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ON THE APPROXIMATION OF FUNCTIONS BY MEANS OF THE OPERATORS OF D.D. STANCU

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Dedicated to Professor D.D.Stancu on his 65th anniversary

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REZUMAT. - Asupra aproximării funcţiilor cu ajutorul operatorilor lui D.D. Stancu. Lucrarea prezintă o sinteză a principalelor rezultate în teoria aproximării uniforme a funcţiilor continue cu ajutorul mai multor clase de operatori liniari şi pozitivi ai lui D.D. Stancu.

Abstract. - The aim of this paper is to present a survey of the principal results obtained in the theory of uniform approximation of continuous functions by means of various classes of linear positive operators of D.D.Stancu. First we consider the original operator S_n^α, of Bernstein type, of D.D.Stancu, which is connected with the probability distribution of Markov-Polya, or with the Vandermonde convolution. We present several representations for them and we investigate the monotonicity properties of the prederivatives of high orders of the sequences of these operators, by making use of a class of positive linear functionals of D.D.Stancu. Estimations of the rate of convergence and evalutions of the remainder terms are also discussed. Further we consider several generalizations of these operators given by different mathematicians. The spline operators included in this paper represent generalizations of the spline operators of Bernstein-Schoenberg type. In the last part of the paper we discuss several multivariate extensions of the operators of

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D.D.Stancu and certain basic approximation results corresponding to some standard regions.

0. **Introduction.** Let us begin by recalling that the D.D.Stancu parameter-dependent linear polynomial operator S_n^α , for the interval $I = [0,1]$ and a function $f:I\to\mathbf{R}$, is defined by

$$(S_n^\alpha f)(x) = S_n^\alpha(f;x) := \sum_{k=0}^{n} w_{n,k}^\alpha(x)\, f\left(\frac{k}{n}\right), \tag{0.1}$$

where, by using the factorial powers, we have

$$w_{n,k}^\alpha(x) := \binom{n}{k} \frac{x^{(k,-\alpha)}\,(1-x)^{(n-k,-\alpha)}}{1^{(n,-\alpha)}} , \tag{0.2}$$

α being a real parameter.

This operator has been introduced and investigated by D.D.Stancu in 1968 in the memoir [37]; it was studied further in his subsequent papers [40] and [44], as well as in several papers published by other authors: [27], [8], [22] and [23].

It is easy to see that the linear operator S_n^α , which maps the space $C(I)$ into itself, is positive if $\alpha\geq 0$ and according to the Vandermonde convolution formula we have

$$S_n^\alpha(1;x) = \sum_{k=0}^{n} w_{n,k}^\alpha(x) = 1 ,$$

so that the norm of this operator is

$$\|S_n^\alpha\| = sup\{\|S_n^\alpha f\|, \|f\|\leq 1\} = 1.$$

If $\alpha<0$ but $-\alpha_n\leq\varepsilon$, with $0\leq\varepsilon<\frac{1}{2}$, then S_n^α is a positive operator on $C([\varepsilon,1-\varepsilon])$ (see [44]) and the following inequality of G.Mühlbach [27] holds: $\| S_n^\alpha \| \le 5$.

One observes that the operator S_n^α is interpolatory at both sides of the interval I and that for $\alpha=0$ it coincides with the Bernstein operator B_n, while for $\alpha=\alpha_n=-1/n$ it becomes, as it has been shown in [37], the Lagrange interpolating operator L_n corresponding to the nodes k/n $(k=0,1,\ldots,n)$.

As limiting cases of S_n^α we can obtain (see [37], [40]) the well known operators of:

i) Favard-Szasz-Mirakyan M_n, defined by

$$(M_n f)(x) := e^{-nx}\sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad x\ge 0$$

(see [9], [25], [58], [61]);

ii) Baskakov P_n, defined by

$$(P_n f)(x) := \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right), \quad x\ge 0$$

(see [1], [13], [59]).

1. **Probabilistic interpretation.** As the Bernstein operator B_n can be generated by starting from the Bernoulli probability distribution, so it is possible to arrive at the operator S_n^α, as D.D.Stancu has shown in 1969 in the paper [40], by using a more general probability distribution, which is connected with the Markov-Polya urn scheme.

Consider an urn containing a white balls and b black balls. One ball is drawn at random from this urn and then it is returned together with a constant number c of identical balls of the same color. This process is repeated n times. Denoting by Z_j the one-zero random variable according as the j-th trial results in white or black, the probability that the total number of white balls: $Z_1 + Z_2 +\ldots+ Z_n$ be equal with k $(0 \le k \le n)$ is given by

$$P(k;n,a,b,c) =$$

$$= \binom{n}{k} \frac{a(a+c)\ldots(a+(k-1)c)b(b+c)\ldots(b+(n-k-1)c)}{(a+b)(a+b+c)\ldots(a+b+(n-1)c)} .$$

If we adopt the notations: $x = a/(a+b)$, $\alpha=c/(a+b)$ and we hold α constant, allowing x to vary, we obtain the discrete probability distribution of Markov-Polya. We see that the probability to have

$$Y_n = \frac{Z_1+Z_2+\ldots+Z_n}{n} = \frac{k}{n}$$

is given just by

$$w_{n,k}^{\alpha}(x) = \binom{n}{k} \frac{x^{(k,\alpha)}(1-x)^{(n-k,-\alpha)}}{1^{(n,-\alpha)}}$$

The corresponding distribution function is

$$F_n^{\alpha}(y;x) = \begin{cases} 0 & \text{if } y<0 \\ \sum_{k=0}^{ny} w_{n,k}^{\alpha}(x) & \text{if } 0\le y\le 1 \\ 1 & \text{if } y>1 \end{cases}$$

If f is a real-valued function, defined on $I = [0,1]$, such that the mean value of $f(Y_n)$ exists for $n\in\mathrm{N}$, then one observes that it is given just by $(S_n^{\alpha}f)(x)$. This probabilistic interpretation of the polynomials $S_n^{\alpha}f$ was given in [40]. In the same paper D.D.Stancu has given a formula for the representation by means of factorial moments and finite differences of a linear positive operator L_n: $C[0,1] \to C[0,1]$ constructed by a probabilistic method $(L_n f)(x) = E(f(Y_n))$, where $Y_n = (X_1 + X_2 + \dots + X_n)/n$, X_j $(j = 1,2,\dots,n)$ being identically distributed random variables, with $E(X_j) = E(Y_n) = x$. If Y_n is of discrete type having on $[0,1]$ the jump points $a_{n,k} = k/n$ and jumps $p_{n,k}(x)$, then we have

$$(L_n f)(x) = \sum_{j=0}^{n} h_{n,j}(x)\left(\Delta_{1/n}^{j} f\right)(0) \;,$$

where - in terms of factorial mements - we have

$$h_{n,j}(x) = \frac{1}{j!}\mu_{n,[j]}(x) = \frac{1}{j!}\sum_{k=0}^{n} k^{[j]} p_{n,k}(x) \;.$$

In the case of Markov-Polya probability distribution we obtain [38]: $\mu_{n,[j]} = n^{[j]}\cdot\; x^{(j,-\alpha)}/1^{(j,-\alpha)}$,
so that we have [37], [40] the following representation, in terms of finite differences,

$$(S_n^{\alpha} f)(x) = \sum_{j=0}^{n} \binom{n}{k} \frac{x^{(j,-\alpha)}}{1^{(j,-\alpha)}} (\Delta_{1/n}^{j} f)(0) \;. \qquad (1.1)$$

In [42] Stancu has established a recurrence relation for the central moments of the operator S_n^{α} :

$$T_{n,m}^{\alpha}(x) = \sum_{k=0}^{n} w_{n,k}^{\alpha}(x)\left(\frac{k}{n} - x\right)^{m} \;,\; m \in \mathbf{N} \;,$$

In particular, from formula (4) of [42], we obtain the following estimation for the forth moment:

$$\frac{n(1+\alpha)}{1+n\alpha}\cdot T_{n,4}^{\alpha}(x) = O\left(\frac{1}{n}+\alpha\right), \quad \alpha\geq 0. \qquad (1.2)$$

Assuming that $n\alpha=O(1)$, Mühlbach [27] has obtained the following relation

$$n^{m}T_{n,m}^{\alpha}(x) = \sum_{i=0}^{[m/2]} A_{m,i}^{\alpha}(x)\,n^{i},$$

where $A_{m,i}^{\alpha}(x)$ are polynomials of degree m. From this relation one obtains the following inequality: $0\leq T_{n,2m}^{\alpha}(x)\leq \text{Const}.n^{-m}$, $\forall\, x\in I$.

2. **Other representations of** S_n^{α}. In 1968 D.D.Stancu, at pag. 1182 of [37], has proved that if the parameter α is positive and $x\in(0,1)$, then S_n^{α} can be represented by means of the Bernstein operator B_n according to the formula

$$(S_n^{\alpha}f)(x) = \frac{1}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}\int_0^1 t^{\frac{x}{\alpha}}(1-t)^{\frac{1-x}{\alpha}}(B_nf)(t)\,dt, \qquad (2.1)$$

where by $B(a,b)$ we denote the Beta function.

It should be noted that if we take into account (0.1) and (0.2) then we see directly that for $x=0$ and $x=1$ we have

$$(S_n^{\alpha}f)(0) = f(0), \quad (s_n^{\alpha}f)(1) = f(1).$$

The idea to give such integral representations has been used later in other papers [43], [48], [28], [54] for other parameter dependent operators.

According to a statement of Mühlbach [27], for a bounded function f on I and any $x\in[0,1]$ there exists a triplet of distinct points x_0,x_1,x_2 in I so that we have

$$(S_n^\alpha f)(x) = (B_n f)(x) + \frac{\alpha x(1-x)}{1+\alpha}[x_0, x_1, x_2; B_n f] ,$$

where the square brackets represent the symbol for divided differences.

We notice that formula (2.1) is very useful in applications, because it permits to transfer to S_n^α many properties of the Bernstein operator. For instance, with the aid of (2.1) it is possible to obtain a representation of $(S_n^\alpha f)(x)$ in terms of divided differences namely

$$(S_n^\alpha f)(x) = \sum_{k=0}^{n} \frac{1^{\left(k,\frac{1}{n}\right)}}{1^{(k,-\alpha)}} \cdot x^{(k,-\alpha)} \cdot \left[0, \frac{1}{n}, \ldots, \frac{k}{n}; f\right] .$$

This representation can be obtained also by using probabilistic tools [53]. Now refering to the preceding formula we can see that if f is a polynomial of degree m $(m \le n)$ then $S_n^\alpha f$ is itself a polynomial of degree m.

By using formula (2.1) and an expression in terms of the secon order divided differences, for the difference of two consecutive terms from the sequence of the Bernstein polynomials [36], it is easy to establish the relation

$$\begin{aligned}(S_{n+1}^\alpha - S_n^\alpha)(f;x) &= \\ &= -\frac{1}{n(n+1)} \sum_{k=0}^{n-1} \frac{(x+k\alpha)(1-x+(n-k-1)\alpha)}{(1+n\alpha)(1+(n-1)\alpha)} \cdot \\ &\cdot w_{n-1,k}^\alpha(x) \left[\frac{k}{n}, \frac{k+1}{n+1}, \frac{k+1}{n}; f\right] ,\end{aligned} \tag{2.2}$$

which was proved directly in [37].

By using this relation we are able to state the following result: if f is convex (concave) of first order on I, then the sequence $(S_n^\alpha f)$ is decreasing (increasing) on I.

We further note that if we assume that $0\le\alpha=\alpha(n)\to 0$, as $n\to\infty$, by extending a result of E.Moldovan [26], we can deduce the converse of the previous implication, namely:

$(S_n^\alpha f)$ nondecreasing (nonincreasing) on I, $\forall\ f\in C(I)$, and $0\le\alpha=\alpha(n)\to 0$ $(n\to\infty)$ $\to f$ nonconcave (nonconvex) of first order on I.

In 1972 D.D.Stancu [47] (see also [21]) has generalized the operator S_n^α , by using a sequence of polynomials depending on a real parameter α: $\left(\varphi_m^{\langle\alpha\rangle}\right)$, as well as the Nörlund difference quotient D_α^k , defined by

$$(D_\alpha^k g)(x) = D_\alpha(D_\alpha^{k-1} g)(x) \ , \ (D_\alpha g)(x) = [g(x+\alpha)-g(x)]\cdot\alpha^{-1}.$$

Assuming that $\varphi_m^{\langle\alpha\rangle}(0)\ne 0$, D.D.Stancu has introduced the general linear operator L_n^α defined, for any function f: $J\to\mathbf{R}$ by

$$(L_n^{\langle\alpha\rangle} f)(x) := \frac{1}{\varphi_n^{\langle\alpha\rangle}(0)}\sum_{k=0}^{\infty}(-1)^k\frac{x^{(k,-\alpha)}}{k!}\left(D_\alpha^k\varphi_n^{\langle\alpha\rangle}\right)(x)\, f(x_{n,k}) \ , \qquad (2.3)$$

where $x_{n,k}\in J$, $k = 0(1)n$, $n\in\mathbf{N}$.

The operator S_n^α can be obtained from $L_n^{\langle\alpha\rangle}$ if we choose $\varphi_n^{\langle\alpha\rangle}(x)=(1-x)^{(n,-\alpha)}$. In the same paper [47] there has been given a general Bernstein-type series, generalizing the operators of Baskakov, Favard Schurer and Sikkema. We notice that in the paper [21] the function $\varphi_n^{\langle\alpha\rangle}$ has been multiplied by the normalization factor $\lambda_n^\alpha=1/\varphi_n^{\langle\alpha\rangle}(0)$ and the operator $L_n^{\langle\alpha\rangle}$,using the nodes $x_{n,k} = k/n$, was further studied.

As a tool for the investigation of the monotonicity properties of the derivatives of the sequence of the Bernstein polynomials, D.D.Stancu has introduced a class of positive linear functionals $T_k^{(\nu)}$. Consider the following points of an interval $[a,b]$ of the real axis: $a_i = a + ih$, $b_j = a + jl$, where $i=0(1)n$, $j=1(1)n$, $0<h\le(b-a)/n$, $0<l\le(b-a)/n$. One associates to each function f, defined on $[a,b]$, the Stancu linear functionals $T_k^{(\nu)}$, $0\le k\le n$, $1\le\nu\le r+1$, defined recursively as follows:

$$T_k^{(2)}(f)=[a_k,a_{k+1},b_{k+1};f]\ ,\ 0\le k\le n-1$$
$$T_k^{(\nu+1)}(f)=T_{k+1}^{(\nu)}(f)-T_k^{(\nu)}(f)\ ,\ 1<\nu\le r\ ,\ 0\le k\le n-r.$$

In [50] there has been suggested to use, for the investigation of the similar monotonicity properties of the sequence $(S_n^\alpha f)$, in place of the differentiation operator D the predorivative operator of Nörlund, with increment $\alpha:D_\alpha$.

In our recent paper [4] we proved that for $\alpha\ge0$, $m\le n$, there holds the following relation

$$D_\alpha^m(S_{n+1}^\alpha-S_n^\alpha)(f;x) =$$
$$= -\frac{1}{n(n+1)(1+(n-1)\alpha)(1+n\alpha)}[(x+m\alpha)(1-x-m\alpha)S_n^{\alpha,m+1}T_\nu^{(m+2)}f(x)+$$
$$+m(1-2x-(2m-1)\alpha)S_n^{\alpha,m}T_\nu^{(m+1)}f(x)-m(m-1)S_n^{\alpha,m-1}T_\nu^{(m)}f(x)],$$

where

$$S_n^{\alpha,r}f(x)=\sum_{\nu=0}^{n-r}(-1)^{\nu+r-1}D_\alpha^{\nu+r-1}\Phi_{n-1}^\alpha(x-\alpha)\frac{(x+r\alpha)^{(\nu,-\alpha)}}{\nu!}f\left(\frac{\nu}{n}\right),$$
$$\Phi_n^\alpha(x)=\frac{(1-x)^{(n,-\alpha)}}{1^{(n,-\alpha)}}\ ,\ T_k^{(2)}f=\left[\frac{k}{n},\frac{k+1}{n+1},\frac{k+1}{n};f\right],$$
$$T_k^{(j+1)}f = T_{k+1}^{(j)}f - T_k^{(j)}f\quad (j\ge2)\ .$$

By using the above formula, which generalized the relation (2.2), one can state monotonicity properties of the sequence $\left(D_\alpha^m S_n^\alpha f\right)$, where $m\le n$, $0\le m\alpha\le1$ (see[4]).

The operator S_n^α of Stancu has many beautiful properties, such as shape-preserving and preservation of the convexity (concavity) of different orders [21]. In a recent paper [5] we have proved that if $f\in \mathrm{Lip}_M\delta$, then also $S_n^\alpha f\in \mathrm{Lip}_M\,\delta$, $\forall\, n\in \mathbf{N}$, $\alpha\geq 0$. Moreover, if $0\leq\alpha=\alpha(n)\to 0$ $(n\to\infty)$, then the converse implication holds too [5].

From the paper [21] it results that if we denote by $v[f]$ the number of changes of sign of the function f on I, then we have

$$v\left[\Delta_\alpha^r S_n^\alpha f\right] \leq v\left[\left\{\Delta_{1/n}^r f\left(\frac{k}{n}\right)\right\}_{k=0}^n\right], \quad r\in\mathbf{N},\ \alpha\geq 0\ , \tag{2.4}$$

where $\Delta_h^r f$ is the r-th order forward difference of f with the step h. From (2.4) there follows that, in particular, S_n^α is a variation-diminishing operator, since in addition it preserves the linear functions.

Ending this section, we mention that by extending a theorem of Freedman and Passow [10], we have proved in [7] that: f is a continuous and piecewise linear function with at most $n-1$ changes of slope, which can occur only at i/n $(i=1,2,\ldots,n-1)$, if and only if

$$S_{nm}^\alpha(f;x) = S_{nm+1}^\alpha(f;x)\ ,\forall\, m\in\mathbf{N}\ ,\ x\in I\ .$$

3. **Evaluation of the remainder term.** In the paper [35] D.D.Stancu has discovered a representation of the remainder of the Bernstein's approximation formula, $f(x) = (B_nf)(x) + (R_nf(x)$, under the form of an average of certain divided differences of second-order on specified points. In [46] he has found a similar

expression for the remainder of the approximation formula: $f(x) = (S_n^\alpha f)(x) + (R_n^\alpha f)(x)$, namely

$$(R_n^\alpha f)(x) = -\sum_{k=0}^{n-1} \frac{(x+k\alpha)(1-x+(n-1-k)\alpha)}{n(1+(n-1)\alpha)} w_{n-1,k}^\alpha(x) \cdot \left[x, \frac{k}{n}, \frac{k+1}{n}; f\right].$$

It should be remarked that this can be expressed also under a form which permits to put into evidence that $S_n^\alpha f$ is interpolatory at the ends of the interval I (see [55]):

$$(R_n^\alpha f)(x) = -\frac{x(1-x)}{n} \cdot \frac{1+n\alpha}{1+\alpha} \sum_{k=0}^{n-1} v_{n-1,k}^\alpha(x) \cdot \left[x, \frac{k}{n}, \frac{k+1}{n}; f\right],$$

where

$$v_{n-1,k}^\alpha(x) = \binom{n-1}{k} \frac{(x+\alpha)^{(k,-\alpha)}(1-x+\alpha)^{(n-1-k,-\alpha)}}{(1+2\alpha)^{(n-1,-\alpha)}}.$$

Since the remainder vanishes for any linear function and $v_{n-1,k}^\alpha(x) \geq 0$ on $[0,1]$, while

$$\sum_{k=0}^{n-1} v_{n-1,k}^\alpha(x) = 1,$$

by applying a known theorem of T.Popoviciu [29], we can write this remainder under "the simple form":

$$(R_n^\alpha f)(x) = -\frac{x(1-x)}{n} \cdot \frac{1+n\alpha}{1+\alpha} [u_n, v_n, w_n; f],$$

where u_n, v_n, w_n are distinct points of I, which might depend on the function f.

If we assume that $f \in C^2(0,1)$, then by applying the known theorem of Peano, we can obtain an integral representation of this remainder [46]:

$$(R_n^\alpha f)(x) = \int_0^1 K_n^\alpha(t;x)\, f''(t)\, dt \ ,$$

where

$$K_n^\alpha(t;x) = (R_n^\alpha \varphi_x)(t) \ , \ \varphi_x(t) = \frac{1}{2}[x-t+|x-t|] \ ,$$

the subscript x indicating that R_n^α is to be performed with respect to x, while t is held fixed. The Peano kernel K_n^α , for a fixed x in I, represents a spline function of first degree, having the knots j/n $(j=0,1,\dots,n)$. Since K_n^α $(t;x)\leq 0$ on $[0,1]$ X $[0,1]$, we can apply the mean value theorem to the above integral and we get finally

$$(R_n^\alpha f)(x) = -\frac{x(1-x)}{2n}\cdot\frac{1+n\alpha}{1+\alpha} f''(\xi_n) \ , \ \xi_n \in (0,1) \ .$$

4. **Estimates of the rate of convergence of** $(S_n^\alpha f)$ Since the monomials e_0 and e_1 are fixed points for the operator S_n^α, while for e_2 we have

$$(S_n^\alpha e_2)(x) = x^2 + \frac{1+\alpha n}{n+\alpha n}\cdot x(1-x) \ ,$$

if we assume that $0\leq\alpha=\alpha(n)\to 0$ $(n\to\infty)$, then S_n^α is a linear positive operator and according to the Bohman-Korovkin criterion we can state that for any function $f\in C[0,1]$ we have $S_n^\alpha f \to f$, uniformly on I.Concerning the measure of the rate of convergence, we can use the first order modulus of continuity

$$\omega_1(f;\delta)=\omega(f;\delta):=\sup\{|f(x+h)-f(x)|: x,x+h\in J, 0\leq h\leq\delta\},$$

or the second order modulus of continuity

$$\omega_2(f;\delta):=\sup\{|f(x+h)-2f(x)+f(x-h)|: x,x\pm h\in J, 0\leq h\leq\delta\},$$

where - in general $J=[a,b]$ and $0\le\delta\le 1/2(b-a)$, with $f\in C(J)$.

The following estimates are known:

$$\|R_n^\alpha f\| \le K\omega\left(f;\sqrt{\frac{1+n\alpha}{n+n\alpha}}\right), \text{ if } f\in C(I) \tag{4.1}$$

$$\|R_n^\alpha f\| \le L\sqrt{\frac{1+n\alpha}{n+n\alpha}}\,\omega\left(f';\sqrt{\frac{1+n\alpha}{n+n\alpha}}\right), \text{ if } f'\in C(I). \tag{4.2}$$

In [37] for K and L were given the values 3/2 and 3/4, while in [12] were found better constants: 5/4, respectively 5/8.

In [12] Gonska has deduced an estimation using the second order modulus of continuity

$$\|R_n^\alpha f\| \le 3\frac{1}{4}\cdot\omega_2\left(f;\sqrt{\frac{1+n\alpha}{n+n\alpha}}\right). \tag{4.3}$$

We notice that the numerical factors from these inequalitices are not optimal and their estimation is an open problem.

It is easy to verify that by starting from (1.2) we can obtain for the approximation of the function f by $S_n^\alpha f$ an asymptotic estimation of Voronovskaya type, namely

$$\lim_{n\to\infty}\frac{n(1+\alpha)}{1+n\alpha}(R_n^\alpha f)(x) = -\frac{x(1-x)}{2}f''(x), \quad x\in I, \tag{4.4}$$

when $0\le\alpha=\alpha(n)\to 0$ $(n\to\infty)$ and for any f having a second derivative at the point x.

If in addition $0\le\alpha n\le 1$, then we can give the following estimation

$$\left|\frac{n(1+\alpha)}{1+n\alpha}(R_n^\alpha f)(x)+\frac{x(1-x)}{2}f''(x)\right| \le \frac{7}{16}\,\omega\left(f'';\frac{1}{\sqrt{n-1}}\right).$$

In [27] Mühlbach, by using a theorem of Mamedov [16], has obtained a more general result than (4.4):

$$\lim_{n\to\infty} n^m\left[(S_n^\alpha f)(x)-\sum_{k=0}^{2m-1}\frac{f^{(k)}(x)}{k!}T_{n,k}^\alpha(x)\right]= \left[\frac{x(1-x)(1+\beta)}{2}\right]^m\cdot\frac{f^{(2m)}(x)}{m!}, \tag{4.5}$$

with $x\in I$ and f having the derivative of order $2m$ at x, under the hypotheses that $\alpha=\alpha(n)$ and $n\alpha\to\beta\in\mathbb{R}$, when $n\to\infty$.

The convergence of the sequence $(S_n^\alpha f)^{(m)}$ of derivative of order m $(0\le m<n)$ of $(S_n^\alpha f)$ was studied in [22] by Mastroianni and Occorsio, who proved the following limiting relation

$$\lim_{n\to\infty}(S_n^\alpha f)^{(m)}(x) = f^{(m)}(x)\ ,\quad (0\le m<n)$$

$\forall\ x\in I$ and $\alpha=\alpha(n)=o(n^{-1})$ $(n\to\infty)$. They gave the estimates

$$\left|(R_n^\alpha f)^{(m)}(x)\right| < 2\omega\left(f^{(m)};\sqrt{\delta_{n,m}(x)}\right)+ \|f^{(m)}\|\left(\frac{1}{(1-n\alpha)^m}-\beta_{n,m}^\alpha\right),\ \forall\ f^{(m)}\in C(I), \tag{4.6}$$

and

$$\left|(R_n^\alpha f)^{(m)}(x)\right| < 2\sqrt{\delta_{n,m}(x)}\cdot\omega\left(f^{(m+1)};\sqrt{\delta_{n,m}(x)}\right)+\frac{m}{n}\|f^{(m+1)}\|+ \left((1-n\alpha)^{-m}-\beta_{n,m}^\alpha\right)\cdot\|f^{(m)}\|\ ,\ \forall\ f^{(m+1)}\in C(I), \tag{4.7}$$

where

$$\delta_{n,m}(x) = \frac{[1+\alpha(n-m)]x(1-x)}{(1+\alpha)(n-m)} + \frac{3m}{n}, \quad \beta^{\alpha}_{n,m} = \prod_{\nu=1}^{m} \frac{1-(\nu-1)/n}{1+(n-\nu)\alpha} .$$

Concerning the sequence $(D_{\alpha}^{m}S_n^{\alpha}f)$, in [21] has been proved the relation $\lim_{n\to\infty}\|D_{\alpha}^{m}S_n^{\alpha}-f^{(m)}\|=0$, $\forall\ f\in C^{m}(I)$, $\alpha=\alpha(n)=O(n^{-1})$.

We further note that in [21] there has been given estimations of the orders of approximation of $f^{(m)}$ by $D_{\alpha}^{m}S_n^{\alpha}f$, using the modulus of continuity of $f^{(m)}$, respectively of $f^{(m+1)}$, under the hypothesis that these are continuous on I. Comparing these estimates with those given at (4.6), respectively at (4.7), we can see that $D_{\alpha}^{m}S_n^{\alpha}f$ converges to $f^{(m)}$, if $f^{(m)}\in C(I)$, with the same rate as $(S_n^{\alpha}f)^{(m)}$. It should be noticed that we have assumed that $\alpha\geq0$. Neverthless, there can be obtained convergence theorems and asymptotic estimates also for $\alpha<0$, under suitable hypotheses (see [27] and [44]).

5. Generalisations of the Stancu operator. We next turn to the extension given in [47] for the operator S_n^{α} , namely to the operator L_n^{α} , mentioned at (2.3). By using, as in [21], a normalization factor for the fundamental polynomials, depending on the parameter α, which are used for the construction of these operators, we can write

$$(L_n^{\alpha}f)(x) := \sum_{k=0}^{\infty} (-1)^k \frac{x^{(k,-\alpha)}}{k!}\left(D_{\alpha}^{k}\varphi_n^{\alpha}(x)\right)f(x_{n,k}),$$

where $\alpha\geq0$, $x\in K=[0,a]$ $(a>0)$, $x_{n,k}\in J\supseteq K$, $f:J\to N$, while $\varphi_n^{\alpha}(x)$ are polinomials in x of degree n, such that

$$\varphi_n^{\alpha}(0)=1 \ , \ (-1)^k D_{\alpha}^k \varphi_n^{\alpha}(x) \geq 0 \ , \ \forall x \in K \ , \ n \in \mathbb{N}.$$

This class of operators introduced and investigated by Stancu [47] was further investigated in [21] and a special case of it in [12]. The operator L_n^{α} contains several special important linear positive operators used in approximation theory of functions, as - for example: the Bernstein and the Baskakov operators, as well as an operator $P_n^{\alpha,\beta}$ introduced and investigated by Stancu himself [41].

The operator L_n^{α} possesses similar properties with those of the operator S_n^{α}. In particular, if $x_{n,k} = k/n$ then $L_n^{\alpha}f$ can be represented [47] in terms of successive differences of the function f; in addition, it has the variation diminishing property - in the sense of Schoenberg [31], and it preserves the order and Lipschitz constant [5]. Estimations of the rate of uniform convergence of $L_n^{\alpha}f$ to the function $f \in C(K)$ can be seen in [21], [47] and [12]. We mention also that the following relation holds

$$\lim_{n\to\infty} \| D_{\alpha}^m L_n^{\alpha} f - f^{(m)} \| = 0 \ , \ \forall \ f^{(m)} \in C(K) \ ,$$

for $0 \leq \alpha = \alpha(n) \to 0 \ (n \to \infty)$; some bounds for the corresponding error were given in [21].

In 1972 D.D.Stancu has introduced [47] a Bernstein-type operator $L_{n,p}^{\alpha,\beta,\gamma}$, defined by the following formula

$$(L_{n,p}^{\alpha,\beta,\gamma} f)(x) := \sum_{k=0}^{n+p} \binom{n+p}{k} \frac{x^{(k,-\alpha)} (1-x)^{(n+p-k,-\alpha)}}{1^{(n+p,-\alpha)}} f\left(\frac{k+\beta}{n+\gamma}\right) , \qquad (5.1)$$

where $\alpha \geq 0$, $0 \leq \beta \leq \gamma$, p being a given non-negative integer.

This operator, depending on the quadruple of parameters (p,α,β,γ), includes some special operators considered previouly by Schurer [33] and Sikkema [34]. In [47] Stancu has proved that the Bernstein-type operator defined at (5.1) can be represented by means of finite differences on the starting point $\beta/(n+\gamma)$ and with the step $1/(n+\gamma)$. Consequently, if f is a polynomial of degree m $(\le n)$, then (5.1) leads us also to a polynomial of degree m. For the convergence we assume that $f\in C[0,1+p/m]$.

In [12] Gonska and Meier, assuming that $f\in C[0,1+1/m]$ and $\delta>0$, gave estimates of the rate of convergence, involving the first, respectively the second order modulus of continuity. The corresponding inequalities are the following

$$
\begin{aligned}
&|(L_{n,p}^{\alpha,\beta,\gamma}f)(x)-f(x)|\le\\
&\le\omega(f;\delta)\{1+\frac{1}{\delta^2}[(\frac{(n+p-1)(n+p)}{(n+\gamma)^2(1+\alpha)}-\frac{2(n+p)}{n+\gamma}+1)x^2+\\
&+(\frac{(1+2\beta)(n+p)}{(n+\gamma)^2}+\frac{(n+p-1)(n+p)\alpha}{(n+\gamma)^2(1+\alpha)}-\frac{2\beta}{n+\gamma})x+\frac{\beta^2}{(n+\gamma)^2}]\},
\end{aligned}
$$

$$
\begin{aligned}
&|(L_{n,p}^{\alpha,\beta,\gamma}f)(x)-f(x)|\le\\
&\le\{3+\max(\frac{1}{\delta^2},1)[(\frac{(n+p-1)(n+p)}{(n+\gamma)^2(1+\alpha)}-\frac{2(n+p)}{n+\gamma}+1)x^2+\\
&+(\frac{(1+2\beta)(n+p)}{(n+\gamma)^2}+\frac{(n+p-1)(n+p)\alpha}{(n+\gamma)^2(1+\alpha)}-\frac{2\beta}{n+\gamma})x+\frac{\beta^2}{(n+\gamma)^2}]\}\omega_2(f;\delta)+\\
&+2\left|\frac{\beta}{n+\gamma}-\frac{\gamma-p}{n+\gamma}x\right|\max(\frac{1}{\delta},1)\,\omega(f;\delta).
\end{aligned}
$$

By using a probabilistic method, D.D.Stancu has introduced and investigated in detail in [52] a new Bernstein type operator $L_{n,r}$, depending on a non-negative integer parameter r $(n>2r)$, defined by

$$(L_{n,r}f)(x) := \sum_{k=0}^{n-r} p_{n-r,k}(x)\left[(1-x)\,f\left(\frac{k}{n}\right)+xf\left(\frac{k+r}{n}\right)\right] . \qquad (5.2)$$

In the same paper is determined the point spectrum of the operator $L_{m,r}$ and is given the quadrature formula which can be constructed by means of this operator.

In a next paper [56] he has used also a probabilistic way, connected with a variant of Markov-Polya probability distribution, for investigating a class of operators more general $L^{\alpha}_{n,r}$, introduced in [54]:

$$(L^{\alpha}_{n,r}f)(x) := \sum_{k=0}^{n-r} p^{\alpha}_{n-r,k}(x)\left\{[1-x+(m-r-k)\alpha]\,f(\frac{k}{n})+(x+k\alpha)\,f\left(\frac{k+r}{n}\right)\right\} ,$$

where

$$p^{\alpha}_{n-r,k}(x) := \binom{n-r}{k}\frac{x^{(k,-\alpha)}(1-x)^{(n-r-k,-\alpha)}}{(1+\alpha)^{(n-r,-\alpha)}} ,$$

α being a non-negative parameter, which may depend on n. One can see that we have $L^{0}_{n,r}=L_{n,r}$. Many properties of this operator, useful in constructive approximation theory, were established, including convergence theorems, error bounds and representations of the remainder term.

In a recent paper [2] there has been introduced and studied an operator $L^{*}_{n,r}$,called "the Stancu operators of integral type", defined by

$$(L_{n,r}^{*}f)(x) := \sum_{k=0}^{n-r} [(1-x)C_{n,r,k}^{-1}\int_0^1 (1-t)p_{n-r,k}(t)f(t)\,dt +$$
$$+ x(C_{n,r,k}^{*})^{-1}\int_0^1 tp_{n-r,k}(t)f(t)\,dt]\,p_{n-r,k}(x) \;,$$

where

$$C_{n,r,k} = \frac{n-r-k+1}{(n-r+2)(n-r+1)} \;, \quad C_{n,r,k}^{*} = \frac{k+1}{(n-r+2)(n-r+1)} \;.$$

The approximation properties of the operator $L_{n,r}$ have been further studied in the paper [3].

In [11] H.H.Gonska gave an evaluation of the order of approximation of a function f by means of $L_{n,r}f$ using the second order modulus of continuity.

In a recent paper [30] is modified the operator S_n^{α} of D.D.Stancu, defined at (0.1)-(0.2), in the sense in which Kantorovich has modified the Bernstein operator. This Stancu-Kantorovich operator is defined by

$$(K_n^{\alpha}f)(x) := (n+1)\sum_{k=0}^{n} w_{n,k}^{\alpha}(x)\int_{x_{n+1,k}}^{x_{n+1,k+1}} f(t)\,dt \;,$$

where $x_{n+1,k} = k/(n+1)$, $x_{n+1,k+1} = (k+1)/(n+1)$.

For $\alpha=0$ it reduces to the original operator K_n of Kantorovich. One shows that K_n^{α} can be looked upon as an average of the operator K_n, with suitable weights chosen. For this purpose the author uses the same technique which was used in [37] for giving the representation (2.1) of S_n^{α} by means of B_n. By using the operator K_n^{α} the author extends the results of D.D.Stancu from [37] to the approximation in the L_1-norm on [0,1] of the Lebesque integrable functions.

We mention that G.Mastroianni and M.R.Occorsio [23] have introduced and investigated the iterates of the operator S_n^α, of D.D.Stancu, defined at (1.1):

$$(S_n^\alpha)^0 = I\ ,\ (S_n^\alpha)^1 = S_n^\alpha\ ,\ (S_n^\alpha)^j = S_n^\alpha(S_n^\alpha)^{j-1}\quad (j>1)\ .$$

By extending a procedure used by Kelinsky and Rivlin [14], used in the case of the Bernstein operators, these authors gave the following representations

$$(S_n^\alpha)^k(e_i;x) = X_i\left(A_i^{-1}, -\alpha A_i, \frac{1}{n}\right)^k u_i$$

$$(S_n^\alpha)^k(\eta_i^{-\alpha};x) = Y_{i,-\alpha}\left(A_{i,\frac{1}{n}}A_{i,-\alpha}^{-1}\right)^k u_i\ ,$$

where

$$X_k = (x, x^2, \dots, x^k)\ ,\ u_k = (0,\dots,0,1)^T$$

$$Y_{k,h} = \left(x, \frac{x^{(2,h)}}{1^{(2,h)}}, \dots, \frac{x^{(k,h)}}{1^{(k,h)}}\right)$$

$$\Lambda_{k,h} = (c_{i,j})_{i,j=1}^k = \begin{cases} 1^{(i,h)} & ,i=j\ , \\ 0 & ,i\neq j \end{cases}$$

$$A_{k,h} = \Lambda_{k,h}S_{k-1,h}$$

$$V_{k,h} = \left(h^k s_k^{k+1}, h^{k-1}s_{k-1}^{k+1}, \dots, hs_1^{k+1}\right)^T$$

$$S_{0,h} = 1\ ,\ S_{k,h} = \begin{pmatrix} S_{k-1,h} & V_{k,h} \\ 0 & 1 \end{pmatrix}\quad (k>0)\ ,$$

s_i^j being the Stirling numbers of second order.

By using these formulas, in [23] is proved the following relation

$$\lim_{k\to\infty}(S_n^\alpha)^k(f;x) = f(0) + (f(1) - f(0))x,$$

uniformly on [0,1], for any $\alpha\geq 0$. In the same paper [23] it is introduced the operator $S_{n,k}^{\alpha}$, defined by the following operatorial formula

$$S_{n,k}^{\alpha} = I - (I - S_{n}^{\alpha})^{k} ,$$

where I is the identity operator.

One proves that we have

$$\lim_{k\to\infty}(S_{n,k}^{\alpha}f)(x) = (L_{m}f)(x) ,$$

where $L_{n}f$ is the Lagrange interpolating polynomial corresponding to the function f and the nodes i/n $(i=0,1,\ldots,n)$.

This operator is not of positive type, but it gives better approximation than $S_{n}^{\alpha}= S_{n,1}^{\alpha}$ for sufficiently smooth functions. For illustration, we present such a result: If $f^{(p+2k)}\in C(I)$, then we have

$$\frac{d^{p}}{dx^{p}}\left(f(x) - S_{n,k+1}^{\alpha}(f;x)\right) = O(n^{-k}) .$$

Following a procedure used by Mastroianni and Occorsio in [24], we have introduced in [6] the operators $S_{n,\lambda}^{\alpha}$ defined by

$$S_{n,\lambda}^{\alpha}(f;x) := \left(I-(I-S_{n}^{\alpha})^{\lambda}\right)(f;x) = \sum_{i=1}^{\infty}\binom{\lambda}{i}(S_{n}^{\alpha})^{i}(f;x) , \quad \lambda\in\mathbf{R}^{+} .$$

We notice that for $\lambda=1$ we have $S_{n,1}^{\alpha}=S_{n}^{\alpha}$ and that for $\lambda=k\in\mathbf{N}$ we arrive at the previous operator $S_{n,k}^{\alpha}$.

If $\lambda\in(0,1]$ then the linear operator $S_{n,\lambda}^{\alpha}$ is of positive type and it possesses many nice properties. For example, it can be represented under the following matrix form

$$S_{n,\lambda}^{\alpha}(f;x) = \sum_{k=0}^{\infty} \gamma_{n,k}^{\alpha} \cdot z_{n,k}^{\alpha}(x;\lambda)\left[0, \frac{1}{n}, \ldots, \frac{k}{n}; f\right], \tag{5.1}$$

where

$$z_{n,k}^{\alpha}(x;\lambda) = h_{k,-\alpha} W_{k,n,\alpha} F_{k,\alpha,\gamma} W_{k,n,\alpha}^{-1} u_k ,$$

$$h_{k,-\alpha} = (x, x^{(2,-\alpha)}, \ldots, x^{(k,-\alpha)}) ,$$

$$\Gamma_{k,-\alpha} = (a_{i,j})_{i,j=1}^{k} = \begin{cases} \gamma_{n,i}^{\alpha} , & i=j \\ 0 , & i \neq j \end{cases} ,$$

$$T_{k,h} = \left(h^k s_k^{k+1}, h^{k-1} s_{k-1}^{k+1}, \ldots, h s_1^{k+1}\right)^T ,$$

$$D_{k,h} = \begin{pmatrix} D_{k-1,h} & T_{k,h} \\ 0 & 1 \end{pmatrix} \quad (k>0) , \quad D_{0,h}=1 ,$$

$$N_{k,\frac{1}{n},-\alpha} = \Gamma_{k,\alpha} D_{k-1}, \frac{1}{n} D_{k-1,-\alpha}^{-1} .$$

By $W_{k,n,\alpha}$ is denoted the upper triangular matrix having as columns the eigenvalues of $N_{k,1/n,-\alpha}$, normalized so that the elements of the main diagonal are 1, i.e.,

$$N_{k,1/n,-\alpha} = W_{k,n,\alpha} \Gamma_{k,\alpha} W_{k,n,\alpha}^{-1}$$

and

$$F_{k,\alpha,\lambda} = \sum_{j=0}^{\infty} (-1)^j \binom{\lambda}{j+1} (\Gamma_{k,\alpha})^j$$

represents the diagonal matrix whose elements are

$$\gamma_{n,i}^{\alpha}\left[1-(1-\gamma_{n,i}^{\alpha})^{\lambda}\right] , \quad (i=1,2,\ldots,k) .$$

The representation (5.1) of $S_{n,\lambda}^{\alpha} f$ permits us to see that if f is a polynomial of degree m ($m<n$), then also $S_{n,\lambda}^{\alpha} f$ is a polynomial of the same degree. Moreover, if $\lambda \in (0,1)$ then $S_{n,\lambda}^{\alpha}$ preserves the convexity (concavity) of any order of the function f and we have the implication

$$f \in \mathrm{Lip}_M \delta \Rightarrow S_{n,\lambda}^{\alpha} f \in \mathrm{Lip}_M \delta .$$

In [6] we have studied the monotonicity properties of the sequence $(S_{n,\lambda}^{\alpha} f)$, for $m\alpha \le 1$, $m \le n$, $\lambda \in (0,1]$.

For the remainder of approximation formula of the function f by $S_{n,\lambda}^{\alpha} f$ we found a representation by means of the second order divided differences:

$$R_{n,\lambda}\alpha (f;x) = -x(1-x)\left[\frac{n\alpha+1}{n(1+\alpha)}\right]^{\lambda} \cdot [\xi_1, \xi_2, \xi_3; f] ,$$

where $\lambda \in (0,1]$ and ξ_1, ξ_2, ξ_3 are suitable points of I, which might depend on f.

We have obtained also quantitative estimates of the corresponding approximation

$$\|f - S_{n,\lambda}^{\alpha} f\| \le \frac{5}{4}\omega(f;\delta_{n,\lambda}^{\alpha}) , \quad \forall f \in C(I),$$
$$\|f - S_{n,\lambda}^{\alpha} f\| \le \frac{3}{4}\delta_{n,\lambda}^{\alpha} \cdot \omega(f';\delta_{n,\lambda}^{\alpha}) , \quad \forall f' \in C(I) ,$$

where

$$\delta_{n,\lambda}^{\alpha} = \left[\frac{n\alpha+1}{n(1+\alpha)}\right]^{\frac{\lambda}{2}} .$$

It should be noted that if λ is an integer greater than 1, then the corresponding operator was studied earlier in [23].

For $\lambda = j+\delta$, $j \in N$, $\delta \in (0,1)$, we have

$$\lim_{\lambda \to \infty} S_{n,\lambda}^{\alpha}(f;x) = L_n(f;x) ,$$

where $L_n f$ is the Lagrange interpolation polynomial using the nodes k/n $(k=0,1,\ldots,n)$.

6. **Spline-type operators of approximation.** In the papers [51], [57] D.D.Stancu has introduced and studied a spline-type linear positive operator, depending on two non-negative parameters, generalizing the operator considered first in 1965 by I.J.Schoenberg [32] and investigated in detail in 1966 -in a joint paper - by M.J.Marsden and I.J.Schoenberg [20] and later in two papers of M.J.Marsden [17], [18].

Considering two integers m $(m\geq 1)$, n $(n\geq 0)$ and two real parameters α and β satisfying the condition: $0\leq\alpha\leq\beta$, one constructs a spline-type linear positive operator, in the sense of Schoenberg, defined, for any $f:[0,1]\to\mathbf{R}$, by the formula

$$(S_{m,n}^{\alpha,\beta}f)(x):=\sum_{j=0}^{m+n}N_{m,n,j}(x)\,f(\xi_{m,j}^{\alpha,\beta})\quad , \tag{6.1}$$

where

$$\begin{aligned} &0=x_{-m}=\ldots x_{-1}=x_0<x_1<\ldots<x_n<x_{n+1}=\ldots=x_{n+m+1}=1\ ,\\ &\xi_{m,n}^{\alpha,\beta}:=\frac{1}{m+\beta}\,(x_{-m+1+j}+\ldots+x_{-1}+x_0+x_1+\ldots+x_j+\alpha)\ , \end{aligned} \tag{6.2}$$

$$N_{m,n,j}(x):=(x_{j+1}-x_{-m+j})\,[x_{-m+j},\ldots,x_{-1},x_0,x_1,\ldots,x_{j+1};(t-x)_+^m] \tag{6.3}$$

the brackets representing the symbol for divided differences.

The points $x_1,x_2,\ldots,x_n$ are the knots, the abscissas (6.2) are the nodes, while the functions defined at (6.3) are the fundamental spline functions, representing normalized B-splines.

For $\alpha=\beta=0$ one obtains the Schoenberg original operator: $S_{m,n}=S_{m,n}^{0,0}$,which is interpolatory at both sides of the interval [0,1], since

$$(S_{m,n}^{\alpha,\beta}f)(0)=f\left(\frac{\alpha}{m+\beta}\right)\ ,\quad (S_{m,n}^{\alpha,\beta}f)(1)=f\left(\frac{m+\alpha}{m+\beta}\right)\ .$$

The approximation formula

$$f(x)\ =\ (S_{m,n}^{\alpha,\beta}f)(x)\ +\ (R_{m,n}^{\alpha,\beta}f)^{\cdot}(x)$$

has the highest degree of exactness $N=1$ just for $\alpha=\beta=0$.

If we take into account that

$$\xi_{m,j}^{\alpha,\beta}=\left(1+\frac{\beta}{m}\right)^{-1}\left(\xi_{m,j}+\frac{\alpha}{m}\right)\ ,\quad \xi_{m,j}=\xi_{m,j}^{0,0}\ ,$$

and that the operator $S_{m,n}$ reproduces the linear functions, we can write

$$(S_{m,n}^{\alpha,\beta}e_0)(x)=1\ ,\quad (S_{m,n}^{\alpha,\beta}e_1)(x)=x+\frac{\alpha-\beta x}{m+\beta}$$

$$(S_{m,n}^{\alpha,\beta}e_2)(x)=\left(1+\frac{\beta}{m}\right)^{-2}\left[(S_{m,n}e_2)(x)+\frac{2\alpha x}{m}+\frac{\alpha^2}{m^2}\right]\ .$$

A theorem of Marsden [18] states that a necessary and sufficient condition that $S_{m,n}e_2\to e_2$, uniformly on [0,1], is: $m^{-1}\|\Delta\|\to 0$ where $\|\Delta\|$ is the norm of the partition of the interval [0,1] by the points x_j. It will occur if either $m\to\infty$, or m is bounded and $\|\Delta\|\to 0$. So that we can distinguish two cases:

(i) $m\geq n\geq 0$ and (ii) $n>m>0$.

In the first case we assume that $m\to\infty$, in order to have uniformly on [0,1]:

$$\lim_{m\to\infty}S_{m,n}e_j\ =\ e_j\qquad (j=0,1,2)\ .$$

According to the Bohman-Korovkin convergence criterion we can state the following result: If the parameters α and β satisfy the conditions $0\leq\alpha\leq\beta$ then for any $f\in C[0,1]$ and a given natural number n, we have

$$\lim_{m\to\infty} S_{m,n}^{\alpha,\beta} f = f ,$$

uniformly on [0,1].

In the case when we have no knots (n=0), in [51] it is proved that we have

$$(S_{m,0}^{\alpha,\beta} f)(x) = \sum_{j=0}^{m} \Big[\underbrace{0,\ldots,0}_{m+1-j},\underbrace{1,\ldots,1}_{j+1};(t-x)_+^m\Big] f\left(\frac{j+\alpha}{m+\beta}\right) =$$
$$= \sum_{j=0}^{m} \binom{m}{j} x^j (1-x)^{m-j} f\left(\frac{j+\alpha}{m+\beta}\right) ,$$

which is just the Bernstein-type polynomial $B_m^{\alpha,\beta} f$ on x, depending on the parameters α and β, investigated in [41].

In the case (ii), considering that m = 1, one obtains

$$(S_{1,n}^{\alpha,\beta} f)(x) = \frac{(x_1-x)_+}{x_1} f\left(\frac{\alpha}{\beta+1}\right) +$$
$$+\sum_{j=1}^{n} (x_{j+1}-x_{j-1})[x_{j-1},x_j,x_{j+1};(t-x)_+] f\left(\frac{x_j+\alpha}{\beta+1}\right) + \frac{(x-x_n)_+}{1-x_n} f\left(\frac{\alpha+1}{\beta+1}\right) .$$

This is the equation of the interpolatory polygonal line for the abscissas $x_0,x_1,\ldots,x_n$.

In [57] it is shown that in the general case of broken lines interpolation we can give also the formula

$$P_n^f(x) = f(x_0) + [x_0,x_1;f](x-x_0)_+ +$$
$$+ \sum_{j=1}^{n} (x_{j+1}-x_{j-1})[x_{j-1},x_j,x_{j+1};f)(x-x_j)_+ ,$$

where $a = x_0 < x_1 < \ldots < x_n < x_{n+1} = b$, $f \in C[a,b]$.

It is known that if we take $x_j = a + jh$ $(j = 0,1,\ldots,n+1)$, $h = (b-a)/(n+1)$, then if $f \in C[a,b]$, we have

$$\|f - P_n^f\| \le 2\omega\left(\frac{b-a}{n+1}\right).$$

The next step leads to the cubic spline interpolation. In [57] is given an explicit expression for $S_{3,n}f$, by using the divided differences of the first four orders.

We mention that for the remainder of the approximation formula of the function f by $S_{m,n}f$, D.Leviatan [15] has given an integral form, by using the known theorem of Peano.

7. Extensions to two variables of the Stancu operators. By starting from the bidimensional Sfeffensen interpolation formula (see [35]), in the paper [49] D.D.Stancu has associated to a function f, defined on a polygonal domain Ω, an operator $L^{\alpha,\beta}$ defined by

$$(L^{\alpha,\beta}f)(x) := [\Phi^{\alpha,\beta}(0,0)]^{-1}\cdot \sum_{i=0}^{m}\sum_{j=0}^{n_i}(-1)^{i+j}\frac{x^{(i,-\alpha)}\,y^{(j,-\beta)}}{i!\,j!}D_{\alpha,\beta}^{i,j}\Phi^{\alpha,\beta}(x,y)\cdot f(x_i,y_j)\,, \tag{7.1}$$

where $(x_i,y_j)\in\Omega$ and $\Phi^{\alpha,\beta}$ is a bivariate polynomial, whose coefficients might depend on two real parameters α and β, while $D_{\alpha,\beta}^{i,j}=D_{1,\alpha}^{i}\cdot D_{2,\beta}^{j}$ is expressed in terms of iterated Nörlund difference quotions.

If one assumes that $\alpha\ge 0$, $\beta\ge 0$ and that

$$(-1)^{i+j}D_{\alpha,\beta}^{i,j}\Phi^{\alpha,\beta}(x,y)\ge 0 \quad (i=O(1)m,\ j=O(1)n_i)\,,$$

then $L^{\alpha,\beta}$ represents a linear positive operator.

We mention two remarkable special cases of these operators:

$$\Omega = S = [0,1]\times[0,1]\,,\ n_i=n,\ \Phi^{\alpha,\beta}(x,y) = \varphi^{\alpha}_{m}(x)\,\Psi^{\beta}_{n}(y)\ , \tag{7.2}$$

$$\Omega = T = \{(x,y)\,|x\geq 0, y\geq 0, x+y\leq 1\}\,,\ \beta=\alpha,\ n_i=m-i,\ \Phi^{\alpha,\alpha}=\Phi^{\alpha}_{m}\ . \tag{7.3}$$

In the first case, for any $(x,y)\ \epsilon\ S$ and for the nodes $(i/m, j/n)$, the corresponding approximating polynomial can be represented in terms of finite differences, in the following form

$$\begin{aligned}(L^{\alpha,\beta}_{m,n}f)\,(x,y) &= [\varphi^{\alpha}_{m}(0)\,\Psi^{\beta}_{n}(0)]^{-1}\cdot \\ &\cdot\sum_{i=0}^{m}\sum_{j=0}^{n} p^{\alpha,\beta}_{m,n,i,j}(x,y)\left(\Delta^{i,j}_{\frac{1}{m},\frac{1}{n}}\,f\right)(0,0)\ ,\end{aligned} \tag{7.4}$$

where

$$p^{\alpha,\beta}_{m,n,i,j}(x,y) = (-1)^{i+j}\frac{x^{(i,-\alpha)}y^{(j,-\beta)}}{i!\,j!}\left(D^{i}_{1,\alpha}\varphi^{\alpha}(0)\right)\left(D^{j}_{2,-\beta}\Psi^{\beta}_{n}(0)\right)\ .$$

In the case (7.3), for any $(x,y)\ \epsilon\ T$ we have the representation

$$\begin{aligned}(L^{\alpha}_{m}f)\,(x,y) &= \\ &= [\Phi^{\alpha}_{m}(0,0)]^{-1}\sum_{i=0}^{m}\sum_{j=0}^{m-i} q^{\alpha}_{m,i,j}(x,y)\left(\Delta^{i,j}_{\frac{1}{m},\frac{1}{m}}\,f\right)(0,0)\ ,\end{aligned} \tag{7.5}$$

where

$$q^{\alpha}_{m,i,j}(x,y) = (-1)^{i+j}\frac{x^{(i,-\alpha)}y^{(j,-\alpha)}}{i!\,j!}\left(D^{i,j}_{-\alpha,-\alpha}\Phi^{\alpha}_{m}(0,0)\right)\ .$$

If, in particular, we assume that

$$\varphi^{\alpha}_{m}(x) = (1-x)^{(m,-\alpha)}\quad,\ \Psi^{\beta}_{n}(y) = (1-y)^{(n,-\beta)}\ , \tag{7.6}$$

then in the case (7.2) formula (7.1) leads us to the Bernstein-type operator defined by

$$(S^{\alpha,\beta}_{m,n}f)\,(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} w^{\alpha}_{m,i}(x)\,w^{\beta}_{n,j}(y)\,f\left(\frac{i}{m},\frac{j}{n}\right)\ , \tag{7.7}$$

where is used the following notation

$$w^{\gamma}_{p,k}(t) = \binom{p}{k} t^{(k,-\gamma)}(1-t)^{(p-k,-\gamma)}/1^{(p,-\gamma)} .$$

If in the case (7.3) we select

$$\Phi^{\alpha}_{m}(x,y) = (1-x-y)^{(m,-\alpha)} , \tag{7.8}$$

then we obtain the following two-dimensional operator of Stancu

$$(S^{\alpha}_{m}f)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{m-i} w^{\alpha}_{m,i,j}(x,y)\, f\left(\frac{i}{m},\frac{j}{m}\right) , \tag{7.9}$$

where

$$w^{\alpha}_{m,i,j}(x,y) = \binom{m}{i}\binom{m-i}{j} x^{(i,-\alpha)} y^{(j,-\alpha)} \frac{(1-x-y)^{(m-i-j,-\alpha)}}{1^{(m,-\alpha)}} \tag{7.10}$$

In the case (7.7) formula (7.4) leads us to the following representation in terms of finite differences

$$(S^{\alpha,\beta}_{m,n}f)(x,y) = $$
$$= \sum_{r=0}^{m}\sum_{s=0}^{n} \binom{m}{r}\binom{n}{s} \frac{x^{(r,-\alpha)}y^{(s,-\beta)}}{1^{(r,-\alpha)}1^{(s,-\beta)}} \left(\Delta^{r,s}_{\frac{1}{m},\frac{1}{n}} f\right)(0,0) .$$

For the remainder $R^{\alpha,\beta}_{m,n} = I - S^{\alpha,\beta}_{m,n}$, corresponding to the approximation of a function $f:S \to \mathbb{R}$, there can be given the following simpie form expression by means of partial divided differences of second order:

$$(R^{\alpha,\beta}_{m,n}f)(x) =$$
$$= -\frac{x(1-x)(1+m\alpha)}{m(1+\alpha)} [\xi_1,\xi_2,\xi_3; f(t,y)]_t - \frac{y(1-y)(1+n\beta)}{n(1+\beta)} \cdot$$
$$[\eta_1,\eta_2,\eta_3; f(x,z)]_z \quad \frac{x(1-x)y(1-y)(1+m\alpha)(1+n\beta)}{mn(1+\alpha)(1+\beta)} \begin{bmatrix} \xi_1,\xi_2,\xi_3 \\ \eta_1,\eta_2,\eta_3 \end{bmatrix}; f \Bigg] ,$$

where ξ_i and η_j are suitable points from (0,1), generally depending on f.

In [45] is given the following estimations for the orders of approximation of the function $f \in C(S)$ by means of the operator defined at (7.7):

$$\|f - S_{m,n}^{\alpha,\beta} f\| \leq 2\omega(\delta_m^\alpha, \delta_n^\beta) \ , \tag{7.11}$$

$$\|f - S_{m,n}^{\alpha,\beta} f\| \leq \delta_m^\alpha \omega_{1,0}(\delta_m^\alpha, \delta_n^\beta) + \delta_n^\beta \omega_{0,1}(\delta_m^\alpha, \delta_n^\beta) \ , \tag{7.12}$$

where

$$\delta_m^\alpha = \sqrt{\frac{1+\alpha m}{m+\alpha m}} \ , \quad \delta_n^\beta = \sqrt{\frac{1+\beta n}{n+\beta n}} \ .$$

The inequality (7.12) is true when the function f has continuous first partial derivatives on S; by $\omega_{1,0}$ and $\omega_{0,1}$ are denoted the moduli of continuity of f'_x respectively of f'_y .

We mention that the extension of the Stancu operator S_n^α to the hypercube s-dimensional was investigated also in [45].

The representation by finite differences of the operator defined at (7.9) - (7.10), according to the formula (7.5), is the following

$$(S_m^\alpha f)(x,y) = \sum_{r=0}^{m} \sum_{s=0}^{m-r} \binom{m}{r}\binom{m-r}{s} \frac{x^{(r,-\alpha)} y^{(s,-\alpha)}}{1^{(r+s,-\alpha)}} \left(\Delta^{r,s}_{\frac{1}{m},\frac{1}{m}} f\right)(0,0) \ .$$

From this representation we can see immediately that $S_n^\alpha f$ takes the same values as the function f on the three vertices of T.

Assuming that α is positive, if we consider the Dirichlet double integral

$$B(p,q,r) = \iint_T u^{p-1} v^{q-1} (1-u-v)^{r-1} du\, dv \ ,$$

for $x > 0$, $y > 0$ and $x+1 < 1$, then Stancu has proven [39] that the two-dimensional operator S_n^α can be represented as an average mean of the Bernstein two-dimensional operator:

$$(S_m^\alpha f)(x,y) =$$
$$= [D(x,y;\alpha)]^{-1} \iint_T u^{\frac{x}{\alpha}-1} v^{\frac{y}{\alpha}-1} (1-u-v)^{\frac{1-x-y}{\alpha}-1} (B_m f)(u,v)\, du\, dv \ ,$$

where

$$D(x,y;\alpha) = B\left(\frac{x}{\alpha}, \frac{y}{\alpha}, \frac{1-x-y}{\alpha}\right) .$$

By using the modulus of continuity ω defined with respect to the metric $d((x_1,y_1),(x_2,y_2)) = |x_2-x_1| + |y_2-y_1|$, in [39] there has been given the following estimate, in sup norm, of the order of approximation:

$$\|f-S_m^\alpha\| \le 2\omega\left(f; \sqrt{\frac{1+m\alpha}{m+m\alpha}}\right) ,$$

provided f is a continuous function on the standard triangle T.

An extension to the standard simplex T_s, as well as the basic approximation properties of the corresponding operator have been investigated in the memoire [39].

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NEW INTERPOLATION PROCEDURE IN TRIANGLES

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Dedicated to Professor D.D.Stancu on his 65th anniversary

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REZUMAT. - Noi procedee de interpolare în triunghi. Problema construirii unor funcţii interpolatoare care să coincidă cu o funcţie dată pe laturile unui triunghi este larg studiată în ultima perioadă. În această notă se studiază problema construirii unor funcţii interpolatoare care să interpoleze funcţia dată nu numai pe laturile triunghiului dar şi pe anumite linii interioare acestuia (în cazul de faţă se consideră medianele). În final este dată şi o aplicaţie practică a formulelor de interpolare construite.

0. Begining with the paper by Barnhill, Birkhoff and Gordon [1], the interpolation problem to boundary data on a triangle was largely studied. Using boolean sum of some Lagrange's, Hermite's or Birkhoff's univariate operators, there were constructed interpolants that interpolate a given function and certain of its directional derivatives on the boundary ∂T of a given triangle T.

Following the discrete case, where the interpolating nodes lie not only on the boundary ∂T but also in the interior of the triangle T, it is natural to put such a problem in the transfinite interpolation case.

In this note, there are contructed some interpolants that interpolate a given function on the sides of a triangle T and on one of its median. Certainly, instead of median we can take others line in triangle. Finally some practical applications are given.

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1. For algebraic simplicity, one considers the standard triangle $T_a = \{(x,y) \in \mathbb{R}^2 | \ x \geq 0, \ y \geq 0, \ x+y \leq a\}$, (Fig.1) with the vertex $V_1 = (a,0)$, $V_2 = (0,a)$, $V_3 = (0,0)$ and with the side opposite the vertex V_k denoted by E_k. Any other triangle can be obtained by an affine transformation of this standard triangle.

Let $f:T_a \to \mathbb{R}$ be a given function.

One denotes by L_1^y and L_1^x the liniar interpolation operators along the parallels to the side E_1 respectivey E_2 of T_a, i.e.:

$$(L_1^y f)(x,y) = \frac{a-x-y}{a-x} f(x,0) + \frac{y}{a-x} f(x,a-x)$$

$$(L_1^x f)(x,y) = \frac{a-x-y}{a-y} f(0,y) + \frac{x}{a-y} f(a-y,y) .$$

As, it is well known, the main properties of these operators are that each of then interpolates the function f along on two of the sides of T_a:

$$(L_1^y f)(x,0) = f(x,0), \quad (L_1^y f)(x,a-x) = f(x,a-x), \quad x \in [0,a]$$

and

$$(L_1^x f)(0,y) = f(0,y), \quad (L_1^x f)(a-y,y) = f(a-y,y), \quad y \in [0,a] .$$

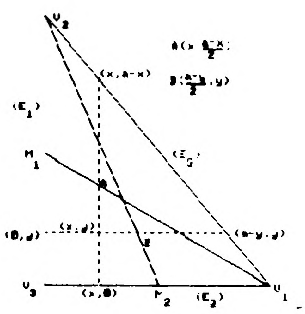

Fig. 1

Now, let us denote by L_2^y and L_2^x the quadratic interpolation operators along on the parallels to E_1 respectively E_2, that interpolates the function f at the points $(x,0), \left(x, \frac{a-x}{2}\right), (x,a-x)$, respectively $(0,y), \left(\frac{a-y}{2}, y\right), (a-y,y)$, where $\left(x, \frac{a-x}{2}\right)$ lie on the median V_1M_1 and $\left(\frac{a-y}{2}, y\right)$ lie the median V_2M_2 (Fig.1).

We have

$$(L_2^y f)(x,y) = \frac{(a-x-y)(a-x-2y)}{(a-x)^2} f(x,0) + \frac{4y(a-x-y)}{(a-x)^2} f\left(x, \frac{a-x}{2}\right) + \frac{y(x+2y-a)}{(a-x)^2} f(x,a-x)$$

and

$$(L_2^x f)(x,y) = \frac{(a-x-y)(a-2x-y)}{(a-y)^2} f(0,y) + \frac{4x(a-x-y)}{(a-y)^2} f\left(\frac{a-y}{2}, y\right) + \frac{x(2x+y-a)}{(a-y)^2} f(a-y,y).$$

So, L_2^y interpolates the function f on the sides E_2 and E_3 and on the median V_1M_1 respectively L_2^x interpolates f on E_1, E_3 and V_2M_2.

In order to obtain interpolation operators that interpolate the function f on all ∂T_a, one considers $L_2^y \oplus L_1^x$ and $L_2^x \oplus L_1^y$. We hawe:

THEOREM 1. *If* $f \in C(T_a)$ *then*

$$L_2^y \oplus L_1^x f = f \quad \text{on} \quad \partial T_a \cup V_1M_1$$

and

$$L_2^x \oplus L_1^y f = f \quad \text{on} \quad \partial T_a \cup V_2M_2.$$

Proof. Taking into account that

$$\begin{aligned}(L_2^y \oplus L_1^x f)(x,y) &= \frac{(a-x-y)}{(a-x)^2}\left[(a-x-2y) f(x,0) + 4y f\left(x, \frac{a-x}{2}\right)\right] + \\ &+ \frac{1}{a-y}[(a-x-y) f(0,y) + x f(a-y,y)] - \\ &- \frac{(a-x-y)(a-x-2y)}{a(a-x)}\left[f(0,0) + \frac{x}{a-x} f(a,0)\right] - \\ &- \frac{4y(a-x-y)}{a^2-x^2}\left[f\left(0, \frac{a-x}{2}\right) + \frac{2x}{a-x} f\left(\frac{a+x}{2}, \frac{a-x}{2}\right)\right]\end{aligned} \tag{1}$$

and the symmetric expression of $L_2^x \oplus L_1^y$, the proof follows by direct substitution.

THEOREM 2. $L_2^y \bigoplus L_1^x f = f$ *and* $L_2^x \bigoplus L_1^y f = f$ *for any* $f \in P_2^2$ *(the set of all polynomials of the degree less or equal to two).*

Proof. As, $L_2^y \bigoplus L_1^x$ and $L_2^x \bigoplus L_1^y$ are linear operators, it must be verified the two equalities for the basic functions e_{ij}, $i + j \le 2$, where $e_{ij}(x,y) = x^i y^j$. This way, the proof is a straightforward computation.

Using the boolean sum operators we can consider now the following approximation formulas

$$f = L_2^y \bigoplus L_1^x f + R_{21}^{yx} f$$

and

$$f = L_2^x \bigoplus L_1^y f + R_{21}^{xy} f$$

where $R_{21}^{yx} f$ respectively $R_{21}^{xy} f$ are the corresponding remainder terms.

THEOREM 3. *If* $f \in B_{12}(0,0)$ [5,p.175] *then*

$$R_{21}^{yx} f)(x,y) = \int_0^a K_{30}(x,y,s) f^{(3,0)}(s,0)\,ds + \int_0^a K_{21}(x,y,s) f^{(2,1)}(s,0)\,ds +$$

$$+ \int_0^a K_{03}(x,y,s) f^{(0,3)}(0,t)\,dt + \iint_{T_a} K_{12}(x,y,s,t) f^{(1,2)}(s,t)\,ds dt$$

where

$$K_{30}(x,y,s) = \frac{y(x+2y-a)}{2(a-x)^2}(x-s)_+^2 - \frac{x}{2(a-y)}(a-y-s)_+^2 -$$

$$+ \frac{xy(a-x-y)}{(a+x)(a-x)^2}(a+x-2s)_+^2 + \frac{x(a-x-y)(a-x-2y)}{2a(a-x)^2}(a-s)^2,$$

$$K_{21}(x,y,s) = \frac{y(x+2y-a)}{a-x}(x-s)_+ - \frac{xy}{a-y}(a-y-s)_+ +$$

$$+ \frac{2xy(a-x-y)}{a^2-x^2}(a+x-2s)_+,$$

$$K_{03}(x,y,t) = 0,$$

$$K_{12}(x,y,s,t) = (x-s)^0_+(y-t)_+ - \frac{2y(a-x-y)}{(a-x)^2}(x-s)^0_+(a-x-2t)_+ -$$

$$\frac{x}{a-y}(a-y-s)^0_+(y-t)_+ + \frac{4xy(a-x-y)}{(a+x)(a-x)^2}(a+x-2s)^0_+(a-x-2t)_+$$

Proof. Taking into account that $R_{21}^{yx}f = f$ for any $f \in P_2^2$, the proof follows by the Sard kernel theorem in triangles [2].

Remark. An analoguos theorem can be given for the remainder $R_{21}^{xy}f$.

To illustrate the approximation procedure, in Fig.2 is given the graph of the function $f(x,y) = 4-x^2-y^2$ while the Fig.3 is the graph of its approximation from (1).

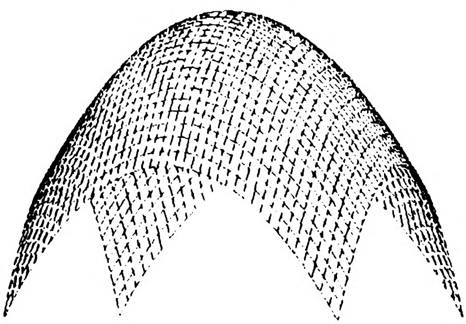

Fig. 2

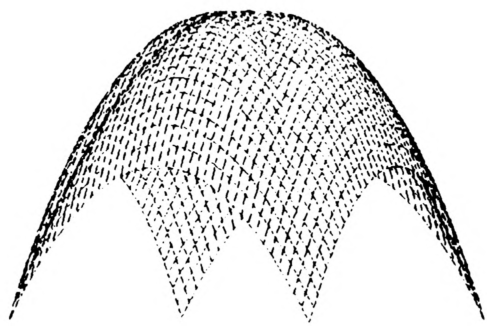

Fig. 3

2. Applications. Using the interpolation function from (1) we can construct surfaces that satisfied sowe given conditions. Such is a roof surface, say $F = F(x,y)$, with the basic properties: $F = 0$ on E_3 (it lies on the side E_3) and $F(0,0) = h$, $h \in \mathbb{R}_+$. Now, if F is extended, by symmetry, to the domain $D_a = \{(x,y) \in \mathbb{R} \mid |x| + |y| \leq a\}$, one obtains a surface that lies on the border of D_a (∂D_a) and take the value h in the cercer of D_a [4].

Surfaces that satisfies these two conditions are obtained from (1) and from the expression of $L_2^x \bigoplus L_1^y$, for $f(0,0) = h$, $f(x,a-x) = 0$, $x \in [0,a]$ and $f(a-y,y) = 0$ for $y \in [0,a]$. It follows that

$$F_1(x,y) = \frac{(a-x-y)(a-x-2y)}{(a-x)^2} f(x,0) + \frac{4y(a-x-y)}{(a-x)^2} f\left(x, \frac{a-x}{2}\right) + \frac{a-x-y}{a-y} f(0,y) - \frac{4y(a-x-y)}{a^2-x^2} f\left(a, \frac{a}{2}\right) - \frac{(a-x-y)(a-x-2y)}{a(a-x)} f(0,0) \tag{2}$$

respectively

$$F_2(x,y) = \frac{(a-x-y)(a-2x-y)}{(a-y)^2} f(0,y) + \frac{4x(a-x-y)}{(a-y)^2} f\left(\frac{a-y}{2}, y\right) + \frac{a-x-y}{a-x} f(x,0) - \frac{4x(a-x-y)}{a^2-y^2} f\left(\frac{a}{2}, 0\right) - \frac{(a-x-y)(a-2x-y)}{a(a-y)} f(0,0) \tag{3}$$

Really, we have: $F_1(0,0) = f(0,0):=h$ and $F_1(x,a-x) = 0$ for all $x \in [0,a]$ respectively $F_2(0,0) = f(0,0):=h$ and $F_2(a-y,y) = 0$ for all $y \in [0,a]$.

We also have: $F_1(x,0) = f(x,0)$, $F_1(0,y) = f(0,y)$ and $F_1\left(x, \frac{a-x}{2}\right) = f\left(x, \frac{a-x}{2}\right)$, for $x,y \in [0,a]$ i.e. F_1 interpolates the function f on $E_1 \cup E_2 \cup V_1M_1$ respectively $F_2(x,0) = f(x,0)$, $F_2(0,y) = f(0,y)$ and $F_2\left(\frac{a-y}{2}, y\right) = f\left(\frac{a-y}{2}, y\right)$, $x,y \in [0,a]$, i.e. F_2 interpolates f on $E_1 \cup E_2 \cup V_2M_2$. In other words, if $g_1(x) = f(x,0)$, $g_2(y) = f(0,y)$ and $g_{31}(x) = f\left(x, \frac{a-x}{2}\right)$, $g_{32}(y) = f\left(\frac{a-y}{2}, y\right)$, then we have the two families of surfaces

$$F_i = F_i(g_1, g_2, g_{3i}), \quad i = 1,2. \tag{4}$$

Hence, for each given g_1, g_2 and g_{31} or g_{32} one obtains a surface from the corresponding family.

What is new here is that we can control the surfaces not only on the boundary of T_a but also on its median.

It is obviously that the function g_1, g_2 and g_{3i} must satisfies the natural conditions: $g_1(0) = g_2(0) = h$, $g_1(a) = g_2(a) = 0$ and $g_{31}(a) = g_{32}(a) = 0$. So, they can be some interpolation functions (polynomials).

Next, one considers some of such surfaces.

A. Let us consider first

$$g_1(x) := \frac{a-x}{a} f(0,0)$$

$$g_2(y) := \frac{(a-y)(a-2y)}{a^2} f(0,0) + \frac{4y(a-y)}{a^2} f\left(0, \frac{a}{2}\right)$$

$$g_{31}(x) := \frac{a-x}{a} f\left(0, \frac{a}{2}\right)$$

in the family F_1 respectively

$$g_1(x) := \frac{(a-x)(a-2x)}{a^2} f(0,0) + \frac{4x(a-x)}{a^2} f\left(\frac{a}{2}, 0\right)$$

$$g_2(y) := \frac{a-y}{a} f(0,0)$$

$$g_{32}(y) := \frac{a-y}{a} f\left(0, \frac{a}{2}\right).$$

in F_2.

From (2) and (3) one obtains:

$$F_{11}(x,y) = \frac{(a-x-y)(a-2y)}{a^2} h + \frac{4y(a-x-y)(a^2+ax-x^2)}{a^2(a^2-x^2)} f\left(0, \frac{a}{2}\right)$$

respectively

$$F_{21}(x,y) = \frac{(a-x-y)(a-2x)}{a^2} h + \frac{4x(a-x-y)(a^2+ay-y^2)}{a^2(a^2-y^2)} f\left(\frac{a}{2}, 0\right)$$

where it must be given $f\left(0, \frac{a}{2}\right)$ and $f\left(\frac{a}{2}, 0\right)$.

An example of surface is in Fig.4 for the function

$$G_1 = \frac{1}{2}(F_{11} + F_{21}) \ . \tag{5}$$

B. In the second case one of the function g_1 or g_2 is taken as an Hermite interpolation polynomial, in order to can control the derivative of the surfaces (the directional tangents) along the axis ox or oy.

So, in the family F_1, we take:

$$g_1(x) = \frac{(x-a)^2}{a^2} g_1(0) + \frac{x(2a-x)}{a^2} g_1(a) + \frac{x(x-a)}{a} g_1'(a)$$

As, $g_1(0) = f(0,0)$ and $g_1(a) = 0$, one obtains

$$g_1(x) = \frac{(x-a)^2}{a^2} f(0,0) + \frac{x(x-a)}{a} g_1'(a) \ .$$

The function g_2 and g_{31} are taken as in the first case:

$$g_2(y) = \frac{(a-y)(a-2y)}{a^2} f(0,0) + \frac{4y(a-y)}{a^2} f\left(0, \frac{a}{2}\right)$$

$$g_{31}(x) = \frac{a-x}{a} f\left(0, \frac{a}{2}\right) \ .$$

It follows that

$$\begin{aligned} F_{12}(x,y) = {} & \frac{(a-x-y)\,[\,(a-x)^2+2y(2x-a)\,]}{a^2(a-x)} f(0,0) + \\ & + \frac{4y(a-x-y)(ax+a^2-x^2)}{a^2(a^2-x^2)} f\left(0, \frac{a}{2}\right) + \\ & + \frac{x(a-x-y)(a-x-2y)}{a(a-x)} f^{(1,0)}(a,0) \ . \end{aligned} \tag{6}$$

On a analogous way, one obtains (from (4)):

$$\begin{aligned} F_{22}(x,y) = {} & \frac{(a-x-y)\,[\,(a-y)^2+2x(2y-a)\,]}{a^2(a-y)} f(0,0) + \\ & + \frac{4x(a-x-y)(ay+a^2-y^2)}{a^2(a^2-y^2)} f\left(\frac{a}{2}, 0\right) + \\ & + \frac{y(a-x-y)(a-2x-y)}{a(a-y)} f^{(0,1)}(0,a) \ . \end{aligned} \tag{7}$$

Finally, we also consider the surface

$$G_2 = \frac{1}{2}(F_{12} + F_{22}) . \tag{8}$$

From (6)-(8), for given $f(0,0)$, $f\left(0,\frac{a}{2}\right)$, $f\left(\frac{a}{2},0\right)$, $f^{(1,0)}(a,0)$ and $f^{(0,1)}(0,a)$, one obtains surfaces, say F, with the basic properties $F(0,0) = h := f(0,0)$ and $F|_{\partial D_a} = 0$. In Fig.5 is an example for the surface G_2 .

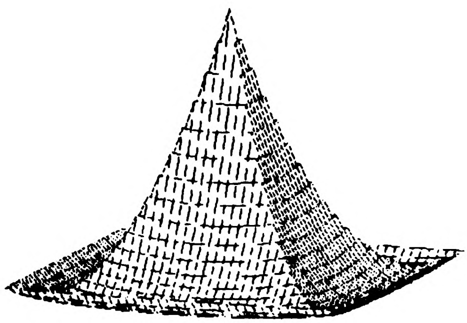

Fig. 4

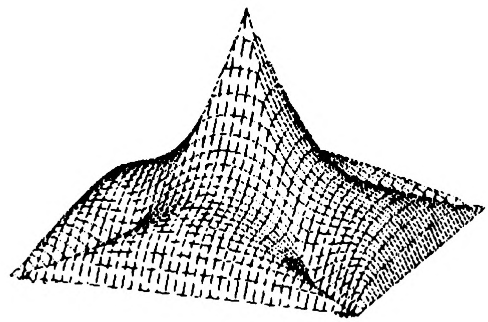

Fig. 5

R E F E R E N C E S

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A NOTE ON A CLASS OF TURAN TYPE QUADRATURE FORMULAS WITH GENERALIZED GEGENBAUER WEIGHT FUNCTIONS

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Dedicated to Professor D.D.Stancu on his 65^{th} anniversary

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REZUMAT. - Notă asupra unei clase de formule de cuadratură de tip Turán cu funcţii pondere Gegenbauer generalizate. Se construiesc formule de cuadratură de tip Turán pentru evaluarea numerică a integralelor cu funcţii pondere Gegenbauer generalizate. Proprietăţile cele mai relevante ale acestor formule constă în independenţa nodurilor de multiplicităţile lor. Se prezintă un algoritm pentru calcularea coeficienţilor procesului de cuadratură prezentat şi se enunţă o teoremă de convergenţă a şirurilor formulelor obţinute. Se dau anumite exemple numerice.

Abstract. Some Turán type quadrature rules are constructed for the numerical evaluation of weighted integrals, with generalized Gegenbauer weight functions. The most relevant features of these formulas is the independence of the nodes of their multiplicity. An algorithm for calculating the weights of the quadrature process is presented and a convergence theorem for the sequences of the obtained formulas is stated. Some numerical examples are also given.

1. Introduction. The quadrature rules for evaluating the integral

$$\int_a^b w(x)\, f(x)\, dx = \sum_{i=1}^{m} \sum_{h=0}^{2s} A_{hi} f^{(h)}(x_{i,m}) + R(f) , \qquad (1.1)$$

considered by Turán [20], have m multiple nodes $x_{i,m}$, $i=1,2,\dots,m$, each having the same multiplicity.

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It is well known ([4], pg. 131) that the degree of precision of (1.1) is $N-1 = 2m(s+1)-1$ when the nodes are placed at the zeros of the polynomial $P_{m,s}(x)$ satisfying the power orthogonality conditions:

$$\int_a^b w(x)\, x^k [P_{m,s}(x)]^{2s+1}\, dx = 0, \quad k=0,1,\ldots,m-1. \qquad (1.2)$$

The polynomials satisfying (1.2) are called s-orthogonal in $[a,b]$ with respect to w [2], [3], [4]. The case $s = 0$ gives rise to the polynomials orthogonal in $[a,b]$ with respect to w.

It is well known that the polynomials satisfying (1.2) can also be seen as polynomials of minimal L_p norm, $p = 2s+2$ [18], [20], [4]; thus some of their properties derive from this interpretation. Let us remark that, in spite of their importance in the field of approximation theory, the polynomials of minimal L_p norm are explicity known only for particular values of p and/or specific weight functions (for istance, see [16]).

For recent results on polynomials of minimal norm we refer to [6], [9], [10].

The zeros of $P_{m,s}$ are real, distinct and are contained in (a,b) [4] and, generally, they depend not only on m but also on s.

Here we shall consider only the case $[a,b]$ finite assuming in particular $[a,b] = [-1,1]$.

Although the nodes $x_{i,m}$, can be evaluated following some stable methods [5], [13], the corresponding algorithms become considerably expensive under the computational aspect, when m or s increases.

Thus, an interesting goal seems to be the construction of Turán type quadrature rules, where the nodes are independent of s, at least for some values of m and for suitable weight functions.

This paper is devoted to study a case in which some quadrature rules, having the above mentioned invariance property with respect to s, can be obtained.

In Section 2 some preliminaries are given and the quadrature rules are constructed; in Section 3 an algorithm for evaluating the weights is presented and some remarks are made about the round-off error propagation; Section 4 contains a convergence result of the rules and in Section 5 numerical examples are carried out.

2. **On some particular Turán type quadrature rules.** In this paper we consider weight functions w of generalized Gegenbauer type (G.G.W.)

$$w(x)=|x|^{\tau}(1-x^2)^{\mu},\quad \mu,\tau>-1,\quad x\in[-1,1]. \tag{2.1}$$

The corresponding orthogonal polynomials were introduced in [8] and some applications are given in different contexts of approximation theory [14], [21]. As to the polynomials s-orthogonal with respect to (2.1), we remark that they are involved, for istance, in the construction of quadrature rules with preassigned nodes y_j, $j=1,2,\ldots,n$ [7], [19]

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{j=1}^{n}\ \sum_{k=0}^{\alpha_j} B_{kj} f^{(k)}(y_j) + \sum_{i=1}^{m}\ \sum_{h=0}^{2s} A_{hi} f^{(h)}(x_{i,m}) \ , \tag{2.2}$$

where $n=3$, $y_1=-1$, $y_2=0$, $y_3=1$, α_2 is an odd positive integer and

$\alpha_1=\alpha_3$ are positive integers; then the highest degree of precision of (2.2) is achieved when the nodes $x_{i,m}$ are placed at the zeros of the polynomial of degree m, s-orthogonal with respect to a weight function (2.1), with $\tau=\alpha_2+1$, $\mu=\alpha_1+1$.

In the following theorem an invariance property of the polynomials $P_{m,s}$ s-orthogonal with respect to $w\epsilon$G.G.W. is stated.

THEOREM 1. *Let $w\epsilon$G.G.W., then the polynomials $P_{m,s}$ are invariant with respect to s, if and only if $\tau=2\mu+1$, $\mu=-1/2$, that is*

$$w(x) = |x|^{2\mu+1}(1-x^2)^{\mu}. \tag{2.3}$$

Proof. If $\tau=2\mu+1$, $\mu=-1/2$ then w reduces to $(1-x^2)^{-1/2}$, in which case the corresponding s-orthogonal polynomials are the Chebyshev polynomials of first kind for each s, as it is well known [3], [1].

The inverse is also true; in fact, due to the symmetry of w, $P_{m,s}$ are even or odd functions, according with the even or odd values of m and denoting by $\pm\sqrt{\alpha_{s,k}}(\mu,\tau)$ the zeros of $P_{m,s}$, then one has

$$P_{m,s}(x;\mu,\tau) = x^{\epsilon}\prod_{k=1}^{[m/2]}(x^2-\alpha_{s,k}(\mu,\tau)),$$

where $\epsilon=0,1$ according to m even or odd. Admit now that $P_{2,s}(x;\mu,\tau)$ be independent of s, which implies that the following s-ortogonality conditions must be fulfilled simultaneously

$$\int_{-1}^{1}|x|^{\tau}(1-x^2)^{\mu}(x^2-\alpha(\mu,\tau))^{2s+1}dx=0,\ s=0,1 \tag{2.4}$$

with $\alpha(\mu,\tau)=:\ \alpha_{0,1}(\mu,\tau) = \alpha_{1,1,}(\mu,\tau)$.

Denote by B the Beta function (or Euler function of the first kind) and by $m_k(\mu,\tau)$, $k=0,1,\ldots$, the moments of w, given by

$$m_k(\mu,\tau) = B((\tau+k+1)/2,\ \mu+1)/2,$$

then from (2.4) it follows

$$\alpha_\mu(\mu,\tau) = m_2(\mu,\tau)/m_0(\mu,\tau),$$

$$2[m_2(\mu,\tau)]^3-3m_4(\mu,\tau)m_2(\mu,\tau)m_0(\mu,\tau) + m_6(\mu,\tau)[m_0(\mu,\tau)]^2 = 0.$$

Using well known properties of the Beta function yields $\tau=2\mu+1$. The same reasoning applied to the case $m=3$ gives $\mu=-1/2$, $\tau=o$, so leading to the Chebyshev polynomials of the first kind.∎

From the above proof, considering $\tau=2\mu+1$, one derives also that

$$P_{2,s}(x;\mu) = x^2 - 1/2,$$

for each μ.

This result immediately leads to the construction of a class of integration rules relative to weight functions (2.3)

$$\int_{-1}^{1}|x|^{2\mu+1}(1-x^2)^\mu f(x)\,dx= \tag{2.5}$$

$$=\sum_{0}^{2s}{}_h\,[A_{h1}(\mu)f^{(h)}(-1/\sqrt{2})+A_{h2}(\mu)f^{(h)}(1/\sqrt{2})]+R_s(f),$$

where

$$R_s(f)=0,\quad f\epsilon P_{N-1},\ N=4s+4.$$

The subscript s in $R_s(f)$ is meant to stress the dependence on s of the remainder.

Let us remark that, since no condition is required for μ, (2.5) represents a class of quadrature rules, each having the same nodes, while the weights depend on μ.

For the symmetry of w, one has [4]

$$A_{h2}(\mu) = (-1)^h A_{h1}(\mu) =: A_h(\mu), \quad h=0,1,\ldots,2s.$$

The evaluation of the weights $A_h(\mu)$ will be carried out by a suitable algorithm presented in Section 3.

We recall that an useful expression of the weights in (1.1) is also given by

$$A_{hi} = (-1)^{N-h-1}\, \Phi^{(N-h-1)}(x)\Big|_{x_{i,m}^-}^{x_{i,m}^+}, \quad i=\overline{1,m};\ h=\overline{0,2s}, \tag{2.6}$$

where the function Φ is the Peano kernel or influence function [1], [17] related to (1.1). The Peano kernel is a generalized monospline having the nodes of the quadrature rule as knots and is defined by

$$\Phi(x) = \int_{-1}^{1} w(t)\,(x-t)_+^{N-1}/(N-1)!\,dt - \sum_{j=1}^{m} \sum_{h=0}^{2s} (-1)^{N-h-1} A_{hj}\,(x-x_{j,m})_+^{N-h-1}/(N-h-1)!\ . \tag{2.7}$$

We shall also use the notation

$$\Phi(x) = \Phi_i(x), \quad x\in[x_{i,m}, x_{i+1,m}], \quad i=0,1,\ldots,m; \quad x_0 := -1, x_{m+1} := 1.$$

The salient properties of Φ, meaningful in the context of the present paper, are the following

$$\Phi(x) > 0 \quad \text{in } (-1,1), \tag{2.8}$$

$$\Phi^{(h)}(1) = \Phi^{(h)}(-1) = 0, \quad h=0,1,\ldots,N-1, \tag{2.9}$$

$$\Phi(x) = \Phi(-x). \tag{2.10}$$

The function Φ allows also to express the remainder, for $f\in C^N[-1,1]$, by

$$R(f) = \int_{-1}^{1} w(x)\,\Phi(x)\,f^{(N)}(x)\,dx \tag{2.11}$$

3. **On the evaluation of the weights.** The weights (2.5) can be computed by (2.6), or, taking in account the interpolatory character of Turán quadratures, by integration of suitable Hermite-Lagrange polynomials. But due to the nature of the weight function (2.3) it true out to be more convenient to get them following the Theorem below.

THEOREM 2. *Let* w *be of type* (2.3), *then the weights of* (2.5) *are the solution of the upper triangular system:*

$$2^{2(\mu+1+n)}\sum_{h=n}^{M} C_{n,h}(\sqrt{2})^{-h}A_h(\mu) = \begin{cases} 0, & n=2k+1,\ k=\overline{0,s-1} \\ B(k+1/2,\mu+1)/(2k)!, & n=2k,\ k=\overline{0,s}, \end{cases} \tag{3.1}$$

where $M=\min(2n,2s)$, B *denotes the Beta function and*

$$C_{n,h} = h!/[(2n-h)!(h-n)!]. \tag{3.2}$$

Proof. Since the degree of precision of (2.5) is $N-1 = 4s+3$, the weights $A_h(\mu)$ can be evaluated solving the system

$$\int_{-1}^{1} w(x)\,g_n(x)\,dx = \sum_{h=0}^{2s} A_h(\mu)\,[(-1)^h g_n^{(h)}(-1/\sqrt{2}) + g_n^{(h)}(1/\sqrt{2})]$$

where

$$g_n(x) = (2x^2-1)^n/n!,\quad n=0,1,\dots,2s.$$

It is easy to show that

$$\int_{-1}^{1} w(x)\,g_n(x)\,dx = \begin{cases} 0, & n=2k+1,\ k=\overline{0,s-1}, \\ B(k+1/2,\mu+1)/(2k)!, & n=2k,\ k=\overline{0,s}. \end{cases}$$

Moreover it is possible to prove, by induction, that, for $h=0,1,\dots,2n$:

$$g_n^{(h)}(x) = \sum_{j=0}^{[h/2]} 2^{2(h-2j)} x^{h-2j} D_{j,h}(2x^2-1)^{n-h+j}/(n-h+j)!\ ,$$

where

$$D_{j,h} = 2^j h!/[j!(h-2j)!].$$

Taking into account that $g_n \epsilon P_{2n}$ is even and has zeros of multiplicity n at $\pm 1/\sqrt{2}$, one gets (3.1), by some easy manipulations.

In the particular case when μ is a nonnegative integer, the corollary below holds:

COROLLARY 1. *If $\mu \epsilon N$, then the weights $A_h(\mu)$ in the quadrature formula* (2.5) *are rational if h is even and are rational up to $\sqrt{2}$ if h is odd.*

In order to get some information on the propagation error and to analyse the convergence of the obtained rules, it is of interest the result of the following theorem.

THEOREM 3. *Let w be of type* (2.3), *then the weights of* (2.5) *satisfy the following bounds*

$$|A_h(\mu)| \leq \delta_0/[h!(\sqrt{2})^h], \tag{3.3}$$

with $\delta_0 = \int_{-1}^{1} w(x)\,dx$.

Proof. As it was shown in [11], [16], the function $\phi^{(N-k)}$ has exactly k zeros, say $y_{j,N-k}$ $(j=1,\ldots,k)$, in $(x_{1,2}, x_{2,2})$, which fulfil the relation

$$x_{1,2} < y_{1,N-k} < y_{1,N-k+1},\ k=2,\ldots,2s+2. \tag{3.4}$$

Due to the symmetry property of w, one has, in particular

$$y_{1,N-1} = 0.$$

Since $\Phi_1^{(N)} = w$, $x \in [x_{1,2}, x_{2,2}]$, we can write

$$\Phi_1^{(N-k)}(x)=\int_{y_{1,N-k}}^{x} dt_{k-1}\int_{y_{1,N-k+1}}^{t_{k-1}} dt_{k-2}\ \dots \int_{y_{1,N-1}}^{t_1} w(t)\,dt.$$

It follows from (3.4)

$$|\Phi_1^{(N-k)}(x_1)| \le \left|\int_{x_1}^{y_{1,N-1}} dt_{k-1}\int_{t_{k-1}}^{y_{1,N-1}} dt_{k-2}\dots\int_{t_1}^{y_{1,N-1}} W(t)\,dt\right| =$$
$$=\left|\int_{x_1}^{0} w(t)\,(x_1-t)^{k-1}/(k-1)!\,dt\right| \le \delta_0/[2(k-1)!(\sqrt{2})^{k-1}].$$

Again from (2.7) one derives

$$|\Phi_0^{(N-k)}(x_1)| = \left|\int_{-1}^{x_1} w(t)\,(x_1-t)^{k-1}/(k-1)!\,dt\right| \le \delta_0/[2(k-1)!(\sqrt{2})^{k-1}],$$

and then, using the last two relations, (3.3) follows. ∎

Among other things, the bounds (3.3) allow us to obtain information about the absolute global error E_{max} due to the round-off errors on the data $f^{(h)}(x_{i,2})$. More precisely, assuming that the computed values f_{hi} are such that $|f_{hi} - f^{(h)}(x_{i,2})| < \epsilon$, $i=1,2$, one gets:

$$E_{max} \le 2\epsilon\sum_{0}^{2s}{}_h |A_h(\mu)| \le 2\epsilon\delta_0 e^{1/\sqrt{2}} =$$
$$= \epsilon e^{1/\sqrt{2}}\sqrt{\pi}\,\Gamma(\mu+1)/[2^{2\mu}\Gamma(\mu+3/2)]$$

From the behaviour of the Gamma function it is easy to deduce that E_{max} is a decreasing function of μ.

4. Behaviour of the remainder. This section is addressed to get estimates of the remainder term of the particular Turán integration rules (2.5), and to investigate its convergence when $s \to \infty$.

We recall that results on the convergence of Turán quadratures (1.1) are contained in [15], where the case of fixed s and $m \to \infty$ is dealed with.

The case of m fixed and $s \to \infty$ is analysed in [12] where the convergence is proved under the assumption that f is analytic in a suitable region of the complex plane containing [-1,1]. This result is reached considering the following expression of the error term

$$R(f) = \int_{-1}^{1} w(x)\,[f(x) - p_n(x)]\,dx, \tag{4.1}$$

where p_n is the polynomial of Hermite-Lagrange, interpolating $f(x)$ at the multiple nodes $x_{i,m}$ of (1.1).

Here we shall present a result of convergence which holds for functions $f \in C^{\infty}[-1,1]$ and is obtained by the use of an expression of R, different from (4.1); actually, $R_s(f)$ can also be written in the form (2.11).

The following Theorem 4 can be stated

THEOREM 4. *Let $f \in C^{\infty}[-1,1]$, let $|f^{(k)}(x)| \le M_k$, $x \in [-1,1]$, and assume*

$$\lim_{k\to\infty} \frac{M_{2k}}{(k+1)!\,2^k} = 0, \tag{4.2}$$

then

$$\lim_{s\to\infty} R_s(f) = 0.$$

Proof. Let us remark that, since w is an even function, such is also Φ, then (2.8), (2.9) yield $\Phi(x) \le \Phi(0)$, $x \in [-1,1]$; furthermore from (2.11), (2.8) one derives

$$|R_s(f)| \le M_N \cdot \Phi(0)\,\delta_0.$$

Relation (2.7), jointly with (3.3), gives

$$\Phi(0) \le \delta_0/(N-1)! + 2\delta_0 \sum_{0}^{2s}{}_h (\sqrt{2})^{-h}/h! \quad (\sqrt{2})^{h-N+1}/(N-h-1)! \le$$
$$\le \delta_0/(4s+3)! + 2\ (\sqrt{2})^{1-N}\delta_0/(2s+3)! \sum_{0}^{2s}{}_h 1/h! \le$$
$$\le c\delta_0/[(2s+3)!2^{2s+2}],$$

with $c = 4(2+e\sqrt{2})$.

Then the claim follows from (4.2).

COROLLARY 1. *Let* $f\epsilon C^{\infty}[-1,1]$, $|f^{(k)}(x)|\le M$, $x\epsilon[-1,1]$, $k=0,1,...,$ *then*

$$\lim_{s\to\infty} R_s[f] = 0.$$

5. Some remarks and numerical examples. System (3.1) can be written as

$$\sum_{n}^{N}{}_h C_{n,h}X_h(\mu) = b_n,\ n=0,1,...,2s, \tag{5.1}$$

where

$$X_h(\mu) = 2^{2(\mu+1)}(\sqrt{2})^{-h}A_h(\mu),$$

$$b_n = \begin{cases} 0, & n=2k+1, \\ 2^{-2n}B(k+1/2,\mu+1)/(2k)!, & n=2k. \end{cases}$$

It is easy to obtain the following recursion to evaluate b_n, $n=0,1,...,2s$

$$b_0 = B(1/2,\mu+1),$$
$$b_{2k} = b_{2(k-1)}/[32k(2k+2\mu+1)],\ k=1,...,s \tag{5.2}$$

Denoting by b_{2k} the exact values and by b'_{2k} the calculated ones, the absolute errors $e_{2k}=|b_{2k}-b'_{2k}|$ satisfy the difference equation (5.2), whose solution, obtained by induction is given by

$$e_{2k} = e_0 \Big/ \left[32^k k! \prod_{j=0}^{k} (2j+2\mu+1)\right], \quad k\geq 1,$$

then the initial error e_0 fastly decreases when k increases.

The starting value in the back sostitution process to solve the system (3.1)

$$\begin{cases} X_{2s} = b_{2s}, \\ X_r = b_r - \sum_{h=r+1}^{M} C_{h-r}^{(h)} X_h, \quad r=2s-1, 2s-2, \dots, 0, \end{cases}$$

can be assumed correct in the machine precision.

The quadrature formula (2.5) is now written, using x_h:

$$\int_{-1}^{1} |x|^{2\mu+1} (1-x^2)^{\mu} f(x)\, dx =$$
$$= 2^{-2(2\mu+1)} \left[\sum_{h=0}^{s} 2^h X_{2h} [f^{(2h)}(-1/\sqrt{2}) + f^{(2h)}(1/\sqrt{2})] + \right.$$
$$\left. + \sqrt{2} \sum_{h=0}^{s-1} 2^h X_{2h+1} [-f^{(2h+1)}(-1/\sqrt{2}) + f^{(2h+1)}(1/\sqrt{2})] \right]$$

In Table 5.1-5.5 the values of x_h are quoted for different choices of μ. Tables 5.6-5.8 contain the absolute errors obtained comparing the exact value and the approximation given by (2.5), when (2.5) is applied for evaluating respectively the following integrals

$$\int_{-1}^{1} |x|^3 (1-x^2) \sin(\alpha x+\beta)\, dx, \quad (\alpha,\beta) = (1,1.2),\ (2,3.2),\ (1,2.14),$$

$$\int_{-1}^{1} |x| \log(x+\alpha)\, dx, \quad \alpha=2,\ 4.5, \qquad \int_{-1}^{1} |x| e^{\alpha x} dx, \quad \alpha=3,\ -2.5,\ \pm 0.5.$$

s	μ = -0.5		
1	.157079632679D+01	-.490873852123D-01	.245436926062D-01
2	.157079632679D+01	-.605922411215D-01	.302961205607D-01
	-.115048559091D-02	.958737992429D-04	
3	.157079632679D+01	-.656256155817D-01	.328128077909D-01
	-.170974941983D-02	.165781777857D-03	-.499342704390D-05
	.166447568130D-06		
4	.157079632679D+02	-.768261845304D-01	.384130922652D-01
	-.293298300099D-02	.310708195577D-03	-.140830247097D-04
	.412217805447D-06	-.910260138211D-08	.162546453252D-09
5	.157079632679D+01	-.762854080849D-01	.381427040424D-01
	-.293319366119D-02	.333345877575D-03	-.198109829212D-04
	.101418421310D-05	-.294696719746D-07	.564848925050D-09
	-.914323799542D-11	.101591533282D-12	
6	.157079632679D+01	-.958214192782D-01	.479107096391D-01
	-.517429117190D-02	.639894166541D-03	-.468566961504D-04
	.255802452024D-05	-.635554804211D-07	.169984858329D-08
	-.341760267433D-10	.479914170246D-12	-.582034826097D-14
	.440935474316D-16		

Table 5.1

s	μ = 0		
1	.200000000000D+01	-.416666666667D-01	.208333333333D-01
2	.200000000000D+01	-.494791666667D-01	.247395833333D-01
	-.781250000000D-03	.651041666667D-04	
3	.200000000000D+01	-.524088541667D-01	.262044274833D-01
	-.110677083333D-02	.105794270833D-03	-.290643601190D-05
	.968812003968D-07		
4	.200000000000D+01	-.587463378906D-01	.293731689453D-01
	-.179777599516D-02	.187235029917D-03	-.793214828249D-05
	.224037775918D-06	-.470950279707D-08	.840982642334D-10
5	.200000000000D+01	-.578397115072D-01	.289198557536D-01
	-.172983805339D-02	.191042158339D-03	-.104281006667D-04
	.525151611005D-06	-.147171962408D-07	.273319358758D-09
	-.430047942102D-11	.477831046780D-13	
6	.200000000000D+01	-.672682921092D-01	.336341460546D-01
	-.281094710032D-02	.338739156723D-03	-.234241208071D-04
	.126270390061D-05	-.401980146415D-07	.789914449209D-09
	-.154996445799D-10	.212037527009D-12	-.252699111278D-14
	.191438720665D-16		

Table 5.2

.314159265359D+01	-.490873852123D-01	.245436926062D-01
.314159265359D+01	-.567572891518D-01	.283786445759D-01
-.766990393943D-03	.639158661619D-04	
.314159265359D+01	-.592739763819D-01	.296369881909D-01
-.104662230840D-02	.988698554692D-04	-.249671352195D-05
.832237840650D-07		
.314159265359D+01	-.645930997047D-01	.322965498524D-01
-.162579742766D-02	.166827276645D-03	-.663189529268D-05
.181531878992D-06	-.364104055284D-08	.650185813008D-10
.314159265359D+01	-.634183998938D-01	.317091999469D-01
-.152194194737D-02	.163845361960D-03	-.821872253326D-05
.407674632078D-06	-.110369041758D-07	.199119405234D-09
-.304774599847D-11	.338638444275D-13	
.314159265359D+01	-.702749978186D-01	.351374989093D-01
-.230781294491D-o2	.271089280528D-03	-.176325698959D-04
.939131436054D-06	-.291638814594D-07	.554753266430D-09
-.106353636405D-10	.141956026417D-12	-.166295664599D-14
.125981564090D-16		

Table 5.3

	$\mu=1$	
.133333333333D+01	-.166666666667D-01	.833333333333D-02
.133333333333D+01	-.188988095238D-01	.944940476190D-02
-.223214285714D-03	.186011904762D-04	
.133333333333D+01	-.195498511905D-01	.977492559524D-02
-.295552248677D-03	.276434358466D-04	-.645874669312D-06
.215291556437D-07		
.133333333333D+01	-.208994063120D-01	.104497031560D-01
-.442327267278D-03	.447994817502D-04	-.167707229475D-05
.446485330112D-07	-.856273235831D-09	.152905934970D-10
.133333333333D+01	-.205136980329D-01	.102568490165D-01
-.406062872494D-03	.427384626527D-04	-.197299115070D-05
.965065808561D-07	-.252765272500D-08	.444015311162D-10
-.661612218619D-12	.735124687355D-14	
.133333333333D+01	-.220408313877D-01	.110204156938D-01
-.581030061058D-03	.665915004861D-04	-.406225050409D-05
.213892378917D-06	-.648441724720D-08	.119632353808D-09
-.224304920229D-11	.292518365176D-13	-.336932148371D-15
.255251627554D-17		

Table 5.4

s	μ=2		
1	.106666666667D+01	-.952380952381D-02	.476190476190D-02
2	.106666666667D+01	-.105158730159D-01	.525793650794D-02
	-.992063492063D-04	.826719576720D-05	
3	.106666666667D+01	-.107526154401D-01	.537630772006D-02
	-.125511063011D-03	.115552849928D-04	-.234863516114D-06
	.782878387045D-08		
4	.106666666667D+01	-.112285663555D-01	.561428317776D-02
	-.177176519950D-03	.175567253204D-04	-.588287941804D-06
	.149424384450D-07	-.263468687948D-09	.470479799907D-11
5	.106666666667D+01	-.110438626857D-01	.552193134285D-02
	-.158888330275D-03	.161086167251D-04	-.634077858809D-06
	.302466758562D-07	-.744299043453D-09	.124677146975D-10
	-.176429924965D-12	.196033249961D-14	
6	.106666666667D+01	-.114722140635D-01	.573610703177D-02
	-.207933569941D-03	.227832624817D-04	-.121648362691D-05
	.626953147959D-07	-.181517333743D-08	.316635499895D-10
	-.569289206413D-12	.711341241587D-14	-.792781525578D-16
	.600592064832D-18		

Table 5.5

s	sin(x+1.2)	sin(2x+3.2)	sin(x+2.14)
1	.20073D-07	.29584D-06	.18141D-07
2	.21902D-12	.57297D-10	.21454D-12
3	.17666D-13	.26869D-14	.63838D-15
4		.71384D-15	

Table 5.6 (μ=1)

s	log(x+2)	log(x+4.5)
1	.10515D-04	.10273D-07
2	.10241D-06	.31346D-11
3	.12210D-08	.44409D-15
4	.16327D-10	
5	.23537D-12	
6	.37748D-14	
7	.33307D-15	
8	.22204D-15	

Table 5.7 (μ=0)

s	e^{3x}	$e^{-2.5x}$	$e^{\pm 0.5x}$
1	.26006D-02	.56157D-03	.12195D-08
2	.29398D-05	.31299D-06	.19984D-14
3	.10186D-08	.52928D-10	.66613D-15
4	.14122D-12	.31086D-14	
5	.17764D-14	.44409D-15	
6	.88818D-15		

Table 5.8 ($\mu=0$)

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SOME EVEN-DEGREE SPLINE INTERPOLATION

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Dedicated to Professor D.D.Stancu on his 65th anniversary

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REZUMAT. - **Asupra interpolării cu funcţii spline polinomiale de grad par.** Este studiată interpolarea cu funcţii spline polinomiale de grad $2m$. Funcţia spline interpolează valorile derivatelor pînă la ordinul m pe noduri echidistante. Este demonstrată existenţa, unicitatea şi se dă estimarea erorii. Rezultatele generalizează pe cele din lucrarea [2].

0. Summary. The problem of interpolating a given function by a polynomial spline of degree $2m$ is considered. The spline interpolation is constructed such that it interpolates the derivatives up to the order m at the knots of a uniform partition. This problem was stated and solved in [2] when $m = 1,2,3,4,5$. The case $m = 2$ with some different assumptions was also considered in [3]. We study this problem for an arbitrary positive integer m. Construction, existence, uniqueness, and error bounds are given for the spline interpolation formula. At the same time we make out the conjectures emphasized in [2].

1. Statement of the problem. For any integer $n \geq 1$ let $\Delta_n: a=x_0<x_1<x_2<\ldots<x_n<x_{n+1}=b$ be a uniform partition, i.e. the knots of Δ_n are given by $x_k=a+kh, k=\overline{0,n+1}$, $h=(b-a)/(n+1)$. We denote $S_{2m}(\Delta_n)$ the space of polynomial splines of degree $2m$ with deficiency m in each knot of Δ_n. Therefore $s \in S_{2m}(\Delta_n)$ if and only if $s \in C^m[a,b]$ and its restriction to any subinterval $[x_k,x_{k+1}]$ is

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a polynomial of degree $2m$.

One considers the problem of approximating a function f on $[a,b]$ by a spline function $s_f \in S_{2m}(\Delta_n)$ such that

$$s_f(x_0)=f_0\ ,\ s_f^{(i)}(x_k)=f_k^{(i)}\ ,\ k=\overline{0,n+1},\ i=\overline{1,m},$$

when $f_0 = f(x_0)$, $f_k^{(i)}=f^{(i)}(x_k)$ are given. Certainly, the function f is so that the involved derivatives exist.

2. Two-point Hermite interpolating polynomial. There exists explicit formula for the Hermite interpolating polynomial of degree $p+q+1$ which matches g and its first derivatives at the node α, and also g and its first q derivatives at the node $\beta\neq\alpha$ [4]. Namely, it has the expression

$$H_{p+q+1}(g;x)=\sum_{i=0}^{p} h_{\alpha,i}(x)\,g^{(i)}(\alpha)+\sum_{j=0}^{q} h_{\beta,j}(x)\,g^{(j)}(\beta)\ ,$$

where the fundamental Hermite interpolating polynomials are given by

$$h_{\alpha,i}(x)=\frac{(x-\alpha)^i}{i!}\left(\frac{x-\beta}{\alpha-\beta}\right)^{q+1}\sum_{\nu=0}^{p-i}\binom{q+\nu}{\nu}\left(\frac{x-\alpha}{\beta-\alpha}\right)^{\nu},\ i=\overline{0,p},$$

and

$$h_{\beta,j}(x)=\frac{(x-\beta)^j}{j!}\left(\frac{x-\alpha}{\beta-\alpha}\right)^{p+1}\sum_{\mu=0}^{q-j}\binom{p+\mu}{\mu}\left(\frac{x-\beta}{\alpha-\beta}\right)^{\mu}\ ,\ j=\overline{0,q}.$$

These polynomials satisfy $h_{\alpha,i}^{(k)}(\alpha)=\delta_{ik}$, $i,k=\overline{0,p}$, $h_{\alpha,i}^{(k)}(\beta)=0$ $i=\overline{0,p}$, $k=\overline{0,q}$ and respectively $h_{\beta,j}^{(k)}(\beta)=\delta_{jk}$, $j,k=\overline{0,q}$, $h_{\beta,j}^{(k)}(\alpha)=0$, $j=\overline{0,q}$, $k=\overline{0,p}$.

In the following some particular cases are necessary.

1^0. $\rho=m$, $q=m-1$, $\alpha=x_k$, $\beta=x_{k+1}$. Setting $x=x_k+th$, $0\le t\le 1$ we have

$$h_{x_k,i}(x)=\frac{h^i t^i(1-t)^m}{i!}\sum_{\nu=0}^{m-i}\binom{m+\nu-1}{\nu}t^\nu\ ,\ i=\overline{0,m},$$

and

$$h_{x_{k+1},j}(x)=\frac{h^j(t-1)^j t^{m+1}}{j!}\sum_{\mu=0}^{m-j-1}\binom{m+\mu}{\mu}(1-t)^\mu\ ,\ j=\overline{0,m-1}.$$

2^0. $\rho=m$, $q=m-1$, $\alpha=0$, $\beta=1$. One denotes $A_{m,i}(t)=h_{0,i}(t)$ and $B_{m-1,j}(t)=h_{1,j}(t)$. Then we have

$$A_{m,i}(t)=\frac{t^i(1-t)^m}{i!}\sum_{\nu=0}^{m-i}\binom{m+\nu-1}{\nu}t^\nu\ ,\ i=\overline{0,m}, \tag{1}$$

and

$$B_{m-1,j}(t)=\frac{(t-1)^j t^{m+1}}{j!}\sum_{\mu=0}^{m-j-1}\binom{m+\mu}{\mu}(1-t)^\mu\ ,\ j=\overline{0,m-1}. \tag{2}$$

Taking into account these formulas we obtain that

$$h_{x_k,i}(x)=h^iA_{m,i}(t)\ \text{and}\ h_{x_{k+1},j}(t)=h^jB_{m-1,j}(t)\ . \tag{3}$$

Remarks on $A_{m,i}(t)$ and $B_{m-1,j}(t)$:

1. Writing (1) in the form

$$A_{m,i}(t)=\frac{t^i(1-t)^m}{i!}\sum_{\nu=0}^{m-i}a_m^{(\nu)}t^\nu\ ,\ i=\overline{0,m},$$

where $a_m^{(\nu)}=\binom{m+\nu-1}{\nu}$, it results these coefficients satisfy following recurrence relations

$$\begin{cases} a_m^{(0)} = 1 \ , \\ a_m^{(\nu)} = a_{m-1}^{(\nu)} + a_m^{(\nu-1)} \ , \quad \nu = \overline{1, m-1} \\ a_m^{(m)} = \sum_{\nu=0}^{m-1} a_m^{(\nu)} \ . \end{cases} \tag{4}$$

Recurrence relations (4) were conjectured in [2].

2. We can rewrite (2) in an equivalent form whether the binomial formula is used. Namely, we have successively

$$B_{m-1,j}(t) = \frac{(t-1)^j t^{m+1}}{j!} \sum_{\mu=0}^{m-j-1} \binom{m+\mu}{\mu} \left[\sum_{\nu=0}^{\mu} \binom{\mu}{\nu} (-1)^\nu t^\nu \right] =$$

$$= \frac{(t-1)^j t^{m+1}}{j!} \sum_{\mu=0}^{m-j-1} (-1)^\mu \left[\sum_{\nu=\mu}^{m-j-1} \binom{m+\nu}{\nu} \binom{\nu}{\mu} \right] t^\mu =$$

$$= \frac{(t-1)^j t^{m+1}}{j!} \sum_{\mu=0}^{m-j-1} (-1)^\mu \binom{m+\mu}{\mu} \left[\sum_{\nu=\mu}^{m-j-1} \binom{m+\nu}{m+\mu} \right] t^\mu \ .$$

Finally one obtains

$$B_{m-1,j}(t) = \frac{(t-1)^j t^{m+1}}{j!} \sum_{\mu=0}^{m-j-1} b_{m-1,j}^{(\mu)} t^\mu \ , \tag{5}$$

where

$$b_{m-1,j}^{(\mu)} = (-1)^\mu \binom{m+\mu}{\mu} \binom{2m-j}{m+\mu+1} \ , \quad j = \overline{0, m-1} \ , \quad \mu = \overline{0, m-j-1} \ .$$

It can easily see these coefficients satisfy following recurrence relations:

$$\begin{cases} b_{m-1,0}^{(m-1)} = (-1)^{m-1} a_m^{(m)} \ , \\ b_{m-1,0}^{(\mu-1)} = -\dfrac{\mu(m+\mu+1)}{m^2-\mu^2} b_{m-1,0}^{(\mu)} , \quad \mu = m-1, m-2, \ldots, 1 , \\ b_{m-1,j}^{(\mu)} = \dfrac{m-j-\mu}{2m-j+1} b_{m-1,j-1}^{(\mu)} , \quad j = \overline{1, m-2} , \quad \mu = \overline{0, m-j-1} , \\ b_{m-1,m-1}^{(0)} = 1 \ . \end{cases} \tag{6}$$

Recurrence relations (6) were also conjectured in [2].

3. Evidently, $A_{m,i}^{(k)}(0)=\delta_{ik}$, $i,k=\overline{0,m}$, $A_{m,i}^{(k)}(1)=0$, $i=\overline{0,m}$, $k=\overline{0,m-1}$. Also, it is easily seen that

$$A_{m,i}^{(m)}(1)=\frac{(-1)^m(2m-i)!}{i!\,(m-i)!}\ ,\ i=\overline{0,m}. \tag{7}$$

4. Analogously, $B_{m-1,j}^{(k)}(1)=\delta_{jk}$, $j,k=\overline{0,m-1}$, $B_{m-1,j}^{(k)}(0)=0$, $k=\overline{0,m}$, $j=\overline{0,m-1}$. A similar formula with (7) holds

$$B_{m-1,j}^{(m)}(1)=\frac{(-1)^{m-j-1}(2m-j)!}{j!\,(m-j)!}\ ,\ j=\overline{0,m-1}. \tag{8}$$

This is obtained in the following manner. Using (5) we have

$$\begin{aligned}B_{m-1,j}^{(m)}(1)&=\binom{m}{j}\sum_{\mu=0}^{m-j-1}(-1)^\mu\binom{m+\mu}{\mu}\binom{2m-j}{m+\mu+1}\frac{(m+\mu+1)!}{(\mu+j+1)!}=\\&=\frac{(2m-j)!}{j!\,(m-j)!}\sum_{\mu=0}^{m-j-1}(-1)^\mu\binom{m+\mu}{\mu+j+1}\binom{m-j-1}{\mu}.\end{aligned}$$

On the other hand, identifying the coefficients of t^m in the two developments of $(1-t)^{m-j-1}(1-t)^{j-m}=(1-t)^{-1}$ we have in the right side that is 1 and in the left side is given by

$$(-1)^{m-j-1}\sum_{\mu=0}^{m-j-1}(-1)^\mu\binom{m+\mu}{\mu+j+1}\binom{m-j-1}{\mu}.$$

Hence (8) holds.

3. Existence and uniqueness of spline interpolation. Let $s_k=s_f(x_k)$, $k=\overline{0,n+1}$, be considered as unknown parameters, excepting $s_0=f(x_0)$. On each subinterval $[x_k,x_{k+1}]$ there exists uniquely the Hermite interpolating polynomial of degree $2m$ which matches s_k and the first m derivatives of f at x_k, and s_{k+1} and the first $m-1$ derivatives of f at x_{k+1}. The spline interpolation function s_f on

$[x_k, x_{k+1}]$ is considered to coincide with this polynomial. Then, we have

$$s_f(x) = h_{x_k,0}(x)\, s_k + h_{x_{k+1},0} s_{k+1} + \sum_{i=1}^{m-1} \left[h_{x_k,i}(x)\, f_k^{(i)} + h_{x_{k+1},i}(x)\, f_{k+1}^{(i)} \right] + h_{x_k,m}(x)\, f_k^{(m)},$$

when $x \in [x_k, x_{k+1}]$. Taking into account (3) we have the following equivalent expression

$$s_f(x) = A_{m,0}(t)\, s_k + B_{m-1,0}(t)\, s_{k+1} + \sum_{i=1}^{m-1} h^i \left[A_{m,i}(t)\, f_k^{(i)} + B_{m-1,i}(t)\, f_{k+1}^{(i)} \right] + A_{m,m}(t)\, f_k^{(m)},$$

where $x = x_k + th$, $t \in [0,1]$.

The parameters s_k, $k=\overline{1,n}$, will be determinated by $s_f \in C^m[a,b]$, which is equivalent to

$$s_f^{(m)}(x_k - 0) = s_f^{(m)}(x_k + 0), \quad k = \overline{1,n}.$$

Using (7) and (8) we have

$$s_f^{(m)}(x_k-0) = \frac{1}{h^m} \Big\{ A_{m,0}^{(m)}(1)\, s_{k-1} + B_{m-1,0}^{(m)}(1)\, s_k + \sum_{i=1}^{m-1} h^i \left[A_{m,i}^{(m)}(1)\, f_{k-1}^{(i)} + B_{m-1,i}^{(m)}(1)\, f_k^{(i)} \right] + h^m A_{m,m}^{(m)}(1)\, f_{k-1}^{(m)} \Big\} =$$

$$= \frac{(-1)^m}{h^m} \Big\{ \frac{(2m)!}{m!} (s_{k-1} - s_k) + \sum_{i=1}^{m-1} \frac{h^i (2m-i)!}{i!\,(m-i)!} \left[f_{k-1}^{(i)} - (-1)^i f_k^{(i)} \right] + h^m f_{k-1}^{(m)} \Big\}.$$

On the other hand we have

$$s_f^{(m)}(x_k+0) = \frac{1}{h^m} \Big\{ A_{m,0}^{(m)}(0)\, s_k + B_{m-1,0}^{(m)}(0)\, s_{k+1} + \sum_{i=1}^{m-1} h^i \left[A_{m,i}^{(m)}(0)\, f_k^{(i)} + B_{m-1,i}^{(m)}(0)\, f_{k+1}^{(i)} \right] + h^m A_{m,m}^{(m)}(0)\, f_k^{(m)} \Big\}.$$

Therefore the following recurrence formula holds

$$\frac{(2m)!}{m!}(-s_{k-1}+s_k) = \sum_{i=1}^{m} \frac{h^i(2m-1)!}{i!(m-i)!}\left[f_{k-1}^{(i)}-(-1)^i f_k^{(i)}\right], \tag{9}$$

for $k=\overline{1,n+1}$, with $s_0 = f_0 = f(x_0)$.

The expressions of coefficients in recurrence formula (9) were conjectured in [2].

4. **Error bounds.** It is known the following result established in [1]. If $g \in C^{2m}[0,h]$ and $H_{2m-1}(g)$ is the Hermite interpolating polynomial of degree $2m-1$ that matches g and its first $m-1$ derivatives at 0 and h, then

$$\left|g^{(r)}(x)-H_{2m-1}^{(r)}(g;x)\right| \le \frac{h^r[x(h-x)]^{m-r}}{r!(2m-2r)!}\|g^{(2m)}\|, \tag{10}$$

$x \in [0,h]$, $r = \overline{0,m}$, and $\|\cdot\|$ denotes sup-norm.

Since s_f' on $[x_k,x_{k+1}]$ is the Hermite interpolating polynomial of degree $2m-1$ matching $g = f'$ and its first $m-1$ derivatives $g^{(r)} = f^{(r+1)}$ at x_k and x_{k+1}, based on (10) it results that

$$\left|s_f^{(r+1)}(x)-f^{(r+1)}(x)\right| \le \frac{h^r[(x-x_k)(x_{k+1}-x)]^{m-r}}{r!(2m-2r)!}\|f^{(2m+1)}\| \tag{11}$$

$x\in[x_k,x_{k+1}]$, $r=\overline{0,m}$. As $\max\{(x-x_k)(x_{k+1}-x) \mid x \in [x_k,x_{k+1}]\} = \frac{h^2}{4}$, from (11) it results that

$$\left|s_f^{(r+1)}(x)-f^{(r+1)}(x)\right| \le \frac{h^{2m-r}}{4^{m-r}r!(2m-2r)!}\|f^{(2m+1)}\|, \tag{12}$$

$x \in [x_k,x_{k+1}]$, $r = \overline{0,m}$ and $f \in C^{(2m+1)}[a,b]$.

Also we have that

$$\|s_f^{(r+1)}-f^{(r+1)}\| \le \frac{h^{2m-r}}{4^{m-r}r!(2m-2r)!}\|f^{(2m+1)}\|, \quad r=\overline{0,m}.$$

Integrating (11) over $[a,x]$, in the case $r = 0$, and using $s_f(a) = f(a)$, when $f \in C^{2m+1}[a,b]$ we obtain

$$|s_f(x) - f(x)| \le \frac{(b-a)h^{2m}}{4^m(2m)!}\|f^{(2m+1)}\|, \quad x \in [a,b]$$

and

$$\|s_f - f\| \le \frac{(b-a)h^{2m}}{4^m(2m)!}\|f^{(2m+1)}\|$$

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DIRECT NUMERICAL SPLINE METHODS FOR SECOND ORDER FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

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Dedicated to Professor D.D.Stancu on his 65^{th} anniversary

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REZUMAT. - Metode numerice directe cu funcţii spline pentru ecuaţii integro-diferenţiale Fredholm de ordinul doi. Este propusă o metodă de colocaţie cu funcţii spline polinomiale pentru rezolvarea numerică a ecuaţiilor integro-difernţiale Fredholm de ordinul doi. Se stabileşte legătura cu metodele discrete multipas şi într-un caz special se dă estimarea erorii şi se demonstrează convergenţa.

Abstract. For the numerical solution of second order Fredholm integro-differential equations is proposed a direct collocation method with polynomial spline functions. The connection with the discrete multistep methods is established and for a special case the estimation of error and the convergence of the given procedure are investigated. Some numerical examples illustrate the direct spline methods.

1. **Introduction.** Consider the nonlinear second order Fredholm type integro-differential equation of the form

$$y''(x)=f(x,y(x),y'(x),z(x)),\quad z(x)=\int_0^a K(x,t,y(t))\,dt\ ,\quad 0\le x\le a \qquad (1)$$

$$y(0)=\alpha,\quad y'(0)=\beta$$

where f and K are given functions, $\alpha,\beta\in\mathbb{R}$, and y is the unknown function.

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There is a number of papers in the literature which consider special forms of the equation (1). References [1,3,4] investigate the question of stability for problems without an integral term. Chawla [3] deals with problems explicitly containing first derivative. The general problem (1) has received much less attention. Linz [6] for example, solves a linear problem (1) by converting it to a second kind Fredholm problem. Garey and Gladwin [4] solve a first order Fredholm integro-differential equation of the form (1) by directly applying of a multistep method for the differential equation and a quadrature rule for the integral term. Recently in [5] Garey, Gladwin and Shaw present a general class of k-step three-part methods for numerically solving second order Fredholm integro-differential equations, the most results being given for the linear case. Their methods consist of a k-step method for second order initial value problems, a k -step extrapolation rule for first order derivatives and a quadrature rule is introduced for the integral term. They are investigating the problem of stability also.

In this article we propose a direct spline collocation method for the general nonlinear problem (1). The existence, uniqueness and the convergence of the constructed approximate spline solution are investigated.

2. Direct spline collocation method. Suppose throughout in this paper that the problem (1) has a unique solution $y:[0,a]\to\mathbb{R}$ and it is smooth enough. The function $f:[0,a]\ x\ \mathbb{R}^3\to\mathbb{R}$ is assumed to be sufficiently smooth function and it satisfies a Lipschitz condition with respect to its last three arguments:

L1) $|f(x,u_1,v_1,w_1)-f(x,u_2,v_2,w_2)| < L_1[|u_1-u_2|+|v_1-v_2|+|w_1-w_2|]$

$$\forall\ (x,u_1,v_1,w_1),(x,u_2,v_2,w_2)\in[0,a]\mathrm{x}\mathbb{R}^3$$

Also, we assume that the kernel function $K:[0,a]^2\mathrm{x}\mathbb{R}\rightarrow\mathbb{R}$ is a smooth function and satisfies the following Lipschitz condition:

L2) $|K(x,t,y_1) - K(x,t,y_2)| \le L_2|y_1-y_2|$

$$\forall\ (x,t,y_1),(x,t,y_2)\in[0,a]^2\mathrm{x}\mathbb{R}$$

Let $m > 3$ be an integer number and $n > m$ another given integer number. Following the idea of [7] - [9] we shall construct a polynomial spline function s of degree m and class of continuity $C^{m-1}[0,a],(s\in S_m)$ to approximate the exact solution of (1). Let

$$\Delta:0=x_0<x_1<\ldots<x_k<\ldots<x_n=a \quad x_k=kh \quad h=a/n$$

be a uniform partition of the interval $[0,a]$.

The first component of the spline function s on the interval $[0,h]$ is:

$$s_0(x)=y(0)+y'(0)+\frac{y''(0)}{2!}x^2+\ldots+\frac{y^{(m-1)}(0)}{(m-1)!}x^{m-1}+\frac{a_0}{m!}x^m\ ,\quad 0\le x\le h \qquad (2)$$

where:

$$y(0)=\alpha,\ y'(0)=\beta,\ y''(0)=f\left(0,\alpha,\beta,\int_0^a K(0,t,\alpha)\,dt\right)$$

The other coefficients $y'''(0),\ldots,y^{(m-1)}(0)$ are determining by the differentiations of equation (2). The last coefficient a_0 is to be determined from the following collocation condition:

$$s_0''(h)=f(h,s_0(h),s_0'(h),\int_0^a K(h,t,\alpha)\,dt)$$

Now, if the polynomial (2) is determined, define the spline function s on the next interval $[h,2h]$, by

$$s_1(x) = \sum_{j=0}^{m-1} \frac{s_0^{(j)}(h)}{j!}(x-h)^j + \frac{a_1}{m!}(x-h) \quad , \quad h \le x \le 2h$$

where $s_0^{(j)}(h)$, $0 \le j \le m-1$ are left hand limits of derivates as $x \to h$ of the segment of s defined on $[0,h]$ in (2) and a_1 is to be determined from the following collocation condition:

$$s_1''(2h) = f\left(2h, s_1(2h), s_1'(2h), \int_0^a K(2h, t, s_0(t))\,dt\right)$$

Continuing in this way, on the interval $[x_k, x_{k+1}]$ the spline function, approximating the solution of (1) is defined by:

$$s(x) := \sum_{i=0}^{m-1} \frac{s^{(i)}(x_k)}{i!}(x-x_k)^i + \frac{a_k}{m!}(x-x_k)^m \quad , \quad x_k \le x \le x_{k+1} \tag{3}$$

where $s^{(i)}(x_k)$, $0 \le i \le m-1$ are left hand limits of the derivates as $x \to x_k$ of the segment of s defined on $[x_{k-1}, x_k]$ and the parameter a_k is determined such that:

$$\begin{aligned} &s_k''(x_{k+1}) = f(x_{k+1}, s_k(x_{k+1}), s_k'(x_{k+1}), z_k), \\ &z_k = \int_0^a K(x_{k+1}, t, s_{k-1}(t))\,dt \\ &k = \overline{0, n-1} \ , \ s_k := s|I_k \quad I_k := [x_k, x_{k+1}] \end{aligned} \tag{4}$$

This procedure yields a spline function $s \in S_m$ over the entire interval $[0,a]$, with the knots $\{x_k\}_{k=1}^{n}$, $x_k = kh$.

It is to be shown that the parameter a_k may be uniqluy determined by the collocation condition (4).

THEOREM 1. *If the conditions L_1 and L_2 are satisfied and if h is so small that*

$$\frac{h}{m-1}\left(\frac{h}{m}+1\right)L_1 < 1$$

then the spline function s given by the above construction exist *and is unique.*

Proof. Replacing s given by (3) in (4) we have:

$$a_k = \frac{(m-2)!}{h^{m-2}}\left(f\left(x_{k+1}, A_k(x_{k+1}) + \frac{a_k}{m!}h^m, A_k'(x_{k+1}) + \frac{a_k}{(m-1)!}h^{m-1}, z_k\right) - A_k''(x_{k+1})\right) \tag{5}$$

where

$$A_k(x) := \sum_{i=0}^{m-1} \frac{s^{i}(x_k)}{i!}(x-x_k)^i \ , \quad z_k := \int_0^a K(x_{k+1}, t, s_{k-1}(t))\,dt$$

If we denote the equation (5) for brevity by

$$a_k = G_k(a_k)$$

using the assumption L_1 for

$$\frac{h}{m-1}\left(\frac{h}{m}+1\right)L_1 < 1$$

the function $G_k : \mathbb{R} \to \mathbb{R}$ is a contraction and therefore (6) has a unique solution a_k for each k, which can be found by iterations.

For the sake of definiteness we consider applying this method for a problem (1) in the case of not appearing the derivative y' of the unknown function in the equation.

Henceforth let consider the following second order Fredholm integro-differential problem:

$$y''(x) = f\left(x, y(x), \int_0^a K(x,t,y(t))\,dt\right), \quad 0 \le x \le a \tag{6}$$

$$y(0) = \alpha, \quad y'(0) = \beta$$

Keeping the notation $z(x) = \int_0^a K(x,t,y(t))\,dt$ and the assumptions L_1 and L_2, straightway we have the following theorem:

THEOREM 2. *If the condition L_1 and L_2 hold and if*

$h<(m(m-1)/L_1)^{1/2}$ then the spline function s approximating the solution of (6), given by the above construction exists and is unique.

For the purpose of error estimating we shall need the consistency relations which hold for spline function $s \in S_m$ with equidistant knots $x_k=kh$, $k=0,1,\ldots,m-1$, given by.

THEOREM 3. [7,p 807] *For any spline function $s \in S_m (m\geq 3)$ there exists a linear relation between the quantities $s(kh)$ and $s'(kh)$, $k=0,1,\ldots,m-1$ given by*

$$\sum_{k=0}^{m-1} c_k^{(m)} s(kh) = h^2 \sum_{k=0}^{m-1} b_k^{(m)} s''(kh) \tag{7}$$

with the coefficients

$$\begin{aligned} c_k^{(m)} &:= (m-1)!\,[Q_{m-1}(k+1) - 2Q_{m-1}(k) + Q_{m-1}(k-1)] \\ b_k^{(m)} &:= (m-1)!\,Q_{m+1}(k+1) \end{aligned} \tag{8}$$

where

$$Q_{m+1}(X) := \frac{1}{m!}\sum_{i=0}^{m+1}(-1)^i \binom{m+1}{i}(x-i)_+^m$$

THEOREM 4. *The values $s(kh)$, $k=0,1,\ldots,n$ of the spline function constructed above are precisely the values furnished by the discret multistep method described by the recurrence relation*

$$\sum_{j=0}^{m-1} c_j^{(m)} y_{j+k} = h^2 \sum_{j=0}^{m-1} b_j^{(m)} y_{j+k}, \quad k=0,1,2,\ldots,n \tag{9}$$

where the coefficients $c_j^{(m)}, b_j^{(m)}$ are given by (8), if the starting values

$$y_0=s(0),\ y_1=s(h),\ldots,y_{m-2}=s((m-2)h) \tag{10}$$

are used.

Proof. For $h<(m(m-1)/L_1)^{1/2}$ only one sequence $\{y_j\}$, $j=m-1,\ldots,n$ satisfies relation (9) with the starting values (10). By the consistency relation (7) the sequence $\{s(jh)\}$, $j=m-1,\ldots,n$ satisfies (9) and evidently has starting values (10). Thus the values $s(jh)$ must coincide with the values y_j, $j=m-1,\ldots,n$ generated by the corresponding multistep method. Theorem 4 tells us that the approximate spline solution of degree m yields the same values as the discrete method of $(m-1)$-steps on X_k. In the sequel we shall be concerned with estimating of error of approximation of solution of problem (6) by splines as well as with the covergence of the approximation s to the exact solution y for $h\to 0$, for the degree $m=3$ and $m=4$. We now define the step function $s^{(m)}$ at the knots $x_k=kh$, $k=1,2,\ldots,m-1$ by the usual arithmetic mean:

$$s^{(m)}(x_k):=\frac{1}{2}\left[s^{(m)}\left(x_k-\frac{1}{2}h\right)+s^{(m)}\left(x_k+\frac{1}{2}h\right)\right],\quad k=1,2,\ldots,n-1 \tag{11}$$

Let y be the unique solution of (6) and we denote:

$y_k:=y(x_k)$, $y'_k:=y'(x_k)$, $y''_k:=y''(x_k)$, $z_k:=z(x_k)$ and analogously for s where $x_k=kh$ and $k=0,1,2,\ldots$

LEMMA 1. *If* $|s(x_k)-y(x_k)|<Kh^p$, *where* K *is a constant independent of* h, *and if* $s''(x_k)=f(x_k,s(x_k),\int_0^a K(x_k,t,s(t)dt)$ *then there exists a constant* K_2 *independent of* h *such that*

$$|s(x_k)-y(x_k)|<K_2h^p,\quad |s''(x_k)-y''(x_k)|<K_2h^p$$

The proof is similar with the proof of Lemma 1 of [7,p 809].

LEMMA 2.[7,p 809] *Let* $y\in C^{m+1}[0,a]$ *and let* $s\in S_m$ *be a spline function with the knots at the points* $\{x_k\}$, $k=1,2,\ldots,n-1$ *such* that the conditions:

$$|s^{(r)}(x_k)-y^{(r)}(x_k)|=O\,(h^{p_r}),\quad r=0,1,\ldots,m-1,\ k=0,1,\ldots,n-1 \tag{12}$$

$$|s^{(m)}(x)-y^{(m)}(x)| = O(h),\ x_k < x < x_{k+1},\quad k=0,1,\ldots,n-1 \tag{13}$$

are satisfied. Then

$$|s(x) - y(x)| = O(h^p) \tag{14}$$

where

$$p := \min_{r=0,1,\ldots,m} (r+p_r)\ ,\ (p_m=1) \tag{15}$$

and furthermore

$$s^{(m)}(x) - y^{(m)}(x) = O(h),\ x \in [0,a] \tag{16}$$

3. Cubic spline function approximating the solution. Theorem 3 gives for $m = 3$

$$s_{k+1}-2s_k+s_{k-1}=\frac{h^2}{6}\left(s''_{k+1}+4s''_k+s''_{k-1}\right)\ ,\ k=1,2,\ldots,n-1$$

By Theorem 4 the cubic spline function yields the same values on the knots as the discrete 2 - step method based on the recurrence formula

$$\begin{aligned} y_{k+1}-2y_k+y_{k-1} &= (h^2/6)(y_{k+1}+4y_k+y_{k-1})= \\ &=(h^2/6)\,[f(x_{k+1},y_{k+1},z_{k+1})+4f(x_k,y_k,z_k)+f(x_{k-1},y_{k-1},z_{k-1})] \end{aligned} \tag{17}$$

if starting values $y_0=\alpha$ and $y_1:=s(h)$ are used.

The 2-step method (17) has the degree of exactness two provided the starting values y_0, y_1 have second order accuracy. One can check that $|y(h) - s(h)| = O(h^2)$ and therefore we have

$$|s(x_k) - y(x_k)| = O(h^2),\ k = 1,2,\ldots,n$$

From Lemma 1 it results direct that

$$|s''(x_k) - y''(x_k)| = O(h^2)$$

and using the standard Taylor formula we obtain

$$s'(x_k) - y'(x_k) = O(h^2)$$

In a similar manner as in [7] we can prove that

$$s'''(x) - y'''(x) = O(h^2),\ x_k < x < x_{k+1},$$
$$k = 0,1,\ldots,n-1$$

THEOREM 5. *If $f \in C^2([0,a] \times \mathbb{R}^2)$ and s is the cubic spline function approximating the solution of problem (6), then there exists a constant K such that, for $h < (6/L_1)^{1/2}$ and $x \in [0,a]$*

$$|s(x) - y(x)| < Kh^2,\ |s'(x) - y'(x)| < Kh^2$$
$$|s''(x) - y''(x)| < Kh^2,\ |s'''(x) - y'''(x)| < Kh$$

provided $s'''(x_k)$ is given by (11) with $m = 3$.

Proof. The conditions of Lemma 2 are fulfilled for $m = 3$, $p_0 = p_1 = p_2 = 2$. Note that $f \in C^2([0,a]x\mathbb{R}^2)$ implies $y \in C^4[0,a]$. Applying Lemma 2 three times succesively, first for s and then for s' and s'', the first three inequalities of the theorem follow. The last inequality follows from (16) and thus the theorem is proved.

4. Spline function of fourth degree approximating the solution. If $m = 4$, Theorem 3 gives the following consistency relation for spline function of degree four:

$$s_{k+1}-s_k-s_{k-1}+s_{k-2}=\frac{h^2}{12}\left(s''_{k+1}+11s''_k+11s''_{k-1}+s''_{k-2}\right),\ 2\le k\le n-1$$

According to Theorem 4, the spline function of degree four approximating the solution furnishes values which, on the knots coincide with the recurrence relation:

$$y_{k+1}-y_k-y_{k-1}+y_{k-2}=\frac{h^2}{12}(y''_{k+1}+11y''_k+11y''_{k-1}+y''_{k-2}) =$$
$$=\frac{h_2}{12}[f(x_{k+1},y_{k+1},z_{k+1})+11f(x_k,y_k,z_k)+11f(x_{k-1},y_{k-1},z_{k-1})+ \tag{18}$$
$$+f(x_{k-2},y_{k-2},z_{k-2})]$$

provided that the starting values are $y_0=\alpha$, $y_1=s(h)$, $y_2=s(2h)$.

The 3-step method (18) has degree of exactness four, if the starting values have the same exactness.

If is not difficult to show that

$$|s(h) - y(h)| < Ch^4 \text{ and } |s(2h) - y(2h)| < C_1h^4$$

From the fact that the 3-step method (18) has the degree of exactness four and by Lemma 1 for $p = 4$, if follows that:

$$s(x_k) - y(x_k) = O(h^4), \quad s''(x_k) - y''(x_k) = O(h^4)$$

Similary as in [7] using the standard Taylor technique it is easy to show that

$$s'(x_k) - y'(x_k) = O(h^4), \quad s'''(x_k) - y'''(x_k) = O(h^3)$$

If $x \in [x_k,x_{k+1}]$ by a direct calculation it follows that

$$s(x) - y(x) = O(h), \quad x_k < x < x_{k+1}, \ k = 0,1,\dots,n-1$$

THEOREM 6. *If $f \in C^3([0,a]x\mathbb{R}^2)$ and s is the spline function of the fourth degree approximating the solution y of the problem* (6), *then there exists a constant K, such that, for any $h <$ $< (12/L_1)^{1/2}$ and $x \in [0,a]$*

$$|s^{(j)}(x) - y^{(j)}(x)| < Kh^{4-j}, \quad j = 0,1,2,3$$

$$|s^{(4)}(x) - y^{(4)}(x)| < Kh$$

provided $s^{(4)}(x_k)$ is calculated by (11) *with $m = 4$.*

Proof. The conditions of Lemma 2 are fulfilled for $m = 4$, $p_0 = p_1 = p_2 = 4$, $p_3 = 3$. Obviously from $f \in C^3([0,a]x\mathbb{R}^2)$, it follows that $y \in C^4[0,a]$. Applying Lemma 2 for s, then successively s', s'', s''' in the role of s in Lemma, the theorem follows, with the last relation coming from (16).

The method of approximating the solution of problem (6), by a spline function, given here for $m = 3$ and $m = 4$ has some advantages over the standard known methods for second order Fredholm integro-differential equations, producing smooth, accurate and global approximations to the solution of (6) and its derivatives. The step size h can be changed at any step without additional complications. In addition, this direct spline collocation method need no starting values.

Note that in this paper it was assummed that the values z_k are calculated exactly. In practical applications it should be suggested to choose a suitable quadrature formula.

5. Numerical example.

Example 1. (Garey, Gladwin and Shaw [5]).

$$y''(x) = y(x) + e^{-5} - 1 + \int_0^5 y(t)\,dt \;,\quad x \in [0,5]$$
$$y(0) = 1 \;,\quad y'(0) = -1$$

The exact solution is $y(x) = e^{-x}$.

Example 2. (Garey, Gladwin and Shaw [5]).

$$y''(x) = y(x) + \log\frac{x+e^{-3}}{x+1} + \int_0^3 \frac{y(t)}{x+e^{-t}}\,dt$$
$$y(0) = 1 \;,\quad y'(0) = -1$$

The exact solution is $y(x) = e^{-x}$.

For the both examples are constructed the cubic spline functions to approximate the solutions.

For each example the higher order of spline looses accuracy as x increases. To compute effectivly the values of z_k if was used the Newton-Gregory quadrature formula of order three.

The value of $e_n := y(x_n) - s(x_n)$ are giving in the following table:

Example 1. (h = 1/10)

x_n	y_n	e_n
1,0	0,3679	$-2,4.10^{-7}$
2,0	0,1352	$-4,1.10^{-7}$
3,0	0,0497	$-4,9.10^{-7}$
4,0	0,0182	$-5,2.10^{-7}$
5,0	0,0069	$-5,1.10^{-6}$

Example 2. (h = 1/5)

x_n	y_n	e_n
1,0	0,3678	$-4,68.10^{-6}$
1,5	0,2229	$-5,91.10^{-7}$
2,0	0,1358	$2,78.10^{-7}$
2,5	0,0824	$2,41.10^{-7}$
3,0	0,0511	$3,44.10^{-6}$

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THE CONTINUOUS DEPENDENCE OF SOLUTIONS OF HYPERBOLIC EQUATIONS ON INITIAL DATA AND COEFFICIENTS

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REZUMAT. - Dependenţa continuă a soluţiilor ecuaţiilor hiperbolice de datele iniţiale şi de coeficienţi. Este studiată problema lui Cauchy pentru ecuaţii hiperbolice cu coeficienţi variabili din punctul de vedere al dependenţei continue a soluţiei de datele iniţiale şi de coeficienţi.

Abstract. The Cauchy problem for one-dimensional hyperbolic equation with variable coefficients is considered. The continuous dependence of solutions on initial data and coefficients of the equation is investigated.

1. Introduction. Let us consider the Cauchy problem

$$P_\varepsilon(u_\varepsilon) = 0, \quad x \in R^1, \quad t > 0, \tag{1}$$

$$u_\varepsilon|_{t=0} = f_0(x), \quad u_{\varepsilon t}|_{t=0} = f_1(x), \quad x \in R^1, \tag{2}$$

where $P_\varepsilon = \frac{\partial^2}{\partial t^2} - a(\varepsilon,x,t)\frac{\partial^2}{\partial x^2} - a_0(\varepsilon,x,t)\frac{\partial}{\partial t} - a_1(\varepsilon,x,t)\frac{\partial}{\partial x} - a_2(\varepsilon,x,t)$, a and a_k, $k = 0,1,2$, are smooth functions in x and t and ε is real parameter. We suppose that equation (1) is of hyperbolic type, i.e.,

$$a(\varepsilon,x,t) \geq \gamma_0 > 0, \quad x \in R^1, \quad t \geq 0. \tag{3}$$

If f_0 and f_1 are smooth then there exists a unique solution of this problem, cf.[1].

The aim of this paper is to examine the dependence of solutions on the problem data. Two theorems are proved. The first one concerns the continuous dependence of solutions on initial

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data (2) with respect to Sobolev space norms. The second theorem establish the continuous dependence of solutions on the coeffitients of ecuation (1).

Let $C^k(\Omega)$ be the set of functions having continuous derivatives up to order k in Ω.

$$B_k(\Omega)=\{f\in C^k(\Omega)\mid \|f\|_{B_k(\Omega)}=\sup_{|\alpha|\le k,\, x\in\Omega}|\partial^\alpha f(x)|<\infty\},$$

$$B_0(\Omega)=B(\Omega),\quad B_m=B(R^1),\quad L_p(\Omega)=\{f\mid \|f\|_{L_p(\Omega)}<\infty\},\quad L_p=L_p(R^1),$$

where

$$\|f\|_{L_p(\Omega)}=\begin{cases}\left(\int_\Omega|f(x)|^p dx\right)^{\frac{1}{p}}, & 1\le p<\infty,\\ \operatorname{ess\,sup}_{x\in\Omega}|f(x)|, & p=\infty.\end{cases}$$

Let W_p^s ($s\ge 0$, integer) be the Sobolev space of functions $f \in L_p$, whose derivatives up to order s belong to L_p. Let

$$S=\{f\in C^\infty(R^1)\mid \sup_{x\in R^1}|x^\beta\partial^\alpha f(x)|\le C(\alpha,\beta)<\infty\}$$

be the Schwartz space. Define the direct and inverse Fourier transforms of $f\in S$ as

$$\hat f(\xi)=F_{x\to\xi}[f]=(2\pi)^{-\frac{1}{2}}\int f(x)\,e^{-ix\xi}d\xi,\quad F^{-1}_{\xi\to x}[f]=F_{x\to\xi}[f](-x).$$

Let us denote $\Pi_t=\{(x,\tau)\mid x\in R^1,\ 0<\tau<t\}$, $b=(a,a_0,a_1,a_2)$, $\bar a = a(0,x,t)-a(\varepsilon,x,t)$, $\bar a_k=a_k(0,x,t)-a_k(\varepsilon,x,t)$, $k=0,1,2$, $\bar b=(\bar a,\bar a_0,\bar a_1,\bar a_2)$ For b, $\bar b$ and an integer $\nu\ge 0$, define b_ν and $\bar b_\nu$ as follows:

$$b_\nu=\begin{cases}\sup_t\|a\|_{B(\Pi_t)}, & \nu=0,\\ \sup_t\max\{\|a\|_{B_1(\Pi_t)};\ \|a_k\|_{B(\Pi_t)},\ k=0,1\}, & \nu=1,\\ \sup_t\max\{\|a\|_{B_\nu(\Pi_t)};\ \|a_k\|_{B_{\nu-1}(\Pi_t)},\ k=0,1;\ \|a_2\|_{B_{\nu-2}(\Pi_t)}\}, & \nu\ge 2,\end{cases}$$

$\bar b_\nu$ is defined exactly like b_ν but with a and a_k, $k=0,1,2,$

replaced by $\bar{a}$ and $\bar{a}_k$. We shall say that $b \in B_\nu(\Pi_t)$ if $b_\nu < \infty$.

Let $u_{\varepsilon 1}(x,t)\,(u_{\varepsilon 2}(x,t))$ be the solution of problem (1), (2) with initial data $(f,0)$ ($(0,f)$, respectively). Denote the operator that solves problem (1), (2) with data $(u_{\varepsilon j}(x,0), u_{\varepsilon jt}(x,0))$ by $T_j(\varepsilon,t)$, $j = 1,2$, i.e. $u_{\varepsilon j}(x,t) = (T_j(\varepsilon,t)f)(x)$. Then $u_\varepsilon(x,t) = (T_1(\varepsilon,t)f_0)(x) + (T_2(\varepsilon,t)f_1)(x)$. For $\varepsilon = 0$ we shall write $u_{0j} \equiv u_j$, $u_0 \equiv u$, $T_j(0,t) \equiv T_j(t)$, and $\bar{T}_j(\varepsilon,t) = T_j(t) - T_j(\varepsilon,t)$, $j = 1,2$.

Les us formulate the main results.

THEOREM 1. *Suppose that a satisfies condition* (3), *and* $b \in B_{m+6-j}(\bar{\Pi}_t)$. *Then the estimate*

$$\|T_j(t)f\|_{W_p^m} \le C(t,\gamma_0,b_{m+6-j})\|f\|_{W_p^{\max(0,m-j+1)}}, \quad f \in S, \quad t \ge 0, \tag{4}$$

holds for $1 \le p \le \infty$.

THEOREM 2. *Suppose that a satisfies condition* (3), *and* $b \in B_{m+8}(\bar{\Pi}_t)$. *Then*

$$\|\bar{T}_j(\varepsilon,t)f\|_{W_p^m} \le C(t,\gamma_0,b_{m+8})\bar{b}_{m+8}\|f\|_{W_p^{m-j+2}}, \quad f \in S, \quad t \ge 0, \tag{5}$$

for $1 \le p \le \infty$.

Remark 1. For $1 \le p < \infty$, S is dense in W_p^m, therefore it follows from (4) and (5) that $T_j(t)$ and $\bar{T}_j(\varepsilon,t)$ are bounded as operators from $W_p^{\max(0,m-j+1)}$ into W_p^m and from W_p^{m-j+2} into W_p^m, respectively. If $p = \infty$, S is not dense in B_m, but these operators are still bounded when considered from $B_{\max(0,m-j+1)}$ into B_m and from B_{m-j+2} into B_m. This follows from (4), (5) and from the fact that the speed of propagation of initial data for equation (1) is finite (see Lemma 4, below).

Remark 2. It follows from (5) that $T_2(\varepsilon,t)$, as an operator from W_p^m into W_p^m, depends continuously on the coeffitients of equation (1) and likewise for $T_1(\varepsilon,t)$ as an operator from W_p^{m+1} into W_p^m. We emphasize that $T_1(\varepsilon,t)$ doesn't depend continuously on the coeffitients of equation (1), as an operator from W_p^m into W_p^m, i,e. the estimate

$$\|T_1(\varepsilon,t)f\|_{W_p^m} \le C\bar{\delta}_N \|f\|_{W_p^m}\,, \tag{6}$$

is not, in general, true for any N, even for $p = 2$. In fact, let $j = 1$, $P_\varepsilon = \frac{\partial^2}{\partial t^2} - a^2(\varepsilon)\frac{\partial^2}{\partial x^2}$. Then $(T_1(\varepsilon,t)f)(x) = F^{-1}_{\xi\to x}[m(\varepsilon,\xi,t)\hat{f}(\xi)]$, where $m = \cos(a(0)|\xi|t) - \cos(a(\varepsilon)|\xi|t)$. It is known (cf.[2]), that m is an L_2-Fourier multiplier if and only if $|m| < \infty$, and the norm of $T_1(\varepsilon,t)$ as an operator from L_2 into L_2, is $\sup_{\xi\in R^1}|m(\varepsilon,\xi$
Hence, in order that (6) be true, it is necessary that

$$\sup_{\xi\in R^1}|m(\varepsilon,\xi,t)| \le C(t)\bar{\delta}_N\,.$$

But one can observe that such estimate doesn't hold in this case.

It should be noted that there is a lot of works dealing with estimates of type (4) for hyperbolic equations and systems. Not pretending to an ample analysis of all of then we would to mention here only some works carried in this direction, in which one can find a large bibliografy. The article [3] was one of the first in which the problem on existence of estimates of type (4) for the wave equation was aborded. Using the theory of Fourier multipliers it is shown in [4] that condition $(n-1)|1/p-1/2| \le$ $j-1$ is necessary and sufficient for the existence of estimates of type (4) for the solutions of the wave equation. Here n is the number of space variables. Sufficient conditions for such estimates for solutions of second order hyperbolic equations with variable coeffitients are indicated in [5] for small t. The case of equations with a single space variable and constant coeffitients is trivial. In the case of variable coeffitients we have Haar's inequality (cf.[1]):

$$\|u(\cdot,t)\|_{L_\infty} \le C(t)\,(\|u(\cdot,0)\|_{L_\infty} + \|F\|_{B(\Pi_t)})\ , \qquad (7)$$

where u is a solution of the following strictly hyperbolic system $u_t - A(x,t)u_x - B(x,t)u = F(x,t)$. It should be emphasized that estimate (4) is not implied by (7) because reducing of equation (1) to the indicated system leads to the following estimate for $u_1(x,t): \|u_1(\cdot,t)\|_B \le C\|f\|_{B_1}$, whereas (4) gives $\|u_1(\cdot,t)\|_B \le C\|f\|_B$.

Finally, we note that estimates of the type (4) for equation (1) were obtained in [6] whenever the coefficients of equation (1) satisfy an additional agebraic condition. Thanks to [7], this extra condition can be dropped.

The proof of these theorems is based on the representation of solutions of the problem (1), (2) as sum of Fourier integral operators. Such representations appeared in [8], [9] and are often used when studying hyperbolic type equations (cf.[10]). Another key point in the proof is connected with the estimation of remainder terms in the expansion of integral Fourier operators amplitudes, (Lemma 3). For those remainder terms which satisfy hyperbolic equation with large parameter multiplying inferior derivatives, energetic type estimates (uniform relative to the large parameter) are proved in [7]. The claimed estimates are obtained from the last inequalities.

2. Representation of solutions. We shall represent the solutions $u_{\varepsilon j}(x,t)$, $j=1,2$, as a sum of Fourier integral operators. Let $\psi(\xi) \in C^{\infty}(R^1)$, $0 \le \psi(\xi) \le 1$, $\psi(\xi) = 1$ if $|\xi| \le 1$ and $\psi(\xi) = 0$ if $|\xi| \ge 2$, and $f \in S$. According to [10] we shall seek the solution $u_{\varepsilon j}(x,t)$ in the form:

$$u_{\varepsilon j}(x,t) = (T_{0j}(\varepsilon,t)f)(x) + (T_{1j}(\varepsilon,t)f)(x), \quad j = 1,2, \tag{8}$$

where

$$(T_{0j}(\varepsilon,t)f)(x) = \int \psi(\xi) K_j(\varepsilon,x,t,\xi) \hat{f}(\xi) e^{ix\xi} d\xi, \tag{9}$$

$$(T_{1j}(\varepsilon,t)f)(x) = \sum_{k=1}^{2} \int (1-\psi(\xi)) A_{kj}(\varepsilon,x,t,\xi) \hat{f}(\xi) e^{i\varphi_k(\varepsilon,x,t,\xi)} d\xi. \tag{10}$$

The functions K_j, A_{kj}, φ_k will be defined below. Functions φ_k are homogenouse of degree 1 with respect to ξ, i.e. $\varphi_k(\varepsilon,x,t,\theta\xi) = \theta\varphi_k(\varepsilon,x,t,\xi)$, $\theta > 0$, $\xi \in R^1$. Let φ_k be a solution of the problem

$$\varphi_{kt} = (-1)^k \sqrt{a(\varepsilon,x,t)}\, \varphi_{kx}, \quad x \in R^1, \quad t>0 . \tag{11}$$

$$\varphi_{k|t=0} = x \cdot \xi, \quad k=1,2, \tag{12}$$

and let K_j be a solution of the problem

$$P_\varepsilon(K_j) + P_{1\varepsilon}(K_j) = 0, \quad x \in R^1, \quad t > 0, \tag{13}$$

$$K_j|_{t=0} = \delta_{1j}, \quad K_{jt}|_{t=0} = \delta_{2j}, \tag{14}$$

where $P_{1\varepsilon} = a(\varepsilon,x,t)\xi^2 - i\xi\left(2a(\varepsilon,x,t)\frac{\partial}{\partial x} + a_1(\varepsilon,x,t)\right)$, $\delta_{kj} = \begin{cases} 1, & k=j, \\ 0, & k \ne j \end{cases}$.

Suppose that A_{kj} satisfies the equation

$$P_\varepsilon(A_{kj}) + i|\xi| Q_{\varepsilon k}(A_{kj}) = 0, \quad x \in R^1, \quad k,j = 1,2, \tag{15}$$

and the conditions

$$\sum_{k=1}^{2} A_{kj}(\varepsilon, x, 0, \xi) = \delta_{1,j}, \tag{16}$$

$$\sum_{k=1}^{2}\left(\frac{\partial A_{kj}(\varepsilon, x, 0, \xi)}{\partial t} + iA_{kj}(\varepsilon, x, 0, \xi)\,\frac{\partial \varphi_k(\varepsilon, x, 0, \xi)}{\partial t}\right) = \delta_{2j}, \quad j=1,2, \tag{17}$$

where

$$\begin{aligned} Q_{\varepsilon k} &\equiv q_{0k}(\varepsilon, x, t)\frac{\partial}{\partial t} + q_{1k}(\varepsilon, x, t)\frac{\partial}{\partial x} + q_{2k}(\varepsilon, x, t), \\ q_{0k} &= 2(-1)^k\sqrt{a(\varepsilon, x, t)}\,\varphi_{kx}\left(\varepsilon, x, t, \frac{\xi}{|\xi|}\right), \\ q_{1k} &= (-1)^{k+1}\sqrt{a(\varepsilon, x, t)}\,q_{0t}, \\ q_{2k} &= ((\sqrt{a(\varepsilon, x, t)})_t - \sqrt{a(\varepsilon, x, t)}\,a_0(\varepsilon, x, t) + \\ &\quad + (-1)^{k+1}a_1(\varepsilon, x, t) + \frac{(-1)^k}{2}a_x(\varepsilon, x, t))\,q_{0k}. \end{aligned} \tag{18}$$

Then $u_{\varepsilon j}$ defined by (8) satisfies the Cauchy problem (1), (2) with initial data $(u_{\varepsilon j}, u_{\varepsilon jt})|_{t=0}$. For $|\xi| \geq 1$ we shall represent A_{kj} in the form

$$A_{kj} = \sum_{l=0}^{N} a_{kjl}(\varepsilon, x, t)\,\xi^{-(l+j-1)} + R_{kjN}(\varepsilon, x, t, \xi), \quad k, j = 1, 2. \tag{19}$$

If we substitute (19) in (15), (16), (17) and group the terms with the same degree of homogeneity with respect to ξ then we obtain

$$Q_{\varepsilon k}(a_{kjo}) = 0, \quad x \in R^1, \quad t > 0, \tag{20}$$

$$a_{kjo|t=0} = \begin{cases} \dfrac{1}{2}, & j = 1, \\ \dfrac{(-1)^{k+1} i}{2\sqrt{a(\varepsilon, x, 0)}}, & j = 2, \end{cases} \tag{21}$$

$$Q_{\varepsilon k}(a_{kjl}) = i\frac{\xi}{|\xi|}P_{\varepsilon}(a_{kjl-1}),\ x\in R^1,\ t>0, \tag{22}$$

$$a_{kjl}|_{t=0} = g_{kjl}(\varepsilon,x),\quad l = 1,\dots,N, \tag{23}$$

$$P_{\varepsilon}(R_{kjN})+i|\xi|Q_{\varepsilon k}(R_{kjN})=-\xi^{-(N+j-1)}P_{\varepsilon}(a_{kjN}),\ x\in R^1,\ t>0, \tag{24}$$

$$R_{kjN|t=0}=0,\quad \frac{\partial R_{kjN}}{\partial t}|_{t=0}=g_{kjN+1}(\varepsilon,x)\,\xi^{-(N+j-1)}, \tag{25}$$

where $g_{kjl}=\dfrac{(-1)^k i}{2\sqrt{a(\varepsilon,x,t)}}\sum_{\nu=1}^{2}\dfrac{\partial a_{\nu jl-1}(\varepsilon,x,0)}{\partial t}$, $g_{kjN+1}=-\dfrac{\partial a_{kjN}(\varepsilon,x,0)}{\partial t}$. If a, a_k, $k = 0,1,2$ are smooth enough the obtained problems for φ_k, a_{kjl}, $l = 0,\dots,N$ and R_{kjN} have classic solutions.

3. Estimation of phases and amplitudes. In this part we shall obtain the necessary estimates for the phases φ_k and the functions a_{kjl}, $l = 0,\dots,N$ and R_{kjN}.

Let $\overline{\varphi}_k(\varepsilon,x,t) = \varphi_k(0,x,t,1) - \varphi_k(\varepsilon,x,t,1)$.

LEMMA 1. *Suppose that a satisfies condition* (3), *and* $a \in B_{m+1}(\Pi_t)$. *Then* φ_k *and* $\overline{\varphi}_k$ *satisfy the following conditions:*

$$\varphi_k(\varepsilon,x,t,-1) = -\varphi_k(\varepsilon,x,t,1), \tag{26}$$

$$\varphi_{kx}(\varepsilon,x,r,1) \ge C(t,\gamma_0,\|a\|_{B_1(\Pi_t)})>0,\ (x,r)\in \Pi_t, \tag{27}$$

$$\left\|\frac{\partial^m\varphi_k(\varepsilon,\cdot,\cdot,1)}{\partial x^m}\right\|_{B(\Pi_t)} \le C(t,\gamma_0,\|a\|_{B_m(\Pi_t)})>0,\quad m\ge 1, \tag{28}$$

$$\varphi_k(\varepsilon,x,t,1)\to-\infty \text{ when } x\to-\infty,\ \varphi_k(\varepsilon,x,t,1)\to+\infty \text{ when } x\to+\infty, \tag{29}$$

$$\left\|\frac{\partial^m\overline{\varphi}_k}{\partial x^m}\right\|_{B(\Pi_t)} \le C(t,\gamma_0,\|a\|_{B_{m+1}(\Pi_t)})\,\|\overline{a}\|_{B_m(\Pi_t)}. \tag{30}$$

Proof. The relations (26) - (29) are proved in [6]. It follows from (11) and (12) that $\overline{\varphi}_k$ is the solution of the problem

$$\dot{\overline{\varphi}}_{kt} = (-1)^k \sqrt{a(0,x,t)}\,\overline{\varphi}_{kx} + f_k(\varepsilon,x,t), \quad x\in R^1, \quad t>0, \tag{31}$$

$$\overline{\varphi}_{k|t=0} = 0, \quad k=1,2,$$

where $f_k = (-1)^k(\sqrt{a(0,x,t)} - \sqrt{a(\varepsilon,x,t)})\,\varphi_{kx}(\varepsilon,x,t,1)$. Solving this one we obtain

$$\overline{\varphi}_k(\varepsilon,x,t) = \int_0^t f_k(\varepsilon, x_k(\eta,\tau),\tau)\,|_{\eta=\phi_k(x,t)}\,d\tau \ , \tag{32}$$

where $x_k(\eta,\tau)$ is the solution of Cauchy problem

$$\frac{dx_k}{dt} = (-1)^{k+1}\sqrt{a(0,x,t)}, \quad x_k(0) = \eta \ , \tag{33}$$

and $\eta = \phi_k(x,t)$ is the inverse to $x_k\ (\eta,t)$. It exists because

$$\frac{\partial x_k}{\partial\eta} = \exp\left\{(-1)^{k+1}\int_0^t \frac{a_x(0,x_k(\eta,\tau),\tau)}{2\sqrt{a(0,x_k(\eta,\tau),\tau)}}\,d\tau\right\} \neq 0 \ .$$

We deduce (30) in the case $m = 0$ from (28) with $m = 1$ and (32). If we differentiate the problem (31) with respect to x then using (28) and (32) we likewise deduce (30) for $m \geq 1$. Lemma 1 is proved.

Let $\overline{a}_{kjl}(\varepsilon,x,t) = a_{kjl}(0,x,t) - a_{kjl}(\varepsilon,x,t)$, $l=0,\ldots,N$, $k,j=1,2$.

LEMMA 2. *Suppose that a satisfies condition (3), and* $b \in B_{l+\nu+2}\ (\overline{\Pi}_t)$. *Then the following estimates are true for* a_{kjl} *and* $\overline{a}_{kjl}$, *when* $k,j = 1,2$, $l = 0,\ldots,N$:

$$\left\| \frac{\partial^\nu a_{kjl}}{\partial x^\nu} \right\|_{B(\Pi_t)} \leq C(t,\gamma_0,b_{l+\nu+1}) \ , \tag{34}$$

$$\left\| \frac{\partial^\nu \overline{a}_{kjl}}{\partial x^\nu} \right\|_{B(\Pi_t)} \leq C(t,\gamma_0,b_{l+\nu+2})\,\overline{b}_{l+\nu+1} \ , \quad \nu = 0,1,\ldots \ . \tag{35}$$

Proof. Let $q_{\nu k}$ be defined by (18). According to (27) $q_{0k} \neq$

$\neq 0$, therefore we can denote $\alpha_{\nu k} = q_{\nu k} q_{0k}^{-1}$, $k,\nu = 1,2, Q'_{\varepsilon k} = q_{0k}^{-1} Q_{\varepsilon k}$. Let us consider the Cauchy problem

$$\begin{aligned} Q'_{\varepsilon k}(u) &= f(\varepsilon,x,t), \quad x \in R^1, \; t>0, \\ u|_{t=0} &= u_0(\varepsilon,x), \quad x \in R^1. \end{aligned} \tag{36}$$

If f and u_0 are smooth enough then the solution of (36) has the form

$$\begin{aligned} u(\varepsilon,x,t) = (u_0(\varepsilon,\eta) \exp\left\{-\int_0^t \alpha_{2k}(\varepsilon, x_k(\eta,\tau),\tau)\,d\tau\right\} + \\ + \int_0^t f(\varepsilon, x_k(\eta,\tau),\tau) \cdot \\ \cdot \exp\left\{-\int_\tau^t \alpha_{2k}(\varepsilon, x_k(\eta,\tau_1),\tau_1)\,d\tau_1\right\} d\tau)\,|_{\eta=\phi_k(x,t)}, \end{aligned} \tag{37}$$

where $x_k\ (\eta,t)$ and $\phi_k(x,t)$ are defined by the same way as in the problem (33). Note that the problem (20), (21) is of the form (36) with $f = 0$. Hence using (37) we obtain the estimate (34) in the case $\nu = l = 0$. Differentiating the problem (20), (21) with respect to x and using (34) for $l = 0$, from (37) we obtain the estimate (34) for $l = 0$, $\nu = 1$. Similarly we get (34) for $l = 0$, $\nu > 1$. Now we shall estimate a_{kjl} for $l \geq 1$. Let us observe that if $Q_{\varepsilon k}(u) = h$, then

$$\begin{aligned} P_\varepsilon(u) = h_t - \alpha_{1k}h_x - \alpha_{2k}h + (\alpha_{2k}^2 + \alpha_{2kt} - \alpha_{1k}\alpha_{2kx} - a_0\alpha_{2k} + a_2)\,u + \\ + \left(2\alpha_{1k}\alpha_{2k} - \alpha_{1kt} - \frac{1}{2}\left(\alpha_{1k}^2\right)_x + a_1 - a_0\alpha_{1k}\right) u_x \,. \end{aligned} \tag{38}$$

Hence

$$P_\varepsilon(a_{kjl}) = \sum_{\nu=0}^{l} \tilde{Q}_{\varepsilon k}^{(\nu+1)}(a_{kjl-\nu}) \,, \tag{39}$$

where $\tilde{Q}_{\varepsilon k}^{(l)}$ are differential operators of order l with respect to x, whose coefficients depend on $a, a_0, a_1, a_2, \partial^\alpha a, \partial^\beta a_0, \partial^\beta a_1, \partial^\gamma a_2$, $|\alpha| \leq l+1$, $|\beta| \leq l$, $|\gamma| \leq l-1$. We remark also that (23) and

(25) imply that g_{kjl} are polynomials in a and $\partial^\alpha a$, $|\alpha| \le l, a_0, a_1$ and $\partial^\beta a_0, \partial^\beta a_1$, $|\alpha| \le l - 1$, for $l \ge 1$, $a_2, \partial^\gamma a_2$, $|\gamma| \le l - 2$ for $l \ge 2$. Finally, we observe that the problem (22), (23) has the form (36). Bearing this in mind, from (37) we similary obtain the estimate (34) for a_{kjl}, $l = 1,\dots,N$.

Let us now prove (35). It follows from (20)-(23), (38) and (39) that $\bar{a}_{kjl}$ is the solution of the problem

$$Q_{0k}(\bar{a}_{kjl}) = \bar{f}_{kjl}, \quad x \in R^1, \quad t>0, \tag{40}$$

$$\bar{a}_{kjl}|_{t=0} = \bar{g}_{kjl}, \quad k,j = 1,2, \quad l = 0,\dots,N, \tag{41}$$

where $\bar{g}_{kjl} = g_{kjl}(0,x) - g_{kjl}(\varepsilon,x)$,

$$\bar{f}_{kjl} = \begin{cases} (Q_{\varepsilon k} - Q_{0k})(a_{kjo}), & l = 0, \\ \sum_{\nu=0}^{l-1} \left(\tilde{Q}_{0k}^{(\nu+1)}(\bar{a}_{kjl-\nu-1}) + (\tilde{Q}_{0k}^{(\nu+1)} - \tilde{Q}_{\varepsilon k}^{(\nu+1)})(a_{kjl-\nu-1}) \right), & l \ge 1. \end{cases} \tag{42}$$

The problem (40), (41) has the form (36). Acting like in the case of (34) and using (31), (28), (30), (42) and (37) we get the estimate (35). Lemma 2 is proved.

Let $\bar{R}_{kjN}(\varepsilon,x,t,\xi) = R_{kjN}(0,x,t,\xi) - R_{kjN}(\varepsilon,x,t,\xi)$, $k,j = 1,2$.
Let $\gamma(t) > 0$ and (x_0,t_0) an arbitrary point from R^2.

$D(x_0,t_0,t) = \{(x,r) \,|\, 0 < r < t \le t_0, \ \ \gamma(t)(t_0 - r) > |x - x_0|\}$,

$$\Gamma(x_0,t_0,t) = \{(x,r) \,|\, x \in R^1, \ \ r = t\} \cap D(x_0,t_0,t),$$

$$E(u;t,x_0,t_0) = \int_{\Gamma(x_0,t_0,t)} \left\{ |Q'_{\varepsilon k}(u)|^2 + |u|^2 \right\} dx.$$

LEMMA 3. *Suppose that a satisfies condition (3), and* $b \in B_{N+m+4}(\bar{\Pi}_t)$. *Then there exists* $\gamma = \gamma(t, \gamma_0, \|a\|_{B_2(\Pi_t)})$, *so the following estimates are true for* $|\xi| \ge 1$ *and* $\nu, m = 0,1,\dots$

$$E\left(\frac{\partial^{\nu+m}R_{kjN}}{\partial x^{m}\partial\xi^{\nu}};t,x_0,t_0\right)\le C(t,\gamma_0,b_{N+m+2})\,|\xi|^{-2(N+j-m-1)}, \tag{43}$$

$$E\left(\frac{\partial^{\nu+m}\overline{R}_{kjN}}{\partial x^{m}\partial\xi^{\nu}};t,x_0t_0\right)\le C(t,\gamma_0,b_{N+m+4})\,\widetilde{b}^{2}_{N+m+1}|\xi|^{-2(N+j-m-3)}, \tag{44}$$

Proof. Let $\beta_k = \alpha_{1k}a_0 + 2\alpha_{1k}\alpha_{2k} - \alpha_{1kt} + a_x - a_0 - \alpha_{1k}\alpha_{1kx}$, $k = 1,2$, with $\alpha_{\nu k}$ defined in Lemma 2. It follows from [7, Theorem 4] that if $\beta_k = 0$, $(x,\tau)\in\overline{\Pi}(t)$ then there exists $\gamma = \gamma(t,\gamma_0,b_2)$, so the inequality

$$E(u,t,x_0t_0)\le C(t,\gamma_0,b_3)\,(E(u;0,x_0,t_0)+\|f\|^2_{L_2(D(x_0,t_0,t))}), \tag{45}$$

holds for the solutions of the equation $P_\varepsilon(u) + i|\xi|Q_{\varepsilon k}(u) = f$. Therefore from (45) we obtain (43) for R_{kjN}, when $\nu = m = 0$.

Let us now estimate the derivatives $\dfrac{\partial^m R_{kjN}}{\partial x^m}$. We observe that

$$\frac{\partial}{\partial x}P_\varepsilon(u) = P_\varepsilon^{(1)}\left(\frac{\partial u}{\partial x}\right) - a_{0x}Q'_{\varepsilon k}(u) + (a_{0x}\alpha_{2k}-\alpha_{2x})\,u,$$

$$\frac{\partial}{\partial x}Q_{\varepsilon k}(u) = Q_{\varepsilon k}^{(1)}\left(\frac{\partial u}{\partial x}\right) + q_{0kx}q_{0k}^{-1}Q'_{\varepsilon k}(u) + (q_{2kx}-\alpha_{2k}q_{0kx})\,u,$$

where
$$P_\varepsilon^{(1)} = \frac{\partial^2}{\partial t^2} - a\frac{\partial^2}{\partial x^2} - a_0\frac{\partial}{\partial t} - a_1^{(1)}\frac{\partial}{\partial x} - a_2^{(1)},$$
$$Q_{\varepsilon k}^{(1)} = q_{0k}\frac{\partial}{\partial t} + q_{1k}\frac{\partial}{\partial x} + q_{2k}^{(1)},$$
$$a_1^{(1)} = a_1 + a_x,\quad a_2^{(1)} = a_2 + a_{1x} - \alpha_{1x}a_{0x},$$
$$q_{2k}^{(1)} = q_{2k} + q_{1kx} - q_{0kx}\alpha_{1k},\quad \alpha_{1k}^{(1)} = \alpha_{2k} + (q_{1kx} - q_{0kx}\alpha_{1k})\,q_{0k}^{-1}.$$

Hence $v = \dfrac{\partial R_{kjN}}{\partial x}$ is the solution of the problem

$$P_\varepsilon^{(1)}(v) + i|\xi|Q_{\varepsilon k}^{(1)}(v) = f_1(\varepsilon,x,t,\xi),\quad x\in R^1,\quad t>0, \tag{46}$$

$$v|_{t=0} = 0,\quad v_t|_{t=0} = -\xi^{-(N+j-1)}(g_{kjN+1})_x, \tag{47}$$

where

$$f_1 = a_{0x}Q'_{\varepsilon k}(R_{kjN}) + (a_{0x}\alpha_{2k}-\alpha_{2x})R_{kjN} - \xi^{-N-j+1}P_\varepsilon(a_{kjN})$$
$$- i|\xi|\big(q_{0kx}q_{0k}^{-1}Q'_{\varepsilon k}(R_{kjN}) + (q_{2kx}-\alpha_{2k}q_{0kx})R_{kjN}\big).$$

We note that $\beta_k^{(1)} = 0$ when $\beta_k = 0$, where $\beta_k^{(1)}$ is obtained from β_k substituing a_1 by $a_1^{(1)}$ and α_{2k} by $\alpha_{2k}^{(1)}$. Using (45), (34) and (43) for $\nu = m = 0$ we deduce the estimate (43) when $\nu = 0$, $m = 1$ for the solution of (46), (47). In the same way we obtain the estimate (43) for $\nu = 0$, $m \geq 2$.

Now we shall estimate $w = \dfrac{\partial R_{kjN}}{\partial \xi}$. Note that w is the solution of the problem $P_\varepsilon(w) + i|\xi|Q_{\varepsilon k}(w)$

$$= (N+1-j)\xi^{-N-j}P_\varepsilon(a_{kjN}) - i\,\mathrm{sgn}\,\xi\ Q_{\varepsilon k}(R_{kjN}),\ x \in R^1,\ t > 0, \tag{48}$$

$$w|_{t=0} = 0,\ w_t|_{t=0} = -(N+j-1)\xi^{-N-j}g_{kjN+1}(\varepsilon,x),\quad |\xi| \geq 1. \tag{49}$$

Using (45), (34) and (43) for $\nu = m = 0$ we deduce the estimate (43) when $\nu = 1, m = 0$ for the solution of (48), (49). Similarly we obtain estimate (43) in the remaining cases. Thus estimate (43) is proved.

We shall pass to estimate (44). It follows from (24) and (25) that $\bar{R}_{kjN}$ is the solution of the problem

$$P(\bar{R}_{kjN}) + i|\xi|Q_{0k}(\bar{R}_{kjN}) = F(\varepsilon,x,t,\xi),\quad x \in R^1,\quad t > 0, \tag{50}$$

$$\bar{R}_{kjN|t=0} = 0,\quad \bar{R}_{kjNt|t=0} = \xi^{-N-j+1}\bar{g}_{kjN+1}(\varepsilon,x),\quad |\xi| \geq 1, \tag{51}$$

where $P \equiv P_0$,

$$F = -\xi^{-(N+j-1)}P(\bar{a}_{kjN}) - \xi^{-(N+j-1)}(P-P_\varepsilon)(a_{kjN}) - (P-P_\varepsilon)(R_{kjN}) -$$
$$- i|\xi|(Q_{0k}-Q_{\varepsilon k})(R_{kjN}).$$

The estimates (34), (35), (43) and the relations (38) and (39) yield

$$\|F\|^2_{L_2(D(x_0,t_0,t))} \leq C(t,\gamma_0,b_{N+4})\bar{b}^2_{N+1}|\xi|^{-2(N+j-3)},\quad |\xi| \geq 1. \tag{52}$$

Using (52) and (45) we obtain the estimate (44) for the solution of (50), (51) when $\nu = m = 0$. Differentiating the problem (50), (51) with respect to x and ξ we obtain the estimate (44) for $\nu + m \geq 1$. Lemma 3 is proved.

LEMMA 4. *Suppose that a satisfies condition* (3), *and* $b \in B_{N+m+5}(\overline{\Pi}_t)$. *Then the following estimates are true for* R_{kjN} *and* $\overline{R}_{kjN}$, *when* $|\xi| \geq 1$, $\nu, m = 0,1,\ldots$

$$\left\| \frac{\partial^{\nu+m} R_{kjN}}{\partial x^m \partial \xi^\nu} \right\|_{B(\Pi_t)} \leq C(t, \gamma_0, b_{N+m+3}) \, |\xi|^{-(N+j-m-\frac{3}{2})}, \tag{53}$$

$$\left\| \frac{\partial^{\nu+m} \overline{R}_{kjN}}{\partial x^m \partial \xi^\nu} \right\|_{B(\Pi_t)} \leq C(t, \gamma_0, b_{N+m+5}) \, \overline{b}_{N+m+2} |\xi|^{-(N+j-m-\frac{7}{2})}. \tag{54}$$

Proof. We shall get (54). The proof of the estimate (53) is analogous. Let $\Gamma(x_0, t_0, t)$ and $D(x_0, t_0, t)$ be the same as in Lemma 3. We denote by $\Gamma_\delta(x_0, t_0, t)$ and $D_\delta(x_0, t_0, t)$ the δ-vicinity of the sets $\Gamma(x_0, t_0, t)$ and $D(x_0, t_0, t)$. Let $0 < \delta < 1$, $\eta(x) \in C^\infty(R^1)$, $0 \leq \eta(x) \leq 1$, $\eta(x) = 1$ if $x \in \Gamma(x_0, t_0, 0)$ and $\eta(x) = 0$ if $x \notin \Gamma_\delta(x_0, t_0, t)$.

Let $\eta_1(x,t) \in C^2(R^2)$, $0 \leq \eta_1(x,t) \leq 1$, $\eta_1(x,t) = 1$ if $(x,t) \in D(x_0, t_0, 0)$ and $\eta_1(x,t) = 0$ if $(x,t) \notin D_\delta(x_0, t_0, t)$. From (45) it follows the finitness of propagation speed of initial data. Hence if $z(x,t,\xi)$ is the solution of the problem

$$P(z) + i|\xi| Q_{0k}(z) = \eta_1(x,t) F(\varepsilon, x, t, \xi), \quad x \in R^1, \quad t > 0,$$

$$z|_{t=0} = 0, \quad z_t|_{t=0} = \eta(x) \, \overline{g}_{kjN+1}(\varepsilon, x) \, \xi^{-(N+j-1)},$$

then $z(x_0, t_0, \xi) = \overline{R}_{kjN}(\varepsilon, x_0, t_0, \xi)$. But supp $z(x, t_0, \xi) \subset [x_1, x_2]$, where $x_1 = x_0 - 2\gamma(t_0) t_0 - \delta$, $x_2 = x_0 + 2\gamma(t_0) t_0 + \delta$. Therefore

$$|R_{kjN}(\varepsilon,x,t,\xi)|^2 = |z(x_0,t_0,\xi)|^2 \le 2\int_{x_1}^{x_2} |z(x,t_0,\xi)\, z_x(x,t_0,\xi)|\,dx \le$$
$$\le 2\left(E(z;t_0,x_0,T(t_0))\,E(z_x;t_0,x_0,T(t_0))\right)^{\frac{1}{2}}, \tag{55}$$

Where $T(t_0) = \delta(\gamma(t_0))^{-1} + t_0(2\gamma(t_0)+1)$. It is clear that $mes\ \Gamma_\delta(x_0,t_0,0) \le C(t_0+\delta)$ and $mes\ D_\delta(x_0,T(t_0),t_0) \le C(t_0+\delta)^2$. Using (44) for $\nu = 0$, $m = 0,1$ and (55) we deduce the estimate

$$|\bar{R}_{kjN}| \le C(t_0,\gamma_0,b_{N+5})\,\bar{b}_{N+2}|\xi|^{-(N+j-\frac{7}{2})}, \quad |\xi| \ge 1.$$

So (54) is proved for $\nu = m = 0$ because (x_0,t_0) is an arbitrary point and the rigth hand of the last inequality doesn't depend on x_0. Estimates (54) for the remaining ν and m can be obtained differentiating the problem (50), (51) with respect to x and ξ. Lemma 4 is proved.

We denote $\bar{K}_j(\varepsilon,x,t,\xi) = K_j(0,x,t,\xi) - K_j(\varepsilon,x,t,\xi)$.

LEMMA 5. *Suppose that a satisfies condition (3), and* $b \in B_{m+2}(\bar{\Pi}_t)$. *Then the following estimates for* K_j *and* $\bar{K}_j$ *are true*

$$\left\| \frac{\partial^{\nu+m} K_j}{\partial x^m \partial \xi^\nu} \right\|_{B(\Pi_t)} \le C(t,\gamma_0,b_{m+1}), \quad |\xi| \le 2, \tag{56}$$

$$\left\| \frac{\partial^{\nu+m} \bar{K}_j}{\partial x^m \partial \xi^\nu} \right\|_{B(\Pi_t)} \le C(t,\gamma_0,b_{m+2})\,\bar{b}_m, \quad |\xi| \le 2, \quad \nu,m = 0,1,\ldots. \tag{57}$$

Proof. We denote

$$D_0(x_0,t_0,t) = \{(x,r)\,|\,0 < r < t \le t_0,\ \sqrt{b_0}(t_0-r) > |x-x_0|\},$$
$$\Gamma_0(x_0,t_0,t) = \{(x,r)\,|\,x \in R^1,\ r = t\} \cap D_0(x_0,t_0,t),$$

$$E_1(u;t,x_0,t_0) = \int_{\Gamma_0(x_0,t_0,t)} \{|u_t|^2 + a|u_x|^2 + |u|^2\}\,dx.$$

The inequality

$$E_1(u;t,x_0,t_0)\le C(t,\gamma_0,b_1)\,(E_1(u;0,x_0,t_0)+\|f\|^2_{L_2(D_0(x_0,t_0,t))}),\quad |\xi|\le 2,\tag{58}$$

is proved for the solutions of the equation

$$P_\varepsilon(u)\ +\ P_{1\varepsilon}(u)\ =\ f(\varepsilon,x,t,\xi),\ x\in R^1,\ t>0,\tag{59}$$

in [6]. The equation (13) has the form (59) with $f = 0$. The ecuations for $\dfrac{\partial^{\nu+m}K_j}{\partial x^m\partial\xi^\nu}$ have the same form with the free term depending on K_j and its derivatives of order less than $\nu + m$ with respect to ξ and x. The inequality (58) yields

$$E_1\left(\frac{\partial^{\nu+m}K_j}{\partial x^m\partial\xi^\nu};t,x_0,t_0\right)\le C(t,\gamma_0,b_{m+1}),\quad |\xi|\le 2,\quad \nu,m=0,1,\ldots\ .\tag{60}$$

The equation for $\bar{K}_j$ is also of the form (59) with $f = (P_\varepsilon - P)(K_j) + (P_{1\varepsilon} - P_{10})(K_j)$ and with trivial initial data. Therefore (58) and (60) lead us in this case to the estimate

$$E_1\left(\frac{\partial^{\nu+m}\bar{K}_j}{\partial x^m\partial\xi^\nu};t,x_0,t_0\right)\le C(t,\gamma_0,b_{m+2})\,\bar{b}^2_m,\ |\xi|\le 2,\ \nu,m=0,1,\ldots\ .\tag{61}$$

From (60) and (61) as in Lemma 4 we obtain estimates (56) and (57). Lemma 5 is proved.

Let $\Theta_k(\varepsilon,x,t,\xi)=1-e^{-i\bar{\varphi}_k(\varepsilon,x,t)\xi}$, $T_l(y)=F^{-1}_{\xi\to y}[(1-\psi(\xi))\xi^{-l-1}]$, $T_{kl}(\varepsilon,x,t,y)=F^{-1}_{\xi\to y}[(1-\psi(\xi))\xi^{-l-1}\Theta_k(\varepsilon,x,t,\xi)]$, $k=1,2$, $l=0,1,\ldots$, $h(y)=(1+y^2)^{-1}$, $\delta=\sup_{(x,\tau)\in\Pi_{t,\varepsilon}}|\bar{\varphi}_k(\varepsilon,x,\tau)|$, $h_0(y)=\begin{cases}\bar{b}_0 h(y), & |y|\ge\delta,\\ 1, & |y|<\delta\end{cases}$.

LEMMA 6. *For T_l the estimate*

$$|T_l(y)|\ \le\ C\cdot h(y),\ l=0,1,\ldots\tag{62}$$

are true.

If $a\in B_1(\bar{\Pi}_t)$, then

$$|T_{kl}|\le C(t,\gamma_0,\|a\|_{B_1(\Pi_t)})\cdot h_0(y),\quad l=0,1,\ldots\tag{63}$$

Proof. We shall consider the case $l = 0$ which is more difficult. The proof in other cases is analogous.

Because $\psi(\xi)$ is even $T_0(y) = i\int_{|\xi|\ge 1}(1-\psi(\xi))\xi^{-1}\sin(\xi y)\,d\xi$. Hence

$$|T_0(y)| \le C, \quad |y| \le 1. \tag{64}$$

If $|y| \ge 1$ then $T_0(y) = -y^{-2}\int_{|\xi|\ge 1}\left(\frac{1-\psi(\xi)}{\xi}\right)'' e^{i\xi y}d\xi$. Therefore

$$|T_0(y)| \le C\cdot y^{-2}, \quad |y| \ge 1. \tag{65}$$

The estimate (62) follows from (64) and (65).

Let us prove (63). If $|y| \ge 1$ then

$$T_{k0}(y) = -y^{-2}\int_{|\xi|\ge 1}\frac{\partial^2}{\partial\xi^2}\left(\frac{1-\psi(\xi)}{\xi}\Theta_k(\varepsilon,x,t,\xi)\right)e^{i\xi y}d\xi = -y^{-2}\sum_{\nu=0}^{2}C_2^{\nu}I_{k\nu}, \tag{66}$$

where $I_{k\nu}(y) = \int_{|\xi|\ge 1}\frac{\partial^{2-\nu}}{\partial\xi^{2-\nu}}\left(\frac{1-\psi(\xi)}{\xi}\right)\frac{\partial^{\nu}}{\partial\xi^{\nu}}\Theta_k(\varepsilon,x,t,\xi)\,e^{i\xi y}d\xi$, $\nu=0,1,2$. We observe that

$$\Theta_k = 2i\sin(\overline{\varphi}_k\cdot\frac{\xi}{2})\,e^{-i\overline{\varphi}_k\cdot\frac{\xi}{2}}, \tag{67}$$

and

$$|2\xi^{-1}\sin(\overline{\varphi}_k\cdot\frac{\xi}{2})| \le |\overline{\varphi}_k|. \tag{68}$$

Then (67), (68) and (30) yield

$$|I_{k0}| \le C\cdot|\overline{\varphi}_k| \le C(t,\gamma_0,\|a\|_{B_1(\Pi_t)})\cdot\overline{b}_0. \tag{69}$$

Because $\left|\frac{\partial}{\partial\xi}\Theta_k\right| \le |\overline{\varphi}_k|$ we have

$$|I_{k1}| \le C\cdot|\overline{\varphi}_k| \le C(t,\gamma_0,\|a\|_{B_1(\Pi_t)})\cdot\overline{b}_0\ . \tag{70}$$

We observe that $I_{k2} = \overline{\varphi}_k^2 T_0(y-\overline{\varphi}_k)$. Hence

$$|I_{k2}| \le C(t,\gamma_0,\|a\|_{B_1(\Pi_t)})\cdot\overline{b}_0\ . \tag{71}$$

Substituing (69)-(71) in (66) we deduce

$$|T_{k0}| \le C(t,\gamma_0,\|a\|_{B_1(\Pi_t)})\cdot \overline{b}_0 y^{-2}, \quad |y|\ge 1. \tag{72}$$

By the other hand T_{k0} may be written in the form $T_{k0} = I_{k3} + I_{k4}$, where

$$I_{k3} = -\int \psi(\xi)\xi^{-1}\Theta_k(\varepsilon,x,t,\xi)e^{i\xi y}d\xi$$
$$I_{k4} = \int \xi^{-1}\Theta_k(\varepsilon,x,t,\xi)e^{i\xi y}d\xi$$

Using (67) and (68) we obtain the estimate for I_{k3}:

$$|I_{k3}| \le C\cdot|\overline{\varphi}_k| \le C(t,\gamma_0,\|a\|_{B_1(\Pi_t)})\cdot \overline{b}_0, \quad y\in R^1. \tag{73}$$

For I_{k4} we have

$$I_{k4} = \int \xi^{-1}(\sin(\xi y) - \sin\xi(y-\overline{\varphi}_k))d\xi.$$

Because $\int \xi^{-1}\sin(\xi\alpha)d\xi = \pi\cdot sgn\alpha$, the last yields
$I_{k4} = \pi(sgn y - sgn(y-\overline{\varphi}_k))$.
Therefore

$$|I_{k4}| \le \begin{cases} 0, & \text{if } y(y-\overline{\varphi}_k) > 0, \\ 2\pi, & \text{if } y(y-\overline{\varphi}_k) \le 0, \end{cases}$$

that leads to the estimate

$$|I_{k4}| \le C\cdot h_0(y), \quad |y| \le 1. \tag{74}$$

Using (73) and (74) we obtain for T_{k0}

$$|T_{k0}| \le C(t,\gamma_0,\|a\|_{B_1(\Pi_t)})h_0(y), \quad |y|\le 1. \tag{75}$$

The estimate (63) follows from (72) and (75). Lemma 6 is proved.

We denote

$$T_{Nj1m}(\varepsilon,x,t,y) = F^{-1}_{\xi\to y}\left[(1-\psi(\xi))\xi^{-1}\frac{\partial^m R_{kjN}(\varepsilon,x,t,\xi)}{\partial x^m}\right],$$

$$\Phi_{Nj1m}(\varepsilon,x,t,y) = F^{-1}_{\xi\to y}\left[(1-\psi(\xi))\xi^{-1}\frac{\partial^m \overline{R}_{kjN}(\varepsilon,x,t,\xi)}{\partial x^m}\right],$$

$$\Psi_{Njlm}(\varepsilon,x,t,y)=F^{-1}_{\xi\to y}\left[(1-\psi(\xi))\xi^{-l}\Theta_k(\varepsilon,x,t,\xi)\frac{\partial^m R_{kjN}(\varepsilon,x,t,\xi)}{\partial x^m}\right].$$

LEMMA 7. *Suppose that a satisfies condition (3), and* $b\in B_{N+m+5}(\overline{\Pi}_t)$. *Then the following estimates are true*

$$|T_{Njlm}|\le C(t,\gamma_0,b_{N+m+3})h(y),\quad \text{if } l+N+j-m-5/2>0, \tag{76}$$

$$|\Phi_{Njlm}|\le C(t,\gamma_0,b_{N+m+5})\,\overline{b}_{N+m+2}h(y),\quad \text{if } l+N+j-m-9/2>0, \tag{77}$$

$$|\Psi_{Njlm}|\le C(t,\gamma_0,b_{N+m+3})\,\overline{b}_0h(y),\ \text{if } l+N+j-m-7/2>0, \tag{78}$$

Proof. Using (54) for $|y|\le 1$ we have

$$\begin{aligned}|\Phi_{Njlm}|&\le C(t,\gamma_0,b_{N+m+5})\,\overline{b}_{N+m+2}\int_1^\infty \xi^{m-l-N-j+\frac{7}{2}}d\xi\le\\ &\le C(t,\gamma_0,b_{N+m+5})\,\overline{b}_{N+m+2},\quad l+N+j-m-\frac{9}{2}>0.\end{aligned}\tag{79}$$

Let $|y|\ge 1$. Integrating twice by parts and using (54) we obtain

$$|\Phi_{Njlm}|\le C(t,\gamma_0,b_{N+m+5})\,\overline{b}_{N+m+2}y^{-2},\ l+N+j-m-9/2>0. \tag{80}$$

The inequalities (79) and (80) emply (77).

Let us prove (78). From (67), (68), (26) and (53) it follows

$$\begin{aligned}|\Psi_{Njlm}|&\le|\overline{\varphi}_k|\,C(t,\gamma_0,b_{N+m+3})\int_1^\infty \xi^{m-l-N-j+\frac{5}{2}}d\xi\le\\ &\le C(t,\gamma_0,b_{N+m+3})\,\overline{b}_0,\quad |y|\le 1,\quad l+N+j-m-\frac{7}{2}>0.\end{aligned}\tag{81}$$

Integrating twice by parts and using (53), (68) and the estimate $\left|\frac{\partial^\nu}{\partial\xi^\nu}\Theta_k\right|\le C(t,\gamma_0,\|a\|_{B_1(\Pi_t)})\,\overline{b}_0,\ \nu=1,2,\ldots$ we obtain

$$|\Psi_{Njlm}|\le C(t,\gamma_0,b_{N+m+3})\,\overline{b}_0y^{-2},\quad |y|\ge 1,\quad l+N+j-m-7/2>0. \tag{82}$$

The inequalities (81) and (82) emply (78). The estimate (76) can be proved analogously. Lemma 7 is proved.

We denote

$$\mathcal{K}_{jml}(\varepsilon,x,t,y)=F^{-1}_{\xi\to y}\left[\xi^l\psi(\xi)\frac{\partial^m K_j(\varepsilon,x,t,\xi)}{\partial x^m}\right],$$

$$\overline{\mathcal{K}}_{jml}(\varepsilon,x,t,y)=F^{-1}_{\xi\to y}\left[\xi^l\psi(\xi)\frac{\partial^m \overline{K}_j(\varepsilon,x,t,\xi)}{\partial x^m}\right].$$

LEMMA 8. *Suppose that a satisfies condition (3), and $b \in B_{m+2}(\bar{\Pi}_t)$. Then the following estimates are true*

$$|K_{jml}| \le C(t,\gamma_0,b_{m+1})h(y), \tag{83}$$

$$\bar{K}_{jml} \le C(t,\gamma_0,b_{m+2})\bar{b}_m h(y), \quad l,m=0,1,\ldots \tag{84}$$

Proof. Integrating twice by parts and using (57) we obtain

$$|\bar{K}_{jml}| \le C(t,\gamma_0,b_{m+2})\bar{b}_m y^{-2}, \quad |y|\ge 1, \quad l,m=0,1,\ldots \tag{85}$$

For $|y| \le 1$ the estimate

$$|\bar{K}_{jml}| \le C(t,\gamma_0,b_{m+2})\bar{b}_m, \quad l,m=0,1,\ldots \tag{86}$$

is evident. The inequalities (85) and (86) emply (84). The estimate (83) can be proved analogously. Lemma 8 is proved.

4. The proof of the theorems.

Proof of the Theorem 1. Using the Young inequality we obtain from (9) and (83)

$$\left\| \frac{\partial^m}{\partial x^m}(T_{0j}(t)f)(\cdot)\right\|_{L_p} \le C(t,\gamma_0,b_{m+1})\|f\|_{L_p}, \quad 1\le p\le\infty. \tag{87}$$

Let

$$(T_{1kjl}(t)f)(x) = F^{-1}_{\xi\to y}\left[(1-\psi(\xi))\xi^{-l-j+1}a_{kjl}(0,x,t)\hat{f}(\xi)\right](\varphi_k(0,x,t,1)),$$

where $l = 0,\ldots,N$,

$$(T_{1kjN+1}(t)f)(x) = F^{-1}_{\xi\to y}\left[(1-\psi(\xi))R_{kjN}(0,x,t,\xi)\hat{f}(\xi)\right](\varphi_k(0,x,t,1)).$$

Let $j + l - 1 = 1,\ldots,N$. We shall use the estimates (28), (34), (62) and the Young inequality, then we do the non-degenerate (in virtue of (27)) change of variables $s = \varphi_k(0,x,t,1)$. Now using (26) and (29) we get

$$\left\| \frac{\partial^m}{\partial x^m} (T_{1kjl}(t) f)(\cdot) \right\|_{L_p} \leq C(t, \gamma_0, b_{m+l+1} \|f\|_{W_p^{\max(0,m-l-j+2)}}, \quad 1 \leq p \leq \infty. \tag{88}$$

If $l + j = 1$, i.e. $j = 1$, $l = 0$, we have

$$\|F^{-1}_{\xi \to y}[(1-\psi(\xi))\xi^\nu \hat{f}(\xi)](\varphi_k(0,\cdot,t,1))\|_{L_p} \leq \|F^{-1}_{\xi \to y}[\xi^\nu \hat{f}(\xi)](\varphi_k(0,\cdot,t,1))\|_{L_p}$$

$$+ \|F^{-1}_{\xi \to y}.[\psi(\xi)\xi^\nu \hat{f}(\xi)](\varphi_k(0,\cdot,t,1))\|_{L_p} \leq C(\|f\|_{W_p^\nu} + \|f\|_{L_p}).$$

Hence, using (28) and (34) we obtain

$$\left\| \frac{\partial^m}{\partial x^m} (T_{1k10}(t) f)(\cdot) \right\|_{L_p} \leq C(t, \gamma_0, b_{m+1}) \|f\|_{W_p^m}, \quad 1 \leq p \leq \infty. \tag{89}$$

From (88) and (89) it follows for $l = 0,1,\ldots,N$

$$\left\| \frac{\partial^m}{\partial x^m} (T_{1kjl}(t) f)(\cdot) \right\|_{L_p} \leq C(t, \gamma_0, b_{m+N+1}) \|f\|_{W_p^{\max(0,m-j+1)}}, \quad 1 \leq p \leq \infty. \tag{90}$$

Further we observe that

$$\frac{\partial^m}{\partial x^m}\left(R_{kjN}(0,x,t,\xi)\, e^{\varphi_k(0,x,t,\xi)}\right) = \sum_{\nu=0}^{m} \sum_{s=0}^{m-\nu} g_{ks}(x,t)\, \xi^s \frac{\partial^\nu R_{kjN}(0,x,t,\xi)}{\partial x^\nu}$$

where g_{ks} depends on $\dfrac{\partial^\nu \varphi_k(0,x,t,1)}{\partial x^\nu}$, $\quad \nu = 1,\ldots,m-\nu$. Therefore from (28) and (76) it follows

$$\left\| \frac{\partial^m}{\partial x^m} (T_{1kjN+1}(t) f)(\cdot) \right\|_{L_p} \leq C(t, \gamma_0, b_{N+m+3}) \|f\|_{W_p^r}, \quad 1 \leq p \leq \infty,$$

where $r = \max(0, \min(n \geq 0, n\text{-integer} \mid n+N+j-m-5/2>0))$. If $N = 3 - j$, then $r = \max(0, m-j+1)$, hence

$$\left\| \frac{\partial^m}{\partial x^m} (T_{1kj2-j}(t) f)(\cdot) \right\|_{L_p} \leq C(t, \gamma_0, b_{m+6-j} \|f\|_{W_p^{\max(0,m-j+1)}}, \quad 1 \leq p \leq \infty. \tag{91}$$

From (8)-(10), using (87), (89) and (91) we obtain (4). Theorem

1 is proved.

Proof of the Theorem 2. Using the Young inequality and (84) we have for $\overline{T}_{0j}(\varepsilon,t) = T_{0j}(0,t) - T_{0j}(\varepsilon,t)$

$$\left\|\frac{\partial^m}{\partial x^m}(\overline{T}_{0j}(\varepsilon,t)f)(\cdot)\right\|_{L_p} \le C\cdot\overline{b}_m\|h*f\|_{L_p} \le C\cdot\overline{b}_m\|h\|_{L_1}\|f\|_{L_p} \le \\ \le C(t,\gamma_0,b_{m+2})\overline{b}_m\|f\|_{L_p},\ m=0,1,\ldots,\ 1\le p\le\infty. \tag{92}$$

In virtue of (15) we shall write $\overline{T}_{1j}(\varepsilon,t) = T_{1j}(0,t) - T_{1j}(\varepsilon,t)$ in the form

$$\overline{T}_{1j}(\varepsilon,t) = \sum_{k=1}^{2}\left(\sum_{l=0}^{N}(G_{kjl}(\varepsilon,t)+H_{kjl}(\varepsilon,t)) + S_{kjN}(\varepsilon,t)+V_{kjN}(\varepsilon,t)\right), \tag{93}$$

where

$$(G_{kjl}(\varepsilon,t)f)(x) = \int(1-\psi(\xi))\overline{a}_{kjl}(\varepsilon,x,t)\xi^{-l-j+1}\hat{f}(\xi)e^{i\varphi_k(0,x,t,\xi)}d\xi,$$

$$(H_{kjl}(\varepsilon,t)f)(x) = \\ = \int(1-\psi(\xi))a_{kjl}(\varepsilon,x,t)\Theta_k(\varepsilon,x,t,\xi)\xi^{-l-j+1}\hat{f}(\xi)e^{i\varphi_k(0,x,t,\xi)}d\xi,$$

$$(S_{kjN}(\varepsilon,t)f)(x) = \int(1-\psi(\xi))\overline{R}_{kjN}(\varepsilon,x,t,\xi)\hat{f}(\xi)e^{i\varphi_k(0,x,t,\xi)}d\xi,$$

$$(V_{kjN}(\varepsilon,t)f)(x) = \\ = \int(1-\psi(\xi))R_{kjN}(\varepsilon,x,t,\xi)\Theta_k(\varepsilon,x,t,\xi)\hat{f}(\xi)e^{i\varphi_k(0,x,t,\xi)}d\xi.$$

We observe that

$$\frac{\partial^m}{\partial x^m}\left(\overline{a}_{kjl}e^{i\varphi_k(0,x,t,\xi)}\right) = \sum_{\nu=0}^{m} g_{kjl\nu}(x,t)\xi^\nu e^{i\varphi_k(0,x,t,\xi)}$$

where $g_{kjl\nu}$ depends polynomially on $\dfrac{\partial^\nu\varphi_k(0,x,t,1)}{\partial x^\nu}$, $\nu = 1,\ldots,m$, and $\dfrac{\partial^\nu\overline{a}_{kjl}}{\partial x^\nu}$, $|\nu| \le m$. Dealing similary as when proving (88) by means of (28), (35) and (62) we obtain

$$\left\|\frac{\partial^m}{\partial x^m}(G_{kjl}(\varepsilon,t)f)(\cdot)\right\|_{L_p} \le \\ \le \sum_{\nu=0}^{m}\|g_{kjl}\|_{B(\Pi_t)}\left\|F^{-1}_{\xi\to y}[(1-\psi(\xi))\xi^{-l-j+1+\nu}\hat{f}(\xi)](\varphi_k(0,\cdot,t,1))\right\|_{L_p} \le$$

$$\leq C(t,\gamma_0,b_{m+l+2})\,\bar{b}_{m+l+1}\left(\sum_{\nu=0}^{m+2+l-j}\|(\Gamma_l * f)(\varphi_k(0,\cdot,t,1))\|_{L_p}+\right. \tag{94}$$

$$\left.+\sum_{\nu=m+1+l-j}^{m}\left\|\left(\Gamma_{kl} * F^{-1}_{\xi\to y}[\xi^{-l-j+2+\nu}\hat{f}]\right)(\varphi_k(0,\cdot,t,1))\right\|_{L_p}\right)\leq$$

$$\leq C(t,\gamma_0,b_{m+l+2})\,\bar{b}_{m+l+1}\|f\|_{W_p^{\max(0,m+2-j-l)}}\ ,\ \ 1\leq p\leq\infty.$$

Let us estimate $H_{kjl}(\varepsilon,t)$. We observe that

$$\frac{\partial^m}{\partial x^m}\left(a_{kjl}\Theta_k e^{i\varphi_k(0,x,t,\xi)}\right)=\left(\sum_{\nu=0}^{m} g_{kjl\nu}(x,t)\Theta_k\xi^\nu+\sum_{\nu=1}^{m}\xi^\nu g'_{kjl\nu}(x,t)\right)e^{i\varphi_k(0,x,t,\xi)},$$

where $g_{kjl\nu}$ depends polynomially on $\dfrac{\partial^\nu\varphi_k(0,x,t,1)}{\partial x^\nu}$, $\nu=1,\dots,m$, and $\dfrac{\partial^\nu a_{kjl}}{\partial x^\nu}$, $|\nu|\leq m$; and $g'_{kjl\nu}$ depends on the same terms and additionally on $\dfrac{\partial^\nu\bar{\varphi}_k}{\partial x^\nu}$, $\nu=1,\dots,m$.

Dealing similary as above and using (28), (30), (34), (62) and (63) we obtain

$$\left\|\frac{\partial^m}{\partial x^m}(H_{kjl}(\varepsilon,t)f)(\cdot)\right\|_{L_p}\leq$$

$$\leq\sum_{\nu=0}^{m}\|g_{kjl\nu}\|_{B(\Pi_t)}\left\|F^{-1}_{\xi\to y}[(1-\psi(\xi))\xi^{-l-j+1+\nu}\hat{f}(\xi)](\varphi_k(0,\cdot,t,1))\right\|_{L_p}+ \tag{95}$$

$$+\sum_{\nu=1}^{m}\|g'_{kjl\nu}\|_{B(\Pi_t)}\left\|F^{-1}_{\xi\to y}[(1-\psi(\xi))\xi^{-l-j+1+\nu}\Theta_k(\varepsilon,\cdot,t,\xi)\hat{f}(\xi)](\varphi_k(0,\cdot,t,1))\right\|_{L_p}\leq$$

$$\leq C(t,\gamma_0,b_{m+l+1})\,\bar{b}_m\|f\|_{W_p^{\max(0,m+2-j-l)}}\ ,\ \ 1\leq p\leq\infty.$$

Let us estimate $S_{kjN}(\varepsilon,t)$. We observe that

$$\frac{\partial^m}{\partial x^m}\left(\bar{R}_{kjN}\cdot e^{i\varphi_k(0,x,t,\xi)}\right)=\sum_{\nu=0}^{m} g_{kjN\nu}(x,t,\xi)\,\frac{\partial^\nu}{\partial x^\nu}\bar{R}_{kjN}\cdot e^{i\varphi_k(0,x,t,\xi)}$$

where $g_{kjN\nu}$ is a polynomial of order $m-\nu$ in ξ, and whose coefficients depend on $\dfrac{\partial^\nu\varphi_k(0,x,t,1)}{\partial x^\nu}$, $\nu=1,\dots,m-\nu$. Dealing similary as when proving estimation (94), by means of (28) and (77) we obtain

$$\left\| \frac{\partial^m}{\partial x^m}(S_{kjN}f)(\cdot)\right\|_{L_p} \le \tag{96}$$

$$\le C(t,\gamma_0,b_m)\sum_{s=0}^{m}\sum_{\nu=0}^{m-s}\|F^{-1}_{\xi\to y}[(1-\psi(\xi))\xi^s\frac{\partial^\nu}{\partial x^\nu}\overline{R}_{kjN}f(\xi)](\varphi_k(0,\cdot,t,1))\|_{L_p}\le$$

$$\le C(t,\gamma_0,b_{m+N+5})\,\overline{b}_{m+N+2}\|f\|_{W_p^{r_1}},\ 1\le p\le\infty,$$

where $r_1 = \max(0,\min(n\ge o, n\text{-integer}\,|\,n + N + j - m - 9/2 > 0))$.

Let us finally estimate $V_{kjN}(\varepsilon,t)$. We observe that

$$\frac{\partial^m}{\partial x^m}\left(R_{kjN}\Theta_k e^{i\varphi_k(0,x,t,\xi)}\right)=$$

$$=\left(\sum_{\nu=0}^{m} g_{kj\nu}(x,t,\xi)\Theta_k\frac{\partial^\nu}{\partial x^\nu}R_{kjN}+\sum_{\nu=0}^{m-1} g'_{kj\nu}(x,t,\xi)\frac{\partial^\nu}{\partial x^\nu}R_{kjN}\right)e^{i\varphi_k(0,x,t,\xi)}$$

where $g_{kj\nu}$ is a polynomial of order $m-\nu$ in ξ, whose coefficients depend on $\dfrac{\partial^\mu\varphi_k(0,x,t,1)}{\partial x^\mu}$, $\mu = 1,\dots,m-\nu$; and $g'_{kj\nu}$ is also a polynomial of order $m-\nu$ in ξ, whose coefficients depend also on $\dfrac{\partial^\mu\varphi_k(0,x,t,1)}{\partial x^\mu}$, and additionally on $\dfrac{\partial^\mu\Theta_k}{\partial x^\mu}$, $\mu = =1,\dots,m-\nu$. Dealing similarly as above and using (28), (30), (76) and (78) we obtain

$$\left\|\frac{\partial^m}{\partial x^m}(V_{kjN}(\varepsilon,t)f)(\cdot)\right\|_{L_p}\le$$

$$\le C(t,\gamma_0,b_m)\sum_{\nu=0}^{m}\sum_{s=0}^{m-\nu}\|F^{-1}_{\xi\to y}[(1-\psi(\xi))\xi^s\Theta_k(\varepsilon,\cdot,t,\xi)\ \times$$

$$\times\ \frac{\partial^m}{\partial x^m}R_{kjN}(\varepsilon,\cdot,t,\xi)](\varphi_k(0,\cdot,t,1))\|_{L_p}+C(t,\gamma_0,b_{m+1})\,\hat{b}_m\ \times \tag{97}$$

$$\times\sum_{\nu=0}^{m-1}\sum_{s=0}^{m-\nu}\|F^{-1}_{\xi\to y}[(1-\psi(\xi))\xi^s\frac{\partial^\nu}{\partial x^\nu}R_{kjN}(\varepsilon,\cdot,t,\xi)](\varphi_k(0,\cdot,t,1))\|_{L_p}\le$$

$$\le C(t,\gamma_0,b_{N+m+3})\,\overline{b}_m\|f\|_{W_p^{r_1}},\ 1\le p\le\infty.$$

It follows from (93)-(97) or $1 \le p \le \infty$

$$\left\|\frac{\partial^m}{\partial x^m}(\overline{T}_{1j}(\varepsilon,t)f)(\cdot)\right\|_{L_p}\le C(t,\gamma_0,b_{m+N+5})\,\overline{b}_{N+m+2}(\|f\|_{W_p^{r_1}}+\|f\|_{W_p^{m+2-j}}). \tag{98}$$

If $N = 3$ then $r_1 = m + 2 - j$. Therefore the right part of inequality (98) can be estimated by $C(t,\gamma_0,b_{m+8})\,\overline{b}_{m+5}\|f\|_{W_p^{m+2-j}}$.

From (92) and (98) it follows (5). Theorem 2 is proved.

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ANIVERSĂRI

PROFESSOR DIMITRIE D. STANCU AT HIS 65th BIRTHDAY

GH. COMAN*

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Professor Dimitrie D. Stancu, a distinguished Romanian mathematician, was born on February 11, 1927 in Calacea-Timiş, Romania. After finishing secondary school in Arad, in 1947, he studied at the University of Cluj, from which he received a master's degree in 1951 and a doctor degree in mathematics in 1956. His advisor for the doctoral disertation was the famous mathematician Tiberiu Popoviciu (1906-1975), a great master of numerical analysis and approximation theory. After his graduation, in 1951, he was named assistent at the Department of Mathematics, University of Cluj, and -in a normal succesion- he advanced up to the rank of full professor in 1969. He holds a continuous academic career at this university, except for one academic year (1961-1962), when he was visiting at the University of Wisconsin, Madison, Wis., U.S.A.

During his teaching career, professor Stancu had given courses on numerical analysis, approximation theory, probability theory, mathematical analysis, arithmetic and theory of numbers

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Since 1961 he is a member of American Mathematical Society and a reviewer of Mathematical Reviews. He is also a member of the society Gesellschaft für Angewandte Mathematik und Mechanik and a reviewer of Zentralblatt für Mathematik. He is member of several editorial committees of mathematical journals: Calcolo (Italy), Revue d'Analyse Numérique et de Théorie de l'Approximation and Studia Univ. Babeş-Bolyai, Mathematica (Cluj-Napoca).

Professor D.D.Stancu has participated, presenting invited lectures, at several international symposiums and conferences on approximation theory, held in: Gatlinburg, Tennessee, U.S.A.(1963), Lancaster-England (1969), Varna-Bulgaria (1970), Durham-England (1977), Hamburg-Germany (1985 and 1990), Acquafredda di Maratea-Itay (1992). He also has presented contributed papers at several meetings of American Mathematical Society, held in New York, Chicago and Milwaukee (1961-1962) and at five international conferences organized by the Mathematical Research Institute of Oberwolhach-Germany, in the period 1971-1981. He has been invited to present colloquium talks at several universities from Germany (Stuttgart, Hannover, Göttingen, Dortmund, Münster, Siegen, Würzburg and Berlin), at two universities from Holland (Delft and Eindhoven) and at four universities from Italy (Roma, Napoli, Potenza and L'Aquila).

Professor D.D.Stancu exercised a profound influence on many of his students. He gave his own ideas generously to them. A number of 28 doctoral students were working under his guidance.

The outstanding attribute of D.D.Stancu is his devotion to research work. He has made important mathematical contributions in various areas of numerical analysis, approximation theory and probability theory. These fall into the following list of topics (the numbers in the square brackets refer to the items from the list of selected mathematical papers of D.D.Stancu, annexed to this article):

1) **Interpolation theory**: interpolation with multiple nodes [6], [34]; multivariate interpolation [2], [3], [4], [8], [31], [54], [55]; study and use of divided differences [24], [32], [58], [60].

2) **Numerical differentiation**: extension to several variables of the Steffensen theorem on remainders [1], representations of remainders in numerical partial differentiation procedures [25], [32].

3) **Orthogonal polynomials**: a class of symmetric orthogonal polynomials on $(-a,a)$ for a weight function of the form $w(x) = p(x)x^{2s}$ [9]; orthogonal polynomials obtained by some general Christoffel type formulas [7], [10], [13], [17], [64]; power orthogonal polynomials [80], [82].

4) **Numerical quadratures and cubatures**: formulas of high degree o exactness [10], [12], [24], [59], [64]; Gauss-Christoffel quadrature formulas [7], [10], [17], [77]; quadrature formulas obtained by linear positive operators [66], [67], [79]; cubature formulas [3], [5], [11], [13].

5) **Taylor-type expansions**: use of multivariate interpolation for obtaining Taylor-type formulas for several variables [15],

[20], [32]; integral representations for remainders in multivariate Taylor expansions [20], [26], [32].

6) **Approximation by linear positive operators**: operators of Bernstein type [16], [21], [22], [27], [38], [39], [42], [50], [68], [70], [81]; representation of remainders [28], [32], [51], [54], [61], [63], [68], [74]; construction by interpolation methods [27], [52], [55], [56]; multivariate approximation by means of linear positive operators [16], [21], [22], [27], [28], [29], [43], [46], [47], [49], [55], [57], [72], [75]; Bernstein power series [48], [52], [53], [54]; monocity of the derivatives of the sequences of Bernstein polynomials [36], [60].

7) **Representations of remainders in linear approximation formulas in several variables**: representations by divided differences [2], [4], [28], [32], [57]; integral representations of remainders [15], [20], [25], [26], [32].

8) **Probabilistic methods in the theory of uniform approximation of continuous functions**: construction by probabilistic methods of linear positive operators [21], [41], [47], [68], [73].

9) **Use of interpolation and calculus of finite differences in probability theory**: integral representations of the distribution functions of some multivariate discrete distributions [44]; expressions in terms of finite differences for different types of moments of random discrete variables [31], [40], [41], [45], [47], [69].

10) **Spline approximation**: approximation of functions by Schoenberg type spline operators [52], [65].

Ending this article we join the members of the family of mathematicians of Professor D.D.Stancu, his colleagues and students, as well of many friends from Romania and many other countries, congratulating him with esteem on his 65th anniversary, wishing him good health and happiness. May he be granted with many more years with an active life and with new satisfactions in his scientific research work.

A LIST OF SELECTED MATHEMATICAL PAPERS OF PROFESSOR D.D.STANCU

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3. Generalization of some interpolation formulas for functions of several variables and certain considerations on the numerical integration formula of Gauss. Acad. R.P.Române, Bul. St. Sect. Mat. Fiz. **9**(1957), 287-313 (Romanian. Russian and French summaries). MR **20**, No. 1874.

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