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POWER LOGICS

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REZUMAT. - Logici de puteri. Se dau două extinderi ale logicii bivalente P a propoziţiilor, la mulţimea $\wp(P)$ a părţilor sale. Una dintre extensii devine o logică trivalentă iar a doua, tetravalentă.

Preliminaries. Let (P,v) be the (bivalent) propositional logic, where $v:P \to V = \{0,1\}$ ist the valuation map. We try to extend the algebra $(P,-,\wedge,\vee)$ to power $(\wp(P),-,\wedge,\vee)$, where, as in [5] or in [7], the operations are defined by:

$$\overline{X} = \{\overline{x} | x \in X\} \quad \text{for } X \subseteq P \text{ and}$$

$$X \mathbin{\delta} Y = \{x \mathbin{\delta} y \mid x \in X,\ y \in Y\} \text{ for } X,Y \subseteq P \text{ and } \delta \in \{\wedge,\vee\}.$$

1. First we define a trivalent valuation

$$u\colon (P) \to U = \{0,1/2,1\}$$

so that $(\wp(P),u)$ be a (trivalent) logic. This means that u must be a homomorphic map of $(\wp(P),-,\wedge,\vee)$ on $(U,-,\wedge,\vee)$, where the operations in U are defined as in [3]:

−	
0	1
1/2	1/2
1	0

∧	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

∨	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1
1	1	1	1

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Now, define $u: \mathcal{P}(P) \to U$ by:

$$u(X) = \begin{cases} 0 \text{ if } & I \cap X = \phi \text{ and } O \subseteq X \\ 1 \text{ if } & O \cap X = \phi \text{ and } I \subseteq X \\ 1/2 & \text{otherwise} \end{cases}$$

Here I, O are the classes of tautologies, contradictions respectively.

COROLLARY. $u(\phi) = 1/2$.

THEOREM 1. *The valuation map above is homomorphoc, yhat is:*

$$u(\overline{X}) = \overline{u(X)},$$

$$u(X \circ Y) = u(X) \circ (Y), \ \circ \in \{\wedge, \vee\}.$$

Proof. a). $u(X) = 1 \to O \cap X = \phi$ and $I \subset X \to \overline{I} \subset \overline{X} =$
$= \{\overline{x} \mid x \in X\} \to O \subset \overline{X}$. (i)

Suppose $y \in I \cap \overline{X} \neq \phi$, that is, $\overline{y} \in I \cap \overline{\overline{X}} = O \cap X \neq \phi$, a contradiction. Therefore $I \cap \overline{X} = \phi$ (ii).

From (i) and (ii) it results that $u(\overline{X}) = 0 = \overline{u(X)}$.

$u(X) = 0 \to u(\overline{X}) = 1 = \overline{u(X)}$, analogously.

$u(X) = 1/2 \to 0 \neq u(X) \neq 1 \to \overline{0} \neq u(\overline{X}) \neq \overline{1} \to$

$\to 1 \neq u(\overline{X}) \neq 0 \to u(\overline{X}) = 1/2 = \overline{u(X)}$.

b). $u(X) = u(Y) = 1 \to O \cap X = \phi = O \cap Y$ and $I \subseteq X \cap Y \to O \cap (X \wedge Y) = O \cap \{x \wedge y \mid x \in X, y \in Y\} = \phi$ and $I \cap I =$
$= I \subseteq X \wedge Y$.

$$u(X) = 0 \to \left\{ \begin{matrix} O \subseteq X \to O \subseteq X \wedge Y \\ I \cap X = \phi \to I \cap (X \wedge Y) = \phi \end{matrix} \right| \to u(X \wedge Y) = 0.$$

$$u(X)=\frac{1}{2}=u(Y)\rightarrow\left\{\begin{array}{c} O\cap X\neq\emptyset \text{ and } I\cap X\neq\emptyset \\ \text{or} \\ I\not\subset X \text{ and } \emptyset\not\subset X \\ \text{or} \\ O\cap X\neq\emptyset \text{ and } O\not\subset X \\ \ldots \end{array}\right| \rightarrow 0\neq u(X\wedge Y)\neq 1 \rightarrow u(X\wedge Y)=\frac{1}{2}.$$

c) $u(X)=0=u(Y) \rightarrow \left\{\begin{array}{l} I\cap X=\emptyset=I\cap Y \\ O\subseteq X\cap Y \end{array}\right| \rightarrow \left\{\begin{array}{l} I\cap(X\vee Y)=\emptyset \\ O\subseteq X\vee Y \end{array}\right| \rightarrow u(X\vee Y)=0.$

$u(X) = 1 \rightarrow O \cap X = \phi$ and $I \subseteq X \rightarrow O \cap (X \vee Y) = \phi$ and $I \subseteq X \vee Y \rightarrow$ $\rightarrow u(X \vee Y) = 1.$

$u(X) = 1/2$ or/and $u(Y) = 1/2 \rightarrow u(X) \notin V$ or/and $u(Y) \notin V \rightarrow$ $\rightarrow 0 \neq u(X \vee Y) \neq 1 \rightarrow u(X \vee Y) = 1/2.$

2. A more natural extension seems to be a tetravalent logic on (P), as follows.

One says ([2]) that $v: P \rightarrow V$ is homomorphic on $(V, -, \wedge, \vee)$, where the operations are defined by:

$-$	
0	1
1	0

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

Also, each of the operations $-, \wedge, \vee$ on P are mappings, namely,

$$-: P \rightarrow P,\ -(x) = \bar{x},\ \sigma: P \times P \rightarrow P,\ \sigma(x,y) = x \sigma y,$$

σ being one from the operations $\wedge$, $\vee$. The homomorphic properties of v may be expressed (see [8]) by the (commutative) diagramms:

$$\begin{array}{ccccc} P & \xrightarrow{\ -\ } & P & \xleftarrow{\ \sigma\ } & P \times P \\ {\scriptstyle v}\downarrow & & {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v\times v} \\ V & \xrightarrow{\ -\ } & V & \xleftarrow{\ \sigma\ } & V \times V \end{array}$$

where $v \times v(x,y) = (v(x),v(y))$.

The valuation $v:P \to V$ may be naturally extended to

$$w: \wp(P) \to \wp(V),\ w(X) = v(X) = \{v(x) \mid x \in X\}.$$

As $|\wp(V)| = 2^{|V|} = 2^2 = 4$, instead of $\wp(V)$, we consider the set of values $W = \{0,1/3,2/3,1\}$ with the operations $-$, $\wedge$, $\vee$, defined by the tables in [3], or briefly:

$$\bar{x} = 1 - x;\ x \wedge y = \min\{x,y\};\ x \vee y = \max\{x,y\}.$$

THEOREM 2. *$(\wp(P),w)$, with $w:\wp(P) \to W$ given above, becomes a tetravalent logic.*

Proof. It is sufficient to verify the commutativity of the diagramms below:

$$\begin{array}{ccccc}
\wp(P) & \xrightarrow{\;-\;} & \wp(P) & \xleftarrow{\;\sigma\;} & \wp(P) \times \wp(P) \\
\downarrow w & & \downarrow w & & \downarrow w \times w \\
W & \xrightarrow{\;-\;} & W & \xleftarrow{\;\sigma\;} & W \times W
\end{array}$$

where $\sigma \in \{\wedge, \vee\}$.

Remark. With the results above may be characterized the sets of (propositional) formulas, having applications in individual formulas (see [1]).

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A CHARACTERIZATION OF COVERING SUBGROUPS IN FINITE Π - SOLVABLE GROUPS

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REZUMAT. - O caracterizare a subgrupurilor acoperitoare în grupuri finite Π-rezolubile. Lucrarea conţine o caracterizare a subgrupurilor acoperitoare în grupuri finite Π-rezolubile şi unele consecinţe.

Abstract. The paper contains a characterization of covering subgroups in finite Π-solvable groups and some consequences.

1. Preliminaries. It is the aim of this paper to prove a theorem giving necessary and sufficient conditions for a subgroup of a Π-solvable group to be an $\mathcal{H}$-convering subgroup, where $\mathcal{H}$ is a Π-homomorph. Some consequences of this theorem are also given.

All groups considered are finite. We shall denote by Π a set of primes, respectively Π′ the complement to Π in the set of all primes and $O_{\Pi'}(G)$ the largest normal Π′-subgroup of a group G.

The main notions used in the paper are given below.

DEFINITION 1.1. A group G is *Π-solvable* if every chief factor of G is either a solvable Π-group or a Π′-group. If Π is the set of all primes, we obtain the notion of solvable group.

DEFINITION 1.2. a) A class $\mathcal{H}$ of groups is a *homomorph* if $\mathcal{H}$ is closed under homomorphisms.

b) A class χ of groups is said to be *Π-closed* if:

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$$G/O_{\Pi'}(G) \in \chi \Rightarrow G \in \chi .$$

A Π-closed homomorph is called a *Π-homomorph*.

DEFINITION 1.3. Let χ be a class of groups, G a group and H a subgroup of G.

a) H is an *χ-covering subgroup* of G if : (i) $H \in \chi$; (ii) $H \le K \le G$, $K_0 \Delta K$, $K/K_0 \in \chi$ imply $K = HK_0$.

b) H is *χ-maximal* in G if : (i) $H \in \chi$; (ii) $H \le H^* \le G$, $H^* \in \chi$ imply $H = H^*$.

c) H is an *χ-projector* of G if, for any normal subgroup N of G, HN/N is *χ-maximal* in G/N.

DEFINITION 1.4. If G is a group, a subgroup H of G is a *stabilizer* of G if H is a maximal subgroup of G with $\mathrm{core}_G H = 1$, where

$$\mathrm{core}_G H = \cap \{ H^g \mid g \in G \}.$$

Finally, we give a BAER's theorem [1], which is used in the proof of the main result.

THEOREM 1.5. *A solvable minimal normal subgroup of a finite group is abelian.*

2. Covering subgroups. In [2] are studied some aspects of the connection between covering subgroups and projectors in finite groups and, particularly, in finite Π-solvable groups.

First, some results for finite groups.

THEOREM 2.1. ([2]) *If $\mathcal{H}$ is a homomorph and G is a group, the subgroup H of G is an $\mathcal{H}$-covering subgroup of G if and only if H is an $\mathcal{H}$-projector in any subgroup K of G with $H \subseteq K$.*

COROLLARY 2.2. *If $\mathcal{H}$ is a homomorph, G a group and H an*

$\mathcal{H}$-covering subgroup of G, then H is an $\mathcal{H}$-projector of G.

The converse of 2.2. is not true. But we have the following:

THEOREM 2.3. ([2]) *If $\mathcal{H}$ is a homomorph, G a group and H an $\mathcal{H}$-projector of G which is a maximal subgroup of G, then H is an $\mathcal{H}$-covering subgroup of G.*

Remark 2.4. For any class χ of groups, the following conditions on a group G are equivalent : (i) $G \in \chi$; (ii) G is an χ-covering subgroup of G; (iii) G is an χ-projector of G; (iv) G is χ-maximal in G.

From now on, we shall investigate the χ-covering subgroups of a group $G \notin \chi$, where χ is a Π-homomorph.

A first result deals with a converse of 2.3.

THEOREM 2.4. *Let χ be a Π-homomorph, G a Π-solvable group, $G \notin \mathcal{H}$ and H a subgroup of G such that there is a minimal normal subgroup N of G with $HN = G$. Then, if H is an $\mathcal{H}$-covering subgroup of G, it follows*: (i) *H is an $\mathcal{H}$-projector of G;* (ii) *H is maximal in G.*

Proof. (i) follows obviously by 2.2.

(ii) In order to show that H is maximal in G, we notice that $H \neq G$, because $H \in \mathcal{H}$ and $G \notin \mathcal{H}$. Let now H^* a subgroup of G such that $H \leq H^* < G$. We shall prove that $H = H^*$.

We first deduce that N is abelian. Indeed, N being a minimal normal subgroup of the Π-solvable group G, N is either a solvable Π-group or a Π'-group. Supposing that N is a Π'-group, we obtain that $N \leq O_{\Pi'}(G) \leq G$ and

$$G/O_{\Pi'}(G) = (G/N)/(O_{\Pi'}(G)/N), \qquad (1)$$

where

$$G/N = HN/N \simeq H/H\cap N.$$

From $H \in \mathcal{H}$ and $\mathcal{H}$ being a homomorph, $H/H \cap N \in \mathcal{H}$, hence $G/N \in \mathcal{H}$. Then (1) implies $G/0_{\Pi'}(G) \in \mathcal{H}$, which leads, by the closure of $\mathcal{H}$, to $G \in \mathcal{H}$, a contradiction. It follows that N is a solvable Π-group. Applying 1.5., N is abelian.

It is easy to see that $H^* \cap N$ is a normal subgroup of G. Indeed, let $g \in G$ and $x \in H^* \cap N$. Because $G = HN = H^*N$, we have $g = h_1n_1$, where $h_1 \in H^*$ and $n_1 \in N$. Then

$$g^{-1}xg = (h_1n_1)^{-1}x(h_1n_1) = (n_1^{-1}h_1^{-1})\,x\,(h_1n_1) = n_1^{-1}(h_1^{-1}xh_1)n.$$

But $H^*\cap N \triangle H^*$ implies $h_1^{-1}xh_1 \in H^*\cap N$, where $H^*\cap N \subseteq N$. We saw that N is abelian and so

$$g^{-1}xg = n_1^{-1}n(h_1^{-1}xh_1) = h_1^{-1}xh_1 \in H^*\cap N.$$

Thus $g^{-1}xg \in H^*\cap N$.

Furthermore, $H^* \cap N \neq N$, for if we suppose that $H^* \cap N = N$, it follows $N \leq H^*$, hence $G = HN = H^*N = H^*$, in contradiction with the choice of H^*. From $H^*\cap N \triangle G$, $H^*\cap N \subseteq N$ and $H^*\cap N \neq N$, we obtain, by the hypothesis that N is a minimal normal subgroup of G, that $H^* \cap N = 1$.

Let us prove now that $H = H^*$. Suppose $H < H^*$ and let $h^* \in H^*\setminus H$. Because $h^* \in G = HN$, we have $h^* = hn$, where $h \in H$ and $n \in N$. Then $n = h^{-1}h^*$ and so $n \in H^* \cap N = 1$. Hence $n = 1$ and $h^* = h \in H$, in contradiction with $h^* \in H^*\setminus H$. It follows $H = H^*$.■

The main result of this paper comes now from 2.3. and 2.4. and gives the following characterization of covering subgroups in finite Π-solvable groups:

THEOREM 2.5. *Let $\mathcal{H}$ be a Π-homomorph, G a Π-solvable group,*

$G \notin \mathcal{H}$ and H a subgroup of G such that there is a minimal normal subgroup N of G with $HN = G$. Then the following two conditions are equivalent:

(i) *H is an $\mathcal{H}$-covering subgroup of G;*

(ii) *H is both an $\mathcal{H}$-projector of G and maximal in G.*

Finally, from 2.5. follows an interesting consequence we give below.

THEOREM 2.6. *If $\mathcal{H}$ is a Π-homomotph, G a Π-solvable group and H a stabilizer of G, then the following two conditions are equivalent:*

(a) *H is an $\mathcal{H}$-covering subgroup of G;*

(b) *H is an $\mathcal{H}$-projector og G.*

The proof is based on a result given in [2]:

LEMMA 2.7. ([2], 4.5.) *Let $\mathcal{H}$ be a Π-homomorph, G a Π-solvable group and $H < G$ such that H is $\mathcal{H}$-maximal in G. Then, the following conditions are equivalent:*

(1) *for any minimal normal subgroup N of G, we have $HN = G$;*

(2) *H is a stabilizer of G.*

Proof of 2.6. (a) implies (b): obviously follows from 2.2.. (b) implies (a): We are in the hypothesis of 2.5.. Indeed, $G \notin \mathcal{H}$, for if we suppose $G \in \mathcal{H}$, we obtain, by (b), $H = G$, in contradiction with H stabilizer of G. Furthermore, 2.7. shows that, for any minimal normal subgroup N of G, $HN = G$. Since H is both an $\mathcal{H}$-projector of G and minimal in G, 2.5. leads to (a).■

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EXISTENCE OF FINITELY ADDITIVE NONTRIVIAL MEASURES ON THE POWERSET OF A SET

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REZUMAT. - Existenţa măsurilor finit-aditive netriviale pe mulţimea părţilor unei mulţimi. În lucrare se demonstrează existenţa a două tipuri de măsuri finit-aditive netriviale pe mulţimea părţilor unei mulţimi.

Abstract. We prove the existence of two kinds of finitely additive nontrivial measures on the powerset of a set; one which is atomic (two-valued) and the other which is atomless. A second proof of the latter using the Hahn-Banach theorem is also given.

By finitely additive nontrivial measure on the powerset $P(S)$ of a set S, which from now on is simply referred to as a *measure on S*, we mean [4, p.343] a function m from $P(S)$ into the closed unit interval [0,1] of real numbers such that

(i) $m(S) = 1$

(ii) $m(\{x\}) = 0$ *for every* $x \in S$

(iii) $m(A \cup B) = m(A) + m(B)$

for every $A,B \subseteq S$ *with* $A \cap B = \phi$

We would like to emphasize that unlike some classical examples of measures, here, we require that m be defined on *every* subset of S and not on some subsets of S.

As usual, we call the measure m on S *two-valued* [4, p.343] (or *atomic*) iff $m(A) = 1$ or $m(A) = 0$ for every $A \subseteq S$. The existence of an atomic measure on a set S is an immediate

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consequence [1] of the fact that the Boolean algebra $(P(S),\subseteq)$ has a (nonprincipal) ultrafilter U such that $\{x\} \notin U$ for every $x \in S$. Indeed, we have:

THEOREM 1. *Let S be an infinite set and let U be a nonprincipal ultrafilter of the Boolean algebra $(P(S),\subseteq)$. Then the function m from $P(S)$ into $[0,1]$ given by*

$$m(A) = 0 \qquad \text{if } A \notin U \tag{1}$$

$$m(A) = 1 \qquad \text{if } A \in U \tag{2}$$

is a two-valued measure on S.

Proof. We must show that (1) and (2) imply (i), (ii), and (iii). Since $S \in U$, we see that (2) implies (i). Clearly, (1) implies (ii) directly since U is a nonprincipal ultrafilter.

It remains to establish (iii). Since U is an ultrafilter of $(P(S),\subseteq)$, we see that for every $A,B \subseteq S$ with $A \cap B = \phi$, it must be the case that either A and B are elements of the prime ideal $P(S) - U$ or else exactly one of A or B is an element of U (because no two elements of U can be disjoint). Thus in either case, (1) and (2) imply (iii), as desired.

Next, we prove the existence of an *atomless* measure on a set S. By an *atomless* measure m on S we mean [5, p.296] a function m from the powerset $P(S)$ of S into the closed unit interval $[0,1]$ of real numbers such that m (besides (i), (ii) and (iii)) has the property that

(iv) every subset of S of positive measure can be split into two disjoint sets of positive measure.

First we prove the existence of an atomless measure m on the countable infinite set ω of all natural numbers $0,1,2,\ldots$

Our proof is based on the notion of *U-limit* of a sequence of real numbers where U is a nonprincipal ultrafilter on ω i.e., U contains no finite subset of ω as an element.

Let $(s_n)_{n\in\omega}$ be a sequence of real numbers. We define [2] the *U-limit* of $(s_n)_{n\in\omega}$ denoted by U-lim s_n, as the unique real number r (whenever it exists) such that

$$\{n|\ r - \varepsilon < s_n < r + \varepsilon\} \in U \text{ for every } \varepsilon > 0. \quad (3)$$

We remark [2] that, with respect to a given nonprincipal ultrafilter U, it is well known that a bounded sequence of real numbers has a (unique) U-limit and that the U-limit of the sum of two sequences is the sum of their U-limits.

LEMMA 1. *There exists an atomless measure on* ω.

Proof. Let U be a nonprincipal ultrafilter on ω. Let m be a mapping from $P(\omega)$ into $[0,1]$ given by

$$m(A) = U\text{-lim}\,\frac{Card\,(A\cap I(n+1))}{n+1} \quad \text{with } n\in\omega \qquad (4)$$

where as usual, $I(n+1)$ is the initial segment of ω determined by $n + 1$, i.e., $I(n+1) = \{0,1, \ldots, n\}$. Clearly the sequence appearing in (4) is a bounded sequence of real numbers.

We show that m satisfies (i), (ii), (iii), (iv).

Since

$$s_n = \frac{Card\,(\omega\cap I(n+1))}{n+1} = \frac{Card\,(I(n+1))}{n+1} = 1,$$

we have for every $\varepsilon > 0$

$$\{n|\ 1 - \varepsilon < s_n < 1 + \varepsilon\} = \omega \in U.$$

Thus by (3) we see that m satisfies (i).

Again, for every $k \in \omega$,

$$s_n = \frac{Card\ (\{k\} \cap I(n+1))}{n+1} = \frac{1}{n+1}, \quad \text{for } n \geq k.$$

Hence, for every $\varepsilon > 0$ we have

$$N = \{n \mid 0 - \varepsilon < s_n < 0 + \varepsilon\} \in U$$

since N contains all but finitely many elements of ω. Thus $m(\{k\}) = 0$ and hence m satisfies (ii).

Now, let $A, B \subseteq \omega$ with $A \cap B = \phi$. Obviously,

$$\frac{Card\,(A \cap I(n+1))}{n+1} + \frac{Card\,(B \cap I(n+1))}{n+1} = \frac{Card\,((A \cup B) \cap I(n+1))}{n+1}.$$

Therefore, by the previous remark and by (4).

$$m(A + B) = m(A) + m(B).$$

Thus m also satisfies (iii).

Finally let $E \subseteq \omega$ be such that $m(E) > 0$.

Hence

$$m(E) = U\text{-}\lim \frac{(Card\,E \cap I(n+1))}{n+1} > 0. \tag{5}$$

Clearly, E is an infinite subset of ω, say,

$$E = \{e_0, e_1, e_2, e_3, \ldots\} \text{ with } e_0 < e_1 < e_2 < \ldots$$

Thus

$$E = A \cup B \text{ with } A \cap B = \phi \tag{6}$$

where

$$A = \{e_0, e_2, e_4, \ldots\} \text{ and } B = \{e_1, e_3, e_5, \ldots\}. \tag{7}$$

But then by (5) and (7), it follows that

$$m(A) = U\text{-}\lim \frac{(Card\,A \cap I(n+1))}{n+1} = \frac{1}{2}\ \frac{U\text{-}\lim\,(E \cap I(n+1))}{n+1} > 0. \tag{8}$$

Also, from (iii), (6), and (8), it follows that

$$m(B) = \frac{1}{2} m(E) > 0. \tag{9}$$

Thus, (5), (8), (9) imply that every subset E of positive measure of ω can be split into two disjoint sets of positive measure. Thus, m satisfies (iv), as desired.

THEOREM 2. *Let S be an infinite set. There exists an atomless measure on S.*

Proof. Let H be an infinite countable subset of S. Since Card (H) = Card (ω), we can apply Lemma 2 to obtain an atomless measure m on H and we define a function m_1 from $P(S)$ into $[0,1]$ given by

$$m_1(A) = m(A \cap H) \text{ for every } A \subset S.$$

It is trivial to verify that m_1 is an atomless measure on S.

Next we prove the existence of a measure on a countable set using the Hahn-Banach theorem [3].

In what follows, let Q be the (countable) set of all rational numbers in the closed unit interval $[0,1]$ of real numbers. Let V be the vector space of all bounded functions from Q into R. Let E be the subspace of V consisting of all measurable functions with respect to the length measure s defined on algebra of all finite unions of intervals of Q.

Based on the above definitions, we prove:

THEOREM 3. *The Hahn-Banach theorem implies the existence of a measure m on Q where in addition to* (i), (ii), (iii), *m also satisfies* (iv).

Proof. Let p be the function from V into R defined by $p(f) = \sup f$. Obviously, p is a sublinear functional. Let h be a function from E into R defined by $h(f) = \int_Q f\, ds$.

Clearly, h is a linear functional defined on the subspace E of V such that $h(f) \leq p(f)$ for every $f \in V$. By the Hahn-Banach theorem there exists a linear functional $\bar{h}$ defined on the entire V such that $\bar{h}$ is an extension of h and $\bar{h}(f) \leq p(f)$ for every $f \in V$. For every $A \subseteq Q$, we let Ch(A) denote the characteristic function of A. Obviously, Ch$(A) \in V$ for every $A \subseteq Q$. Finally, we define the function m from $P(Q)$ into R given by $m(A) = \bar{h}(\mathrm{Ch}(A))$ for every $A \subseteq Q$.

We show that m is in fact a function from $P(Q)$ into $[0,1]$. Indeed,

$$m(A) = \bar{h}(\mathrm{Ch}(A)) \leq p(\mathrm{Ch}(A)) = \sup(\mathrm{Ch}(A)) \leq 1 \qquad (10)$$

for every $A \subseteq Q$.

On the other hand.

$$\begin{aligned} m(A) = \bar{h}(\mathrm{Ch}(A)) = -(\bar{h}(-\mathrm{Ch}(A)) \geq -p(-\mathrm{Ch}(A)) &= \\ = -\sup(-\mathrm{Ch}(A)) &= 0 \end{aligned} \qquad (11)$$

for every $A \subseteq Q$.

Thus, from (10) and (11) it follows that m is a function from $P(Q)$ into $[0,1]$.

Clearly, $m(Q) = \int_Q \mathrm{Ch}(Q)\, ds = 1$.

Also, for $x \in Q$, we see that Ch$(\{x\})$ is an s-measurable function and therefore $m(\{x\}) = \int_Q \mathrm{Ch}(\{x\})\, ds = 0$.Thus, m satisfies (i) and (ii).

Furthermore, if $A,B \subseteq Q$ with $A \cap B = \phi$, then

$$m(A \cup B) = \bar{h}(\mathrm{Ch}(A \cup B)) = \bar{h}(\mathrm{Ch}(A)) + \bar{h}(\mathrm{Ch}(B)) = m(A) + m(B).$$

Thus m satisfies (iii).

To show that m satisfies (iv), let A be a subset of Q with $m(A) > 0$. We define the function f from [0,1] into [0,1], given by $f(x) = m(A \cap [0,x])$. Clearly,

$$|f(x) - f(y)| = |m(A \cap [0,x]) - m(A \cap [0,y])| \le |x-y|$$

Thus, f is continuous on [0,1]. Since $f(0) = 0$ anf $f(1) = m(A) > 0$, we conclude that for some $q \in [0,1]$ it is the case that $f(q) = m(A \cap [0,q]) = \frac{1}{2} m(A)$. Hence, given any subset A of Q having positive measure, we can always find a real number q such that

$$m(A) = m(A \cap [0,q]) + m(A \cap [q,1])$$

where the two terms in the above sum are positive. Therefore, m also satisfies (iv).

From Theorem 3, we derive the existence of an atomless measure on any infinite set just as we derived the conclusion of Theorem 2 from Lemma 1.

Based on Theorem 1 and Theorem 2 we have

THEOREM 4. *On any infinite set there exists a measure which is neither atomic nor atomless.*

Proof. It is enough to prove that on the set of all natural numbers ω there exists a measure which is neither atomic nor atomless.

By Theorem 1, clearly there exists an atomic measure m_1 on the set E of all even natural numbers such that $m_1(E) = 0.5$. Similarly, by Theorem 2 there exists an atomless measure m_2 on the set D of all odd natural numbers such that $m_2(D) = 0.5$. If can be readily verified that the function m from $P(\omega)$ into [0,1] given by $m(S) = m_1(S \cap E) + m_2(S \cap D)$ for every $S \subseteq \omega$ is a measure on ω which is neither atomic nor atomless.

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TWO SIMPLE SUFFICIENT CONDITIONS FOR CONVEXITY*

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REZUMAT. - Două condiţii simple de convexitate. Fie A_n clasa funcţiilor $f(z)=z+a_{n+1}z^{n+1}+\dots$, $n\geq 1$, care sunt analitice în discul unitate U. Se notează cu K subclasa lui $A=A_1$ formată din funcţiile convexe în U. Se determină numerele pozitive α_n şi M_n astfel încât dacă $f\in A_n$ şi $\mathrm{Re}[zf''(z)]>-\alpha_n$ sau $|f''(z)|<M_n$, atunci $I(f)\in K$, unde $I(f)$ este definită de (1).

1. Introduction. Let A_n denote the class of function $f(z) = z + a_{n+1}z^{n+1}+\dots$, $n \geq 1$, that are analytic in the unit disc $U = \{z \in \mathbf{C} : |z| < 1\}$ and let $A = A_1$.

Let f be analytic in the unit disc U. Then the function f with $f(0) = 0$ and $f'(0) \neq 0$ is said to be starlike (univalent) if it satisfies $\mathrm{Re}[zf'(z)/f(z)] > 0$ in U. The function f with $f'(0) \neq 0$ is said to be convex (univalent) if it satisfies $\mathrm{Re}[1 + zf''(z)/f'(z)] > 0$ in U. Further, we denote by S^* and K the subclasses of A consisting of functions f which are starlike and convex in U, respectively.

In this article we obtain two simple sufficient conditions for convexity, which are expressed only by means of the second derivative of a function $f \in A_n$. Actually we determine two positive numbers α_n and M_n, such that if $f \in A_n$ and $\mathrm{Re}[zf''(z)] > -\alpha_n$, or $|f''(z)| < M_n$ then $I(f) \in K$, where $I: A_n \to A_n$ is the integral operator defined by

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$$I(f)(z) = \frac{c+1}{z^c}\int_0^z f(t)t^{c-1}dt, \quad c > -1. \tag{1}$$

2. Preliminaries. Let F and G be two analytic functions in U. If G is univalent, then we say that F is subordinate to G, written $F \prec G$ or $F(z) \prec G(z)$, if $F(0) = G(0)$ and $F(U) \subset G(U)$.

We will need the following lemmas to prove our main rezults.

LEMMA 1 [6]. *Let h be starlike in U and let $p(z) = 1 + p_n z^n + \ldots$ be analytic in U. If $zp'(z) \prec h(z)$, then $p \prec q$, where*

$$q(z) = 1 + \frac{1}{n}\int_0^z \frac{h(t)}{t}dt$$

and q is convex.

LEMMA 2 [3]. *Let c be a real number, $c > -1$, let q be convex in U, with $q(o) = 1$ and let $P(z) = 1 + P_n z^n + \ldots$ be analytic in U.*

If

$$P(z) + \frac{1}{c+1}zP'(z) \prec q(z),$$

then $P \prec Q$, where

$$Q(z) = \frac{c+1}{nz^{(c+1)/n}}\int_0^z q(t)\cdot t^{(c+1)/n-1}dt$$

and Q is convex.

LEMMA 3 [2]. *Let Ω be a set in the complex plane $\mathbf{C}$ and let n be a positive integer. Suppose that the function ψ: $\mathbf{C}^2 \times U \to \mathbf{C}$ satisfies the condition*

$$\psi(is, t; z) \notin \Omega$$

for all $s, t \le -n/2\ (1 + s^2)$ and $z \in U$.

If the function $p(z) = 1 + p_n z^n + \dots$ *is analytic in* U *and*

$$\psi(p(z),\ zp'(z);\ z) \in \Omega$$

for $z \in U$, *then* $\operatorname{Re} p(z) > 0$ *in* U.

3. Main results.

THEOREM 1. *Let* n *be a positive integer and let* c *be a real number,* $-1 < c \le 1$. *Let*

$$\alpha_n = \alpha_n(c) = \frac{n+2}{C_n(c)}, \tag{2}$$

where

$$C_n(c) = 2\left[c+1+\frac{n+2}{n}\ln 2-\frac{n+2c^2}{n}\int_0^1 \frac{t^{(c+1)/n}}{1+t}\,dt\right]. \tag{3}$$

If $f \in A_n$ *and*

$$\operatorname{Re}[zf''(z)] > -\alpha_n, \tag{4}$$

then $I(f) \in K$, *where* I *is the integral operator defined by* (1).

Proof. Let $f \in A_n$ and let $0 \le \alpha \le \alpha_n$. The inequality $\operatorname{Re}[zf''(z)] > -\alpha$ is equivalent to

$$zf''(z) \prec -\frac{2\alpha z}{1+z} = h(z).$$

Since h is starlike and $f'(z) = 1 + p_n z^n + \dots$, by Lemma 1, we deduce $f' \prec q$, where

$$q(z) = 1+\frac{1}{n}\int_0^z \frac{h(t)}{t}\,dt = 1-\frac{2\alpha}{n}\log(1+z).$$

Since q is convex, we have

$$\operatorname{Re} f'(z) > \beta, \tag{5}$$

where

$$\beta = \beta(\alpha) = 1 - \frac{2\alpha}{n} \ln 2. \tag{6}$$

If we let $P(z) = F'(z)$, where $F = I(f)$ and I is the integral operator defined by (1), then we have

$$P(z) + \frac{1}{c+1} zP'(z) = f'(z) \prec q(z),$$

and by Lemma 2 we deduce $P \prec Q$, where

$$Q(z) = \frac{c+1}{nz^{(c+1)/n}} \int_0^z q(t)\, t^{(c+1)/n-1} dt = 1 - \frac{2\alpha}{n} \log(1+z) + \\ + \frac{2\alpha}{n} \int_0^z \frac{t^{(c+1)/n}}{1+t} dt.$$

Since Q is convex, we have

$$Re\, P(z) > \gamma, \tag{7}$$

$$\gamma = \gamma(\alpha, c) = Q(1) = 1 - \frac{2\alpha}{n} \ln 2 + \frac{2\alpha}{n} \int_0^1 \frac{t^{(c+1)/n}}{1+t} dt. \tag{8}$$

On the other hand, is easy to check the inequality $\frac{2\alpha_n}{n}\left[\int_0^1 \frac{t^{(c+1)/n}}{1+t} dt - 1\right] \le 1$, where α_n is defined by (2) and (3). Then this inequality implies that $\gamma \ge 0$ and Re $F'(z) > 0$, hence the univalence of F.

Let $p(z) = 1 + \frac{zF''(z)}{F'(z)}$, then p is analytic in U and $p(z) = 1 + p_n z^n + \ldots$

A simple calculation yields

$$P(z)[zp'(z) + p^2(z)] = (c+1) zf''(z) + (1-c^2) f'(z) + c^2 P(z).$$

By using (4), (5), (7) we obtain

$$Re\{P(z)[zp'(z) + p^2(z)]\} > -(c+1)\alpha + (1-c^2)\beta + c^2\gamma. \tag{9}$$

In order to show that (10) implies Re $p(z) > 0$, according to Lemma 3, we have to chech the inequality

$$Re\{P(z)(t-s^2)\} \le -(c+1)\alpha + (1-c^2)\beta + c^2\gamma, \qquad (10)$$

for all $s, t \le -\frac{n}{2}(1+s^2)$ and $z \in U$.

By using (7), we deduce

$$(t-s^2)\,Re\,P(z) \le -\frac{n}{2}\,Re\,P(z) \le -\frac{n}{2}\gamma.$$

Hence the inequality (10) holds if

$$(2c^2+n)\gamma + 2(1-c^2)\beta - 2(c+1)\alpha \ge 0, \qquad (11)$$

where β and γ are defined by (6) and (8).

It is easy to show that (11) holds if $\alpha \le \alpha_n$ and by applying Lemma 3, from (9) we deduce Re $p(z) > 0$, which shows that $F \in K$.

If we set $c = 0$, then from Theorem 1, we obtain

COROLLARY 1.1. *Let n be a positive integer and let* $\delta_n = \frac{n+2}{C_n}$ *where*

$$C_n = 2\left[1 + \frac{n+2}{n}\ln 2 - \int_0^1 \frac{t^{1/n}}{1+t}dt\right].$$

If $f \in A_n$ *and*

Re $[zf''(z)] > -\delta_n$,

then $f \in S^*$.

This result was recently obtained by P.T.Mocanu [4].

If we set $c = 1$ and $n = 1$, then from Theorem 1 we obtain

COROLLARY 1.2 *If* $f \in A$ *and*

Re $[zf''(z)] > -3/7 = -0.428...$,

then $I(f) \in K$, *where I is the integral operator defined by*

$$I(f)(z) = \frac{2}{z}\int_0^z f(t)\,dt. \tag{12}$$

If we set $c = 1$ and $n = 2$ in Theorem 1, then we deduce

COROLLARY 1.3. *If $f \in A_2$ and*

$$\text{Re}\,[zf''(z)] > -\frac{3}{8\ln 2} = -0,541\ldots, \text{ for } z \in U,$$

then $I(f) \in K$, where I is the integral operator defined by (12).

THEOREM 2. *Let n be a positive integer and let c be a real number, such that $2c + n > 0$ and*

$$4c^4 + 4c^3n + 2c^2n(2+n) - 4cn - n^2 > 0. \tag{13}$$

If $f \in A_n$ and

$$|f''(z)| \le M_n, \text{ for } z \in U, \tag{14}$$

where

$$M_n = \frac{n(n+2c)}{n(2c+1)+2c}, \tag{15}$$

then $I(f) \in K$, where I is the integral operator defined by (1).

Proof. Let $f \in A_n$ and let $M \le M_n$. The inequality $|f''(z)| \le M$ is equivalent to $zf''(z) \prec Mz$ and by Lemma 1 we deduce $f' \prec q$, where $q(z) = 1 + \frac{M}{n}z$, which implies

$$|f'(z)| > 1 - \frac{M}{n}. \tag{16}$$

If we let $P(z) = F'(z)$, where $F = I(f)$ and I is the integral operator defined by (1), then we have

$$P(z) + \frac{1}{c+1}zP'(z) = f'(z) \prec q(z)$$

and by Lemma 2 we deduce $P \prec Q$, where

$$Q(z) = \frac{c+1}{nz^{(c+1)/n}} \int_0^z q(t) \cdot t^{(c+1)/n-1} dt = 1 + \frac{M}{n} \cdot \frac{c+1}{c+n+1} z,$$

which implies

$$|F'(z) - 1| < \frac{M}{n} \cdot \frac{c+1}{c+n+1}. \tag{17}$$

On the other hand it is easy to check the inequality $M_n \le \frac{n(c+n+1)}{c+1}$, where M_n is defined by (15), and by (17) we deduce $|F'(z) - 1| < 1$, which implies the univalence of F.

If we set $p(z) = 1 + \frac{zF''(z)}{F'(z)}$, a simple calculation yields

$$F'(z)\left[\frac{zp'(z)}{p(z)+c} + p(z) - 1\right] = zf''(z).$$

By using (14) we deduce

$$\left|F'(z)\left[\frac{zp'(z)}{p(z)+c} + p(z) - 1\right]\right| < M. \tag{18}$$

In order to show that (18) implies Re $p(z) > 0$, according to Lemma 3, we have to check the inequality

$$\left|F'(z)\left[\frac{t}{c+is} + is - 1\right]\right| \ge M,$$

for all $s, t \le -\frac{n}{2}(1+s^2)$ *and* $z \in U$.

Since

$$\left|\frac{t}{c+is} + is - 1\right|^2 = \frac{(t-c)^2 - 2ts^2 + s^2(1+c^2+s^2)}{c^2+s^2} \ge$$

$$\ge \frac{s^4\left(\frac{n}{2}+1\right)^2 + s^2\left(\frac{n^2}{2} + nc + n + 1 + c^2\right) + \left(\frac{n}{2}+c\right)^2}{c^2+s^2} = g(s)$$

and by a simple calculation we obtain

$$g'(s) = \frac{s}{2(c^2+s^2)^2}[s^4(n+2)^2+2s^2c^2(n+2)^2+4c^4+4c^3n+2c^2n(2+n)-4cn-n^2].$$

From (13) we deduce that g has the minimum value

$$g(0) = \frac{\left[\frac{n}{2}+c\right]^2}{c^2}.$$

Hence the inequality (18) holds if

$$\left[\left(\frac{n}{2}+c\right)/c\right]\cdot\left[1-\frac{M}{n}\right] \geq M. \tag{19}$$

It is easy to show that (19) holds if $M \leq M_n$, where M_n is defined by (15) and by Lemma 3, from (18) we obtain Re $p(z) > 0$, which shows $F \in K$.

It we set $c = 1$ and $n = 1$, then from Theorem 2 we deduce

COROLLARY 2.1. *If $f \in A$ and*

$$|f''(z)| \leq \frac{3}{5} = 0.6, \quad for \quad z \in U,$$

then $I(f) \in K$, where I is the integral operator defined by (12).

If we set $c = 1$ and $n = 2$, then from Theorem 2, we deduce

COROLLARY 2.2. *If $f \in A_2$ and*

$|f''(z)| \leq 1$, for $z \in U$, then $I(f) \in K$, where I is the integral operator defined by (12).

THEOREM 3. *Let n be a positive integer and let c be a real number such that*

$$4c^4+4c^3n+2c^2n(n+2)-4cn-n^2 \leq 0. \tag{19}$$

If $f \in A_n$ and

$|f''(z)| \leq M_n$, for $z \in U$,

where

$$M_n = \frac{n\sqrt{A(c,n)}}{n+\sqrt{A(c,n)}},$$

$$A(c,n) = \frac{(n+2)\sqrt{\Delta}}{2} + \frac{n^2(1-c^2)+2n(1+c-c^2)+2-2c^2}{2}$$

and

$$\Delta = c^4(n^2+4n) - 4c^3n - 2c^2n(n+2) + 4cn + n^2,$$

then $I(f) \in K$, *where* I *is the integral operator defined by* (1).

Proof. The proof of this theorem is similar to Theorem 2, but in this case the minimum value of $g(s)$ is given by

$$g(s_0) = \frac{(n+2)\sqrt{\Delta}}{2} + \frac{n^2(1-c^2)+2n(1+c-c^2)+2-2c^2}{2} = A(c,n),$$

where

$$s_0 = \sqrt{\frac{\sqrt{\Delta}}{n+2} - c^2}.$$

If we set $c = 0$ in Theorem 3, then we deduce

COROLLARY 3.1. *Let* n *be a positive integer and let*

$$M_n = \frac{n(n+1)}{2n+1}.$$

If $f \in A_n$ *and*

$|f''(z)| \le M_n$, *for* $z \in U$,

then $f \in S^*$.

This result was recently obtained by P.T.Mocanu [4].

4. Examples. The following simple examples point out the usefulness of the above convexity criteria.

Example 1. Let $f(z) = z + \int_0^z \int_0^t \left(\frac{a}{e^{as}-1} - \frac{1}{s} \right) ds\, dt,$ where $0 < a \le 1$.

Since $zf''(z) = \frac{az}{e^{az}-1} - 1$ and the function $h: U \to \mathbf{C}$ $h(\omega) = \frac{\omega}{e^{\omega}-1}$ is convex in U [5], then $Re\,[zf''(z)] > h(1) - 1 = \frac{a}{e^{a}-1} > -\frac{3}{7}$ for $0 < a \le 1$.

Because $f \in A$, from Corollary 1.2 we deduce that the function F is convex, where

$$F(z) = \frac{2}{z}\int_0^z f(t)\, dt = z + \frac{2}{z}\int_0^z \int_0^u \int_0^t \left(\frac{a}{e^{as}-1} - \frac{1}{s} \right) ds\, dt\, du.$$

Example 2. Let $f(z) = z + 2a\int_0^z \int_0^t \frac{ds\, dt}{1+\sqrt{1+at^2}}$, where $0 < a \le \frac{93}{196} = 0.47\ldots$

In this case $zf''(z) = \frac{2}{z}\left(\sqrt{1+az^2} - 1\right)$.

Let the function $h(z) = \frac{2}{z}\left(\sqrt{1+az^2}-1\right) = \frac{2}{z}\int_0^z \frac{at}{\sqrt{1+at^2}}dt$ and let $g(z) = \frac{az}{\sqrt{1+az^2}}$, $z \in U$.

In [3] it is showed that $\mathrm{Re}\left(1 + \frac{\omega g''(\omega)}{g'(\omega)}\right) > -\frac{1}{2}$ for $\omega \in U$ and is well known that if g satisfies the above condition, then h is convex in U.

Since $f \in A$ and $Re[zf''(z)] = Re\left\{\frac{2}{z}\left(\sqrt{1+az^2} - 1\right)\right\} > h(-1) = -\frac{2a}{1+\sqrt{1+a}} \ge -\frac{3}{7}$ for $0 < a \le \frac{93}{196} = 0.47\ldots,$ then from Corollary 1.2 we deduce that the function F is convex in U, where

$$F(z) = \frac{2}{z}\int_0^z F(t)\, dt = z + 4a\int_0^z \int_0^t \int_0^s \frac{du\, ds\, dt}{1+\sqrt{1+au^2}}.$$

Example 3. Let $f(z) = a \sin z + (1-a)z$, where $a \in \mathbf{C}$, $|a| \le \frac{1}{sh1} = 0.85\ldots$

Since $f''(z) = -a \sin z$ and because $f \in A_2$, we deduce

$$|f''(z)| = |a| \cdot |\sin z| < |a| \cdot \mathrm{sh}1 \le 1, \text{ for } |a| \le \frac{1}{sh1} = 0.85\ldots$$

From Corollary 2.2 we deduce that the function F is convex in U, where

$$F(z) = \frac{2}{z}\int_0^z f(t)\,dt = (1-a)z + \frac{2a}{z}(1-\cos z).$$

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BOUNDS ON ARGUMENT OF CERTAIN MEROMORPHIC DERIVATIVE IMPLYING STARLIKENESS

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REZUMAT. - Delimitări ale argumentului derivatei unor anumite funcţii meromorfe implicând stelaritatea. Fie Σ_k, $k \geq 0$, clasa funcţiilor f meromorfe în discul unitate, de forma $f(z) = 1/z + a_k z^k + \ldots$, $0 < |z| < 1$ şi fie $\Sigma^* \subset \Sigma_0$ subclasa funcţiilor stelate. Pentru $c > 0$ se consideră operatorul integral I_c definit de (1). Se determină numerele pozitive $\alpha = \alpha_k$, respectiv $\alpha = \alpha_k(c)$ astfel încât pentru $f \in \Sigma_k$ inegalitatea $|\arg[-z^2 f'(z)]| < \alpha\pi/2$ să implice $f \in \Sigma^*$, respectiv $I_c(f) \in \Sigma^*$.

1. Introduction. Let Σ_k be the class of meromorohic functions f in the unit disc $U = \{z \in \mathbf{C};\ |z| < 1\}$ of the form

$$f(z) = \frac{1}{z} + a_k z^k + \ldots,\ 0 < |z| < 1,\ k \geq 0.$$

A function $f \in \Sigma = \Sigma_0$ is called starlike if

$$\mathrm{Re}\left[-\frac{zf'(z)}{f(z)}\right] > 0, \quad \text{for} \quad z \in U.$$

Let denote by Σ^* the class of starlike functions.

For $c > 0$ let define the integral operator I_c: $\Sigma \to \Sigma$ by $F = I_c(f)$, where

$$F(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)\,dt. \tag{1}$$

In this paper we find $\alpha = \alpha_k$ and $\alpha = \alpha_k(c)$ such that

(i) $\quad f \in \Sigma_k$ and $|\arg[-z^2 f'(z)]| < \alpha\frac{\pi}{2} \Rightarrow f \in \Sigma^*$

and

(ii) $\quad f \in \Sigma_k$ and $|\arg[-z^2 f'(z)]| < \alpha\frac{\pi}{2} \Rightarrow F = I_c(f) \in \Sigma^*$

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respectively.

2. Preliminaries. We will need the following lemmas to prove our main results.

LEMMA 1. *Let n be a positive integer, let $\lambda > 0$ and let $\beta^* = \beta^*(\lambda, n)$ be the solution of the equation $\beta\pi = 3\pi/2 - \arctan(n\lambda\beta)$.*

Let

$$\alpha = \alpha(\beta, \lambda, n) = \beta + \frac{2}{\pi}\arctan(n\lambda\beta),$$

for $0 < \beta < \beta^*$. If $p(z) = 1 + p_n z^n + \dots$ is analytic in U, then

$$p(z) + \lambda z p'(z) \prec \left[\frac{1+z}{1-z}\right]^{\alpha} \Rightarrow p(z) \prec \left[\frac{1+z}{1-z}\right]^{\beta},$$

where $\prec$ denotes subordination.

This lemma was proved in [1, Theorem 5] in the case $n = 1$ and $\lambda = 1$.

LEMMA 2. *Let n be a positive integer, let $\lambda > 0$ and let $\beta^* = \beta^*(\lambda, n)$ be the solution of the equation $\beta\pi - 2\arctan(n\lambda\beta) = 0$.*

Let

$$\alpha = \alpha(\beta, \lambda, n) = \frac{2}{\pi}\arctan(n\lambda\beta) - \beta,$$

for $0 < \beta \le \beta^*$. If $p(z) = 1 + p_n z^n + \dots$ is analytic in U, then

$$p(z) - \lambda z p'(z) \prec \left[\frac{1+z}{1-z}\right]^{\alpha} \Rightarrow p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}.$$

The proof of this lemma is similar to that of Lemma 1.

3. Main results

THEOREM 1. *Let k be a positive integer and suppose that α and β are positive numbers that satisfy*

$$\alpha + \beta = \frac{2}{\pi}\arctan[(k+1)\beta] \tag{2}$$

If $f \in \Sigma_k$ *and*

$$|\arg[-z^2 f'(z)]| < \alpha\frac{\pi}{2},\quad z\in U, \tag{3}$$

then

$$|\arg[zf(z)]| < \beta\frac{\pi}{2},\quad z\in U. \tag{4}$$

Moreover if

$$\beta = \beta_k = \frac{1}{k+1}\sqrt{\frac{2(k+1)}{\pi} - 1}, \tag{5}$$

and

$$\alpha = \alpha_k = \frac{2}{\pi}\arctan\sqrt{\frac{2(k+1)}{\pi} - 1} - \beta_k, \tag{6}$$

then

$$|\arg[-z^2 f'(z)]| < \alpha_k\frac{\pi}{2} \Rightarrow |\arg[zf(z)]| < \beta_k\frac{\pi}{2}$$

Proof. If we let $p(z) = zf(z)$ and $n = k + 1$, then

$$-z^2 f'(z) = p(z) - zp'(z)$$

and the inequality (3) becomes

$$p(z) - zp'(z) \prec \left[\frac{1+z}{1-z}\right]^\alpha.$$

Hence by Lemma 2, with $\lambda = 1$ and $n \geq 2$, we deduce

$$p(z) \prec \left[\frac{1+z}{1-z}\right]^\beta,$$

where α and β satisfy (2).

If in (2) we consider α as a function of β, then

$$\alpha'(\beta) = \frac{2n}{\pi}\frac{1}{1+n^2\beta^2} - 1$$

Since $n \geq 2$ we deduce $\alpha'(0) = \frac{2n}{\pi} - 1 > 0$ and $\alpha'(\beta) = 0$ for $\beta = \beta_k$ given by (5). Hence the bigest value of α is given by (6).■

For $k = 1$, $k = 2$ and $k = 3$ we have

$\alpha_1 = 0.04527...$, $\beta_1 = 0.26136...$

$\alpha_2 = 0.16701\ldots, \ \beta_2 = 0.31795\ldots$

$\alpha_3 = 0.25795\ldots, \ \beta_3 = 0.31089\ldots$

and we deduce

$$f \in \Sigma_1 \text{ and } |\arg[-z^2 f'(z)]| < \alpha_1 \frac{\pi}{2} = 0.0711\ldots (4^\circ.07\ldots) \Rightarrow$$
$$|\arg[zf(z)]| < \beta_1 \frac{\pi}{2} = 0.4105\ldots (23^\circ.52\ldots)$$

$$f \in \Sigma_2 \text{ and } |\arg[-z^2 f'(z)]| < \alpha_2 \frac{\pi}{2} = 0.2623\ldots (15^\circ.03\ldots) \Rightarrow$$
$$|\arg[zf(z)]| < \beta_2 \frac{\pi}{2} = 0.4994\ldots (28^\circ.61\ldots)$$

$$f \in \Sigma_3 \text{ and } |\arg[-z^2 f'(z)]| < \alpha_3 \frac{\pi}{2} = 0.4051\ldots (23^\circ.215\ldots) \Rightarrow$$
$$|\arg[zf(z)]| < \beta_3 \frac{\pi}{2} = 0.4883\ldots (27^\circ.97\ldots)$$

THEOREM 2. *Let k be a positive integer and suppose that α and β are positive numbers that satisfy (2). If $f \in \Sigma_k$ and*

$$|\arg[-z^2 f'(z)]| < \alpha \frac{\pi}{2}, \ z \in U$$

then

$$\left|\arg\left[-\frac{zf'(z)}{f(z)}\right]\right| < \arctan[(k+1)\beta], \ z \in U,$$

hence $f \in \Sigma^*$.

Moreover, if $\alpha = \alpha_k$ is given by (5) and (6), then

$$|\arg[-z^2 f'(z)]| < \alpha_k \frac{\pi}{2} \Rightarrow \left|\arg\left[-\frac{zf'(z)}{f(z)}\right]\right| < \arctan\sqrt{\frac{2(k+1)}{\pi} - 1} \Rightarrow f \in \Sigma^*. \tag{7}$$

Proof. Applying Theorem 1, from (3) and (4) we deduce

$$\left|\arg\left[-\frac{zf'(z)}{f(z)}\right]\right| \le |\arg[-z^2 f'(z)]| + |\arg[zf(z)]| < \gamma \frac{\pi}{2},$$

where

$$\gamma = \alpha + \beta = \frac{2}{\pi} \arctan[(k+1)\beta].$$

If $\alpha = \alpha_k$ then we deduce (7). ∎

For $k = 1$, $k = 2$ and $k = 3$, we deduce

$$f \in \Sigma_1 \text{ and } |\arg[-z^2 f'(z)]| < 0.07\ldots(4^\circ.07\ldots) \Rightarrow$$

$$\left|\arg\left[-\frac{zf'(z)}{f(z)}\right]\right| < 0.481\ldots(27^\circ.59\ldots) \Rightarrow f \in \Sigma^*$$

$$f \in \Sigma_2 \text{ and } |\arg[-z^2 f'(z)]| < 0.262\ldots(15^\circ.03\ldots) \Rightarrow$$

$$\left|\arg\left[-\frac{zf'(z)}{f(z)}\right]\right| < 0.761\ldots(43^\circ.64\ldots) \Rightarrow f \in \Sigma^*$$

$$f \in \Sigma_3 \text{ and } |\arg[-z^2 f'(z)]| < 0.405\ldots(23^\circ,21\ldots) \Rightarrow$$

$$\left|\arg\left[-\frac{zf'(z)}{f(z)}\right]\right| < 0.893\ldots(51^\circ.19\ldots) \Rightarrow f \in \Sigma^*.$$

THEOREM 3. *Let k be a positive integer and let $c > 0$. Suppose that the positive numbers α, β and γ satisfy*

$$\alpha = \frac{2}{\pi} \arctan \frac{(k+1)\beta}{c} + \beta \tag{8}$$

and

$$\beta = \frac{2}{\pi} \arctan[(k+1)\gamma] - \gamma. \tag{9}$$

If $f \in \Sigma_k$ and

$$|\arg[-z^2 f'(z)]| < \alpha\frac{\pi}{2},\ z \in U, \tag{10}$$

then

$$\left|\arg\left[-\frac{zF'(z)}{F(z)}\right]\right| < \arctan[(k+1)\gamma],\ z \in U,$$

hence $F \in \Sigma^*$, where $F = I_c(f)$ is given by (2).

Moreover, if $\beta = \beta_k$ is given by

$$\beta_k = \frac{2}{\pi} \arccos\sqrt{\frac{\pi}{2(k+1)}} - \frac{1}{k+1}\sqrt{\frac{2(k+1)}{\pi} - 1} \tag{11}$$

and $\alpha = \alpha_k(c)$ is given by

$$\alpha_k(c) = \frac{2}{\pi} \arctan \frac{(k+1)\beta_k}{c} + \beta_k, \tag{12}$$

then

$$|\arg[-z^2f'(z)]| < \alpha_k\frac{\pi}{2} \Rightarrow \left|\arg\left[-\frac{zF'(z)}{f(z)}\right]\right| < \arccos\sqrt{\frac{\pi}{2(k+1)}} \Rightarrow \tag{13}$$
$$\Rightarrow F \in \Sigma^*.$$

Proof. From (2) we obtain

$$(c+1)F(z) + zF'(z) = cf(z)$$

and if we denote

$$p(z) = -z^2F'(z) = 1 + p_nz^n + \ldots, \quad z \in U,$$

where $n = k + 1$, then

$$zp'(z) + cp(z) = -cz^2f'(z)$$

and the inequality (10) becomes

$$p(z) + \frac{1}{c}zp'(z) \prec \left[\frac{1+z}{1-z}\right]^{\alpha}.$$

Using Lemma 1, we deduce

$$p(z) \prec \left[\frac{1+z}{1-z}\right]^{\beta},$$

i.e.

$$|\arg[-z^2F'(z)]| < \beta\frac{\pi}{2}, \tag{14}$$

where α and β satisfy

$$\alpha = \beta + \frac{2}{\pi}\arctan\frac{n\beta}{c}.$$

Now, by Theorem 1, the inequality (14) implies

$$|\arg zF(z)| < \gamma\frac{\pi}{2},$$

where

$$\beta + \gamma = \frac{2}{\pi}\arctan(n\gamma)$$

and we deduce

$$\left|\arg\left[-\frac{zF'(z)}{F(z)}\right]\right| \le |\arg[-z^2F'(z)]| + |\arg[zF(z)]| < \delta\frac{\pi}{2},$$

where

$$\delta = \beta + \gamma = \frac{2}{\pi} \arctan(n\gamma) .$$

Hence α and β satisfy (8) and (9).

Since α given by (8) is an increasing of β, the bigest value of α occurs when $\beta' = 0$.

Since

$$\beta = \delta - \frac{1}{n} \tan\left[\delta \frac{\pi}{2}\right],$$

we have

$$\beta' = 1 - \frac{\pi}{2n} \frac{1}{\cos^2\left[\delta \frac{\pi}{2}\right]}$$

$$\text{and } \beta' = 0, \text{ if } \cos\left[\delta \frac{\pi}{2}\right] = \sqrt{\frac{\pi}{2n}} .$$

In this case $\beta = \beta_k$ is given by (11) and $\alpha = \alpha_k(c)$ is given by (12) and we deduce (13).■

COROLLARY. *If k is a positive integer, $f \in \Sigma_k$ and $F = I_c(f)$ is given by* (2), *with*

$$0 < c \leq \frac{(k+1)\beta_k}{\tan\left[(1-\beta_k)\frac{\pi}{2}\right]}, \tag{15}$$

where β_k is defined by (11), *then*

$$\operatorname{Re}[-z^2 f'(z)] > 0 \ (z \in U) \Rightarrow F \in \Sigma^*.$$

Proof. From (12) and (15) we deduce $\alpha_k(c) \geq 1$ and the result follows from Theorem 3. ■

For $c = 1$ and $k = 2$ we obtain $\beta_2 = 0.11961...$, $\alpha_2(1) = 0.3389...$ and from Theorem 3 we deduce that if $f \in \Sigma_2$ and

$$|\arg[-z^2 f'(z)]| < \alpha_2 \frac{\pi}{2} = 0.5324...(30^\circ.50),$$

then

$$\left|\arg\left[-\frac{zF'(z)}{F(z)}\right]\right| < \arccos\sqrt{\frac{\pi}{6}} = 0.7617\ldots\,(43^\circ.64\ldots) \Rightarrow F \in \Sigma^*,$$

where

$$F(z) = \frac{1}{z^2}\int_0^z tf(t)\,dt.$$

For $c = 1/2$ and $k = 2$ we have $\alpha_2\left(\frac{1}{2}\right) = 0.5158\ldots$ and from Theorem 3 we deduce that if $f \in \Sigma_2$ and

$$|\arg[-z^2f'(z)]| < \alpha_2\left(\frac{1}{2}\right)\frac{\pi}{2} = 0.8103\ldots\,(46^\circ.43)$$

then

$$\left|\arg\left[-\frac{zF'(z)}{F(z)}\right]\right| < \arccos\sqrt{\frac{\pi}{6}} \Rightarrow F \in \Sigma^*,$$

where

$$F(z) = \frac{1}{2z^{3/2}}\int_0^z t^{1/2}f(t)\,dt.$$

For $k = 3$, $k = 4$ and $k = 5$, from our corollary we deduce the following particular results:

$$f \in \Sigma_3 \text{ and } \operatorname{Re}[-z^2f'(z)] > 0 \Rightarrow I_c(f) \in \Sigma^*, \text{ if } 0 < c \leq 0.442\ldots$$

$$f \in \Sigma_4 \text{ and } \operatorname{Re}[-z^2f'(z)] > 0 \Rightarrow I_c(f) \in \Sigma^*, \text{ if } 0 < c \leq 0.914\ldots$$

$$f \in \Sigma_5 \text{ and } \operatorname{Re}[-z^2f'(z)] > 0 \Rightarrow I_c(f) \in \Sigma^*, \text{ if } 0 < c \leq 1.532\ldots$$

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LIAPUNOV FUNCTIONS AND ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF DIFFERENTIAL - DIFFERENCE EQUATIONS

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REZUMAT. - Funcţii Liapunov şi comportarea asimptotică a soluţiilor unor ecuaţii diferenţiale. Bazat pe tehnica funcţiilor Liapunov, sunt date criterii pentru existenţa unor margini exponenţiale pentru matricea fundamentală de soluţii a ecuaţiei variaţionale.

Introduction. Estimation of fundamental matrix of solutions of variational system play important role both in characterization of the system and in the theory of nonlinear differential equations with perturbations (when the behaviour of the solutions of this equation is estimate by properties of solutions of unperturbed equation).

Recently Brauer [3-5], Hale [8], Marlin and Struble [9], Brauer and Strauss [5], Fennell and Proctor [7], Athanassov [2], Pachpatte [13], Fozi M.Dannan and S.Elaydi [6], Morchalo [10,11], Ventura [14] and possibly others have obtained results on qualitative behaviour of solutions of perturbed nonlinear differential systems using the nonlinear variation of constants formula of Alekseev [1].

In this paper, base on the Liapunov functions, are given criteria for existence of exponential bounds of fundamental matrix of variational equation. In this case Liapunov function will be defined on defference of two solutions.

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1. In discussing functional differential equations we will use the standard notation introduced by Hale [8].

Let R^n be a vector space of dimension n, with norm $\|\cdot\|$ and $h \geq 0$ a real number. We denote by $C(\langle a,b\rangle, R^n)$ the space of continuous functions with supremum norm. When $\langle a,b\rangle$ is $\langle -h,0\rangle$ we denote $C = C(\langle -h,0\rangle, R^n)$ and $\varphi \in C$, $\|\varphi\|^{(h)} = \sup_{-h\leq s\leq 0} \|\varphi(s)\|$. C_H will denote the set of $\varphi \in C$ for which $\|\varphi\| \leq H$, $C_\infty = \{\varphi \in C: \|\varphi\| < \infty\}$. If $t_0 \geq 0$ and $A > 0$ are real numbers, then for every $t \in \langle t_0, t_0+A\rangle$ and $x \in C(\langle t_0-h, t_0+A\rangle, R^n)$ we let $x_t \in C$ be defined by $x_t(s) = x(t+s)$, $-h \leq s \leq 0$. If $\Omega \subset R \times C$, $R = (-\infty, \infty)$ is open and if $f\colon \Omega \to R^n$ is continuous, then the relation

$$x_t'(0) = f(t, x_t) \tag{1}$$

is the functional differential equation where $x_t'(0)$ denote the right-hand derivative of the functional $x(u)$ at $t = u$.

DEFINITION. Let t_0 be any given number ≥ 0 and let $\varphi \in C$ be any given function. A function $x_t(t_0,\varphi)$ is said to be a solution of (1) with initial function φ at $t = t_0$ if there exists a number $A > 0$ such that

1^0 for each $t, t_0 \leq t \leq t_0+A$, $x_t(t_0,\varphi)$ is defined and $\in C$,

2^0 $x_{t_0}(t_0,\varphi) = \varphi$,

3^0 $x_t'(0) = f(t, x_t)$, $t_0 \leq t \leq t_0+A$.

Our main concerns is the system (1) where f is a nonlinear continuous function with a continuous first derivative with respect to $x_t \in C$.

For every solution $x_t(t_0,\varphi)$ of (1) we can define a nonautonomus linear functional differential equation

$$z_t'(0) = f_x(t, x_t(t_0, \varphi)) z_t \quad \text{for} \quad t \geq t_0 \geq 0 \tag{2}$$

with $z_{t_0}(t_0, \varphi) = \varphi$ which is called the linear variational equation of (1) with respect to the solution $x_t(t_0, \varphi)$.

Let the system (1) admits unique solution $x_t(t_0, \varphi)$ for $t \geq t_0$. Let $\phi(t, t_0, \varphi) = \frac{\partial}{\partial \varphi} x_t(t_0, \varphi)$. It is trivial that $\phi(t, s, \varphi)$ is a solution of (2) [14].

2. In this part we shall prove the theorem gives estimation of fundamental matrix $\phi(t, s, \varphi)$.

LEMMA [14]. *Assume that $x_t(t_0, \varphi_1)$ and $x_t(t_0, \varphi_2)$ are the solutions of (1) through (t_0, φ_1) and (t_0, φ_2) respectively, which exist for $t \geq t_0$ and such that φ_1, φ_2 belong to a convex subset D of R^n, then for all $t \geq t_0$*

$$x_t(t_0, \varphi_1) - x_t(t_0, \varphi_2) = \int_0^1 \phi(t, t_0, \xi(\lambda)) (x_{t_0}(t_0, \varphi_1) - x_{t_0}(t_0, \varphi_2)) \, d\lambda$$

where $\xi(\lambda) = \varphi_2 + \lambda(\varphi_1 - \varphi_2)$.

THEOREM 1. *Let there exist a function $\alpha(t)$, continuous possessing a continuous derivative for $t \geq t_0 \geq 0$, and a positive function $K(t)$, continuous for $t \geq t_0 \geq 0$, then the function $\phi(t, \tau, \psi)$ satisfies*

$$|\phi(t, \tau, \psi)| \leq K(\tau) \exp[\alpha(t) - \alpha(\tau)] \tag{3}$$

for all $t_0 \leq \tau \leq t < \infty$, $\psi \in C$ if and only if there exists a Liapunov function satisfying the following properties:

a) *$V(t, \varphi_1 - \varphi_2)$ is defined and continuous for $<0, \infty) \times C \times C$,*

b) $\|\varphi_1 - \varphi_2\|^{(h)} \leq V(t, \varphi_1 - \varphi_2) \leq K(t) \|\varphi_1 - \varphi_2\|^{(h)}$,

for $t \in <0, \infty)$, $\varphi_1, \varphi_2 \in C$,

c) $|V(t,r_1) - V(t,r_2)| \le K(t)\ \|r_1-r_2\|^{(h)}$ *for* $t\epsilon<0,\infty), r_1, r_2 \epsilon C$,

d) $V'_{(1)}(t,\varphi_1-\varphi_2) \le \alpha'(t)\, V(t,\varphi_1-\varphi_2)$ *for* $t\epsilon<0,\infty)$, $\varphi_1,\varphi_2 \epsilon C$,

where the derivative of V along the solutions of (1) will be denoted by $V'_{(1)}$ *and is defined to be*

$$V'_{(1)} = \overline{\lim_{h_1\to 0}} \frac{1}{h_1} [V(t+h_1, x_{t+h_1}(t_0,\varphi_1) - x_{t+h_1}(t_0,\varphi_2)) - \\ -V(t, x_t(t_0,\varphi_1) - x_t(t_0,\varphi_2))]$$

where $x_t(t_0,\varphi_1)$, $x_t(t_0,\varphi_2)$ *are solutions of (1) with* $x_{t_0}(t_0,\varphi_1) = \varphi_1$, $x_{t_0}(t_0,\varphi_2) = \varphi_2$.

We shall also sometimes write

$$V'_{(1)}(t,\varphi_1-\varphi_2) = \overline{\lim_{h_1\to 0}} \frac{1}{h_1} [V(t+h_1, x_{t+h_1}(t,\varphi_1) - x_{t+h_1}(t,\varphi_2)) - \\ -V(t,\varphi_1-\varphi_2)] \tag{4}$$

where it is understood in this notation that φ_1, φ_2 *denotes the solutions of (1) at time t.*

Proof. Let the condition (3) of Theorem is satisfied. Since

$$\sup_{\tau\ge 0} \|x_{t+\tau}(t,\varphi_1) - x_{t+\tau}(t,\varphi_2)\|^{(h)} \exp[-(\alpha(t+\tau) - \alpha(t))] \ge \\ \ge \|x_t(t,\varphi_1) - x_t(t,\varphi_2)\|^{(h)} = \|\varphi_1 - \varphi_2\|^{(h)}.$$

Hence and (3) we have

$$\|\varphi_1-\varphi_2\|^{(h)} \le \|x_{t+\tau}(t,\varphi_1) - x_{t+\tau}(t,\varphi_2)\|^{(h)} \exp[-(\alpha(t+\tau) - \alpha(t))] \le \\ \le K(t)\ \|\varphi_1-\varphi_2\|^{(h)}. \tag{5}$$

Define the function

$$V(t,\varphi_1-\varphi_2) = \\ = \sup_{\tau\ge 0} \|x_{t+\tau}(t,\varphi_1) - x_{t+\tau}(t,\varphi_2)\|^{(h)} \exp[-(\alpha(t+\tau) - \alpha(t))], \tag{6}$$

where $x_{t+\tau}(t,\varphi)$ is a solution of (1) for $t \geq t_0$, $\varphi \in C$.

Hence (5) b) is also satisfied.

From (5) and the uniqueness of solutions of (1) it follows that $V(t, \varphi_1-\varphi_2)$ is defined on $<0,\infty) \times C \times C$.

Let $r_1 = \varphi_1 - \varphi_2$, $r_2 = \overline{\varphi}_1 - \overline{\varphi}_2$. Then from (6) we have

$$|V(t,r_1) - V(t,r_2)| =$$
$$= \Big|\sup_{\tau\geq 0}\|x_{t+\tau}(t,\varphi_1) - x_{t+\tau}(t,\varphi_2)\|^{(h)}\exp[-(\alpha(t+\tau)-\alpha(t))] -$$
$$- \sup_{\tau\geq 0}\|x_{t+\tau}(t,\overline{\varphi}_1) - x_{t+\tau}(t,\overline{\varphi}_2)\|^{(h)}\exp[-(\alpha(t+\tau)-\alpha(t))]\Big| \leq$$
$$\leq \sup_{\tau\geq 0}\|x_{t+\tau}(t,\varphi_1) - x_{t+\tau}(t,\overline{\varphi}_2) - x_{t+\tau}(t,\overline{\varphi}_1) +$$
$$+ x_{t+\tau}(t,\overline{\varphi}_2)\|^{(h)}\exp[-(\alpha(t+\tau)-\alpha(t))] \leq$$
$$\leq K(t)\|r_1 - r_2\|^{(h)}.$$

Now we shall prove the continuity of $V(t,u)$. Take a $\delta \geq 0$ and $(t+\delta,\ \varphi_2)$, $(t,\varphi_1) \in <t_0,\infty) \times C$.

Then

$$|V(t+\delta,\varphi_2) - V(t,\varphi_1)| \leq |V(t+\delta,\varphi_2) - V(t+\delta,\varphi_1)| +$$
$$+ |V(t+\delta,\varphi_1) - V(t,\varphi_1)| \leq K(t+\delta)\ \|\varphi_2 - \varphi_1\|^{(h)} +$$
$$+ |V(t+\delta,\varphi_1) - V(t,\varphi_1)| \leq K(t+\delta)\ \|\varphi_2 - \varphi_1\|^{(h)} +$$
$$+ |V(t+\delta,\varphi_1 - 0) - V(t,\varphi_1 - 0)| = K(t+\delta)\ \|\varphi_2 - \varphi_1\|^{(h)} +$$
$$+ \Big|\sup_{\tau\geq 0}\|x_{t+\tau}(t,\varphi_1) - x_{t+\tau}(t,0)\|^{(h)}\exp[-(\alpha(t+\tau)-\alpha(t))] -$$
$$- \sup_{\tau\geq 0}\|x_{t+\tau}(t,\varphi_1) - x_{t+\tau}(t,0)\|^{(h)}\exp[-(\alpha(t+\tau)-\alpha(t))]\Big| \leq$$
$$\leq K(t+\delta)\ \|\varphi_2 - \varphi_1\|^{(h)} + \sup_{\tau\geq 0}\|x_{t+\tau}(t,\varphi_1) - x_{t+\tau}(t,0) - x_{t+\tau}(t,\varphi_1) +$$
$$+ x_{t+\tau}(t,0)\|^{(h)}\exp[-(\alpha(t+\tau)-\alpha(t))].$$

Hence

$$\lim_{\varphi_2 \to \varphi_1} V(t+\delta,\varphi_2) = V(t,\varphi_1).$$

Hence, the continuity of $V(t,u)$ is verified.

With the help of (4), (6) and the uniqueness solutions of (1) we have

$$V'_{(1)}(t,\varphi_1-\varphi_2)=\lim_{h_1\to 0}\frac{1}{h_1}[V(t+h_1,x_{t+h_1}(t,\varphi_1)-x_{t+h_1}(t,\varphi_2))-V(t,\varphi_1-\varphi_2)]=$$

$$=\overline{\lim_{h_1\to 0_+}}\frac{1}{h_1}[\sup\|x_{t+h_1+\tau}(t+h_1,x_{t+h_1}(t,\varphi_1))-x_{t+h_1+\tau}(t,\varphi_2))\|^{(h)}\cdot$$

$$\cdot\exp[-(\alpha(t+h_1+\tau)-\alpha(t+h_1))]-\sup_{\tau\ge 0}\|x_{t+\tau}(t,\varphi_1)-x_{t+\tau}(t,\varphi_2)\|^{(h)}\cdot$$

$$\cdot\exp[-(\alpha(t+\tau)-\alpha(t))]=$$

$$=\overline{\lim_{h_1\to 0_+}}\frac{1}{h_1}[\sup_{\tau\ge h_1}\|x_{t+\tau}(t,\varphi_1)-x_{t+\tau}(t,\varphi_2)\|^{(h)}\cdot\exp[-(\alpha(t+\tau)-\alpha(t+h_1))]-$$

$$-\sup_{\tau\ge 0}\|x_{t+\tau}(t,\varphi_1)-x_{t+\tau}(t,\varphi_2)\|^{(h)}\cdot\exp[-(\alpha(t+\tau)-\alpha(t))]\le$$

$$\le\overline{\lim_{h_1\to 0_+}}\frac{1}{h_1}[\sup_{\tau\ge 0}\|x_{t+\tau}(t,\varphi_1)-x_{t+\tau}(t,\varphi_2)\|^{(h)}\exp[-(\alpha(t+\tau)-\alpha(t+h_1))]-$$

$$-\sup_{\tau\ge 0}\|x_{t+\tau}(t,\varphi_1)-x_{t+\tau}(t,\varphi_2)\|^{(h)}\exp[-(\alpha(t+\tau)-\alpha(t))]=$$

$$=\overline{\lim_{h_1\to 0_+}}\frac{1}{h_1}[\sup_{\tau\ge 0}\|x_{t+\tau}(t,\varphi_1)-x_{t+\tau}(t,\varphi_2)\|^{(h)}\exp[-\alpha(t+\tau)-\alpha(t))]\cdot$$

$$\cdot(\exp[\alpha(t+h_1)-\alpha(t)]-1]=\alpha'(t)V(t,\varphi_1-\varphi_2).$$

Conversely: Let the function $V(t,\varphi_1-\varphi_2)$ satisfies the conditions a)-d) of Theorem 1. Then

$$\|x_t(t_0,\varphi_1)-x_t(t_0,\varphi_2)\|^{(h)}\le K(t_0)\|\varphi_1-\varphi_2\|^{(h)}\exp(\alpha(t)-\alpha(t_0))$$

for $t_0\le t\le t_0+\varepsilon$, $\varepsilon>0$.

Hence every solution $x_t(t_0,\varphi)$ is defined for all $t\ge t_0\ge 0$ and

$$\left\|\frac{\partial x_t(t_0,\varphi)}{\partial\varphi}\right\|\le K(t_0)\exp(\alpha(t)-\alpha(t_0))\quad\text{for all } t\ge t_0,\ \varphi\in C$$

or

$$\|\Phi(t,t_0,\varphi)\|\le K(t_0)\ \exp(\alpha(t)-\alpha(t_0))$$

which completes the proof of the Theorem.

Remarks.

1) Let $\alpha(t) = \int_{t_0}^{t} g(s)\,ds$, g is continuous and nonnegative for all $t \geq 0$, then

$$\|\phi(t,\tau,\psi)\| \leq K(\tau)\exp\left(\int_{\tau}^{t} g(s)\,ds\right)$$

for $t_0 \leq \tau \leq t < \infty$, $\psi \in C$,

$$V(t,\varphi_1-\varphi_2) = \sup_{\tau\geq 0} \|x_{t+\tau}(t,\varphi_1) - x_{t+\tau}(t,\varphi_2)\|^{(h)} \exp\left(-\int_{\tau}^{t+\tau} g(s)\,ds\right),$$

for $t_0 \leq \tau \leq t < \infty$, $\varphi_1,\varphi_2 \in C$.

2) Let $\alpha(t) = -\alpha t$, $\alpha > 0$, then

$$\|\phi(t,\tau,\psi)\| \leq K(\tau)\exp[-\alpha(t-\tau)].$$

3) Let $\alpha(t) = \beta t$, $\beta > 0$, then

$$\|\phi(t,\tau,\psi)\| \leq K(\tau)\exp[\beta(t-\tau)].$$

4) Let $\alpha(t) = -\beta(t)$, where $\beta(t)$ is continuous, nonnegative and possessing a continuous derivative for $t \geq t_0 \geq 0$, then

$$\|\phi(t,\tau,\varphi)\| \leq K(\tau)\exp[-(\beta(t)-\beta(\tau))].$$

From this we obtain Lemma II.1 8.

THEOREM 2. *Let the assumptions of Theorem 1 are satisfied where $\alpha(t) = -\alpha t$, $\alpha > 0$, then*

$$\int_{t_0}^{t} \|x_s(t_0,\varphi_1) - x_s(t_0,\varphi_2)\|^{(h)}\,ds < \infty \text{ for } t \geq t_0,$$

where $x_t(t_0,\varphi_1)$ and $x_t(t_0,\varphi_2)$ are solutions of (1).

Proof. By Theorem 1 there exists a function $V(t,\varphi_1-\varphi_2)$ having the four properties a)-d) from this Theorem. Define for $t \geq t_0$

$$\gamma(t) = V(t, x_t(t_0,\varphi_1) - x_t(t_0,\varphi_2)) + \alpha\int_{t_0}^{t} \|x_s(t_0,\varphi_1) - x_s(t_0,\varphi_2)\|^{(h)}\,ds. \quad (7)$$

Then

$$\gamma'(t) = V'_{(1)}(t, x_t(t_0,\varphi_1) - x_t(t_0,\varphi_2)) + \alpha \| x_t(t_0,\varphi_1) - x_t(t_0,\varphi_2) \|^{(h)} \le 0$$

for $t \ge t_0$, $x_t \in C$.

Thus $\gamma(t)$ is nonincreasing on $<t_0,\infty)$. Since $\gamma(t_0) = V(t_0,\varphi_1-\varphi_2)$, we see that $\gamma(t) \le V(t_0,\varphi_1-\varphi_2)$ for all $t \ge t_0$. Hence by (7) we obtain

$$\int_{t_0}^{\infty} \| x_t(t_0,\varphi_1) - x_t(t_0,\varphi_2) \|^{(h)} dt < \frac{1}{\alpha} V(t_0,\varphi_1-\varphi_2)$$

which proves our Theorem.

THEOREM 3. *If*

1^0 *assumptions of Theorem* 1 *are satisfied,*

2^0 $V'_{(1)}(t,\varphi_1-\varphi_2) \le -c(\|\varphi_1-\varphi_2\|^{(h)})^p$, $\varphi_1,\varphi_2 \in C$, $t \ge t_0$

for some $c > 0$, $p > 0$,

then

$$\int_{t_0}^{\infty} \| x_t(t_0,\varphi_1) - x_t(t_0,\varphi_2) \|^p dt < \infty.$$

Proof. Define the function

$$\gamma(t) = V(t, x_t(t_0,\varphi_1) - x_t(t_0,\varphi_2)) + \\ + c\int_{t_0}^{t} (\| x_s(t_0,\varphi_1) - x_s(t_0,\varphi_2) \|^{(h)})^p ds.$$

Thus, this theorem can be proved by following the proof of Theorem 2.

DEFINITION 1. The function $g \in \mathcal{H}$, if $g \in C(<0,r>, (0,\infty))$, $g(0) = 0$, $g(t)$ is strictly increasing with respect to t.

DEFINITION 2. The solution $x_t(t_0,\varphi)$ of the equation (1) is said to be globally uniformly stable in variation if there exists a constant $M>0$ such that

$$\| \Phi(t,t_0,\varphi) \| \le M \text{ for all } t \ge t_0,\ \varphi \in C.$$

DEFINITION 3. The solution $x_t(t_0,\varphi)$ of (1) is said to be generalized exponentially asymptotically stable if there a constant $N>0$ and a function $\alpha(t) \in \mathcal{H}$, $\alpha(t) \to \infty$ as $t \to \infty$ such that

$$\|\Phi(t,t_0,\varphi)\| \leq N\,exp[\alpha(t_0)-\alpha(t)]$$

for all $t \geq t_0$, $\varphi \in C_\infty$. If $\alpha(t) = -\alpha t$, then $x_t(t_0,\varphi)$ is exponentially asymptotically stable.

DEFINITION 4. The solution $x_t(t_0,\varphi_1)$ of (1) is said to be uniformly Lipschitz stable if there exist $M>0$ and $\delta>0$ such that

$$\|x_t(t_0,\varphi_1)-x_t(t_0,\varphi_2)\|^{(h)} \leq M\|x_{t_0}(t_0,\varphi_1)-x_{t_0}(t_0,\varphi_2)\|^{(h)}$$

whenever

$$\|x_{t_0}(t_0,\varphi_1)-x_{t_0}(t_0,\varphi_2)\|^{(h)} \leq \delta \quad \text{and} \quad t \geq t_0.$$

DEFINITION 5. The solution $x_t(t_0,\varphi)$ of (1) is said to be uniformly Lipschitz stable in variation if there exist $M>0$ and $\delta>0$ such that

$$\|\Phi(t,t_0,\psi)\| \leq M \quad \text{for} \quad t \geq t_0 \quad \text{and} \quad \psi \in C_\delta.$$

DEFINITION 6. The solution $x_t(t_0,\varphi)$ of (1) is said to be globally uniformly slowly growing in variation if for every $\beta>0$ there exists a constant $K>0$, possibly depending on β, such that

$$\|\Phi(t,t_0,\psi)\| \leq K\,exp[\beta(t-t_0)]$$

for all $t \geq t_0$ and $\psi \in C_\infty$.

THEOREM 4. *If the solution* $x_t(t_0,\varphi_1)$ *of* (1) *is uniformly Lipschitz stable in variation, then the solution* $x_t(t_0,\varphi_1)$ *of* (1) *is uniformly Lipschitz stable.*

Proof. From equality

$$x_t(t_0,\varphi_1)-x_t(t_0,\varphi_2)=\int_0^1\Phi(t,t_0,\xi(s))(x_{t_0}(t_0,\varphi_1)-x_{t_0}(t_0,\varphi_2))\,ds$$

and Definition 5 we obtain

$$\|x_t(t_0,\varphi_1)-x_t(t_0,\varphi_2)\|^{(h)}\le M\|x_{t_0}(t_0,\varphi_1)-x_{t_0}(t_0,\varphi_2)\|^{(h)}$$

for all $t \ge t_0$ and $\|x_{t_0}(t_0,\varphi_1)-x_{t_0}(t_0,\varphi_2)\|^{(h)}\le\delta$

and the proof is complete.

THEOREM 5. *The solution* $x_t(t_0,\varphi)$ *of* (1) *is exponentially asymptotically stable if and only if there exists a Liapunov function satisfying the conditions* a)- d) *of Theorem* 1 *with* $\alpha(t) = -\alpha t$, $\alpha>0$.

THEOREM 6. *The solution* $x_t(t_0,\varphi)$ *of* (1) *is globally uniformly slowly growing in variation if and only if there exists a Liapunov function satisfying the conditions* a)- d) *of Theorem* 1 *with* $\alpha(t) = \beta t$, $\beta > 0$.

This Theorems can be proved in the same manner as Theorem 1, so we omit the details.

Now consider the perturbed equation

$$x_t'(0) = f(t,x_t) + X(t,x_t) \tag{8}$$

where $f(t,\varphi)$ satisfies the same conditions as $f(t,\varphi)$ in (1) and $X:\Omega \to R^n$ is continuous in t,φ and

$$\|X(t,\varphi_1)-X(t,\varphi_2)\|\le\omega(t,\|\varphi_1-\varphi_2\|^{(h)}) \tag{9}$$

for $\quad t\ge t_0,\ \varphi_1,\varphi_2\in C,\ \omega\in C(\langle t_0,\infty)\times R^+,R^+),$

$$\omega(t,\varphi_1)\le\omega(t,\varphi_2) \quad \text{for any } 0\le\varphi_1\le\varphi_2,\ t\ge t_0. \tag{10}$$

THEOREM 7. *If*

1^0 *assumptions of Theorem 1 are satisfied,*

2^0 $u(t) = u(t,t_0,\varphi)$ *is the maximal solution of the scalar ordinary differential equation*

$$\frac{du}{dt} = \alpha'(t)\,u + K(t)\,\omega(t,u)\,,\quad u_0 = u(t_0) \geq V(t_0,\varphi_1-\varphi_2) \tag{11}$$

where $V(t,\ \varphi_1-\varphi_2)$ *is the function given in Theorem 1,*

3^0 *conditions (9) and (10) are satisfied,*

then the solutions $x_t(t_0,\varphi_1)$, $x_t(t_0,\varphi_2)$ *of the equation (8) satisfy the relation*

$$\|x_t(t_0,\varphi_1) - x_t(t_0,\varphi_2)\|^{(h)} \leq u(t,t_0,\varphi_1-\varphi_2)\,,\ t\geq t_0,\ \|\varphi_1-\varphi_2\|^{(h)} \leq u_0. \tag{12}$$

Proof. Let $y_{t+s}(t,\varphi)$, $x_{t+s}(t,\varphi)$ be the solutions of (8) and (1) respectively with $y_t(t,\varphi) = \varphi$, $x_t(t,\varphi) = \varphi$.

By Theorem 1 there exists a function $V(t,\varphi_1-\varphi_2)$ having the four properties from this Theorem. Then

$$\begin{aligned}
&V'_{(8)}(t,\varphi_1-\varphi_2) = \overline{\lim_{h_1\to 0+}}\frac{1}{h_1}\,[V(t+h_1,y_{t+h_1}(t,\varphi_1) - y_{t+h_1}(t,\varphi_2)) - V(t,\varphi_1-\varphi_2)] = \\
&= \overline{\lim_{h_1\to 0+}}\frac{1}{h_1}\,[V(t+h_1,y_{t+h_1}(t,\varphi_1) - y'_{t+h_1}(t,\varphi_2)) - \\
&- V(t+h_1,x_{t+h_1}(t,\varphi_1) - x_{t+h_1}(t,\varphi_2)) + V(t+h_1,x_{t+h_1}(t,\varphi_1) - x_{t+h_1}(t,\varphi_2)) - \\
&- V(y,\varphi_1-\varphi_2)] \leq V'_{(1)}(t,\varphi_1-\varphi_2) + \overline{\lim_{h_1\to 0+}}\frac{1}{h_1}\,[K(t+h_1)\,\|y_{t+h_1}(t,\varphi_1) - \\
&- y_{t+h_1}(t,\varphi_2) - (x_{t+h_1}(t,\varphi_1) - x_{t+h_1}(t,\varphi_2))\|^{(h)} \leq \\
&\leq V'_{(1)}(t,\varphi_1-\varphi_2) + K(t)\,\omega(t,\|y_t(t,\varphi_1) - y_t(t,\varphi_2)\|^{(h)}) \leq \\
&\leq \alpha'(t)\ V(t,\varphi_1-\varphi_2) + K(t)\,\omega(t,V(t,\varphi_1-\varphi_2))\,,
\end{aligned}$$

from the relation a), d) and the fact that $f(t,x_t) = f(t,y_t)$.

Thus, if $u(t,t_0,\varphi)$ is the maximal solution of (11) then

$$V(t,x_t(t_0,\varphi_1)-x_t(t_0,\varphi_2))\le u(t,t_0,\varphi_1-\varphi_2)$$

for all $t\ge t_0$ [see 12]. Furthermore, from b)

$$\|x_t(t_0,\varphi_1)-x_t(t_0,\varphi_2)\|^{(h)}\le V(t,x_t(t_0,\varphi_1)-x_t(t_0,\varphi_2)),$$

the theorem follows immediately.

COROLLARY. *Let $\alpha(t) = -\beta(t)$, where $\beta(t)$ is continuous, nonnegative and possessing continuous derivative for $t\ge t_0\ge 0$, then we obtain Theorem* II.3[8].

COROLLARY. *Let $f(t,0)\equiv 0$ for all $t\ge t_0$, then we obtain some results given in* [2].

COROLLARY. *Let $\alpha'(t) = -\alpha$, $\alpha > 0$, $K(t) = K =$ const., $\omega(t,\|\varphi_1-\varphi_2\|^{(h)}) = \alpha_1\|\varphi_1-\varphi_2\|^{(h)}$, $\alpha - K\alpha_1 > 0$, then the solutions* (8) *are asymptotically exponentially stable.*

COROLLARY. *Let $\alpha'(t) = -\alpha$, $\alpha > 0$, $K(t)\omega(t,u) = Ng(t)$ for all $t\ge t_0$ and $\|u\| < \infty$, where $g\in C[R^+,R^+]$, $0 < N =$ const. $\int_{t_0}^{\infty} g(s)\exp(\alpha s)\,ds < \infty$, then*

$$\|x_t(t_0,\varphi_1)-\dot{x}_t(t_0,\varphi_2)\|^{(h)}\to 0 \quad \text{as} \quad t\to\infty. \tag{13}$$

Proof. By Theorem 7 is easy to derive the inequality

$$V'_{(8)}(t,\varphi_1-\varphi_2)\le -\alpha V(t,\varphi_1-\varphi_2)+Ng(t).$$

The solution of (11) is given by

$$u(t)=u_0\exp(-\alpha(t-t_0)) + N\exp(-\alpha t)\int_{t_0}^{t} g(s)\exp(\alpha s)\,ds.$$

In the same manner as that for Theorem 7 we obtain (13).

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A PROPERTY OF AN INTERPOLATING OPERATOR

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REZUMAT. - O proprietate a unui operator de interpolare. Se demonstrează conservarea unei proprietăţi de convexitate generalizată pentru funcţii cu valori în spaţii Hilbert, prin aplicarea unui operator liniar de tip interpolator.

Let I be an interval of the real axis and let $(E,<,>)$ be a Hilbert space. Denote by $\mathcal{F}(I,E)$ the set of all the functions $f : I \to E$. For the function $f \in \mathcal{F}(I,E)$ and for the points of $I : x_1 < \ldots < x_{n+1}$, where $n \geq 0$, we denote by:

$$[x_1,\ldots,x_{n+1};f] = \sum_{k=1}^{n+1}\left(\prod_{\substack{i=1\\ i\neq k}}^{n+1}(x_k - x_i)^{-1}\right) f(x_k), \tag{1}$$

the divided difference of order n of the function f on the points x_i.

In [1] we have consider the following definition.

DEFINITION 1. If $n \geq -1$ is an integer we denote by $\mathcal{E}_c^n(I,E)$ the set of the functions $f \in \mathcal{F}(I,E)$ that satisfies the relation:

$$\langle [x_1,\ldots,x_{n+2};f],\ [x_2,\ldots,x_{n+3};f]\rangle \geq 0, \tag{2}$$

for any points $x_1 < \ldots < x_{n+3}$ of I.

This definition may be regarded as a generalization of the convexity of order n, since in the particular case where $E = \mathbb{R}$, a function $f \in \mathcal{F}(I,E)$ is as in Definition 1, if and only if f is either nonconcave of order n or nonconvex of order n on I in the usual sense [2]. This statement as well as many properties that

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are specific for the convex functions, are proved in [1] for the functions as in Definition 1. In this paper we present an operator that preserves the $\mathscr{E}_c^n(I,E)$ class for $n = 0$ and $n = 1$.

DEFINITION 2. If $n \geq 2$ is an integer and $x_1 < \ldots < x_n$ are points of the interval I, we denote by $U_{x_1,\ldots,x_n} : \mathscr{F}(I,E) \to \mathscr{F}(I,E)$ the operator defined by:

$$U_{x_1,\ldots,x_n}[f](x) = \frac{x - x_k}{x_{k+1} - x_k} \cdot f(x_{k+1}) + \frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot f(x_k), \qquad (3)$$

where $f \in \mathscr{F}(I,E)$, $x \in I$, and k is the index, $1 \leq k \leq n-1$ with the property that $x \in [x_k, x_{k+1}]$, or $k = 1$ if $x \in I \cap (-\infty, x_1)$ or $k = n-1$ if $x \in I \cap (x_n, \infty)$, respectively.

We prove:

THEOREM. [1] *If $n = 0$ or $n = 1$, $f \in \mathscr{E}_c^n(I,E)$ and $x_1 < \ldots < x_{n+3}$ are points of the interval I, then $U_{x_1,\ldots,x_{n+3}}[f] \in \mathscr{E}_c^n(I,E)$.*

Proof. Let denote $y_i = f(x_i)$, $(1 \leq i \leq n+3)$, $g = U_{x_1,\ldots,x_{n+3}}[f]$.

First we consider the case where $n = 0$. Let the points of the interval I : $t_1 < t_2 < t_3$. If $t_3 \leq x_2$ or $t_1 \geq x_2$ the points $g(t_i)$ are on a same line in E, and the theorem is obvious. Now, let consider only the case where $t_1 < x_2 \leq t_2$. Let denote: $q = x_2 - t_1$, $r = t_2 - x_2$, $A = [x_1, x_2; g]$ and $B = [x_2, x_3; g]$. We have:

$$\langle [t_1, t_2; g], [t_2, t_3; g] \rangle = \frac{r}{r+q} \|B\|^2 + \frac{q}{r+q} \langle A, B \rangle \geq 0.$$

Let consider now the case $n = 1$. It is sufficient to prove the following inequality:

$$\langle [t_2, t_3; g] - [t_1, t_2; g], [t_3, t_4; g] - [t_2, t_3; g] \rangle \geq 0, \qquad (4)$$

for any points of the interval I.

We denote $A = [x_1, x_2; f]$, $B = [x_2, x_3; f]$ and $C = [x_3, x_4; f]$.

Since $f \in \mathscr{E}_c^1(I,E)$, from (2) we obtain:

$$\langle B-A,\ C-B\rangle \geq 0. \qquad (5)$$

Let denote by p_1, p_2, p_3 the number of the elements of the set $\{t_1, t_2, t_3, t_4\}$ that are contained in the intervals $(-\infty, x_2)$, $[x_2, x_3]$, (x_3, ∞), respectively.

In virtue of the symmetry, we have to consider the cases where (p_1,p_2,p_3) is one of the following vectors: (4,0,0), (3,1,0), (2,2,0), (1,3,0), (0,4,0), (3,0,1), (2,1,1), (1,2,1) and (2,0,2). If one of the numbers p_1, p_2 or p_3 is greater or equal to 3 then one of the following equalities holds: $[t_1,t_2,t_3;g] = 0$ or $[t_2,t_3,t_4;g] = 0$. But in this case the relation (4) is obvious. It remains to verify the cases (2,2,0), (2,1,1), (1,2,1) and (2,0,2). Let denote $D = \langle B-A, C-B\rangle \geq 0$.

Case (2,2,0). We have $t_1 < t_2 < x_2 \leq t_3 < t_4 \leq x_3$. Let denote: $q = x_2 - t_2$, $r = t_3 - x_2$. We have $[t_1,t_2;g] = A$, $[t_2,t_3;g] = \frac{rB+qA}{r+q}$, $[t_3,t_4;g] = B$.

With these notations, the relation (4) becomes:

$$\left\langle \frac{rB+qA}{r+q} - A,\ B - \frac{rB+qA}{r+q} \right\rangle \geq 0,$$

that is equivalent with the obvious inequality $\frac{rq}{(r+q)^2}\|B-A\|^2 \geq 0$.

Case (2,1,1). We have $t_1 < t_2 < x_2 \leq t_3 \leq x_3 < t_4$. We denote: $q = x_2 - t_2$, $r = t_3 - x_2$, $s = x_3 - t_3$, $p = t_4 - x_3$. We have:

$$[t_1,t_2;g] = A,\ [t_2,t_3;g] = \frac{rB+qA}{r+q},\ [t_3,t_4;g] = \frac{pC+sB}{p+s}.$$

Then the left side of the inequality in (4) becomes:

$$<\frac{rB+qA}{r+q}-A,\ \frac{pC+sB}{p+s}-\frac{rB+qA}{r+q}> =$$

$$=\frac{1}{(p+s)(r+q)^2}\{pr(r+q)<B,C> + r(sq-pr)\|B\|^2 + r(pr-2sq-pq)<A,B> -$$

$$-pr(r+q)<A,C> + rq(p+s)\|A\|^2\} =$$

$$=\frac{1}{(p+s)(r+q)^2}\{pr(r+q)D + qr(p+s)\|A-B\|^2\} \geq 0.$$

Case (1,2,1). We have $t_1 < x_2 \leq t_2 < t_3 \leq x_3 < t_4$. Let denote: $p = x_2 - t_1$, $q = t_2 - x_2$, $r = x_3 - t_3$, $s = t_4 - x_3$. We have: $[t_1, t_2; g] = \frac{pA+qB}{p+q}$, $[t_2, t_3; g] = B$, $[t_3, t_4; g] = \frac{sC+rB}{s+r}$. Then the left side in (4) becomes:

$$<B-\frac{pA+qB}{p+q},\ \frac{sC+rB}{s+r}-B> = \frac{sp}{(s+r)(p+q)}D \geq 0.$$

Case (2,0,2). We have $t_1 < t_2 < x_2 < x_3 < t_3 < t_4$. In virtue of the symetry we can suppose that $x_2 - t_2 \geq t_3 - x_3$. Let denote $q = x_2 - t_2$, $l = x_3 - x_2$, and $r = t_3 - x_3$. We have $[t_1,t_2;g] = A$, $[t_2,t_3;g] = = \frac{rC+lB+qA}{l+r+q}$, $[t_3, t_4; g] = C$. Therefore the left side in (4) becomes:

$$<\frac{rC+lB+qA}{l+r+q}-A,\ C-\frac{rC+lB+qA}{l+r+q}> =$$

$$=\frac{1}{(l+r+q)^2}\{(l+q)r\|C\|^2 + (l+q-r)<B,C> - (l+q+r+2qr)<A,C> +$$

$$+(l+r-q)<A,B> + q(l+r)\|A\|^2 - l^2\|B\|^2\} =$$

$$=\frac{1}{(l+r+q)^2}\{(l+q-r)D + (q-r)\|A-B\|^2 + (q+l)r\cdot\|A-C\|^2\} \geq 0.$$

The theorem is proved.

This theorem cannot be extended for $n \geq 2$. More, for $n = 2$ we have.

LEMMA. *If $f \in \mathcal{F}(I,E)$ is not linear on I, and if I is an open interval, then there exist the points $x_1 < \ldots < x_5$ of I such that*

$U_{x_1,\ldots,x_5}[f]$ *does not belong to* $\mathscr{E}_c^2(I,E)$.

Proof. Since f is not linear on I, there exist the points $x_1 < x_2 < x_3$ of I such that the points $f(x_i)$, $(1 \le i \le 3)$ are not on a line in the space E. Then we choose still two arbitrary points x_4, x_5 of I such that $x_3<x_4<x_5$. Let denote $g=U_{x_1,\ldots,x_5}[f]$. In order to show that $g \notin \mathscr{E}_c^2(I,E)$, we take the number $q > 0$ and the points t_i, $(1 \le i \le 5)$ of I such that $x_1 \le t_1 < t_2 < x_2 < < t_3 < t_4 < t_5 \le x_3$, and $t_2 = t_1 + 2q$, $x_2 = t_2 + q$, $t_3 = x_2 + q$, $t_4 = t_3 + q$, $t_5 = t_4 + q$.

We also denote $a = g(t_1+q) - g(t_1)$ and $b = g(t_3) - g(x_2)$. Since the points $f(x_1), f(x_2), f(x_3)$ are not on a line, we obtain $a \ne b$. Let denote $c = g(t_1)$. Then we have $g(t_2) = c + 2a$, $g(t_3) = c + 3a + b$, $g(t_4) = c + 3a + 2b$ and $g(t_5) = c + 3a + 3b$. By the formula (1), we obtain

$$[t_1,t_2,t_3,t_4;g] = \left\{-\frac{c}{40} + \frac{c+2a}{12} - \frac{c+3a+b}{8} + \frac{c+3a+2b}{15}\right\}q^{-3} = \frac{q^{-3}}{120}(b-a),$$

$$[t_2,t_3,t_4,t_5;g] = \left\{-\frac{c+2a}{24} + \frac{c+3a+b}{4} - \frac{c+3a+2b}{3} + \frac{c+3a+3b}{8}\right\}q^{-3} =$$

$= \frac{q^{-3}}{24}(a-b)$. Therefore $< [t_1,t_2,t_3,t_4;g], [t_2,t_3,t_4,t_5;g] > =$
$= \frac{-1}{120\cdot 24} q^{-6}\|a-b\|^2 < 0$. Hence g does not belong to $\mathscr{E}_c^2(I,E)$.

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ON THE APPROXIMATION OF THE DERIVATIVES OF FUNCTIONS BY GENERALIZED SZASZ-MIRAKJAN TYPE OPERATOR

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REZUMAT. - **Asupra aproximării derivatelor funcţiilor prin operatori de tip Szasz-Mirakjan generalizaţi.** În această lucrare se studiază operatori de tip Szasz-Mirakjan obţinuţi cu ajutorul polinoamelor Appel. Se dau estimări a ordinului de aproximare ale derivatelor unei funcţii prin derivatele acestor operatori.

Abstract. In this paper one studies a generalized Szasz-Mirakjan type operator obtained by means of the Appell polynomials. One gives estimates of the order of approximation of the derivatives of a function by the derivatives of this operators.

Key words: Appell polynomials, the derivatives of generalized Szasz-Mirakjan operator, order of approximation

1. Using the Appell polynomials, A.Jakimovski and D.Leviatan [2] have obtained an Szasz-Mirakjan type operator and they have studied its approximation properties.

Let $g(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function defined in the disk $|z|<R, R>1$ and suppose $g(1) \neq 0$. Define the Appell polynomials $p_k(x) \equiv p_k(x,g)$, $k \geq 0$, by

$$g(u)\, e^{ux} \equiv \sum_{k=0}^{\infty} p_k(x)\, u^k \tag{1}$$

To each function f defined in $[0,\infty)$ the following operators are

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associated

$$P_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx)\, f\left(\frac{k}{n}\right) \tag{2}$$

We suppose that $p_k(x,g) \geq 0$ for $0 \leq x < \infty$, $k = 0,1,\ldots$

If $a_n \geq 0$ for any $n \in \mathrm{N}$ then supposition is fulfilled because

$$p_k(x) = \sum_{\nu=0}^{k} \frac{x^{k-\nu}}{(k-\nu)!} a_\nu$$

For $g(z) \equiv 1$, the operators P_n become of the Szasz-Mirakjan operator

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \tag{3}$$

2. In this section we are concerned with the estimate of the order of approximation of the derivatives of a function $f \in C^1[0,a]$ by means of the derivatives of the linear positive operator P_n. We shall use the modulus of continuity, defined by

$$\omega(f;\delta) = \sup |f(x'') - f(x')|,$$

where x' and x'' are points from $[0,a]$ so that $|x' - x''| < \delta$, δ being a positive number.

THEOREM 2.1. *If $f \in C^1[0,a]$, then for any $x \in [0,a]$ we have*

$$|P_n'(f;x) - f'(x)| \leq \left(1 + \sqrt{a + \frac{1}{n}\,\frac{g''(1) + g'(1)}{g(1)}}\right) \omega\left(f'; \frac{1}{\sqrt{n}} + \frac{1}{n}\right) + \omega\left(f'; \frac{1}{n}\right) \tag{4}$$

Proof: It is known [3] that the first order derivative of P_n is

$$P_n'(f;x) = \frac{ne^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx)\, \Delta f\left(\frac{k}{n}\right) =$$

$$= \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx)\left[\frac{k}{n}, \frac{k+1}{n}; f\right]$$

We denote by h the first order divided difference of the function f, $h(t) = \left[t, t+\frac{1}{n}; f\right]$ and so we can write

$$P_n'(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx)\, h\left(\frac{k}{n}\right) = P_n(h;x)$$

By using these notations we have

$$|P_n'(f;x) - f'(x)| \le |P_n(h;x) - h(x)| + |h(x) - f'(x)|$$

It was proved in [1], that:

$$|P_n(h;x) - h(x)| \le \left(1 + \sqrt{a + \frac{1}{n} \cdot \frac{g''(1) + g'(1)}{g(1)}}\right) \omega\left(h; \frac{1}{\sqrt{n}}\right) \qquad (5)$$

Hence

$$|P_n'(f;x) - f'(x)| \le \left(1 + \sqrt{a + \frac{1}{n} \cdot \frac{g''(1) + g'(1)}{g(1)}}\right) \omega\left(h; \frac{1}{\sqrt{n}}\right) + \qquad (6)$$
$$+ |h(x) - f'(x)|$$

But using the mean Theorem for the divided differences and the properties of the modulus of continuity, we have

$$|h(x) - f'(x)| = \left|\left[x, x+\frac{1}{n}; f\right] - f'(x)\right| = \left|f'\left(x + \frac{\theta}{n}\right) - f'(x)\right| \le$$
$$\le \omega\left(f'; \frac{1}{n}\right), \quad \text{where } \theta \in (0,1)$$

By making use of the mean theorem, we want to express $\omega\left(h; \frac{1}{\sqrt{n}}\right)$ by means of the modulus of continuity of f'.

We have

$$|h(t+\delta) - h(t)| = \left| \left[t+\delta,\ t+\delta+\frac{1}{n};\ f\right] - \left[t,\ t+\frac{1}{n};\ f\right] \right| =$$

$$= n\left| f\left(t+\delta+\frac{1}{n}\right) - f(t+\delta) - f\left(t+\frac{1}{n}\right) + f(t) \right| =$$

$$= \left| f'\left(t+\delta+\frac{\theta_1}{n}\right) - f'\left(t+\frac{\theta_2}{n}\right) \right| \le \omega\left(f';\ \delta + \frac{|\theta_1-\theta_2|}{n}\right) \le \omega\left(f';\ \delta+\frac{1}{n}\right)$$

where $\theta_1, \theta_2 \in (0,1)$.

Now, if we set $\delta = \frac{1}{\sqrt{n}}$, it results $\omega\left(h; \frac{1}{\sqrt{n}}\right) \le \omega\left(f'; \frac{1}{\sqrt{n}} + \frac{1}{n}\right)$ and we obtain finally the desired inequality.

If $f \in C^2[0,a]$, it is known that $\omega\left(f'; \frac{1}{n}\right) \le \frac{1}{n}\|f''\|$ and we obtain

COROLLARY 2.1. *If $f \in C^2[0,a]$, we have in the maximum norm over $[0,a]$:*

$$|P_n(f';x) - f'(x)| \le \left(1+\sqrt{a+\frac{1}{n}\cdot\frac{g''(1)+g'(1)}{g(1)}}\right)\omega\left(f';\ \frac{1}{\sqrt{n}}+\frac{1}{n}\right) + \frac{1}{n}\|f''\| \tag{7}$$

For $g(z) \equiv 1$, one obtains the estimate of the order of approximation by the derivative of the Szasz-Mirakjan operator. If $f \in C^1[0,a]$, we have

$$|S_n'(f;x) - f'(x)| \le (1+\sqrt{a})\ \omega\left(f';\ \frac{1}{\sqrt{n}}+\frac{1}{n}\right) + \omega\left(f';\ \frac{1}{n}\right)$$

and if $f \in C^2[0,a]$

$$|S_n'(f;x) - f'(x)| \le (1+\sqrt{a})\ \omega\left(f';\ \frac{1}{\sqrt{n}}+\frac{1}{n}\right) + \frac{1}{n}\|f''\|.$$

3. We consider now $f \in C^k[0,a]$. We shall prove the following

[illegible]esult:

THEOREM 3.1. *If* $f \in C^k[0,a]$, [illegible] $[0,a]$ *we have*

$$[illegible] (f;x) [illegible]^{(k)}(x)| \le \left(1 + \sqrt{a + \frac{1}{n} \cdot \frac{g''(1) [illegible] g'(1) [illegible]}{g(1)}} [illegible] \frac{1}{\sqrt{n}} + \frac{k}{n}\right) + \omega\left(f^{(k)}; \frac{k}{n}\right) \tag{8}$$

Proof: It is known [3] that the k order derivative of P_n is

$$P_n^{(k)}(f;x) = \frac{n^k e^{-nx}}{g(1)} \sum_{i=0}^{\infty} p_i(nx)\, \Delta^k f\left(\frac{i}{n}\right)$$

where $\Delta f\left(\frac{i}{n}\right) = f\left(\frac{i+1}{n}\right) - f\left(\frac{i}{n}\right)$ and $\Delta^k f\left(\frac{i}{n}\right) = \Delta\, \Delta^{k-1} f\left(\frac{i}{n}\right)$, $k = 2, 3, \ldots$

By making use of the relation between the divided differences and the finite differences, we have

$$P_n^{(k)}(f;x) = k!\, \frac{e^{-nx}}{g(1)} \sum_{i=0}^{\infty} p_i(nx)\, \frac{\Delta^k f\left(\frac{i}{n}\right)}{k! \left(\frac{1}{n}\right)^k} =$$

$$= k!\, \frac{e^{-nx}}{g(1)} \sum_{i=0}^{\infty} p_i(nx) \left[\frac{i}{n}, \frac{i+1}{n}, \ldots, \frac{i+k}{n}; f\right] =$$

$$= k!\, \frac{e^{-nx}}{g(1)} \sum_{i=0}^{\infty} p_i(nx)\, h\left(\frac{i}{n}\right) = k!\, P_n(h;x)$$

where we have denoted

$$h(t) = \left[t, t + \frac{1}{n}, \ldots, t + \frac{k}{n}; f\right]$$

We want give the estimate of the order of approximation of the $f^{(k)}$ by means of the k derivative of the operators P_n

$$\begin{aligned} |P_n (f;x) [illegible]^{(k)}(x)| &= |k!\, P_n(h;x) - f^{(k)}(x)| \le \\ &\le [illegible] h(x)| + |k!\, h(x) - f^{(k)}(x)| \end{aligned} \tag{9}$$

Here we shall [illegible] and to get to the modulus of continuity of the $f^{(k)}$ function, we shall use the mean theorem and the properties of the modulus of continuity. We have

$$|h(t+\delta)-h(t)| = \left|\left[t+\delta, t+\delta+\frac{1}{n}, \ldots, t+\delta+\frac{k}{n}; f\right] - \right.$$
$$\left. - \left[t, t+\frac{1}{n}, \ldots, t+\frac{k}{n}; f\right]\right| = \frac{1}{k!}\left|f^{(k)}\left(t+\delta+\theta_1\frac{k}{n}\right) - f^{(k)}\left(t+\theta_2\frac{k}{n}\right)\right| \le$$
$$\le \frac{1}{k!}\,\omega\left(f^{(k)}; \delta+\frac{k}{n}|\theta_1-\theta_2|\right) \le \frac{1}{k!}\,\omega\left(f^{(k)}; \delta+\frac{k}{n}\right)$$

where $\theta_1, \theta_2 \in (0,1)$.

For $\delta = \frac{1}{\sqrt{n}}$, we obtain

$$\omega\left(h; \frac{1}{\sqrt{n}}\right) \le \frac{1}{k!}\,\omega\left(f^{(k)}; \frac{1}{\sqrt{n}}+\frac{k}{n}\right)$$

On the other hand, we have

$$|k!\,h(x) - f^{(k)}(x)| = \left|k!\left[x, x+\frac{1}{n}, \ldots, x+\frac{k}{n}; f\right] - f^{(k)}(x)\right| =$$
$$= \left|f^{(k)}\left(x+\theta_3\frac{k}{n}\right) - f^{(k)}(x)\right| \le \omega\left(f^{(k)}; \theta_3\frac{k}{n}\right) \le \omega\left(f^{(k)}; \frac{k}{n}\right)$$

where $\theta_3 \in (0,1)$.

Coming back to (9) we obtain the desired estimation.

If $f \in C^{k+1}[0,a]$, it is known that $\omega\left(f^{(k)}; \frac{k}{n}\right) \le \frac{k}{n}\|f^{(k+1)}\|$ and it results

COROLLARY 3.1. *If $f \in C^{k+1}[0,a]$, we have*

$$|P_n^{(k)}(f;x) - f^{(k)}(x)| \le$$
$$\le\left(1+\sqrt{a+\frac{1}{n}\cdot\frac{g''(1)+g'(1)}{g(1)}}\right)\omega\left(f^{(k)}; \frac{1}{\sqrt{n}}+\frac{k}{n}\right)+\frac{k}{n}\|f^{(k+1)}\|$$

For $g(z) \equiv 1$, we obtain for Szasz-Mirakjan operator: if $f \in C^k[0,a]$, then

$$\left|S_n^{(k)}(f;x) - f^{(k)}(x)\right| \le (1+\sqrt{a})\ \omega\left(f^{(k)};\ \frac{1}{\sqrt{n}} + \frac{k}{n}\right) + \omega\left(f^{(k)};\ \frac{k}{n}\right);$$

if $f \in C^{k+1}[0,a]$, then

$$\left|S_n^{(k)}(f;x) - f^{(k)}(x)\right| \le (1+\sqrt{a})\ \omega\left(f^{(k)};\ \frac{1}{\sqrt{n}} + \frac{k}{n}\right) + \frac{k}{n}\left\|f^{(k+1)}\right\|.$$

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ON THE REMAINDER IN APPROXIMATION FORMULA BY GENERALIZED FAVARD-SZASZ OPERATORS

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REZUMAT. - Asupra restului în formula de aproximare prin operatori Favard-Szasz generalizaţi. Lucrarea studiază un operator generalizat de tip Favard-Szasz, obţinut de către A. Jakimovski şi D.Leviatan cu ajutorul polinoamelor Appell. Se dă o reprezentare integrală a restului în formula de aproximare a unei funcţii $f \in C[0,\infty)$, de tip exponenţial, prin acest operator.

1. A. Jakimovski and D. Leviatan [2] considered a generalization of the well known Favard-Szasz operator [1], [4]. Let

$$g(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an analytic function in the disk $|z| < R$, $(R > 1)$ and suppose $g(1) \neq 0$. Define the Appell polynomials $p_k(x) = p_k(x,g)$ $(k \geq 0)$ by the expansion

$$g(u)\,e^{ux} = \sum_{k=0}^{\infty} p_k(x)\,u^k \qquad (1)$$

To each function f defined in $[0,\infty)$ is associated the operator P_n, according to the formula

$$(P_n f)\,(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx)\, f\left(\frac{k}{n}\right), \qquad (2)$$

where n is a natural number.

B.Wood [5] has proved that this operator is positive in

$[0,\infty)$ if and only if $a_n/g(1) \geq 0$ for $n \in \mathbb{N}$. The case $g(z) \equiv 1$ yields the classical operators of Favard-Szasz,

$$(S_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right).$$

A.Jakimovski and D.Leviatan established several results analogue to those obtained by Szasz, as well as certain other new approximation results. They proved that if f is of exponential type, i.e. $|f(t)| \leq e^{At}$ for any $t \geq 0$ and certain A finite, then $\lim_{n\to\infty} P_n f = f$, the convergence being uniform in any compact interval of the real axis.

2. It is the purpose of this paper to give an integral representation of the remainder in the approximation formula

$$f(x) = (P_n f)(x) + (R_n f)(x), \tag{3}$$

where we suppose that $f \in C^2[0,\infty)$.

The values of the operator P_n for the test functions e_0, e_1, e_2, where $e_i(t) = t^i$, $i \in \{0,1,2\}$, are given by

$$\begin{aligned} (P_n e_0)(x) &= 1 \\ (P_n e_1)(x) &= x + \frac{1}{n}\frac{g'(1)}{g(1)} \\ (P_n e_2)(x) &= x^2 + \frac{1}{n}\left(1 + 2\frac{g'(1)}{g(1)}\right) + \frac{1}{n^2}\frac{g''(1)+g'(1)}{g(1)}. \end{aligned} \tag{4}$$

By making use of the Mac-Laurin's formula, with the remainder having an integral representation, we have

$$f(x) = f(0) + xf'(0) + g(f;x),$$

where

$$g(f;x) = g(x) - \int_0^x (x-t)\, f''(t)\, dt.$$

Since P_n is linear operator, it follows that R_n is linear, so that we can write

$$(R_n f)(x) = f(0)\,(R_n e_0)(x) + f'(0)\,(R_n e_1)(x) + (R_n g)(x), \tag{5}$$

where, according to (4), we have

$$(R_n e_0)(x) = 0 \text{ and } (R_n e_1)(x) = -\frac{1}{n}\,\frac{g'(1)}{g(1)}.$$

It follows that

$$\begin{aligned}(R_n f)(x) &= -\frac{1}{n} f'(0)\,\frac{g'(1)}{g(1)} + (R_n g)(x) = \\ &= -\frac{1}{n} f'(0)\,\frac{g'(1)}{g(1)} + g(x) - (P_n g)(x).\end{aligned} \tag{6}$$

Now let us calculate $(P_n g)(x)$. We can write successively:

$$(P_n g)(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx)\, g\left(\frac{k}{n}\right) =$$

$$= \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \int_0^{k/n} \left(\frac{k}{n} - t\right) f''(t)\, dt =$$

$$= \frac{e^{-nx}}{g(1)} \left[p_1(nx) \int_0^{1/n} \left(\frac{1}{n} - t\right) f''(t)\, dt + p_2(nx) \int_0^{2/n} \left(\frac{2}{n} - t\right) f''(t)\, dt + \ldots \right.$$

$$\left. + p_k(nx) \int_0^{k/n} \left(\frac{k}{n} - t\right) f''(t)\, dt + \ldots \right] =$$

$$= \frac{e^{-nx}}{g(1)} \sum_{k=1}^{\infty} \int_{(k-1)/n}^{k/n} \sum_{j=k}^{\infty} p_j(nx) \left(\frac{j}{n} - t\right) f''(t)\, dt.$$

By using (4), we can write

$$\sum_{j=k}^{\infty} p_j(nx)\left(\frac{j}{n}-t\right)=\frac{1}{n}\left(g'(1)\,e^{nx}+nxg(1)\,e^{nx}-\sum_{j=0}^{k-1} jp_j(nx)\right)-$$

$$-\,t\left(g(1)\,e^{nx}-\sum_{j=0}^{k-1} p_j(nx)\right).$$

It follows that

$$(P_n g)(x)=\sum_{k=1}^{\infty}\int_{(k-1)/n}^{k/n}\left[\frac{1}{n}\frac{g'(1)}{g(1)}+x-t-\frac{e^{-nx}}{g(1)}\sum_{j=0}^{k-1}\left(\frac{j}{n}-t\right)p_j(nx)\right]f''(t)\,dt$$

Replacing in (6) we obtain

$$(R_n f)(x)=-\frac{1}{n}f'(0)\,\frac{g'(1)}{g(1)}+\int_0^x (x-t)\,f''(t)\,dt-$$

$$-\sum_{k=1}^{\infty}\int_{(k-1)/n}^{k/n}\left[\frac{1}{n}\frac{g'(1)}{g(1)}+x-t-\frac{e^{-nx}}{g(1)}\sum_{j=0}^{k-1}\left(\frac{j}{n}-t\right)p_j(nx)\right]f''(t)\,dt$$

We can write this results as follows

$$(R_n f)(x)=-\frac{1}{n}f'(0)\,\frac{g'(1)}{g(1)}+\int_0^{\infty}G_n(t;x)\,f''(t)\,dt,$$

where, if we suppose that $x\in\left[\frac{i-1}{n},\frac{i}{n}\right]$, $i\in\mathbf{N}$, we have:

(a) if $t\in\left[\frac{k-1}{n},\frac{k}{n}\right]$, $1\le k\le i-1$,

$$G_n(t;x)=\frac{e^{-nx}}{g(1)}\sum_{j=0}^{k-1}\left(\frac{j}{n}-t\right)p_j(nx)$$

(b) if $t\in\left[\frac{i-1}{n},x\right]$

$$G_n(t;x)=\frac{e^{-nx}}{g(1)}\sum_{j=0}^{i-1}\left(\frac{j}{n}-t\right)p_j(nx)$$

$t \in \left[x, \frac{i}{n}\right]$

$$G_n(t;x) = -\frac{1}{n}\frac{g'(1)}{g(1)} + t - x + \frac{e^{-nx}}{g(1)}\sum_{j=0}^{i-1}\left(\frac{j}{n} - t\right)p_j(nx)$$

d) if $t \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$, $k \ge i+1$

$$G_n(t;x) = -\frac{1}{n}\frac{g'(1)}{g(1)} + t - x + \frac{e^{-nx}}{g(1)}\sum_{j=0}^{k-1}\left(\frac{j}{n} - t\right)p_j(nx)$$

It is easy to see that in the cases (a) and (b) we have $G_n(t;x) \le 0$.

On the other hand, for each $m \in \mathbf{N}$ we have

$$-\frac{1}{n}\frac{g'(1)}{g(1)} + t - x + \frac{e^{-nx}}{g(1)}\sum_{j=0}^{m-1}\left(\frac{j}{n} - t\right)p_j(nx) =$$

$$= -\frac{1}{n}\frac{g'(1)}{g(1)} + t(P_ne_0)(x) - \left[(P_ne_1)(x) - \frac{1}{n}\frac{g'(1)}{g(1)}\right] +$$

$$+ \frac{e^{-nx}}{g(1)}\sum_{j=0}^{m-1}\left(\frac{j}{n} - t\right)p_j(nx) = t\frac{e^{-nx}}{g(1)}\sum_{j=0}^{\infty}p_j(nx) -$$

$$- \frac{e^{-nx}}{g(1)}\sum_{j=0}^{\infty}p_j(nx)\frac{j}{n} + \frac{e^{-nx}}{g(1)}\sum_{j=0}^{m-1}\left(\frac{j}{n} - t\right)p_j(nx) =$$

$$= \frac{e^{-nx}}{g(1)}\sum_{j=0}^{\infty}\left(t - \frac{j}{n}\right)p_j(nx) + \frac{e^{-nx}}{g(1)}\sum_{j=0}^{m-1}\left(\frac{j}{n} - t\right)p_j(nx) =$$

$$= -\frac{e^{-nx}}{g(1)}\sum_{j=m}^{\infty}\left(\frac{j}{n} - t\right)p_j(nx).$$

Consequently, we can state that for any $t \in [0,\infty)$ we have $G_n(t;x) \le 0$, x being a fixed point in $[0,\infty)$. For this reasons we can apply the mean theorem to the integral and we can write

$$(R_nf)(x) = -\frac{1}{n}f'(0)\frac{g'(1)}{g(1)} + f''(\xi)\int_0^{\infty}G_n(t;x)\,dt,$$

where $\xi \in (0,\infty)$.

If we choose $f(x) = x^2$, we get

$$x^2 = (P_n t^2)(x) + (R_n t^2)(x),$$

so that we obtain

$$x^2 = x^2 + \frac{x}{n}\left(1 + 2\frac{g'(1)}{g(1)}\right) + \frac{1}{n^2}\frac{g''(1) + g'(1)}{g(1)} + 2\int_0^\infty G_n(t;x)\,dt$$

It follows that

$$\int_0^\infty G_n(t;x)\,dt = -\frac{1}{2}\left[\frac{x}{n}\left(1 + 2\frac{g'(1)}{g(1)}\right) + \frac{1}{n^2}\frac{g''(1) + g'(1)}{g(1)}\right].$$

In this way we arrive to the following representation of the remainder:

$$(R_n f)(x) =$$

$$-\frac{1}{n}\frac{g'(1)}{g(1)}f'(0) - \frac{1}{2}f''(\xi)\left[\frac{x}{n}\left(1 + 2\frac{g'(1)}{g(1)}\right) + \frac{1}{n^2}\frac{g''(1) + g'(1)}{g(1)}\right].$$

If $g(z) \equiv 1$, we obtain a result given by F.Stancu [3], for the remainder in the approximation formula by means of Favard-Szasz operators:

$$(R_n f)(x) = -\frac{1}{2}f''(\xi)\cdot\frac{x}{n}, \quad 0 < \xi < \infty.$$

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SOME REMARKS ON BESSEL'S INEQUALITY IN INNER PRODUCT SPACES

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REZUMAT. - Observaţii asupra inegalităţii lui Bessel în spaţii pre-Hilbertiene. Se dau cîteva generalizări ale inegalităţii lui Bessel, care extind rezultate ale lui Bombieri, Selberg şi Heilbronn.

Abstract. - Some generalizations of Bessel's inequality in inner product spaces containing the results of Bombieri, Selberg and Heilbronn are given.

1. Introduction. In the recent paper [3], J.E.Pečarić proved the following result connected with the well known Bessel's inequality which holds in inner product spaces $(X;(\ ,\ ))$.

THEOREM A. *Suppose that* x, y_r $(r=\overline{1,n})$ *are vectors in* X *and* c_r $(r=\overline{1,n})$ *are arbitrary complex (real) numbers. Then the following inequality*

$$\left|\sum_{r=1}^{n} c_r(x,y_r)\right|^2 \le \|x\|^2 \sum_{r,s=1}^{n} |c_r|^2 \,|(y_r,y_s)| \tag{1}$$

holds.

He showed that this inequality improves the following result which is important in Number Theory (see [2]):

THEOREM B. *If* x, y_r, c_r $(r=\overline{1,n})$ *are as above, then*

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$$\left|\sum_{r=1}^{n}(x,y_r)\right|^2 \le \|x\|^2\left(\sum_{r=1}^{n}|c_r|^2\right)\max_{1\le r\le n}\left\{\sum_{s=1}^{n}|(y_r,y_s)|\right\}$$

which contains the Bombieri's result (see also [2]):

$$\sum_{r=1}^{n}|(x,y_r)|^2 \le \|x\|^2 \max_{1\le r\le n}\left\{\sum_{s=1}^{n}|(y_r,y_s)|\right\}. \tag{2}$$

Note that (2) is a natural generalization of Bessel's inequality

$$\sum_{r=1}^{n}|(x,e_r)|^2 \le \|x\|^2 \quad \text{for all} \quad x\in X,$$

where $\{e_r\}_{r=\overline{1,n}}$ is an orthonormal family in X.

In [3] it is also showed that (1) contains as particular cases other generalizations of Bessel's inequality due to Selberg (see [2]) and Heilbronn [1] (see also [2]):

THEOREM C. *If x, y_r $(r=\overline{1,n})$ are as above, then one has the inequality*

$$\sum_{r=1}^{n}|(x,y_r)|^2\left(\sum_{s=1}^{n}|(y_r,y_s)|\right)^{-1} \le \|x\|^2$$

and

THEOREM D. *In the above assumptions one has:*

$$\sum_{r=1}^{n}|(x,y_r) \le \|x\|\left(\sum_{r,s=1}^{n}|(y_r,y_s)|\right)^{1/2}$$

respectively.

In this paper, we will give a generalization of (1). Some other connected results and applications will be also pointed out.

2. The main results. The first result is embodied in the following theorem:

THEOREM 1. *Let $x_i, y_i \in X$ and $\alpha_i, \beta_i \in K\,(i=\overline{1,n})$. Then one has the inequality:*

$$\left|\sum_{i,j=1}^{n} \alpha_i \beta_j (x_i, y_j)\right|^2 \le \min\{A, B, C\} \tag{3}$$

where

$$A := \sum_{i,j=1}^{n} |\alpha_i|^2 \,|(x_i, x_j)| \sum_{i,j=1}^{n} |\beta_i|^2 \,|(y_i, y_j)|$$

$$B := \sum_{i=1}^{n} |\alpha_i|^2 \sum_{i=1}^{n} |\beta_i|^2 \left(\sum_{i,j=1}^{n} |(x_i, x_j)|^2 \sum_{i,j=1}^{n} |(y_i, y_j)|^2 \right)^{1/2}$$

and

$$C := \sum_{i=1}^{n} |\alpha_i| \sum_{i=1}^{n} |\beta_i| \max_{1\le i\le n} \left\{ \sum_{j=1}^{n} |\alpha_j|\,|(x_i, x_j)| \right\} \max_{1\le i\le n} \left\{ \sum_{j=1}^{n} |\beta_j|\,|(y_i, y_j)| \right\},$$

respectively.

Proof. We have

$$\left|\sum_{i,j=1}^{n} \alpha_i \beta_j (x_i, y_j)\right|^2 = \left|\left(\sum_{i=1}^{n} \alpha_i x_i, \sum_{j=1}^{n} \overline{\beta}_j y_j\right)\right|^2 \le$$

$$\le \left\|\sum_{i=1}^{n} \alpha_i x_i\right\|^2 \left\|\sum_{i=1}^{n} \overline{\beta}_i y_i\right\|^2 = \left|\sum_{i,j=1}^{n} \alpha_i \overline{\alpha}_j (x_i, x_j)\right| \left|\sum_{i,j=1}^{n} \overline{\beta}_i \beta_j (y_i, y_j)\right|$$

$$\le \sum_{i,j=1}^{n} |\alpha_i|\,|\alpha_j|\,|(x_i, x_j)| \sum_{i,j=1}^{n} |\beta_i|\,|\beta_j|\,|(y_i, y_j)|.$$

Using the Cauchy-Buniakowski-Schwarz's inequality for double sums, one has:

$$\sum_{i,j=1}^{n} |\alpha_i||\alpha_j||(x_i,x_j)| \le \left(\sum_{i,j=1}^{n} |\alpha_i|^2|(x_i,x_j)|\right)^{1/2}\left(\sum_{i,j=1}^{n} |\alpha_j|^2|(x_i,x_j)|\right)^{1/2}$$
$$= \sum_{i,j=1}^{n} |\alpha_i|^2\,|(x_i,x_j)|$$

and

$$\sum_{i,j=1}^{n} |\beta_i|\,|\beta_j|\,|(y_i,y_j)| \le \sum_{i,j=1}^{n} |\beta_i|^2\,|(y_i,y_j)|$$

and the first inequality in (3) is proved.

For the second inequality, we observe, by the same inequality, that

$$\sum_{i,j=1}^{n} |\alpha_i|\,|\alpha_j|\,|(x_i,x_j)| \le \left(\sum_{i,j=1}^{n} |\alpha_i|^2|\alpha_j|^2\right)^{1/2}\left(\sum_{i,j=1}^{n} |(x_i,x_j)|^2\right)^{1/2} =$$
$$= \sum_{i=1}^{n} |\alpha_i|^2\left(\sum_{i,j=1}^{n} |(x_i,x_j)|^2\right)^{1/2}$$

and

$$\sum_{i,j=1}^{n} |\beta_i||\beta_j||(y_i,y_j)| \le \sum_{i=1}^{n} |\beta_i|^2\left(\sum_{i,j=1}^{n} |(y_i,y_j)|^2\right)^{1/2}$$

which proves the statement.

For the last inequality, we observe that:

$$\sum_{i,j=1}^{n} |\alpha_i|\,|\alpha_j|\,|(x_i,x_j)| \le \sum_{i=1}^{n} |\alpha_i| \max_{1\le i\le n}\left\{\sum_{j=1}^{n} |\alpha_j|\,|(x_i,x_j)|\right\}$$

and

$$\sum_{i,j=1}^{n} |\beta_i|\,|\beta_j|\,|(y_i,y_j)| \le \sum_{i=1}^{n} |\beta_i| \max_{1\le i\le n}\left\{\sum_{j=1}^{n} |\beta_j|\,|(y_i,y_j)|\right\}$$

and the proof of the theorem is finished.

Remark 1. If we choose in inequality

$$\left|\sum_{i,j=1}^{n} \alpha_i\beta_j (x_i, y_j)\right|^2 \le \sum_{i,j=1}^{n} |\alpha_i|^2 \,|(x_i, x_j)|\, \sum_{i,j=1}^{n} |\beta_i|^2 \,|(y_i, y_j)| \tag{4}$$

$\alpha_1 = 1,\ \alpha_2 = \ldots = \alpha_n = 0;\ \ x_1 = x,\ x_2 = \ldots = x_n = 0$, we deduce for $\beta_i = c_i\ (i = \overline{1,n})$, the inequality (1).

Now, if we assume that $(x_i)_{i=\overline{1,n}}$, $(y_i)_{i=\overline{1,n}}$ are orthonormal families in X, then

$$\left|\sum_{i,j=1}^{n} \alpha_i\beta_j (x_i, y_j)\right|^2 \le \sum_{i=1}^{n} |\alpha_i|^2 \sum_{i=1}^{n} |\beta_i|^2 \tag{5}$$

which gives us, for $x_i = y_i\ \ (i=\overline{1,n})$ the usual Cauchy-Buniakowski-Schwarz's inequality for real or complex numbers.

Remark 2. If in the inequality

$$\left|\sum_{i,j=1}^{n} \alpha_i\beta_j (x_i, y_j)\right|^2 \le \\ \le \sum_{i=1}^{n} |\alpha_i|^2 \sum_{i=1}^{n} |\beta_i|^2 \left(\sum_{i,j=1}^{n} |(x_i, x_j)|^2 \sum_{i,j=1}^{n} |(y_i, y_j)|^2\right)^{1/2} \tag{6}$$

we put: $\alpha_1 = 1,\ \alpha_2 = \ldots = \alpha_n = 0$ and $x_1 = x,\ x_2 = \ldots = x_n = 0$, we deduce the inequality

$$\left|\sum_{j=1}^{n} \beta_j (x, y_j)\right|^2 \le \|x\|^2 \sum_{i=1}^{n} |\beta_i|^2 \left(\sum_{i,j=1}^{n} |(y_i, y_j)|^2\right)^{1/2}. \tag{7}$$

If in this inequality we choose $\beta_j = \overline{(x, y_j)}\ \ (j=\overline{1,n})$, then we obtain

$$\sum_{j=1}^{n} |(x, y_j)|^2 \le \|x\|^2 \left(\sum_{i,j=1}^{n} |(y_i, y_j)|^2\right)^{1/2}$$

Remark 3. If in the inequality

$$\left|\sum_{i,j=1}^{n}\alpha_i\beta_j(x_i,y_j)\right|^2\le$$
$$\le\sum_{i=1}^{n}|\alpha_i|\sum_{i=1}^{n}|\beta_i|\max_{1\le i\le n}\left\{\sum_{j=1}^{n}|\alpha_j|\,|(x_i,x_j)|\right\}\max_{1\le i\le n}\left\{\sum_{j=1}^{n}|\beta_j|\,|(y_i,y_j)|\right\} \tag{8}$$

we put $\alpha_1 = 1$, $\alpha_2 = \ldots = \alpha_n = 0$ and $x_1 = x$, $x_2 = \ldots = x_n = 0$, we deduce the inequality

$$\left|\sum_{j=1}^{n}\beta_j(x,y_j)\right|^2\le\|x\|^2\sum_{i=1}^{n}|\beta_i|\max_{1\le i\le n}\left\{\sum_{j=1}^{n}|\beta_j|\,|(y_i,y_j)|\right\} \tag{9}$$

which gives, for $\beta_j=\overline{(x,y_j)}$ $(j=\overline{1,n})$, that

$$\sum_{j=1}^{n}|(x,y_j)|^2\le\|x\|\left(\sum_{i=1}^{n}|(x,y_i)|\right)^{1/2}\left(\max_{1\le i\le n}\left\{\sum_{j=1}^{n}|(x,y_j)|\,|(y_i,y_j)|\right\}\right)^{1/2}.$$

Now, if we assume that in this last inequality, the family $(y_i)_{i=\overline{1,n}}$ is orthonormal, then we obtain

$$\sum_{j=1}^{n}|(x,y_j)|^2\le\|x\|\left(\max_{1\le i\le n}\{|(x,y_i)|\}\sum_{i=1}^{n}|(x,y_i)|\right)^{1/2} \tag{10}$$

which is another inequality of Bessel's type.

If in (8) we consider that $(x_i)_{i=\overline{1,n}}$, $(y_i)_{i=\overline{1,n}}$ are orthonormal, then we have:

$$\left|\sum_{i,j=1}^{n}\alpha_i\beta_j(x_i,y_j)\right|^2\le\sum_{i=1}^{n}|\alpha_i|\sum_{i=1}^{n}|\beta_i|\max_{1\le i\le n}\{|\alpha_i|\}\max_{1\le i\le n}\{|\beta_i|\}.$$

The second result is embodied in the next theorem:

THEOREM 2. *Let* $x, x_i\in X$ $(i=\overline{1,n})$ *and* $c_i\in K$ $(i=\overline{1,n})$. *Then one has the inequality*

$$\left|\sum_{i=1}^{n} c_i (x, x_i)\right| \le \frac{1}{2} \left(\|x\|^2 + \min \{A', B', C'\} \right) \tag{11}$$

where

$$A' := \sum_{i,j=1}^{n} |c_i|^2 \, |(x_i, x_j)|, \quad B' := \sum_{i=1}^{n} |c_i|^2 \left(\sum_{i,j=1}^{n} |(x_i, x_j)|^2 \right)^{1/2}$$

and

$$C' := \sum_{i=1}^{n} |c_i| \max_{1 \le i \le n} \left\{ \sum_{j=1}^{n} |c_j| \, |(x_i, x_j)| \right\}$$

respectively.

Proof. We have

$$0 \le \left\| x - \sum_{i=1}^{n} \bar{c}_i x_i \right\|^2 = \|x\|^2 - 2\, Re\left[\left(x, \sum_{i=1}^{n} \bar{c}_i x_i \right) \right] + \left\| \sum_{i=1}^{n} \bar{c}_i x_i \right\|^2$$

$$\le \|x\|^2 - 2 \left| \sum_{i=1}^{n} c_i (x, x_i) \right| + \sum_{i,j=1}^{n} \bar{c}_i c_j (x_i, x_j)$$

from where we get

$$\left| \sum_{i=1}^{n} c_i (x, x_i) \right| \le \frac{1}{2} \left(\|x\|^2 + \sum_{i,j=1}^{n} \bar{c}_i c_j (x_i, x_j) \right) \le$$

$$\le \frac{1}{2} \left(\|x\|^2 + \sum_{i,j=1}^{n} |c_i| \, |c_j| \, |(x_i, x_j)| \right).$$

Now, by an argument similar to that embodied in the proof of Theorem 1, we obtain the inequality (11). We will omit the details.

Remark 4. In the above assumptions, for $c_i = \overline{(x, x_i)}$ $(i = \overline{1, n})$, we have the inequality

$$\sum_{i=1}^{n} |(x,x_i)|^2 \left(2-\sum_{j=1}^{n} |(x_i,x_j)|\right) \le \|x\|^2 , \quad x \in X. \qquad (12$$

Indeed, by the inequality

$$\left|\sum_{i=1}^{n} c_i (x,x_i)\right| \le \frac{1}{2}\left(\|x\|^2 + \sum_{i=1}^{n} |c_i|^2 \sum_{j=1}^{n} |(x_i,x_j)|\right)$$

from (11), we obtain for $c_i = \overline{(x,x_i)}$ $(i=\overline{1,n})$ that

$$\sum_{i=1}^{n} |(x,x_i)|^2 \le \frac{1}{2}\left(\|x\|^2 + \sum_{i=1}^{n} |(x,x_i)|^2 \sum_{j=1}^{n} |(x_i,x_j)|\right)$$

which is equivalent with (12).

Now, if we assume in (12) that $(x_i)_{i=\overline{1,n}}$ is orthonormal, then we deduce Bessel's inequality.

Remark 5. By the inequality

$$\left|\sum_{i=1}^{n} c_i (x,x_i)\right| \le \frac{1}{2}\left(\|x\|^2 + \sum_{i=1}^{n} |c_i| \max_{1\le i\le n} \left\{\sum_{j=1}^{n} |c_j|\, |(x_i,x_j)|\right\}\right)$$

we obtain for $c_i = \overline{(x,x_i)}$ $(i=\overline{1,n})$ that

$$\sum_{i=1}^{n} |(x,x_i)|^2 \le \frac{1}{2}\left(\|x\|^2 + \sum_{i=1}^{n} |(x,x_i)| \max_{1\le i\le n} \left\{\sum_{j=1}^{n} |(x,x_j)|\, |(x_i,x_j)|\right\}\right)$$

which gives for $(x_i)_{i=\overline{1,n}}$ orthonormal, another inequality of Bessel's type

$$\sum_{i=1}^{n} |(x,x_i)|^2 \le \frac{1}{2}\left(\|x\|^2 + \max_{1\le i\le n} \{|(x,x_i)|\} \sum_{i=1}^{n} |(x,x_i)|\right).$$

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NONLINEAR PERTURBATIONS OF THE LINEAR ASYMPTOTICALLY CONTROLLABLE SYSTEMS

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REZUMAT. - Perturbaţii neliniare ale sistemelor asimptotic controlabile. În această lucrare se studiază modul cum se păstrează proprietatea de controlabilitate asimptotică a unui sistem liniar la o anumită clasă de perturbări neliniare.

1. Introduction. In the sequel we shall restrict our attention about the systems:

(A,B) $\dot{X} = A(t)x + B(t)u$

(A,B,f) $\dot{X} = A(t)x + B(t)u + f(t,x)$

where $x \in R^n$ is the state vector, $u \in R^m$ is the control vector, the matrices $A(t)$ and $B(t)$ are of dimensions $n\times n$ and $n\times m$ respectively and their entries are real continuous functions on $J = [t_0, \infty)$, and $f: J \times R^n \to R^n$ is a continuous function. Here R^p is the real Euclidean space of dimension p of all vectors $z = (z_1,\dots,z_p)$ endowed whith the norm $|z| = (z_1^2 + \dots + z_p^2)^{1/2}$.

We denote by $\phi(t,s)$ the evolution matrix generated by $A(t)$ and by $x(t;t_0,x_0,u)$ the solution of (A,B) corresponding to control u which satisfies the initial condition $x(t_0) = x_0$. By $BC(J,R^n)$ we denote the Banach space of bounded and continuous mappings $x:J \to R^n$ endowed with the norm:

$$\|x\| = \sup_{t\ge t_0} |x(t)|$$

We recall the next definition ([3]):

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DEFINITION. The system (A,B) is said to be *asymptotically controllable* if for every $x_0, x_1 \in R^n$ there exists a continuous function $u: J \to R^m$ such that:

$$\lim_{t\to\infty} x(t; t_0, x_0, u) = x_1$$

By analogy with the above definition, we say that the system (A,B,f) is *asymptotically controllable* if for every $x_0, x_1 \in R^n$ there exists a continuous function $u: J \to R^m$ and a corresponding solution $x(t;\ t_0, x_0, u)$ such that:

$$\lim_{t\to\infty} x(t; t_0, x_0, u) = x_1 .$$

Our problem is to found some conditions on the term $f(t,x)$ such that if the system (A,B) is asymptotically controllable then the perturbed system (A,B,f) is also asymptotically controllable.

In the case of usual controllability this problem was investigated by Muntean ([6]) for the perturbing nonlinear term not depending upon the control vector. More general systems with the perturbing nonlinear term depending upon t,x and u are investigated in [1] and [4].

The main result of this paper and the method of proof are in the same spirit as the related results of Balachandran and Dauer ([2]) and Tonkov ([7]). The main tool used is Banach's fixed-point argument.

2. Main result.

THEOREM. *Assume there are fulfilled the following condition:*

(i) $\int_{t_0}^{t} \| \Phi(t,s) \| ds \le K_1$ *and* $\int_{t_0}^{t} \| \Phi(t_0,s) \| ds \le K_2,\ t \ge t_0$

where $K_1 > 0$, $K_2 > 0$,

(ii) $f(t,0) = 0$, $t \geq t_0$

(iii) $|f(t,x_1) - f(t,x_2)| \leq a(t)\cdot|x_1-x_2|$, $x_1,x_2 \in R^n$, $t \geq t_0$

where $a \in BC(J,R_+)$.

If the system (A,B) *is asymptotically controllable and* $\alpha K_1 < 1$, where $\alpha = \sup_{t\geq t_0} a(t)$, *then the perturbed system* (A,B,f) *is also asymptotically controllable.*

Proof. We first define on the Banach space $BC(J,R^n)$ the operator T, as below:

$$(Tx)(t) = \int_{t_0}^{t}\phi(t,s)\,f(s,x(s))\,ds$$

We observe that T is continuous and from (ii) and (iii) we have $t \geq t_0$:

$$|(Tx)(t)| \leq \int_{t_0}^{t}\|\phi(t,s)\|\cdot|f(s,x(s))|ds \leq \int_{t_0}^{t}\|\phi(t,s)\|\cdot a(s)\,|x(s)|ds \leq$$

$$\leq (\sup_{t\geq t_0} a(t))\cdot\|x\|\cdot\int_{t_0}^{t}\|\phi(t,s)\|ds \leq \alpha K_1\cdot\|x\|.$$

Consequently Tx is bounded and thus:

$T(BC(J,R^n)) \subseteq BC(J,R^n)$.

For all $y_1,y_2 \in BC(J,R^n)$ and $t \geq t_0$ we have:

$$|(Ty_1)(t) - (Ty_2)(t)| \leq \int_{t_0}^{t}\|\phi(t,s)\|\cdot|f(s,y_1(s)) - f(s,y_2(s))|ds \leq$$

$$\leq\int_{t_0}^{t}\|\phi(t,s)\|\cdot a(s)\cdot|y_1(s)-y_2(s)|ds\leq\alpha\|y_1-y_2\|\cdot\int_{t_0}^{t}\|\phi(t,s)\|ds\leq\alpha K_1\|y_1-y_2\|,$$

$$\Rightarrow \|Ty_1 - Ty_2\| \leq \alpha K_1\|y_1 - y_2\|.$$

Therefore T is a contraction.

We next denote by **B** the set all bounded solutions of (A,B) and $\mathbf{B}_f$ the set all bounded of (A,B,f) on the interval J, where each pair $(x,\tilde{x}) \in \mathbf{B}\times\mathbf{B}_f$ corresponds to the same control u.

For each $x \in \mathbf{B}$ we define the operator

$T_x: BC(J,R^n) \to BC(J,R^n)$

$(T_x y)(t) = x(t) + (Ty)(t)$ (1)

From here we have

$\|T_x y_1 - T_x y_2\| = \|Ty_1 - Ty_2\|$

and therefore T_x is also a contraction.

Applying the well-known Banach's theorem, T_x will have, for each $x \in \mathbf{B}$, one fixed point $\tilde{x} \in BC(J,R^n)$, and according to (1) we can write:

$$\tilde{x}(t) = x(t) + \int_{t_0}^{t} \Phi(t,s)\, f(s,\tilde{x}(s))\, ds, \quad t \geq t_0. \tag{2}$$

Differentiating the relation (2), we observe that $\tilde{x}$ verifies the system (A,B,f), more precisely $\tilde{x} \in \mathbf{B}_f$.

Now, let $x_0, x_1 \in R^n$ be arbitrarily chosen and fixed. Since (A,B) is asymptotically controllable, there is $u: J \to R^m$ continuous so that the solution $x(\cdot) = x(\cdot;\, t_0, x_0, u)$ satisfies:

$$\lim_{t\to\infty} x(t) = x_1 \tag{3}$$

As we previously showed, for this control u there is a unique bounded solution $\tilde{x}(\cdot)$ of (A,B,f), which verifies the relation (2).

Also from (2) we observe that $\tilde{x}(t_0) = x(t_0) = x_0$.

It remains to shown that $\lim_{t\to\infty} \tilde{x}(t) = x_1$.

From (2) we have:

$$|\tilde{x}(x) - x(t)| \le \int_{t_0}^{t} \|\Phi(t,s)\| \cdot |f(s,\tilde{x}(s))|\,ds =$$

$$= \int_{t_0}^{t} \|\Phi(t,t_0)\,\Phi(t_0,s)\| \cdot |f(s,\tilde{x}(s))|\,ds \le \tag{4}$$

$$\le \|\Phi(t,t_0)\| \cdot \int_{t_0}^{t} \|\Phi(t_0,s)\| \cdot a(s)\,|\tilde{x}(s)|\,ds \le$$

$$\le \alpha K_2 \|\tilde{x}\| \cdot \|\Phi(t,t_0)\|.$$

We denote

$$\varphi(t) = \|\Phi(t,t_0)\|^{-1}, \quad t \ge t_0 \tag{5}$$

and we have:

$$\Phi(t,t_0) \cdot \int_{t_0}^{t} \varphi(s)\,ds = \int_{t_0}^{t} \Phi(t,s)\,\Phi(s,t_0)\,\varphi(s)\,ds,$$

from where

$$\|\Phi(t,t_0)\| \cdot \int_{t_0}^{t} \varphi(s)\,ds \le \int_{t_0}^{t} \|\Phi(t,s)\| \cdot \frac{1}{\varphi(s)}\,\varphi(s)\,ds \le K_1 \tag{6}$$

and from here it follows:

$$\|\Phi(t,t_0)\| \le \frac{K_1}{\int_{t_0}^{t} \varphi(s)\,ds} \tag{7}$$

Now, we denote

$$\psi(t) = \int_{t_0}^{t} \varphi(s)\,ds, \quad t \ge t_0$$

and therefore

$$\psi'(t) = \varphi(t), \quad t \ge t_0.$$

From (5) we obtain:

$$\|\Phi(t,t_0)\| = \frac{1}{\psi'(t)}, \quad t \ge t_0.$$

Then, from (7) it follows:

$$\frac{\psi(t)}{\psi'(t)} \le K_1, \quad \text{namely} \quad \frac{\psi'(t)}{\psi(t)} \ge \frac{1}{K_1}$$

Integrating over $[t_1, t]$, $t_1 > t_0$, we obtain:

$$\ln[\psi(t)]_{t_1}^{t} \ge \frac{t-t_1}{K_1}, \quad \text{namely} \quad \ln\frac{\psi(t)}{\psi(t_1)} \ge \frac{t-t_1}{K_1}$$

$$\Rightarrow \psi(t) \ge \psi(t_1)\cdot e^{\frac{t-t_1}{K_1}}, \quad t \ge t_0$$

From here we obtain $\lim_{t\to\infty} \psi(t) = \infty$ and from (7) it follows:

$$\lim_{t\to\infty} \|\Phi(t, t_0)\| = 0$$

Passing at limit for $t \to \infty$ in (4), we get:

$$\lim_{t\to\infty} |\tilde{x}(t) - x(t)| = 0 \tag{8}$$

On the other hand:

$$|\tilde{x}(t) - x_1| \le |\tilde{x}(t) - x(t)| + |x(t) - x_1|,$$

from where it results, according to (3) and (8):

$$\lim_{t\to\infty} \tilde{x}(t) = x_1.$$

Consequently, we showed that the asymptotically controllability of (A,B) implies the asymtotically controllability of the perturbed system (A,B,f).

The proof is complete.

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ON COMPLETENESS OF METRIC SPACES

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REZUMAT. - Asupra completitudinii spaţiilor metrice. În această lucrare se studiază existenţa metricilor complete pe un spaţiu topologic local compact, Hausdorff şi cu bază numărabilă şi existenţa metricilor Riemann complete pe o varietate diferenţiabilă finit dimensională. La sfîrşitul lucrării se dau aplicaţii în teoria geodezicelor.

1. Introduction. Many authors obtained interesting results concerning the existence of a complete metric on metrisable topological spaces and on differentiable manifolds see [1], [2], [3], [4] and [5].

Let (M,τ) be a locally compact topological space with countable base. In this paper we consider the space of all metrics on M generating the topology τ, endowed with the compact-open topology. The main result states that the set of complete metrics of the above space is dense. If M is a finite dimensional differentiable manifold we obtain that the set of complete Riemannian (or Finslerian) metrics is dense in the space of all Riemannian (respectively Finslerian) metrics on M.

Let (M,τ) and (M',τ') two topological spaces and we introduce the following notation, $\text{Top}(M,M') = \{f: M \to M' | f \text{ is a continuous function}\}$.

Let $K \subseteq M$ be a compact subset and $D \subseteq M'$ an open subset and we use the notation $B(K,D) = \{f \in \text{Top}(M,M')/f(K) \subseteq D\}$.

DEFINITION 1.1. On the set $\text{Top}(M,M')$ we consider the

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topology for which the family $\{B(K,D)\}$ is a subbase, where $K \subseteq M$ is compact and $D \subset M'$ is an open subset. This topology is called the compact-open topology and the topological space thus obtained is denoted by $\mathrm{Top}_{CO}(M,M')$.

DEFINITION 1.2. A continuous function $f\colon M \to M'$ is called a proper function if for every $K \subseteq M'$ compact set, $f^{-1}(K)$ is a compact subset of M.

We need the following results from [1], [4] and [6].

PROPOSITION 1.1. *Let (M,τ) be a Hausdorff, locally compact topological space which satisfies the second axiom of countability and $K \subseteq M$ a compact subset. If $g\colon M \to R$ is a continuous function, then there exists a proper function $f\colon M \to R$ such that $f|K = g|K$.*

PROPOSITION 1.2. *Let (M,d) be a metric space and $f\colon M \to R$ a continuous proper function. Then $d'\colon M\times M \to R_+$, where $d'(p,q) = d(p,q) + |f(q) - f(q)|$ for every $p,q \in M$, is a complete metric on M which generates the same topology on M as d.*

PROPOSITION 1.3. *Let M be a finite dimensional, differentiable manifold of class C^k $(k \geq 1)$ and $K \subseteq M$ a compact subset. If $g\colon M \to R$ is a C^k differentiable function, then there exists a C^k proper function $f\colon M \to R$ such that $f|K = g|K$.*

PROPOSITION 1.4. *Let M be a Riemann manifold of class $C^k (k \geq 1)$ with metric tensor $g = (g_{ij})$ and let $f\colon M \to R$ be a proper function of class C^k. Then M is complete with respect to the metric $\tilde{g} = (\tilde{g}_{ij})$ where $\tilde{g}_{ij} = g_{ij} + \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}$.*

2. Let (M,τ) be a topological space. It is known that if

(M,τ) is a locally compact, Hausdorff topological space satisfying the second axiom of countability then M is metrisable. Therefore we can consider the space of metrics on M, which generate the topology τ, i.e. let $\mathbf{M} = \{\rho: M\times M \to R_+ | \rho$ is a metric and $\tau_\rho = \tau\}$ and $\mathbf{M}$ is endowed with the compact-open topology. Now, let $\mathbf{M}_0$ be the subset of $\mathbf{M}$ consisting the complete metrics on M which generate the topology τ.

We have the following result.

THEOREM 2.1. *$\mathbf{M}_0$ is dense in $\mathbf{M}$ in the compact-open topology.*

Proof. Let $\rho_0 \in \mathbf{M}$ a metric on M and $V \in V_{\mathbf{M}}(\rho_0)$. It suffices to prove $V \cap \mathbf{M}_0 \neq \phi$. Using the fact that the family $\{B(K,D)\}$ is a subbase for the compact-open topology of $\mathbf{M}$, where $K \subset M\times M$ is compact and $D \subset R$ is an open subset, it follows the existence of compact subsets $K_1, K_2, \ldots, K_r \subset M\times M$ and open subsets $D_1, D_2, \ldots, D_r \subset R$ such that $\rho_0 \in \bigcap_{i=1}^{r} B(K_i, D_i) \subseteq V$.

Let $\Pi_1,\Pi_2: M\times M \to M$ the canonical projections, i.e. $\Pi_1(x,y) = x$ and $\Pi_2(x,y) = y$ for every $(x,y) \in M\times M$.

For every $i \in \{1,2,\ldots,r\}$ we have: $K_i \subsetneq \Pi_1(K_i)\times\Pi_2(K_i) \subseteq$ $\subseteq (\Pi_1(K_i) \cup \Pi_2(K_i))\times(\Pi_1(K_i) \cup \Pi_2(K_i))$ and $\Pi_1(K_i) \cup \Pi_2(K_i) \subseteq M$ is a compact set. Put $K_i^* := \Pi_1(K_i) \cup \Pi_2(K_i) \subseteq M$ and, then we have $K_i \subseteq K_i^* \times K_i^*$ for every $i \in \{1,2,\ldots,r\}$. Since $K_1^* \cup \ldots \cup K_r^* \subset M$ is a compact set and using Proposition 1.1. we may define a proper function $f: M \to R$ such that $f|_{K_1^*\cup\ldots\cup K_r^*} = 0$.

Define a metric $\rho_i: M\times M \to R$ as follows: $\rho_i(x,y) = \rho_0(x,y) + |f(x) - f(y)|$, for every $x,y \in M$. Using Proposition 1.2. we have that ρ_1 is a complete metric and $\tau_{\rho_1} = \tau_{\rho_0} = \tau$. The theorem will be proved once we show that $\rho_1 \in V$. Indeed, for every

$i \in \{1,2,\ldots,r\}$, and for every $(x,y) \in K$, we have $\rho_0(x,y) \in D_i$. But $(x,y) \in K_i$ implies that $(x,y) \in K_i^* \times K_i^*$ and we have $f(x) = = f(y) = 0$, then $\rho_1(x,y) = \rho_0(x,y) \in D_i$ and $\rho_1(K_i) \subseteq D_i$ for every $i \in \{1,2,\ldots,r\}$. In conclusion the relation $\rho_1 \in \bigcap_{i=1}^{x} B(K_i, D_i) \subseteq V$ holds.

In the following let M be a differentiable manifold of class C^k, where $k \in N^*$. We begin by recalling some definitions. For every $x \in M$ let $T_{2,x}^0 = T_x^*(M) \otimes T_x^*(M) = L(T_xM, T_xM, R)$ denote the space of tensors of type(0,2) which are tangent at x to M, and let $T_2^0(M) = \bigcup_{x \in M} T_{2,x}^0(M)$ denote the bundle of tensors of type(0,2) on M.

Recall that a Riemannian metric on M is a symmetric tensor field of type(0,2), $g: M \to T_2^0(M)$. It is known that every Riemannian metric generates a structure of metric space on M, with the metric $d_g: M \times M \to R_+$, $d_g(x,y) = \inf_{\gamma} L_\gamma(x,y)$, for every $x,y \in M$ (infinimum of the set of arc lengths of paths joining x to y).

DEFINITION 2.1. A Riemannian metric g is said to be complete if either of the following two equivalent conditions hold:

1^0 (M, d_g) is a complete metric space.

2^0 Enery geodesic arc $c = c(t)$ can be extended for all values of $t \in R$.

Now we consider the space of function $\mathfrak{R} = \{g: M \to T_2^0(M) \mid g$ is a Riemannian metric$\}$ endowed with the compact-open topology and the subspace $\mathfrak{R}_0 = \{g: M \to T_2^0(M) / g$ is a complete Riemann metric$\}$.

THEOREM 2.2. *$\mathfrak{R}_0$ is dense in $\mathfrak{R}$ with respect to the compact-open topology.*

Proof. Let $g_0 \in \mathfrak{R}$ be a fixed Remannian metric on M and

$V \in \mathbf{V}_{\mathfrak{R}}(g_0)$ a fixed neighbourhood. We shall prove that $V \cap \mathfrak{R}_0 \neq \emptyset$. From the definition of the compact open topology on $\mathfrak{R}$ there exist compact subsets $K_1, \ldots, K_r \subseteq M$ and the open subsets $D_1, \ldots, D_r \subset T_2^0(M)$ such that $g_0 \in \bigcap_{i=1}^{r} B(K_i, D_i) \subseteq V$. We consider the compact set $K^* = \bigcup_{i=1}^{r} K_i \subseteq M$. Since M is a locally compact space it folows that there exists an open subset $G \subseteq M$, such that $\overline{G}$ is compact and $K^* \subseteq G$. Using Proposition 1.3. there exists a C^k proper function $f: M \to R$ such that $f|_{\overline{G}} = 0$.Next we consider a new Riemannian metric $\tilde{g}$ on M, which is defined in the following manner. If $(U,\varphi) = (U,x^1,\ldots,x^n)$ is a local chart at p, and $g_p = g_{ij}(p)dx^i dx^j$, we define $\tilde{g}_p = g_p + \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}dx^i dx^j$, for every $p \in U$. We obtain that $\tilde{g}$ is a Riemannian metric on M, and using Proposition 1.4. we have that $\tilde{g}$ is a complete metric.

The theorem will be proved once we show that $\tilde{g} \in V$.

Indeed for every $i \in \{1,\ldots,r\}$ and for every $p \in K_i$ we have $\frac{\partial f}{\partial x^i} = \frac{\partial f}{\partial x^j} = 0$ and from the relation $\tilde{g}_p = g_p + \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}dx^i dx^j$ results that $\tilde{g}_p = g_p \in D_i$. Therefore $\tilde{g} \in B(K_i, D_i)$ for every $i \in \{1,2,\ldots,r\}$ and we have $\tilde{g} \in \bigcap_{i=1}^{r} B(K_i, D_i) \leq V$.

Remark 2.1. In this proof we obtain that if g is a Riemannian metric on M, and $K \subseteq M$ is a compact subset then there exists $\tilde{g}$

a complete. Riemannian metric on M such that $g|_k = \tilde{g}|_k$.

3. In this section we give some applications of the Theorem 2.2. to geodesics.

PROPOSITION 3.1. *Let (M,g) be a (not necessarily complete) Riemannian manifold of class C^k. For every point $p \in M$, and for* *a geodesic* $c: I \to M$ $(0 \in \mathrm{int} I)$ *whith respect to the metric* g *such that* $c(0) = p$, *there exists a complete Riemannian metric* $\tilde{g}$ *on* M *and an arc of geodesic* c *such that this arc is a geodesic arc with respect to the metric* $\tilde{g}$, *too.*

Proof. Let $(U,\varphi) = (U,x^1,\ldots,x^n)$ be a local chart in p, such that $\bar{U}$ is compact. Also, let $\varepsilon > 0$ be a positive real number such that $[-\varepsilon,\varepsilon] \subset I$ and $c([-\varepsilon,\varepsilon]) \subseteq U$. The differential equation of geodesics with respect to the metric g is as follows:

$$\frac{d^2c^k(t)}{dt^2} + \sum_{i,j=1}^{n} \Gamma^k_{ij}(c(t)) \frac{dc^i(t)}{dt} \frac{dc^j(t)}{dt} = 0, \qquad (1)$$

for every $k = \overline{1,n}$, where Γ^k_{ij} are the Christoffel symbols of the second kind.

Using the proof of Theorem 2.2. we obtain a complete Riemannian metric $\tilde{g}$ such that $g|_U = \tilde{g}|_U$.

Therefore the differential equation of geodesics with respect to the metric $\tilde{g}$ is also (1). Hence we obtain that $c: [-\varepsilon,\varepsilon] \to M$ is a geodesic arc with respect to the metric $\tilde{g}$.

Remark 3.2. c can be extended as a geodesic with respect to the metric $\tilde{g}$, for all values of $t \in R$.

PROPOSITION 3.2. *Let* (M,g) *a Riemannian manifold and* $c: [a,b] \to M$ *a geodesic with respect to the metric* g. *Then there exists a complete Riemannian metric* $\tilde{g}$ *on* M *such that* c *is a geodesic are with respect to the metric* g.

Proof. Let $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ be a subdivision of $[a,b]$, such that for every $k \in \{1,2,\ldots,n\}$, $c([t_{k-1},t_k])$ is contained in a coordinate neighbourhood on M. We suppose that for every $k = \overline{1,n}$ we have $c([t_{k-1},t_k]) \subseteq U_k$, where (U_k,φ_k) is a local chart on M and $\bar{U}_k$ is compact. We consider again a complete Riemannian metric $\tilde{g}$ on M such that $g|_{\bar{U}_1 \dots \bar{U}_n} = \tilde{g}|_{\bar{U}_1 \dots \bar{U}_n}$. Therefore $c|_{[t_{k-1}, t_k]} : [t_{k-1}, t_k] \to M$ is a geodesic arc with respect to the metric $\tilde{g}$, and finally we obtain that c: $[a,b] \to M$ is a geodesic arc with respect to $\tilde{g}$.

Remark 3.3. Theorem 2.2. and Propositions 3.1, 3.2. are true for Finsler spaces.

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ON A METHOD FOR POTENTIAL FUNCTION DETERMINATION FROM MOTION EQUATIONS

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REZUMAT. - Asupra unei metode de determinare a funcţiei de forţă pornind de la ecuaţiile mişcării. Nota de faţă prezintă o metodă de determinare a funcţiei de forţă care generează un câmp de forţe, date fiind ecuaţiile mişcării unui punct material de masă unitate în acest câmp, în spaţiul euclidian n-dimensional.

Many problems of celestial mechanics, in which the force is known as deriving from a potential function, require the determination of this last one. The way to find the force function starting from a known orbit (or family of orbits) is ensured by Szebehely's [4] renowned equation. (It is to be however mentioned that the Romanian astronomer C.Drâmbă [2] found an equivalent equation some decade before V.Szebehely.)

The purpose of this note is to point out a method for the determination of the potential function starting this time from the equations of motion. So, consider a material point of unit mass moving in the n-dimensional Euclidean space R^n, in a force field generated by the potential function U_n (the index added to U signifies hereafter simply the dimension of the space we deal with), and write the equations of motion with respect to an inertial frame in the form

$$\ddot{x}_i = f_i(x_1, \ldots, x_n), \quad i = \overline{1,n},$$

where $(x_1, \ldots, x_n) \in R^n$ is the position vector of the point mass and f_i are known real-valued functions. We search for a function

$U_n = U_n(x_1,\ldots,x_n)$ of class $C^2(I)$, $I \subseteq R^n$, which verifies the system

$$\partial U_n/\partial x_i = f_i(x_1,\ldots,x_n),\quad i=\overline{1,n}, \tag{1}$$

the existence condition

$$\partial^2 U_n/\partial x_i\,\partial x_j = \partial^2 U_n/\partial x_j\,\partial x_i,\quad i,j=\overline{1,n},$$

namely

$$\partial f_i/\partial x_j = \partial f_j/\partial x_i,\quad i,j=\overline{1,n}, \tag{2}$$

being fulfilled. We shall suppose, without loss of generality, that $U_n(x_1^0,\ldots,x_n^0) = 0$, where the superscript marks the initial values.

We shall prove by induction that

$$U_n = \sum_{i=1}^{n}\int_{x_i^0}^{x_i} f_i(x_1,\ldots,x_{i-1},x_i',x_{i+1}^0,\ldots,x_n^0)\,dx_i', \tag{3}$$

where the prime indicates the variable with respect to which the integration is performed.

In the case $n = 1$, equations (1) reduce to

$$dU_1/dx_1 = f_1(x_1),$$

with the solution

$$U_1 = \int_{x_1^0}^{x_1} f_1(x_1')\,dx_1',$$

hence (3) is true.

Suppose that (3) is true for n and prove for $n+1$. The system (1) becomes

$$\partial U_{n+1}/\partial x_i = f_i(x_1, \ldots, x_n, x_{n+1}), \quad i = \overline{1, n+1}. \tag{4}$$

The integration of the last equation in (4) with respect x_{n+1} (cf. [3]) leads to

$$U_{n+1} = \int_{x_{n+1}^0}^{x_{n+1}} f_{n+1}(x_1, \ldots, x_n, x'_{n+1})\, dx'_{n+1} + V_n(x_1, \ldots, x_n), \tag{5}$$

where the expression of the function V_n (of only n variables) is not determined yet. Replacing (5) in the rest of equations (4), and taking into account condition (2), we get

$$\int_{x_{n+1}^0}^{x_{n+1}} (\partial f_i/\partial x_{n+1})(x_1, \ldots, x_n, x'_{n+1})\, dx'_{n+1} + \partial V_n/\partial x_i =$$
$$= f_i(x_1, \ldots, x_n, x_{n+1}), \quad i = \overline{1, n},$$

which lead to

$$\partial V_n/\partial x_i = g_i(x_1, \ldots, x_n), \quad i = \overline{1, n}, \tag{6}$$

where we denoted

$$g_i(x_1, \ldots, x_n) = f_i(x_1, \ldots, x_n, x_{n+1}^0), \quad i = \overline{1, n}.$$

One sees that the functions g_i fulfil condition (2). The system (6) is of the form (1), for which (3) holds. Writing hence (3) for (6), and substituting it into (5), we get

$$U_{n+1} = \sum_{i=1}^{n} \int_{x_i^0}^{x_i} g_i(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}^0, \ldots, x_n^0)\, dx'_i +$$
$$+ \int_{x_{n+1}^0}^{x_{n+1}} f_i(x_1, \ldots, x_n, x'_{n+1})\, dx'_{n+1}.$$

Lastly, we replace the expresion of g_i in the above formula, obtaining

$$U_{n+1} = \sum_{i=1}^{n+1} \int_{x_i^0}^{x_i} f_i(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}^0, \ldots, x_{n+1}^0)\, dx'_i,$$

and the proof is complete.

To give a simple example, let the motion be described by the equations

$$\ddot{x}_i = -\alpha\mu r^{-(\alpha+2)} x_i, \quad i = \overline{1,n}, \tag{7}$$

where α is a positive constant,

$$r = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

is the radius vector of the point mass, and we call μ (constant) attraction parameter. The functions f_i (right-hand sides of the system (7)) fulfil condition (2), hence (3) holds and yields

$$U_n = \mu r^{-\alpha}.$$

This is the potential function which generates an attraction obeying an inverse $(\alpha + 1)$-law (see e.g. [1]). Placing a point mass in the origin of the frame, and concretizing $n = 3$, $\alpha = 1$, we have the relative motion in the classic 3-spatial two-body problem, where μ is the gravitational parameter of the dynamic system, and $U_3 = \mu/r$ is the well-known Newtonian force function.

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FIRST AND SECOND ORDER PERTURBATIONS OF INITIALLY CIRCULAR ORBITS IN MARS' ATMOSPHERE

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REZUMAT. - Perturbaţiile de ordinul întâi şi doi ale unei orbite iniţial circulare în atmosfera planetei Marte. Se studiază mişcarea iniţial circulară a unui orbiter în atmosfera simetrică şi fără rotaţie a planetei Marte. Adoptându-se un model analitic pentru distribuţia densităţii în atmosfera marţiană în gama de înălţimi 100-1000 km, se determină perturbaţiile de ordinele întâi şi al doilea în cinci elemente orbitale independente pe durata unei perioade nodale.

1. Introduction. The first analytic models for the density distribution with height in the Martian atmosphere were proposed in [6,7], on the basis of the numerical data listed by the *MA-87* model. Two altitude ranges are covered: 0-100 km [6] and 100-1000 km [7]. The motion of an orbiter in the lower altitude range was studied in [3].

For heights between 100-1000 km, the density profile of the Martian atmosphere is represented by the formula [7]:

$$\varrho = \varrho_0 \exp(A_0 + A_1/h), \tag{1}$$

where ϱ and $\varrho_0 = 1$ are expressed in kg/m^3, h is the altitude in km above Mars' surface, A_0 and A_1 are constants separately determined for minimal, nominal, and maximal density profiles (some numerical estimates of ours the nominal model with $A_0 = -$ $- 37.936$, $A_1 = 2376.1$).

The motion of an orbiter in this height range was approached in [1,2,4,5], under more or less restrictive conditions as to planet, atmosphere, and orbit. In this paper we shall estimate

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analytically the first and second order perturbations undergone over one nodal period by five independent orbital parameters

$$y \in Y = \{p,\ q = e \cos \omega,\ k = e \sin \omega, \Omega, i\} \qquad (2)$$

of an orbiter moving in the same region of the Martian atmosphere. Here p = semilatus rectum, e = eccentricity, ω = argument of periastron, Ω = longitude of ascending node, i = inclination (all with respect to a frame originated in Mars' mass centre). We shall work with the following hypotheses:

(i) the atmosophere is spherically symmetric;

(ii) the athmospheric rotation is neglected;

(iii) the initial orbit is circular.

2. Basic equations. Since the nodal period was chosen as basic time interval, let us denote: μ = gravitational parameter of the planet, r = planetocentric radius vector, u = argument of latitude, $A = \cos u$, $B = \sin u$, $C = \cos i$, $D = \sin i$, and describe the perturbed motion by means of the Newton-Euler equations written in the form [1, 4, 5]:

$$
\begin{aligned}
dp/du &= 2(Z/\mu)r^3T,\\
dq/du &= (Z/\mu)(r^3kBCW/(pD) + r^2T(r(q + A)/p + A) + r^2BS),\\
dk/du &= (Z/\mu)(-r^3qBCW/(pD) + r^2T(r(k + B)/p + B) - r^2AS),\\
d\Omega/du &= (Z/\mu)r^3BW/(pD), \qquad (3)\\
di/du &= (Z/\mu)r^3AW/p,\\
dt/du &= Zr^2(\mu p)^{-1/2},
\end{aligned}
$$

where S, T, W are the radial, transverse, and binormal components of the perturbing acceleration, respectively, and

The variations of the orbital elements (2) in the interval

$$Z = (1 - r^2 C\dot{\Omega}/(\mu p)^{1/2})^{-1}. \qquad (4)$$

$[u_0, u]$, where the subscript (as hereafter for $y \in Y$) means initial value, are determined from

$$\Delta y = \int_{u_0}^{u} (dy/du)\, du, \quad y \in Y. \qquad (5)$$

These integrals can be estimated from (3) by successive approximations.

3. Perturbing acceleration. Taking into account hypothesis (ii), the components of the perturbing acceleration will be

$$\begin{aligned} S &= -\rho\,\delta\, v_{rel} v_r, \\ T &= -\rho\,\delta\, v_{rel} v_n, \\ W &= 0, \end{aligned} \qquad (6)$$

where ϱ = drag parameter of the orbiter, v_{rel} = orbiter speed with respect to the atmospheric flow, v_r, v_n = radial and trasverse components of the orbiter velocity with respect to the planet centre. Using the orbit equation in polar coordinates under the form

$$r = p/(1 + Aq + Bq), \qquad (7)$$

and the well-known expressions of the velocities v_{rel}, v_r and v_n, given, e.g., in [4], one finds easily (to first order in eccentricity, or, equivalently, in q and k):

$$\begin{aligned} S &= -\rho\,\delta\,(\mu/p)\,(Bq - Ak), \\ T &= -\rho\,\delta\,(\mu/p)\,(1 + 2Aq + 2Bk), \\ W &= 0. \end{aligned} \qquad (8)$$

By the fourth equation (3) and $W = 0$, follows that $Z = 1$.

Now, using (7) and (8), the equations of motion become (also to first order in q and k):

$$\begin{aligned} dp/du &= -2\rho\delta p^2(1 - Aq - Bk), \\ dq/du &= -2\rho\delta p(A + B^2q - ABk), \\ dk/du &= -2\rho\delta p(B - ABq + A^2k), \\ d\Omega/du &= 0, \\ di/du &= 0, \\ dt/du &= p^{3/2}\mu^{-1/2}(1 - 2Aq - 2Bk). \end{aligned} \tag{9}$$

4. Expression of the density. We have to express the density as function of u (through A and B). With hypothesis (i), $h = r-R$ (where R = 3380 km is Mars' radius). Using again the expression (7) and introducing its expansion in (1), we find after some calculations (which can be found e.g. in [1,4] and will not be repeated here) that the density can be written under the form (with the same accuracy of first order in q and k):

$$\varrho = X(1 + (b + 1)Aq + (b + 1)Bk), \tag{10}$$

where we denoted

$$X = \varrho_0 \exp(A_0 + A_1/(p - R)), \tag{11}$$

$$b = A_1p/(p - R)^2 - 1. \tag{12}$$

With (10), equations (9) become (again to first order in q and k):

$$\begin{aligned} dp/du &= -2X\delta p^2(1 + bAq + bBk), \\ dq/du &= -2X\delta p(A + (1 + bA^2)q + bABk), \\ dk/du &= -2X\delta p(B + bABq + (1 + bB^2)k), \\ d\Omega/du &= 0, \\ di/du &= 0, \\ dt/du &= p^{3/2}\mu^{-1/2}(1 - 2Aq - 2Bk). \end{aligned} \tag{13}$$

5. Changes of the orbital elements. Equations (13) will be solved for the elements (2) by successive approximations. Using (5) and (13), and hypothesis (iii), we find

$$\begin{aligned} \Delta p &= -2X\delta p_0^2 (u - u_0)\,, \\ \Delta q &= -2X\delta p_0 (B - B_0)\,, \\ \Delta k &= 2X\delta p_0 (A - A_0)\,, \\ \Delta\Omega &= 0, \\ \Delta i &= 0. \end{aligned} \tag{14}$$

Here $A_0 = A(u_0)$ (do not confuse with the numerical coefficient A_0 which appears in (1) and will no more appear).

Expressions (14) lead immediately to the first order changes of the orbital parameters (2) over one nodal period:

$$\begin{aligned} \Delta_1 p &= -4\pi X\delta p_0^2, \\ \Delta_1 y &= 0, \quad y \in Y - \{p\}\,. \end{aligned} \tag{15}$$

Using (13) and (14), we also found the second order changes

$$\begin{aligned} \Delta_2 p &= 16\pi X^2\delta^2 p_0^3 (\pi - u_0)\,, \\ \Delta_2 q &= -6\pi X^2\delta^2 p_0^2 b B_0, \\ \Delta_2 k &= -2\pi X^2\delta^2 p_0^2 (4 - 3bA_0)\,, \\ \Delta_2\Omega &= 0, \\ \Delta_2 i &= 0. \end{aligned} \tag{16}$$

Observe that only p undergoes first order perturbations, while the second order perturbations affect q and k, too. As to the fact that orbit is planar, this was to be espected, having in view equations (6).

Of course, the other quoted papers (see Section 1) work with less restrictive hypotheses (e.g. small but nonzero initial eccentricity, oblate planet, rotating atmosphere; one, some, or all), but they deal with first order perturbations only. From this viewpoint, this paper is the first to determine second order perturbations, too. Our hypotheses (i)-(iii) can obviously be relaxed, obtaining generalizations of our present results, but this subject will be dealt with elsewhere.

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