (1.11) *be in the class* $R[\alpha,\beta]$, *and let* c *be a real number such that* $c > -1$. *Then the function* $F(z)$ *defined by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\,dt \tag{2.1}$$

also belongs to the class $R[\alpha,\beta]$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{n=2}^{\infty} b_n z^n, \tag{2.2}$$

where

$$b_n = \left(\frac{c+1}{c+n}\right) a_n . \tag{2.3}$$

Therefore

$$\begin{aligned}\sum_{n=2}^{\infty} (n-\beta)\, C(\alpha,n)\, b_n &= \sum_{n=2}^{\infty} (n-\beta)\, C(\alpha,n) \left(\frac{c+1}{c+n}\right) a_n \\ &\le \sum_{n=2}^{\infty} (n-\beta) C(\alpha,n)\, a_n \le 1-\beta,\end{aligned} \tag{2.4}$$

since $f(z) \in R[\alpha,\beta]$. Hence, by Lemma 1, $F(z) \in R[\alpha,\beta]$.

THEOREM 2. *Let the function* $F(z) = z - \sum_{n=2}^{\infty} a_n z^n$ $(a_n \ge 0)$ *be in the class* $R[\alpha,\beta]$, *and let* c *be a real number such that* $c > -1$. *Then the function* $f(z)$ *defined by* (2.1) *is univalent in* $|z| < r_1^*$, *where*

$$r_1^* = \inf_n \left[\frac{(n-\beta)\, C(\alpha,n)\,(c+1)}{n(1-\beta)(c+n)}\right]^{\frac{1}{n-1}} \quad (n \ge 2). \tag{2.5}$$

The result is sharp.

Proof. From (2.1), we have

$$f(z) = \frac{z^{1-c}\,(z^c F(z))'}{(c+1)} \quad (c > -1) \tag{2.6}$$

$$= z - \sum_{n=2}^{\infty} \left(\frac{c+n}{c+1}\right) a_n z^n . \tag{2.7}$$

In order to obtain the required result it suffices to show $|f'(z) - 1| < 1$ in $|z| < r_1^*$.

Now

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} \frac{n(c+n)}{(c+1)} a_n |z|^{n-1}.$$

Thus $|f'(z) - 1| < 1$, if

$$\sum_{n=2}^{\infty} \frac{n(c+n)}{(c+1)} a_n |z|^{n-1} < 1. \tag{2.8}$$

But Lemma 1 confirms that

$$\sum_{n=2}^{\infty} \frac{(n-\beta)\,C(\alpha,n)}{(1-\beta)} a_n \leq 1. \tag{2.9}$$

Thus (2.8) will be satisfied if

$$\frac{n(c+n)|z|^{n-1}}{(c+1)} \leq \frac{(n-\beta)\,C(\alpha,n)}{(1-\beta)}$$

or if

$$|z| < \left[\frac{(n-\beta)\,C(\alpha,n)\,(c+1)}{n(1-\beta)\,(c+n)}\right]^{\frac{1}{n-1}} \quad (n \geq 2). \tag{2.10}$$

Therefore $f(z)$ is univalent in $|z| < r_1^*$. Sharpness follows if we take

$$f(z) = z - \frac{(1-\beta)(c+n)}{(n-\beta)\,C(\alpha,n)(c+1)} z^n \quad (n \geq 2). \tag{2.11}$$

In the same way, we can prove the following theorems using Lemma 2 instead of Lemma 1.

THEOREM 3. *Let the function $f(z)$ defined by* (1.11) *be in the class $C[\alpha,\beta]$, and let c be a real number such that $c > -1$. Then the function $F(z)$ defined by* (2.1) *also belongs to the class $C[\alpha,\beta]$.*

THEOREM 4. *Let the function $F(z) = z - \sum_{n=2}^{\infty} a_n z^n$ $(a_n \geq 0)$ be in the class $C[\alpha,\beta]$, and let c be a real number such that $c > -1$. Then the function $f(z)$ defined by* (2.1) *is univalent in $|z| < r_2^*$, where*

$$r_2^* = \inf_n \left[\frac{(n-\beta)\,C(\alpha,n)\,(c+1)}{(1-\beta)\,(c+n)}\right]^{\frac{1}{n-1}} \quad (n \geq 2). \tag{2.12}$$

The result is sharp for the function

$$f(z) = z - \frac{(1-\beta)(c+n)}{n(n-\beta)C(\alpha,n)(c+1)} z^n \quad (n \geq 2). \tag{2.13}$$

3. Modified Hadamard Product. Let the functions $f_i(z)$ $(i = 1,2)$ be defined by

$$f_i(z) = z - \sum_{n=2}^{\infty} a_{n,i} z^n \quad (a_{n,i} \geq 0). \tag{3.1}$$

The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = z - \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n. \tag{3.2}$$

THEOREM 5. *Let the functions* $f_i(z)$ $(i=1,2)$ *defined by* (3.1) *be in the class* $R[\alpha,\beta]$ *with* $0 \leq \alpha \leq \frac{1}{2}$, *and* $0 \leq \beta < 1$. *Then* $f_1 * f_2(z)$ *belongs to the class* $R[\alpha,\gamma(\alpha,\beta)]$, *where*

$$\gamma(\alpha,\beta) = 1 - \frac{(1-\beta)^2}{2(1-\alpha)(2-\beta)^2 - (1-\beta)^2}. \tag{3.3}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [6], we need to find the largest $\gamma = \gamma(\alpha,\beta)$ such that

$$\sum_{n=2}^{\infty} \frac{(n-\gamma)C(\alpha,n)}{(1-\gamma)} a_{n,1} a_{n,2} \leq 1. \tag{3.4}$$

Since

$$\sum_{n=2}^{\infty} \frac{(n-\beta)C(\alpha,n)}{(1-\beta)} a_{n,1} \leq 1 \tag{3.5}$$

and

$$\sum_{n=2}^{\infty} \frac{(n-\beta)C(\alpha,n)}{(1-\beta)} a_{n,2} \leq 1, \tag{3.6}$$

by the Cauchy-Schwarz inequality we have

$$\sum_{n=2}^{\infty} \frac{(n-\beta)C(\alpha,n)}{(1-\beta)} \sqrt{a_{n,1} a_{n,2}} \leq 1. \tag{3.7}$$

Thus it is sufficient to show that

$$\frac{(n-\gamma)C(\alpha,n)}{(1-\gamma)} a_{n,1} a_{n,2} \leq \frac{(n-\beta)C(\alpha,n)}{(1-\beta)} \sqrt{a_{n,1} a_{n,2}} \quad (n \geq 2). \tag{3.8}$$

that is, that

$$\sqrt{a_{n,1} a_{n,2}} \leq \frac{(1-\gamma)(n-\beta)}{(1-\beta)(n-\gamma)}. \tag{3.9}$$

Note that

$$\sqrt{a_{n,1}\, a_{n,2}} \leq \frac{(1-\beta)}{(n-\beta)\, C(\alpha, n)} \quad (n \geq 2). \tag{3.10}$$

Consequenly, we need only to prove that

$$\frac{(1-\beta)}{(n-\beta)\, C(\alpha, n)} \leq \frac{(1-\gamma)(n-\beta)}{(1-\beta)(n-\gamma)} \quad (n \geq 2). \tag{3.11}$$

or, equivalently, that

$$\gamma \leq 1 - \frac{(n-1)(1-\beta)^2}{C(\alpha, n)(n-\beta)^2 - (1-\beta)^2} \quad (n \geq 2). \tag{3.12}$$

Since

$$A(n) = 1 - \frac{(n-1)(1-\beta)^2}{C(\alpha, n)(n-\beta)^2 - (1-\beta)^2} \tag{3.13}$$

is an icreasing function of $n(n \geq 2)$ for $0 \leq \alpha \leq \frac{1}{2}$, and $0 \leq \beta < 1$, letting $n = 2$ in (2.13), we obtain

$$\gamma \leq A(2) = 1 - \frac{(1-\beta)^2}{2(1-\alpha)(2-\beta)^2 - (1-\beta)^2}, \tag{3.14}$$

which completes the proof of Theorem 5.

Finally, by taking the functions

$$f_i(z) = z - \frac{1-\beta}{2(1-\alpha)(2-\beta)} z^2 \quad (i = 1,2) \tag{3.15}$$

we can see that the result in Theorem 5 is sharp.

THEOREM 6. *Let the functions* $f_i(z)$ $(i = 1,2)$ *defined by* (3.1) *be in the class* $C[\alpha,\beta]$ *with* $0 \leq \alpha \leq \frac{1}{2}$, *and* $0 \leq \beta < 1$. *Then* $f_1 * f_2(z)$ *belongs to the class* $C[\alpha,\gamma(\alpha,\beta)]$, *where*

$$\gamma(\alpha, \beta) = 1 - \frac{(1-\beta)^2}{4(1-\alpha)(2-\beta)^2 - (1-\beta)^2}. \tag{3.16}$$

The result is sharp for the functions

$$f_i(z) = z - \frac{1-\beta}{4(1-\alpha)(2-\beta)} z^2 \quad (i = 1,2). \tag{3.17}$$

COROLLARY 1. *For* $f_i(z)$ $(i = 1,2)$ *as in Theorem 5, we have*

$$h(z) = z - \sum_{n=2}^{\infty} \sqrt{a_{n,1}\, a_{n,2}}\; z^n \tag{3.18}$$

belongs to the class $R[\alpha,\beta]$.

The result follows from the inequality (3.7). It is sharp for the same functions as in Theorem 5.

THEOREM 7. *Let the function* $f_1(z)$ *defined by* (3.1) *be in the class* $R[\alpha,\beta]$ *with* $0 \le \alpha \le \frac{1}{2}$, *and* $0 \le \beta < 1$, *and the function* $f_2(z)$ *defined by* (3.1) *be in the class* $R[\alpha,\tau]$ *with* $0 \le \alpha \le \frac{1}{2}$, *and* $0 \le \tau < 1$, *then* $f_1 * f_2(z) \in R[\alpha, \xi(\alpha,\beta,\tau)]$, *where*

$$\xi(\alpha,\beta,\tau) = 1 - \frac{(1-\beta)(1-\tau)}{2(1-\alpha)(2-\beta)(2-\tau)-(1-\beta)(1-\tau)}. \tag{3.19}$$

The result is sharp.

Proof. Proceeding as the proof of Theorem 5, we get

$$\xi \le B(n) = 1 - \frac{(n-1)(1-\beta)(1-\tau)}{C(\alpha,n)(n-\beta)(n-\tau)-(1-\beta)(1-\tau)} \quad (n \ge 2). \tag{3.20}$$

Since the function $B(n)$ is an increasing function of $n (n \ge 2)$ for $0 \le \alpha \le \frac{1}{2}$, $0 \le \beta < 1$ and $0 \le \tau < 1$, letting $n = 2$ in (3.20) we obtain

$$\xi \le B(2) = 1 - \frac{(1-\beta)(1-\tau)}{2(1-\alpha)(2-\beta)(2-\tau)-(1-\beta)(1-\tau)}. \tag{3.21}$$

which evidently proves Theorem 7.

Finally the result is best possible for the functions

$$f_1(z) = z - \frac{1-\beta}{2(1-\alpha)(2-\beta)} z^2 \tag{3.22}$$

and

$$f_2(z) = z - \frac{1-\tau}{2(1-\alpha)(2-\tau)} z^2. \tag{3.23}$$

In the same way, we can prove the following theorem using Theorem 6 instead of Theorem 5.

THEOREM 8. *Let the function* $f_1(z)$ *defined by* (3.1) *be in the class* $C[\alpha,\beta]$ *with* $0 \le \alpha \le \frac{1}{2}$, *and* $0 \le \beta < 1$, *and the function* $f_2(z)$ *defined by* (3.1) *be in the class* $C[\alpha,\tau]$ *with* $0 \le \alpha \le \frac{1}{2}$, *and* $0 \le \tau < 1$, *then* $f_1 * f_2(z) \in C[\alpha, \xi(\alpha,\beta,\tau)]$, *where*

$$\xi(\alpha,\beta,\tau) = 1 - \frac{(1-\beta)(1-\tau)}{4(1-\alpha)(2-\beta)(2-\tau)-(1-\beta)(1-\tau)}. \tag{3.24}$$

The result is sharp for the functions

$$\left.\begin{aligned} f_1(z) &= z - \frac{1-\beta}{4(1-\alpha)(2-\beta)} z^2 \\ &\text{and} \\ f_2(z) &= z - \frac{1-\tau}{4(1-\alpha)(2-\tau)} z^2 . \end{aligned}\right\} \tag{3.25}$$

COROLLARY 2. *Let the functions* $f_i(z)$ $(i = 1,2,3)$ *defined by* (3.1) *be in the class* $R[\alpha,\beta]$, *with* $0 \le \alpha \le \frac{1}{2}$, *and* $0 \le \beta < 1$, *then* $f_1 * f_2 * f_3(z) \in R[\alpha, \eta(\alpha,\beta)]$, *where*

$$\eta(\alpha,\beta) = 1 - \frac{(1-\beta)^3}{4(1-\alpha)^2(2-\beta)^3-(1-\beta)^3}. \tag{3.26}$$

The result is best possible for the functions

$$f_i(z) = z - \frac{1-\beta}{2(1-\alpha)(2-\beta)} z^2 \quad (i = 1,2,3). \tag{3.27}$$

Proof. From Theorem 5, we have $f_1 * f_2(z) \in R[\alpha, \gamma(\alpha,\beta)]$, where γ is given by (3.3). We use now Theorem 7, we get $f_1 * f_2 * f_3(z) \in R[\alpha, \eta(\alpha,\beta)]$, where

$$\eta(\alpha,\beta) = 1 - \frac{(1-\beta)(1-\gamma)}{2(1-\alpha)(2-\beta)(2-\gamma)-(1-\beta)(1-\gamma)}$$

$$= 1 - \frac{(1-\beta)^3}{4(1-\alpha)^2(2-\beta)^3-(1-\beta)^3}.$$

This completes the proof of Corollary 2.

COROLLARY 3. *Let the functions* $f_i(z)$ $(i = 1,2,3)$ *defined by* (3.1) *be in the class* $C(\alpha,\beta)$, *with* $0 \le \alpha \le \frac{1}{2}$, *and* $0 \le \beta < 1$, *then* $f_1 * f_2 * f_3(z) \in C[\alpha,\eta(\alpha,\beta)]$, *where*

$$\eta(\alpha,\beta) = 1 - \frac{(1-\beta)^3}{16(1-\alpha)^2(2-\beta)^3-(1-\beta)^3}. \tag{3.28}$$

The result is best possible for the functions

$$f_i(z) = z - \frac{1-\beta}{4(1-\alpha)(2-\beta)} z^2 \quad (i = 1,2,3). \tag{3.29}$$

THEOREM 9. *Let the functions* $f_i(z)$ $(i = 1,2)$ *defined by* (3.1) *be in the class* $R[\alpha,\beta]$, *with* $0 \le \alpha \le \frac{1}{2}$, *and* $0 \le \beta < 1$. *Then the function*

$$h(z) = z - \sum_{n=2}^{\infty} \left[a_{n,1}^2 + a_{n,2}^2 \right] z^n \tag{3.30}$$

belongs to the class $R[\alpha, \varphi(\alpha,\beta)]$, *where*

$$\varphi(\alpha, \beta) = 1 - \frac{(1-\beta)^2}{(1-\alpha)(2-\beta)^2 - (1-\beta)^2}. \tag{3.31}$$

The result is sharp for the functions $f_i(z)$ $(i = 1,2)$ *defined by* (3.15).

Proof. By virtue of Lemma 1, we obtain

$$\sum_{n=2}^{\infty} \left[\frac{(n-\beta)\,C(\alpha,n)}{1-\beta}\right]^2 a_{n,1}^2 \le \left[\sum_{n=2}^{\infty} \frac{(n-\beta)\,C(\alpha,n)}{1-\beta}\, a_{n,1}\right]^2 \le 1 \tag{3.32}$$

and

$$\sum_{n=2}^{\infty} \left[\frac{(n-\beta)\,C(\alpha,n)}{1-\beta}\right]^2 a_{n,2}^2 \le \left[\sum_{n=2}^{\infty} \frac{(n-\beta)\,C(\alpha,n)}{1-\beta}\, a_{n,2}\right]^2 \le 1. \tag{3.33}$$

It follows from (3.32) and (3.33) that

$$\sum_{n=2}^{\infty} \frac{1}{2}\left[\frac{(n-\beta)\,C(\alpha,n)}{1-\beta}\right]^2 \left[a_{n,1}^2 + a_{n,2}^2\right] \le 1. \tag{3.34}$$

Therefore, we need to find the largest $\varphi = \varphi(\alpha,\beta)$ sush that

$$\frac{(n-\varphi)\,C(\alpha,n)}{1-\varphi} \le \frac{1}{2}\left[\frac{(n-\beta)\,C(\alpha,n)}{1-\beta}\right]^2 \quad (n \ge 2), \tag{3.35}$$

that is, that

$$\varphi \le 1 - \frac{2(n-1)(1-\beta)^2}{C(\alpha,n)(n-\beta)^2 - 2(1-\beta)^2} \quad (n \ge 2). \tag{3.36}$$

Since

$$D(n) = 1 - \frac{2(n-1)(1-\beta)^2}{C(\alpha,n)(n-\beta)^2 - 2(1-\beta)^2} \tag{3.37}$$

is an increasing function of $n(n \ge 2)$ for $0 \le \alpha \le \frac{1}{2}$, and $0 \le \beta < 1$, we readily have

$$\varphi \le 1 - \frac{(1-\beta)^2}{(1-\alpha)(2-\beta)^2 - (1-\beta)^2}, \tag{3.38}$$

which completes the proof of Theorem 9.

THEOREM 10. *Let the functions* $f_i(z)$ $(i = 1,2)$ *defined by* (3.1) *be in the class* $C[\alpha,\beta]$, *with* $0 \le \alpha \le \frac{1}{2}$, *and* $0 \le \beta < 1$. *Then the function* $h(z)$ *defined by* (3.30) *belongs to the class* $C[\alpha,\varphi(\alpha,\beta)]$, *where*

$$\varphi(\alpha, \beta) = 1 - \frac{(1-\beta)^2}{2(1-\alpha)(2-\beta)^2 - (1-\beta)^2}. \tag{3.39}$$

The result is sharp for the functions $f_i(z)$ $(i = 1,2)$ *defined by* (3.17).

THEOREM 11. *Let* $f_1(z) \in R[\alpha,\beta]$ *and* $f_2(z) \in R[\alpha,\tau]$, *with* $0 \leq \alpha \leq \frac{1}{2}$, *and* $0 \leq \beta < 1$, $0 \leq \tau < 1$, *then* $f_1 * f_2(z) \in C[\alpha,\psi(\alpha,\beta,\tau)]$, *where*

$$\psi(\alpha, \beta, \tau) = 1 - \frac{(1-\beta)(1-\tau)}{(1-\alpha)(2-\beta)(2-\tau) - (1-\beta)(1-\tau)}. \tag{3.40}$$

The result is sharp.

Proof. Since $f_1(z) \in R[\alpha,\beta]$ and $f_2(z) \in R[\alpha,\tau]$, therefore

$$\sum_{n=2}^{\infty} (n-\beta)\, C(\alpha, n)\, a_{n,1} \leq 1-\beta \tag{3.41}$$

and

$$\sum_{n=2}^{\infty} (n-\tau)\, C(\alpha, n)\, a_{n,2} \leq 1-\tau. \tag{3.42}$$

It follows that

$$\sum_{n=2}^{\infty} (n-\beta)(n-\tau)\,[C(\alpha, n)]^2\, a_{n,1}\, a_{n,2} \leq (1-\beta)(1-\tau). \tag{3.43}$$

We want to find the largest $\psi = \psi(\alpha,\beta,\tau)$ such that

$$\sum_{n=2}^{\infty} n(n-\psi)\, C(\alpha, n)\, a_{n,1}\, a_{n,2} \leq 1-\psi. \tag{3.44}$$

This will certainly be satisfied if

$$\frac{n(n-\psi)\, C(\alpha, n)}{(1-\psi)} \leq \frac{(n-\beta)(n-\tau)[C(\alpha, n)]^2}{(1-\beta)(1-\tau)} \quad (n \geq 2) \tag{3.45}$$

or if

$$\psi \leq 1 - \frac{n(n-1)(1-\beta)(1-\tau)}{C(\alpha, n)(n-\beta)(n-\tau) - n(1-\beta)(1-\tau)} \quad (n \geq 2). \tag{3.46}$$

Since

$$E(n) = 1 - \frac{n(n-1)(1-\beta)(1-\tau)}{C(\alpha, n)(n-\beta)(n-\tau) - n(1-\beta)(1-\tau)} \tag{3.47}$$

is an increasing function of $n (n \geq 2)$ for $0 \leq \alpha \leq \frac{1}{2}$, anf $0 \leq \beta,\tau < 1$, letting $n = 2$ in (3.47) we get

$$\psi \leq 1 - \frac{(1-\beta)(1-\tau)}{(1-\alpha)(2-\beta)(2-\tau) - (1-\beta)(1-\tau)}, \tag{3.48}$$

which completes the proof of the theorem. Finally, the result is sharp for the functions $f_i(z)$

($i = 1,2$) defined by (3.22) and (3.23), respectively.

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A GENERALIZATION IN n-DIMENSIONAL COMPLEX SPACE OF AHLFORS' AND BECKER'S CRITERION FOR UNIVALENCE

Paula CURT*

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REZUMAT. - O generalizare în spaţiul complex n-dimensional a criteriului de univalenţă al lui Ahlfors şi Becker. În acest articol se obţine varianta n-dimensională a criteriului de univalenţă Ahlfors şi Becker.

1. Introduction. In this note we obtain a sufficient condition for univalence. Pfalzgraff [3] extend in $\mathbf{C}^n$ the theory of subordination chains and he obtained the n-dimensional version of the generalized Loewner differential equation. In his paper, Pfalzgraff used only normalized subordination chains. There is no reason to use only normalized chains. From every subordination chain, following a similar method with that used in complex plane, we obtain a normalized chain. On the other hand, in order to avoid the passage to a normalized chain, we obtain a version of Theorem 2.3 [3] for nenormalized subordination chains. Finally, we present the n-dimensional generalization of Ahlfors' and Becker's criterion for univalance.

2. Preliminaries. Let $\mathbf{C}^n$ denote the space of n complex variables $z = (z_1, \ldots, z_n)$ with the usual inner product, $<\cdot,\cdot>$, and euclidian norm $\|\cdot\|$.

Let B^n denote the open unit ball in $\mathbf{C}^n$.

We denote by $\mathfrak{L}(\mathbf{C}^n)$ the space of continuous linear operators from $\mathbf{C}^n$ into $\mathbf{C}^n$, i.e. the n×n complex matrices $A = (A_{jk})$, with the standard operator norm:

* *"Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, 3400 Cluj-Napoca, Romania*

$$\|A\| = sup\,\{\|Az\| : \|z\| \le 1\},\ A \in \mathcal{L}(\mathbf{C}^n)$$

The class of holomorphic mappings $(f_1(z), \ldots, f_n(z))$, $z \in B^n$ from B^n into $\mathbf{C}^n$ is denoted by $\mathcal{H}(B^n)$. We say that $f \in \mathcal{H}(B^n)$ is locally biholomorphic (locally univalent) in B^n if f has a local holomorphic inverse at each point in B^n, or equivalently, if the derivative

$$Df(z) = \left(\frac{\partial f_k(z)}{\partial z_j}\right)_{1 \le j,k \le n}$$

is nonsingular at each point $z \in B^n$.

Let us denote by M the class of mappings:

$$M = \{h \in \mathcal{H}(B^n),\ h(0) = 0,\ Dh(0) = I,\ Re\,\langle h(z), z\rangle \ge 0,\ z \in B^n\}$$

A mapping $v \in \mathcal{H}(B^n)$ is called a Schwarz function if $\|v(z)\| \le \|z\|$ for all $z \in B^n$.

If $f, g \in \mathcal{H}(B^n)$, we say that f is subordinate to g (in B^n) if there exists a Schwarz function v such that $f(z) = g(v(z))$, $z \in B$, and we shall write $f \prec g$ to indicate that f is subordinate to g.

A subordination chain is a function $L: B^n \times [0, \infty) \to \mathbf{C}^n$ such that for each $t \ge 0$, $L(\cdot,t) \in \mathcal{H}(B^n)$, $L(0,t) = 0$ and there exist Schwarz functions $v = v(z,s,t)$ such that $L(z,s) = L(v(z,s,t),t)$, $0 \le s \le t$, $z \in B^n$ for all $0 \le s \le t < \infty$.

An univalent subordination chain is a subordination chain $L = L(z,t)$ such for each $t \ge 0$, $L(\cdot,t)$ is univalent in B^n.

A subordination chain $L = L(z,t)$, $z \in B^n$, $t \ge 0$ ic called a normalized subordination chain if $DL(0,t) = e^t I$ for $t \ge 0$.

We shall need the following theorem to prove our results.

THEOREM 1 [3]. *Let $L(z,t) = e^t z + \ldots$ be a function from $B^n \times [0,\infty)$ into $\mathbf{C}^n$ such that:*

(i) *For each $t \ge 0$, $L(\cdot,t) \in \mathcal{H}(B^n)$.*

(ii) *$L(z,t)$ is a locally absolutely continuous function of t locally uniformly with respect* *to* $z \in B^n$.

Let $h(z,t)$ *be a function from* $B^n \times [0,\infty)$ *into* $\mathbf{C}^n$ *that:*

(iii) *For each* $t \geq 0$, $h(\cdot,t) \in M$.

(iv) *For each* $T > 0$ *and* $r \in (0,1)$ *there is a number* $K = K(r,T)$ *such that* $\|h(z,t)\| \leq K(r,T)$ *where* $\|z\| \leq r$ *and* $0 \leq t \leq T$.

(v) *For each* $z \in B^n$, $h(z,t)$ *is a measurable function of* t *on* $[0,\infty)$.

Suppose $h(z,t)$ *satisfies:*

$$\frac{\partial L(z,t)}{\partial t} = DL(z,t)h(z,t) \text{ a.e. } t \geq 0, \forall z \in B^n \tag{1}$$

Further, suppose there is a sequence $\{t_m\}_m$, $t_m > 0$ *increasing to* ∞ *such that:*

$$\lim_{m\to\infty} e^{-t_m} f(z,t_m) = F(z) \tag{2}$$

locally uniformly in B^n.

Then for each $s \geq 0$, $L(\cdot,s)$ *is univalent on* B^n.

3. Main results. Using the following change of parameter

$$\theta(t) = \arg a_1(t),\ t^* = \log|a_1(t)| \tag{3}$$

we shall pass from a nenormalized subordination chain $L(z,t) = a_1(t)z + \dots$ to a normalized one:

$$L_1(z,t^*) = L\left(e^{-i\theta(t)}z,\ t\right) \tag{4}$$

We next present a version of Theorem 1 [3] for nenormalized subordination chains. This version is not the most generally possible. The condition for the first coefficient of the function $L(z,t)$ appears too strong, but it is sufficient for our purpose.

THEOREM 2. *Let* $L(z,t) = a_1(t)z + \dots$, $a_1(t) \neq 0$ *be a function from* $B^n \times [0,\infty)$ *into* $\mathbf{C}^n$ *such that:*

(i) *For each* $t \geq 0$, $L(\cdot, t) \in \mathcal{H}(B^n)$.

(ii) *$L(z,t)$ is a locally absolutely continuous function of t locally uniformly with respect to* $z \in B^n$.

Let $h(z,t)$ be a function from $B^n \times [0,\infty)$ into $\mathbf{C}^n$ such that:

(iii) *For each* $t \geq 0$, $h(\cdot, t) \in \mathcal{H}(B^n)$.

(iv) *For each* $z \in B^n$, $h(z, \cdot)$ *is a measurable function on* $[0,\infty)$

(v) *For each* $t \geq 0$, $h(0,t) = 0$ *and* $\mathrm{Re} < h(z,t), z > \geq 0$, $\forall z \in B^n$.

(vi) *For each* $T > 0$ *and* $r \in (0,1)$ *there is a number* $K = K(r,T)$ *such that* $\|h(z,t)\| \leq K(r,T)$ *where* $\|z\| \leq r$ and $0 \leq t \leq T$.

Suppose $h(z,t)$ satisfies:

$$\frac{\partial L(z,t)}{\partial t} = DL(z,t)h(z,t) \text{ a.e. } t \geq 0, \ \forall z \in B^n \tag{5}$$

Further, suppose:

(a) $|a_1(t)| \to \infty$ *where* $t \to \infty$, $a_1(t) \in C^1[0,\infty)$

(b) *There is a sequence* $\{t_m\}$, $t_m > 0$, $t_m \to \infty$ *such that*

$$\lim_{m\to\infty} \frac{L(z,t_m)}{a_1(t)} = F(z) \tag{6}$$

locally uniformly in B^n.

Then for each $s \geq 0$, $L(\cdot,s)$ is univalent on B^n.

Proof. We shall show that $L_1(z,t^*)$ from (4) satisfies the requirements of Theorem 1.

It is obviously that $L_1(z,t^*) \in \mathcal{H}(B^n)$ for each $t^* \in [0,\infty)$ and $L_1(z,t^*)$ is a locally absolutely continuous function on $[0,\infty)$ locally uniform with respect to $z \in B^n$. Using the change of parameter (3) relation (1) becomes:

$$\frac{\partial L_1(z,t^*)}{\partial t^*} = \frac{dt}{dt^*}\left[DL\left(ze^{-i\theta(t)},t\right)\left(ze^{-i\theta(t)}\right)\left(-i\frac{d\theta(t)}{dt}\right) + \frac{\partial L\left(e^{-i\theta(t)}z,t\right)}{\partial t}\right]$$

Next, applying (1) and since $DL_1(z,t^*) = DL\left(e^{-i\theta(t)}z,t\right)e^{-i\theta(t)}$ we have.

$\frac{\partial L_1(z,t^*)}{\partial t^*} = DL_1(z,t^*)h_1(z,t^*)$ where, $h_1:B^n \times [0,\infty) \to C^n$ is the function defined by:

$$h_1(z,t^*) = \frac{dt}{dt^*}\left[h\left(ze^{-i\theta(t)},t\right)e^{i\theta(t)} - i\frac{d\theta(t)}{dt}z\right] \tag{7}$$

The condition (a) and the properties of function h imply that the function h_1 satisfies (iii), (iv), (v) from Theorem 1.

Further, we shall show that $h_1(\cdot,t^*) \in M$ for each $t^* \in [0,\infty)$. It is obviously that $h_1(0,t^*) = 0$. Using (1) it follows that $Dh(0,t) = \frac{a_1'(t)}{a_1(t)}$ and $Dh_1(0,t^*) = I$. Since $\mathrm{Re} < h_1(z,t^*), z > = \frac{dt}{dt^*}\mathrm{Re} < h\left(ze^{-i\theta(t)},t\right), ze^{-i\theta(t)} > \geq 0$ for all $z \in B^n$ then $h_1(\cdot,t^*) \in M$.

By Theorem 1 it follows that function $L_1(z,t^*)$ is univalent in B^n for all $t^* \in [0,\infty)$ and so $L(\cdot,t)$ is an univalent function in B^n.

Next, using Theorem 2 we shall obtain the n-dimensional generalization of Ahlfors' and Becker's univalence criterion.

THEOREM 3. *Let $f \in \mathcal{H}(B^n)$, $f(0) = 0$, $Df(0) = I$ be locally univalent in B^n and let $c \in \mathbf{C} \setminus \{-1\}$ with $|c| \leq 1$.*

If

$$\| c\|z\|^2 I - (1 - \|z\|^2)(Df(z))^{-1}D^2f(z)(z,\cdot)\| \leq 1,\ \forall\ z \in B^n \tag{8}$$

then f is an univalent function in B^n.

Proof. We shall show that (8) enables us to imbed f as the initial element $f(z) = L(z,0)$ in a suitable subordination chain.

We define:

$$L(z,t) = f(e^{-t}z) + \frac{1}{1+c}(e^t - e^{-t})Df(ze^{-t})(z),\ t \in [0,\infty),\ z \in B^n \tag{9}$$

Since $a_1(t) = \frac{e^t(1+ce^{-2t})}{1+c}$ we deduce that $a_1(t) \neq 0$, $|a_1(t)| \to \infty$ when $t \to \infty$ and $a_1(t) \in C^1([0,\infty))$.

It is easy to check that:

$L(z,t) = a_1(t)z$ + (holomorphic term), thus $\lim_{t\to\infty} \frac{L(z,t)}{a_1(t)} = z$ locally uniform with respect to B^n and thus (6) holds with $F(z) = z$.

Obviously $L(z,t)$ satisfies the absolute continuity requirements of Theorem 2.

From (9) we obtain:

$$DL(z,t) = \frac{1}{1+c} e^t Df(ze^{-t}) \times \\ \times \left[I + ce^{-2t}I + (1-e^{-2t})\left(Df(ze^{-t})\right)^{-1} D^2 f(ze^{-t})(ze^{-t}, \cdot)\right] \quad (10)$$

If we let, for each fixed $(z,t) \in B^n \times [0,\infty)$, $E(z,t)$ the linear operator defined by:

$$E(z,t) = -ce^{-2t}I - (1-e^{-2t})\left(Df(ze^{-t})\right)^{-1} D^2 f(ze^{-t})(ze^{-t}, \cdot) \quad (11)$$

then (10) becomes:

$$DL(z,t) = \frac{1}{1+c} e^t Df(ze^{-t})(I - E(z,t)) \quad (12)$$

Next, we shall show that for each $z \in B^n$ and $t \in [0,\infty)$, $I - E(z,t)$ is an invertible operator.

For $t = 0$, $E(z,0) = -cI$, we have $I - E(z,t) = (1+c)I$ and since $1+c \neq 0$ it follows that $I - E(z,t)$ is an invertible operator.

For $t > 0$, since $E(\cdot,t)\colon \bar{B}^n \to \mathfrak{L}(\mathbf{C}^n, \mathbf{C}^n)$ is holomorphic using the weak maximum modulus theorem [2] we obtain that $\|E(z,t)\|$ can have no maximum in B^n unless $\|E(z,t)\|$ is of constant value throughout $\bar{B}^n$

If $z = 0$ and $t > 0$ we have

$$\|E(0,t)\| = \|ce^{-2t}I\| = |c|e^{-2t} < 1 \quad (13)$$

Also, we have

$$\|E(z,t)\| \le \max_{\|w\|=1} \|E(w,t)\| \quad (14)$$

Let now $u = e^{-t}w$, where $\|w\| = 1$, then $\|u\| = e^{-t}$ and so:

$$E(w,t) = -c\|u\|^2 I - (1-\|u\|^2)(Df(u))^{-1} D^2 f(u)(u, \cdot) \quad (15)$$

Using (8), (15), (13) and (14) it follows:

$$\|E(z,t)\| < 1,\ t > 0. \tag{16}$$

Hence for $t > 0$ I-$E(z,t)$ is an invertible operator too.

Further calculation shows that

$$\frac{\partial L(z,t)}{\partial t} = DL(z,t)(I - E(z,t))^{-1}(I + E(z,t))(z)$$

Hence $L(z,t)$ satisfies the differential equation (5) for all $z \in B^n$ and $t \geq 0$ where

$$h(z,t) = (I - E(z,t))^{-1}(I + E(z,t))(z) \tag{17}$$

It remains to show that $h(z,t)$ in (17) satisfies the conditions of Theorem 2. Clearly, $h(z,t)$ satisfies the holomorphy and measurability requirements, and $h(0,t) = 0$.

Furthermore, the inequality

$$\|h(z,t) - z\| = \|E(z,t)(h(z,t) + z)\| \leq \|E(z,t)\|\,\|h(z,t) + z\| < \|h(z,t) + z\|$$

implies that $\mathrm{Re} < h(z,t), z > \geq 0\ \forall\ z \in B^n,\ t \geq 0$.

By using inequality:

$$\|(I - E(z,t))^{-1}\| \leq (1 - \|E(z,t)\|)^{-1}$$

we obtain:

$$\|h(z,t)\| \leq \frac{1 + \|E(z,t)\|}{1 - \|E(z,t)\|} \cdot \|z\|$$

Hence $h(z,t)$ satisfies the condition (vi) of Theorem 2 and it follows that the functions $L(z,t)$ ($t \geq 0$) are univalent in B^n. In particular $f(z) = L(z,0)$ is univalent in B^n.

Remark. For $c = 0$, Theorem 3 becomes the n-dimensional version of Becker's univalence criterion [3].

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ON A FAVARD-SZASZ TYPE OPERATOR

Alexandra CIUPA* and Ioan GAVREA*

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REZUMAT. - Asupra unui operator de tip Favard-Szasz. În lucrare se introduce un operator generalizat de tip Favard-Szasz. Se studiază proprietăţile de aproximare a unei funcţii $f \in L_1[0,\infty)$ prin şirul de operatori P_n şi se dă evaluarea ordinului de aproximare cu ajutorul modulului de continuitate de ordinul doi.

1. Following a method given by I.L. Durrmeyer [2] and M.M. Derriennic [1], S.M. Mazhar and V. Totik [6] have obtained the following Favard-Szasz type operator:

$$(M_n f)(x) = n\sum_{k=0}^{\infty}\left(\int_0^{\infty} f(t)p_{n,k}(t)dt\right)p_{n,k}(x)$$

where $p_{n,k}(x) = e^{-nx}\dfrac{(nx)^k}{k!}$ and $f \in L_1[0,\infty)$

Similarly, we will modify a sequence of Favard-Szasz type operators, which are introduced by A. Jakimovski and D. Leviatan [4]. Let $g(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in the disk $|z| < R$, $R > 1$, and suppose $g(1) \neq 0$. Define the Appell polynomials $p_k(x)$, $k \geq 0$ by:

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k. \tag{1.1}$$

To each function f defined in $[0,\infty)$ associate the operators:

$$(L_n f)(x) = \frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) \tag{1.2}$$

We suppose $p_k(x) \geq 0$ for $x \in [0,\infty)$, $k = 0,1,2,\ldots$ If $a_n \geq 0$, $n \in N$, this supposition is satisfied, because

$$p_k(x) = \sum_{i=0}^{k} a_i \frac{x^{k-i}}{(k-i)!} \tag{1.3}$$

* *Technical University, Department of Mathematics, 3400 Cluj-Napoca, Romania*

A. Jakimovski, D.Leviatan [4] and B.Wood [7] have studied the properties of this sequence of operators.

For $f \in L_1[0,\infty)$, we replace $f\left(\frac{k}{n}\right)$ into L_n, by $\frac{n}{d_k(1)}\int_0^\infty p_k(nt)f(t)\,dt$ where $d_k(x) = \sum_{i=0}^{k} a_i x^i$, $a_0 = 0$ and so we obtain a class of positive linear operators:

$$(P_n f)(x) = \frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\frac{n}{d_k(1)}\int_0^\infty p_k(nt)e^{-nt}f(t)\,dt \tag{1.4}$$

Next, we will denote by E the class of functions of exponential type, which have the property that $|f(t)| \le e^{At}$, for each $t \ge 0$ and some finite number A.

The following lemma is essential to study the convergence of the sequence $P_n f$.

LEMMA 1.1. *For all $x \ge 0$, we have*

$$(P_n e_0)(x) = 1 \tag{1.5}$$

$$(P_n e_1)(x) = x + \frac{1}{n}\left(\frac{g'(1)}{g(1)} + 1\right) - \frac{1}{n}\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\frac{d_k'(1)}{d_k(1)} \tag{1.6}$$

$$(P_n e_2)(x) < x^2 + \frac{2x}{n}\left(2 + \frac{g'(1)}{g(1)}\right) + \frac{1}{n^2}\left(\frac{g''(1) + 4g'(1)}{g(1)} + 2\right) \tag{1.7}$$

where $e_i(x) = x^i$, $i \in \{0,1,2\}$

Proof. Next, we will use the properties of the gamma function and the values of the operator L_n defined at (1.2) for the test function e_i. We have

$$(P_n e_0)(x) = \frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\frac{n}{d_k(1)}\int_0^\infty p_k(nt)e^{-nt}\,dt$$

But, using (1.3), we have

$$\int_0^\infty p_k(nt)e^{-nt}\,dt = \sum_{i=0}^{k} a_i\frac{1}{(k-i)!}\int_0^\infty (nt)^{k-i}e^{-nt}\,dt =$$

$$= \sum_{i=0}^{k} a_i\frac{1}{(k-i)!}\frac{1}{n}\Gamma(k-i+1) = \frac{1}{n}\sum_{i=0}^{k} a_i = \frac{1}{n}d_k(1).$$

Therefore,

$$(P_n e_0)(x) = \frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx) = (L_n e_0)(x) = 1$$

Similarly, one calculates $(P_n e_1)(x)$. By making use of

$$(L_n e_1)(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{k}{n} = x + \frac{1}{n} \frac{g'(1)}{g(1)},$$

one obtains (1.6).

To calculate $(P_n e_2)(x)$, we will use

$$(L_n e_2)(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{k^2}{n} = x^2 + \frac{x}{n}\left(1 + 2\frac{g'(1)}{g(1)}\right) + \frac{1}{n^2}\left(\frac{g''(1) + g'(1)}{g(1)}\right) \quad (1.8)$$

We have

$$(P_n e_2)(x) = \frac{e^{-nx}}{g(1)} n \sum_{k=0}^{\infty} \frac{p_k(nx)}{d_k(1)} \int_0^{\infty} e^{-nt} p_k(nt) t^2 dt$$

But

$$\int_0^{\infty} e^{-nt} p_k(nt) t^2 dt = \frac{1}{n^3} \sum_{i=0}^{k} a_i \frac{1}{(k-i)!} \Gamma(k-i+3) = \frac{1}{n^3} \sum_{i=0}^{k} a_i (k-i+1)(k-i+2) =$$

$$= \frac{1}{n^3}(k+1)(k+2) \sum_{i=0}^{k} a_i - \frac{2}{n^3}(k+1) \sum_{i=0}^{k} i a_i + \frac{1}{n^3} \sum_{i=1}^{k} i(i-1) a_i =$$

$$= \frac{1}{n^3}(k+1)(k+2) d_k(1) - \frac{2}{n^3}(k+1) d_k'(1) + \frac{1}{n^3} d_k''(1).$$

Replacing in $(P_n e_2)(x)$ and using (1.8), we obtain

$$(P_n e_2)(x) = x^2 + \frac{x}{n}\left(4 + 2\frac{g'(1)}{g(1)}\right) + \frac{1}{n^2}\left(\frac{g''(1) + 4g'(1)}{g(1)} + 2\right) + \frac{1}{n^2} A(n,x)$$

where we have denoted

$$A(n,x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{d_k(1)} \left(d_k''(1) - 2(k+1) d_k'(1)\right)$$

But $A(n,x)$ is negative, because

$$d_k''(1) - 2(k+1) d_k'(1) = \sum_{i=2}^{k} i(i-1) a_i - 2(k+1) \sum_{i=1}^{k} i a_i = \sum_{i=1}^{k} i(i-2k-3) a_i$$

and so we obtain (1.7).

We will prove that the expresion $\frac{1}{n^2} A(n,x)$ is bounded. Taking account that

$$d_k(x) = \sum_{i=0}^{k} a_i x^i, \; d_k'(x) = \sum_{i=1}^{k} i a_i x^{i-1} \text{ and } d_k''(x) = \sum_{i=2}^{k} i(i-1) a_i x^{i-2}$$

and using the Lagrange theorem, we have

$$|d_k'(1) - d_k'(0)| = |d_k'(\eta)|, \text{ where } \eta \in (0,1), \text{ or}$$

$$d_k'(1) - a_1 = d_k'(\eta) \le g''(\eta) \le g''(1) = M_1$$

One results that $\dfrac{d_k'(1)}{d_k(1)} \le \dfrac{M_1 + a_1}{d_k(1)} < \dfrac{M_1 + a_1}{a_0}$

Similarly, we have:

$$|d_k'(1) - d_k'(0)| = |d_k''(\xi)|, \text{ where } \xi \in (0,1), \text{ or}$$

$$d_k'(1) - 2a_2 = d_k''(\xi) \le g'''(\xi) \le g''(1) = M_2$$

and so we obtain $\dfrac{d_k'(1)}{d_k(1)} \le \dfrac{M_2 + 2a_2}{d_k(1)} < \dfrac{M_2 + 2a_2}{a_0}$

By making use of this estimations we obtain:

$$|A(n,x)| < \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{M_2 + 2a_2}{a_0} + 2\frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx)(k+1)\frac{M_1 + a_1}{a_0} =$$

$$= \frac{M_2 + 2a_2}{a_0} + \frac{2(M_1 + a_1)}{a_0}\left[1 + n\left(x + \frac{1}{n}\frac{g'(1)}{g(1)}\right)\right]$$

THEOREM 1.1. *If* $f \in L_1[0,\infty) \cap E$, *then* $\lim\limits_{n\to\infty}(P_n f)(x) = f(x)$, *the convergence being uniform in each compact* $[0,a]$.

Proof. According to Lemma 1.1 and noticing that the expression

$$\frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} P_k(nx) \frac{d_k'(1)}{d_k(1)}$$

is bounded, one results that $\lim\limits_{n\to\infty}(P_n e_i)(x) = e_i(x)$, $i \in \{0, 1, 2\}$, where $e_i(x) = x^i$ are the test functions. In accordance with the Bohman-Korovkin theorem, we obtain the desired result.

2. In this section we are concerned with the estimate of the order of approximation of a function $f \in C[0,a]$ by means of the linear positive operator P_n. We will use the second order modulus of continuity, defined as

$$\omega_2(f;h) = \sup\{|f(x+t) - 2f(x) + f(x-t)| : 0 \le t \le h,\ x-t,\ x+t \in [a,b]\}$$

For $g \in C^2[0,a]$ let as consider the Taylor expansion of the following form:

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\,du$$

By applying the operator P_n to the function $g(t)$, we obtain

$$(P_n g)(x) - g(x) = g'(x)(P_n(t-x))(x) + \left(P_n\left(\int_x^t (t-u)\,g''(u)\,du\right)\right)(x)$$

But

$$\int_x^t (t-u)\,g''(u)\,du = \int_0^a [(t-u)_+ - (x-u)_+]\,g''(u)\,du$$

Therefore, we can write

$$\left(P_n\left(\int_x^t (t-u)\,g''(u)\,du\right)\right)(x) = \int_0^a (P_n[(t-u)_+ - (x-u)_+])\,g''(u)\,du =$$

$$= g''(c)\int_0^a (P_n[(t-u)_+ - (x-u)_+])(x)\,du, \text{ where } c \in (0,a).$$

For $g(x) = x^2$, we obtain

$$(P_n e_2)(x) - x^2 = 2x(P_n(t-x))(x) + 2\int_0^a (P_n[(t-u)_+ - (x-u)_+])(x)\,du$$

and so it results that

$$\int_0^a (P_n[(t-u)_+ - (x-u)_+])(x)\,du = \frac{1}{2}(P_n(t-x)^2)(x)$$

Therefore, for all function $g \in C^2[0,a]$, we have:

$$\begin{aligned} |(P_n g)(x) - g(x)| &\le \|g'\|(P_n(|t-x|)(x) + \left|P_n\left(\int_x^t (t-u)\,g''(u)\,du\right)(x)\right| \le \\ &\le \|g'\|(P_n(|t-x|))(x) + \frac{1}{2}\|g''\|(P_n(t-x)^2)(x) \le \\ &\le \|g'\|\sqrt{(P_n(t-x)^2)(x)} + \frac{1}{2}\|g''\|(P_n(t-x)^2)(x) \end{aligned} \tag{2.1}$$

THEOREM 2.1. *If $f \in C[0,a]$, then for any $x \in [0,a]$ we have*

$$|(P_n f)(x) - f(x)| \le \frac{h}{a}\|f\| + \frac{3}{4}\omega_2(f;h)\cdot\left(3 + \frac{a}{h}\right)$$

where $h = \sqrt{(P_n(t-x)^2)(x)}$ *and*

$$(P_n(t-x)^2)(x) = 2\frac{x}{n} + \frac{1}{n^2}\left(\frac{g''(1) + 4g'(1)}{g(1)} + 2\right) + \frac{1}{n^2}\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\frac{d_k''(1)}{d_k(1)} +$$

$$+ \frac{2}{n}\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\frac{d_k'(1)}{d_k(1)}\left(x - \frac{k+1}{n}\right)$$

Proof. Let f_h be the Steklov function attached to the function f. We will use the following result of V.V.Juk [5]: if $f \in C[0,a]$ and $h \in \left(0, \frac{b-a}{2}\right)$, then $\|f - f_h\| \le \frac{3}{4}\omega_2(f;h)$ and $\|f_h''\| \le \frac{3}{2}\frac{1}{h^2}\omega_2(f;h)$.

Since $(P_n e_0)(x) = e_0$ we can write

$$|(P_n f)(x) - f(x)| \le |P_n(f - f_h)(x)| + |(P_n f_h)(x) - f_h(x)| +$$
$$+ |f_h(x) - f(x)| \le 2\|f - f_h\| + |(P_n f_h)(x) - f_h(x)|$$

By making use of the relation (2.1) for the function $f_h \in C^2[0,a]$, it results:

$$|(P_n f_h)(x) - f_h(x)| \le \|f_h'\| \sqrt{(P_n(t-x)^2)(x)} + \frac{1}{2}\|f_h''\|(P_n(t-x)^2)(x)$$

In according with a results from [3] and [5], we obtain:

$$\|f_h'\| \le \frac{1}{a}\|f_h\| + \frac{a}{2}\|f_h''\| \le \frac{1}{a}\|f\| + \frac{a}{2}\|f_h''\| \le \frac{1}{a}\|f\| + \frac{3a}{4}\frac{1}{h^2}\omega_2(f;h)$$

and so results that

$$|(P_n f_h)(x) - f_h(x)| \le \left(\frac{1}{a}\|f\| + \frac{3a}{4}\frac{1}{h^2}\omega_2(f;h)\right)\sqrt{(P_n(t-x)^2(x)} +$$
$$+ \frac{1}{2}\frac{3}{2}\frac{1}{h^2}\omega_2(f;h)(P_n(t-x)^2)(x)$$

By inserting into it $h = \sqrt{(P_n(t-x)^2)(x)}$ we obtain

$$|(P_n f_h)(x) - f_h(x)| \le \frac{1}{a}\|f\| h + \frac{3a}{4}\frac{1}{h}\omega_2(f;h) + \frac{3}{4}\omega_2(f;h)$$

now we can write that

$$|(P_n f)(x) - f(x)| \le \frac{3}{2}\omega_2(f;h) + \frac{h}{a}\|f\| + \frac{3a}{4}\frac{1}{h}\omega_2(f;h) + \frac{3}{4}\omega_2(f;h) =$$
$$= \frac{h}{a}\|f\| + \frac{3}{4}\omega_2(f;h)\left(3 + \frac{a}{h}\right)$$

and so the theorem is proved.

Remarks: 1. Let us denoting by

$$B(n,x) = \frac{1}{n^2}\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\frac{d_k''(1)}{d_k(1)} + \frac{2}{n}\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\frac{d_k'(1)}{d_k(1)}\left(x - \frac{k+1}{n}\right)$$

In accordance with the notices from the proof of Lemma 1.1, we have

$$B(n,x) \le \frac{1}{n^2}\frac{M_2 + 2a_2}{a_0} + \frac{2}{n}\frac{M_1 + a_1}{a_0}\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\left|x - \frac{k+1}{n}\right|$$

Next, applying the Cauchy inequality, we obtain

$$\sum_{k=0}^{\infty} p_k(nx)\left|x - \frac{k+1}{n}\right| \le \sqrt{\sum_{k=0}^{\infty} p_k(nx)}\ \sqrt{\sum_{k=0}^{\infty} p_k(nx)\left(x - \frac{k+1}{n}\right)^2} =$$

$$= e^{nx} g(1)\sqrt{\frac{x}{n} + \frac{1}{n^2}\left(1 + \frac{g''(1) + 3g'(1)}{g(1)}\right)}$$

Therefore, we can write that

$$B(n,x) \le \frac{1}{n^2}\frac{M_2 + 2a_2}{a_0} + \frac{2}{n}\frac{M_1 + a_1}{a_0}\sqrt{\frac{x}{n} + \frac{1}{n^2}\left(1 + \frac{g''(1) + 3g'(1)}{g(1)}\right)}$$

and so we obtain

$$(P_n(t-x)^2)(x) \le 2\frac{x}{n} + \frac{1}{n^2}\left(\frac{g''(1) + 4g'(1)}{g(1)} + 2\right) + \frac{1}{n^2}\frac{M_2 + 2a_2}{a_0} +$$

$$+ \frac{2}{n}\frac{M_1 + a_1}{a_0}\sqrt{\frac{x}{n} + \frac{1}{n^2}\left(1 + \frac{g''(1) + 3g'(1)}{g(1)}\right)} = C(n,x)$$

Choosing $h_1 = \sqrt{C(n,x)}$, we obtain the following order of approximation:

$$|(P_n f)(x) - f(x)| \le \frac{h_1}{a}\|f\| + \frac{3}{4}\omega_2(f;h_1)\left(3 + \frac{a}{h_1}\right)$$

2. If $f \in C^2[0,a]$, from the relation (2.1) we obtain

$$|(P_n f)(x) - f(x)| \le \|f'\|\sqrt{(P_n(t-x)^2)(x)} + \frac{1}{2}\|f''\|(P_n(t-x)^2)(x)$$

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PERIODIC SOLUTIONS FOR AN INTEGRAL EQUATION FROM BIOMATHEMATICS VIA LERAY-SCHAUDER PRINCIPLE

Radu PRECUP*

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REZUMAT. - Soluţii periodice pentru o ecuaţie integrală din biomatematică via principiul lui Leray-Schauder. Rezultatele stabilite în această lucrare se referă la existenţa, unicitatea şi aproximarea monoton-iterativă a soluţiilor periodice netriviale pentru ecuaţia integrală (1). Demonstraţiile se bazează pe principiul de continuare al lui Leray-Schauder şi pe tehnica iteraţiilor monotone.

Abstract. The main results of this paper concern the existence, the uniqueness and the monotone iterative approximation of periodic nontrivial solutions for the delay integral equation $x(t) = \int_{t-\tau}^{t} f(s, x(s))\,ds$. The proofs are achieved by the Leray-Schauder continuation principle and the monotone iterative technique.

1. Introduction. The delay integral equation

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))\,ds \tag{1}$$

is a model for the spread of certain infectious diseases with a contact rate that varies seasonally. In this equation $x(t)$ represents the fraction of infectives in the total population at time t, τ is the leangth of time an individual is infective and $f(t, x(t))$ is the proportional of new infectives per unit of time.

In [1-4, 6-11] sufficient conditions were given for the existence of nontrivial ω-periodic continuous solutions to Eq. (1) in case of a ω-periodic contact rate

* *"Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, 3400 Cluj-Napoca, Romania*

$$f(t+\omega,x)=f(t,x),\quad f(t,0)=0.$$

The tools were Banach's fixed point theorem, topological fixed point principles, fixed point index theory, monotonicity technique.

In [7] we used the Leray-Schauder continuation principle (in Granas' approach) to prove the existence of positive continuous solutions $x(t)$ for Eq. (1) on a given interval $[-\tau,T]$, when it is known the proportion $\phi(t)$ of infectives for $-\tau \le t \le 0$, i.e.

$$x(t)=\phi(t)\quad \text{for } -\tau \le t \le 0. \tag{2}$$

Clearly, we had to assume that ϕ satisfies the following condition

$$\phi_0=\phi(0)=\int_{-\tau}^{0} f(s,\phi(s))\,ds. \tag{3}$$

Under condition (3) the problem (1)-(2) is equivalent with the initial values problem

$$x'(t)=f(t,x(t))-f(t-\tau,x(t-\tau))\quad \text{for } 0 \le t \le T \tag{4}$$

$$x(t)=\phi(t)\quad \text{for } -\tau \le t \le 0.$$

We made use of the following hypotheses:

(i) $f(t,x)$ is nonnegative and continuous for $-\tau \le t \le T$ and $x \ge 0$.

(ii) $\phi(t)$ is continuous, $0 < a \le \phi(t)$ for $-\tau \le t \le 0$ and satisfies (3).

(iii) there exists an integrable function $g(t)$ such that

$$f(t,x) \ge g(t)\quad \text{for } -\tau \le t \le T \text{ and } x \ge a$$

and

$$\int_{t-\tau}^{t} g(s)\,ds \ge a\quad \text{for } 0 \le t \le T.$$

(iv) there exists a positive function $h(x)$ such that $1/h(x)$ is locally integrable on $[a,\infty)$,

$$f(t,x) \le h(x)\quad \text{for } 0 \le t \le T \text{ and } x \ge a$$

and

$$T < \int_{\phi_0}^{\infty} (1/h(x))\,dx.$$

THEOREM A [7]. *Suppose that* (i)-(iv) *are satisfied. Then* Eq. (1) *has at least one continuous solution* $x(t)$, $x(t) \geq a$, *for* $-\tau \leq t \leq T$, *which satisfies* (2).

Approximation schema to solve (1)-(2) under assumptions (i)-(iv), based on the monotone iterative method of Lakshmikantham (see [5]), were described in [8] for the cases where $f(t,x)$ is nondecreasing or nonincreasing with respect to x.

In this paper we shall use a similar technique based upon the Leray-Schauder continuation principle, to establish a new existence result for the periodic solutions of Eq. (1). Finally, the monotone iterative technique is used to prove the uniqueness of the solution and to approximate it, in case $f(t,x)$ is nonincreasing with respect to x.

In [4] (see also [6]) the following conditions are used:

(h_1) $f(t,x)$ is nonnegative and continuous for $-\infty < t < \infty$ and $x \geq 0$.

(h_2) $f(t,0) = 0$ for $-\infty < t < \infty$ and there exists $\omega > 0$ such that $f(t+\omega, x) = f(t,x)$ for $-\infty < t < \infty$ and $x \geq 0$.

(h_3) there exist $0 < a < R$ and a nonnegative locally integrable function $g(t)$ with period ω such that

$$f(t,x) \geq g(t) \quad \text{for } 0 \leq t \leq \omega \text{ and } a \leq x \leq R,$$

and

$$\int_{t-\tau}^{t} g(s)\,ds \geq a \quad \text{for } 0 \leq t \leq \omega.$$

(h_4) $f(t,x) \leq R/\tau$ for $0 \leq t \leq \omega$ and $a \leq x \leq R$.

One of the results in [4] is the following theorem whose proof is based on Schauder's fixed point theorem.

THEOREM B [4]. *If* (h_1)-(h_4) *are satisfied, then* Eq. (1) *has at least one positive and continuous solution* $x(t)$ *with period* ω *and*

$$a \le \inf_{-\infty<t<\infty} x(t) \le \sup_{-\infty<t<\infty} x(t) \le R.$$

Let us remark that for a given function $f(t,x)$ satisfying (h_1) and (h_2) could exist several intervals $[a,R]$ such that (h_3) and (h_4) hold. If these intervals are disjoint, then Theorem B ensures that the corresponding solutions are distinct. For example, if $f(t,x) = x/\tau$ for $-\infty < t < \infty$ and $x \ge 0$, we may take arbitrary $\omega > 0$, $a > 0$ and $R > a$. Clearly, in this case, any nonnegative constant is a solution.

Assumption (h_4) is essential for the domain invariance in Schauder's fixed point theorem. We shall replace (h_4) by another condition which guarantes that the Leray-Schauder boundary condition is satisfied.

We shall see that there are cases where our main existence result, Theorem 1, applies and Theorem B does not, and conversely.

2. Main existence result. We use instead of (h_4) the following hypothesis:

(h_5) there exists a positive function $h(x)$ such that $1/h(x)$ is integrable for $a \le x \le R$, and a number b such that $a < b < R$,

$$f(t,x) \le h(x) \quad \text{for } 0 \le t \le \omega \text{ and } a \le x \le R, \tag{5}$$

$$\int_b^R (1/h(x))\,dx \ge \omega \tag{6}$$

and

$$f(t,x) < b/\tau \quad \text{for } 0 \le t \le \omega \text{ and } b \le x \le R. \tag{7}$$

THEOREM 1. *Suppose that* (h_1)-(h_3) *and* (h_5) *are satisfied. Then,* Eq. (1) *has at least one continuous solution* $x(t)$ *with period* ω *and*

$$a \le \inf_{-\infty<t<\infty} x(t) < b \text{ and } \sup_{-\infty<t<\infty} x(t) < R. \tag{8}$$

Proof. Let E be the real Banach space of all continuous and ω-periodic functions $x(t)$, with norm

$$\|x\| = \sup_{-\infty<t<\infty} |x(t)| = \max_{0\le t\le\omega} |x(t)|.$$

Let $K = \{x \in E\,;\ a \le x(t) \text{ for } 0 \le t \le \omega\}$ and

$$U = \{\, x \in K;\ \min_{0\le t\le\omega} x(t) < b \ \text{ and } \ \|x\| < R\,\}.$$

Obviously, K is a closed convex subset of E and U is bounded and open in K. Consider the homotopy

$$H : [0,1] \times \bar{U} \to K,$$

$$H(\lambda,x)(t) = (1-\lambda)a + \lambda\int_{t-\tau}^{t} f(s,x(s))\,ds$$

for $0 \le \lambda \le 1$, $x \in \bar{U}$ and $0 \le t \le \omega$. It is easy to show that, by (h_1)-(h_3), H is well-defined and completely continuous. We claim now that H satisfies the Leray-Schauder condition on the boundary ∂U of U with respect to K, i.e. $H(\lambda,\cdot)$ is fixed point free on ∂U for each $\lambda \in [0,1]$. Assume, by contradiction, that there would exist $\lambda \in (0,1]$ and $x \in \partial U$ such that $H(\lambda,x)=x$, that is

$$x(t) = (1-\lambda)a + \lambda\int_{t-\tau}^{t} f(s,x(s))\,ds \quad \text{for } -\infty < t < \infty. \tag{9}$$

Since x is on ∂U, we have either

$$\|x\|=R \ \text{ and } \ \min_{0\le t\le\omega} x(t) < b, \tag{10}$$

or

$$\|x\| \le R \ \text{ and } \ \min_{0\le t\le\omega} x(t) = b. \tag{11}$$

First, suppose (10). Then, by differentiating (9), we obtain

$$x'(t) = \lambda f(t,x(t)) - \lambda f(t-\tau, x(t-\tau)).$$

Hence, by (h_1) and (5), we have

$$x'(t) \le \lambda f(t,x(t)) \le \lambda h(x(t)) \le h(x(t)).$$

Let $0 \le t_0 \le \omega$ be such that $x(t_0) = \min_{0\le t\le\omega} x(t)$. Integration from t_0 to t yields

$$\int_{t_0}^{t}(x'(s)/h(x(s)))\,ds \le t-t_0 \le \omega \quad \text{for} \quad t_0 \le t \le t_0+\omega.$$

Thus,

$$\int_{x(t_0)}^{x(t)}(1/h(u))\,du \le \omega \quad \text{for} \quad t_0 \le t \le t_0+\omega.$$

Since $x(t_0) < b$, by (6), we deduce that $x(t) < R$ for all $t_0 \le t \le t_0+\omega$, equivalently for all $-\infty < t < \infty$. Therefore, $\|x\| < R$, a contradiction. Next, suppose (11). Let $0 \le t_0 \le \omega$ be such that $x(t_0) = \min_{0 \le t \le \omega} x(t) = b$. Then, by (9) and (7), we obtain

$$\begin{aligned} b = x(t_0) &= (1-\lambda)a + \lambda\int_{-\tau}^{t_0} f(s,x(s))\,ds < \\ &< (1-\lambda)b + \lambda b = b, \end{aligned} \tag{12}$$

again a contradiction. Thus, H is an admissible homotopy on $\bar{U}$. On the other hand, the mapping $H(0,\cdot)$ is essential (its fixed point index $i(H(0,\cdot), U, K)$ equals 1) because $H(0,\cdot)=a$ and the constant function a belongs to U. Consequently, by the Leray-Schauder principle, $H(1,\cdot)$ is essential too. Therefore, there exists at least one fixed point of $H(1,\cdot)$ in U, that is a continuous solution with period ω for Eq. (1) satisfying (8). Thus, Theorem 1 is proved.

Remark 1. Theorem 1 remains true if instead of (7) we only assume that there exists a locally integrable function $l(t)$ with period ω such that

$$f(t,x) \le l(t) \quad \text{for} \quad 0 \le t \le \omega \quad \text{and } b \le x \le R$$

and

$$\int_{-\tau}^{t} l(s)\,ds < b \quad \text{for} \quad 0 \le t \le \omega.$$

Indeed, under this more general assumption, the strict inequality (12) also holds.

Remark 2. Here is an example for which Theorem 1 applies but Theorem B does not. Let $\tau = \omega = 1$ and let $f(t,x) = h_1(x)$ $(-\infty < t < \infty)$, where

$$\begin{aligned} h_1(x) &= 5x && \text{for } 0 \le x \le 1 \\ &= -4x+9 && \text{for } 1 \le x \le 2 \\ &= 1 && \text{for } 2 \le x \le 3 \\ &= 3x-8 && \text{for } 3 \le x \le 5 \\ &= x+2 && \text{for } 5 \le x. \end{aligned}$$

Conditions (h_1)-(h_3) and (h_5) are fulfilled with $a = 1$, $b = 2$, $R = 3$, $g(t) = 1$ and $h(x) = h_1(x)$, but for any $R > 0$ there is no $a < R$ such that (h_3)-(h_4) be satisfied.

Remark 3. For a given function $f(t,x)$ satisfying (h_1)-(h_2) there could exist several intervals $[a, R]$ such that (h_3) and (h_5) hold. If these intervals are disjoint, then the corresponding solutions by Theorem 1 are distinct. Here is an example: Let τ and $h_1(x)$ be as in Remark 2 and let

$$f(t,x) = g_1(t)h_n(x) \text{ for } -\infty < t < \infty,\ 4(n-1) \le x \le 4n,\ n = 1, 2, \ldots,$$

where $h_n(x) = 4(n-1) + h_1(x - 4(n-1))$ and $g_1(t)$ is any continuous nonnegative function with a period $\omega > 0$ such that

$$\int_{t-1}^{t} g_1(s)\,ds \ge 1 \text{ for } 0 \le t \le \omega.$$

It is easy to see that if $\omega \cdot \max\limits_{0 \le t \le \omega} g_1(t) \le (4(n-1)+1)^{-1}$, all the assumptions of Theorem 1 are fulfilled for $a = 4(n-1)+1$, $b = 4(n-1)+2$, $R = 4(n-1)+3$, $g(t) = (4(n-1)+1)g_1(t)$ and $h(x) = \max\limits_{0 \le t \le \omega} g_1(t)\,h_n(x)$. Therefore, for each nonnull natural number n so that $4(n-1) + 1 \le (\omega \max\limits_{0 \le t \le \omega} g_1(t))^{-1}$, Eq. (1) has at least one continuous solution $x_n(t)$ with period ω, such that

$$4(n-1) + 1 \le \inf_{-\infty < t < \infty} x_n(t) < 4(n-1) + 2$$

and

$$\sup_{-\infty < t < \infty} x_n(t) < 4(n-1) + 3.$$

For example, in case $g_1(t) = 1$, such solutions are the following constant ones

$$x_n(t) = 4(n-1) + 9/5,\ n = 1, 2, \ldots.$$

Notice that none of these constant solutions can be obtained by means of Theorem B.

Example 1. Let us give another function which satisfies the assumptions of Theorem 1:

$$f(t,x) = g_0(t)h(x),\ -\infty < t < \infty,\ x \ge 0,$$

where $h(x) = (x-1/2)(x-2)(x-3)+3$ and $g_0(t)$ is a continuous function with period

$$\omega \le \int_{2.6}^{2.7} (1/h(x))\,dx$$

and satisfies the following conditions

$$0 \le g_0(t) \le 1 \text{ for } 0 \le t \le \omega,$$

$$\int_{t-1}^{t} g_0(s)\,ds \ge 1/h((11+\sqrt{19})/6) \text{ for } 0 \le t \le \omega.$$

For this function we take $\tau = 1$, $a = 1$, $b = 2.6$, $R = 2.7$ and $g(t) = h((11+\sqrt{19})/6)\,g_0(t)$, where $h((11+\sqrt{19})/6) > 1$ is the minimum of $h(x)$ for $1 \le x \le 2.7$.

3. Uniqueness and monotone iterative approximation schema. Under the assumptions of Theorem 1, denote by A the completely continuous operator from $P = \{x \in E;\ 0 \le x(t) \text{ for } 0 \le t \le \omega\}$ into P,

$$Ax(t) = \int_{t-\tau}^{t} f(s, x(s))\,ds, \quad -\infty < t < \infty,\ x \in P.$$

Also define the following sequence of functions in P:

$$v_0(t) = R,\ v_n(t) = Av_{n-1}(t),\ n = 1, 2, \ldots.$$

THEOREM 2. *Let* (h_1)-(h_3) *and* (h_5) *hold and suppose that* $f(t,x)$ *is nonicreasing in* x *for* $a \le x \le R$ *and there exists* $\alpha \in (-1, 0)$ *such that*

$$f(t, \gamma x) \le \gamma^{\alpha} f(t, x) \tag{13}$$

for all $t \in [0, \omega]$, $\gamma \in (0, 1)$ *and* $x \in [a, R]$ *with* $\gamma x \in [a, R]$. *If*

$$A^2(R)(t) \le R \quad \text{for } 0 \le t \le \omega, \tag{14}$$

then Eq. (1) *has a unique continuous solution* $x^*(t)$ *of period* ω *such that* $a \le x^*(t) \le R$ *for* $0 \le t \le \omega$. *Moreover,*

$$a \le v_1(t) \le v_3(t) \le \ldots \le v_{2n+1}(t) \le \ldots \le x^*(t) \le \ldots$$
$$\le v_{2n}(t) \le \ldots \le v_4(t) \le v_2(t) \le v_0(t) = R \text{ for } t \in [0, \omega], \tag{15}$$

$v_n(t) \to x^*(t)$ *uniformly for* $0 \le t \le \omega$ *as* $n \to \infty$.

Proof. By Theorem 1, there exists at least one continuous solution $x(t)$ of period ω for Eq. (1), such that $a \le x(t) \le R$ for $0 \le t \le \omega$. Now let $x(t)$ be any solution of this type for Eq. (1). Since $f(t, x)$ is nonicreasing in x for $a \le x \le R$, from

$$a \le x(t) \le R = v_0(t) \text{ for } 0 \le t \le \omega,$$

we get

$$a \le A(R)(t) \le x(t) \text{ for } 0 \le t \le \omega.$$

It follows that

$$a \le A(R)(t) \le x(t) \le A^2(R)(t) \text{ for } 0 \le t \le \omega.$$

By (14), this yields

$$a \le A(R)(t) \le A^3(R)(t) \le x(t) \le A^2(R)(t) \le R \text{ for } 0 \le t \le \omega.$$

Finally, we obtain

$$\begin{aligned} &a \le v_1(t) \le v_3(t) \le \dots \le v_{2n+1}(t) \le \dots \le x(t) \le \dots \\ &\le v_{2n}(t) \le \dots \le v_2(t) \le v_0(t) = R \text{ for } 0 \le t \le \omega. \end{aligned} \tag{16}$$

By the complete continuity of A^2, the sequence $(v_{2n+1}(t))_{n \ge 0}$ contains a subsequence uniformly convergent to some $x_*(t)$ in K and, similarly, $(v_{2n}(t))_{n \ge 0}$ contains a subsequence convergeing uniformly to some $x^*(t)$ in K. From (16), we see that the entier sequences $(v_{2n+1}(t))_{n \ge 0}$ and $(v_{2n}(t))_{n \ge 0}$ converge uniformly to $x_*(t)$ and $x^*(t)$, respectively, for $0 \le t \le \omega$, and that

$$a \le x_*(t) \le x(t) \le x^*(t) \le R \quad \text{for} \quad 0 \le t \le \omega. \tag{17}$$

Obviously, we have

$$x_*(t) = A\,x^*(t) \text{ and } x^*(t) = Ax_*(t).$$

Now we prove that under assumption (13), we have indeed $x_*(t) = x^*(t)$ for all $0 \le t \le \omega$.

To this end, let

$$\gamma_0 = \min_{0 \le t \le \omega} (x_*(t)/x^*(t)).$$

From (17), we see that $0 < a/R \le \gamma_0 \le 1$. We show that $\gamma_0 = 1$. Suppose $\gamma_0 < 1$. Since $x_*(t) \ge \max\{a, \gamma_0 x^*(t)\} = \gamma_0 \max\{a/\gamma_0, x^*(t)\} \ge a$ for $0 \le t \le \omega$, by (13), we get that

$$x^*(t) = Ax_*(t) \le A(\gamma_0 \max\{a/\gamma_0, x^*\})(t) \le$$

$$\le \gamma_0^{\alpha} A(\max\{a/\gamma_0, x^*\})(t).$$

One has $x^*(t) \le \max\{a/\gamma_0, x^*(t)\} \le R$ and so,

$$A(\max\{a/\gamma_0, x^*\})(t) \le Ax^*(t) = x_*(t).$$

It follows $x^*(t) \le \gamma_0^{\alpha} x_*(t)$ for $0 \le t \le \omega$. Hence $\gamma_0^{-\alpha} \le \gamma_0$ or, equivalently, $\alpha \le -1$, a contradiction. Thus, $\gamma_0 = 1$ as claimed. Consequently, $x_*(t) = x(t) = x^*(t)$ for $0 \le t \le \omega$ and the proof is complete.

Remark 4. A sufficient condition for (14) is that

$$A(a)(t) \le R \quad \text{for} \quad 0 \le t \le \omega. \tag{18}$$

Indeed, from $a \le A(R)(t) \le A(a)(t)$, we get that

$$A^2(a)(t) \le A^2(R)(t) \le A(a)(t) \le R,$$

whence (14).

If in Theorem 2 we use (18) instead of (14), then we have in addition that for any continuous function $x_0(t)$ with period ω satisfying $a \le x_0(t) \le R$, one has

$$x_n(t) \to x^*(t) \text{ uniformly for } 0 \le t \le \omega \tag{19}$$

as $n \to \infty$, where $x_n = Ax_{n-1}$, $n = 1, 2, \ldots$. Indeed, in this case, from $a \le x_0(t) \le R$, we obtain

$$a \le v_1(t) \le x_1(t) \le A(a) \le R = v_0(t)$$

whence,

$$a \le v_1(t) \le x_2(t) \le v_2(t) \le A(a) \le R = v_0(t)$$

and, in general,

$$a \le v_1(t) \le v_3(t) \le \dots \le v_{2[(n-1)/2]+1}(t) \le$$

$$\le x_n(t) \le v_{2[n/2]}(t) \le \dots \le v_2(t) \le v_0(t) = R,$$

$n = 1,2, \dots$. Since $v_n(t) \to x^*(t)$, it follows that $x_n(t) \to x^*(t)$, as claimed.

Remark 5. If $f(t,x)$ satisfies (h_4) then (18) holds. Indeed, from $f(t,a) \le R/\tau$ for any t, by integrating, we obtain (18).

Conversely, if $f(t,x)$ is constant in $t(f(t,x) = h(x))$ and satisfies all assumptions of Theorem 2 and (18), then (h_4) is fulfilled. Indeed, for any $x_0 \in [a, R]$, we have $A(R) \le \le A(x_0) \le A(a) \le R$. Hence, $h(x_0) = \tau^{-1}A(x_0) \le R/\tau$.

The next theorem completes the results in [4].

THEOREM 3. *Let* (h_1)-(h_4) *hold and suppose that* $f(t,x)$ *is nonincreasing in* x *for* $a \le x \le R$ *and there exists* $\alpha \in (-1, 0)$ *such that* (13) *is satisfied. Then* Eq. (1) *has a unique continuous solution* $x^*(t)$ *of period* ω *such that* $a \le x^*(t) \le R$ *for* $0 \le t \le \omega$. *Moreover*, (19) *holds.*

The proof is similar with that of Theorem 2, so we omit the details.

Example 2. Suppose that $f(t,x)$ satisfies (h_1)-(h_2) and

$$f(t,x) = g(t)(3/x)^{1/2} \text{ for } -\infty < t < \infty \text{ and } 1 \le x \le 3,$$

where $g(t)$ is a continuous function with period ω such that

$$0 \le g(t) \le \sqrt{3} \quad \text{for } 0 \le t \le \omega$$

and

$$\int_{t-1}^{t} g(s)\,ds \ge 1 \quad \text{for } 0 \le t \le \omega.$$

The assumptions of Theorem 3 are fulfilled with $\tau = 1$, $a = 1$, $R = 3$ and $\alpha = -1/2$.

We conclude with a simple example of functions which satisfy all assumptions of

Theorem 2, but not (h_4).

Example 4. Suppose that $f(t,x)$ satisfies (h_1)-(h_2) and

$$f(t,x) = M(3/x)^{1/2} \text{ for } -\infty < t < \infty \text{ and } 1 \le x \le 3,$$

where M is any constant such $\sqrt{3} < M < 5\sqrt{30}/12$. All assumptions of Theorem 2 are satisfied with $\tau = 1$, $a = 1$, $b = 2.5$, $R = 3$, provided that $M\omega \le 2 - 5\sqrt{30}/18$, while (h_4) does not hold. Therefore, there are cases where Theorem 3 fails and Theorem 2 applies.

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SUR LES COUPLES DE CONNEXIONS FINSLER - PROJECTIVE CONJUGUEES

Gh. MURĂRESCU*

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REZUMAT. - **Asupra perechilor de conexiuni Finsler - proiective conjugate. În această notă este studiată relaţia de conjugare pentru conexiunile proiective Finsler.**

Introduction. La rélation de conjugaison a été introduite par Norden [7] pour les connexions affines. Vedernicov [10] et puis Gawcarzewicz [2] ont généralisé cette rélation, fondent une théorie globale (en utilisant la théorie des espaces fibrés). R. Miron [4], V. Cruceanu [1], D. Opris [8], l'auteur [5], [6] ont développé l'étude de la rélation de conjugaison entre les connexions: metriques, linéaires, projectives, tangentielles, de type Cartan et Finsler. Dans cette note nous étudions la rélation de conjugaison pour les connexions Finsler projectives.

Définition de la rélation de conjugaison pour les connexions linéaires.

DÉFINITION 1.[2]. Deux connexions ∇ et $\bar{\bar{\nabla}}$ définies sur un espace fibré principal $P(M,G)$ sont dites h-conjuguées, où $h: G \to G$ est un endomorphisme du groupe G, s'il existe un espace fibré réduit $P_0(M,H^h)$ de $P(M,G)$ de groupe structural $H^h = \{g \in G : h(g) = g\}$ tel que pour toute section $\sigma: U \to P_0$, on a: $\sigma^* \bar{\bar{\omega}} = \mathfrak{h} \circ \sigma^*\omega$, où ω, $\bar{\bar{\omega}}$ sont les formes de conexion de ∇ et de $\bar{\bar{\nabla}}$ ($\mathfrak{h}$ est l'endomorphisme de l'algèbre de Lie induit par h). Pour le groupe $GL(n,R)$ sont connues deux types de conjugaison:

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* *University of Craiova, Faculty of Mathematics and Computer Science, 1100 Craiova, Romania*

- la - conjugaison définie par un tenseur de type (1,1). (plus general une ε-densité de type (1,1)) (c^α_β):

- la - conjugaison définie par un tenseur de type (2,0), (plus general une ε-densité de type (2,0)) ($c^{\alpha\beta}$).

DÉFINITION 2 a) Les connexions ∇ et $\bar{\nabla}$ sont φ-conjuguées si entre leurs coefficients (dans les bases locales) éxiste les rélations:

$$\bar{\gamma}^\alpha_{\beta\gamma} = c^\alpha_\lambda \gamma^\lambda_{\beta\mu} \bar{c}^\mu_\gamma, \quad (c^\alpha_\lambda \bar{c}^\lambda_\beta = \delta^\alpha_\beta).$$

b) Les connexions ∇ et $\bar{\nabla}$ sont ψ-conjuguées si entre leurs coefficients éxiste les rélations:

$$\bar{\gamma}^\alpha_{\beta\gamma} = -c^{\alpha\lambda} \gamma^\mu_{\beta\lambda} \bar{c}_{\mu\gamma}, \quad (c^{\alpha\lambda} \bar{c}_{\lambda\beta} = \delta^\alpha_\beta).$$

Définition de la connexion Finsler projective. Soit $\xi = (E,p,M)$ un fibré vectoriel linéaire (à la fibre type $F = R$) doté à une connexion nonlinéaire N définie par la distribution horizontale $H: z \in E \to H_z E$ (donc $T_z E = H_z E \oplus V_z E$); soit puis $\{U_\alpha\} = \{U_i = \partial/\partial x_i$, $U_0 = \partial/\partial y\}$ la base naturelle et $\{X_\alpha\} = \{X_i = l$, $x_0 = \partial/\partial y\}$ la base adaptée dans T_z ($\alpha,\beta = 1, ..., n, 0$: $i,j = 1,..,n$). La transformation de la carte locale sur E est de la forme:

$$x^{i'} = x^{i'}(x^1, ..., x^n), \; y' = f(x^1, ..., x^n)\cdot y. \tag{1}$$

Une connexion linéaire ∇ s'appelle d-connexion si elle conserve par parallélisme des distributions H et V. Les coefficients locals d'une telle connexion par rapport à la base X sont notés par $\Gamma^\gamma_{\alpha\beta}(x,y)$ et par rapport à la base $\{U_\alpha\}$ par $\gamma^\lambda_{\alpha\beta}(x,y)$.

DÉFINITION 3. 9 Une d-connexion linéaire sur E s'appelle quasi-projective normale si:

$$\Gamma^0_{0\omega} = w\Gamma^i_{i\omega}, \; w \in R \tag{2}$$

Cette définition a un caractére géométrique si et seulement si la fonction $f(x^1 \ldots x^n)$ des formules (1) est de la forme.

$$f(x^1, \ldots, x^n) = \Delta^{-w}, \text{ où } \Delta = \left| \frac{\partial x^{i'}}{\partial x^i} \right|.$$

Prof. Stavre P. [9] montre que une transformation $\tau : \nabla \to \bar{\nabla}$ de type:

$$\bar{\nabla}_{XY} = \nabla_{XY} + \alpha(X) \cdot hY + \alpha(Y) \cdot hX, \text{ (avec les motations usuelles)} \tag{3}$$

conserve la classe des d-connexions linéaires quasi-projective normale.

Les coefficients de type Weyl $J^\alpha_{\beta\gamma}$ qui sont inveriants à les transformations de type (3), pour $W = -\frac{1}{n+1}$, essenticls restent:

$$J^i_{jk} = \Gamma^i_{jk} + \left(\Gamma^0_{0k} \cdot \delta^i_j + \Gamma^0_{0j} \cdot \delta^i_k \right). \tag{4}$$

On définissent comme dans le cas clasique les coefficients J^i_{jkh} et $J_{jk} = J^i_{jki}$ (le dernier est symétrique).

DÉFINITION 4. Soit ∇ une connexion quasi-projective normale, avec les coefficients locals $\gamma^\gamma_{\alpha\beta}$, associée a l'application de connexion homogene K (pour $y > 0$); la connexion projective normale généralisée, associée à ∇, notée par $\overset{*}{\nabla}$, est une connexion linéaire dont les coefficients locals $\overset{*}{\gamma}^\gamma_{\alpha\beta}$ (dans les bases naturelles) sont définis par:

$$\overset{*}{\gamma}^i_{jk} = \Gamma^i_{jk} + y^{-1}(N_j\delta^i_k + N_k\delta^i_j) = J^i_{jk} \,,$$

$$\overset{*}{\gamma}^i_{00} = 0 \,;\ \overset{*}{\gamma}^i_{0k} = y^{-1}\delta^i_k \,;\ \overset{*}{\gamma}^i_{k0} = y^{-1}\delta^i_k \,,\ \overset{*}{\gamma}^0_{00} = 0 \,,$$

$$\overset{*}{\gamma}^0_{0k} = 0 \,,\ \overset{*}{\gamma}^0_{k0} = 0 \,,\ \overset{*}{\gamma}^0_{jk} = (y/n-1) \cdot J_{jk}$$

Connexions Finsler - projectives conjuguées. Ainsi qu'il résulte de la définition 4, une connexion Finsler projective est une connexion linéaire donc se peut appliquer, pour la rélation de la conjugaison, la définition 2.

Pour φ-conjugaison, d'après la rélation $\overset{*}{\gamma}^\alpha_{\beta\gamma} = c^\alpha_\lambda \overset{*}{\gamma}^\lambda_{\beta\mu} c^\mu_\gamma$ et les expressions des coefficients $\overset{*}{\gamma}$ données par les formules (5), s'obtient les rélations:

$$\bar{J}^i_{jk} = c^i_h J^h_{je} c^e_k + \frac{y}{n-1} c^i_0 J_{ie} c^e_k + y^{-1} c^i_j c_k, \tag{6}$$

$$^i_j J^j_{kh} c^h_0 + \frac{y}{n-1} c^i_0 J_{kh} c^h_0 + y^{-1} c^i_k c^0_0 = y^{-1} \delta^i_k, \tag{7}$$

$$c^0_i J^i_{kj} c^j_0 + \frac{y}{n-1} c^0_0 J_{jk} c^j_0 + y^{-1} c^0_k c^0_0 = 0 \tag{8}$$

$$\frac{y}{n-1} \bar{J}_{jk} = c^0_i J^i_{jh} c^h_k + \frac{y}{n-1} c^0_0 J_{jh} c^h_k + y^{-1} c^0_j c^0_k: \tag{9}$$

la matrice du tenseur de φ-conjugaison (et sa inverse) est de la forme

$$C = (c^\alpha_\beta) = \begin{bmatrix} (c^i_j) & & c^0_1 \\ & & c^0_n \\ 0 & \dots 0 & c^0_0 \end{bmatrix}, \ \bar{C} = \begin{bmatrix} (\bar{c}^i_j & & \bar{c}^0_1 \\ & & \bar{c}^0_n \\ 0 & \dots 0 & \bar{c}^0_0 \end{bmatrix}, \ C \cdot \bar{C} = E$$

Pour ψ - conjugaison, d'aprés la rélation $\overset{*}{\gamma}{}^\alpha_{\beta\gamma} = -c^{\alpha\lambda} \overset{*}{\gamma}{}^\mu_{\beta\lambda} c_{\mu\gamma}$ et les expressions des coefficients $\overset{*}{\gamma}$ données par les formules (5), s'obtient les rélations:

$$-\bar{J}^i_{jk} = c^{ih} J^e_{jh} \bar{c}_{ek} + y^{-1} c^{i0} \cdot \bar{c}_{jk} + \frac{1}{n-1} y c^{ih} J_{jh} \bar{c}_{0k}, \tag{10}$$

$$-y^{-1} \delta^i_k = c^{ih} J^e_{kh} \bar{c}_{e0} + y^{-1} c^{i0} \cdot \bar{c}_{0k} + \frac{y}{n-1} c^{ih} J_{kh} \bar{c}_{00}, \tag{11}$$

$$-\frac{y}{n-1} \bar{J}_{jk} = c^{0i} J^h_{ji} \bar{c}_{hk} + y^{-1} c^{00} \bar{c}_{jk} + \frac{y}{n-1} c^{0i} J_{ji} \bar{c}_{0k}, \tag{12}$$

$$c^{0i} J^j_{ki} \bar{c}_{j0} + \frac{y}{n-1} c^{0i} J_{ki} \bar{c}_{00} + y^{-1} c^{00} \cdot \bar{c}_{k0} = 0. \tag{13}$$

Le tenseur de ψ-conjugaison $C(c^{\alpha\beta})$ satisfait les conditions: $c^{i0} \cdot \bar{c}_{0u} = 2 \cdot \delta^i_u$.

***Remarque.* Les rélations de φ-conjugaison (6)-(9) et les rélations de ψ-conjugaison (10)-(13) sont caractérisées par les coefficients *J* de type Weyl.**

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COMMON FIXED POINTS OF WEAK COMPATIBLE MAPPINGS

H.K. PATHAK* and V. POPA**

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REZUMAT. - Puncte fixe comune pentru aplicaţii slab compatibile. În lucrare este dată o teoremă de punct fix pentru perechi de aplicaţii slab compatibile, care extinde unele rezultate ale lui Ding, Diviccaro-Sessa, Jungck şi Kang-Cho-Jungck. Apoi, acest rezultat este extins pentru şiruri de aplicaţii.

Abstract. In this paper, first, we present a common fixed point theorem for two pairs of weak compatible mappings, which extends the results of Ding, Diviccaro-Sessa, Jungck and Kang-Cho-Jungck for non-proper maps. Secondly we extend our result for sequence of mappings.

Introduction. In [6], the concept of compatible mappings was introduced as a generalization of commuting mappings ($AS=SA$). The utility of compatibility in the context of fixed point theory was demonstrated by extending a theorem of Park-Bae [12]. Jungck [8], extened a result of Singh-Singh [18] by employing compatible mappings in lieu of commuting mappings and by using four functions, as opposed to three. Most recently Kang-Cho-Jungck [9] extened the result of Ding [3], Diviccaro-Sessa [4] and Jungck [7] by employing compatibility in lieu of commuting and weakly commuting mappings respectively.

In this paper, we extend the result of Ding [2], Diviccaro-Sessa [4], Jungck [7] and

* *Department of Mathematics, Kalyan Mahavidyalaya, Bhilai Nagar (M.P.), India.*

** *University of Bacău, Department of Mathematics, 5500 Bacău, Romania*

Kang-Cho-Jungck [9] for two pairs of weak compatible mappings which are not necessarily proper.

Sessa [16] generalized commuting mappings by calling mappings A and S from a metric space (X,d) into itself a weakly commuting pairs, if $d(ASx, SAx) \leq d(Ax, Sx)$ for all x in X (see also [17]). Subsequently, Jungck [6] defined the following:

DEFINITION 1.1. Let A and S be mappings from a metric space (X,d) into itself. Then the mappings A and S are said to be compatible iff

$$\lim_n d(ASx_n, SAx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_n Ax_n = \lim_n Sx_n = \xi$ for some ξ in X.

However, since elementary functions which are not 'proper' are not compatible (see example 2.5[8]), it is desirable to introduce a less restrictive concept-a concept, we shall call weak compatibility of mappings. This new class of mappings includes in its domain all the compatible mappings as well as those mappings which are not necessarily 'proper'.

DEFINITION 1.2. [13,14]. Let A and S be mappings from a metric space (X,d) into itself. Then the pair $\{A,S\}$ is said to be S-weak compatible, iff the followings limits exist and satisfy

(i) $$\lim_n d(SAx_n, ASx_n) \leq \lim_n d(ASx_n, Ax_n)$$

(ii) $$\lim_n d(SAx_n, Sx_n) \leq \lim_n d(ASx_n, Ax_n),$$

whenever there exists a sequence $\{x_n\} \subseteq x$ such that $\lim_n Ax_n = \lim_n Sx_n = \xi$ for some ξ in X.

Clearly two compatible maps A and S are S-weak compatible (as well as A-weak compatible), but the converse is not true. Observe example 2.8 of this note. It can also be seen that two weakly commuting mappings are compatible, but the converse is false. Examples supporting this fact and other related results can be found in [6,7,8,14].

In order to add validity and weight to the argument that our concept of weak compatible maps is viable, meaningful and potentially productive generalization of compatible maps, a series of propositions has been given. The following proposition help us to recognise the weak compatible pair of maps.

PROPOSITION 1.3 *Every compatible pair of maps* $\{A,S\}$ *is A-weak (S-weak) compatible.*

Proof. In case, the pair of maps $\{A,S\}$ is compatible we have that $\lim_n d(ASx_n, SAx_n) = 0$, whenever there exists a sequence $\{x_n\}$ in X such that $\lim_n Ax_n = \lim_n Sx_n = t$ for some t in X. Then $\lim_n d(ASx_n, SAx_n) \le \lim_n d(SAx_n, Sx_n)$ is obvious. Now

$$\lim_n d(ASx_n, Ax_n) \le \lim_n d(ASx_n, SA_n) + \lim_n d(SAx_n, Sx_n) + \lim_n d(Sx_n, Ax_n)$$

i.e., $\lim_n d(ASx_n, Ax_n) \le \lim_n d(SAx_n, Sx_n)$, as desired. Similar conclusion can be drawn in favour of S-weak compatibility of the pair $\{A,S\}$.

The converse of above assertion is not true. However, the following propositions show that the definitions 1.1 and 1.2 are equivalent under some conditions.

PROPOSITION 1.4. *Let A and S be self-maps of a metric space* (X,d). *Let A,S are A-weak (S-weak) compatible and let* $\lim_n d(SAx_n, Sx_n) = 0$ $(\lim_n d(ASx_n, Ax_n) = 0)$, *whenever there exists a sequence* $\{x_n\}$ *in X such that* $\lim_n Ax_n = \lim_n Sx_n = t$ *for some t in X. Then A,S are compatible.*

Proof. Since A, S are A-weak compatible and $\lim_n d(SAx_n, Sx_n) = 0$, it follows that $\lim_n d(ASx_n, SAx_n) = 0$. Therefore, the maps A,S are compatible and $\lim_n d(ASx_n, Ax_n) = 0$. This completes the proof.

As a direct consequence of Propositions 1.3 and 1.4, we have the following:

PROPOSITION 1.5. *Let A, S be selfmaps of a metric space (X,d). Let $\lim_n d(SAx_n, Sx_n) = 0$ ($\lim_n d(ASx_n, Ax_n) = 0$) whenever there exists a sequence $\{x_n\}$ in X such that $\lim_n Ax_n = \lim_n Sx_n = t$ for some t in X. Then A,S are compatible iff they are A-weak (S-weak) compatible.*

By recalling that a mapping $f{:}X \to Y$ between topological spaces is proper iff $f^{-1}(C)$ is compact in X when C is compact in X, we can say:

PROPOSITION 1.6. *Suppose that A and S are continuous selfmaps of a metric space X and that S is proper. If $Ax = Sx$ implies $ASx = SAx$, then the pair (A,S) is A-weak (S-weak) compatible.*

Proof. By sufficient condition of Theorem 2.2 of [8], the pair of maps (A,S) is compatible. Hence by Proposition 1.3, the pair (A,S) is A-weak (S-weak) compatible.

PROPOSITION 1.7. *Let A and S be continuous selfmaps of a metric space X and that S be proper. Let $Ax = Sx$ implies $ASx = SAx$. Then A,S are compatible iff they are A-weak (S-weak) compatible.*

PROPOSITION 1.8. *Suppose that A and S are continuous selfmaps of a metric space and that S is proper. If $Ax = Sx$ implies $x = Sx$, then the pair (A,S) is A-weak (S-weak) compatible.*

Proof. By corollary 2.6 of [8], the pair of maps (A,S) is compatible. The pair (A,S) is therefore A-weak (S-weak) compatible by Proposition 1.3.

PROPOSITION 1.9. *Suppose A and S are continuous selfmaps of a metric space, the pair (A,S) is A-weak (S-weak) compatible and S is proper. If $Ax = Sx$ implies $Sx = SAx$ ($Ax = ASx$), then the pair (A,S) is compatible.*

Proof. If $Ax = Sx$ implies $Sx = SAx$ ($Ax = ASx$), then the continuity of A and S and

A-weak (S-weak) compatibility of the pair $\{A,S\}$ says that $d(ASx, SAx) \leq 0$ which implies $ASx = SAx$ and therefore the pair $\{A,S\}$ is compatible by sufficient condition of Theorem 2.2 of [8].

PROPOSITION 1.10. *Suppose A and S are continuous selfmaps of a metric space X, the pair $\{A,S\}$ is A-weak (S-weak) compatible and $S(A)$ is proper. If $Ax = Sx$ implies $x = Ax$ ($x = Sx$) and $A(S)$ is injective map, then the pair $\{A,S\}$ is compatible.*

Proof. If $Ax = Sx$ implies $x = Ax$, then the continuity of A and S and the A-weak (S-weak) compatibility of the pair $\{A,S\}$ says that $d(ASx, Sx) \leq 0$ and so $ASx = Sx = Ax$. The injectiveness of A implies $x = Sx$. Then calling corollary 2.6 of [8], the pair $\{A,S\}$ is compatible.

As a direct consequence of Propositions 1.3, 1.9 and 1.10, we also have:

PROPOSITION 1.11. *Let A and S be continuous self-maps of a metric space X and that S be proper. Then A and S are compatible iff they are A-weak (S-weak) compatible, whenever either of the following conditions hold:*

(a_1) *$Ax = Sx$ implies $Sx = SAx$ ($Ax = ASx$);*

(a_2) *$Ax = Sx$ implies $x = Ax$ and A is injective.*

An analogous proposition holds if A is proper instead of S is proper.

Next, we give some properties of weak compatible maps in metric space for our main theorem.

PROPOSITION 1.12. *Let A and S be continuous maps from a metric space (X,d) into itself. Let S be proper map. Then the pair $\{A,S\}$ is A-weak compatible iff $Ax = Sx$ implies $d(ASx, SAx) \leq d(SAx, Sx)$ and $d(ASx, Ax) \leq d(SAx, Sx)$.*

Proof. Suppose that $\{x_n\}$ is a sequence in X defined by $\{x_n\} = x, n = 1,2,\ldots,$ and $Ax =$

Sx. Then $\lim_n Ax_n = \lim_n Sx_n = Sx_n$, $\lim_n ASx_n = AS$ and $\lim_n SAx_n = Sx$. Since the pair $\{A,S\}$ is A-weak compatible, we have

$$d(ASx, SAx) \le d(SAx, Sx) \text{ and } d(ASx, Ax) \le d(SAx, Sx).$$

Hence the necessity of the condition follows.

To prove sufficiency let there be a sequence $\{x_n\}$ in X and suppose that

(b_1) $\lim_n Ax_n = \lim_n Sx_n = t$ for some t in X.

Then $M = \{Sx_n : n \in N\} \cup \{t\}$ is compact, so that $S^{-1}(M)$ is compact since S is proper. Consequently, $\{x_n\}$ has a subsequence $\{x_{k_n}\}$ which converges to an element $\xi \in X$. Since A and S are continuous, $Ax_{k_n} \to A\xi$ and $Sx_{k_n} \to S\xi$. Then (b_1) implies

(b_2) $Ax_n, Sx_n \to t = A\xi = S\xi$,

and $d(AS\xi, SA\xi) \le d(SA\xi, S\xi)$, $d(AS\xi, A\xi) \le d(SA\xi, S\xi)$ by hypothesis.

By continuity of A,S and (b_2) we have that $ASx_n \to AS\xi$ and $SAx_n \to SA\xi$. Therefore, $\lim_n d(ASx_n, SAx_n) \le \lim_n d(SAx_n, Sx_n)$, $\lim_n d(ASx_n, Ax_n) \le \lim_n d(SAx_n, Sx_n)$ as desired.

Remark 1.13. Proposition 1.12 shows that weak compatible maps do not necessarily exhibits the commutativity of maps at their considence points in contrast to weak commuting and compatible maps.

2. Common fixed points of four mappings: Throughout this section, suppose that the function $\phi: [0,\infty)^5 \to [0,\infty)$ satisfies the following conditions:

(i') ϕ is non-decreasing and upper semi-continuous in each co-ordinate variable.

(ii') For each $t > 0$, $\psi(t) = \max\{\phi(t,0,t,t,t), \phi(0,t,t,t,t), \phi(t,t,t,0,2t), \phi(t,t,t,2t,0)\} < t$.

Our main result is the following theorem:

THEOREM 2.1. *Let A,B,S and T be mappings from a complete metric space (X,d) into* *itself. Suppose that one of A,B,S and T is continuous and satisfying the conditions:*

(2.1) $A(X) \subseteq T(X)$ *and* $B(X) \subseteq S(X)$;

(2.2) *The pairs* $\{A,S\}$ *and* $\{B,T\}$ *are S-weak and T-weak compatible respectively;*

(2.3) $d(Ax, By) \le \phi(d(Ax, Sx), d(By, Ty), d(Sx, Ty), d(Ax, Ty), d(By, Sx))$ *for all x,y in X,*

where ϕ *satisfies* (i') *and* (ii'), *then A,B,S and T have a unique common fixed point in X.*

Let x_0 be an arbitrary point in X. Since (2.1) holds we can choose x_1 in X such that $y_1 = Tx_1 = Ax_0$ and, for this point x_1, there exists a point x_2 in X such that $y_2 = Sx_2 = Bx_1$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$y_{2n+1} = Tx_{2n+1} = Ax_{2n}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for $n = 0,1,2,\ldots$. Let $d_n = d(y_n, y_{n+1})$.

The following lemmas will shorten the proof of our theorem.

LEMMA 2.2([11]). *Suppose* ϕ: $[0,\infty)^5 \to [0,\infty)$ *is non-decreasing and upper semi-continuous from the right. If* $\psi(t) < t$ *for every* $t > 0$, *then* $\lim_n \psi^n(t) = 0$ *where* $\psi^n(t)$ *denotes the composition of* $\psi(t)$ *with itself n-times.*

Using lemma 2.2 and condition (2.3) Kang et al. [8] have established the following lemma:

LEMMA 2.3([9]). $\lim_n d_n = 0$ *and* $\{y_n\}$ *is a Cauchy sequence.*

Proof of theorem 2.1. Since $\{y_n\}$ is a Cauchy sequence in the complete metric space X, it converges to some point ξ in X. Consequently, the subsequences $\{Ax_{n2n}\}$, $\{Sx_{2n}\}$, $\{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ converge to . Suppose that S is continuous. Since, the pair $\{A,S\}$ is S-weak compatible

$$d(ASx_{2n}, SSx_{2n}) \le d(ASx_{2n}, SAx_{2n}) + d(SAX_{2n}, SSx_{2n})$$

i.e.,

(2.4) $d(ASx_{2n}, SSx_{2n}) \le d(ASx_{2n}, Ax_{2n}) + \varepsilon_n$

where $a_n \to 0$ as $n \to \infty$. Again, i.e.,

(2.5) $d(ASx_{2n}, Tx_{2n-1}) \le d(ASx_{2n}, Ax_{2n}) + \alpha_n$ where $\alpha_n \to 0$ as $n \to \infty$.

By (2.3), (2.4) and (2.5), we have

$$d(ASx_{2n}, Ax_{2n}) \le d(ASx_{2n}, Bx_{2n-1}) + d(Bx_{2n-1}, Ax_{2n})$$

$$\le \phi\,(d(ASx_{2n}, SSx_{2n}), d(Bx_{2n-1}, Tx_{2n-1}), d(SSx_{2n}, Tx_{2n-1}),$$

$$d(ASx_{2n}, Tx_{2n-1}), d(Bx_{2n-1}, SSx_{2n}) + d(Bx_{2n-1}, Ax_{2n})$$

$$\le \phi\,(d(ASx_{2n}, Ax_{2n}) + a_n, d(Bx_{2n-1}, Tx_{2n-1}), d(SSx_{2n}, Tx_{2n-1}),$$

$$d(ASx_{2n}, Ax_{2n}) + \alpha_n, d(Bx_{2n-1}, SSx_{2n})) + d(Bx_{2n-1}, Ax_{2n}).$$

Letting $n \to \infty$ and using S-weak compatibility of the pair $\{A,S\}$, we obtain

$$d(S\xi, \xi) = \lim_n d(SAx_{2n}, Sx_{2n}) \le \lim_n d(ASx_{2n}, Ax_{2n})$$

$$\le \phi\,(\lim_n d(ASx_{2n}, Ax_{2n}), 0, d(S\xi, \xi), \lim_n d(ASx_{2n}, Ax_{2n}), d(\xi, S\xi),$$

$$< \lim_n d(ASx_{2n}, Ax_{2n}),$$

since ϕ is upper semicontinuous, and so we arrive at contradiction. Therefore, $\lim_n d(SAx_{2n}, ASx_{2n}) \le \lim_n d(ASx_{2n}, Ax_{2n}) = 0$. i.e,

$$S\xi = \lim_n SAx_{2n} = \lim_n ASx_{2n} = \lim_n Ax_{2n} = \xi.$$

By (2.3), we also obtain

$$d(A\xi, Bx_{2n-1}) \le \phi(d(A\xi, S\xi), d(Bx_{2n-1}, Tx_{2n-1}), d(S\xi, Tx_{2n-1}), d(A\xi, Tx_{2n-1}), d(Sx_{2n-1}, S\xi))$$

Letting $n \to \infty$ and using the fact that $S\xi = \xi$, we have

$$d(A\xi, \xi) \le \phi(d(A\xi, \xi), 0, 0, d(A\xi, \xi), 0),$$

So that $A\xi = \xi$. Since $A(x) \subseteq T(X)$, $\xi \in T(X)$, there exists a point η in X such that $\xi = A\xi = T\xi$. Now

$$d(\xi, B) = d(A\xi, B\xi) \le \phi\,(0, d(B\xi, T\xi), d(S\xi, T\xi), d(A\xi, T\xi), d(B\xi, \xi))$$

which implies that $\xi = B\eta$. Since, the pair $\{B,T\}$ is T-weak compatible, and $B\eta = T\eta = \xi$,

$d(TB\eta, BT\eta) = d(BT\eta, B\eta)$, $d(TB\eta, T\eta) \leq d(BT\eta, B\eta)$, and hence $d(T\xi, B\xi) \leq d(B\xi, \xi)$, $d(T\xi, \xi) \leq d(B\xi, \xi)$. Moreover, by (2.3) we have

$$d(\xi, B\xi) = d(A\xi, B\xi) \leq \phi(d(A\xi, S\xi), d(B\xi, T\xi), d(S\xi, T\xi), d(A\xi, T\xi), d(B\xi, S\xi))$$
$$= \phi(0, d(B\xi, \xi), d(B\xi, \xi), d(B\xi, \xi), d(B\xi, \xi)),$$

so that $\xi = B\xi$. Therefore, ξ is a common fixed point of A,B,S and T. Similarly, it can be prove that ξ is a common fixed point of A,B,S and T, if T is continuous instead of S.

Now, suppose that A is continuous. Then we have that $AAx_{2n}, ASx_n \to A\xi$. By (2.3) and S-weak compatibility of the pair $\{A,S\}$, we have

$$d(AAx_{2n}, Bx_{2n-1}) \leq \phi(d(AAx_{2n}, SAx_{2n}), d(Bx_{2n-1}, Tx_{2n-1}), d(SAx_{2n}, Tx_{2n-1}),$$
$$d(AAx_{2n}, Tx_{2n-1}), d(Bx_{2n-1}, SAx_{2n}))$$
$$\leq \phi(d(AAx_{2n}, ASx_{2n}) + d(ASx_{2n}, SAx_{2n}), d(Bx_{2n-1}, Tx_{2n-1}),$$
$$d(SAx_{2n}, Sx_{2n}) + d(Sx_{2n}, Tx_{2n-1}), d(AAx_{2n}, Tx_{2n-1}),$$
$$d(Bx_{2n-1}, Sx_{2n}) + d(Sx_{2n}, SAx_{2n}))$$
$$\leq \phi(d(AAx_{2n}, ASx_{2n}) + d(ASx_{2n}, Ax_{2n}), d(Bx_{2n-1}, Tx_{2n-1}),$$
$$d(ASx_{2n}, Ax_{2n}) + d(Sx_{2n-1}), d(AAx_{2n}, Tx_{2n-1}),$$
$$d(Bx_{2n-1}, Sx_{2n}) + d(ASx_{2n}, Ax_{2n})).$$

Letting $n \to \infty$, we obtain

$$d(A\xi, \xi) \leq \phi(d(A\xi, \xi), 0, d(A\xi, \xi), d(A\xi, \xi), d(A\xi, \xi)),$$

so that $\xi = A\xi$. Since $A(X) \subseteq T(X)$, there exists a point ξ' in X such that $\xi = A\xi = T\xi'$. Again using (2.3) and S-weak compatibility of the pair $\{A,S\}$, we have

$$d(AAx_{2n}, B\xi') \leq \phi(d(AAx_{2n}, SAx_{2n}), d(B\xi', T\xi'), d(SAx_{2n}, T\xi'),$$
$$d(AAx_{2n}, T\xi'), d(B\xi', SAx_{2n}))$$
$$\leq \phi(d(AAx_{2n}, ASx_{2n}) + d(ASx_{2n}, Ax_{2n}), d(B\xi', T\xi'),$$

$$d(ASx_{2n}, Ax_{2n}) + d(Ax_{2n}, T\xi'), d(AAx_{2n}, T\xi'),$$

$$d(B\xi', Sx_{2n}) + d(ASx_{2n}, Ax_{2n})).$$

Letting $n \to \infty$, we have

$$d(\xi, B\xi') \leq \phi(0, d(B\xi', T\xi'), d(\xi, T\xi'), d(\xi, T\xi'), d(B\xi', \xi))$$

which implies that $\xi = B\xi'$. Since B and T are T-weak compatible and $T\xi' = B\xi' = \xi$,

$d(TB\xi', BT\xi') \leq d(BT\xi', d(TB\xi', T\xi') \leq d(BT\xi', B\xi')$, and hence

$d(T\xi, B\xi) \leq d(B\xi, \xi)$, $d(T\xi, \xi) \leq d(B\xi, \xi)$.

Moreover by (2.3), we have

$d(Ax_{2n}, B\xi) \leq \phi\, d(Ax_{2n}, Sx_{2n}), d(B\xi, T\xi), d(Sx_{2n}, T\xi), d(Ax_{2n}, T\xi), d(B\xi, Sx_{2n}))$.

Letting $n \to \infty$, we have

$$d(\xi, B\xi) \leq \phi(0, d(B\xi, T\xi), d(\xi, T\xi), d(\xi, T\xi), d(B\xi, \xi))$$

$$\leq \phi(0, d(B\xi, \xi), d(B\xi, \xi), d(B\xi, \xi), d(B\xi, \xi)),$$

so that $\xi = B\xi$ and thus $T\xi = B\xi = \xi$. Since $B(X) \subseteq S(X)$, there exists a point η in X, such that $\xi = B\xi = S\eta$. Again using (2.3), we have

$$d(A\eta, \xi) = d(A\eta, B\xi) \leq \phi(d(A\eta, \xi), 0, 0, d(A\eta, \xi), 0),$$

so that $A\eta = \xi$. Since A and S are S-weak compatible and $A\eta = S\eta = \xi$, $d(SA\eta, AS\eta) \leq d(AS\eta, A\eta), d(SA\eta, S\eta) \leq d(AS\eta, A\eta)$, and hence

$d(S\xi, A\xi) \leq d(A\xi, \xi)$, $d(S\xi, \xi) \leq d(A\xi, \xi)$.

Moreover by (2.3), we have

$$d(A\xi, \xi) = d(A\xi, B\xi) \leq \phi(d(A\xi, S\xi), 0, d(S\xi, \xi), d(A\xi, \xi), d(\xi, S\xi))$$

$$\leq \phi(d(A\xi, \xi), 0, d(A\xi, \xi), d(A\xi, \xi), d(A\xi, \xi)).$$

Case I. When $0 \leq x \leq 1$, $|Ax - Sx| = \dfrac{x^2}{x+1} \to 0$ iff $x \to 0$ and $SA(0) = S(0)\ AS(0) = A(0) = 0$. Again, $|Bx - Tx| = \dfrac{x}{2(x+2)} \to 0$ iff $x \to 0$ and $TB(0) = T(0) = BT(0) = B(0) = 0$

Case II. When $1 \le x \le 2$, neither $|Ax - Sx| = \frac{1}{x+1}$ nor $|Bx - Tx| = \frac{x}{2(x+2)}$ tends to 0.

Case III. When $x \ge 2$, $|Ax - Sx|$ as well as $|Bx - Tx|$ tend to 0 iff $x \to \infty$. Then as $x \to \infty$, $Ax = Sx$ which implies $d(SAx, ASx) \le d(ASx, Ax)$, $d(SAx, Sx) \le d(ASx, Ax)$. As $x \to \infty$, $Bx = Tx$ which implies $d(TBx, BTx) \le d(BTx, Bx)$, $d(TBx, Tx) \le d(BTx, Bx)$. To see these, merely let $x_n = x$ for all n. Hence the pair $\{A,S\}$ and $\{B,T\}$ are S-weak and T-weak compatible respectively.

But if $x_n = n$ for $n \in \mathbf{N}$, then $\lim_n Sx_n = \lim_n Ax_n = 1$, $\lim_n Tx_n = \lim_n Bx_n = 1$ whereas $\lim_n |SAx_n - ASx_n| = 1/2 \ne 0$, $\lim_n |TBx_n - BTx_n| = 2/3 \ne 0$. Therefore, the pairs $\{A,S\}$ and $\{B,T\}$ are not compatible. However, if $x_n = n$ for $n \in \mathbf{N}$, when

$\lim_n |SAx_n - ASx_n| = \lim_n |ASx_n - Ax_n| = 1/2$, $0 = \lim_n |SAx_n - Sx_n| < \lim_n |ASx_n - Ax_n| = 1/2$ and $\lim_n |TBx_n - BTx_n| = \lim_n |BTx_n - Bx_n| = 2/3$, $0 = \lim_n |TBx_n - Ax_n| < \lim_n |BTx_n - Bx_n| = 2/3$.

Here, none of the four maps is proper. Finally, we see that the pairs $\{A,S\}$ and $\{B,T\}$ are not even weakly commuting, for

$$|S(4) - A(4)| = \frac{1}{5} < \frac{3}{10} = |SA(4) - AS(4)| \text{ and}$$
$$|T(100) - B(100)| = \frac{1}{51} < \frac{8}{153} = |TB(100) - BT(100)|.$$

Further, let $\phi(t_1, t_2, t_3, t_4, t_5) = \max\{\frac{t_3}{1+t_3}, |t_5 - t_4|, t_4 - t_2\}$. now we shall discuss different possibilities in the following manner:

Case 1. If $0 \le x \le 1$, $0 \le y \le 2$ then $d(Ax, By) = \frac{|2x - y|}{2+2x+y+xy} \le \frac{|2x - y|}{2+|2x + y|}$

$$= \frac{\left|x - \frac{y}{2}\right|}{1 + \left|x - \frac{y}{2}\right|} = \frac{d(Sx, Ty)}{1 + d(Sx, Ty)} = \psi\left(\left|x - \frac{y}{2}\right|\right) = \psi(d(Sx, Ty))$$

Case 2. If $x \le 1$, $0 \le y \le 2$, then $d(Ax, By) = \frac{2x - y}{(x+1)(y+2)} = \left(1 - \frac{y}{y-2}\right) - \left(1 - \frac{x}{x-1}\right) =$

$$= d(By, Sx) - d(Ax, Ty) \leq \psi(d(By, Sx)).$$

Case 3. If $0 \leq x \leq 1$, $y \leq 2$, then $d(Ax, By) = \dfrac{y - 2x}{(x+1)(y+2)} = \left(1 - \dfrac{x}{x+1}\right) - \left(1 - \dfrac{y}{y+2}\right) = d(Ax, Ty) - d(By, Ty) \leq \psi(d(Ax, Ty))$.

Case 4. If $x \geq 1$, $y \geq 2$, then $d(Ax, By) = \left|\dfrac{x}{x+1} - \dfrac{y}{y+2}\right| = \left|\left(1 - \dfrac{y}{y+2}\right) - \left(1 - \dfrac{x}{x+1}\right)\right| = |d(By, Sx) - d(Ax, Ty)| \leq \psi(\max\{d(By, Sx), d(Ax, Ty)\})$.

Therefore, all the assumptions of Theorem 2.1 are satisfied and 0 is the unique common fixed point of A, B, S and T.

3. Common fixed point for sequence of mappings. In this section, we extend Theorem 2.1 for sequence of mappings. In the sequel, we need the following definition.

DEFINITION 3.1. For $f: (X,d) \to (X,d)$ denote $F_f = \{x \in X : x = f(x)\}$.

The main result of this section is prefaced by the following:

THEOREM 3.2. *Let A,B,S and T be mappings from a complete metric space (X,d) into itself. If the inequality* (2.3) *holds for all x and y in X, then* $(F_S \cap F_T) \cap F_A = (F_S \cap F_T) \cap F_B$.

Proof. Let $x \in (F_S \cap F_T) \cap F_A$. Then

$$d(x, Bx) = d(Ax, Bx) \leq \phi(d(Ax, Sx), d(Bx, Tx), d(Sx, Tx), d(Ax, Tx, d(Bx, Sx))$$

$$= \phi(0, d(Bx, x), 0, 0, d(Bx, x))$$

$$\leq \phi(0, d(Bx, x), d(Bx, x), d(Bx, x), d(Bx, x))$$

$$< \psi(d(Bx, x)).$$

By Lemma 2 of [2] we have $Bx = x$. Thus $(F_S \cap F_T) \cap F_A \subset (F_S \cap F_T) \cap F_B$. Similarly, we have $(F_S \cap F_T) \cap F_B \subset (F_S \cap F_T) \cap F_A$. Finally, we have the following:

THEOREM 3.3. *Let S,T and $\{f_i\}_{i \in N}$ be mappings from a complete metric space (X,d) into itself such that:*

(3.1) *one of S,T or f_i (or some f_i) is continuous,*

(3.2) $f_i(X) \subset S(X) \cap T(X)$ *for each* $i \in \mathbf{N}$,

(3.3) *the pair* (f_i,S) *and* (f_i,T) *are S-weak and T-weak compatible respectively for each* $i \in \mathbf{N}$,

(3.4) $d(f_i x, f_{i+1} y) \leq \phi(d(f_i x, Sx), d(f_{i+1} y, Ty), d(Sx, Ty), d(f_i x, Ty), d(f_{i+1} y, Sx))$

$\forall\ i \in \mathbf{N}$ *for each x and y in X, where* ϕ *satisfy* (i') *and* (i"), *then S,T and* $\{f_i\}_{i\in\mathbf{N}}$ *have a unique common fixed point in X.*

Proof. By Theorem 2.1, S, T, f_1 and f_2 have a unique common fixed point z. We prove that z is the unique common fixed point for S,T and f_1 and for S,T and f_2.

Let w be a second fixed point of f_1 and S. Using the inequality (3.4) we have $d(w,z) = d(f_1 w, f_2 z) \leq \phi(d(f_1 w, Sw), d(f_2 z, Tz), d(Sw, Tz), d(f_1 w, Tz), d(f_2 z, Sw)) = \phi(0, 0, d(wz), d(w,z), d(w,z)) \leq \phi(0, d(w,z), d(w,z), d(w,z), d(w,z)) < \psi(d(w,z))$ and by Lemma 2 [2] it follows that $w = z$.

Thus z is the unique common fixed point of f_1, S, T. By Theorem 3.2 $(F_S \cap F_T) \cap F_{f_1} = (F_S \cap F_T) \cap F_{f_2}$ and thus z is the unique common fixed point for S,T and f_2. By (3.4) and Theorem 3.2 it follows that z is the unique common fixed point for f_i, S and T, $\forall\ i \in \mathbf{N}$.

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THE INFLUENCE OF ATMOSPHERIC DENSITY DIURNAL VARIATION ON ARTIFICIAL SATELLITE NODAL PERIOD

V. MIOC*, E. RADU*, L. MIRCEA* and A. CRISTEA*

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REZUMAT. - Influenţa variaţiei diurne a densităţii atmosferice asupra perioadei nodale a sateliţilor artificiali. Utilizându-se modelul termosferic TD pentru descrierea distribuţiei densităţii atmosferei terestre, se studiază efectul separat al variaţiei diurne a densităţii asupra perioadei nodale a sateliţilor artificiali. Se iau în considerare diferite cazuri particulare (orbite iniţial circulare, neglijarea rotaţiei atmosferei, orbite iniţial polare).

1. Introduction. Lots of papers deal with the analytic study of the perturbed orbital motion of artificial satellites in the terrestrial atmosphere. Almost all these papers resort to a much too simple law to describe the atmospheric density distribution. In this way important effects are neglected. A profitable way to avoid such shortcomings consists of expressing the distribution of the density by means of the TD thermospheric model with its variants TD 86 [10, 12] and TD 88 [11, 13]. This model offers a much more complete image of the density dependence on various factors.

The TD model expresses the density as

$$\rho = X_0 \sum_{n=1}^{7} h_n g_n , \tag{1}$$

where the factor X_0 features the general dependence of the density on solar and geomagnetic activity, the term $h_1 g_1$ expresses the altitude-dependence of the mean density, while the other terms allow for various effects of density distribution (see below).

* *Astronomical Observatory, 3400 Cluj-Napoca, Romania*

Denoting

$$\rho_n = X_0 h_n g_n, \quad n = \overline{1,7}, \tag{2}$$

(1) becomes

$$\rho = \rho_1 + \sum_{n=2}^{7} \rho_n . \tag{3}$$

In other words, we consider an atmosphere generally influenced by solar and geomagnetic activity, featured by a height-dependent mean density (ρ_1) corrected for: individual dependence on the mean solar radio flux (ρ_2), North-South asymmetry (ρ_3), annual (ρ_4) and semiannual (ρ_5) variation, diurnal (ρ_6) and semidiurnal (ρ_7) variation. To explicit these corrections, see [1-3, 10-13].

The influence of these effects in artificial satellite motion was studied considering them either together (globally [2-4] or partially [9, 14, 15]) or separately [5-8]. Out of the corrections ρ_n, $n = \overline{2,7}$, ρ_3, ρ_6 and ρ_7 are changing during a revolution of the satellite; the perturbations they cause in satellite motion were called rapid effects. The other corrections can be considered constant over one revolution of the satellite (varying very slowly with time); they cause slow effects in satellite motion.

The rapid effects were studied together in [9]. Out of them, the semidiurnal variation influence was considered in both orbital elements and nodal period [7], while the influence of the North-South asymmetry and that of the diurnal effect were considered only in orbital elements ([5] and [8]), respectively).

In this paper we shall study the effect of the diurnal variation in atmospheric density distribution on the nodal period of an artificial satellite. More exactly, we shall estimate analytically the difference (due to this factor) between the real (perturbed) nodal period and the corresponding Keplerian one, under the following hypotheses:

(i) only the first order difference will be determined;

(ii) only satellites moving in quasi-circular orbits will be considered.

2. Equations of motion. Let us describe the perturbed motion of an artificial satellite by means of the Newton-Euler equations written as (e.g.[2])

$$\begin{aligned} dp/du &= 2(Z/\mu)\, r^3 T, \\ dq/du &= (Z/\mu)\,(r^3 kBCW/(pD) + r^2 T(r(q+A)/p + A) + r^2 BS), \\ dk/du &= (Z/\mu)\,(-r^3 qBCW/(pD) + r^2 T(r(k+B)/p + B) - r^2 AS), \\ d\Omega/du &= (Z/\mu) r^3 BW/(pD), \\ di/du &= (Z/\mu)\, r^3 AW/p, \\ dt/du &= Zr^2(\mu p)^{-1/2}, \end{aligned} \tag{4}$$

where $Z = (1 - r^2 C\dot{\Omega}/(\mu p)^{1/2})^{-1}$, μ = Earth's gravitational parameter, r = geocentric radius vector of the satellite, p = semilatus rectum, $q = e\cos\omega$, $k = e\sin\omega$ (e = eccentricity, ω = argument of perigee), Ω = longitude of ascending node, i = inclination ($C = \cos i$, $D = \sin i$), u = argument of latitude ($A = \cos u$, $B = \sin u$), S, T, W = radial, transverse, and binormal components of the perturbing acceleration, respectively.

Consider the orbital parameters

$$z = z(u) \in Y = \{p, q, k, \Omega, i\}, \tag{5}$$

and write them as $z = z_0 + \Delta z$, with $z_0 = z(u_0)$, u_0 being the initial position. The change of z in the interval $[u_0, u]$ is given by

$$\Delta z = \int_{u_0}^{u} (dz/du)\, du, \quad z \in Y, \tag{6}$$

the integrands being provided by equations (4). According to hypothesis (i), the integrals (6) will be estimated by successive approximations, with $Z \sim 1$, stopping the process after the first order approximation. In such a way, the elements $z \in Y$ will be considered constant (and equal to z_0) in the right-hand side of equations (4), and these ones will be separately integrated.

In the following sections, for sake of brevity, we shall no longer use the subscript "0" to mark the initial values of $z \in Y$ and of functions of them. Every other unspecified index "O" constitutes a simple notation and does not refer to the initial u. In fact, every quantity which does not depend on u (explicitly or through A and B) will be considered constant over one revolution of the satellite and equal to its value for $u = u_0$.

Observing hypothesis (ii), hence using expansions to first order in q and k (as throughout of this paper), the orbit equation in polar coordinates leads to

$$r^n = p^n(1 - nAq - nBk). \tag{7}$$

In the same approximation, and considering as unique perturbing factor the diurnal variation in the atmospheric density distribution, the components of the perturbing acceleration will be written as (cf.[2, 3, 6])

$$\begin{aligned} S &= -(\mu/p)\rho_6\delta(Bq - Ak), \\ T &= -(\mu/p)\rho_6\delta(1 + 2Aq + 2Bk) + (\mu p)^{1/2}\rho_6\delta Cw, \\ W &= -(\mu p)^{1/2}\rho_6\delta ADw, \end{aligned} \tag{8}$$

where δ = satellite drag parameter and w (constant) = angular velocity of rotation of the atmosphere with respect to the Earth's axis.

By virtue of the considerations made above, we shall separately have in view the first five equations (4). Because we take $Z \approx 1$ for the analytic calculation of the integrals (6), we shall write these equations omitting in advance the factor Z. So, with (7) and (8), and introducing the abbreviations

$$\begin{aligned} b &= p^{5/2}\mu^{-1/2}\delta, \\ x &= Cw, \\ y &= -p^{-3/2}\mu^{1/2}, \end{aligned} \tag{9}$$

the mentioned equations acquire the form

$$\begin{aligned} dp/du &= pb(2(x+y) - 2(3x+y)Aq - 2(3x+y)Bk)\rho_6, \\ dq/du &= b(2(x+y)A + ((x+2y) - (5x+2y)A^2)q - 2(3x+y)ABk)\rho_6, \\ dk/du &= b(2(x+y)B - 2(2x+y)ABq + (-4x + (5x+2y)A^2)k)\rho_6, \\ d\Omega/du &= b(x/C)(-AB + 3A^2Bq + 3AB^2k)\rho_6, \\ di/du &= b(Dx/C)(-A^2 + 3A^3q + 3A^2Bk)\rho_6. \end{aligned} \tag{10}$$

3. Expression of the density. In order to write equations (10) in a suitable form to perform the integrals (6), we must express ρ_6 as function only of u (through A and B).

The TD model [10-13] gives g_6 under the form

$$g_6 = (a_7 f_m + 1)\sin(t - p_6)\cos\varphi, \tag{11}$$

where φ = latitude, t = local time (in hours), the constant a_7 and the phase p_6 are given in the model, while f_m has the expression

$$f_m = (F_b - 60)/160, \tag{12}$$

with F_b = radio solar flux on 10.7 cm wavelength averaged for three solar rotations.

Taking into account the fact that

$$\begin{aligned} \sin(\alpha - \Omega) &= CB/\cos\varphi, \\ \cos(\alpha - \Omega) &= A/\cos\varphi, \end{aligned} \tag{13}$$

where α = right ascension of the satellite, and introducing the notation

$$L = \Omega - \alpha_\odot - p_6 + \pi, \tag{14}$$

where $\alpha_\odot$ = Sun's right ascension, one obtains for g_6 (see also [1, 2])

$$g_6 = (a_7 f_m + 1)(CB\cos L + A\sin L), \tag{15}$$

or, keeping the notations used in [3]

$$g_6 = G_4 A + G_5 B, \tag{16}$$

where we abbreviated

$$\begin{aligned} G_4 &= (a_7 f_m + 1)\sin L, \\ G_5 &= (a_7 f_m + 1) C\cos L. \end{aligned} \tag{17}$$

Now we have to express h_6 in terms of u only. The TD model gives

$$h_6 = \sum_{j=0}^{3} k'_{6j} \exp((120 - h) B_j), \tag{18}$$

where $k'_{6j} = 10^8 k_{6j}$ (k_{6j} = numerical constants tabulated in the model) are at most of order unity and were introduced in order to assign to X_6 the part of the small parameter σ (see Section 4 below); h = height (in km); $B_0 = 0$, $B_j = (40\,j)^{-1}$ in the TD 86 model, $B_j = (29\,j)^{-1}$ in the TD 88 model ($j = \overline{1,3}$).

Consider for h the expression

$$h = r - R + \varepsilon R \sin^2\varphi, \tag{19}$$

with R = mean equatorial terrestrial radius, ε = Earth's flattening. Replacing $\sin\varphi$ = DB and (7) in (19), and observing hypothesis (ii), we derive the following expansion to first order in q and k (see [2])

$$\exp((120 - h)B_j) = A_j(1 + \varepsilon R D^2 B_j A^2 + B_j p A q + B_j p B k), \; j = \overline{0,3}, \tag{20}$$

where we abbreviated

$$A_j = \exp(B_j(120 - p + R - \varepsilon R D^2)), \; j = \overline{0,3}. \tag{21}$$

Replacing (20) in (18), this expression becomes

$$h_6 = K_{60} + K_{61} A^2 + K_{62} A q + K_{62} B k, \tag{22}$$

in which we have introduced the notations

$$K_{60} = \sum_{j=0}^{3} k'_{6j} A_j ,$$

$$K_{61} = \sum_{j=0}^{3} k'_{6j} a R D^2 A_j B_j , \tag{23}$$

$$K_{62} = \sum_{j=0}^{3} k'_{6j} A_j B_j p .$$

Making $n = 6$ in (2) and using (16) and (22), we get

$$\begin{aligned} \rho_6 = X_0(&K_{60} G_4 A + K_{61} G_4 3 + K_{60} G_5 B + K_{61} G_5 A^2 B + \\ &+ (K_{62} G_4 A^2 + K_{62} G_5 AB) q + (K_{62} G_5 - K_{62} G_5 A^2 + \\ &+ K_{62} G_4 AB) k) . \end{aligned} \tag{24}$$

In this formula X_0 and all coefficients of A^n, $A^n B$ are constant over one revolution of the satellite, as we have already mentioned.

4. Variation of the nodal period. In order to estimate the difference ΔT_Ω between the real (perturbed) nodal period and the corresponding Keplerian one, difference caused by an arbitrary perturbing factor (in our case the diurnal variation of the density) featured by a small parameter σ, we shall use a method due primarily to I.D.Zhongolovich (for the principle see e.g. [3, 6]). According to this method, and observing hypotheses (i) and (ii), the respective difference can be written as

$$\Delta T_\Omega = \sum_{n=1}^{4} I_n , \tag{25}$$

with

$$
\begin{aligned}
I_1 &= (3/2)\,(p/\mu)^{1/2}\int_0^{2\pi}(1-2Aq-2Bk)\,\Delta p\,du\,,\\
I_2 &= -2p(p/\mu)^{1/2}\int_0^{2\pi}(1-3Aq-3Bk)\,A\,\Delta q\,du\,,\\
I_3 &= -2p(p/\mu)^{1/2}\int_0^{2\pi}(1-3Aq-3Bk)\,B\,\Delta k\,du\,,\\
I_4 &= \int_0^{2\pi}(\partial(r^4C\dot{\Omega}/(\mu p))/\partial\sigma)\,\sigma\,du\,.
\end{aligned}
\tag{26}
$$

In these formulae, as specified in Section 2, the subscript "O" for p, q, k, C was dropped. The factors Δz, $z \in \{p, q, k\}$, under integrals in I_1, I_2, I_3 are given by (6).

To obtain ΔT_Ω we proceed according to the following steps:

(a) Replace (24) in (10), and perform the calculations observing hypothesis (ii), obtaining in this way the integrands of (6) as functions of u only.

(b) Perform the integrals (6) for p, q, k, obtaining the changes Δz, $z \in \{p, q, k\}$, in terms of u and constants of integration.

(c) Replace Δz, $z \in \{p, q, k\}$, obtained at the step (b) in (26), and perform the calculations observing hypothesis (ii), obtaining in this way the integrands of the first three formulae (26).

(d) Perform the integrals (26) (for the fourth one, use (4) and (10), and consider X_0 to be the small parameter σ), then perform the sum (25).

After all these calculations we obtain

$$
\begin{aligned}
\Delta T_\Omega = \pi X_0(p^4/\mu)\delta(&xH_1 + yJ_1 + f_1^0 + (xH_2 + yJ_2 + f_2^0)q\, +\\
&+ (xH_3 + yJ_3 + f_3^0)k)\,,
\end{aligned}
\tag{27}
$$

where b, which appeared in (10), was replaced by its expression (9), and we used the abbreviating notations

$$H_1 = (4K_{60} + K_{61})G_5,$$

$$H_2 = -(12\pi(4K_{60} + 3K_{61} - 4K_{62})G_4 - (51K_{60} + 23K_{61} - 28K_{62})G_5)/16, \tag{28}$$

$$H_3 = -((140K_{60} + 81K_{61} + 28K_{62})G_4 + 12\pi(4K_{60} + K_{61} - 4K_{62})G_5)/16,$$

and

$$J_1 = (4K_{60} + K_{61})G_5,$$

$$J_2 = (12\pi(4K_{60} + 3K_{61} + 4K_{62})G_4 + (88K_{60} + 7K_{61} - 24K_{62})G_5)/16, \tag{29}$$

$$J_3 = -((232K_{60} + 129K_{61} + 28K_{62})G_4 - 12\pi(4K_{60} + K_{61} + 4K_{62})G_5)/16.$$

In (27) x and y are given by (9), while f_n^0, $n = \overline{1,3}$, are constants of integration (or combinations of such constants) obtained at the step (b). Their expressions (see also [3]), which depend on the Earth, atmospheric model, and satellite orbit at $u = u_0$, are very long and will not be reproduced here.

5. Particular cases. Suppose that the orbit is initially circular ($q = k = 0$). Imposing this condition to equation (27) and taking into account notations (9), we obtain

$$\Delta T_\Omega = \pi X_0(p^4/\mu)\,\delta(CwH_1 - 2\pi J_1/T_0 + f_1^0), \tag{30}$$

where p stands for the radius of the initial orbit and we used the well-known relation $T_0 = 2\pi p^{3/2}\mu^{-1/2}$ (true for circular and quassi-circular orbits), T_0 denoting the Keplerian period corresponding to $u = u_0$.

Suppose now that we neglect the atmospheric rotation ($w = 0$). In this case, taking into account (9), equation (27) becomes

$$\Delta T_{\Omega} = \pi X_0 (p^4/\mu)\delta(\gamma J_1 + f_1^0 + (\gamma J_2 + f_2^0)q + (\gamma J_3 + f_3^0)k). \quad (31)$$

If the initial orbit is circular and the atmospheric rotation is neglected, equation (27) reduces to

$$\Delta T_{\Omega} = \pi X_0 (p^4/\mu)\delta(\gamma J_1 + f_1^0). \quad (32)$$

If the initial orbit is polar ($C = 0$), one obtains again expression (31); with the supplementary condition of initially circular orbit, one recovers (32).

A mention must be made here. Imposing supplementary conditions to describe particular cases, the integration constants f_n^0, $n = \overline{1,3}$, also change their expressions, according to the restrictions we imposed. However, by abuse of notation, we kept the same symbols in formulae (30)-(32) for these integration constants.

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ON MEASURE OF THE AMOUNT OF INFORMATION CONSIDERED AS A GENERALIZED MEAN VALUE

Ion MIHOC*

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REZUMAT. - Asupra măsurii cantităţii de informaţie considerată ca şi o valoare medie generalizată. În această lucrare se prezintă unele proprietăţi ale măsurării cantităţii de informaţie asociată unei variabile aleatoare generalizate, respectiv, distribuţiei de probabilitate generalizate asociată acesteia.

1. Introduction and notations. Let (Ω, K, P) be a probability space, that is, Ω an arbitrary set, called the set of elementary events; K a σ-algebra of subsets of Ω, containing Ω itself, the elements of K being called events; and P o probability measure, that is, a nonnegative and additive set function, defined on K, for which $P(\Omega) = 1$.

Let

$$\Delta_N^* = \left\{ \mathcal{P} = (p_1, p_2, \ldots, p_N) : p_i \geq 0,\ i = \overline{1,N},\ \sum_{i=1}^{N} p_i = 1 \right\} \tag{1.1}$$

be the set of all probability distributions associated with a discrete finite random variable X.

Let $\mathcal{P} = (p_1, p_2, \ldots, p_N)$ be a finite discrete probability distribution, that is, $\mathcal{P} \in \Delta_N^*$.

DEFINITION 1. [8] The amount of uncertainty of the distribution $\mathcal{P}$, that is, the amount of uncertainty concerning the outcome of an experiment, the possible results of which have the probabilities $p_1, p_2, \ldots, p_N$, is called the entropy of the distribution $\mathcal{P}$ and is usually measured by the quantity

$$H(\mathcal{P}) = H(p_1, p_2, \ldots, p_N) = H(X) = -\sum_{i=1}^{N} p_i \log_2 p_i \tag{1.2}$$

* *"Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, 3400 Cluj-Napoca, Romania*

In conection with the notion of uncertainty we also have to mention the concept of information. If we receive some information, the previously existing uncertainty will be diminished. The meaning of information is precisely this diminishing of uncertainty. The uncertainty with respect to an outcome of an experiment may be considered as numerically equal to the information furnished by the occurrence of this outcome; thus uncertainty can also be measured. To speak about information or about uncertainty means essentialy the same thing: in the first case we consider an experiment which has been performed, in the second case an experiment not yet performed [7].

Also, if X is a random variable assuming the distinct values $x_1, x_2, \ldots, x_N$ with probabilities $p_1, p_2, \ldots, p_N$, we can say that $H(\mathcal{P})$ is the information contained in the value of X and we may write $H(X)$ instead of $H(\mathcal{P})$.

Let us consider $\mathcal{P} = (p_1, p_2, \ldots, p_N) \in \Delta_N^*$, $\mathcal{Q} = (q_1, q_2, \ldots, q_N) \in \Delta_N^*$.

Let us denote by $\mathcal{P}*\mathcal{Q}$ the direct product of distributions $\mathcal{P}$ and $\mathcal{Q}$, that is,

$$\mathcal{P}*\mathcal{Q} = (p_1 q_1, \ldots, p_1 q_N, \ldots, p_N q_1, \ldots, p_N q_N) \in \Delta_{NN}^*. \qquad (1.3)$$

Then we have from (1.2)

$$H(\mathcal{P}*\mathcal{Q}) = H(\mathcal{P}) + H(\mathcal{Q}) \qquad (1.4)$$

which expresses one of the most important proprieties of entropy, namely, its additivity: the entropy of a combined experiment consisting of the performance of two independent is equal to the sum of the entropies of these two experiments.

Rényi [6] introduced a generalization of the notion of a random variable.

DEFINITION 2. An incomplete random variable X is a function $X = X(\omega)$ measurable with respect to the measure on K and defined on a subset Ω_1 of Ω, where $\Omega_1 \in K$ and $P(\Omega_1)>0$.

The only difference between an ordinary random variable (X is an ordinary or complete random variable if $P(\Omega_1) = 1$) and an incomplete random variable is thus that latter is not necessary defined for every $\omega \in \Omega$. Therefore, for an incomplete random variable we have $0 < P(\Omega_1) < 1$.

DEFINITION 3. If $0 < P(\Omega_1) \leq 1$, then the random variable X, defined on the Ω_1, is a generalized random variable. The distribution of a generalized random variable X will called a generalized probability distribution.

In this sense, the ordinary distribution can be considered as a particular case of a latter.

We denote by

$$w(\mathbf{P}) = \sum_{i=1}^{N} p_i \tag{1.5}$$

the weight of the distribution $\mathbf{P}$.

Using the above definitions it follows that:

- if $w(\mathbf{P}) = 1$, then $\mathbf{P}$ is an ordinary distribution;

- if $0 < w(\mathbf{P}) < 1$, then $\mathbf{P}$ is an incomplete distribution;

- if $0 < w(\mathbf{P}) \leq 1$, then $\mathbf{P}$ is a generalized probability distribution.

Also, we denote by

$$\Delta_N = \left\{ (p_1, p_2, \ldots, p_N) : p_i \geq 0,\ i = \overline{1,N};\ 0 < w(\mathbf{P}) \leq 1 \right\} \tag{1.6}$$

the set of all finite discrete generalized probability distributins.

DEFINITION 4. [6] The measure of the amount of information of order α of the generalized probability distribution $\mathbf{P}$ has the form

$$H_\alpha(\mathbf{P}) = \frac{1}{1-\alpha} \log_2 \left(\frac{\sum_{i=1}^{N} p_i^\alpha}{w(\mathbf{P})} \right),\ \alpha > 0,\ \alpha \neq 1,\ \mathbf{P} \in \Delta_n . \tag{1.7}$$

If $\mathcal{P}$ is an ordinary distribution, that is, $\mathcal{P} \in \Delta_N^*$, then (1.7) reduces to

$$H_\alpha^*(\mathcal{P}) = \frac{1}{1-\alpha} \log_2 \left(\sum_{i=1}^{N} p_i^\alpha \right), \ \alpha > 0, \ \alpha \neq 1, \ \mathcal{P} \in \Delta_N^*. \tag{1.8}$$

and $H_\alpha^*(\mathcal{P})$ will be called Rényi's information of order α of the ordinary probability distribution $\mathcal{P}$.

It is worth mentioning that

$$\lim_{\alpha \to 1} H_\alpha(\mathcal{P}) = H_1(\mathcal{P}) = \frac{1}{w(\mathcal{P})} \sum_{i=1}^{N} p_i \log_2 \frac{1}{p_i}, \tag{1.9}$$

where $H_1(\mathcal{P})$ is Shannon's information or information of order 1; respectively, that

$$\lim_{\alpha \to 1} H_\alpha^*(\mathcal{P}) = H_1^*(\mathcal{P}) = -\sum_{i=1}^{N} p_i \log_2 p_i, \tag{1.10}$$

where $H_1^*(\mathcal{P}) = H(\mathcal{P})$ is Shannon's measure of entropy of the probability distribution $\mathcal{P}$, $\mathcal{P} \in \Delta_N^*$.

DEFINITION 5. [3] The information generating function associated to the Rényi information of order α has the following form

$$G_\alpha(u) = \left(\frac{\sum_{i=1}^{N} p_i^\alpha}{w(\mathcal{P})} \right)^{\frac{k \cdot u}{1-\alpha}}, \ \alpha > 0, \ \alpha \neq 1, \ u \in R \tag{1.11}$$

if $\mathcal{P} \in \Delta_N$, respectively, the form

$$G_\alpha^*(u) = \left(\sum_{i=1}^{N} p_i^\alpha \right)^{\frac{k \cdot u}{1-\alpha}}, \ \alpha > 0, \ \alpha \neq 1, \ u \in R \tag{1.12}$$

if $\mathcal{P} \in \Delta_N^*$, where $k = \log_2 e$.

For these information generating functions we have the following properties

$$G_\alpha'(u)\big|_{u=0} = \log_e(G_\alpha(1)) = H_\alpha(\mathcal{P}) \tag{1.11a}$$

$$\lim_{\alpha \to 1} (\log_e G_\alpha(1)) = H_1(\mathcal{P}), \tag{1.11b}$$

if $\mathcal{P} \in \Delta_N$, respectively,

$$\frac{d}{du}\left(G_\alpha^*(u)\right)\big|_{u=0} = \log_e G_\alpha^*(1) = H_\alpha^*(\mathcal{P}) \tag{1.12a}$$

$$\lim_{\alpha\to 1} \frac{d}{du}\left(G_\alpha^*(u)\right)\Big|_{t=0} = H_1^*(\mathbf{P}) = H(\mathbf{P}), \tag{1.12b}$$

if $\mathbf{P} \in \Delta_N^*$.

2. Generalized measures of the amount of information. In this section we define a new measure of the amount of information when $\mathbf{P} \in \Delta_N$.

DEFINITION 6. [2] The measure of the amount of information of order $\frac{\alpha}{n}$ and of type $\{\beta + a_i\}$, associated to the generalized probability distribution $\mathbf{P}$, respectively, to a generalized drandom variable X, has the form

$$H_{\frac{\alpha}{n}}^{(\beta+a_i)}(\mathbf{P}) = \frac{n}{n-\alpha} \log_2 \frac{\sum_{i=1}^{N} p_i^{\frac{\alpha}{n}+\beta+a_i-1}}{\sum_{i=1}^{N} p_i^{\beta+a_i}} \tag{2.1}$$

where

$$\alpha > 0,\ \alpha \neq n,\ n \geq 1,\ \beta + a_i \geq 1,\ i = \overline{1,N} \tag{2.1a}$$

Remark 1. If we denote by

$$q_i = \frac{p_i^{\beta+a_i}}{\sum_{i=1}^{N} p_i^{\beta+a_i}},\ i = \overline{1,N},\ \sum_{i=1}^{N} q_i = 1, \tag{2.2a}$$

$$\alpha^* = \frac{\alpha - n}{n},\ \alpha^* \in (-1, 0) \cup (0, \infty) \tag{2.2b}$$

and suppose

$$\alpha > 0,\ \alpha \neq n,\ n \geq 1,\ \beta + a_i \geq 1,\ i = \overline{1,N} \tag{2.2a}$$

respectively, $\mathbf{P} \in \Delta_N$, then we obtain a new form for (2.1), namely,

$$H_{\alpha^*}(\mathbf{P}) = H_{\alpha^*}(X) = -\frac{1}{\alpha^*} \log_2 \left(\sum_{i=1}^{N} q_i \cdot p_i^{\alpha^*} \right) \tag{2.2}$$

DEFINITION 7. The information generating function of order $\frac{\alpha}{n}$ and of type $\{\beta +$

$a_i\}$ can be defined as follows

$$G^{(\beta+a_i)}_{\frac{\alpha}{n}}(u) = \left(\sum_{i=1}^{N} q_i \cdot p_i^{\alpha^*}\right)^{-\frac{k \cdot u}{\alpha^*}}, \quad u \in R \tag{2.3}$$

where

$$\alpha^* = \frac{\alpha - n}{n}, \ \alpha^* \in (-1, 0) \cup (0, \infty); \ \alpha > 0, \ \alpha \neq n, \ n \geq 1, \tag{2.3a}$$

respectively,

$$\beta + a_i \geq 1, \ i = \overline{1, N}; \ \mathcal{P} \in \Delta_N, \ k = \log_2 e. \tag{2.3b}$$

This information generating function has very important properties, namely,

$$\frac{d}{du}\left(G^{(\beta+a_i)}_{\frac{\alpha}{n}}(u)\right)\Big|_{u=0} = \log_e G^{(\beta+a_i)}_{\frac{\alpha}{n}}(1) = H_{\alpha^*}(\mathcal{P}); \tag{2.4}$$

$$\lim_{\alpha \to n} H^{(\beta+a_i)}_{\frac{\alpha}{n}}(\mathcal{P}) = \lim_{\alpha^* \to 0} H_{\alpha^*}(P) = -\sum_{i=1}^{N} q_i \cdot \log_2 p_i \tag{2.5}$$

THEOREM 1. *For the amount of information $H_{\alpha^*}(\mathcal{P})$, when $0 < \alpha < n$, we have the following double inequality*

$$H_1^*(Q) \leq H_1^{(\beta+a_i)}(\mathcal{P}) \leq H^{(\beta+a_i)}_{\frac{\alpha}{n}}(\mathcal{P}) = H_{\alpha^*}(\mathcal{P}), \tag{2.6}$$

where

$$H_1^*(Q) = -\sum_{i=1}^{N} q_i \cdot \log_2 q_i \tag{2.7}$$

is Shannon's measure of information or the information of order 1 of ordinary (or complete) probability distribution $Q = (q_1, q_2, \ldots, q_N)$, that is, $Q \in \Delta_N^*$ and

$$H_1^{(\beta+a_i)}(\mathcal{P}) = -\sum_{i=1}^{N} q_i \cdot \log_2 p_i \tag{2.8}$$

is Rathie's measure of information of order $\{\beta + a_i\}$, [5].

Proof. The first inequality follows immediatly if we take into account that

$$H_1^{(\beta+a_i)}(\mathcal{P}) = -\sum_{i=1}^{N} q_i \cdot \log_2 p_i = -\frac{\sum_{i=1}^{N} p_i^{\beta+a_i} \cdot \log_2 p_i}{\sum_{i=1}^{N} p_i^{\beta+a_i}} \tag{2.8a}$$

as well as the inequality of Gibbs

$$H_1^*(Q) = -\sum_{i=1}^{N} q_i \cdot \log_2 q_i \leq -\sum_{i=1}^{N} q_i \cdot \log_2 p_i = H_1^{(\beta+a_i)}(\mathbf{P}). \tag{2.9}$$

The second inequality follows if we apply Jensen's inequality

$$\sum_{i=1}^{N} q_i \cdot f(x_i) \leq f\left(\sum_{i=1}^{N} q_i \cdot x_i\right), \tag{2.10}$$

where $-f(x)$ is a convex function on an interval (a,b); $x_1, x_2, \ldots, x_N$ are arbitrary real numbers $a < x_i < b$, $i = \overline{1,N}$ and $q_1, q_2, \ldots, q_N$ are positive numbers $\sum_{i=1}^{N} q_i = 1$.

Indeed, by putting

$$q_i = \frac{p_i^{\beta+a_i}}{\sum_{i=1}^{N} p_i^{\beta+a_i}}, \ i = \overline{1,N} \tag{2.10a}$$

$$x_i = p_i, \ i = \overline{1,N}, \tag{2.10b}$$

respectively,

$$f(x_i) = \log_2\left(p_i^{\alpha^*}\right), \ i = \overline{1,N}, \tag{2.10c}$$

we can derive from (2.10) a new inequality

$$\sum_{i=1}^{N} q_i \cdot \log_2 p_i \leq \log_2\left(\sum_{i=1}^{N} q_i \cdot p_i^{\alpha^*}\right). \tag{2.11}$$

Now, if we have in view that $0 < \alpha < n$, that is, $\alpha^* = \frac{\alpha - n}{n} < 0$, from (2.1) and (2.11) we obtain just the double inequality (2.6).

3. The measure of information as a generalized mean value. In what follows we are concerned with the others properties of measure of the information of order $\frac{\alpha}{n}$ and of type $\{\beta + a_i\}$, associated to a generalized random variable X, that is, $\mathbf{P} \in \Delta_N$.

If we denote by J the interval $(0,1]$, then the set Δ_N can be written as

$$\Delta_N = J^N = \{(p_1, p_2, \ldots, p_N) : p_i \in J, \ i = \overline{1,N}, \ 0 < w(\mathbf{P}) \leq 1\}. \tag{3.1}$$

DEFINITION 8. [1] The mean value in the set J^N is defined by

$$M : J^N \to J \tag{3.2}$$

$$M(\rho) = M_\varphi[\rho]_f = \varphi^{-1}\left(\frac{\sum_{i=1}^{N} f(p_i)\cdot\varphi(p_i)}{\sum_{i=1}^{N} f(p_i)}\right) \tag{3.3}$$

where:

1° $\varphi(t)$ is continuous and strictly monotonic if $t \in J$;

2° $f(t)$ is positive and bounded function in J;

This mean value is called the mean value of the set J^N constitued wich the weight function f and wich the representation function φ.

DEFINITION 9. [1] The generalized measure of the amount of information in the Daroczy's sense has form

$$H_\varphi(\rho)_f = -\log_2 M_\varphi[\rho]_f, \tag{3.4}$$

where $M_\varphi[\rho]_f$ represents the weighted mean associated to the generalized probability distribution ρ and if are satisfed the following conditions:

o$_1$) $M_\varphi[\rho]_f$ depends only by the probabilities p_i, $i = \overline{1, N}$;

o$_2$) $\varphi(t)$ is continuous and strictly monotonic on the interval J;

o$_3$) $f(t)$ is a positive and bounded function in J;

o$_4$) if $\rho \in \Delta_N$, $R \in \Delta_N$ and

$$\rho * R = (p_1 r_1, \ldots, p_1 r_N, \ldots, p_N r_1, \ldots, p_N r_N) \in \Delta_{NN} \tag{3.5}$$

then

$$M_\varphi[\rho * R]_f = M_\varphi[\rho]_f \cdot M_\varphi[R]_f. \tag{3.6}$$

THEOREM 2. *The measure of the amount of information of order* $\frac{\alpha}{n}$ *and of type* $\{\beta + a_i\}$, $H_{\frac{\alpha}{n}}^{(\beta + a_i)}(\rho)$ *has the following form*

$$H_{\frac{\alpha}{n}}^{(\beta+a_i)}(\mathcal{P}) = H_{\alpha^*+1}^{(\beta+a_i)}(\mathcal{P}) = -\log_2\{M_{\alpha^*}[\mathcal{P}]\}, \tag{3.7}$$

where

$$M_{\alpha^*}[\mathcal{P}] = \left(\sum_{i=1}^{N} q_i \cdot p_i^{\alpha^*}\right)^{\frac{1}{\alpha^*}} \tag{3.8}$$

and

$$\alpha^* = \frac{\alpha - n}{n}, \ \alpha^* \in (-1,0) \cup (0,\infty) \tag{3.8a}$$

$$q_i = \frac{p_i^{\beta+a_i}}{\sum_{i=1}^{N} p_i^{\beta+a_i}}, \ i = \overline{1,N}; \ w(Q) = 1, \ Q \in \Delta_N^*; \tag{3.8b}$$

$$w(\mathcal{P}) = \sum_{i=1}^{N} p_i \leq 1, \ \mathcal{P} \in \Delta_N. \tag{3.8c}$$

More, the weighted mean associated $M_{\alpha^*}[\mathcal{P}]$ *satisfies all conditions from the Definition 9, that is, the measure* (3.7) *is a particular case of the amount of information in the Daroczy' sense.*

Proof. The representation (3.7) follows imediatly from (3.8a), (3.8b), (3.8c) and from the following representations

$$H_{\frac{\alpha}{n}}^{(\beta+a_i)}(\mathcal{P}) = -\frac{\log_2\left(\sum_{i=1}^{N} q_i \cdot p_i^{\alpha^*}\right)}{\alpha^*} = H_{\alpha^*+1}^{(\beta+a_i)}(\mathcal{P}) \tag{3.9}$$

$$M_{\alpha^*}[\mathcal{P}] = \left(\frac{\sum_{i=1}^{N} p_i^{\beta+a_i} \cdot p_i^{\alpha^*}}{\sum_{i=1}^{N} p_i^{\beta+a_i}}\right)^{\frac{1}{\alpha^*}} \tag{3.10}$$

when the weight function $f(t)$ has the form

$$f(t) = t^{\beta+a_i}, \ \beta + a_i \geq 1, \ i = \overline{1,N}. \tag{3.11}$$

In the next, we will examine the following equality

$$\varphi^{-1}\left(\frac{\sum_{i=1}^{N} f(p_i)\cdot\varphi(p_i)}{\sum_{i=1}^{N} f(p_i)}\right) = \left(\frac{\sum_{i=1}^{N} p_i^{\beta+a_i}\cdot p_i^{\alpha^*}}{\sum_{i=1}^{N} p_i^{\beta+a_i}}\right)^{\frac{1}{\alpha^*}} \tag{3.12}$$

where

$$\varphi(t) = t^{\alpha^*},\ t \in J = (0,1], \tag{3.13}$$

$$f(t) = t^{\beta+a_i},\ t \in J;\ \beta + a_i \geq 1,\ i = \overline{1,N}; \tag{3.14}$$

In fact, from (3.12) it follows

$$\varphi\varphi^{-1}\left(\frac{\sum_{i=1}^{N} f(p_i)\cdot\varphi(p_i)}{\sum_{i=1}^{N} f(p_i)}\right) = \frac{\sum_{i=1}^{N} f(p_i)\cdot\varphi(p_i)}{\sum_{i=1}^{N} f(p_i)} =$$

$$= \varphi\left[\left(\frac{\sum_{i=1}^{N} p_i^{\beta+a_i}\cdot p_i^{\alpha^*}}{\sum_{i=1}^{N} p_i^{\beta+a_i}}\right)^{\frac{1}{\alpha^*}}\right] = \varphi(t^*) \tag{3.12a}$$

and, from here, if we have in view (3.13), we obtain

$$\varphi(t^*) = (t^*)^{\alpha^*} \tag{3.13a}$$

and, respectively, just the equality (3.12).

Now, we must to verify that the weighted mean $M_{\alpha^*}[\mathcal{P}]$, as, the functions φ and f fulfill all conditions from the Definition 9.

Thus:

c_1) $M_{\alpha^*}[\mathcal{P}]$, when β and a_i are fixed such that $\beta + a_i \geq 1$, $i = \overline{1,N}$, dependends only upon the probabilities $p_1, p_2, \ldots, p_N$;

c_2) Because the representation function φ has the form (3.13) it follows that, for α^* -

fixed, $\alpha^* \in (-1,0) \cup (0,\infty)$, $\varphi(t)$ is a power function, that is, it is a continuous and strictly monotonic in the interval J;

c_3) The weight function f has the form $f(t) = t^{\beta+a_i}$, where $\beta + a_i \geq 1$, $\beta + a_i$ - fixed, $i = \overline{1,N}$, that is, it is a power function which, evidently, is a positive and bounded function for all $t \in J$,

c_4) In order to verify this condition c_4) we consider two generalized probability distribution $\mathcal{P}$ and R, that is,

$$\mathcal{P} = (p_1, p_2, \ldots, p_N) \in \Delta_N, \quad R = (r_1, r_2, \ldots, r_N) \in \Delta_N, \tag{3.14}$$

respectively, their direct product

$$\mathcal{P} * R = (p_1 r_1, \ldots, p_1 r_N, \ldots, p_N r_1, \ldots, p_N r_N) \in \Delta_{NN} \tag{3,15}$$

Then, the mean values corresponding then will be

$$M_{\alpha^*}[\mathcal{P}] = \left(\sum_{i=1}^{N} q_i \cdot p_i^{\alpha^*}\right)^{\frac{1}{\alpha^*}}; \tag{3.14a}$$

$$M_{\alpha^*}[R] = \left(\sum_{j=1}^{N} t_j \cdot r_j^{\alpha^*}\right)^{\frac{1}{\alpha^*}}; \tag{3.14b}$$

$$M_{\alpha^*}[\mathcal{P} * R] = \left(\sum_{i=1}^{N}\sum_{j=1}^{N} (q_i t_j)(p_i r_j)^{\alpha^*}\right)^{\frac{1}{\alpha^*}} \tag{3.15a}$$

where

$$\alpha^* \in (-1, 0) \cup (0, \infty) \tag{3.14c}$$

and

$$q_i = \frac{p_i^{\beta+a_i}}{\sum_{i=1}^{N} p_i^{\beta+a_i}}, \quad i = \overline{1,N}, \quad \sum_{i=1}^{N} q_i = 1 \tag{3.15b}$$

$$t_j = \frac{r_j^{\beta + a_j}}{\sum_{j=1}^{N} r_j^{\beta + a_j}}, \quad j = \overline{1,N}, \quad \sum_{j=1}^{N} t_j = 1 \tag{3.15c}$$

Also, we obtain

$$M_{\alpha'}[\rho] \cdot M_{\alpha'}[R] = \left[\left(\sum_{i=1}^{N} q_i \cdot p_i^{\alpha'}\right)\left(\sum_{j=1}^{N} t_j \cdot r_j^{\alpha'}\right)\right]^{\frac{1}{\alpha'}} =$$

$$= \left(\sum_{i=1}^{N}\sum_{j=1}^{N} (q_i t_j)(p_i r_j)^{\alpha'}\right)^{\frac{1}{\alpha'}} = M_{\alpha'}[\rho * R],$$

that is, just the equality

$$M_{\alpha'}[\rho * R] = M_{\alpha'}[\rho] \cdot M_{\alpha'}[R], \tag{3.16}$$

which expresses one of the important properties of entropy, namely, its additivity. This completes the proof of Theorem 2.

COROLLARY 1. *The measure of the amount of information $H_{\alpha'}(X)$, where X is a generalized random variable (respectively, the distribution of the random variable X is a generalized probability distribution), satisfies the additivity property, namely*

$$H_{\frac{\alpha}{\pi}}^{(\beta+a_i)}(\rho * R) = H_{\frac{\alpha}{\pi}}^{(\beta+a_i)}(\rho) + H_{\frac{\alpha}{\pi}}^{(\beta+a_i)}(R) \tag{3.17}$$

where

$$\rho \in \Delta_N, \quad R \in \Delta_N, \quad \rho * R \in \Delta_{NN}.$$

This property is a consequence of the relations (3.14a), (3.14b), (3.15a), respectively, of (3.16).

COROLLARY 2. *Suppose the conditions* (3.8a) - (3.8c) *are satisfied. Then*

$$\lim_{\alpha' \to 0} H_{\alpha'+1}^{(\beta+a_i)}(\rho) = -\log_2\left(\sum_{i=1}^{N} p_i^{q_i}\right)$$

where

$$M_o[\rho] = \prod_{i=1}^{N} p_i^{q_i} \tag{3.18a}$$

is just geometric mean of the generalized probability distribution $\rho = (p_1, p_2, \ldots, p_N)$, that is, $\rho \in \Delta_N$ while the probabilities q_i, $i = \overline{1,N}$ are taken as weigts and we have $w(Q) = 1$

COROLLARY 3. *The measure of the amount of information* $H_{\alpha^*}(\mathcal{P})$ *verifies the following limit relation*

$$\lim_{\alpha^* \to -1} H_{\alpha^*+1}^{\beta+a_i}(\mathcal{P}) = -\log_2 \left(\frac{1}{\sum_{i=1}^{N} \frac{q_i}{p_i}} \right) \tag{3.19}$$

where

$$M_{-1}[\mathcal{P}] = \lim_{\alpha^* \to -1} M_{\alpha^*}[\mathcal{P}] = \frac{1}{\sum_{i=1}^{N} \frac{q_i}{p_i}} \tag{3.19a}$$

represents the harmonic mean associated to the generalized probability distribution $\mathcal{P}$.

COROLLARY 4. *If* $\alpha^* = 1$ *then we obtain*

$$H_2^{(\beta+a_i)}(\mathcal{P}) = -\log_2 M_1[\mathcal{P}] \tag{3.20}$$

where

$$M_1[\mathcal{P}] = \sum_{i=1}^{N} q_i \cdot p_i, \quad \mathcal{P} \in \Delta_N, \quad Q \in \Delta_N^* \tag{3.20a}$$

is just the aritmetic mean associated to the generalized probability $\mathcal{P}$ *and probabilities* q_i, $i = \overline{1,N}$ *are taken as weights.*

COROLLARY 5. *If* $\alpha^* = 1$, *that is,* $\alpha = 2n$, $n \geq 1$, *then in the following conditions* $\beta + a_i = 1$, $i = \overline{1,N}$, *the mean* $M_1[\mathcal{P}]$, *defined by* (3.20a), *can be written in the form*

$$E_N^*(\mathcal{P}) = \sum_{i=1}^{N} p_i^2, \quad \text{if } \mathcal{P} \in \Delta_N^*, \tag{3.21}$$

respectively, in the form

$$E_N(\mathcal{P}) = \frac{1}{w(\mathcal{P})} \cdot \sum_{i=1}^{N} p_i^2, \quad \text{if } \mathcal{P} \in \Delta_N. \tag{3.22}$$

These measures represints just the Onicescu's informational energy [4].

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Tehnoredactare computerizată: **M. TOPLICEANU, L. SĂCELEAN**

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