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METRIC CONVEXITY IN GRAPHS

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REZUMAT. - Convexitatea metrică în grafuri. În această lucrare se prezintă o sinteză a unor rezultate recente în domeniul convexităţii metrice în grafuri. Sînt analizate diferite proprietăţi ale mulţimilor şi funcţiilor convexe în grafuri, caracterizările unor clase de grafuri cu ajutorul convexităţii.

Contents

1. Introduction. It is well-known that the ideas and results of convex analysis are of high importance for many mathematical disciplines. Convex analysis has shown itself as a powerful instrument usefut for applications. Therefore the development of mathematical structures and the enlargement of their applications lead to the creation of distinct analogies and generalizations of the notions of convex sets and convex functions (see, for instance, [31], [47], [73]).

Among them the notion of metric convexity introduced by K.Menger [52] is one of the most developed. Recall that a set A

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in a metric space (X,d) is called *convex* provided for every pair of points $x,y \in A$, the metric interval

$$[x,y] = \{z \in X : d(x,z) + d(z,y) = d(x,y)\}$$

is contained in A. For any set $B \subset X$, its *convex hull* conv$B$ is defined in a standard way to be the intersection of all convex sets in X containing B. Since the intersection of any family of convex sets is again a convex set, conv$B$ is the least convex set in X containing B.

The notions of metric convex set and covex hull became fruitful in general topology, differential geometry and functional analysis. (A sufficiently complete list of results and references on metric convexity in metric spaces and linear normed spaces can be found in [13], [73].)

Later the notion of a convex function on a metric space (X,d) was defined (see [70], [71]): a real-valued function f on X is called *convex* provided

$$f(z) \le \frac{d(z,y)}{d(x,y)} \cdot f(x) + \frac{d(x,z)}{d(x,y)} \cdot f(y)$$

for all points $x,y \in X$ $(x \neq y)$ and $z \in [x,y]$.

The actual period in the development of metric convexity is connected with investigations of discrete structures and of some extreme problems on them (see, for instance, [61], [62]). At the same time, a considerable part of the results on convexity in discrete spaces is concentrated around metric convexity in graphs. It is interesting to mention that the notions of convex set and convex function in graphs appeared previously in connection with some location problems (see [25], [66], [68],

[69], [82]). And only later, due to the development of generalized convexity theory, some properties of metric convexity and related metric behaviour of graphs where studied by distinct authors.

In this article, we deal with metric convexity in ordinary (may be, infinite) graphs. Since this topic became too wide to be described compact, we will be concentrated below on some results closely connected with the author's interests in this field. Some additional information on metric convexity in graphs can be found in the literature placed at the end of the paper.

For the convenience, we mention here some necessary definitions connected with graphs.

Everywhere below $G = (X,U)$ denotes a graph with vertex-set X and edge-set U. A graph G is called *finite* if the set X is finite. If card$X = n$, then G will be denoted by G_n. By a *subgraph* H of G we mean the induced one, i.e., two vertices x,y are adjacent in H if and only if they are adjacent in G. For any set $Y \subset X$, the subgraph in G induced by Y is denoted by $G(Y)$.

A graph G is *connected* if for any two vertices u, v in G, there exists a finite chain containing u, v. *We assume that all the considered below graphs are connected.*

In order to consider metric convexity in G, we assume that G is equipped with standard metric: for any vertices $x, y \in X$, denote by $d(x,y)$ the least number of edges in a chain connecting x, y. Is easily seen that d indeed is a metric on X, i.e., $d(x,y)$ satisfies the following conditions:

1) $d(x,y) \geq 0$, with $d(x,y) = 0$ if and only if $x = y$,

2) $d(x,y) = d(y,x)$,

3) $d(x,y) \le d(x,z) + d(z,y)$.

If vertices x, y belong to a connected subgraph H of G, then $d_H(x,y)$ denotes the distance between x, y in the graph H. A connected subgraph H is called an *isometric subgraph* of G if $d_H(x,y) = d_G(x,y)$ for every pair of vertices x, y in H.

A *clique* in G is a vertex-set having every two distinct vertices adjacent. If X is a clique, then G is called a *complete* graph. K_n denotes a complete graph with n vertices. The supremum of the cardinality of a clique in G is called the *density* of G, and is denoted by φ.

A vertex z in G is called *simplicial* provided the set $O(z)$ of all vertices in G adjacent with z form a clique. The *degree* $\deg(z)$ of z is the number of all vertices neighbor to z. Put $\Sigma(z) = O(z) \cup \{z\}$.

A sequence $l = (\ldots, v_{i-1}, v_i, v_{i+1}, \ldots)$ of vertices in G such that every two consecutive vertices are adjacent is called a *chain*. A chain l is *finite* if it is of the form $l = (v_1, \ldots, v_n)$; it is *one-side infinite* provided it has one of the forms $l = (v_1, v_2, \ldots)$, $l = (\ldots, v_2, v_1)$; l is *infinite* if it has no end-vertex. A chain is *simple* if all its vertices are distinct. A *circuit* of length n in G is a chain of the form $(v_1, v_2, \ldots, v_n, v_1)$. A circuit is *simple* if all its vertices $v_1, \ldots, v_n$ are distinct. Let C_n denote the simple circuit of length n.

A simple chain $l = (\ldots, v_{i-1}, v_i, v_{i+1}, \ldots)$ of vertices in G is called *geodesic* if any two vertices of the form v_{i-1}, v_{i+1}

are not adjacent in G. A *segment* (a *ray*, a *line*) is a finite (respectively, one-side infinite, both-side infinite) geodesic chain in G.

A *disconnecting* vertex-set in a graph G is a set $Y \subset X$ such that the induced graph $G(X \setminus Y)$ is disconnected. A graph without disconnecting vertices is called a *block*. A *tree* is a connected graph without circuits.

A *bipartite* graph is a graph containing no circuit of odd length. The vertex-set of a bipartite graph X can be partitioned into two disjoint sets Y, Z such that every edge in G joins a vertex in Y and a vertex in Z.

Also recall that G is named a *chord* graph provided it contains no simple circuit of the length greater than three as an induced subgraph. A *Husimi* tree is a graph such that each its block is a complete subgraph.

A graph G is called *planar* if it can be placed in the plane such that every vertex of G is a point and every edge of G is a rectifiable arc with end-points in X satisfying the properties: 1) every vertex x of G is an end of each arc incident with x, 2) a common point of two arcs is a vertex for both of them.

2. **Extremal structure of convex sets.** In this section some analogies of Krein-Mil'man's theorem about extremal points of convex sets in linear space are studied. Recall that Krein-Mil'man's theorem [50] states that every compact convex set in Hausdorff linear topological space is the closed convex hull of its extremal points.

Since every vertex-set in G is closed, the closed convex hull in G is identical with the convex hull, and a set of vertices in G is compact if and only if it is finite. Therefore we will discuss below the following problem. To determine necessary and sufficient conditions for the implementation of the assertion: for every finite set A of vertices in G, its convex hull coincides with the convex hull of extremal vertices of A.

By analogy with the linear space, we introduce the following definition. A vertex z of a set $A \subset X$ is called *extremal* in A if $z \notin [x,y]$ for all $x, y \in A \setminus \{z\}$, where $[x,y]$ is the metric interval with the ends x, y. By ext$A$ the set of all extremal vertices in A will be denoted.

It will be shown below that extremal vertices are closely related with simplicial vertices. The following well-known result (see [27], [51]) gives a sufficient condition for the existence of simplicial vertices in a graph.

LEMMA 2.1. *Any nonempty finite chord graph G contains at least one simplicial vertex; if G is not complete, then it contains at least two nonadjacent simplicial vertices.*

The following theorem strengths this assertion.

THEOREM 2.2. [74]. *For a graph $G = (X,U)$ the following conditions are equivalent:*

1) *every nonempty finite set in X contains at least one extremal vertex,*

2) *every nonempty finite subgraph in G contains at least one simplicial vertex,*

3) *G is a chord graph.*

The relation between extremal and simplicial vertices is shown in the following lemma.

LEMMA 2.3. *Every extremal vertex of a set* $A \subset X$ *is simplicial in the subgraph* $G(A)$. *If* A *is convex, then every simplicial vertex of* $G(A)$ *is extremal in* A.

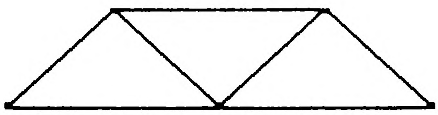
Now we can formulate an assertion analogous to Krein-Mil'man's theorem.

THEOREM 2.4. [74]. *For a graph* $G = (X,U)$ *the following conditions are equivalent:*

1) $\mathrm{conv}A = \mathrm{conv}(\mathrm{ext}A)$ *for every finite set* $A \subset X$,

2) $\mathrm{conv}A = \cup\{[x,y]: x, y \in \mathrm{ext}A\}$ *for every finite set* $A \subset X$,

3) *G is a chord graph containing no subgraph*



(1)

Note that for finite graphs, the equivalence of items 1) and 3) in Theorem 2.4 was established independently in [37], [47], and [72], [73].

In connection with Theorem 2.4, we mention two interesting lemmas. We say that a segment $(v_1,\ldots,v_n)$ is a shortest path provided $d(v_1,v_n) = n - 1$.

LEMMA 2.5. [47]. *Let G be a finite chord graph. Then every its vertex belongs to a segment whose ends are simplicial vertices in G.*

LEMMA 2.6. [45]. *For a graph G the following conditions are* *equivalent:*

1) *every segment in G is a shortest path,*

2) *G is a chord containing no subgraph* (1).

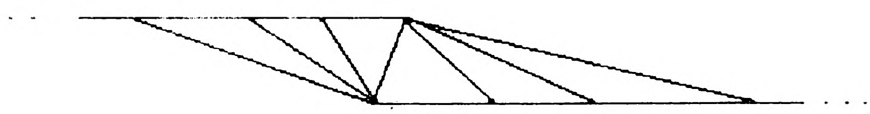
The following result shows some conditions for a graph G to satisfy conditions 2) and 3) of Theorem 2.4 for subgraphs and sets of any (may be, infinite) cardinality. These conditions are sufficiently cumbersome in the general case. Therefore, for the compactness of the description, we will restrict our attention on the class of graphs which contain no infinite complete subgraphs. Denote this class of graphs by K.

THEOREM 2.7. [74]. *For a graph* $G = (X,U) \in K$ *the following conditions are equivalent:*

1) *every nonempty set in X contains at least one extremal vertex,*

2) *every nonempty subgraph of G contains at least one simplicial vertex,*

3) *G is a chord graph containing no line and no subgraph*



THEOREM 2.8. [74]. *For a graph* $G = (X,U) \in K$ *the following conditions are equivalent:*

1) $\mathrm{conv}A = \mathrm{conv}(\mathrm{ext}A)$ *for every set* $A \subset X$,

2) $\mathrm{conv}A = \cup\{[x,y]: x, y \in \mathrm{ext}A\}$ *for every set* $A \subset X$,

3) $A = \mathrm{conv}(\mathrm{ext}A)$ *for every convex set* $A \subset X$,

4) $A = \cup \{[x,y]: x, y \in \text{ext}A\}$ *for every convex set* $A \subset X$,

5) *G is a chord graph containing no ray and no subgraph* (1).

Another well-known result on extremal structure of convex sets in linear space belongs to S.Straszewicz [81]: every compact convex set in finite-dimensional linear topological space is the closed convex hull of its exposed points. Recall that a boundary point x of a convex set A in a linear space is called exposed if there exists a hyperplane H such that $A \cap H = \{x\}$.

In order to formulate the respective analogous result for graphs, we need some definitions. A vertex-set H in $G = (X,U)$ is called a half-space provided both H and $X \setminus H$ are convex. A vertex z of a set $A \subset X$ is called *exposed* in A provided $\{z\} = A \cap H$ for some half-space $H \subset X$. Denote by exp$A$ the set of all exposed vertices of A. It is easily seen that any exposed vertex of a set is also extremal for the set, i.e., $\text{exp}A \subset \text{ext}A$ for every set $A \subset X$.

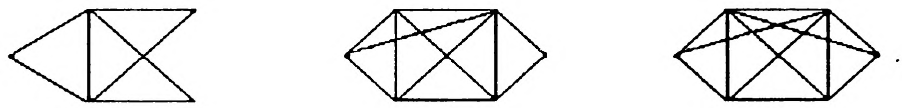
The following result is analogous to Straszewicz's theorem.

THEOREM 2.9. *For a graph* $G = (X,U)$ *the following conditions are equivalent:*

1) $\text{conv}A = \text{conv}(\text{exp}A)$ *for every finite set* $A \subset X$,

2) $\text{conv}A = \cup\{[x,y]: x,y \in \text{exp}A\}$ *for every finite set* $A \subset X$,

3) *G is a chord graph containing no subgraph* (1) *and none of*



(2)

THEOREM 2.10. *For a graph* $G = (X,U) \in K$ *the following conditions are equivalent:*

1) conv$A$ = conv(exp$A$) *for every set* $A \subset X$,

2) conv$A$ = $\cup$ {[x,y]: x, $y \in$ exp$A$} *for every set* $A \subset X$,

3) A = conv(exp$A$) *for every convex set* $A \subset X$,

4) A = $\cup$ {[x,y]: x, $y \in$ exp$A$} *for every convex set* $A \subset X$,

5) *G is a chord graph containing no ray and none of* (1) *or* (2).

At the end of this section we formulate two open problems.

PROBLEMS 2.11. *To describe the family of graphs* $G = (X,U)$ *satisfying at least one of the conditions:*

1) exp$A \neq \phi$ *for every nonempty convex set* $A \subset X$,

2) ext$A$ = exp$A$ *for every convex set* $A \subset X$.

3.**Convexity of balls, ball neighborhoods, and diametrally maximal sets.** It is well-know that some classes of convex sets are of special interest in the convexity theory. These are balls, ball neighborhoods, diametrally maximal sets, etc. Below we establish conditions under which these sets are convex in a graph. Recall that a set of the form

$$\Sigma_r(z) = \{ x \in X : d(x,z) \le r\}$$

is called the ball with center x and radius r. A set of the form

$$\Sigma_r(A) = \{ x \in X : d(x,A) \le r\}$$

is called the r-neighborhood of a set $A \subset X$. A set A in X is called diametrally maximal if diam$(z \cup A)$ > diam$A$ for every vertex $z \in X \setminus A$, where diam$K$ denotes the diameter of a set $K \subset X$.

Let M be a connected set in X. (M is called connected if the subgraph $G(M)$ is a connected component in G.) For any vertex $x \in M$, put

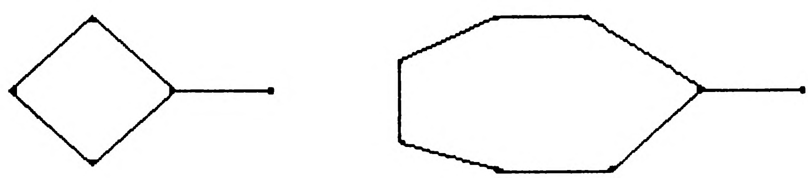
$$Q_M(x) = \{y \in M : \text{diam}M = d(x,y) = d_M(x,y)\}.$$

A pair $\{x,y\}$ is called *diametral* in M provided $y \in Q_M(x)$ (or $x \in Q_M(y)$, which is the same).

THEOREM 3.1. [76]. *Every diametrally maximal set in G is convex if and only if the following conditions are fulfilled:*

1) *G contains no simple circuit isometric to C_6 or C_n, $n \ge 8$,*

2) *G contains no subgraph isometric to one of*



3) *if $Q_T(y) = \{x\}$ for some vertices x, y in a simple circuit $T \ne C_4$, then x is simplicial in the subgraph G(T).*

THEOREM 3.2. [76]. *Every ball in G is convex if and only if the following conditions are fulfilled:*

1) *G contains no simple circuit isometric to C_4 or C_n, $n \ge 6$,*

2) *if $Q_T(y) = \{x\}$ for some vertices x, y in a simple circuit*

T in G, then x is simplicial in the subgraph G(T).

THEOREM 3.3. [76]. *For a graph G the following conditions are equivalent:*

1) *for any convex set $A \subset X$ and $r \geq 0$, the r-neighborhood $\Sigma_r(A)$ is convex,*

2) *for any vertices $a, b \in X$, the 1-neighborhood $\Sigma_1(\mathrm{conv}\{a,b\})$ is convex,*

3) *G contains no simple circuit isometric to C_n, $n \geq 4$.*

Note that Theorems 3.2 and 3.3 are repeated in [37] in an equivalent form.

COROLLARY 3.4. *If a graph G contains no simple circuit isometric to C_n, $n \geq 4$, then the following conditions are equivalent:*

1) *every diametrally maximal set in G is convex,*

2) *every ball in G is convex,*

3) *every neighborhood of a convex set in G is convex,*

4) *G is a tree.*

4. **Convex functions.** Recall that a real-valued function f on X is called *convex* provided

$$f(z) \leq \frac{d(z,y)}{d(x,y)} \cdot f(x) + \frac{d(x,z)}{d(x,y)} \cdot f(y)$$

for all vertices $x, y \in X$ $(x \neq y)$ and $z \in [x,y]$. One can state the following simple properties of convex functions on X.

THEOREM 4.1. 1) *For any convex functions f, g and real number $\lambda \geq 0$, the functions $f + g$ and λf are convex,*

2) *the least upper bound of any family of convex functions*

is a convex function,

3) *the limit of any pointwise convergent sequence of convex functions is a convex function,*

4) *for any convex function f and real number λ, the sets*

$$\{z \in X : f(z) \le \lambda\},\ \{z \in X : f(z) < \lambda\}$$

are convex.

Similarly to the case of linear space, we can define an *affine* function f on X as a real-valued function such that both functions f and $-f$ are convex. In other words, f is affine if

$$f(z) = \frac{d(z,y)}{d(x,y)}\cdot f(x) + \frac{d(x,z)}{d(x,y)}\cdot f(y)$$

for all vertices x, $y \in X$ $(x \ne y)$ and $z \in [x,y]$. From this definition follows immediately

COROLLARY 4.2. 1) *For any affine functions f_1, f_2 and real numbers λ_1, λ_2, the function $\lambda_1 f_1 + \lambda_2 f_2$ is affine,*

2) *the limit of any pointwise convergent sequence of affine functions is an affine function,*

3) *for any affine function f and real number λ, the sets*

$$\{z \in X : f(z) \le \lambda\},\ \{z \in X : f(z) < \lambda\}$$

are half-spaces.

A function $f : X \to R$ is called *quasiconvex* if for every real number λ, the set $\{z \in X : f(z) \le \lambda\}$ is convex. Equivalently, f is quasiconvex if

$$f(z) \le \max\{f(x),f(y)\}$$

for all vertices x, $y \in X$ and $z \in [x,y]$.

THEOREM 4.3. 1) *For any quasiconvex function f and real numbers $\lambda \ge 0$, $\mu \in R$, the function $\lambda f + \mu$ is quasiconvex,*

2) *the least upper bound of any family of quasiconvex functions is a quasiconvex function,*

3) *the limit of any pointwise convergent sequence of quasiconvex functions is a quasiconvex function.*

Similarly, a function $f : X \to R$ is called *quasiaffine* if both functions f and $-f$ are quasiconvex, i.e., f is quasiaffine if

$$\min\{f(x),f(y)\} \leq f(z) \leq \max\{f(x),f(y)\}$$

for all vertices x, $y \in X$ and $z \in [x,y]$.

COROLLARY 4.4. 1) *For any quasiaffine functions f_1, f_2 and any real numbers λ_1, λ_2, the function $\lambda_1 f_1 + \lambda_2 f_2$ is quasiaffine,*

2) *the limit of every pointwise convergent sequence of quasiaffine functions is a quasiaffine function,*

3) *a function f is quasiaffine if and only if for every real number λ, the sets*

$$\{z \in X : f(z) \leq \lambda\}, \{z \in X : f(z) < \lambda\}$$

are half-spaces.

Below we study some properties of the classes of convex, affine, quasiconvex, and quasiaffine functions on X. Let $\mathbf{A}$, $\mathbf{D}$, $\mathbf{CA}$, and $\mathbf{CD}$ denote, respectively, the collection of all affine, convex, quasiaffine, and quasiconvex functions on X, and let $\mathbf{F}$ (respectively, $\mathbf{I}$) denote the family of all real (constant) functions on X. Trivially,

$$\begin{array}{ccccc} & & \mathbf{CA} & \subset & \mathbf{CD} & \subset & \mathbf{F} \\ & & \cup & & \cup & & \\ \mathbf{I} & \subset & \mathbf{A} & \subset & \mathbf{D} & & \end{array}$$

THEOREM 4.5. [65], [75]. 1) *The following conditions are equivalent:* $\mathbf{A} = \mathbf{F}$, $\mathbf{D} = \mathbf{F}$, $\mathbf{CA} = \mathbf{F}$, $\mathbf{CA} = \mathbf{CD}$, $\mathbf{CD} = \mathbf{F}$, $\mathbf{A} = \mathbf{CD}$,

$D = CD$, G is a complete graph,

2) any two of the classes A, D, CA coincide if and only if the two classes are trivial, i.e., are equal to I or to F.

THEOREM 4.6. [65], [75]. 1) *$A \neq I$ if and only if the graph $G = (X,U)$ can be decomposed into at most countable family of pairwise disjoint complete subgraphs G_i such that every vertex z in G_i is adjacent to all the vertices in $G_{i-1} \cup G_{i+1}$ and only to them,*

2) *for a finite graph G, one has $D \neq I$ if and only if X contains a convex set Y with connected complement $X \setminus Y$ such that every vertex $z \in Y$ adjacent in $X \setminus Y$ is adjacent to all the vertices in $X \setminus Y$,*

3) *$CA \neq I$ if and only if X contains at least one half-space,*

4) *$CD \neq I$ (provided* card$X > 1$).

For any family H of functions on X, let H_+ denote the collection of all functions which are the sums of finite subfamilies of H. We have the relations

$$\begin{array}{ccccccccc} & & D & = & D_+ & \subset & CD & \subset & CD_+ = F \\ & & \cup & & \cup & & \cup & & \cup \\ I & \subset & A & = & A_+ & \subset & CA & \subset & CA_+ \end{array}$$

THEOREM 4.7. [65], [75]. *The following implications hold:*

1) *$CA_+ = A \leftrightarrow CA = A$,*

2) *$CA_+ = D \leftrightarrow CA = D$,*

3) *$CD_+ = CD$ holds if and only if G is a complete graph,*

4) *$CA_+ = CA$ holds if and only if the intersection of every collection of half-spaces in G is either empty or a half-space.*

The supremum properties of convex functions play an important role in convex analysis. For example, at the base of the finite dimensional theory of duality of convex functions lies the famous theorem by Minkowski: every convex function is a pointwise supremum of affine functions. Below we investigate an analogous assertion for convex functions on a graph.

For any family H of functions on X, let H_s denote the collection of all finite functions which are pointwise supreme of subfamilies of H. It is easily seen that the following relations are valid:

$$\begin{array}{ccccccccc} & & CA & \subset & CA_s & \subset & CD & = & CD_s & \subset & F \\ & & \cup & & \cup & & \cup & & \cup & & \\ I & \subset & A & \subset & A_s & \subset & D & = & D_s & & \end{array}$$

In our notations, the analogous assertion to Minkowski's theorem for convex functions on graphs looks as in item 4) of Theorem 4.8.

THEOREM 4.8. [65], [75]. 1) *The following conditions are equivalent:* $A = CA$, $A = CA_s$, $A_s = CA$, $A_s = CA_s$,

2) $A = A_s \leftrightarrow$ *either* $A = I$ *or* $A = F$,

3) $CA = CA_s$ *holds if and only if the intersection of every collection of half-spaces in* G *is either empty or a half-space,*

4) $A_s = D$ *holds if and only if* G *is either a complete graph or a simple chain,*

5) $A_s = CD$ *holds if and only if* G *is a complete graph,*

6) $CA_s = D \leftrightarrow CA = D$,

7) *if* G *is a finite graph, then* $CA_s = CD$ *holds if and only if the intersection of every collection of half-spaces in* G *is either empty or a half-space.*

As a logical consequence of this circle of questions, we will consider the family H_* which is the smallest collection of functions on X containing a family H of functions and is closed with respect to taking finite sums and finite supreme. We have the relations

$$\begin{array}{ccccccccc} & & D & = & D_* & \subset & CD & \subset & CD_* & \subset & F \\ & & \cup & & \cup & & \cup & & \cup & & \\ I & \subset & A & \subset & A_* & & CA & \subset & CA_* & & \end{array}$$

THEOREM 4.9. [75]. *The following implications hold:*

1) $A_* = A \longleftrightarrow A_s = A$,

2) $A_* = D \longleftrightarrow A_s = D$,

3) $CA_* = A \longleftrightarrow CA = A$,

4) $CA_* = CA \longleftrightarrow CA_s = CA$,

5) $A_* = CA \longleftrightarrow A_* = CA_* \longleftrightarrow A = CA$,

6) $A_* = CD$ *holds if and only if* G *is a complete graph,*

7) *the following conditions are equivalent:* $A = CD_*$, $D = CD_*$ $CA = CD_*$, G *is a complete graph.*

In connection with the above results, we formulate some open problems.

PROBLEMS 4.10. 1) *To determine conditions for the feasibility of any of the relations:*

a) $D \neq I$, $CA_+ = CA$, $CA_+ = CD$, $CA_+ = CD_+$, $CA_+ = F$, $CD_+ = F$,

b) $A_* = CD_*$, $CA_* = CD$, $CA_* = CD_*$, $CD_* = CD$, $CD_* = F$,

2) *to determine conditions for the feasibility of the following property: the intersection of every collection of half-spaces in* G *is either empty or is a half-space.*

The remained part of this section is devoted to the study of separation properties of convex functions on X. Below we consider a graph G to be finite. A family H of functions on X will be said to have *separation property* if for any disjoint convex sets $Y, Z \subset X$ there exists a function $f \in H$ such that

$$\inf \{f(x): x \in Y\} > \sup \{f(x): x \in Z\}.$$

Il the set Y (respectively, Z) is a singleton, then we will speak about *upper (lower) separation property.* If both sets Y and Z are singletons, we will say that *H separates vertices.*

THEOREM 4.11. [75]. 1) *For the family **A**, separation property, upper separation property, lower separation property, and separation property for vertices are equivalent and hold if and only if the graph G is either complete or a simple chain,*

2) *the following properties are equivalent:*

a) *D separates vertices,*

b) *D has lower separation property,*

c) *G is a chord graph,*

3) *the following conditions are equivalent:*

a) *D has separation property,*

b) *D has upper separation property,*

c) *G is a chord graph containing no subgraph* (1).

4) a) ***CA** separates vertices if and only if any two vertices of G can be separated by some complementary half-spaces,*

b) ***CA** has lower separation property $\leftrightarrow$ **CA** has upper separation property $\leftrightarrow$ convexity in G is regular,*

c) ***CA** has separation property if and only if convexity in G is normal,*

5) ***CD** has separation property.*

Sometimes it is necessary to know about the existence in a given class of a function satisfying the respective separation condition. We say that a real-valued function f on X satisfies *separation condition* if for any disjoint convex sets Y, Z in X one of the inequalities

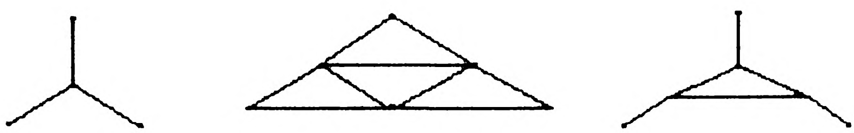
$$\inf \{f(x): x \in Y\} > \sup \{f(x): x \in Z\},$$

$$\inf \{f(x): x \in Z\} > \sup \{f(x): x \in Y\}$$

holds. If one of the sets Y, Z is a singleton, we speak about *weak separation condition*.

THEOREM 4.12. [75]. 1) *There exists a function $f \in A$ separating vertices in X if and only if G is either a complete graph or a simple chain,*

2) *there exists a function $f \in CA$ separating vertices in X if and only if G is a chord graph containing none of the subgraphs*



3) *the following conditions are equivalent:*

a) *there is a function $f \in D$ separating vertices in X,*

b) *there is a function $f \in CD$ separating vertices in X,*

c) *G is a chord graph,*

4) *the following conditions are equivalent:*

a) *each of the classes A, D, CA and CD contains a function satisfying separation condition,*

b) *each of the classes A, D, CA and CD contains a function satisfying weak separation condition,*

c) *graph G is a simple chain.*

5. **Convexity of Steiner functions.** As we know, Steiner's problem (or Weber's problem, in a different terminology) on a graph consists in finding a minimum of a function

$$f(z) = \Sigma\mu(x)\cdot d(z,x), \tag{3}$$

where $\mu(x) \geq 0$ and the sum is taken over the set of all verces $x \in X$. Unlike to the case of Euclidean space, functions (3) have no "good" properties like convexity, which guarantee the absence of local minima different from the global one. Therefore it is reasonable to find the class of all graphs for which Steiner's problem is confined to the scheme of convex analysis. An analogous problem will be studied below for functions

$$F(z) = \Sigma\mu(A)\cdot d(z,A), \tag{4}$$

where $\mu(A) \geq 0$ and the sum is taken over the family of all convex sets A in X.

THEOREM 5.1. [72], [79]. *For a graph $G = (X,U)$ the following conditions are equivalent:*

1) *every function (3) is convex,*

2) *for every vertex $x \in X$, the function $p(z) = d(z,x)$ is convex,*

3) *every function (3) is quasiconvex,*

4) *for any vertices x_1, $x_2 \in X$, the function*

$$p(z) = \mu_1 d(z,x_1) + \mu_2 d(z,x_2), \quad \mu_1,\mu_2 \geq 0$$

is quasiconvex,

5) *G is a chord graph containing no subgraph of the form (1).*

From Theorem 3.2 follows

COROLLARY 5.2. *Every function $p(z) = d(z,x)$, $x \in X$ is quasiconvex if and only if the following conditions are fulfilled:*

1) *G contains no simple circuit isometric to C_4 or C_n, $n \geq 6$,*

2) *if $Q_T(y) = \{x\}$ for some vertices x,y in a simple circuit $T \subset G$, then x is simplicial in the subgraph $G(T)$.*

THEOREM 5.3. [79]. *For a graph $G = (X,U)$ the following conditions are equivalent:*

1) *every function (4) is convex,*

2) *for every convex set $A \subset X$ with* card*$A \leq 2$, the function $p(z) = d(z,A)$ is convex,*

3) *every function (4) is quasiconvex,*

4) *for any convex set A_1, $A_2 \subset X$, with* card*$A_1 \leq 2$ and* card*$A_2 \leq 2$, the function $p(z) = d(z,A_1) + d(z,A_2)$ is quasiconvex,*

5) *G is a Husimi tree.*

From Theorem 3.3 follows

COROLLARY 5.4. *For a graph $G = (X,U)$ the following conditions are equivalent:*

1) *for every convex set $A \subset X$, the function $p(z) = d(z,A)$ is quasiconvex,*

2) *for every convex set $A \subset X$ with* card*$A \leq 2$, the function $p(z) = d(z,A)$ is quasiconvex,*

3) *G contains no simple circuit isometric to C_n, $n \geq 4$.*

A function $f : X \to R$ is called *strictly convex* (respectively, *strictly quasiconvex*) provided it is convex (respectively, quasiconvex) and

$$f(z) < \frac{d(z,y)}{d(x,y)} \cdot f(x) + \frac{d(x,z)}{d(x,y)} \cdot f(y)$$

respectively, $f(z) < \max\{f(x), f(y)\}$

for all vertices x, $y \in X$ $(x \neq y)$ and $z \in [x,y] \setminus \{x,y\}$ in case $f(x) \neq f(y)$.

THEOREM 5.5. *For a graph* $G = (X,U)$ *the following conditions are equivalent:*

1) *for every vertex* $x \in X$, *the function* $p(z) = d(z,x)$ *is strictly convex,*

2) *for any vertices* x_1, $x_2 \in X$, *the function* $p(z) = d(z,x_1) + d(z,x_2)$ *is strictly convex,*

3) *for every convex set* $A \subset X$ *with* $\operatorname{card}A \leq 3$, *the function* $p(z) = d(z,A)$ *is strictly quasiconvex,*

4) *G is a complete graph.*

THEOREM 5.6. *For a graph* $G = (X,U)$ *the following conditions are equivalent:*

1) *for every vertex* $x \in X$, *the function* $p(z) = d(z,x)$ *is strictly quasiconvex,*

3) *for every convex set* $A \subset X$ *with* $\operatorname{card}A \leq 2$, *the function* $p(z) = d(z,A)$ *is strictly quasiconvex,*

4) *G is a Husimi tree.*

THEOREM 5.7. [79]. *For a graph G with at most countable number of vertices, the folloving conditions are equivalent:*

1) *every finite function* (3) *with* $\mu(x) > 0$ *for all* $x \in X$ *is strictly convex,*

2) *G is a chord graph containing no subgraph* (1).

THEOREM 5.8. *The following conditions are equivalent:*

1) *every finite function* (4) *with* $\mu(A) > 0$ *for all convex sets* A *in* X *strictly convex,*

2) G *is a Husimi tree.*

At the end of this section we put the following problem.

PROBLEM 5.9. *For a graph* $G = (X,U)$, *to determine conditions for the feasibility of the following property: the function* $f(z) = \Sigma \{d(z,x) : x \in Y\}$ *is convex for every finite set* $Y \subset X$.

6. **Convex sets in chord graphs.** It was shown above that chord graphs play a special role for metric convexity. In this connection, we collect here different properties of convex sets in chord graphs.

We say that convexity in a graph $G = (X,U)$ has *join property* provided

$$\mathrm{conv}(A \cup B) = \cup \{[a,b] : a \in A,\ b \in B\}$$

for any convex sets A,B in X, and that it has *cone property* if

$$\mathrm{conv}(a \cup B) = \cup \{[a,b] : b \in B\}$$

for every vertex a and every convex set B in X.

For any set $A \subset X$, put $P(A) = \cup \{[x,y] : x,\ y \in A\}$.

THEOREM 6.1. [77]. *For a chord graph* $G = (X,U)$ *the following conditions are equivalent:*

1) *convexity in* G *has join property,*

2) *convexity in* G *has cone property,*

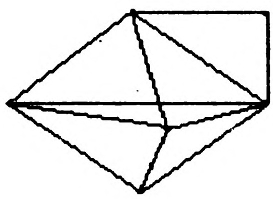
3) $\mathrm{conv}\{x,y,z\} = \cup \{[x,v] : v \in [y,z]\}$ *for any vertices* $x,\ y,\ z \in X$ *such that* $\mathrm{diam}\{x,y,z\} \le 2$,

4) $\mathrm{conv}A = P(A)$ *for every set* $A \subset X$,

5) $\mathrm{conv}A = P(A)$ *for every set* $A \subset X$ *with* $\mathrm{card}A \le 3$ *and*

diam*A* ≤ 2,

6) *G contains none of the subgraphs*



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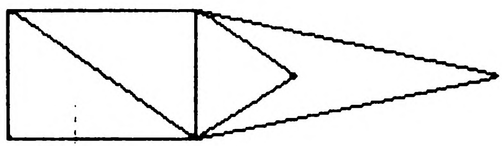

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The following results complete Theorem 6.1.

THEOREM 6.2. [64]. *For a chord graph* $G = (X,U)$ *the following conditions are equivalent:*

1) $\mathrm{conv}(a \cup B) = \cup \{[a,b] : b \in B\}$ *for every vertex* $a \in X$ *and every set* $B \subset X$ *of diameter one,*

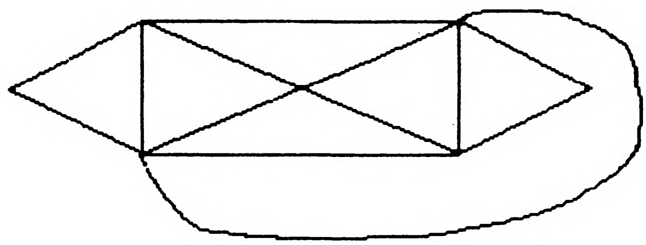
2) $\mathrm{conv}\{a,b,c\} = [a,b] \cup [a,c]$ *for every vertex* $a \in X$ *and every edge* $(b,c) \in U$ *such that* $\mathrm{diam}\{a,b,c\} \le 2$,

3) *G contains no subgraph* (5).

THEOREM 6.3. [77]. *For a chord graph* $G = (X,U)$ *the following conditions are equivalent:*

1) *for any vertices* $x, y \in X$, *the interval* $[x,y]$ *is convex,*

2) *G contains no isometric subgraph*



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LEMMA 6.4. [64]. *Let $G = (X,U)$ be a chord graph. For any pair of vertices $x, y \in X$ such that $d(x,y) \leq 2$, the interval $[x,y]$ is convex.*

For any sets A, B in X, the sets

$$A/B = \{z \in X : B \cap [\cup [z,a] : a \in A] \neq \phi\},$$

$$A//B = \{z \in X : B \cap \text{conv}(z \cup A) \neq \phi\}$$

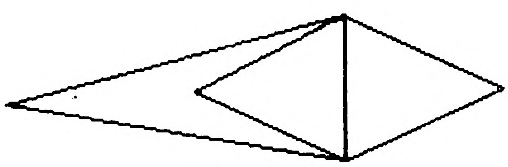
are called, respectively, weak and strong shadows of A relative to B.

THEOREM 6.5. [64]. *For a chord graph $G = (X,U)$ the following conditions are equivalent:*

1) *for any vertices $a, b \in X$, the set a/b is convex,*

2) *for any adjacent vertices $a, b \in X$, the set a/b is convex,*

3) *G contains no isometric subgraph (7) and none*



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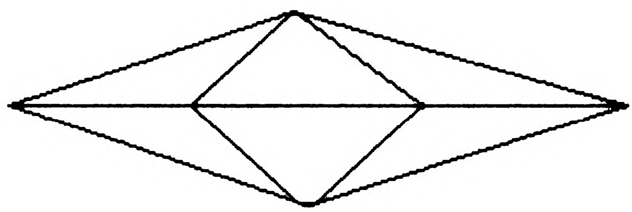
THEOREM 6.6. [64]. *For a chord graph $G = (X,U)$ the following conditions are equivalent:*

1) *for every convex set $B \subset X$ and every vertex $a \in X \setminus B$, the set a/B is convex,*

2) *for any pairwise adjacent vertices a, b, $c \in X$, the set $a/\{b,c\}$ is convex,*

3) *G contains no isometric subgraphs* (7), (8), *and no subgraph*



THEOREM 6.7 [64]. *For a chord graph $G = (X,U)$ the following conditions are equivalent:*

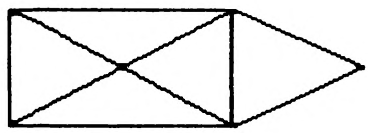
1) *for any disjoint convex sets A, $B \subset X$, the set A/B is convex,*

2) *for every convex set $A \subset X$ and every vertex $b \in X \setminus A$, the set A/b is convex,*

3) *for every edge $(a,c) \in U$ and every vertex $b \in X \setminus \{a,c\}$ such that $\max\{d(a,b), d(b,c)\} \le 2$, the set $\{a,c\}/b$ is convex,*

4) *for any pairwise adjacent vertices $a,b,c \in X$, the sets a/b and $\{a,c\}/b$ are convex,*

5) *G contains none of the subgraph* (8) *and*



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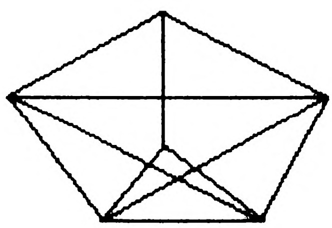
THEOREM 6.8. [64]. *For a chord graph $G = (X,U)$ the following conditions are equivalent:*

1) *for any pairwise adjacent vertices a, b, $c \in X$, the set $\{a,c\}/b$ is convex,*

2) *G contains none of the subgraph (6), (9), and*



THEOREM 6.9. [64]. *For a chord graph $G = (X,U)$ the following conditions are equivalent:*

1) *for any convex sets A, $B \subset X$, the set A/B is convex,*

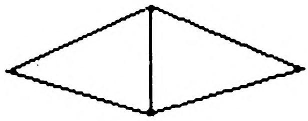
2) *for every convex set $A \subset X$ and every vertex $b \in X$, the set A/b is convex,*

3) *for every convex set $A \subset X$ of diameter two and every vertex $b \in X$, the set A/b is convex,*

4) *for every vertex $a \in X$ and every convex set $B \subset X$, the set a/B is convex,*

5) *for every edge $(a,b) \in U$, the set $a/\{a,b\}$ is convex,*

6) *G contains no subgraph*



Now we will discuss some separation properties of convex sets. A *half-space* in X is any convex set $A \subset X$ with convex complement $X \setminus A$. We say that two complementary half-spaces P, Q *separate* sets A, B if $A \subset P$ and $B \subset Q$. Convexity in X is called:

i) *separating,*

ii) *regular,*

iii) *normal,*

provided it is possible to separate by complemantary half-spaces, respectively:

i) any two distinct points,

ii) any convex set and any its exterior point,

iii) any two disjoint convex sets.

THEOREM 6.10. [64]. *For a chord graph G the following conditions are equivalent:*

1) *every semispace in G is a half-space,*

2) *G contains none of the subgraphs* (2).

THEOREM 6.11. [77]. *For a finite chord graph G the following conditions are equivalent:*

1) *every half-space in G is a semispace,*

2) *G is a tree.*

COROLLARY 6.12. *For a chord graph G the following conditions are equivalent:*

1) *convexity in G is regular,*

2) *for every set $B \subset X$ of diameter one and every vertex $a \in B$, the sets $\{a\}$ and $B \setminus \{a\}$ are separated by complementary half-spaces,*

3) *for every set $B \subset X$ of diameter one with at most four vertices and for every vertex $a \in B$, the sets $\{a\}$ and $B \setminus \{a\}$ are separated by complementary half-spaces,*

4) *G contains no subgraph (2).*

THEOREM 6.13. [77]. *For a chord graph $G = (X,U)$ the following conditions are equivalent:*

1) *convexity in G is normal,*

2) *any two disjoint edges $(a,b),(c,d) \in U$ such that the set $\{a,b,c,d\}$ has at most one pair of nonadjacent vertices, are separated by complementary half-spaces,*

3) *any two disjoint parts of a set with at most four vertices in X are separated by complementary half-spaces,*

4) *G contains none of the subgraphs (8), (9).*

Note that some sufficient conditions for the separability of vertices in a chord graph by complementary half-spaces are studied in [63].

We continue with some combinatorial problems on convex sets in chord graphs. Further $\mathcal{S}$ denotes the family of all convex sets in G. Put

$$\mathcal{S}_k = \{A \in \mathcal{S} : \text{card}A = k\},\ k = 0,1,\ldots$$

THEOREM 6.14. [77]. *For a chord graph G_n with n vertices, one has* $\text{card}\mathcal{S}_k \geq n - k + 1$.

1) $\text{card}\mathcal{S}_2 = n - 1$ *if and only if G_n is a tree,*

2) *for $3 \leq k \leq n - 1$, the equality $\text{card}\mathcal{S}_k = n - k + 1$ holds*

if and only if G_n *is a simple chain.*

For any vertex $z \in X$, call by a *semispace* corresponding to z any convex set in $X \setminus \{z\}$ maximal with respect to inclusion. It is known that the family of sets consisting of X and of all semispaces in X forms *the least base* B of convexity; i.e., every convex set in X can be represented as the intersection of some elements from B, and every proper subfamily of B does not satisfy this property.

THEOREM 6.15. [77]. *If* B *is the least base of convexity in a chord graph* G_n, *then* card$B \geq n + 1$. *The equality* card$B = n + 1$ *holds if and only if* G_n *is a complete graph.*

Denote by P the family of all half-spaces in a graph G.

THEOREM 6.16. *For a chord graph* G_n, $n \geq 4$, *one has* card$P \geq 6$. *For* $n = 4$, *the equality* card$P = 6$ *holds if and only if* G_n *is either a chain or a star, and for* $n \geq 5$, *one has* card$P =$ $= 6$ *if and only if* G_n *contains a complete subgraph* K_{n-3} *such that every vertex in* $G_n - K_{n-3}$ *is adjacent to all vertices in* K_{n-3} *and only to them.*

For a set $A \subset X$, put

$$P_0(A) = P(A),\ P_{k+1}(A) = P(P_k(A)),\ k = 0,1,\ldots$$

It is easy to prove that

$$A \subset P_1(A) \subset P_2(A) \subset \ldots \subset \text{conv}A = \cup \{P_k(A) : k \geq 0\}.$$

This method of convex hull construction gives us the following characteristic number for convex hulls: for any set $A \subset X$, denote by $\beta(A)$ the least natural number k such that conv$A = P_k(A)$.

THEOREM 6.17. [77]. *For any vertex-set* A *in a chord graph*

G_n, $n \geq 5$, *one has* $\beta(A) \leq n - 4$, and $\beta(A) \leq 1$ *in case* $n \leq 5$.

Classical Helly [43], Caratheodory [15], and Radon [59] theorems about convex sets in linear space became a starting point for the following definitions. The *Helly number of* X is the least natural number h satisfying the property: any finite family of convex sets in X has a common point if and only if each its h-membered subfamily has a common point. The *Radon number* of X is the least natural number r such that every set $A \subset X$ containing at least r vertices can be divided in two disjoint subsets whose convex hulls have a common point. The *Caratheodory number* in X is the least natural number c such that for every set $A \subset X$

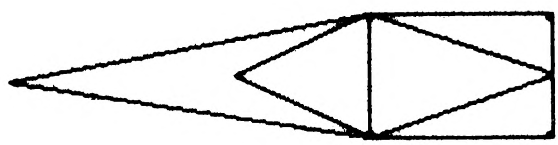
$$\text{conv}A = \cup \{\text{conv}B : B \subset A, \text{card}B \leq c\}.$$

THEOREM 6.18. [21]. *The Helly number of convexity in a chord graph* G *equals the density of* G.

THEOREM 6.19. [21], [72]. *If* G *is a chord graph with density* φ, *then for the Radon number* r *in* G, *one has:*

1) $3 \leq r \leq 4$ *if* $\varphi = 2$, *and* $r = 3$ *if and only if* G *is a simple chain,*

2) $4 \leq r \leq 5$ *if* $\varphi = 3$, *and* $r = 4$ *if and only if* G *contains no subgraph*



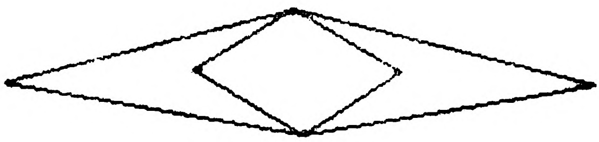
THEOREM 6.20. [77]. *For a chord graph* G *the following*

conditions are equivalent:

1) *the Caratheodory number of convexity in G is at most two,*

2) *G contains none of the subgraphs* (5), (6).

7. **Convex simple and quasisimple planar graphs.** From the point of view of generalized convexity theory, graphs with the most poor collection of convex sets are of certain interest. Since any vertex, any pair of end vertices of an edge, and the whole vertex-set X in a graph are convex, it is interesting to study those graphs which contain no other convex set. In [59] the following definition is introduced. A graph $G = (X,U)$ is called convex simple if every proper convex set in G (i.e., a set different from empty set and the whole X) has at most two vertices. For example, the graph shown below is convex simple.



The class of all convex simple graphs is too large to have a suitable description. Therefore we concentrate our attention on planar convex simple graphs. The following theorem was first proved in [60] for finite graphs.

THEOREM 7.1. [20]. *A planar graph G different from the graph of cube Q_3 is convex simple if and only if it contains no convex set of three vertices.*

The following theorem gives an interesting characterization of convex simple planar graphs. Recall that S (S_k) means the family of all (respectively, all k-membered) convex sets in G.

THEOREM 7.2. [20], [72], [78]. *For a finite graph G_n,*

$$\text{card}S \geq 3n - 2, \quad \text{card}S_2 + \text{card}S_3 \geq 2n - 4.$$

The following conditions are equivalent:

1) $\text{card}S = 3n - 2$,

2) *G_n is different from the graph of cube Q_3 and* $\text{card}S_2 + \text{card}S_3 = 2n - 4$,

3) *G_n is planar and convex simple.*

THEOREM 7.3. [19]. *A planar graph $G = (X,U)$ with* $\text{card}X \geq 5$ *different from the graph of octahedron F_3 is convex simple if and only if it contains at least one vertex of* degree ≥ 2, *and every such a vertex has a unique dual vertex in G (a vertex z is dual for x provided $O(z) = O(x)$).*

Now we are going to describe convex simple planar graphs. Denote by T any tree with at least three vertices, and let T_0 be a copy of a subtree formed by all the interior vertices in T. Denote by $L(T,T_0)$ the graph containing $T \cup T_0$ with the following additional edges: any vertex z in T_0 is adjacent to all the vertices in $O(\bar{z})$ and only to them (here $\bar{z} \in T$ is a copy of z and $O(z) = \{v \in T : v \text{ is adjacent to } \bar{z}\}$).

THEOREM 7.4. [18]. *For any planar convex simple graph G there is a tree T such that $G = L(T,T_0)$.*

In connection with the previous theorem, there appears the problem to describe those trees T for which the graph $L(T,T_0)$ is planar and convex simple.

THEOREM 7.5. [19]. *For any tree T with at least three vertices, the graph $L(T,T_0)$ is convex simple.*

The class of all trees T for which the graph $L(T,T_0)$ is planar is not described. We know only one particular result.

THEOREM 7.6. [19]. *If a tree T has at most countable number of vertices, then the graph $L(T,T_0)$ is planar.*

We are interested to know about the uniqueness of the representation of a planar convex simple graph in the form $L(T,T_0)$ for a suitable tree T.

THEOREM 7.7. [19]. *For any trees S and T, the graphs $L(S,S_0)$ and $L(T,T_0)$ are isomorphic if and only if S and T are isomorphic.*

The obtained results permit a description of a more wide class of graphs. By definition (see [11]), a graph G is called *convex quasisimple* if every proper convex set in G generates a complete subgraph. In other words, a graph is convex quasisimple if the diameter of every proper convex vertex-set in G is at most one.

THEOREM 7.8. [19]. *A planar graph G is convex quasisimple if and only if it contains no convex vertex-set inducing one of the following subgraphs:*

THEOREM 7.9. [17]. *Any planar convex quasisimple graph G contains a convexly simple subgraph.*

Let T be a tree with at least three vertices. Denote by $R(T)$ the family of graphs R obtained from T by the addition of some new edges in correspondence with the following rules:

1) the distance (in T) between the ends of any new edge (x,y) is equal to two,

2) any new edge is incident to at least one end-vertex of T,

3) for any end-vertex of T, its degree in R is at most three,

4) if one of the vertices of a new edge (x,y) is interior for T and a vertex z lies between x and y in T, then $\deg_T z = 2$,

5) if T is not a star, then R contains no simple circuit containing the end-vertices of T only,

6) if T is a star and R contains a simple circuit containing the end-vertices of T only, then this circuit contains all the end vertices of T.

Let T_0 be a subtree, consisting of all the interior vertices of a tree T. For any graph $R \in \mathcal{R}(T)$, denote by $L(R,T_0)$ the graph containing R, T_0 and the following edges: every vertex $z \in T_0$ is adjacent to all the vertices in $O(\bar{z})$, where $\bar{z} \in T$ means the copy of z and $O(\bar{z}) = \{v \in T : v \text{ is adjacent to } \bar{z}\})$.

THEOREM 7.10. [19]. *For any planar convex quasisimple graph* G *with* card$X \geq 4$ *different from complete graph* K_4, *there is a tree* T *and a graph* $R \in \mathcal{R}(T)$ *such that* $G = L(R,T_0)$.

THEOREM 7.11. [19]. *For any tree* T *with at least three vertices and for any graph* $R \in \mathcal{R}(T)$, *the graph* $L(R,T_0)$ *is convex quasisimple.*

THEOREM 7.12. [19]. *If a tree* T *has at most countable number of vertices, then for any graph* $R \in \mathcal{R}(T)$ *the graph* $L(R,T_0)$ *is* planar.

THEOREM 7.13. [19]. *Let S and T be some trees, $Q \in R(S)$ and $R \in R(T)$. The graphs $L(Q,S_0)$ and $L(R,T_0)$ are isomorphic if and only if S and T are isomorphic.*

8. Characterization of hypercubes and Hamming graphs by means of convexity. Let S be any set. The graph of *hypercube* $H(S)$ is defined as follows (see [28]): the vertex-set of $H(S)$ consists of all finite subsets in S (the empty set inclusively); two vertices A, B in $H(S)$ are adjacent if and only if the symmetric difference $(A \setminus B) \cup (B \setminus A)$ of the sets A, B is a one-point set.

Below we assume that any graph isomorphic to a graph of hypercube also is called a graph of hypercube. Observe, that for a finite set S with $\text{card}S = k$, $H(S)$ is the graph of k-dimensional cube.

It is not hard to prove that the function

$$d(A,B) = \text{card}[(A \setminus B) \cup (B \setminus A)]$$

is the induced metric on $H(S)$; i.e., $d(A,B)$ is equal to the number of edges in a shortest path in $H(S)$ with the ends A,B.

A graph G is called *median* if for any vertices $x, y, z \in X$ their "median" $[x,y] \cap [y,z] \cap [x,z]$ consists of a vertex.

THEOREM 8.1. [80]. *For a graph G the following conditions are equivalent:*

1) *G is a hypercube,*

2) *G contains no three-vertex convex set, and any two disjoint convex sets in G are separated by complementary half-spaces,*

3) *G contains no three-vertex convex set, and any two vertices in G are separated by complementary half-spaces,*

4) *G contains no three-vertex convex set and convexity in G satisfies cone condition,*

5) *G is a median graph and contains no three-vertex convex set.*

For proof of Theorem 8.1 we use the following lemmas.

LEMMA 8.2. [4]. *For a bipartite graph G the following conditions are equivalent:*

1) *G is a hypercube,*

2) *every interval [x,y] in G generates a hypercube.*

LEMMA 8.3. [54]. *Any median graph G is bipartite. Every interval [x,y] in a median graph G is a convex set.*

The relation between hypercubes and median graphs is shown in the following lemma.

LEMMA 8.4. [4]. *A graph G is a hypercube if and only if it is median and any two vertices in G have either two common adjacent vertices or have no common adjacent vertices.*

Let $\{S_\omega\}$, $\omega \in I$ be a family of pairwise disjoint sets. The *Hamming* graph H is defined as follows: the family of vertices in H consists of all finite subsets $A \subset \cup S_\omega$ such that card $(A \cap S_\omega) \leq 1$ for each $\omega \in I$; two distinct vertices A, B in H are adjacent if and only if the symmetric difference $(A \setminus B) \cup (B \setminus A)$ of the sets A, B is contained in one of the sets S_ω, $\omega \in I$. If $\{S_\omega\}$ is a finite family of finite sets, then H is the Cartesian product of complete graphs.

It is easily seen that for any vertices A, B in the Hamming graph H, the induced distance $d(A,B)$ looks as

$$d(A,B) = \sum_{\omega} \text{sign card}([(A \setminus B) \cup (B \setminus A)] \cap S_{\omega}).$$

Recall that for any vertex $x \in X$ and for any set $M \subset X$, the value $d(x,M) = \min\{d(x,u) : u \in M\}$ is called the distance from x to M, and the set

$$N_x(M) = \{z \in M : d(x,z) = d(x,M)\}$$

is named the *metric projection* of x on M. For $z \in M$, put

$$W_z(M) = \{x \in X : N_x(M) = \{z\}\}.$$

A set $M \subset X$ is named *Chebishev* provided $N_x(M)$ is a one-vertex set for every $x \in X$.

THEOREM 8.5. [80]. *A graph G is a Hamming graph if and only if the following conditions are fulfilled:*

1) every three-vertex set in G induced a complete subgraph,

2) every clique in G is a Chebishev set,

3) for every clique C in G and for every vertex $z \in C$, the set $W_z(C)$ is convex.

REFERENCES

1. B.D.Acharya, S.P.Rao Hebbare, M.N.Vartak, *Distance convex sets in graphs*, Proc. Symp. Optimization, Design of Experiments and Graph Theory. Indian Inst. of Technology, Bombay, 1986. 335-342.
2. H.-J.Bandelt, *Graphs with intrinsic S_3 convexities*, J.Graph Theory 13(1989), 215-228.
3. H.-J.Bandelt, J.P.Barthelemy, *Medians in median graphs*, Discrete Appl. Math. 8(1984), 131-142.
4. H.-J.Bandelt, H.M.Mulder, *Infinite median graphs*, (0,2)-graphs, and hypercubes, J.Graph Theory 7(1983), 487-497.
5. H.-J.Bandelt, H.M.Mulder, *Distance-heriditary graphs*, J.Comb.Theory B41(1986), 182-208.
6. H.-J.Bandelt, H.M.Mulder, *Regular Pseudo-Median Graphs*, J.Graph Theory 12(1988), 533-549.
7. H.-J.Bandelt, H.M.Mulder, *Three interval conditions for graphs*, ARS Combinatoria B29(1990), 213-223.
8. H.-J.Bandelt, H.M.Mulder, *Helly theorems for distmantable graphs and pseudo-modular graphs*. In: R.Bodendiek, R.Henn (Eds.) Topics in

Combinatorics and Graph Theory. Physica-Verlag, Heidelberg, 1990,65-71.

9. H.-J.Bandelt, E.Pesch, *A Radon theorem for Helly graphs*, Arch. Math. 52(1989), 95-98.
10. H.-J.Bandelt, E.Prisner, *Clique graphs and Helly graphs*, J.Comb. Theory B51(1991), 34-45.
11. L.M.Batten, *Geodesic subgraphs*, J.Graph Theory 7(1983), 159-163.
12. M.Bern, M.Klawe, A.Wong, *Bounds on the convex label number of trees*, Combinatorica 7(1987), 221-230.
13. V.G.Boltyanskii, P.S.Soltan. *Combinatorial geometry of various classes of convex sets*. Shtiintsa, Kishinev, 1978 (Russian).
14. G.Burosch, I.Havel, J.-M.Laborde, *Some intersection theorems and a characterization of hypercubes*. In: Graphs, Hypergraphs and Appl. Proc. Conf. Graph Theory. Leipzig, 1985, 23-26.
15. C.Caratheodory, *Über den Variabilitätsbereich der Koeffizienten von Potenzreihen*, die gegebene Werte nicht annehmen, Math. Ann. 64(1907), 95-115.
16. S.G.Cataranciuc, *d-Convex simple planar graphs*, In: Investigations in numerical methods and theoretical cybernetics. Shtiintsa, Kishinev, 1985, 68-75 (Russian).
17. S.G.Cataranciuc, *On properties of vertices of d-convex quasisimple planar graphs*, Kishinev State Univ. Shtiintsa, Kishinev, 1988. 23 pp. The paper is registered in MoldNIINTI 29.09.1988, No 1023-M88 (Russian).
18. S.G.Cataranciuc, V.D.Cepoi, *Construction and isomorphism of d-convexly simple planar graphs*, Mat. Issled. No 96(1987), 64-68 (Russian).
19. S.G.Cataranciuc, V.P.Soltan, *d-Convex simple and d-convex quasisimple planar graphs*, Kishinev State Univ. Shtiintsa, Kishinev, 1988. 23 pp. The paper is registered in MoldNIINTI 29.09.1989, No 1022-M88 (Russian).
20. V.D.Cepoi, *Two theorems on d-convex simple planar graphs*, In:Investigations in numerical methods and theoretical cybernetics. Stiintsa, Kishinev, 1985, 120-126 (Russian).
21. V.D.Cepoi, *Some properties of d-convexity in triangulated graphs*, Mat. Issled. No 87 (1986), 164-177 (Russian).
22. V.D.Cepoi, *Geometric properties of d-convexity in bipartite graphs*. In: Modelirov. inform. sistem. Stiintsa, Kishinev, 1986, 88-100 (Russian).
23. V.D.Cepoi, *Isometric subgraphs of Hamming graphs and d-convexity*, Kibernetika (Kiev) No 1(1988), 6-9 (Russian).
24. V.D.Cepoi, *d-Convexity and local conditions on graphs*, Issled. po prikl. matem. inform. Shtiintsa, Kishinev, 1990, 184-191 (Russian).
25. P.M.Dearing, R.L.Francis, T.J.Lowe, *Convex location problems on tree networks*, Oper. Res. 24(1976), 628-634.
26. J.M.Delire, *Graphs with high Radon numbers*, Bull. Cl. Sci. Acad. Roy. Belg. 70(1984), 14-24.
27. G.A.Dirac, *On rigid circuit graphs*, Abh. Math. Semin. Univ. Hamburg 25(1961), 71-76.
28. D.Z.Djokovic, *Distance-preserving subgraphs of hypercubes*, J.Combin. Theory B14 (1973), 263-267.
29. F.F.Dragan, Eccentricity, *Helly propery, and flowers in chord graphs*, Kishinev State Univ. Shtiintsa, Kishinev, 1987. 14 pp. The paper is registered in MoldNIINTI 27.11.1987, No 908-M87 (Russian).
30. F.F.Dragan, V.D.Cepoi, *Medians in quasimedian graphs*, Kishinev State Univ. Shtiintsa, Kishinev, 1988. 23 pp. The paper is registered in MoldNIINTI 25.02.1988, No 949-M88 (Russian).
31. P.Duchet, *Convexity in combinatorial structures*, Rend. Circ. Mat. Palermo 36, Suppl. 14(1986), 261-293.
32. P.Duchet, *Convex sets in graphs*, II. Minimal path convexity, J.Comb. Theory, B44(1988), 307-316.
33. Y.Egawa, *Characterization of the Cartesian product of complete graphs by convex subgraphs*, Discrete Math. 58(1986), 307-309.
34. M.G.Everett, S.B.Seidman, *The hull number of a graph*, Discrete Math. 57(1985), 217-223.
35. M.Farber, *Bridged graphs and geodesic convexity*, Discrete Math. 66(1987), 249-257.
36. M.Farber, *On diametres and radii of bridged graphs*, Discrete Math.

Discr. Math. 73(1989), 249-260.
37. M.Farber, R.E.Jamison, *On local convexity in graphs*, Discrete Math. 66(1987), 231-247.
38. M.Farber, R.E.Jamison, *Convexity in graphs and hypergraphs*, SIAM J. Algebraic Discrete Methods 7(1986), 433-444.
39. L.F.German, O.I.Topale, *Starshapedness*, the Radon number and Minty graphs, Kibernetika (Kiev) No 2(1987), 1-5 (Russian).
40. F.Harary, J.Nieminen, *Convexity in graphs*, J.Diff. Geom. 16(1981), 185-190.
41. S.P.R.Hebbare, *A class of distance convex simple graphs*, ARS Combinatoria 7(1979), 19-26.
42. S.P.R.Hebbare, *Another characterization and properties of planar distance convex simple graphs*, Proc. Symp. Optimization, Design of Experiments and Graph Theory. Indian Inst. of Technology, Bombay, 1986, 346-353.
43. E.Helly, *Über Mengen konvexer Körper mit gemeinschaftlichten punkten*, Jber. Deutsch. Math. Verein. 32(1932), 175-176.
44. W.A.Horn, *Three results for trees*, using mathematical induction, J.Res. Nat. Bur. Standards B76 (1972), 39-42.
45. E.Howorka, *On metric properties of certain clique graphs*, J.Comb. Theory B27(1979), 67-74.
46. E.Howorka, *A characterization of Ptolematic graphs*, J.Graph Theory 5(1981), 323-331.
47. R.E.Jamison, *A perspective on abstract convexity: classifying aligments by varietes.* In: Proc. conf. Convexity and related combinatorial geometry. Lect. Notes Pure Appl. Math. 76(1982), 113-150.
48. R.E.Jamison, *Convexity and block graphs*, Congressus Numeratium 33(1981), 129-142.
49. D.C.Kay, G.Chartrand, *A characterization od certain Ptolematic graphs*, Canad. J.Math. 17(1965), 342-346.
50. M.G.Krein, D.P.Mil'man, *On extreme points of regularly convex sets*, Studia Math. 9(1940), 133-138.
51. G.C.Lekkerkerker, J.C.Boland, *Representation of a finite graph by a set of intervals on the real line*, Fund. Math. 51(1962), 45-64.
52. K.Menger, *Metrische Untersuchungen*, Ergebnisse eines math. Kolloq. Wien. 1(1931), 20-27.
53. H.M.Mulder, *The structure of median graphs*, Discrete Math. 24(1978), 197-204.
54. H.M.Mulder, *The interval function on a graph*. Math. Centre Tracts. No 132, 1980.
55. J..Nieminen, *Distance center and centroid of a median graph*, J.Franklin Inst. 323(1987), 89-94.
56. J.Nieminen, *The center and the distance of a Ptolematic graph*, Oper. Res. Lett. 7(1988), 91-94.
57. C.F.Prisacaru, A.V.Prisacaru, *On minimal covering of graph vertices by d-convex sets*, Mat. Issled. No 96(1987), 114-118 (Russian).
58. C.F.Prisacaru, V.P.Soltan, *On d-convexity of Lame functions on graphs*, Mat. Issled. No 66(1982), 136-140.
59. J.Radon, *Mengen konvexen Körper, die einen gemeinsamen punkt enhalten*, Math. Ann. 83(1921), 113-115.
60. S.B.Rao, S.P.R.Hebbare, *Characterization of planar distance convex simple graphs*, Proc. Symp. Graph Theory. ISI Calcutta, 1976. pp. 138-150.
61. I.V.Sergienko, T.T.Lebedeva, V.A.Roschin, *Approximation methods for solving discrete optimization problems*. Naukova Dumka, Kiev, 1980.
62. I.V.Sergienko, *Mathematical models and methods for solving discret optimization problems*. Naukova Dumka, Kiev, 1988.
63. A.I.Sochirca, *Separability of the vertices of a triangulated graph by complementary d-convex halfspaces*, Mat. Issled. No 100(1988), 104-114 (Russian).
64. A.I.Sochirca, V.P.Soltan, *Joins and penumbras of d-convex sets in triangulated graphs*, Trans. Inst. Math. Tbilisi 85(1987), 40-51 (Russian).
65. A.I.Sochirca, V.P.Soltan, *d-Convex functions on graphs*, Mat. Issled. No

110(1988), 93-106 (Russian).
66. P.S.Soltan, *A joint solution of some Steiner's problems on graphs*, Dokl. Akad. Nauk SSSR 202(1972), 294-297 (Russian).
67. P.S.Soltan, V.D.Cepoi, *Solution of Weber's problem for discrete median metric spaces*, Trans. Inst. Math. Tbilisi 85(1987), 52-76 (Russian).
68. P.S.Soltan, C.F.Prisacaru, *Steiner's problem on graphs*, Dokl. Akad. Nauk SSSR 198(1971), 46-49 (Russian).
69. P.S.Soltan, D.C.Zambitschi, C.F.Prisacaru. *Extremal problems on graphs and algorithms of their solution*. Stiintsa, Kishinev, 1973 (Russian).
70. P.S.Soltan, V.P.Soltan, *d-Convex functions*, Dokl. Akad. Nauk SSSR 249(1979), 555-558 (Russian).
71. V.P.Soltan, *Some properties of d-convex functions*, I, II, Bull. Acad. Sci. Moldova. Ser. Phis.-Techn. Math. Sci. No 2(1980), 27-31; No 1(1981), 21-26 (Russian).
72. V.P.Soltan, *d-Convexity in graphs*, Dokl. Akad. Nauk SSSR 272(1983), 535-537 (Russian).
73. V.P.Soltan, *Introduction to the axiomatic convexity theory*. Shtiintsa, Kishinev, 1984 (Russian).
74. V.P.Soltan, *Simplicial vertices and an analogue of Krein-Mil'man's theorem for graphs*, Metody Diskret. Analiz. No 48(1989), 73-84 (Russian).
75. V.P.Soltan, V.D.Cepoi, *Some classes of d-convex functions in graphs*, Dokl. Akad. Nauk SSSR 273(1983), 1314-1317 (Russian).
76. V.P.Soltan, V.D.Cepoi, *Conditions for invariance of diameters under d-convexitation in a graph*, Kibernetika (Kiev) No 6(1983), 14-18 (Russian).
77. V.P.Soltan, V.D.Cepoi, *d-Convex sets in triangulated graphs*, Mat. Issled. No 78(1984), 105-124 (Russian).
78. V.P.Soltan, V.D.Cepoi, *The number of d-convex sets in a graph*, Bull. Acad. Sci. Moldavian SSR. Ser. Phis.-Techn. Math. Sci. No 2(1984), 19-24 (Russian).
79. V.P.Soltan, V.D.Cepoi, *d-Convexity and Steiner functions on a graph*, Dokl. Akad. Nauk Belorussian SSR 29(1985), 407-408 (Russian).
80. V.P.Soltan, V.D.Cepoi, *Characterization of hypercubes and Hamming graphs by means of d-convexity*, Metody Diskret. Analiz. No 45(1987), 77-93 (Russian).
81. S.Straszewicz, *Über exponierte Punkte abgeschlossener Punktmengen*, Fund. Math. 24(1935), 139-143.
82. B.C.Tansel, R.L.Francis, T.J.Lowe, *Location on networks: a survey*, Manag. Sci. 29(1983), 482-511.
83. O.I.Topale, *Starshapedness in graphs*, Mat. Issled. No 78(1984), 130-133 (Russian).
84. M.Van de Vel, *Matching binary convexities*, Topol. and Appl. 16(1983), 207-235.
85. M.Van de Vel, *Abstract, topological, and uniform convex structures*, Vrije Universiteit, Amsterdam. Rapportnr. WS-353, 1989.
86. C.P.Vanden, *A characterization of the n-cube by convex subgraphs*, Discrete Math. 41(1982), 109-110.
87. C.P.Vanden, *A convexity problem in 3-polytopial graphs*, Arch. Math. 43(1984), 84-88.
88. C.P.Vanden, *A convex characterization of the graphs of the dodecahedron and icosahedron*, Discrete Math. 50(1984), 99-105.
89. S.V.Yushmanov, *On metric properties of chord and Ptolematic graphs*, Dokl. Akad. Nauk SSSR 300(1988), 296-299 (Russian).
90. S.V.Yushmanov, *On median of Ptolematic graph*, Issled. operatsii i ASU. Kiev, No 32(1988), 67-70 (Russian).
91. S.V.Yushmanov, *A general method for estimating metric characteristics of a graph that are associated with the eccentricity*, Dokl. Akad. Nauk SSSR 306(1989), 52-54 (Russian).

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MEANS AND CONVEXITY

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REZUMAT. - Medii şi convexitate. În lucrare se consideră o noţiune de convexitate în raport cu o medie de puteri, numită r-convexitate. Se generalizează inegalitatea lui Hermite-Hadamard pentru funcţii cu inversă r-convexă aşa cum în [3] s-a procedat pentru funcţii cu inversă logaritmic convexă.

1. Introduction. In this paper we consider a notion of convexity with respect to a power mean called r-convexity. We generalize Hermite-Hadamard's inequality for functions with r-convex inverse. Then we apply it for the study of the monotony of the "relative growth" of generalized logarithmic means. We try to analyse so the position of the mean values of two numbers between those numbers.

As moust of the definitions and results which we need may be found in the book of P. S. Bullen, D. S. Mitrinović and P. M. Vasić [1] we content ourself to refer mainly at it.

2. Means. We shall use in what follows some means of two positive numbers $0<a<b$. They all belong to the familly of extended mean values defined by K. B. Stolarsky (see [1], p.345) for $r\neq s$, $rs\neq 0$ by:

$$E_{rs}(a,b)=((r/s)(b^s-a^s)/(b^r-a^r))^{1/(s-r)}$$

the definition for other values being obtained by taking limits. As special cases we have the power means:

$$P_r=E_{r,2r} \quad \text{for } r\neq 0$$

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$$P_r=E_{r,2r} \quad \text{for } r\neq 0$$

and

$$P_0(a,b)=G(a,b)=(a\cdot b)^{1/2}$$

then the generalized logarithmic means defined by:

$$L_r=E_{1,r+1}, \text{ for } r\neq -1,\ r\neq 0$$

but

$$L_{-1}(a,b)=L(a,b)=(b-a)/(\log b-\log a)$$

and

$$L_0(a,b)=I(a,b)=(1/e)(b^b/a^a)^{1/(b-a)}.$$

Also we use weighted power means defined for $0\le t\le 1$ by:

$$P_{rt}(a,b)=(ta^r+(1-t)b^r)^{1/r} \text{ if } r\neq 0$$

and

$$P_{0t}(a,b)=G_t(a,b)=a^t b^{1-t}.$$

For $t=1/2$ we get the usual power means and for $r=1$ the weighted arithmetic mean $P_{rt}=A_t$.

Among the properties of these means we are interested in their monotony with respect to the parameter. So we have (see [1], p.159) for $r<s$:

$$P_{rt}(a,b)<P_{st}(a,b),\ 0<t<1 \tag{1}$$

and also (see [1] p. 347):

$$L_r(a,b)<L_s(a,b). \tag{2}$$

3. **r-Convexity**. Let us consider the following notion: we said that the positive function $f:[a,b]\to\mathbb{R}$ is r-convex if:

$$f(A_t(x,y))\le P_{rt}(f(x),f(y)),\ \forall x,y\in[a,b],\ t\in[0,1].$$

As we can remark, this notion differs from a similar one given in [1] called r-mean convexity.

From (1) we deduce that if f is r-convex then it is also s-convex for every $s>r$. Also from the definition we deduce that f is r-convex if and only if: a) f^r is convex, for $r>0$; b) $\log f$ is convex, for $r=0$ and c) f^r is concave for $r<0$. Thus 0-convexity is in fact logarithmic convexity.

The paper [3] deals with functions which have logarithmic convex inverse. We consider also functions with r-convex inverse. Let us denote by $K_r^-[a,b]$ the set of positive, strictly increasing functions with r-convex inverse defined on [a,b]. We have:

$$K_r^-[a,b] \subset K_s^-[a,b] \ , \quad \text{for} \quad r < s \ . \tag{3}$$

It is also easy to check the following:

LEMMA 1. *If the positive function f is twice differentiable then it belongs to* $K_r^-[a,b]$ *if and only if:*

$$f'(x)>0 \text{ and } 1+xf''(x)/f'(x)\le r, \ \forall \ x\in[a,b] \tag{4}$$

Integrating the differential equation obtained from (4) we get functions which can be considered to be r-linear. As a special case we have:

LEMMA 2. *The function* f_r *defined by:*

$$f_r(x) = \begin{cases} x^r - a^r , & r > 0 \\ \log x - \log a , & r = 0 \\ a^r - x^r , & r < 0 \end{cases} \tag{5}$$

has the properties:

$$f_r(x)\ge 0 \ , \ f_r'(x)>0 \ , \ 1+xf_r''(x)/f_r'(x) = r \ , \ \forall \ x\ge a \ .$$

4. Hermite-Hadamard's inequality. For a function $f:[a,b]\to\mathbb{R}$ consider the integral arithmetic mean defined by:

$$A(f;a,b) = \int_a^b f(x)\,dx/(b-a) \ .$$

Hermite-Hadamard's inequality (see[1], p.30) gives for a concave function f the evaluation:

$$(f(a)+f(b))/2 \le A(f;a,b) \le f((a+b)/2). \tag{6}$$

Also H.-J.S e i f f e r t proved in [3] that for a function f from $K_0^-[a,b]$ holds:

$$A(f;a,b) \le f(I(a,b)). \tag{7}$$

We remark that from (2) it follows:

$$I(a,b)=L_0(a,b)<L_1(a,b)=(a+b)/2$$

thus (7) improves the right side of (6) for this special case. We can do the same thing for functions from $K_r^-[a,b]$ with $r\neq 0$.

In the proof of the relation (7) it is used the following result, proposed as a problem by R. E u l e r in [2]:

$$\lim_{n\to\infty}\left(\prod_{i=1}^{n}(c+(i-1)/n)\right)^{1/n} = I(c,c+1) \ , \ \forall\ c>0 \ . \tag{8}$$

The expression from the first member of (8) is a geometric mean (of n numbers). We can prove a similar relation to (8) for an arbitrary power mean.

LEMMA 3. *If $r\neq 0$ and $c>0$ then:*

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{\left(c+\frac{i-1}{n}\right)^{r}}{n}\right)^{1/r} = L_r(c,c+1) \ . \tag{9}$$

Proof. If $r>0$, the mean value theorem of the differential calculus applied to the function $f(x)=(x+1)^{r+1}$, $x>0$, gives:

$$((x+1)^{r+1}-x^{r+1})/(r+1)<(x+1)^r<((x+2)^{r+1}-(x+1)^{r+1})/(r+1). \tag{10}$$

For $n>1/c$, we get by addition:

$$L_r\left(c-\frac{1}{n}, c+1-\frac{1}{n}\right) < \left(\sum_{i=1}^{n}\left(c+\frac{\left(\frac{i-1}{n}\right)^r}{n}\right)\right)^{\frac{1}{r}} < L_r(c, c+1)$$

hence (9). For $r<0$, $r\neq-1$, we have to do minor changes in the proof, while for $r=-1$ we must replace (10) by:

$$\log(x+2)-\log(x+1)<(x+1)^{-1}<\log(x+1)-\log x.$$

Finally we remark that the case $r=0$, excepted from (9), is contained in (8).

Replacing (8) by (9) in the proof of (7) given in [3] we get:

THEOREM 1. *If the function f belongs to* $K_r^-[a,b]$ *then:*

$$A(f;a,b)\leq f(L_r(a,b)). \tag{11}$$

Let us remark that the function f_r *defined by (5) verifies:*

$$A(f_r;a,b)=f_r(L_r(a,b)). \tag{12}$$

We can improve also the left inequality from (6) for the same class of functions.

THEOREM 2. *If the function f belongs to* $K_r^-[a,b]$ *then:*

$$A(f;a,b)\geq \geq(f(a)(b^r-L_r^r(a,b))+f(b)(L_r^r(a,b)-a^r))/(b^r-a^r), \tag{13}$$

if $r\neq0$ *and*

$$A(f;a,b)\geq(f(a)(L(a,b)-a)+f(b)(b-L(a,b)))/(b-a) \tag{14}$$

if $r=0$.

Proof. For $t\in[a,b]$ we have:

$$f(t)=\frac{f(b)-f(t)}{f(b)-f(a)}f(a)+\frac{f(t)-f(a)}{f(b)-f(a)}f(b) \ . \tag{15}$$

So, if $r>0$, $(f^{-1})^r$ being convex:

$$t^r \le \frac{f(b)-f(t)}{f(b)-f(a)}a^r+\frac{f(t)-f(a)}{f(b)-f(a)}b^r$$

or

$$f(t) \ge \frac{f(b)-f(a)}{b^r-a^r}t^r+\frac{b^r f(a)-a^r f(b)}{b^r-a^r} \ .$$

It is also valid for $r<0$. By integration we get (13). For $r=0$; $\log(f^{-1})$ is convex and (15) gives:

$$\log t \le \frac{f(b)-f(t)}{f(b)-f(a)}\log a+\frac{f(t)-f(a)}{f(b)-f(a)}\log b \ .$$

Isolating $f(t)$ and integrating we get (14).

5. **The relative growth.** We consider the following expression:

$$D_r(a,b)=\begin{cases}\dfrac{L_r^r(a,b)-a^r}{b^r-a^r} \ , & r\neq 0\\[2mm] \dfrac{b-L(a,b)}{b-a} \ , & r=0 \ .\end{cases}$$

which we call relative growth of L_r. It is easy to see that:

$0\le D_r(a,b)\le 1, \ \forall r; \ D_1(a,b)=1/2.$

THEOREM 3. *If $r<s$ and $0<a<b$ then:*

$$D_r(a,b)\ge D_s(a,b). \tag{16}$$

Proof. As the function f_r given by (5) belongs to $K_r^-[a,b]$ and $r<s$, from (3) it follows that it is also in $K_s^-[a,b]$ and so (12), (13) and (14) implies:

$A(f_r;a,b)=f_r(L_r(a,b))\ge f_r(b)D_s(a,b)$

which gives (16). In fact we must consider separately the cases: $0<r<s$, $0=r<s$, $r<s=0$ and $r<s<0$.

Remark 1. From (2) it follows that the evaluation given by (11) is improved by decreasing the value of the parameter r. The same conclusion is valid for (13) and (14) if we take into account (16). On the other hand, from (4) we deduce that for a strictly increasing and continuously twice differentiable function, there is a sufficiently large r for which (11) and (13) be valid.

Remark 2. An inequality similar to (16) for power means was proved by A.J.Goldman (see [1], p.203). On the other hand we remark that (16) contains many inequalities between means. For example, for $r>1$ it is equivalent with $L_r(a,b)\geq P_r(a,b)$ and for $0<r<1$ it gives $L_r(a,b)\leq P_r(a,b)$. For $r<0<s$ we get:

$$E_{r,r+1}(a,b)\leq L(a,b)\leq E_{s,s+1}(a,b).$$

All these relations may be found in [1]. We also have:

$L_{r,r+1}(a,b)L(a,b)\leq G^2(a,b)$, for $r<-1$ but the converse inequality for $-1<r<0$.

Remark 3. From $0\leq D_r(a,b)\leq 1$ we deduce that it may be preferable to use instead D_r the differences $D_r-1/2$, that is:

$$\frac{L_r^r(a,b)-P_r^r(a,b)}{b^r-a^r}\ \text{for } r\neq 0\ ;\ \frac{A(a,b)-L(a,b)}{b-a}\ ,\ r=0$$

where $A=A_{1/2}$. These are between $-1/2$ and $1/2$ and are decreasing upon r, as D_r is.

REFERENCES

1. Bullen, P.S., Mitrinović, D.S., Vasić, P.M., *Means and their Inequalities*, D.Reidel Publ. Comp., Dordrecht, 1988.
2. Euler, R., *Problem 1178*, Math. Mag. 56(1983), 326.
3. Seiffert, H.-J., *Eine Integralgleichung für streng monotone Funktionen mit logaritmisch konkaver Umkehrfunktion*, El. Math. 44(1989), 16-18.

THÉORÈMES DE POINT FIXE DANS LES ESPACES AVEC MÉTRIQUE VECTORIELLE

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REZUMAT. - Teoreme de punct fix în spaţii cu metrică vectorială. În această lucrare se stabilesc trei teoreme de punct fix în spaţii cu metrică vectorială analoge teoremelor de punct fix pe spaţii metrice demonstrate în lucrarea [3].

1. Notions préliminaires

DEFINITION 1.1. Soit X un ensemble ordonné. Une suite $\{x_n\}_{n\in N}$ d'élèments de X *(o)-converge* vers un élèment $x\in X$ s'il existe deux suites $\{a_n\}_{n\in N}$ et $\{b_n\}_{n\in N}$ d'élèments de X, telles que $a_n\le x_n\le b_n$ $(\forall)$ $n\in N$ et $a_n\uparrow x$, $b_n\downarrow x$.

Nous désignons par $x = (0) - \lim x_n$ ou $x_n \xrightarrow{o} x$.

DEFINITION 1.2. Un ensemble ordonné X s'appele *ensemble réticulé* si pour tous $x,y\in X$ (donc aussi pour tout nombre fini d'élèments) il existe $x\vee y$ et $x\wedge y$.

DEFINITION 1.3. Un ensemble réticulé X s'appelle *ensemble réticulé relativement complet* si pour tout sous-ensemble denombrable borné de X il existe la borne supérieure et la borne inférieure.

DEFINITION 1.4. On appelle *espace linéaire complétement réticulé* tout espace linéaire ordonné qui est un ensemble réticulé relativement complet.

DEFINITION 1.5. Un espace linéaire ordonné X est appellé *espace archimédien* si $\bigwedge_{n\in N} \frac{1}{n}x = 0$ pour tout $x>0$, $x\in X$.

DEFINITION 1.6. Dans un espace linéaire réticulé archimédien

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X une suite $\{x_n\}_{n\in N}$ d'élèments de X *ρ-converge* (ou converge avec régulateur) vers un élèment x, s'il existe $v>0$ (appellé régulateur de convergeance) tel que: pour tout nombre $\varepsilon>0$ il existe $n_\varepsilon \in N$ de maniére que:

$$|x_n - x| \le \varepsilon v \quad \text{si} \quad n \ge n_\varepsilon$$

On note $x = (\rho) - \lim_n x_n$ (ou $x_n \xrightarrow{\rho} x$)

Si $x = (\rho) - \lim_n x_n$ alors $x = (0) - \lim_n x_n$

DEFINITION 1.7. On appelle *espace régulier* tout espace linéaire réticulé archimedien tel que: tout suite (o)-convergente est (ρ)-convergeante.

2. **Définitions et notations.** Soit X un espace linéaire complétement réticulé et $Z\neq\emptyset$ un ensemble. On définit une métrique vectorielle d: $Z\times Z\to X$ et pour $A\subset Z$ on note le diamètre de A par $\delta(A) = \sup\{d(z_1,z_2)/z_1,z_2\in A\}$.

DEFINITION 2.1. On dit que l'ensemble $B\subset Z$ est *d-fermé* si tout $\{z_n\}_{n\in N}$, $z_n\in B$, $z_n \xrightarrow{d} z$ implique $z\in B$.

$$\left(\,!!\; z_n \xrightarrow{d} z \;\rightarrow\; d(z_n, z) \xrightarrow{\rho} 0 \right)$$

DEFINITION 2.2. Soit $B\subset Z$. Définissons par:

$$\overline{B} = \{ z\in Z \,/\, z = d\text{-}\lim_n z_n \,,\; z_n\in B \}$$

LEMME 2.1. *Si Z est d-complet la suite $\{B_n\}\downarrow$, $B_n\subset Z$, B_n d-fermé et $\delta(B_n)\downarrow 0$ alors il existe $z_0\in B$ unique tel que*

$$\bigcap_{n\in N} B_n = \{z_0\} \,.$$

DEFINITION 2.3. Soit l'ensemble $Z\neq\emptyset$ d-complet. Une application f: $Z\to Z$ s'appelle *application de Picard* s'il existe $z^*\in Z$ telle que $\mathrm{Fix}(f) = \{z^*\}$ et la suite $\{f^n(z_0)\}_{n\in N}$ d-converge vers z^* pour tout $z_0\in Z$.

DEFINITION 2.4. Soit $Z\neq\emptyset$ un certain ensemble. Une application $f{:}Z\to Z$ est *une application de Janos* si

$$\bigcap_{n\in N} f^n(Z) = \{z^*\} \quad \text{où} \quad \{z^*\} = \text{Fix}(f)$$

DEFINITION 2.5. Soit $Z\neq\emptyset$ un certain ensemble, $f, f_n: Z\to Z$, $n\in N$. La suite est *assimptotiquement uniformément convergente* (designons par $f_n \overset{a}{\rightrightarrows} f$) s'il existe $v>0$, $v\in X$ tel que pour tout $\varepsilon>0$ il existe $n_0(\varepsilon)$, $m_0(\varepsilon)\in N$ tels que $d(f_n^m(z), f^m(z)) < \varepsilon v$ pour tout $n>n_0$, $m>m_0$ et $z\in Z$.

3. Théorèmes de point fixe dans les espaces avec métrique vectorielle.

THÉORÈME 3.1. *Soit X un espace linéaire complétement réticulé, $Z\neq\emptyset$ d-complet, $f: Z\to Z$ et $\varphi: X_+\to X_+$. Nous supposons que:*

(i) φ est $\downarrow$ et $\varphi^n(t) \xrightarrow{0} 0$ pour $t > 0$ et $n\to\infty$

(c'est - à - dire: φ est function de comparaison)

(ii) *$\delta(f(A))\le\varphi(\delta(A))$ pour tout $A\subset Z$ tel que $f(A)\subset A$*

(c'est - à - dire: f est (δ,φ) - contraction generalisée)

Alors:

(a) f est application de Picard

(b) f est application de Janos

Démonstration a) Soit $A_1 = \overline{f(Z)}$, $A_2 = \overline{f(A_1)}, \ldots, A_{n+1} = \overline{f(A_n)}$ Alors nous avons: $A_{n+1}\subset A_n$, $A_n=\bar{A}_n$ et $f(A_n)\subset A_n$ pour tout $n\in N$. D'autre part:

$$\delta(A_{n+1}) = \delta(\overline{f(A_n)}) = \delta(f(A_n)) \le \varphi(\delta(A_n)) \le \varphi^2(\delta(A_{n-1})) \le \ldots \le \varphi^n(\delta(Z)) \xrightarrow{0} 0$$

Donc $\delta(A_{n+1})\downarrow 0$. Alors d'après le LEMME 2.1 il existe $z^*\in Z$ unique tel que $\bigcap A_n = \{z^*\}$ et $f\left(\bigcap_{n=1} A_n\right) \subset \bigcap_{n=1} A_n$, donc $\text{Fix}(f)=\{z^*\}$. Soit $z_0\in Z$ et $B_n = \{f_2^n(z_0), f^{n+1}(z_0), \ldots, z^*\}$. Comme $f(B_n) = \{f^{n+1}(z_0), f^{n+2}(z_0), \ldots, z^*\} = B_{n+1}\subset B_n$ et $\delta(B_n) = \delta(f(B_n)) \le \varphi(\delta(B_n))$ il résulte que $\delta(B_n)\downarrow 0$ pour $n\to\infty$, c'est - à - dire $f^n(z_0)\to z^*$, $n\to\infty$

b) $z^*\in\bigcap_{n\in N} f^n(Z) \subset \bigcap_{n=1} A_n = \{z^*\}$ et donc $\bigcap_{n\in N} f^n(Z) = \{z^*\}$ q.e.d.

THÉORÈME 3.2. *Soit X un espace linéaire complétement réticulé $Z\neq\emptyset$ d - complet, $f: Z\to Z$ une application ayant la proprieté suivante: il existe $n_k\in N^*$ tel que f^{n_k} soit une (δ,φ) contraction generalisée.*

Alors:

a) f est une application de Picard

b) f est une application de Janos

Démonstration (a) + (b). Dans le théorème 3.1. nous avons Fix $(f^{n_k}) = \{z^*\}$ et $\delta(f^{n_k}(Z)) \downarrow 0$ pour $k\to\infty$ D'autre part:

$$Z \supset f(Z) \supset f^2(Z) \supset \ldots \supset f^{n_k}(Z) \supset \ldots \quad \text{donc} \quad \bigcap_{n=1}^{\infty} f^{n_k}(Z) = \{z^*\} \qquad \text{q.e.d.}$$

THEOREM 3.3. *Soit X un espace linéaire complétement réticulé et régulier, $Z \neq \emptyset$ d - complet et f, f_n: $Z\to Z$ $n\in$N. Supposons que:*

i) f est une application de Picard (On note Fix (f) = $\{z^*\}$)

ii) $f_n \xrightarrow{a} f$

iii) Fix $(f_n) \neq \emptyset$ *pour tout* $n\in N$ *(On note Fix* (f_n) = $\{z_n^*\}$)

Alors: $z_n^* \xrightarrow{d} z^*$

Démonstration. Nous avons:

$$d(z_n^*, z^*) = d(f_n(z_n^*), z^*) = d(f_n^m(z_n^*), z^*) \leq$$
$$\leq d(f_n^m(z^*), f^m(z_n^*)) + d(f^m(z_n^*), z^*)$$

D'après (ii) il en résulte qu'il existe $v>0$, $v\in X$ tell que pour tout $\varepsilon>0$ ils existent $n_0(\varepsilon)$, $m_0(\varepsilon)\in$N:

$$d(f_n^m(z_n^*), f^m(z_n^*)) < \frac{\varepsilon}{2}v \quad \text{quel que soit } n>n_0(\varepsilon);\ m>m_0(\varepsilon).$$

D'après (i) il en résulte que pour tout $n\in$N nous avons:

$$d(f^m(z_n^*), z^*) \xrightarrow{o} 0, \quad \text{pour } m\to\infty.$$

L'espace *X* etant régulier il existe un régulateur de convergeance $w\geq v$ tel que: $(\forall)\varepsilon>0$ il existe $m(\varepsilon,n)\geq m_0(\varepsilon)$ tel que:

$$d(f^m(z_n^*), z^*) \leq \frac{\varepsilon}{2}w \quad (\forall)\ m\geq m(\varepsilon,n)$$

Donc on obtient:

$$d(z_n^*, z^*) \leq \frac{\varepsilon}{2}v + \frac{\varepsilon}{2}v \leq \varepsilon w \quad \text{pour tout } n\geq n_0(\varepsilon)$$

Donc: $z_n^* \xrightarrow{d} z^*$

q.e.d.

B I B L I O G R A P H I E

1. Amman H., *Order structures and fixed points*, Math. Inst. Ruhr.-Universitate D-4630 Bochner Germany (1977).
2. Cristescu R., *Ordered vector spaces and Linear operators*, Ed. Acad. Abacus Press, Kent (1976).
3. Rus I.A., *Tehnique of the fixed point structures*, Sem. on fixed point theory. Preprènt Nr.3 (1987) Cluj-Napoca.
4. Voicu Florica, *Applications contractives dans les espaces ordonnés*, Seminar en Differential Equations Preprènt Nr. 3 (1989) Cluj-Napoca.

A NEW DOUBLE FOURIER EXPONENTIAL-LAGUERRE SERIES FOR FOX'S H-FUNCTION

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REZUMAT. - O nouă serie Fourier exponenţial-Laguerre pentru H-funcţiile lui Fox. În această notă este prezentată o nouă serie Fourier exponenţial-Laguerre pentru H-funcţiile lui Fox, în două variabile.

1. Introduction. The object of this paper is to introduce a new class of double Fourier Exponential-Laguerre series for Fox's H-function [4] and present one double Fourier series of this class.

In what follows for sake brevity:

$$\sum_{j=1}^{p} e_j-\sum_{j=1}^{q} f_j=A,\quad \sum_{j=1}^{n} e_j-\sum_{j=n+1}^{p} e_j+\sum_{j=1}^{m} f_j-\sum_{j=n+1}^{q} f_j=B.$$

The following formulas are required in the proof:

The integral [1,p.704,(2.2)]:

$$\int_0^{\pi}\cos 2ux\left(\sin\frac{x}{2}\right)^{-2w_1} H_{p,q}^{m,n}\left[z\left(\sin\frac{x}{2}\right)^{-2h}\,\middle|\,\begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right]dx=$$

$$=\sqrt{\pi}\,H_{p+2,q+2}^{m+1,n+1}\left[z\,\middle|\,\begin{matrix}(1-w_1-2u,h),(a_p,e_p),(1-w_1+2u,h)\\(\frac{1}{2}-w_1,h),(b_q,f_q),(1-w_1,h)\end{matrix}\right],\qquad(1.1)$$

where $h>0$, $A\le 0$, $B>0$, $|\arg z|< 1/2B\pi$, $\mathrm{Re}[1-2w_1+2h(1-a_j)/e_j]>0$, $(j=1,\ldots,n)$.

The integral [2,p.711,(2.3)]:

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$$\int_0^\infty y^{w_2+a}e^{-y}L_v^a(y)\,H_{p,q}^{m,n}\left[zy^k \,\middle|\, \begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right]dy=$$

$$=\frac{(-1)^v}{v!}H_{p+2,q+1}^{m,n+2}\left[z\,\middle|\,\begin{matrix}(-w_2-a,k),(-w_2,k),(a_p,e_p)\\(b_q,f_q),(v-w_2,k),\end{matrix}\right], \tag{1.2}$$

where $h>0$, $A\leq 0$, $B>0$, $|\arg z|<1/2B\pi$, $\mathrm{Re}[w_2+a+kb_j]>-1$ $(j=1,2,\ldots,m)$, $\mathrm{Re}a>-1$.

The orthogonality property of Laguerre polynomials [3,p.292-293, (2) & (3)]:

$$\int_0^\infty x^a e^{-x}L_m^a(x)\,L_n^a(x)\,dx=\begin{cases}0, & m\neq n,\ \mathrm{Re}a>-1;\\ \dfrac{\Gamma(a+n+1)}{n!}, & m=n,\ \mathrm{Re}a>0.\end{cases} \tag{1.3}$$

The following orthogonality property:

$$\int_0^\pi e^{2imx}\cos 2nx\,dx=\begin{cases}0, & m\neq n\\ \dfrac{\pi}{2}, & m=n\neq 0\\ \pi, & m=n=0.\end{cases} \tag{1.4}$$

2. Double Fourier Exponential-Laguerre series. The double Fourier Exponential-Laguerre series to be established is

$$(\sin\frac{x}{2})^{-2w_1}y^{w_2}H_{p,q}^{m,n}\left[z(\sin\frac{x}{2})^{-2h}y^k\,\middle|\,\begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right]=$$

$$=\frac{2}{\sqrt{(\pi)}}\sum_{r=-\infty}^{\infty}\sum_{t=0}^{\infty}\frac{(-1)^t}{\Gamma(a+t+1)}e^{2irx}L_t^a(y)\ . \tag{2.1}$$

$$x\,H_{p+4,q+3}^{m+1,n+3}\left[z\,\middle|\,\begin{matrix}(1-w_1-2r,h),(-w_2-a,k),(-w_2,k),(a_p,e_p),(1-w_1+2r,h)\\(\frac{1}{2}-W_1,h),(b_q,f_q),(1-w_1,h),(t-w_2,k)\end{matrix}\right],$$

valid under the conditions of (1.1), (1.2), and (1.3).

Proof. To establish (2.1), let

$$f(x,y)=(\sin\frac{x}{2})^{-2w_1}y^{w_2}H_{p,q}^{m,n}\left[z(\sin\frac{x}{2})^{-2h}y^{k}\left|\begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right.\right]=$$

$$=\sum_{r=-\infty}^{\infty}\sum_{t=0}^{\infty}A_{r,t}\,e^{2irx}L_t^{a}(y)\ . \tag{2.2}$$

Equation (2.2) is valid, since f(x,y) is continuous and of bounded variation in the region $0 < x < \pi$, $0 < y < \infty$.

Multiplying both sides of (2.2) by $y^{a}e^{-y}L_v^{a}(y)$, integrating with respect to y from 0 to ∞, and using (1.2) and (1.3), we obtain

$$(-1)^{v}(\sin\frac{x}{2})^{-2w_1}H_{p+2,q+2}^{m,n+2}\left[z(\sin\frac{x}{2})^{-2h}\left|\begin{matrix}(-w_2-a,k),(w_2,k),(a_p,e_p)\\(b_q,f_q),(v-w_2,k)\end{matrix}\right.\right]=$$

$$=\sum_{r=-\infty}^{\infty}A_{r,v}\Gamma(a+v+1)\,e^{2irx}\ . \tag{2.3}$$

Multiplying both sides of (2.2) by $\cos 2ux$, integrating with respect to x from 0 to π and using (1.1) and (1.4), we get

$$A_{u,v}=\frac{2(-1)^{v}}{\sqrt{(\pi)}\,\Gamma(a+v+1)} \tag{2.4}$$

$$\times H_{p+4,q+3}^{m+1,n+3}\left[z\left|\begin{matrix}(1-w_1-2u,h),(-w_2-a,k),(-w_2,k),(a_p,e_p),(1-w_1+2u,h)\\(\frac{1}{2}-w_1,h),(b_q,f_q),(1-w_1,h),(v-w_2,k)\end{matrix}\right.\right],$$

except that $A_{0,v}$ is one-half of the above value.

From (2.2) and (2.4), the formula (2.1) is obtained.

Since on specializing the parameters the H-function yields almost all special functions appearing in applied mathematics and physical sciences. Therefore, the result presented in this paper is of a general character and hence may encompass several casses of interest.

REFERENCES

1. Bajpai,S.D. *Fourier series of generalized hypergeometric functions*. Proc. Camb. Phil. Soc. 65(1969), 703-707.
2. Bajpai,S.D. *An integral involving Fox's H-function and Whittaker functions*. Proc. Camb. Soc. 65(1969), 709-712.
3. Erdélyi,A. *Tables of integral transforms*, Vol.2. McGraw-Hill, New York (1954).
4. Fox,C. *The G and H-functions as symmetrical Fourier kernels*. Trans. Amer. Math. Soc. 98(1961), 395-429.

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THE CLOSE-TO-CONVEXITY RADIUS OF SOME FUNCTIONS

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REZUMAT. - Razele de aproape convexitate ale unor funcţii. În lucrare sînt determinate razele de aproape convexitate ale funcţiilor sinus integral şi sinus hiperbolic integral, folosindu-se condiţia de aproape convexitate a lui Kaplan.

1. Preliminaries. Let f be an analytic function in the unit disk U. The function f is said to be convex if it is univalent in U and if $f(U)$ is a convex domain. The function f is said to be close-to-convex if there is a convex function φ on U such that $\mathrm{Re}(f'(z)/\varphi'(z)) > 0$ for $z\in U$. Using a well known criterion of univalence, due to Ozaki and Kaplan, it follows from definition that every close-to-convex function is univalent in U. The following theorem is also due to Kaplan and gives a necessary and sufficient analytic condition for close-to-convexity.

THEOREM 1 [1]. *An analytic function f in U is close-to-convex if and only if f' is a nonvanishing function in U and*

$$\int_{t_1}^{t_2} \mathrm{Re}\left(\frac{zf''(z)}{f'(z)}+1\right)dt > -\pi\ , \quad z=re^{it}\ ,$$

for every $r\in(0,1)$ and $0\le t_1<t_2<2\pi$.

For a function f which is analytic around the origin we define its close-to-convexity radius as being the radius of the largest disk centered at 0 in which f is close-to-convex. It is obvious that the problem of finding the close-to-convexity radius of f is the same with that of determining the maximum value of

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the real and positive parameter λ for which the function $g(z)=f(\lambda z)$ is close-to-convex in U.

Finding the close-to-convexity radius of a function is important as an independent problem and also because in this way is obtained a lower bound for the radius of univalence.

2. **Main problem.** We deal in this note with the problem of finding the close-to-convexity radii for the functions

$$\mathrm{Si}(z) = \int_0^z \frac{\sin t}{t} dt , \quad z \in \mathbf{C}$$

$$\mathrm{Shi}(z) = \int_0^z \frac{\mathrm{sh}\, t}{t} dt , \quad z \in \mathbf{C}$$

Note first that these functions have the same close-to-convexity radius, denoted by r_0. This becomes clear from the relation $\mathrm{Shi}(z)=\mathrm{Si}(iz)/i$. The nonvanishing condition for the derivative implies that $r_0 \le \pi$.

Letting $\Delta = \{(t_1,t_2):0\le t_1<t_2<2\pi\}$ Theorem 1 applied to these functions now gives

$$I_1(t_1,t_2) = \int_{t_1}^{t_2} \mathrm{Re}(z \,\mathrm{ctg}\, z)\, dt > -\pi \qquad (1)$$

$$I_2(t_1,t_2) = \int_{t_1}^{t_2} \mathrm{Re}(z \,\mathrm{cth}\, z)\, dt > -\pi \qquad (2)$$

where $z=re^{it}$, for every $r \in (0,r_0)$ and $(t_1,t_2) \in \Delta$.

If we put

$x=x(t)=r\cos(t)$, $y=y(t)=r\sin(t)$

$g_1(t)=y\,\mathrm{sh}(2y)+x\sin(2x)$, $g_2(t)=x\,\mathrm{sh}(2x)+y\sin(2y)$

$h_1(t)=\mathrm{Re}(z \text{ ctg } z)=g_1(t)/(\mathrm{ch}(2y)-\cos(2x))$

$h_2(t)=\mathrm{Re}(z\ \mathrm{cth}\ z)=g_2(t)/(\mathrm{ch}(2x)-\cos(2y))$

then the functions g_1, h_1, g_2, h_2 are even, periodical of period π and verify the relations

$$g_j(t)=g_j(\pi-t),\ h_j(t)=h_j(\pi-t),\ \mathrm{sgn}\ h_j=\mathrm{sgn}\ g_j,\ j=1,2 \qquad (3)$$
$$g_2(t)=g_1(t-\pi/2),\ h_2(t)=h_1(t-\pi/2).$$

Using the well-known inequalitis $\sin(a)/a \le 1 \le \mathrm{sh}(b)/b$, $\cos(a) \le 1 \le \mathrm{ch}(b)$, $a,b\in\mathbb{R}^*$ and the sign of g_1' if follows that g_1 increases on $[0,\pi/2]$, decreases on $[\pi/2,\pi]$, and $r\sin(2r) \le g_1(t) \le r\,\mathrm{sh}(2r)$. Consequently relations (1), (2) are fulfiled for $r \le \pi/2$ because g_j and h_j are positive, so $r_0 \in (\pi/2,\pi]$.

Using the sign of h_1 it follows that the minimum points (t_1,t_2) of I_1 with respect to $\bar{\Delta}$ verify $t_1 \in \{0,\pi-t_0,2\pi-t_0\}$, $t_2 \in \{t_0,\pi+t_0,2\pi\}$ where $t_0 = t_0(r)$ is the unique root of the equation $g_1(t) = 0$ situated in $[0,\pi/2]$. Aplying relations (3) we find

$$I_1(0,t_0)=I_1(2\pi-t_0,2\pi)=I_1(\pi-t_0,\pi+t_0)/2<0$$
$$I_1(\pi-t_0,2\pi)=I_1(0,\pi+t_0).$$

So the minimum points (t_1,t_2) of I_1 with respect to $\bar{\Delta}$ satisfy the relation $(t_1,t_2)\in\{(0,\pi+t_0),(\pi-t_0,\pi+t_0),(0,2\pi)\}$. We distinguish two cases:

a) If $I_1(0,\pi-t_0)\ge 0$ then $I_1(\pi-t_0,\pi+t_0)\le I_1(0,\pi+t_0)\le I_1(0,2\pi)$ so $\min\{I_1(t_1,t_2):(t_1,t_2)\in\Delta\}=\min\{I_1(t_1,t_2):(t_1,t_2)\in\bar{\Delta}\}=I_1(\pi-t_0,\pi+t_0)$.

b) If $I_1(0,\pi-t_0)<0$ then $I_1(0,2\pi)<I_1(0,\pi+t_0)<I_1(\pi-t_0,\pi+t_0)$ so $\min\{I_1(t_1,t_2):(t_1,t_2)\in\bar{\Delta}\}=\inf\{I_1(t_1,t_2):(t_1,t_2)\in\Delta\}=I_1(0,2\pi)$.

Consequently, with a previous use of relations (3), the close-to-convexity condition (1) for the function Si and for a fixed r becomes

$I_1(0,t_0)>-\pi/2$ and $I_1(0,\pi/2)\geq-\pi/4$. (4)

Consider now the case of the function Shi. Denoting by $t_0'=t_0'(r)$ the root of the equation $g_2(t)=0$ situated in $[0,\pi/2]$ we have by (3) that $t_0+t_0'=\pi/2$. Using again (3) and proceeding in an analogous way as before it follows that the minimum value of I_2 with respect to Δ may be

$$I_2(t_0',\pi-t_0') = 2I_2(t_0',\pi/2) = 2I_1(0,t_0)$$

or

$$I_2(t_0',2\pi-t_0') = I_2(0,\pi)+2I_2(t_0',\pi/2) = 2[I_1(0,t_0)+I_1(0,\pi/2)].$$

So, the close-to-convexity condition (2) for the function Shi and for a fixed r becomes

$I_1(0,t_0)>-\pi/2$ and $I_1(0,t_0) + I_1(0,\pi/2)>-\pi/2$. (5)

It follows now, from (4) and (5), that the following conditions are fulfiled when r equals r_0:

$I_1(0,t_0)=-\pi/2$ or $I_1(0,\pi/2)=-\pi/4$ (6)

$I_1(0,t_0)=-\pi/2$ or $I_1(0,t_0) + I_1(0,\pi/2)=-\pi/2$. (7)

Presuming that $I_1(0,t_0)>-\pi/2$ for $r=r_0$ we obtain from (6) and (7) that $I_1(0,\pi/2)=-\pi/4$ and $I_1(0,t_0) + I_1(0,\pi/2)=-\pi/2$, so $I_1(0,t_0)=I_1(0,\pi/2)=-\pi/4$ which is impossible because h_1 is negative on $[o,t_0]$ and positive on $[t_0,\pi/2]$.

Finally, the close-to-convexity radius r_0 of the functions Si and Shi is the smallest root, situated in $(\pi/2,\pi]$, of the equation

$$I_1(0,t_0(r)) = \int_0^{t_0(r)} \frac{y\,\mathrm{sh}(2y)+x\sin(2x)}{\mathrm{ch}(2y)-\cos(2x)}\,dt = -\frac{\pi}{2},$$

where $x=r\cos(t)$, $y=r\sin(t)$ and $t_0(r)$ is the unique root of the equation $y\operatorname{sh}(2y) + x\sin(2x) = 0$ situated in $(0,\pi/2)$.

An approximative value obtained for r_0 is $r_0 \approx 3.1411\ldots$

R E F E R E N C E S

1. P.L.Duren, *Univalent Functions*, Springer-Verlag, New-York Berlin Heidelberg Tokyo, 1983.

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GENERALIZED PRE-STARLIKE FUNCTIONS

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REZUMAT. - Funcţii prestelare generalizate. Lucrarea se ocupă cu funcţii prestelare cu mai mulţi parametri, de ordinul α şi tipul β. Sînt stabilite unele inegalităţi privind coeficienţii acestor funcţii.

Introduction. A function $f(z)$ normalised by $f(0)=f'(0)-1=0$ is said to be in the class S if it is analytic and univalent in the unit disc $U=\{z:|z|<1\}$. A function $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ is said to be in the class of functions starlike of order α, $0\leq\alpha<1$, denoted by $S^*(\alpha)$, if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}\geq\alpha \qquad (z\in U)$$

Further we say that f in S belongs to the class $S(\alpha,\beta)$ if f satisfies

$$\left|\frac{\frac{zf'(z)}{f(z)}-1}{\frac{zf'(z)}{f(z)}+1-2\alpha}\right|<\beta$$

where $\beta\in(0,1]$, $0\leq\alpha<1$.

The convolution or Hadamard product of two power series $f(z)=\sum_{n=0}^{\infty} a_n z^n$ and $g(z)=\sum_{n=0}^{\infty} b_n z^n$ is defined as the power series $f*g(z)=\sum_{n=0}^{\infty} a_n b_n z^n$. A normalised analytic function is said to be in the class of functions prestarlike of order α and type β,

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$0 \le \alpha < 1$, $\beta \in (0,1]$, denoted by R_α^β, if $f * S_{\alpha,\beta} \in S(\alpha,\beta)$ where $S_{\alpha,\beta} = z(1-\beta z)^{-2(1-\alpha)}$. For $\beta=1$ we get the class R_α introduced by Ruscheweyh [2].

Main Results. We need the following lemma due to Kulkarni S.R. [1]:

LEMMA: *Let f be in $S(\alpha,\beta)$, then for z in $\overline{U}$*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1-\beta(1-2\alpha)}{1+\beta} . \tag{1}$$

We also need,

LEMMA: *Let*

$$S_{\alpha,\beta} = z(1-\beta z)^{-2(1-\alpha)} = z + \sum_{n=2}^{\infty} \gamma(n,\alpha,\beta) z^n, \quad \text{then}$$

$$\gamma(n,\alpha,\beta) = \frac{\prod_{k=2}^{n} [\beta(k-2\alpha)]}{(n-1)!} \quad \text{for } n=2,3,4,\ldots \tag{2}$$

Proof: We have

$$S = z(1-\beta z)^{-2(1-\alpha)} =$$

$$= z\left\{1 + \sum_{n=2}^{\infty}\left(\frac{\prod_{k=2}^{n}[\beta(k-2\alpha)]}{(n-1)!}\right) z^{n-1}\right\} =$$

$$= z + \sum_{n=2}^{\infty}\left(\frac{\prod_{k=2}^{n}[\beta(k-2\alpha)]}{(n-1)!}\right) z^{n}$$

Hence the result follows.

THEOREM 1. *Let f be in R_α^β, then*

$$\operatorname{Re}\{G(z)\} > \frac{1}{1+\beta} \tag{3}$$

where

$$G(z) = \frac{f*\left(\dfrac{z}{(1-\beta z)^{3-2\alpha}}\right)}{f*\left(\dfrac{z}{(1-\beta z)^{2-2\alpha}}\right)}$$

Proof : Since f is in R_α^β, $F=f*S_{\alpha,\beta}$ belongs to $S(\alpha,\beta)$

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1-\beta(1-2\alpha)}{1+\beta}$$

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}+1-2\alpha\right\} > \frac{2(1-\alpha)}{1+\beta}$$

We have

$$F(z) = f*S_{\alpha,\beta}(z)$$

$$zF'(z) = f*z(S_{\alpha,\beta}(z))'$$

$$\frac{zF'(z)}{F(z)}+1-2\alpha = 2(1-\alpha)\frac{f*(z(1-\beta z)^{-(3-2\alpha)})}{f*(z(1-\beta z)^{-(2-2\alpha)})}$$

Hence the result follows.

THEOREM 2. *If $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be in R_α^β and* $S_{\alpha,\beta} = z(1-\beta z)^{-2(1-\alpha)}$ *then*

$$|a_n| \le \frac{1+\sum_{k=2}^{n-1}|a_k|\gamma(k,\alpha,\beta)}{\beta^{n-1}+\sum_{k=2}^{n}\beta^{n-k}\gamma(k,\alpha,\beta)}\ .$$

Proof : In view of Theorem 1, we can write

with $|b_n| \le 1$.

$$f*(z(1-\beta z)^{-(3-2\alpha)}) = (f*S_{\alpha,\beta})\left(1+\sum_{n=1}^{\infty} b_n z^n\right) \tag{4}$$

Equating the coefficients of z^n in the power series expansion of (4), we have,

$$a_n\left(\beta^{n-1}+\sum_{k=2}^{n} \beta^{n-k}\gamma(k,\alpha,\beta)\right)=$$
$$=b_{n-1}+\sum_{k=2}^{n-1} a_k\gamma(k,\alpha,\beta)\, b_{n-k}+a_n\gamma(n,\alpha,\beta)$$

Whence the result.

Note : For $\beta=1$, we get the result of Silverman and Silvia [3].

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REFERENCES

1. Kulkarni,S.,R. : *Some problems connected with univalent functions*. Ph.D thesis. Shivaji Univ. Maharashtra State, India.(1982)
2. Ruscheweyh : *Linear operators between classes of prestarlike functions*. Comm. Math. Helv. 52(1977) pp. 497-509.
3. Silverman & Silvia : *The influence of the second coefficient on prestarlike functions*. Rocky Mountain Journal of Maths. vol 10, no.3(1980).

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ON A PARTICULAR n-α-CLOSE-TO-CONVEX FUNCTION

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REZUMAT. - Asupra unor funcţii n-α-aproape convexe. În lucrare sînt stabilite cîteva proprietăţi ale unor funcţii n-α-convexe.

1. Introduction. Let A be the class of functions $f(z)$ which are analytic in the unit disc $U=\{z\in C : |z|<1\}$, with $f(0)=f'(0)-1=0$. In [2] the author defined the class $K_{n,\alpha}(\delta)$, the class of functions $f\in A$ which satisfy

$$\mathrm{Re}\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^nf(z)}+\alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right]>\delta\ ,\ z\in U$$

where $\alpha\geq 0$, $\delta<1$ and $D^nf(z)=\frac{z}{(1-z)^{n-1}}*f(z)=\frac{z(z^{n-1}f(z))^{(n)}}{n!}$,

where (*) stands for the Hadamard product (convolution) of power series, i.e. if $r(z)=\sum_{j=0}^{\infty}r_jz^j$ and $s(z)=\sum_{j=0}^{\infty}s_jz^j$, then $(r*s)(z)=\sum_{j=0}^{\infty}r_js_jz^j$.

Note that the classes $K_{n,\alpha}(\delta)$ and $Z_n(\delta)\equiv K_{n,0}(\delta)$ were studied in [2] and the classes $K_{n,\alpha}(1/2)$ and $Z_n(1/2)$ were introduced by H.S.Al-Amiri [1] and S.Ruscheweyh [7] respectively.

We denote by $AC_n(\delta)$ (the class of n-close-to-convex functions of order δ) the class of functions $f\in A$ which satisfy

$$\mathrm{Re}\frac{D^{n+1}f(z)}{D^{n+1}g(z)}>\delta\ ,\ z\in U$$

where $g\in Z_{n+1}(\delta)$, $\delta<1$ and let $C_{n,\alpha}(\delta)$ (the class of n-α-close-to-

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convex functions of order δ) the class of functions $f\in A$ which satisfy

$$\mathrm{Re}\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^{n+1}g(z)}+\alpha\frac{D^{n+2}f(z)}{D^{n+2}g(z)}\right]>\delta\ ,\ z\in U$$

where $g\in Z_{n+2}(\delta)$, $\delta<1$. These classes were introduced in [3] and we have presented in [4] some properties by using sharp subordination results from [5] and [6], and the classes $C_{n,\alpha}(1/2)$, $AC_n(1/2)$ were studied in [1].

Let $\gamma\in C$ with Re $\gamma>-1$ and $b_\gamma(z)=\sum_{j=1}^{\infty}\frac{\gamma+1}{\gamma+j}z^j$. In [7], S.Ruscheweyh showed that if Re $\gamma\geq(n-1)/2$ and $f\in Z_n(1/2)$, then $f*b_\gamma\in Z_n(1/2)$.

In [4] we presented some new results concerning this function and in this paper we will give other new properties of the function $b_\gamma(z)$.

2. **Preliminaries.** We will need the next lemmas to prove our results.

LEMMA A. [4, Theorem 3]. *Let* $\gamma>-1$ *and*

$$\delta_0=\max\left\{\frac{n-\gamma}{n+1}\ ,\ \frac{2n-\gamma}{2(n+1)}\right\}\leq\delta<1\ .$$

If $f\in Z_n(\delta)$ *then* $f*b_\gamma\in Z_n(\tilde{\delta}(n,\gamma,\delta))$ *where*

$$\tilde{\delta}(n,\gamma,\delta)=\frac{1}{n+1}\left[\frac{\gamma+1}{F(1,2(n+1)(1-\delta),\gamma+2;1/2)}-\gamma+n\right]$$

and this result is sharp.

LEMMA B. [4, Theorem 4]. *Let* $\gamma>-1$ *and*

$$\max\left\{\frac{n-\gamma+1}{n+2}, \frac{2n-\gamma+2}{2(n+2)}\right\} \le \delta \le \frac{2n-\gamma+3}{2(n+2)}.$$

If $f\in AC_n(\delta)$ related to $g\in Z_{n+1}(\delta)$ then $f*b_\gamma\in AC_n(\delta)$ related to $g*b_\gamma\in Z_{n+1}(\delta)$.

3. Main results.

THEOREM 1. *If $-1<\gamma\le 0$, then $f\in Z_n\left(\frac{n-\gamma}{n+1}\right)$ implies that*

$$f*b_\gamma\in Z_n\left(\tilde{\delta}\left(n,\gamma,\frac{n-\gamma}{n+1}\right)\right), \text{ where}$$

$$\tilde{\delta}\left(n,\gamma,\frac{n-\gamma}{n+1}\right) = \frac{1}{(n+1)\sqrt{\pi}}\,\frac{\Gamma(\gamma+3/2)}{\Gamma(\gamma+1)} + \frac{n-\gamma}{n+1}$$

and the result is sharp.

Proof. If $-1<\gamma\le 0$, then $\max\left\{\frac{n-\gamma}{n+1}, \frac{2n-\gamma}{2(n+1)}\right\} = \frac{n-\gamma}{n+1}$ and by using Lemma A for $\delta = (n-\gamma)/(n+1)$ and a simple calculus we obtain our result.

THEOREM 2. *If $\gamma\ge 0$, then $f\in Z_n\left(\frac{2n-\gamma}{2(n+1)}\right)$ implies that*

$$f*b_\gamma\in Z_n\left(\frac{2n-\gamma+1}{2(n+1)}\right)$$

and this result is sharp.

Proof. If $\gamma\ge 0$ then $\max\left\{\frac{n-\gamma}{n+1}, \frac{2n-\gamma}{2(n+1)}\right\} = \frac{2n-\gamma}{2(n+1)}$; taking $\delta = \frac{2n-\gamma}{2(n+1)}$ in Lemma A and using the well-known relation $F(1,a,a;z)=1/(1-z)$ we have $\tilde{\delta}\left(n,\gamma,\frac{2n-\gamma}{2(n+1)}\right) = \frac{2n-\gamma+1}{2(n+1)}$ and we obtain our result.

Taking $\gamma = 0$ in Lemma A we obtain the next result.

COROLLARY 1. *Let $\frac{n}{n+1}\le\delta<1$ and $f\in Z_n(\delta)$; then*

$$f*b_0\in Z_n(\tilde{\delta}(n,0,\delta))\ ,\ where$$

$$\tilde{\delta}(n,0,\delta)=\begin{cases}\dfrac{1}{n+1}\left[\dfrac{1-2(n+1)(1-\delta)}{2-2^{2(n+1)(1-\delta)}}+n\right], & for\ \delta\neq\dfrac{2n+1}{2(n+1)}\\ \dfrac{1}{n+1}\left[\dfrac{1}{2\ln 2}+n\right], & for\ \delta=\dfrac{2n+1}{2(n+1)}\end{cases}$$

and this result is sharp.

Taking $n=0$ in the above corollary we obtain:

COROLLARY 2. *Let $0\le\delta<1$ and $f\in A$ with* $\mathrm{Re}\dfrac{zf'(z)}{f(z)}>\delta$, $z\in U$. *Then* $\mathrm{Re}\dfrac{zF'(z)}{F(z)}>\tilde{\delta}$, $z\in U$ *where*

$$\tilde{\delta}=\begin{cases}\dfrac{2\delta-1}{2-2^{2(1-\delta)}}, & for\ \delta\neq\dfrac{1}{2}\\ \dfrac{1}{2\ln 2}, & for\ \delta=\dfrac{1}{2}\end{cases}$$

and $F(z)=f(z)*b_0(z)$, and this result is sharp.

Considering $n=0$ and $\delta=0$ in Lemma B we obtain the following result:

COROLLARY 3. *Let $2\le\gamma\le3$ and $f,g\in A$. Then* $\mathrm{Re}\dfrac{f'(z)}{g'(z)}>0$, $z\in U$, *where* $\mathrm{Re}\left(1+\dfrac{zg''(z)}{g'(z)}\right)>-1$, $z\in U$ *implies* $\mathrm{Re}\dfrac{F'(z)}{G'(z)}>0$, $z\in U$ *where* $\mathrm{Re}\left(1+\dfrac{zG''(z)}{G'(z)}\right)>-1$, $z\in U$ *and*

$$F(z)=f(z)*b_\gamma(z)\ ,\ G(z)=g(z)*b_\gamma(z).$$

Taking $n=0$ and $\delta=1/2$ in Lemma B we obtain the next result:

COROLLARY 4. *Let $0\le\gamma\le1$ and $f,g\in A$. Then*
$\mathrm{Re}\dfrac{f'(z)}{g'(z)}>\dfrac{1}{2}$, $z\in U$ *where* $\mathrm{Re}\left(1+\dfrac{zg''(z)}{g'(z)}\right)>0$, $z\in U$ *implies*
$\mathrm{Re}\dfrac{F'(z)}{G'(z)}>\dfrac{1}{2}$, $z\in U$ *where* $\mathrm{Re}\left(1+\dfrac{zG''(z)}{G'(z)}\right)>0$, $z\in U$ *and*

$F(z)=f(z)*b_\gamma(z)$, $G(z)=g(z)*b_\gamma(z)$.

Considering $n=0$, $\gamma=0$ in Lemma B and $\gamma=0$ in Theorem 2 we obtain the next two results concerning $f*b_0$ respectively.

COROLLARY 5. *Let* $1/2 \le \delta \le 3/4$ *and* $f,g\in A$. *Then* $\mathrm{Re}\frac{f'(z)}{g'(z)}>\delta$, $z\in U$ *where* $\mathrm{Re}\left(1+\frac{zg''(z)}{g'(z)}\right)>2\delta-1$, $z\in U$ *implies* $\mathrm{Re}\frac{F'(z)}{G'(z)}>\delta$, $z\in U$ *where* $\mathrm{Re}\left(1+\frac{zG''(z)}{G'(z)}\right)>2\delta-1$, $z\in U$ *and* $F(z)=f(z)*b_0(z)$, $G(z)=g(z)*b_0(z)$.

COROLLARY 6. *If* $f\in Z_n\left(\frac{n}{n+1}\right)$ *then* $f*b_0\in Z_n\left(\frac{2n+1}{2(n+1)}\right)$ *and this result is sharp.*

R E F E R E N C E S

1. H.S. Al-Amiri, *Certain analogy of the α-convex functions*, Rev.Roum.Math. Pures Appl., XXIII,10(1978), 1449-1454.
2. T.Bulboacă, *Applications of the Briot-Bouquet differential subordination*, /to appear/.
3. T.Bulboacă, *Classes of n-α-close-to-convex functions*, /to appear/.
4. T.Bulboacă, *New subcasses of analytic functions*, Seminar on Geometric Function Theory, Preprint 5(1986), "Babeş-Bolyai" Univ. Cluj-Napoca, 13-24.
5. S.S.Miller and P.T.Mocanu, *Differential subordinations and univalent functions*, Michigan Math.J., 28(1981), 151-171.
6. P.T.Mocanu, D.Ripeanu and I.Şerb, *The order of starlikeness of certain integral operators*, Mathematica (Cluj), 23(46), No.2(1981), 225-230.
7. S.Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc., 49(1975), 109-115.

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ON A MARX-STROHHÄCKER DIFFERENTIAL SUBORDINATION

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REZUMAT. - Asupra unei subordonări diferenţiale Marx-Strohhäcker. Fie A clasa funcţiilor f, $f(0) = f'(0) - 1 = 0$, analitice în discul unitate U. Fie q o funcţie univalentă în U, cu $q(0) = 1$. Presupunem că sînt verificate condiţiile (5) şi (6), unde $h(z) = q(z) + zq'(z)/q(z)$. Rezultatul principal al lucrării afirmă că dacă $f \in A$ şi $zf''(z)/f'(z) \prec zk''(z)/k'(z)$ atunci $zf'(z)/f(z) \prec zk'(z)/k(z)$, unde k este definită de (9). Se consideră cazul particular $k(z) = (e^{\lambda z} - 1)/\lambda$ unde $|\lambda| \le 4$.

1. Introduction. If the function f with $f'(0) \neq 0$ is analytic in the unit disc U, then f is convex in U (i.e. f is univalent and $f(U)$ is a convex domain) if and only if $\mathrm{Re}[zf''(z)/f'(z)+1]>0$ in U. Let A denote the class of analytic functions f in U, which are normalized by $f(0) = 0$ and $f'(0)=1$. A function f in A is starlike in U (i.e. f is univalent and $f(U)$ is starlike with respect to the origin) if and only if $\mathrm{Re}[zf'(z)/f(z)] > 0$. If $\mathrm{Re}[zf'(z)/f(z)] > \alpha$, $0 \le \alpha < 1$, then f is called starlike of order α. A classic result due to Marx [2] and Strohhäcker [7] asserts that a convex function f in A is starlike of order 1/2, i.e.

$$f \in A,\ \mathrm{Re}\frac{zf''(z)}{f'(z)}+1 > 0\ (z \in U) \Rightarrow \mathrm{Re}\frac{zf'(z)}{f(z)} > \frac{1}{2}\ (z \in U). \qquad (1)$$

If F and G are analytic functions in U and G is univalent then we say that F is subordinate to G, written $F \prec G$, or $F(z) \prec G(z)$, if $F(0) = G(0)$ and $F(U) \subset G(U)$.

If we let $k(z) = z/(1-z)$, then the implication (1) can be

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rewritten as

$$f \in A,\ \frac{zf''(z)}{f'(z)} \prec \frac{zf''(z)}{k'(z)} \Rightarrow \frac{zf'(z)}{f(z)} \prec \frac{zk'(z)}{k(z)}. \tag{2}$$

In [4] S.S.Miller and the present author determined certain general sufficient conditions on the function k in A, for which the implication (2) holds.

In this paper we determine other sufficient conditions on k in A, for which (2) holds. For example these new conditions are satisfied if $k(z) = (e^{\lambda z} - 1)/\lambda$, where $|\lambda| \le 4$. This example is an improvement of a recent result of V.Anisiu and the author [1]. In particular we offer a new and more simple proof of the starlikeness condition obtained in [1].

2. **Preliminaries.** We shall use the following lemmas to prove our results.

LEMMA 1. *Let G be an analytic and univalent function on $\bar{U}$, with $G'(\zeta) \neq 0$, for $\zeta \in \partial U$. Let F be analytic in U, with $F(0) = G(0)$. If F is not subordinate to G, then there exist points $z_0 \in U$ and $\zeta \in \partial U$, and an $m \ge 1$, for which*

(i) $F(z_0) = G(\zeta)$ *and*

(ii) $z_0 F'(z_0) = m\zeta G'(\zeta)$.

More general forms of this lemma may be found in [3]. A recent survey on the theory and applications of differential subordinations is given in [5].

LEMMA 2.[1]. *The radius of univalence of the function $f(z) = (e^z - 1)/z$ is given by $r = 4.83\ldots$, where r satisfies the system*

$$\begin{cases} e^{r\cos t}\sin(r\sin t-t)+\sin t = 0 \\ e^{r\cos t}[r\cos(r\sin t)-\cos(r\sin t-t)]+\cos t = 0 \, . \end{cases}$$

LEMMA 3. *If* $|z| < r_0 = 4.046...$, *where* r_0 *is the root in the interval* $(0,2\pi)$ *of the equation* $r[1 + \mathrm{ctg}(r/2)] = 2$, *then*

$$\mathrm{Re}\left(1+\frac{1}{e^z-1}-\frac{1}{z}\right) > 0 .$$

Proof. It is well known that

$$\frac{z}{e^z-1} = 1-\frac{z}{2}+\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}z^{2n} \, , \quad |z| < 2\pi,$$

where B_{2n} are the Bernoulli numbers. Therefore we have

$$1+\frac{1}{e^z-1}-\frac{1}{z} = \frac{1}{2}+\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}z^{2n-1}$$

and we deduce

$$\mathrm{Re}\left(1+\frac{1}{e^z-1}-\frac{1}{z}\right) \geq \frac{1}{2}-\left|\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}z^{2n-1}\right| .$$

Using the well known formula

$$\frac{z}{2}\mathrm{ctg}\frac{z}{2} = 1-\sum_{n=1}^{\infty}\frac{|B_{2n}|}{(2n)!}z^{2n} \, , \quad |z| = r < 2\pi ,$$

we easily obtain

$$\mathrm{Re}\left(1+\frac{1}{e^z-1}-\frac{1}{z}\right) \geq \frac{1}{2}-\frac{1}{r}+\frac{1}{2}\mathrm{ctg}\frac{r}{2}$$

and from this last inequality we deduce the desired result.

LEMMA 4.[6]. *The inequality*

$$\operatorname{Re}\frac{\lambda z}{e^{\lambda z}-1} > 0$$

holds for all $z \in U$, *if and only if* $|\lambda| \le r^*$, *where* $r^* = 2.832\ldots$ *is given by*

$$r^* = \sqrt{1+y_0^2} \tag{3}$$

and y_0 *is the smallest positive root of the equation*

$$y\sin y + \cos y = \frac{1}{e}\ . \tag{4}$$

We note that r^* is the radius of starlikeness of the function $f(z) = e^z - 1$.

3. **Main results.**

THEOREM 1. *Let* q *be univalent in* U, *with* $q(0) = 1$. *Let*

$$h(z) = q(z) + \frac{zq'(z)}{q(z)}$$

and suppose that

$$h \text{ is convex in } U \tag{5}$$

$$\operatorname{Re}\left[\frac{h'(z)}{q'(z)}q(z)\right] > 0\ ,\ z\in U\ . \tag{6}$$

If P *is an analytic function in* U, *such that*

$$P(z) \prec h(z), \tag{7}$$

then the analytic solution p *of the differential equation*

$$zp'(z) + P(z)p(z) = 1 \tag{8}$$

satisfies $p \prec 1/q$.

Proof. Condition (6) implies $q(z) \neq 0$ in U, hence the function $1/q$ is analytic and univalent.

Without loss of generality we can assume q is univalent, with $q(z) \neq 0$ on $\bar{U}$ and $q'(z) \neq 0$ for $z \in \partial U$. If not, then we can replace q, h, P, and p by $q_r(z) = q(rz)$, $h_r(z) = h(rz)$, $P_r(z) = P(rz)$ and $p_r(z) = p(rz)$ respectively, where $0 < r < 1$. These new functions satisfy the conditions of the Theorem on $\bar{U}$. We would then prove $p_r \prec 1/q_r$ and by letting $r \to 1$ we obtain $p \prec 1/q$.

Now assume that $p \not\prec 1/q$. From Lemma 1 there exist points $z_0 \in U$ and $\zeta \in \partial U$ and $m \geq 1$ such that $p(z_0) = 1/q(\zeta)$ and $z_0 p'(z_0) = -m\,\zeta\, q'(\zeta)/q^2(\zeta)$. Therefore from (8) we obtain

$$P(z_0) = \frac{1}{p(z_0)} - \frac{z_0 p'(z_0)}{p(z_0)} = q(\zeta) + \frac{m\zeta q'(\zeta)}{q(\zeta)}$$
$$= h(\zeta) + \frac{(m-1)\zeta q'(\zeta)}{q(\zeta)} .$$

If we let

$$\delta = \frac{P(z_0) - h(\zeta)}{\zeta h'(\zeta)} = \frac{(m-1)\, q'(\zeta)}{q(\zeta)\, h'(\zeta)} ,$$

then from (6) and the fact that $m \geq 1$ we deduce $\mathrm{Re}\,\delta > 0$, or equivalenty $|\arg \delta| < \pi/2$. Since $\zeta\, h'(\zeta)$ is the outward normal to the boundary of the convex domain $h(U)$ we deduce that $P(z_0) \notin h(U)$, which contradicts (5). Hence we have $p \prec 1/q$.

THEOREM 2. *Let q satisfy the conditions (5) and (6) of Theorem 1 and let*

$$k(z) = z \exp \int_0^z \frac{q(t) - 1}{t}\, dt . \tag{9}$$

If $f \in A$ and

$$\frac{zf''(z)}{f'(z)} \prec \frac{zk''(z)}{k'(z)} , \tag{10}$$

then $zf'(z)/f(z)$ is analytic in U and

$$\frac{zf'(z)}{f(z)} \prec \frac{zk'(z)}{k(z)} .$$

Proof. From (9) we obtain

$$q(z) = \frac{zk'(z)}{k(z)} \quad \text{and} \quad h(z) = q(z) + \frac{zq'(z)}{q(z)} = 1 + \frac{zk''(z)}{k'(z)} .$$

Since condition (10) implies $f'(z) \neq 0$, the function $P(z) = 1 + zf''(z)/f'(z)$ is analytic in U and satisfies (7). For this particular P equation (8) has the analytic solution $p(z) = f(z)/[zf'(z)]$. Thus all conditions of Theorem 1 are satisfied and we deduce $p \prec 1/q$. Since $1/q(z) \neq 0$, this implies $p(z) \neq 0$ and so $1/p(z) = zf'(z)/f(z)$ is analytic in U. In addition from $p \prec 1/q$ and $q(z) \neq 0$ we obtain $1/p \prec q$, i.e. $zf'(z)/f(z) \prec zk'(z)/k(z)$.

4. **A particular case.** If we let

$$q(z) = \frac{\lambda z}{e^{\lambda z}-1} ,$$

then from Lemma 2 we deduce that q is univalent in U if $|\lambda| \le 4.83...$ and in this particular case we have

$$k(z) = \frac{1-e^{-\lambda z}}{\lambda} \quad \text{and} \quad h(z) = 1-\lambda z .$$

On the other hand we have

$$\frac{q'(z)}{h'(z)\,q(z)} = 1 + \frac{1}{e^{\lambda z}-1} - \frac{1}{\lambda z}$$

and by using Lemma 3 we deduce that

$$\operatorname{Re}\frac{q'(z)}{h'(z)\,q(z)} > 0\ ,\quad \text{if}\quad |\lambda| \le r_0 = 4.046\ldots$$

Thus if $|\lambda| \le r_0$ all conditions of Theorem 2 are satisfied and we obtain the following result.

THEOREM 3. *Let $r_0 = 4.046\ldots$ be the root in the interval $(0,2\pi)$ of the equation $r[1 + \operatorname{ctg}(r/2)] = 2$. If $f \in A$ and*

$$\left|\frac{f''(z)}{f'(z)}\right| \le M \le r_0\ ,\quad \text{for}\quad z\in U\ ,$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{\lambda z}{e^{\lambda z}-1}\ ,\quad \text{for}\quad |\lambda| = M\ .$$

This theorem is an improvement of a result in [1].

By using Lemma 4, from Theorem 3 we deduce the following sufficient condition of starlikeness, which was obtained in [1].

THEOREM 4. *Let $r^* = 2.83\ldots$ be given by* (3) *and* (4). *If $f\in A$ and*

$$\left|\frac{f''(z)}{f'(z)}\right| \le r^*\ ,\quad \text{for}\quad z\in U\ ,$$

then f is starlike in U and this result is sharp.

Example. Let $f \in A$ be defined by

$$f(z) = \int_0^z e^{\lambda t^2}dt\ .$$

From Theorem 4 we deduce that f is starlike if $|\lambda|\le r^*/2=1.41\ldots$

In particular, if we denote by ρ the radius of starlikeness of the error function

$$\operatorname{erf}(z) = \int_0^z e^{-t^2}dt ,$$

then $\rho \geq \sqrt{\frac{r^*}{2}} = 1.19\ldots$ We note that the inequality $\rho \geq r^*/2$ in [1] has to be corrected by $\rho \geq \sqrt{\frac{r^*}{2}}$.

REFERENCES

1. V.Anisiu, P.T.Mocanu, *On a simple sufficient condition of starlikeness*, Mathematica, 31(54), 2(1989), 97-101.
2. A.Marx, *Untersuchungen über schlichte Abbildungen*, Math. Ann. 107(1932/33), 40-67.
3. S.S.Miller, P.T.Mocanu, *Differential subordinations and univalent functions*, Michigan Math.J., 28(1981), 151-171.
4. S.S.Miller, P.T.Mocanu, *Marx-Strohhäcker differential subordination systems*, Proc. Amer. Math. Soc., 99,2(1987), 527-534.
5. S.S.Miller, P.T.Mocanu, *The theory and applications of second-order differential subordinations*, Studia Univ. Babeş-Bolyai, Math., 34,4(1989), 3-33.
6. P.T.Mocanu, *Asupra razei de stelaritate a funcţiilor univalente*, Stud. Cerc. Mat. (Cluj), 11(1960), 337-341.
7. E.Strohhäcker, *Beitrage zur Theorie der schlichten Funktionen*, Math.Z., 37(1933), 356-380.

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CONVOLUTION OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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REZUMAT. - Convoluţii de funcţii univalente cu coeficienţi negativi. În lucrare sînt stabilite unele proprietăţi ale convoluţiilor de funcţii stelate de ordin α şi tip β cu coeficienţi negativi.

1. Let A denote the class of functions of the form

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad n=2,3,\ldots$$

that are analytic in the unit disc $U = \{z\in \mathbf{C};\ |z|<1\}$. The function $f\in A$ is said to be starlike of order α, $\alpha\in[0,1)$, with negative coefficients, if

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > \alpha, \quad z\in U.$$

We denote this class by $S^*(\alpha)$. Let $\alpha\in[0,1)$ and $\beta\in(0,1]$; we define the class $S^*(\alpha,\beta)$ of starlike functions of order α and type β with negative coefficients by

$$S^*(\alpha,\beta) = \{f\in A;\ J(f(z);\alpha)<\beta,\ z\in U\},$$

where

$$J(f(z);\alpha) = \left|\frac{\frac{zf'(z)}{f(z)}-1}{\frac{zf'(z)}{f(z)}+1-2\alpha}\right|.$$

Remark 1. Let D be the disc with the center at $a = (1-2\alpha\beta^2+ \beta^2)/(1-\beta^2)$ and the radius $r = 2\beta(1-\alpha)/(1-\beta^2)$ when $\beta\in(0,1)$ and

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$\alpha\in[0,1)$, and let $D = \{w\in\mathbb{C};\ \mathrm{Re}\ w>\alpha\}$, when $\beta=1$ and $\alpha\in[0,1)$. Then for $z\in U$ we have

$$J(f(z);\alpha) \prec \beta \Rightarrow \frac{zf'(z)}{f(z)} \in D \qquad (1)$$

and we deduce that if $f\in S^*(\alpha,\beta)$, then

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > \sigma\ ,\quad z\in U\ ,$$

where $\sigma=\sigma(\alpha,\beta)$ and

$$\sigma(\alpha,\beta) = \frac{1+2\alpha\beta-\beta}{1+\beta}\ .$$

We obtain $S^*(\alpha,1) = S^*(\alpha)$ and $S^*(\alpha,\beta)\subset S^*(\sigma)$, where $\sigma=\sigma(\alpha,\beta)$.

Remark 2. By using (1) we also obtain

a) if $0\le\alpha_1<\alpha_2<1$, then $S^*(\alpha_2,\beta)\subset S^*(\alpha_1,\beta)$;

b) if $0<\beta_1<\beta_2\le 1$, then $S^*(\alpha,\beta_1)\subset S^*(\alpha,\beta_2)$.

Let f and g be two functions in A,

$$f(z) = z-\sum_{n=2}^{\infty} a_n z^n \ \text{ and }\ g(z) = z-\sum_{n=2}^{\infty} b_n z^n\ .$$

Then we define the (modified) Hadamard product or convolution of f and g by

$$(f*g)(z) = z-\sum_{n=2}^{\infty} a_n b_n z^n\ .$$

In this paper we show that if f, $g\in S^*(\alpha,\beta)$, then $f*g\in S^*(\alpha,\gamma)\cap S^*(\delta,\beta)$, where $0<\gamma<\beta$ and $\alpha<\delta<1$.

We will use the following result due to V.P.Gupta and P.K.Jain [1].

THEOREM A. *A function f,*

$$f(z) = z-\sum_{n=2}^{\infty} a_n z^n, \quad a_n \ge 0, \quad n=2,3,\ldots$$

is in $S^*(\alpha,\beta)$ *if and only if*

$$\sum_{n=2}^{\infty} \frac{n-1+\beta(n+1-2\alpha)}{2\beta(1-\alpha)} a_n \le 1 .$$

The result is sharp.

2. THEOREM 1. *Let* $f,g\in S^*(\alpha,\beta)$, $\alpha\in[0,1)$, $\beta\in(0,1]$. *Then* $f*g\in S^*(\alpha,\gamma(\alpha,\beta))$, *where*

$$\gamma(\alpha,\beta) = \frac{2\beta^2(1-\alpha)}{(3-2\alpha)(\beta+1)^2-2(1-\alpha)}$$

and $0<\gamma(\alpha,\beta)<\beta$. *The result is sharp.*

Proof. From Theorem A we know that if $f,g\in S^*(\alpha,\beta)$ and

$$f(z) = z-\sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z-\sum_{n=2}^{\infty} b_n z^n,$$

then

$$\sum_{n=2}^{\infty} \frac{n-1+\beta(n+1-2\alpha)}{2\beta(1-\alpha)} a_n \le 1 \tag{2}$$

and

$$\sum_{n=2}^{\infty} \frac{n-1+\beta(n+1-2\alpha)}{2\beta(1-\alpha)} b_n \le 1 . \tag{3}$$

From Theorem A we also know that $f*g\in S^*(\alpha,\gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{n-1+\gamma(n+1-2\alpha)}{2\gamma(1-\alpha)} a_n b_n \le 1 \tag{4}$$

and we wish to find the smallest $\gamma=\gamma(\alpha,\beta)$ such that (4) holds.

From (2) and (3) we get by means of the Cauchy-Schwarz inequality

$$\sum_{n=2}^{\infty} \frac{n-1+\beta(n+1-2\alpha)}{2\beta(1-\alpha)} \sqrt{a_n b_n} \le 1 \tag{5}$$

which implies

$$\sqrt{a_n b_n} \le \frac{2\beta(1-\alpha)}{n-1+\beta(n+1-2\alpha)}, \quad n=2,3,\dots \tag{6}$$

We observe that the inequalities

$$\frac{n-1+\gamma(n+1-2\alpha)}{2\gamma(1-\alpha)} a_n b_n \le \frac{n-1+\beta(n+1-2\alpha)}{2\beta(1-\alpha)} \sqrt{a_n b_n}, \quad n=2,3,\dots \tag{7}$$

imply (4). We also observe that (7) is equivalent to

$$\frac{n-1+\gamma(n+1-2\alpha)}{\gamma} \sqrt{a_n b_n} \le \frac{n-1+\beta(n+1-2\alpha)}{\beta}, \quad n=2,3,\dots \tag{8}$$

By using (6) we obtain

$$\frac{n-1+\gamma(n+1-2\alpha)}{\gamma} \sqrt{a_n b_n} \le \frac{2\beta(1-\alpha)(n-1+\gamma(n+1-2\alpha))}{\gamma(n-1+\beta(n+1-2\alpha))}, \quad n=2,3,\dots$$

In order to obtain (8) it will be sufficient to show that

$$\frac{2\beta(1-\alpha)(n-1+\gamma(n+1-2\alpha))}{(n-1+\beta(n+1-2\alpha))\gamma} \le \frac{n-1+\beta(n+1-2\alpha)}{\beta}, \quad n=2,3,\dots$$

These last inequalities are equivalent to

$$\frac{n-1}{\gamma}+n+1-2\alpha \le \frac{(n-1+\beta(n+1-2\alpha))^2}{2\beta^2(1-\alpha)}, \quad n=2,3,\dots$$

or

$$\gamma \geq \gamma(n) = \frac{2\beta^2(1-\alpha)(n-1)}{(n-1+\beta(n+1-2\alpha))^2-2\beta^2(1-\alpha)(n+1-2\alpha)}, \quad n=2,3,\ldots$$

We note that $\gamma(n) \leq \gamma(2)$ for all $n = 2,3,\ldots$ and then we choose

$$\gamma(\alpha,\beta) = \gamma(2) = \frac{2\beta^2(1-\alpha)}{(3-2\alpha)(\beta+1)^2-2(1-\alpha)}.$$

We have $\gamma(2)>0$ because $2\beta^2(1-\alpha)>0$ and

$(3-2\alpha)(\beta+1)^2 - 2(1-\alpha) = 2(1-\alpha)(\beta+1)^2 - 2(1-\alpha) +$

$+ (\beta+1)^2 = 2(1-\alpha)(\beta^2+2\beta) + (\beta+1)^2 > 0.$

We also have

$$\beta-\gamma(\alpha,\beta) = \frac{(1-\beta)^2+8\beta(1-\alpha)+4\alpha\beta(1-\beta)+2\alpha\beta^2}{(3-2\alpha)(\beta+1)^2-2(1-\alpha)} > 0 .$$

By Theorem A the function

$$f_2(z) = z-\frac{2\beta(1-\alpha)}{1+\beta(3-2\alpha)}z^2 \tag{9}$$

is an element of $S^*(\alpha,\beta)$ and

$$(f_2*f_2)(z) = z-\frac{4\beta^2(1-\alpha)^2}{(1+\beta(3-2\alpha))^2}\cdot z^2 \in S^*(\alpha,\gamma(\alpha,\beta)),$$

because

$$\frac{4\beta^2(1-\alpha)^2}{(1+\beta(3-2\alpha))^2} = \frac{2\gamma(1-\alpha)}{1+\gamma(3-2\alpha)}, \quad \text{when } \gamma = \gamma(\alpha,\beta).$$

Then the functions $f = g = f_2$ are extremal functions for this theorem.

COROLLARY 1.1. *If* $f,g\in S^*(\alpha,\beta)$, *then* $f*g\in S^*(\alpha,\beta)$.

Proof. We use Theorem 1 and Remark 2 b).

COROLLARY 1.2. *If* $f,g\in S^*(\alpha,\beta)$, *then* $f*g\in S^*(\rho(\alpha,\beta))$, *where*

$$\rho(\alpha,\beta) = 1 - \frac{4\beta^2(1-\alpha)^2}{(\beta+1)(5\beta+1-4\alpha\beta)} \ . \tag{10}$$

Proof. If $f,g\in S^*(\alpha,\beta)$, then $f*g\in S^*(\alpha,\gamma(\alpha,\beta))\subset S^*(\sigma(\alpha,\gamma(\alpha,\beta)))$, where

$$\sigma(\alpha,\gamma(\alpha,\beta)) = \frac{1+2\alpha\gamma(\alpha,\beta)-\gamma(\alpha,\beta)}{1+\gamma(\alpha,\beta)} = \rho(\alpha,\beta)$$

and $\rho(\alpha,\beta)$ is given by (10).

COROLLARY 1.3. *If* $f,g\in S^*(\alpha)$, *then* $f*g\in S^*\left(\frac{2-\alpha^2}{3-2\alpha}\right)$.

Proof. We know that $S^*(\alpha) = S^*(\alpha,1)$ (see Remark 1) and by using Corollary 1.2 we obtain $f*g\in S^*(\rho(\alpha,1))$ and $\rho(\alpha,1)=\frac{2-\alpha^2}{3-2\alpha}$.

The preceding result (Corollary 1.3) are due to A. Schild and H. Silverman [2].

3. THEOREM 2. *Let* $\alpha\in[0,1)$ *and* $\beta\in(0,1]$. *If* $f,g\in S^*(\alpha,\beta)$, *then* $f*g\in S^*(\delta(\alpha,\beta),\beta)$, *where*

$$\delta(\alpha,\beta) = 1 \quad \frac{2\beta(1-\alpha)^2}{5\beta+1-4\alpha\beta} \ ,$$

and $\alpha<\delta(\alpha,\beta)<1$. *The result is sharp.*

Proof. If $f,g\in S^*(\alpha,\beta)$, then (6) holds. By Theorem A we know that $f*g\in S^*(\delta,\beta)$ if and only if

$$\sum_{n=2}^{\infty} \frac{n-1+\beta(n+1-2\delta)}{2\beta(1-\delta)} a_n b_n \le 1 \tag{11}$$

and we wish to find the largest $\delta=\delta(\alpha,\beta)$ such that (11) be satisfied.

We note that the next inequalities

$$\frac{n-1+\beta(n+1-2\delta)}{1-\delta}\sqrt{a_n b_n} \le \frac{n-1+\beta(n+1-2\alpha)}{1-\alpha} \ , \quad n=2,3,\ldots \tag{12}$$

implies (11).

By using (6) we have

$$\frac{n-1+\beta(n+1-2\delta)}{1-\delta}\sqrt{a_n b_n} \le \frac{2\beta(1-\alpha)}{1-\delta} \cdot \frac{n-1+\beta(n+1-2\delta)}{n-1+\beta(n+1-2\alpha)}, \quad n=2,3,\dots$$

and we deduce that

$$\frac{2\beta(1-\alpha)}{1-\delta} \cdot \frac{n-1+\beta(n+1-2\delta)}{n-1+\beta(n+1-2\alpha)} \le \frac{n-1+\beta(n+1-2\alpha)}{1-\alpha}, \quad n=2,3,\dots$$

or

$$2\beta(n-1+\beta(n+1-2\delta))(1-\alpha)^2 \le (1-\delta)(n-1+\beta(n+1-2\alpha))^2, \quad n=2,3,\dots \qquad (13)$$

implies (12).

The inequalities (13) are equivalent to

$A\delta \le B$,

where

$$A = -4\beta^2(1-\alpha)^2 + (n-1)^2 + 2\beta(n-1)(n+1-2\alpha) +$$
$$+ \beta^2(n+1-2\alpha)^2 = (n-1)(\beta+1)((n-1)(\beta+1) + 4\beta(1-\alpha)) > 0$$

and

$$B = (n-1)^2 + 2\beta(n-1)(n+1-2\alpha) + \beta^2(n+1-2\alpha)^2 -$$
$$- 2\beta(1-\alpha)^2(n-1) - 2\beta^2(1-\alpha)^2(n+1) =$$
$$= (n-1)(\beta+1)((n-1)(\beta+1) + 4\beta(1-\alpha) - 2\beta(1-\alpha)^2).$$

We obtain

$$\delta \le \frac{B}{A} = \frac{(n-1)(\beta+1)+4\beta(1-\alpha)-2\beta(1-\alpha)^2}{(n-1)(\beta+1)+4\beta(1-\alpha)} =$$
$$=1-\frac{2\beta(1-\alpha)^2}{(n-1)(\beta+1)+4\beta(1-\alpha)} = \delta(n)$$

We have

$$\delta \le \delta(2) \le \delta(n), \quad n = 2,3,\dots,$$

because $\delta(n)$ is an icreasing function of n.

Now we choose

$$\delta(\alpha,\beta) = \delta(2) = 1 - 2\beta(1-\alpha)^2/(5\beta+1-4\alpha\beta).$$

We have $\delta(\alpha,\beta) > \alpha$ because

$$\delta(\alpha,\beta)-\alpha = \frac{(3\beta+1)(1-\alpha)+4\alpha^2\beta}{4\beta(1-\alpha)+\beta+1} > 0$$

and $\delta(\alpha,\beta) < 1$, because

$$1-\delta(\alpha,\beta) = \frac{2\beta(1-\alpha)^2}{4\beta(1-\alpha)+\beta+1} > 0 .$$

The extremal functions are $f = g = f_2$ given by (9).

Remark 3. By using Theorem 2 and Remark 2.a) we obtain again Corollary 1.1.

Remark 4. Since $\sigma(\delta(\alpha,\beta),\beta) = \rho(\alpha,\beta)$, where $\rho(\alpha,\beta)$ is given by (10), we obtain $S^*(\delta(\alpha,\beta),\beta)\subset S^*(\rho(\alpha,\beta))$. So we can prove Corollary 1.2 by using 2 and Remark 1.

Remark 5. We have $\delta(\alpha,1) = (2-\alpha^2)/(3-2\alpha)$, hence we can obtain Corollary 1.3 by using Theorem 2 and Remark 1.

Remark 6. For given α and β, $\alpha \in [0,1)$, $\beta \in (0,1]$, the classes $S^*(\alpha,\gamma(\alpha,\beta))$ and $S^*(\delta(\alpha,\beta),\beta)$ are inclused in $S^*(\rho(\alpha,\beta))$, but they are generally distinct.

R E F E R E N C E S

1. V.P.Gupta and P.K.Jain, *Certain classes of univalent functions with negative coefficients*, Bull. Austral. Math. Soc., vol. 14(1976), 409-416.
2. A.Schild and H.Silverman, *Convolutions of univalent functions with negative coefficients*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A, XXIX, 12(1975), 99-107.

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UNIVALENCY CRITERIA OF KUDRIASOV'S TYPE

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REZUMAT. - Un criteriu de univalenţă de tip Kudriasov. În lucrare se obţin condiţii de univalenţă similare cu cele date de Kudriasov, condiţii care folosesc şi coeficientul a_2.

Let A be the class of regular functions in $U = \{z:|z|<1\}$, $f(z) = z + a_2z^2 + \ldots$ and $f(z)/z \neq 0$ for all $z \in U$.

THEOREM A [3]. *Let $f(z)$ be a regular function in U, $f(z) = z + a_2z^2 + \ldots$*

If

$$\left|\frac{f''(z)}{f'(z)}\right| \le M \tag{1}$$

for all $z \in U$, where $M = 3{,}05$, then the function $f(z)$ is univalent in U.

In Kudriasov's results the constant M doesn't depend from a_2. The result could be improved for valours of $|a_2|$ approaching to 0.

In this paper we obtain the conditions of univalence similar to the result of Kudriasov's type, conditions which use coefficient a_2 too.

THEOREM 1. *If $f(z)$ is a regular function in U, $f(z) = a + a_3z^3 + \ldots$, and*

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$$\left|\frac{f''(z)}{f'(z)}\right| \le 4 \tag{2}$$

for all z ∈ U, then the function f(z) is univalent in U.

Proof. Let's consider the function $g(z) = \frac{1}{4}\frac{f''(z)}{f'(z)}$. Using Schwarz's lemma [2] and Becker's univalence criterion [1], for the function $g(z)$, we obtain

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| = (1-|z|^2)\,|z|\cdot 4\,|g(z)| \le 4(1-|z|^2)\,|z|^2 \le 1,$$

and, hence, it results that the function $f(z)$ is univalent in U.

THEOREM 2. *Let α be a complex number,* Re α>0 *and the function f(z) belongs to the class A.*

If

$$\left|\frac{f''(z)}{f'(z)}\right| \le M \tag{3}$$

for all z ∈ U, where the constant M verifies the condition

$$M \le \frac{1}{\max\limits_{|z|\le 1}\left[\dfrac{(1-|z|^{2\operatorname{Re}\alpha})}{\operatorname{Re}\alpha}\,|z|\,\dfrac{|z|+\dfrac{2|a_2|}{M}}{1+\dfrac{2|a_2|}{M}|z|}\right]} \tag{4}$$

then, for every complex number β, Re β≥Re α *the function*

$$F_\beta(z) = \left[\beta\int_0^z u^{\beta-1}f'(u)\,du\right]^{\frac{1}{\beta}} \tag{5}$$

is regular and univalent in U.

Proof. Let's consider the function $F:[0,1] \to \mathbb{R}$,

$$F(x) = \frac{(1-x^{2\mathrm{Re}\,\alpha})}{\mathrm{Re}\,\alpha}\, x\, \frac{x+\frac{2|a_2|}{M}}{1+\frac{2|a_2|}{M}x} \;;\; x = |z|$$

Because $F\left(\frac{1}{2}\right) \neq 0$ it results that $\max\limits_{x\in[0,1]} F(x) > 0$. Let's consider the function $g(z) = \frac{1}{M}\frac{f''(z)}{f'(z)}$. Using the generalization of Schwarz's lemma [2] for the function $g(z) = \frac{1}{M}\frac{f''(z)}{f'(z)}$, where M is a real positive constant which verifies the inequality (4), we obtain

$$\left|\frac{1}{M}\frac{f''(z)}{f'(z)}\right| \le \frac{|z|+\frac{2|a_2|}{M}}{1+\frac{2|a_2|}{M}|z|} \tag{6}$$

for all $z \in U$, and, hence we have

$$\frac{(1-|z|^{2\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le$$
$$\le M \cdot \max_{|z|\le 1}\left|\frac{(1-|z|^{2\mathrm{Re}\,\alpha})}{\mathrm{Re}\,\alpha}\,|z|\,\frac{|z|+\frac{2|a_2|}{M}}{1+\frac{2|a_2|}{M}|z|}\right| \tag{7}$$

From (4) and (7) we obtain

$$\frac{(1-|z|^{2\mathrm{Re}\,\alpha})}{\mathrm{Re}\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \tag{8}$$

for all $z \in U$ and from Pascu's univalence criterion [4], it results that the function $F_\beta(z)$ is regular and univalent in U.

COROLLARY 1. *If the function $f(z)$ belongs to the class A and*

$$\left|\frac{f''(z)}{f'(z)}\right| \le M \tag{9}$$

for all $z \in U$, where the constant M verifies the inequality

$$M \le \frac{1}{\max\limits_{|z|\le 1}\left[(1-|z|^2)\,|z|\,\dfrac{|z|+\dfrac{2|a_2|}{M}}{1+\dfrac{2|a_2|}{M}|z|}\right]}; \tag{10}$$

then the function $f(z)$ is univalent in U.

Proof. From the THEOREM 2, for $\alpha = 1$ and $\beta = 1$, we obtain the COROLLARY 1.

Observation. From Kudriasov's result it doesn't result the THEOREM 1,but from COROLLARY 1 for $a_2=0$ we obtain the THEOREM 1.

R E F E R E N C E S

1. J.Becker, *Löwnersche Differentialgleichung und Schlichtheits-Kriterion*, Math. Ann. 202, 4(1973), 321-335.
2. G.M.Goluzin, *Gheometriceskaia teoria funktii kompleksnogo peremenogo*, Moscova, 1952.
3. S.N.Kudriasov, *O nekotorih priznakah odnolistnosti analiticeschih funcţii*, Matematiceschie zametki, T.13, Nr. *3*(1973), 359-366.
4. N.N.Pascu, *An improvement of Becker's univalence criterion*, Sesiunea comemorativă Simion Stoilow, Braşov, Preprint (1987), 43-48.

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TEST SETS IN QUANTITATIVE KOROVKIN APPROXIMATION

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REZUMAT. - Mulţimi test în aproximarea Korovkin cantitativă. Lucrarea conţine rezultate cantitative de tip Korovkin în care - în afară de funcţiile liniare - este utilizată ca funcţie test o singură funcţie convexă.

1. Let (X,d) be a compact metric space and let $B(X)$ denote the space of all real-valued bounded functions on X.

Let $C(X)$ be the subspace of $B(X)$ consisting of all continuous functions on X. For $f \in B(X)$ and $\delta > 0$ let

$$\omega(f,\delta) = \sup \{|f(x) - f(y)| : x,y \in X,\ d(x,y) \le \delta\}.$$

Suppose that there exists a constant $\mu > 0$ such that

$$\omega(f,t\delta) \le (1 + \mu t)\omega(f,\delta) \tag{1}$$

for all $f \in B(X)$ and all $t,\delta > 0$.

Let F be a nonnegative function in $B(X^2)$ such that

$$F(\cdot,y) \in C(X) \text{ for each } y \in X. \tag{2}$$

Suppose that there exist constants $q \ge 1$ and $k > 0$ such that

$$d^q(x,y) \le kF(x,y) \text{ for all } x,y \in X. \tag{3}$$

T.Nishishiraho [2] has proved

THEOREM 1. *Let $T:C(X) \to B(X)$ be a positive linear operator such that $T1 = 1$. Then*

$$|Tf(x) - f(x)| \le (1 + \mu k\delta^{-q}\, TF(\cdot,x)(x))\omega(f,\delta)$$

for all $f \in C(X)$, $x \in X$ and $\delta > 0$.

2. Let E be a normed real space and E' the dual of E endowed

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with the usual norm. Let X be a compact convex subset of E.

For $f \in C(X)$, $h_1, \dots, h_m \in E'$ and $\delta > 0$ let us denote

$$\omega(f;h_1,\dots,h_m) = \sup\{|f(x) - f(y)| : x,y \in X,$$

$$\sum_{i=1}^{m} (h_i(x) - h_i(y))^2 \le 1\},$$

$$\Omega(f,\delta) = \inf\{\omega(f;h_1,\dots,h_m) : m \ge 1,\ h_1,\dots,h_m \in E',$$
$$\sum_{i=1}^{m} \|h_i\|^2 = \delta^{-2}\}.$$

In what follows let $L:C(X) \to C(X)$ be a positive linear operator such that $L1 = 1$ and $Lh = h$ for all $h \in E'$.

For $x \in X$ and $\delta > 0$ let us denote

$$\tau(\delta,x) = \sup\{\sum_{i=1}^{m} (Lh_i^2(x) - h_i^2(x)) : m\ge 1,\ h_1,\dots,h_m \in E',\ \sum_{i=1}^{m} \|h_i\|^2 = \delta^{-2}\}.$$

Then we have (see [1], [5, Th.1.4]):

THEOREM 2. *Let $f \in C(X)$, $x \in X$, $\delta > 0$. Then*

(i) $0 < \delta_1 \le \delta_2$ *implies* $\Omega(f,\delta_1) \le \Omega(f,\delta_2)$ *and*

$$\tau(\delta_1,x) \ge \tau(\delta_2,x)$$

(ii) $\lim_{\delta \to 0} \Omega(f,\delta) = 0$

(iii) $|Lf(x) - f(x)| \le (1 + \tau(\delta,x))\Omega(f,\delta)$.

In Theorem 1 the test set is $\{1\} \cup \{F(\cdot,x) : x \in X\}$.

In Theorem 2 it is $\{1\} \cup E' \cup \{h^2 : h \in E'\}$.

Suppose now that there exists a constant $c > 0$ such that

$$2(\|x\|^2 + \|y\|^2) - \|x + y\|^2 \ge c\|x - y\|^2 \qquad (4)$$

for all $x,y \in E$ (See[6,p.86], [7]). In this case we shall obtain quantitative results in which- besides the linear functions - - only one convex function is involved as test function.

Let us remark that $c \le 1$; moreover, $c = 1$ if and only if E is an inner - product space. Condition (4) implies that E is uniformly convex (see[5]).

THEOREM 3. *Let* $f \in C(X)$, $x \in X$, $\delta > 0$. *Then:*

$$|Lf(x) - f(x)| \le (1 + LF(\cdot,x)(x)/c\delta^2)\omega(f,\delta) \quad (5)$$

$$|Lf(x) - f(x)| \le (1 + (Le - e)(x)/c\delta^2)\Omega(f,\delta) \quad (6)$$

where $e(x) = \|x\|^2$ *and* $F(x,y) = 2(e(x) + e(y)) - e(x+y)$, $x,y \in X$.

Proof. In this case (1) holds with $\mu = 1$ (see [2,Lemma 3]). By virtue of (4) we can choose $q = 2$ and $k = 1/c$; so (2) and (3) are also satisfied. Now (5) is a consequence of Th.1.

For $x,y \in X$, $a \in [0,1]$ and $f \in C(X)$ let us denote

$$(x,a,y;\ f) = (1 - a)f(x) + af(y) - f((1 - a)x + ay)$$

From (4) it follows (see[4]) that $(x,a,y;\ e) \ge ca(1-a)e(x-y)$ for all $x,y \in X$. Let $x \in X$. Then $f \in C(X) \to Lf(x)$ defines a probability Radon measure on X with barycenter x. It has been proved in [3] that for all $f \in C(X)$ there exist $u,v \in X$, $u \ne v$ and $a \in (0,1)$ such that

$$Lf(x) - f(x) = (Le(x) - e(x))(u,a,v;\ f)/(u,a,v;\ e).$$

Let $h \in E'$. Then we have $Lh^2(x) - h^2(x) = (Le(x) -$

$- e(x))(u,a,v;h^2)/(u,a,v;e) \le (Le(x) - e(x))(h(u) - h(v))^2/$

$/ce(u-v) \le (Le - e)(x)\ \|h\|^2/c$

It follows that $\tau(\delta,x) \le (Le - e)(x)/c\delta^2$ and thus (6) is a consequence of Th.2.

Let us remark that in (6) the test functions are the constant function 1, the linear functions and the convex function e. On the other hand it is easy to verify that $\omega \le \Omega$.

REFERENCES

1. M.Campiti, *A generalization of Stancu-Muhlbach operators*, Constr. Approx. 7(1991), 1-18.
2. T.Nishishiraho, *Convergence of positive linear approximation processes,*

Tôhoku Math.J. **35**(1983), 441-458.
3. I.Rasa, *On the barycenter formula*, Anal. Numer. Theor. Approx. **13**(1984), 163-165.
4. I.Rasa, *Convexity properties in normed linear spaces*, in: "Proc. Second Symp. Math. Appl.", Traian Vuia Polytechn. Inst., Timişoara, 1987, pp. 106-108.
5. I.Rasa, *Korovkin approximation and parabolic functions*, Conf. Sem. Mat. Univ. Bari **236**(1990).
6. L.Schwartz, *Geometry and Probability in Banach spaces*, Lect. Notes Math. **852**, Springer-Verlag, 1981.
7. R.Smarzewski, *Asymptotic Chebyshev centers*, J.Approx. Theory **59**(1989), 286-295.

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