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CONSIDERATIONS CONCERNING POWER ALGEBRA

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Dedicated to Professor I.Muntean on his 60^{th} anniversary

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RESUMAT. - **Consideraţii privind algebra părţilor.** Se dau două teoreme de caracterizare a extensiilor unei algebre universale şi anume:
1) Condiţia necesară şi suficientă ca o algebră de părţi să fie extensia unei algebra date.
2) Condiţia necesară şi suficientă ca o algebră de părţi să fie extensia unei algebre dintr-o varietate dată.

Let (A,Ω) be an universal algebra, $P(A)$ the Power set of A and $n\in N$. Note $\Omega_n = \{\omega\in\Omega \mid \omega \text{ is n-ary operation}\}$ and we have

$$\Omega = \bigcup_{n\in N} \Omega_n .$$

The operations from (A,Ω) are extended to operations in $P(A)$, as follows: if $\omega \in \Omega_n$ and $X_i \subseteq A$, $i = 1,\ldots,n$, then

$$\omega(X_1,\ldots,X_n) = \{\omega(x_1,\ldots,x_n) \mid x_i\in X_i, i=1,\ldots,n\} \qquad (1)$$

$(P(A),\Omega)$ is called the Power Algebra of (A,Ω).

Here we continue the study of $(P(A),\Omega)$, considering the variety of algebras. For more details see the bibliography.

The following theorem gives a necessary and sufficient condition for a structure of Ω -algebra on $P(A)$ to be obtained from an Ω -algebra on A, by (1).

THEOREM 1. *Let* $(P(A),\Omega)$ *be a structure of universal algebra on Power set of* A. *There exists a structure of universal algebra* (A,Ω) *on* A *so that* $(P(A),\Omega)$ *may be obtained from* (A,Ω), *by* (1), *if and only if* $(P(A),\Omega)$ *verify the conditions:*

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i) *The subset* $\mathbf{A} = \{\{x\} \mid x \in A\}$ *is a subalgebra of the algebra* $(P(A),\Omega)$.

ii) *If* $n \in N^*$, $\omega \in \Omega_n$; $X_1,\dots,X_n \subseteq A$, $i = 1,\dots,n$, *then*

$$X_i = \phi \Rightarrow \omega(X_1,\dots,X_n) = \phi.$$

iii) *If* $n \in N^*$, $\omega \in \Omega_n$, $i = 1,\dots,n$; $X_1,\dots,X_{i-1},X_{i+1},\dots,X_n \subseteq A$ *and* $X_{ij} \subseteq A$, $j \in J$, *then*

$$\omega(X_1,\dots,X_{i-1}, \bigcup_{j\in J} X_{ij},X_{i+1},\dots,X_n) = $$

$$= \bigcup_{j\in J} \omega(X_1,\dots,X_{i-1},X_{ij},X_{i+1},\dots,X_n),$$

thas is, each (not nullary) operation $\omega \in \Omega$ *is distributive relatively to union, on each from components.*

Proof. Suppose that there is a structure of Ω-algebra on A, so that $(P(A),\Omega)$ is the Power algebra of (A,Ω). From (1) it results that for every $n \in N, \omega \in \Omega_n$ and $x_1,\dots,x_n \in A$, we have:

$$\omega(\{x_1\},\dots,\{x_n\}) = \{\omega(x_1,\dots,x_n)\}.$$

Therefore $\mathbf{A}$ is a subalgebra of the algebra $(P(A),\Omega)$, and the condition i). Also from (1) follows the condition ii).
Using (1), we hawe:
$x \in \omega(X_1,\dots,X_{i-1}, \bigcup_{j\in J} X_{ij},X_{i+1},\dots,X_n) \Leftrightarrow$ there exist $(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_n) \in X_1\times\dots\times X_{i-1} \times \bigcup_{j\in J} X_{ij}\times X_{i+1}\times\dots$ $\dots\times X_n$, so that $x = \omega(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_n) \Leftrightarrow$
$\Leftrightarrow$ there exists $j \in J$ and $(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_n) \in$
$\in X_1\times\dots\times X_{i-1}\times X_{ij}\times X_{i+1}\times\dots\times X_n$ so that $x =$
$=\omega(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_n) \Leftrightarrow$ there is $j \in J$ so that
$x \in \bigcup_{j\in J} \omega(X_1,\dots,X_{i-1},X_{ij},X_{i+1},\dots,X_n)$, and so, the condition iii) is verified too.

Conversely, suppose that $(P(A),\Omega)$ verifies the conditions i), ii), iii). From i) it results that for each $n \in N$, $\omega \in \Omega_n$ and $x_1,\ldots,x_n \in A$, the subset $\omega(\{x_1\},\ldots,\{x_n\})$ contains the only element x, that is, $\omega(\{x_1\},\ldots,\{x_n\}) = \{x\}$. Using this fact, we may define the n-ary operation $\omega: A^n \to A$ by

$$\omega(x_1,\ldots,x_n) = x. \tag{2}$$

So is defined on A a structure of Ω-algebra. The symbol ω denotes both an operation in the gived algebra $(P(A),\Omega)$ and the corresponding operation defined by (2) in A. Let ω' be the operation defined in P(A) by (1), starting from ω in (A,Ω). To close the proof, it is sufficient to prove the coincidence of the operations ω and ω' on $(P(A),\Omega)$. Give $X_i \subseteq A$, $i = 1,\ldots,n$.
If $X_i = \{x_i\}$ $(i = 1,\ldots,n)$, then from (1) and (2) it results:

$$\omega'(\{x_1\},\ldots,\{x_n\}) = \{\omega(x_1,\ldots,x_n)\} = \omega(\{x_1\},\ldots,\{x_n\}). \tag{3}$$

If one from the sets $X_1,\ldots,X_n$ is empty, then from (1) and (i) follows: $\omega'(X_1,\ldots,X_n) = \phi = \omega(X_1,\ldots,X_n)$.
If each of the sets $X_1,\ldots,X_n$ is nonempty then, from iii) and (3) follows:

$$\omega(X_1,\ldots,X_n) = \omega\Big(\bigcup_{x_1\in X_1}\{x_1\},\ldots,\bigcup_{x_n\in X_n}\{x_n\}\Big) =$$
$$= \bigcup\{\omega(\{x_1\},\ldots,\{x_n\}) \mid (x_1,\ldots,x_n)\in X_1\times\ldots\times X_n\} =$$
$$= \bigcup\{\omega'(\{x_1\},\ldots,\{x_n\}) \mid (x_1,\ldots,x_n)\in X_1\times\ldots\times X_n\} =$$
$$= \omega'\Big(\bigcup_{x_1\in X_1}\{x_1\},\ldots,\bigcup_{x_n\in X_n}\{x_n\}\Big) = \omega'(X_1,\ldots,X_n).$$

Therefore $\omega = \omega'$, that is, the Power-algebra of the algebra (A,Ω) coincides with $(P(A),\Omega')$.

The set Ω of the operation's symbols, together with the "arity"-function $\Omega \to N$, is called **signature**. The fact that ω is the symbol of a n-ary operation is designet by $\omega \in \Omega_n$.

To give an algebra of signature Ω on the set A means to give a function which associates to each $\omega \in \Omega_n$, an n-ary operation on A (designed by $\dot{\omega}$ too). If Ω is a signature and X is a so called alphabet (from Ω_o disjoint set), then we may define the **algebra of Ω-words** (or the algebra of polynomials) over X, inductively, by the folloving:

(i) The elements from $X \cup \Omega_o$ are words (polynomials).

(ii) If $n \in N^*$, $\omega \in \Omega_n$ and $p_1,\ldots,p_n$ are words (polynomials) then $\omega(p_1,\ldots,p_n)$ is a word (a polynomial).

If the Ω -words over the same alphabet X are expressed by $x_1,\ldots,x_n \in X$, then the formal equality

$$p(x_1,\ldots,x_n) = q(x_1,\ldots,x_n)$$

is called **Ω-identity**. This identity is verified in the algebra (A,Ω), if for every replacing of x_i by $a_i \in A$, $i = 1,\ldots,n$ is obtained in A the equality:

$$p(a_1,\ldots,a_n) = q(a_1,\ldots,a_n).$$

Let Λ be a set of Ω -identities. The class of all the Ω -algebras which verify each of the identities in Λ is called variety of Ω -algebras defined by Λ, denoted by (Ω,Λ).

It is shown that, even (A,Ω) is in the variety (Ω,Λ), the Power algebra $(P(A),\Omega)$ is not necessarily in the same variety. The following theorem characterizes the Ω -algebras on P(A), defined by (1) from (A,Ω), which belong to a given variety (Ω,Λ).

THEOREM 2. *Let* A *be a set,* $(P(A),\Omega)$ *an universal algebra on* P(A) *and* (Ω,Λ) *a variety of* Ω -algebras. *There is a structure* (A,Ω) *of universal algebra on* A, *in the variety* (Ω,Λ), *so that* $(P(A),\Omega)$ *is the Power algebra of* (A,Ω), *if and only if the algebra* $(P(A),\Omega)$ *verifies the conditions* i),ii),iii) *from Theorem*

1 and the condition

iv) *The subalgebra* $\mathbf{A} = \{\{x\} | x \in A\}$ *of* $(P(A),\Omega)$ *belongs to the variety* (Ω,Λ).

Proof. From Theorem 1 it follows that there is a structure of Ω -algebra on A so that $(P(A),\Omega)$ is the Power algebra of (A,Ω) if and only if $(P(A),\Omega)$ verifies the conditions i), ii) and iii). In accordance with i), $\mathbf{A}$ is a subalgebra of $(P(A),\Omega)$. The function $f:A \to P(A), f(x) = \{x\}$, mappes izomorfely the algebra (A,Ω) on the subalgebra $\mathbf{A}$. Therefore $(A,\Omega)\in(\Omega,\Lambda)$ if and only if $\mathbf{A}\in(\Omega,\Lambda)$.

COROLLARIES. 1) *Let* A *be a set and* $(P(A),\cdot)$ *a groupoide. There is a structure of semigroup* $(A,\cdot)$ *on* A *so that*
(4). $X.Y = \{x.y | x \in X, y \in Y\}$, *for each* $X,Y \in P(A)$, *if and only if* $(P(A),\cdot)$ *verifies the conditions:*

i) $\mathbf{A}=\{\{x\} | x \in A\}$ *is a subgroupoide of* $(P(A),\cdot)$.

ii) $X = \phi$ *or* $Y = \phi \Rightarrow X.Y = \phi$.

iii) $(X \cup X').Y = (X.Y) \cup (X'.Y)$,

$X.(Y\cup Y') = (X.Y) \cup (X.Y')$, *for each* $X,X',Y,Y' \subseteq A$.

iv) $(\mathbf{A},\cdot)$ *is a semigroup.*

The groupoide which verifies the conditions above is a semigroup.

2) *Let* A *be a set and* $(P(A),\cdot)$ *a groupoide. There is a group structure on* A, $(A,\cdot)$ *so that the operation in* $(P(A),\cdot)$ *is defined by* (4), *if and only if* $(P(A),\cdot)$ *verifies the above conditions* i),ii),iii) *and* $(\mathbf{A},\cdot)$ *is a group.*

3) *Let* A *be a set and* $(P(A),+,.)$ *a structure with two binary operations. There is a structure of ring on* A, $(A,+,.)$ *so that*

$X+Y = \{x+y \mid x \in X \text{ and } y \in Y\}$

$X.Y = \{x.y \mid x \in X \text{ and } y \in Y\}$ for each $X,Y \subseteq A$,

if and only if $(P(A),+,.)$ *verifies the conditions:*

i) $\{x\} + \{y\}$ *and* $\{x\}.\{y\}$ *contain all a single element, that is,* A *is subalgebra of* $(P(A),+,.)$.

ii) $X = \phi$ or $Y = \phi \Rightarrow X+Y = \phi$ *and* $X.Y = \phi$.

iii) $(X \cup X')+Y = (X+Y) \cup (X'+Y)$

$X+(Y \cup Y') = (X+Y) \cup (X+Y')$

$(X \cup X').Y = (X.Y) \cup (X'.Y)$

$X.(Y \cup Y') = (X.Y) \cup (X.Y')$, for each $X,X',Y,Y' \subseteq A$.

iv) $(A,+,.)$ *is a ring.*

B I B L I O G R A P H Y

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ON THE JENSEN - HADAMARD INEQUALITY

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REZUMAT. - Despre inegalitatea lui Jensen-Hadamard. În această notă obţinem o generalizare pentru funcţionalele liniare şi pozitive a inegalităţii lui Jensen-Hadamard; extensii comune pentru o inegalitate a lui J. Sándor [3] şi H. Alzer [1]; precum şi alte rezultate înrudite.

1. Introduction. The famous Jensen-Hadamard inequality states that for a (continuous) convex function $f:[a,b] \to \mathbb{R}$ (with $a<b$) one has

$$(b-a)f\left(\frac{a+b}{2}\right) \le \int_a^b f(x)dx \le (b-a)\left[\frac{f(a)+f(b)}{2}\right] \tag{1}$$

For relation (1) many applications in different branches of mathematics have been obtained (See e.g. [2],[3],[4],[5],[6]) and certain extensions and generalizations are also known ([3], [1], [2]). The aim of this paper is to obtain some new generalizations and other relations related to the Jensen-Hadamard inequality.

2. A generalization. Let $f:[a,b] \to \mathbb{R}$ be a (continuous) convex function and $L:C[a,b] \to \mathbb{R}$ a positive, linear functional on $C[a,b]$ - the space of all continuous functions defined on $[a,b]$. Let us denote by $e_k(x) = x^k$, $x \in [a,b]$, $k \in \mathbb{N}$.

THEOREM 1. *If the above conditions are satisfied, with* $L(e_0) = 1$ *, then:*

$$f(L(e_1)) \le L(f) \le L(e_1)\left[\frac{f(b)-f(a)}{b-a}\right] + \frac{bf(a)-af(b)}{b-a} \tag{2}$$

Proof. Since f is convex, it is well-known that

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$$f(x)-f(y)\geq f'_{+}(y)(x-y),\ x,y\in[a,b]$$

By setting $y=L(e_1)$ and applying the positive linear functional L we get $L(f)\geq f(L(e_1))\cdot L(e_0)+f'_{+}(L(e_1))\cdot(L(e_1)-L(e_1))=f(L(e_1))$ by $L(e_0)=1$. This gives the left side of (2), where clearly, from $a\leq e_1(x)\leq b$ we have $aL(e_0)\leq L(e_1)\leq bL(e_0)$, i.e. $L(e_1)\in[a,b]$.

For the right side of (2), let us consider the inequality

$$f(x)\leq(x-a)\frac{f(b)}{b-a}+(b-x)\frac{f(a)}{b-a}$$

which means intuitively that the graph of f on [a,b] is below the line segment joining $(a,f(a))$ and $(b,f(b))$. From $e_1(x)=x$, $e_0(x)=1$, $x\in[a,b]$ by application of L, after simple calculations we get the desired result ∎

Remarks. 1) For $L(f)=\frac{1}{b-a}\int_a^b f(t)dt$ we have $L(e_0)=1$ and L is positive linear functional. For this L, relation (2) gives exactly inequality (1).

2) Let $w_i\geq 0$ $(i=1,\ldots,n)$ with $\sum_{i=1}^n w_i=1$, and let $a_i\in[a,b]$, $i=1,\ldots,n$. Let us define $L(f)=\sum_{i=1}^n w_i f(a_i)$

Then clearly L is positive linear functional, so by (2) we get:

$$f\left(\sum_{i=1}^n w_i a_i\right)\leq\sum_{i=1}^n w_i f(a_i)\leq\left(\sum_{i=1}^n w_i a_i\right)\left[\frac{f(b)-f(a)}{b-a}\right]+\frac{bf(a)-af(b)}{b-a}\qquad(3)$$

for a convex function $f[a,b]\to\mathbb{R}$. The left side of this relation is the well-known Jensen inequality for n numbers.

3. On an inequality of Sándor and Alzer. In this section we shall obtain a unified method to prove certain generalization of

(1) discovered by J.Sándor [3] and H.Alzer [1]. First we state two lemmas.

LEMMA 1. *For* $x \in [a,b]$ *one has*

$$(b-a)^n/2^{n-1} \le (x-a)^n+(b-x)^n \le (b-a)^n, \quad n\ge 1 \tag{4}$$

Proof. We consider the functions $h:[a,b] \to \mathbb{R}$ defined by $h(x) = (x-a)^n + (b-x)^n$. Here $h(a) = h(b) = (b-a)^n$ and $h(\frac{a+b}{2}) = (b-a)^n / 2^{n-1}$. Obviously, $h'(x) = n[2x-(a+b)]\cdot q(x)$, with $q(x) = (x-a)^{n-2} + (x-a)^{n-3}(b-x) + \ldots + (b-x)^{n-2} > 0$, so $h'(x) \le 0$ for $x \le (a+b)/2$; and $h'(x) \ge 0$ for $x \ge (a+b)/2$. We get $h(x)\le h(a)$ for $x \in [a, \frac{a+b}{2}]$ and $h(x)\le h(b)$ for $x \in [\frac{a+b}{2}, b]$ In all cases $h(x) \le (b-a)^n$ and $h(x) \ge (b-a)^n / 2^{n-1}$ ■

LEMMA 2. *For* $f \in C^n[a,b]$ *and* $t \in [a,b]$ *one has*

$$(-1)^n\int_a^b f(x)\,dx = \sum_{i=1}^{n}\left[\frac{(t-a)^i-(t-b)^i}{i!}\right]f^{(i-1)}(t)\cdot(-1)^{n-i+1} + \frac{1}{n!}\left[\int_a^t (x-a)^n f^{(n)}(x)\,dx + \int_t^b (x-b)^n f^{(n)}(x)\,dx\right] \tag{5}$$

Proof. Applying the generalized partial integration formula (called also "Green-Lagrange identity") we can write:

$$\int_a^t (x-a)^n f^{(n)}(x)\,dx = (t-a)^n f^{(n-1)}(t) - n(t-a)^{n-1}f^{(n-2)}(t) + n(n-1)\cdot (t-a)^{n-2}f^{(n-3)}(t) - \ldots + (-1)^k n(n-1)\ldots(n-k+1)(t-a)^{n-k}\cdot f^{(n-k+1)}(t) + \ldots + (-1)^{n-1}n!(t-a)f(t) + (-1)^n\int_a^t n!f(x)\,dx$$

and likewise

$$\int_t^b (x-b)^n f^{(n)}(x)\,dx = -(t-b)^n f^{(n-1)}(t) + n(t-b)^{n-1}f^{(n-2)}(t) - n(n-1)\cdot (t-b)^{n-2}f^{(n-3)}(t) + \ldots + (-1)^{k+1}n(n-1)\ldots(n-k+1)(b-a)^{n-k}\cdot f^{(n-k+1)}(t) + \ldots + (-1)^n n!(t-b)f(t) + (-1)^n\int_t^b n!f(x)\,dx$$

Adding these two relations and dividing with n! we obtain (5) ■

We now prove the following:

THEOREM 2. *Let* $f \in C^{2k}[a,b]$ *with* $f^{(2k)}(x) \geq 0$ *for* $x \in (a,b)$ ($k \geq 1$, *positive integer*) *and let* $t \in [a,b]$ *be arbitrary. Then:*

$$\int_a^b f(x)\,dx \geq \sum_{i=1}^{2k}\left[\frac{(t-a)^i-(t-b)^i}{i!}\right](-1)^{i-1}f^{(i-1)}(t) + \frac{1}{(2k)!}\left\{\frac{(b-a)^{2k}}{2^{2k-1}}\cdot[f^{(2k-1)}(t) - f^{(2k-1)}(a)] + S_{k,a,b}(t)\right\} \tag{6}$$

and

$$\int_a^b f(x)\,dx \leq \sum_{i=1}^{2k}\left[\frac{(t-a)^i-(t-b)^i}{i!}\right](-1)^{i-1}f^{(i-1)}(t) + \frac{1}{(2k)!}\cdot \{(b-a)^{2k}\cdot[f^{(2k-1)}(t) - f^{(2k-1)}(a)] + S_{k,a,b}(t)\} \tag{7}$$

where $S_{k,a,b}(t) = \int_a^b (b-x)^{2k}f^{(2k)}(x)\,dx - 2\int_a^b (b-x)^{2k}f^{(2k)}(x)\,dx$.
If $f^{(2k)}(x) > 0$, *the inequalities are strict.*

Proof. We apply Lemma 2 for $n = 2k$ and first the right side of (4), then the left side of (4). Since $\int_t^b = \int_a^b - \int_t^a$, the theorem follows by simple computations. From the proofs of (4) and (5) we can see that for $f^{(2k)}(x) > 0$, $x \in (a,b)$, the inequalities in (6) and (7) are strict ∎

THEOREM 3. *Under the same conditions,*

$$\sum_{j=0}^{k-1}\frac{(b-a)^{2j+1}}{2^{2j}(2j+1)!}f^{(2j)}\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)\,dx \leq \sum_{j=0}^{k-1}\frac{(b-a)^{2j+1}}{2^{2j}(2j+1)!}\cdot f^{(2j)}\left(\frac{a+b}{2}\right) + \frac{1}{(2k)!\,2^{2k}}(b-a)^{2k}\,[f^{(2k-1)}(b) - f^{(2k-1)}(a)] \tag{8}$$

Proof. Let us apply Lemma 2 with $t = (a+b)/2$. Since $\left(\frac{b-a}{2}\right)^i - \left(\frac{a-b}{2}\right)^i = 2\left(\frac{b-a}{2}\right)^i$, for i odd; $= 0$, for i even; with the notation $i=2j+1$ we easily can find the left side of (8). In order to prove the right-hand side inequality, we can remark that $(x-a)^{2k} \le \left(\frac{b-a}{2}\right)^{2k}$, if $x \in \left[a, \frac{a+b}{2}\right]$, and $(b-x)^{2k} \le \left(\frac{b-a}{2}\right)^{2k}$, for $x \in \left[\frac{a+b}{2}, b\right]$. So, in all cases the second term is less than

$$\frac{1}{(2k)!}\cdot\frac{(b-a)^{2k}}{2^{2k}}\int_a^b f^{(2k)}(x)\,dx = \frac{1}{(2k)!}\cdot\frac{(b-a)^{2k}}{2^{2k}}[f^{(2k-1)}(b)-f^{(2k-1)}(a)] \ \blacksquare$$

Remark. The left side of Theorem 3 is due to J. Sándor [3].

THEOREM 4. *With the same conditions,*

$$\frac{1}{2}\sum_{i=1}^{2k}\frac{(b-a)^i}{i!}[f^{(i-1)}(a)+(-1)^{i-1}f^{(i-1)}(b)] + \frac{(b-a)^{2k}}{2^{2k-2}(2k)!}[f^{(2k-1)}(b) - f^{(2k-1)}(a)] \le \int_a^b f(x)\,dx \le \frac{1}{2}\sum_{i=1}^{2k-1}\frac{(b-a)^i}{i!}[f^{(i-1)}(a)+(-1)^{i-1}f^{(i-1)}(b)] \tag{9}$$

If $f^{(2k)}(x) > 0$, *the inequalities are strict.*

Proof. Setting t=a and t=b in (6), after addition we get the left side inequality. By doing the same thing with (7) we get the right side of (9) ■

Remark. The right side of (9) is due to H.Alzer [1].

4. **Some related inequalities.** Finally, we will prove two related results.

THEOREM 5. *If* $f \in C^n[a,b]$, *then:*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)}\int_a^b f(x)\,dx\right| \le \sum_{j=1}^{\left[\frac{n-1}{2}\right]}\frac{(b-a)^{2j}}{2^{2j}}\left|f^{(2j)}\left(\frac{a+b}{2}\right)\right| + \frac{1}{2^n\cdot n!}\int_a^b |f^{(n)}(x)|\,dx \tag{10}$$

Proof. We apply Lemma 2 with t=(a+b)/2. The modulus-inequality for sums and integrals implies at once the result if we observe that $\left(\frac{b-a}{2}\right)^i - \left(\frac{a-b}{2}\right)^i = 0$, i even; $= 2\left(\frac{b-a}{2}\right)^i$, otherwise. Remark that $i = 2j + 1 \le n \Rightarrow j \le \left[\frac{n-1}{2}\right]$, where $[x]$ denotes the integer part of x ∎

Remark. For $n = 1$ we obtain the inequality

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{2}\int_a^b |f'(x)|\,dx \tag{11}$$

for $f \in C^1[a,b]$. This improves the relation

$$\left|f\left(\frac{a+b}{2}\right)\right| \le \frac{1}{b-a}\int_a^b f(x)\,dx + \frac{1}{2}\int_a^b |f'(x)|\,dx \tag{12}$$

known as "Gallagher - Sobolev inequality" ([7])

THEOREM 6. *If* $f \in C^n[a,b]$ *and* $|f^{(n)}(t)| \le M$ *for all* $t \in [a,b]$, *then*

$$\left|\int_a^b f(x)\,dx + \sum_{i=1}^{n}\left[\frac{(t-a)^i-(t-b)^i}{i!}\right](-1)^i f^{(i-1)}(t)\right| \le \frac{M(b-a)^{n+1}}{n!} \tag{13}$$

Proof. The result follows by an application of Lemma 2 and the remark that

$$\int_a^t (x-a)^n dx + \int_t^b (b-x)^n dx \le \int_a^b [(x-a)^n + (b-x)^n]\,dx \le (b-a)^{n+1}$$

by Lemma 1 ∎

COROLLARY. *Under the same conditions,*

$$\left|2\int_a^b f(x)\,dx + \sum_{i=1}^{n}\frac{(b-a)^i}{i!}[(-1)^i f^{(i-1)}(b) - f^{(i-1)}(a)]\right| \le \frac{2M(b-a)^{n+1}}{n!} \tag{14}$$

Proof. Using (13) for $t = a$ and $t = b$, respectively, from the modulus inequality we get relation (14) ∎

ON THE JENSEN - HADAMARD INEQUALITY

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ON CERTAIN DIFFERENTIAL AND INTEGRAL INEQUALITIES FOR ANALYTIC FUNCTIONS

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REZUMAT. - Asupra unor inegalităţi diferenţiale şi integrale pentru funcţiile analitice. Fie A_n clasa funcţiilor f analitice în discul unitate $U = \{z;\ |z| < 1\}$ care admit dezvoltarea de forma $f(z) = z + a_{n+1}z^{n+1}+\dots,\ z \in U$, unde $n\geq 1$. Fie $g \in A_n$ şi fie $f(z) = \int_0^z g(t)/t\, dt$. Se arată că dacă g satisface inegalitatea $|g'(z) - 1| < M_n$ unde M_n este dat de (2), atunci $|zf'(z)/f(z) - 1| < 1$, care este echivalentă cu $\mathrm{Re}\int_0^1 g(uz)/ug(z)\, du > \frac{1}{2}$ pentru $z \in U$. In cazul $n = 1$ acest rezultat a fost obţinut în [3].

1. Introduction. Let A_n denote the class of functions f which are analytic on the unit disc $U=\{z;\ |z|<1\}$, of the form $f(z) = z+a_{n+1}z^{n+1} + \dots,\ z\in U$ where n is a positive integer. In a recent paper the first author obtained the following result: If $g \in A = A_1$ satisfies $|g'(z) - 1| < M_1 = 8/(2 + \sqrt{15})$ then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \text{ for } z \in U \quad \text{where}$$
$$f(z) = \int_0^z \frac{g(t)}{t}\, dt = \int_0^1 \frac{g(uz)}{u}\, du$$

In the present paper we extend the above result to the class A_n, for all $n\geq 1$. This new result allow us to improve some of the particular examples given in [3].

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2. Preliminaries. If f and g are analytic functions on U, and g is univalent then we say that f is subordonate to g, written $f \prec g$, or $f(z) \prec g(z)$ if $f(0) = g(0)$ and $f(U) \subset g(U)$. We shall use the following lemmas to prove our results.

LEMMA 1. [1, p.192]. *Let h be a convex function on* U *(i.e h is univalent and* h(U) *is a convex domain). If p is analytic in* U, *of the form* $p(z) = 1 + p_n z^n + \dots$, $z \in U$, $n \geq 1$ *and* p *satisfies the differential subordonation* $p(z) + z \cdot p'(z) \prec q(z)$, *then* $p(z) \prec$ $\prec q(z)$, *where*

$$q(z) = \frac{1}{nz^{1/n}} \int_0^z h(t)\, t^{1/n-1}\, dt$$

LEMMA 2. [2, p.201] *Let* E *be a set in the complex plane* **C** *and let* q *be an analytic and univalent function on* U. *Suppose that the function* $H: \mathbf{C} \times U \to \mathbf{C}$ *satisfies*

$H[q(\zeta),\ m\ \zeta\ q'(\zeta)\ z] \notin E$,

whenever $m \geq n$, $|\zeta| = 1$ and $z \in U$. *If* p *is analytic on* U *of the form* $p(z) = q(0) + p_n z^n + \dots$, *and* p *satisfies*

$H[p(z),\ zp'(z); z] \in E$, *for* $z \in U$, *then* $p \prec q$.

3. Main results.

THEOREM 1. *If* $f \in A_n$, $n \geq 1$, *satisfies*

$$|f'(z) + zf''(z) - 1| \leq M,\ z \in U, \tag{1}$$

where $M \leq M_n$, *with*

$$M_n = (n+1)^2 \frac{\sqrt{(n+1)^6 - 4n} - n(n+3)}{(n+1)^4 - n(n+4)} \tag{2}$$

then

$$\left|\frac{zf'(z)}{f(z)}-1\right| < 1, \quad z \in U \tag{3}$$

Proof. Since the inequality (1) can be rewritten as

$$f'(z) + z\cdot f''(z) \prec 1 + Mz,$$

by using Lemma 1, we deduce $f'(z) \prec 1 + Mz/(n+1)$ and

$$\frac{f(z)}{z} \prec 1+\frac{Mz}{(n+1)^2} \tag{4}$$

Let $p(z) = z\cdot f'(z)/f(z)$ and $P(z) = f(z)/z$. Since $M_n \le (n+1)^2$, from (4) we deduce $P(z) \neq 0$; which shows that the function p is analytic in U and the inequality (1) becomes

$$|P(z)[\, zp'(z) + p^2(z)\,] - 1| < M, \; z \in U \tag{5}$$

The inequality (3) is equivalent to

$$p(z) \prec 1+z = q(z) \tag{6}$$

and to prove (6), by Lemma 2, it is sufficient to check the inequality

$$|P(z)\;[\, m\zeta + (1 + \zeta)^2] - 1| \ge M \tag{7}$$

for all $m \ge n$, $|\zeta| = 1$ and $z \in U$. If we let $\zeta = e^{i\theta}$, then

$$\begin{aligned} L(m,\theta,z) &= |P(z)[\, m\zeta + (1 + \zeta)^2] - 1|^2 = \\ &= |P(z)\cdot\zeta\cdot(\zeta + \overline{\zeta} + m + 2) - 1|^2 = \\ &= (2\cos\theta + m + 2)\{(2\cos\theta + m + 2)|P(z)|^2 - \\ &- 2\,\mathrm{Re}[e^{i\theta}P(z)]\} + 1. \end{aligned}$$

From (4) we deduce $|P(z) - 1| < M/(n+1)^2$ and $|P(z)| > 1-M/(n+1)^2$. For $m \ge n$ we have

$$\frac{1}{2}\cdot\frac{\partial L}{\partial m} = (2\cos\theta + m+2)\cdot|P(z)|^2-\mathrm{Re}\,[e^{i\theta}P(z)] =$$

$$= (m+2)\cdot|P(z)|^2+\mathrm{Re}\{e^{i\theta}P(z)[2\overline{P(z)}-1]\}\geq$$

$$\geq|P(z)|\cdot\{(n+2)\cdot|P(z)|-|2P(z)-1|\}\geq$$

$$\geq|P(z)|\cdot\left\{n+2-\frac{n+2}{(n+1)^2}M-\frac{2M}{(n+1)^2}-1\right\}=$$

$$=|P|\cdot\left\{n+1-\frac{n+4}{(n+1)^2}M\right\}>0\ ,$$

which shows that L is an increasing function of n.

Hence we deduce

$L(m,\theta,z)\geq L(n,\theta,z) = (2\cos\theta+n+2)\{(2\cos\theta+n+2)|P|^2 - 2\mathrm{Re}[e^{i\theta}P]\}+1\geq$

$\geq (2\cos\theta+n+2)\{(n+2)\cdot|P|^2 + 2\mathrm{Re}[e^{i\theta}P(\overline{P}-1)]\}+1 \geq$

$\geq (2\cos\theta+n+2)\{(n+2)\cdot|P|^2-2|P|\cdot|P-1|\}+1 =$

$= (2\cos\theta+n+2)|P|\{(n+2)|P|-2|P-1|\}+1 \geq$

$$\geq n\left[1-\frac{M}{(n+1)^2}\right]\left\{(n+2)\left[1-\frac{M}{(n+1)^2}\right]-\frac{2}{(n+1)^2}M\right\}+1=K(M)$$

Since $0<M<M_n$, where M_n given by (2) is the positive root of the equation $K(M)=M^2$, we deduce $L(m,\theta,z)\geq M^2$, which yields (7). Hence the subordonation (6) holds and we obtain (3), which completes the proof of Theorem 1.

The following two theorems are versions of Theorem 1.

THEOREM 2. *If* $g\in A_n$ *satisfies* $|g'(z)-1|<M_n$, *where* M_n *is given by* (2) *then*

$$\left|\frac{z\cdot f'(z)}{f(z)}-1\right|<1\ ,\ \text{for } z\in U$$

where $f(z)=\int_0^z\frac{g(t)}{t}dt = \int_0^1\frac{g(uz)}{u}du.$

THEOREM 3. *If* $g \in A_n$ *satisfies* $|g'(z) - 1| < M_n$, *where* M_n *is given by* (2) *then*

$$\mathrm{Re} \int_0^1 \frac{g(uz)}{ug(z)} du > \frac{1}{2}, \text{ for } z \in U.$$

From (2) we deduce the following particular values of M_n:

$$M_1 = \frac{8}{\sqrt{15}+2} = 1.362 \ldots$$
and
$$M_2 = \frac{81}{\sqrt{721}+10} = 2.198 \ldots$$

4. Examples.

Example 1. If we let $g(z) = (\sin\lambda z)/\lambda$ than $g \in A_2$ and if

$$|\lambda| \le \ln[1+M_2+\sqrt{M_2(M_2+2)}] = 1.830\ldots$$

then we deduce

$$|g'(z)-1| = 2\left|\sin^2\frac{\lambda z}{2}\right| \le 2\,\mathrm{sh}^2\frac{|\lambda z|}{2} < 2\,\mathrm{sh}^2\frac{|\lambda|}{2} \le M_2 = 2.198\ldots,$$

for $z \in U$ and by Theorem 3 we obtain

$$\mathrm{Re}\frac{\mathrm{Si}(z)}{\sin z} > \frac{1}{2}, \text{ for } |z| < 1.830\ldots, \tag{8}$$

where

$$\mathrm{Si}(z) = \int_0^1 \frac{\sin uz}{u} du = \int_0^z \frac{\sin t}{t} dt.$$

We note that in [3] the inequality (8) was proved only for $|z| < 1.504\ldots$

Exemple 2. If we let $g(z) = (\tan\lambda z)/\lambda$, then $g \in A_2$ and if $|\lambda| \le \arctan\sqrt{M_2} = 0.977\ldots$

then we deduce $|g'(z)-1| = |\tan^2\lambda z| \le \tan^2|\lambda z| < \tan^2|\lambda| \le M_2$, for $z \in U$ and by Theorem 3 we obtain

$$\mathrm{Re} \int_0^1 \frac{\tan uz}{u \tan z}\, du > \frac{1}{2} \quad , \text{ for } |z| < 0.977\ldots \qquad (9)$$

We note that in [3] the inequality (9) was proved only for $|z| < 0.862\ldots$

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ON THE UNIVALENCE OF AN INTEGRAL OPERATOR

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REZUMAT. - Asupra univalenţei unui operator integral. In această notă se obţine o condiţie suficientă de univalenţă pentru funcţii de forma (1).

In this note we obtain a sufficient condition for univalence of function

$$F_\alpha(z) = \left[\alpha\int_0^z f^{\alpha-1}(u)\,du\right]^{\frac{1}{\alpha}} \tag{1}$$

where $f(z) = z + a_2z^2 + \ldots$, is a regular function in $U = \{z:|z|<1\}$ and α is a complex number. The following Lemma is due to Ch. Pommerenke ([1]).

LEMMA ([1]). *Let* $f(z,t) = a_1(t)z+\ldots$, $a_1(t) \neq 0$ *be regular for each* $t\in I = [0,+\infty)$ *in* U, *and locally absolutely continuous in* I, *locally uniform with respect to* U.

For almost all $t\in I$ *suppose*

$$z\frac{\partial f(z,t)}{\partial z} = \frac{\partial f(z,t)}{\partial t}\, p(z,t)\ ,\quad z\in U$$

where $p(z,t)$ *is regular in* U *and satisfies* $\mathrm{Re}\, p(z,t)>0$, $z\in U$. *If* $|a_1(t)|\to\infty$, *for* $t\to\infty$ *and* $f(z,t)/a_1(t)$ *forms a normal family in* U, *then for each* $t\in I$, $f(z,t)$ *can be continued regularly in* U *and gives univalent function.*

THEOREM 1. *Let* α *be a complex number,* $\mathrm{Re}\alpha>0$ *and* $f(z) =$

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$= z+\dots$, *be a regular function in* U. *If*

$$(1-|z|^2)\left|(\alpha-1)\ \frac{zf'(z)}{f(z)}\right| \le 1 \tag{2}$$

for all $z \in U$, *then function* (1) *is regular and univalent in* U.

Proof. Let $U_r = \{z:|z|<r\}$, $r>0$. Because $f'(0) = 1$ there exists $r_0 \in (0,1]$ such that $\frac{f(z)}{z} = 1 + b_1 z + \dots \neq 0$ for all $z \in U_{r_0}$. It results that the function

$$g(z) = \left[\frac{f(z)}{z}\right]^{\alpha-1} = 1+c_1 z+\dots, \tag{3}$$

is regular in U_{r_0} for all $\alpha \in C$ and hence

$$g_\alpha(z) = \alpha\int_0^z u^{\alpha-1}g(u)\,du = z^\alpha+c_1\frac{\alpha}{\alpha+1}z^{\alpha+1}+\dots=z^\alpha h(z)$$

where $h(z)=1+c_1\frac{\alpha}{\alpha+1}z+\dots,$
is a regular function in U_{r_0}. Because $h(0) = 1$ there exists $r_1 \in (0,r_0]$ such that $h(z) \neq 0$ in U_{r_1} and hence the function $F_\alpha(z) = z(h(z))^{1/\alpha} = z+\dots$, is regular in U_{r_1} for all $\alpha \in C-\{0\}$. It results that the function

$$H(z,t) = \left(\frac{F_\alpha(e^{-t}z)}{e^{-t}z}\right)^\alpha + \alpha(e^{2t}-1)\left(\frac{f(e^{-t}z)}{e^{-t}z}\right)^{\alpha-1} = b_1(t)+b_2(t)z+\dots \tag{4}$$

is regular in U_{r_1} for all $t\in I$ and $\alpha\in C-\{0\}$. Since $b_1(t) = 1+\alpha(e^{2t}-1)\neq 0$ for all $t\in I$, if $\mathrm{Re}\,\alpha>0$, there exists $r_2 \in (0,r_1]$ such that $H(z,t)\neq 0$ in U_{r_2} for all $t\in I$ and hence the function

$$f(z,t) = (e^{-t}z)[H(z,t)]^{1/\alpha} = a_1(t)z+a_2(t)z^2+\dots \tag{5}$$

where $a_1(t) = e^{-t}[1+\alpha(e^{2t}-1)]^{1/\alpha}$ is regular in U_{r_2} for all

$t\in I$ (for $a_1(t)$ we choose a fixed branch). We observe that $a_1(t)\neq 0$ for all $t\in I$ and $\lim_{t\to\infty}|a_1(t)| = \infty$.

If $p(z,t)$ is the function, defined by

$$p(z,t) = z\frac{\partial f(z,t)}{\partial z} \Big/ \frac{\partial f(z,t)}{\partial t} \tag{6}$$

then in order to prove that the function $p(z,t)$ is regular and with positive real part in U it is sufficient to prove that the function

$$w(z,t) = \frac{p(z,t)-1}{p(z,t)+1} \tag{7}$$

is regular and $|w(z,t)|<1$ for all $z\in U$ and $t\in I$. A simple calculation yields:

$$w(z,t) = (1-e^{-2t})(\alpha-1)\frac{e^{-t}zf'(e^{-t}z)}{f(e^{-t}z)} \tag{8}$$

For $t=0$, $w(z,0) = 0$ for all $z\in U$.

For $t>0$, if $|z|=1$ then $|e^{-t}z|<1$ and hence by maximum principle we obtain

$$|w(z,t)| < \max_{|z|=1}|w(z,t)| = |w(e^{i\theta},t)| \tag{9}$$

where θ is a real number.

If $u = e^{-t+i\theta}$ then $|u|=e^{-t}$ and hence

$$|w(e^{i\theta},t)|=\left|(1-|u|^2)(\alpha-1)\frac{uf'(u)}{f(u)}\right| \tag{10}$$

Because $|u|<1$, from (2), (9) and (10) we conclude that $|w(z,t)|<1$ for all $z\in U$ and $t>0$. Because $w(z,0) = 0$ for all $z\in U$ and $|w(z,t)|<1$ for all $z\in U$ and $t>0$ it results that $|w(z,t)|<1$ for all $z\in U$ and $t\in I$. From Lemma, for $t = 0$, it results that the function $f(z,0)$ defined by (1) is regular and univalent in U.

THEOREM 2. *If* $f(z) = z+\dots$ *is a regular and univalent function in* U *and* α *is a complex number such that,*

$$|\alpha-1| \le 1/4, \tag{11}$$

then the function (1) *is regular and univalent in* U.

Proof. If $f(z) = z+\dots$ is regular and univalent in U, then we have

$$\left|\frac{zf'(z)}{f(z)}\right| \le \frac{1+|z|}{1-|z|} \quad \text{for all } z \in U, \quad \text{and hence,}$$

$$(1-|z|^2)\left|(\alpha-1)\ \frac{zf'(z)}{f(z)}\right| \le (1-|z|^2)\,|\alpha-1|\,\frac{1+|z|}{1-|z|} = \tag{12}$$

$$= |\alpha-1|\ (1+|z|)^2 < 4|\alpha-1|$$

for all $z\in U$.

By (11) and (12) it results that $(1-|z|^2)\left|(\alpha-1)\frac{zf'(z)}{f(z)}\right| \le 1$, for all $z\in U$.

From Theorem 1 it results that function $F_\alpha(z)$ is regular and univalent in U.

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ON A SUFFICIENT CONDITION FOR UNIVALENCE

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REZUMAT. - **Asupra unei condiţii suficiente de univalenţă.** In lucrare este dat un criteriu de univalenţă care generalizează rezultate obţinute de mai mulţi autori.

An interesting extension of the well-known condition for univalence due to Becker was obtained by S.Ruscheweyh ([3]).

V.Sing and P.N.Chichra in [4] generalize this result. Z.Lewandowski proved in [1] the next extension of univalence condition due to Sing and Chichara.

THEOREM A. *Let* $f(z) = z+\dots$ *and* $h(z) = c_0 + c_1 z + \dots$ *be analytic in* $U = \{z:|z|<1\}$ *and* $f'(z) \neq 0$ *in* U. *If there exists the numbers* $a>1/2$, $s = \alpha+i\beta$, $\alpha>0$, $\beta\in\mathbb{R}$, $k=a/\alpha$, *such that*

$$\left|\frac{zf'(z)}{f(z)g(z)} - \frac{as}{\alpha}\right| \leq \frac{a|s|}{\alpha} \tag{1}$$

and

$$\left||z|^{2k}\frac{zf'(z)}{f(z)g(z)} + (1-|z|^{2k})\left(\frac{zf'(z)}{f(z)} + s\frac{zg'(z)}{g(z)}\right) - \frac{as}{\alpha}\right| \leq \tag{2}$$

$$\leq \frac{a|s|}{\alpha}$$

for all $z\in U$, *then the function* $f(z)$ *is univalent in* U.

The aim of this note is to generalize the Theorem A.

We denote by U_r the disc $\{z:|z|<r\}$, $r>0$, $U_1 = U$ and by I the

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interval $[0,\infty)$ of the real axis. The next Theorem is due to Ch. Pommerenke ([2]).

THEOREM B. *Let* $f(z,t) = a_1(t)z + \dots$, $a_1(t) \neq 0$ *be analytic for all* $t\in I$ *in* $z\in U_{r_0}$, $0<r_0\leq 1$ *and locally absolutely continuous in* I, *locally uniformly with respect to* U_{r_0}. *For almost all* $t\in I$ *suppose*

$$z\frac{\partial f(z,t)}{\partial z} = p(z,t)\frac{\partial f(z,t)}{\partial t}, \quad z\in U_{r_0} \tag{3}$$

where $p(z,t)$ *is analytic and satisfies* $\operatorname{Re} p(z,t)>0$ *in* U. *If* $\lim_{t\to\infty}|a_1(t)| = \infty$ *and* $f(z,t)/a_1(t)$ *forms a normal family in* U_{r_0}, *then for all* $t\in I$, $f(z,t)$ *can be continued analytically in* U *and gives an univalent function.*

THEOREM 1. *Let* $f(z)=z+\dots$ *be anaytic and* $f'(z) \neq 0$ *in* U. *If there exists a function* $g(z) = 1+c_1z+\dots$ *analytic in* U *and the complex numbers* a,c,λ, $s = \alpha+i\beta$, $\alpha>0$, $\beta\in\mathbb{R}$ *such that*

$$\operatorname{Re}(2a\lambda-s)>0, \quad \operatorname{Re}a>0 \tag{4}$$

$$c[1+c(e^{2at}-1)] \neq 0 \text{ for all } t\in I \tag{5}$$

and

$$\left|\frac{zf'(z)}{f(z)g(z)} \quad c\lambda k\right| < |c\lambda k|, \quad k = \frac{a}{\alpha} \tag{6}$$

$$\left||z|^{2k}\left(\frac{zf'(z)}{f(z)g(z)} - c\lambda k\right) + (1-|z|^{2k})\left(c\frac{zf'(z)}{f(z)} + c\lambda\frac{zg'(z)}{g(z)} - c\lambda k\right)\right| \leq$$
$$\leq |c\lambda k| \tag{7}$$

for all $z\in U$, *then the function* $f(z)$ *is univalent in* U.

Proof. The function

$$h(z,t) = 1+c(e^{2at} - 1)\, g(e^{-at} z) = b_0(t) + b_1(t)z + \dots \tag{8}$$

is analytic in U for all $t\in I$. Because $\mathrm{Re}\, a>0$, from (5) we conclude that there exists a number $\rho>0$ such that

$|b_0(t)| = |1 + c(e^{2at} - 1)| \geq \rho$ for all $t\in I$.

It results that there exists a number r_1, $0<r_1\leq 1$ such that $h(z,t)\neq 0$ for all $z\in U_{r1}$ and $t\in I$, and hence the function

$$f(z,t) = f(e^{-at}z)\, [h(z,t)]^{\lambda} = a_1(t)z+ \dots \tag{9}$$

is analytic in U_{r1} for all $t\in I$ (in (9) we choose a fixed branch)

By (4) and (5) it results that

$$a_1(t) = e^{(2a\lambda-a)t}[c+(1-c)e^{-2at}]^{\lambda} \neq 0 \tag{10}$$

for all $t\in I$ and $\lim\limits_{t\to\infty} |a_1(t)| = \infty$. Thus $f(z,t)/a_1(t)$ forms a normal family of analytic functions in U_{r0}, $r_0 = r_1/2$.

By uniform continuity of the function $\partial f(z,t)/\partial t$ on $U_{r0} \times [0,T]$, where $T>0$ is an arbitrarily fixed number, it results that $f(z,t)$ is local absolutely continuous in I, uniformly with respect to U_{ro}.

In order to prove that the function

$$p(z,t) = z\frac{\partial f(z,t)}{\partial z} \bigg/ \frac{\partial f(z,t)}{\partial t} \tag{11}$$

is analytic and with positive real part in U or all $t\in I$, it is sufficient to prove that the function

$$w(z,t) = \frac{p(z,t) - 1}{p(z,t) + 1} \tag{12}$$

is analytic in U and $|w(z,t)|<1$ for all $z\in U$ and $t\in I$.

By a simple calculation we obtain

$$w(z,t) = \frac{(1+s)\ A(z,t)\ +1-2a+s}{(1-s)\ A(z,t)\ +1+2a-s} \tag{13}$$

$$\begin{aligned} A(z,t) &= e^{-2at}\frac{e^{-st}zf'(e^{-st}z)}{c\lambda f(e^{-st}z)g(e^{-st}z)} - 1 + \\ &+ (1-e^{-2at})\left(\frac{e^{-st}zf'(e^{-st}z)}{\lambda f(e^{-st}z)} + \frac{e^{-st}zg'(e^{-st}z)}{g(e^{-st}z)}\right) \end{aligned} \tag{14}$$

Because $f'(z)\neq 0$ in U and from (6) it results that $f(z)g(z)/z\neq 0$in U, we conclude that the function $A(z,t)$ is analytic in U for all $t\in I$.

The ineqality $|w(z,t)|<1$ is equivalent to the inequality

$$|A(z,t) + 1 - k| < |k|, \quad k = \frac{a}{\alpha} \tag{15}$$

or

$$\begin{aligned} &\left|e^{-2at}\left(\frac{e^{-st}zf'(e^{-st}z)}{c\lambda f(e^{-st}z)g(e^{-st}z)} - k\right) + \right. \\ &\left. + (1-e^{-2at})\left(\frac{e^{-st}zf'(e^{-st}z)}{\lambda f(e^{-st}z} + \frac{e^{-st}zg'(e^{-st}z)}{g(e^{-st}z)} - k\right)\right| < |k|. \end{aligned} \tag{16}$$

For $t = 0$ by (6) it results that

$$|A(z,0) + 1 - k| = \left|\frac{zf'(z)}{c\lambda f(z)g(z)} - k\right| < |k|$$

for all $z\in U$.

If $t>0$, because $\alpha = \operatorname{Re} s>0$, we have $|e^{-st} z| < 1$ for all z, $|z| = 1$, and hence

$$\begin{aligned} |A(z,t) + 1 - k| &\leq \max_{|z|=1} |A(z,t) + 1 - k| = \\ &= |A(e^{i\theta},t) + 1 - k|, \end{aligned} \tag{17}$$

where θ is a real number. If $u = e^{-st+i\theta}$, then $|u| = e^{-\alpha t}$ and $e^{-2at} = |u|^{2k}$, $k=a/\alpha$.

By (14), (16), (7) and (17), because $u \in U$ we obtain

$$|A(z,t) + 1 - k| < \Big||u|^{2k}\left(\frac{uf'(u)}{c\lambda f(u)g(u)} - k\right) +$$

$$+ (1 - |u|^{2k})\left(\frac{uf'(u)}{\lambda f(u)} + \frac{ug'(u)}{g(u)} - k\right)\Big| \le k$$

It results that the ineqality (15) holds true for all $z\in U$ and $t\in I$. Because the function $A(z,t)$ is analytic and $|w(z,t)|<1$ for all $z\in U$ and $t\in I$, by (13) it results that the function $w(z,t)$ is analytic in U for all $t\in I$.

By Theorem B, for $t=0$, it results that function $f(z)$ is univalent in U.

Remarks 1. If Re $c>1/2$ then condition (5) of the Theorem 1 holds true.

2. If in Theorem 1, a is a real number, $a>1/2$, $\lambda=s$ and $g(z)=h(z)/c$, where $h(z) = c+c_1z+\ldots$ is an analytic function in U, then from Theorem 1 we obtain Theorem A.

THEOREM 2. *Let* $f(z) = z + \ldots$ *be an analytic function and* $f'(z) \neq 0$ *in* U. *If there exists the complex numbers* λ, k *such that* Re$k>0$, Re$(k\lambda)>1/2$, $|k\lambda - 1|<1$ *and*

$$(1 - |z|^{2\mathrm{Re}k})\left|(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right) - \lambda k\right| \le |\lambda|\mathrm{Re}k \qquad (18)$$

for all $z\in U$, *then the function* $f(z)$ *is univalent in* U.

Proof. If in Theorem 1, $c=1/(\lambda k)$, $g(z)=zf'(z)/f(z)$, then Re$(2a\lambda-s)= \alpha$Re$(2k\lambda-s/\alpha)>0$, Rec = Re$1/(\lambda k) > 1/2$, because $\left|1 - \frac{1}{\lambda k}\right| < \left|\frac{1}{\lambda k}\right|$, $\frac{zf'(z)}{f(z)g(z)} - c\lambda k = 0$ and hence the conditions (4), (5) and (6) hold true. Replacing in the inequality (7) c with $1/\lambda k$ and the function $g(z)$ with $zf'(z)/f(z)$ we obtain

$$\left|(1-|z|^{2k})\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)-\lambda k\right]\right| \le |\lambda k| \qquad (19)$$

For $z = 0$ and $\mathrm{Re}k>0$ the inequality (19) holds true. If $z \neq 0$, $z\in U$ and $\mathrm{Re}k >0$ we have

$$|1-|z|^{2k}| = |1-e^{2k\ln|z|}| = \left|2k\ln|z|\int_0^1 e^{2kt\ln|z|}dt\right| \le$$

$$\le |2k\ln|z|| \int_0^1 |e^{2kt\ln|z|}|dt = |2k\ln|z|| \int_0^1 e^{2t\mathrm{Re}k\ln|z|}dt| =$$

$$= |k|\frac{1-e^{2\mathrm{Re}k\ln|z|}}{\mathrm{Re}k} = \frac{|k|}{\mathrm{Re}k}(1-|z|^{2\mathrm{Re}k})$$

and hence

$$|1-|z|^{2k}| \le \frac{|k|}{\mathrm{Re}k}(1-|z|^{2\mathrm{Re}k}) \qquad (20)$$

By (18) and (20) it results that

$$|1-|z|^{2k}|\cdot\left|(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1+\frac{zf''(z)}{f'(z)}\right) - \lambda k\right| \le \frac{|k|}{\mathrm{Re}k}|\lambda|\mathrm{Re}k = |\lambda k|$$

and hence the inequality (19) holds true.

From Theorem 1 it results that the function $f(z)$ is univalent in U.

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SUFFICIENT CONDITIONS FOR CONVEXITY OF ORDER α AND UNIVALENCE RELATED TO THE SCHWARZIAN DERIVATIVE

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Dedicated to Professor I. Muntean on his 60^{th} anniversary

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REZUMAT. - Condiţii suficiente de convexitate de ordin α şi de univalenţă care folosesc derivata lui Schwarz. Fie f o funcţie olomorfă pe discul unitate U din planul complex, de forma

$$f(z) = z+a_{n+1}z^{n+1}+\dots\ ,\ n\geq 1$$

Se notează cu $K(\alpha)$, $\alpha<1$, clasa funcţiilor de forma de mai sus, cu n = 1, care sînt convexe de ordin α. Atunci $K(0) = K$ este clasa funcţiilor convexe. Fie $\beta \in C$. În lucrare se determină domenii D, D', care depind de α, β şi n, pentru care au loc proprietăţile

$$\varphi(f,\beta;z) = \beta\frac{zf''(z)}{f'(z)} + z^2\{f;z\}\in D,\quad z\in U \rightarrow f\in K(\alpha)$$

$$\varphi(f,\beta;z) = \left(\frac{zf''(z)}{f'(z)}+1\right)^2 + \beta z^2\{f;z\}\in D',\quad z\in U \rightarrow f\in K$$

unde prin $\{f;z\}$ s-a notat derivata lui Schwarz a lui f în z. Particularizînd apoi pe β şi luînd $\alpha = -1/2$ sînt obţinute condiţii suficiente de univalenţă.

1. Introduction. Let U be the unit disk in the complex plane. We define A to be the class of all analytic functions f on U normalized by $f(0)=0$ and $f'(0)=1$. An analytic function f on U is said to be convex order α, $\alpha<1$, if the following inequality is satisfied

$$\text{Re}\ [\ zf''(z)/f'(z) + 1\ ] > \alpha,\quad z \in U.$$

We denote by $K(\alpha)$ the class of all such functions f which belong to A. Note that $K(0)=K$ is the class of convex functions. It is well known that a function $f\in K(\alpha)$ is univalent if $\alpha \geq -1/2$.

For an analytic function f on U and for $z\in U$ let $\{f;z\}$ be the

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Schwartzian derivative of f in z

$$\{f;z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

The following theorem was obtained in [1]: if $f \in A$ and $\rho(u,v)$, $u,v \in \mathbf{C}$ is a complex - valued function which satisfies some given conditions then

$$\mathrm{Re}\ \rho(zf''(z)/f'(z) + 1,\ z^2\{f;z\}) > 0 \Rightarrow f \in K.$$

Some particular cases of this result, which were obtained by particularizing the function ρ, are also given in [1]. We insist here on two of them:

$$\mathrm{Re}\ \beta \ge 0,\ \mathrm{Re}\ [zf''(z)/f'(z)+1+\beta z^2\{f;z\}] > 0 \Rightarrow f \in K \qquad (1)$$

$$\mathrm{Re}\ [(zf''(z)/f'(z) + 1)^2 + z^2\{f;z\}] > 0 \Rightarrow f \in K \qquad (2)$$

The purpose of this paper is to find domains $D \subseteq \mathbf{C}$, $D' \subseteq \mathbf{C}$, for which the following statements are true:

$$\varphi(f,\beta;z) = \beta zf''(z)/f'(z)+z^2\{f;z\} \in D \Rightarrow f \in K(\alpha) \qquad (3)$$

$$\psi(f,\beta;z) = (1 + zf''(z)/f'(z))^2 + \beta z^2\{f;z\} \in D' \Rightarrow f \in K \qquad (4)$$

where $f \in A$, $\beta \in \mathbf{C}$ and $\alpha<1$. The domains D' obtained for the assertion (4) are larger than the right half-plane However, for some casses of the assertion (3), depending on α and β, it is necessary to consider f of the special form $f(z) = z+a_{n+1}z^{n+1}+\ldots$, with $n\ge1$. This stronger hypothesis assures in some casses that D contains the right half-plane, so (3) generalizes (1), or in some other casses even the existence of D. The idea to consider f of this form for the reasons mentioned above belongs to professor P.T.Mocanu, to whom the author is indebted.

2. Preliminaries. Let p and q be analytic functions on U. We say that p is subordinated to q and write $p(z) \prec q(z)$ if q is univalent, $p(0) = 0$ and $p(U) \subset q(U)$. Let P be the class of all analytic functions p on U with positive real part normalized by $p(0) = 1$. It is obvious that $p \in P$ if and only if $p(z) \prec (1+z)/(1-z)$. The folowing lemma will be repeatedly used:

LEMMA 1 ([2]). *Let p be analytic on U. If p has the form $p(z) = 1+p_n z^n+\ldots$, $n \geq 1$, and $p \notin P$ then there exists a point $z_0 \in U$ such that $p(z_0) = is$, $s \in \mathbb{R}$, $z_0 p'(z_0) \leq -n(1+s^2)/2$.*

We shall also use the next lemmas.

LEMMA 2. *If $\varphi(f,\beta;z)$ is analytic on U, where φ is defined by relation (3) and $f \in A$, $\beta \in C$ then $f'(z) \neq 0$ for all $z \in U$.*

Proof. Let us presume that there exists a zero z_0 of order $m \geq 1$ for f'. Then $z_0 \neq 0$ and $f''(z)/f'(z) = m/(z-z_0)+\ldots$, so

$$\varphi(f,\beta;z) = -\frac{z_0^2 m(m+2)}{(z-z_0)^2}+\ldots$$

It follows that $m(m+2) = 0$, contradiction.

LEMMA 3. *Let β be a complex number satisfying $\beta \neq 2m/(m+2)$ for every integer m, $m \geq 1$. If the function $\psi(f,\beta;z)$ defined by (4) is analytic on U, where $f \in A$, then f' is a nonvanishing function on U.*

Proof. Presuming again that z_0 is a zero of order $m \geq 1$ for f' we obtain

$$\psi(f,\beta;z) = z_0^2 m\left(m-\beta-\frac{m\beta}{2}\right)\frac{1}{(z-z_0)^2}+\ldots$$

so $\beta = 2m/(m+2)$ which is impossible.

3. Main results. Let $f \in A$ be a function of the form

$$f(z) = z + a_{n+1}z^{n+1} + \dots, \ n \geq 1,$$

(5) and let α be real, $\alpha < 1$ and $\beta = \beta_1 + i\beta_2 \in \mathbf{C}$. Let $\varphi(f,\beta;z)$ be the function defined by (3). We denote by $\gamma(s)$, $s \in \mathbf{R}$, the following curve

$$\gamma(s) = (1 - \alpha)(\gamma_1(s) + i\gamma_2(s))$$
$$\gamma_1(s) = \frac{1 - n - \alpha}{2}s^2 - \beta_2 s + \frac{1 + \alpha - n}{2} - \beta_1$$
$$\gamma_2(s) = (\beta_1 - \alpha)s - \beta_2$$

Consider now the closed set $G(\alpha,\beta,n)$ defined by

$G(\alpha,\beta,n) = \{u+iv \in \mathbf{C}: \exists s \in \mathbf{R}$ such that $u \leq (1-\alpha)\gamma_1(s)$, $v = (1-\alpha)\gamma_2(s)\}$, under the hypothesis that α,β,n are such that $G(\alpha,\beta,n)$ does not contain the origin. Then we may define the domain $D(\alpha,\beta,n)$ as being the connected component of the complement of $G(\alpha,\beta,n)$ which contains the origin. As is easy to see, $G(\alpha,\beta,n)$ may be a closed set which has a parabola as border, or a line, or a half-line, while $D(\alpha,\beta,n)$ is the interior or the exterior of parabola, or a half-plane, or a plane with a slit along a half-line. Under these hypotheses and considerations the following theorem is thrue.

THEOREM 1. *If $\varphi(f,\beta;z)$ belongs to $D(\alpha,\beta,n)$ for every $z \in U$ then $f \in K(\alpha)$.*

Proof. Let p be defined the relation $(1-\alpha)p(z) + \alpha = zf''(z)/f'(z) + 1$. Then

$$\varphi(f,\beta;z) = \beta(1-\alpha)(p(z)-1) + \frac{1-\alpha}{2}[2zp'(z) - 2\alpha p(z) - (1-\alpha)p^2(z) + 1 + \alpha] \tag{6}$$

Since $\varphi(f,\beta;z)$ is analytic it follows from Lemma 2 that f is a nonvanishing function, so p is analytic on U. Considering (5) it is obvious that p has the form $p(z) = 1 + p_n z^n + \ldots$ It is also easy to see that $f \in K(\alpha)$ if and only if $p \in P$. Let us presume that $p \notin P$. Applying Lemma 1 we get a point $z_0 \in U$ such that $p(z_0) = is$ and $z_0 p'(z_0) \le -n(1 + s^2)/2$, $s \in \mathbb{R}$. If we let $\phi(f,\beta;z_0) = u + iv$ then relation (6) gives $u \le (1-\alpha)\gamma_1(s)$, $v = (1-\alpha)\gamma_2(s)$ so $\phi(f,\beta;z_0) \in G(\alpha,\beta,n)$.

But this last assertion contradicts hypothesis and the proof is finished. We concentrate now on the assertion (4).

THEOREM 2. *Let f be in A and let β be real and positive, $\beta \ne 2m/(m+2)$ for all integers $m \ge 1$. If $\psi(f,\beta;z)$ is defined by (4) and*

$$\psi(f,\beta;z) \prec \left(\frac{1+z}{1-z}\right)^2$$

then $f \in K$.

Proof. From Lemma 3 it follows that $f'(z) \ne 0$ for all $z \in U$ so $p(z) = 1 + zf''(z)/f'(z)$ is analytic on U and $p(0) = 1$. We have $f \in K$ if and only if $p \in P$. Using p the function ψ gets the form

$$\psi(f,\beta;z) = p^2(z) + \frac{\beta}{2}[2zp'(z) + 1 - p^2(z)] \qquad (7)$$

Presuming that $p \notin P$ we find by Lemma 1 a point $z_0 \in U$ such that $p(z_0) = is$, $z_0 p'(z_0) = t \le -(1+s^2)/2$. So, by (7),

$$\psi(f,\beta;z_0) = -s^2 + \frac{\beta}{2}(2t + 1 + s^2) \le -s^2 \le 0$$

which contradicts the subordination from hypothesis and the proof is finished.

Remark. If we state in the hypotheses of Theorem 2 which concern β only that $\beta \geq 0$ the theorem remains valid under the supplemental assumption that $f'(z) \neq 0$ for all $z \in U$.

The next theorem can be obtained in an analogous way as Theorem 2.

THEOREM 3. *Let* $f \in A$ *and let* $\beta = \beta_1 + i\beta_2$ *with* $\beta_1 \geq 0$ and $\beta_2 \neq 0$. *If* $\psi(f,\beta;z)$ *is defined by* (4) *then the following assertions are true:*

a) If $\beta_2 > 0$ *and* $\psi(f,\beta;z) \in D_1$ *for all* $z \in U$ *then* $f \in K$.

b) *If* $\beta_2 < 0$ *and* $\psi(f,\beta;z) \in D_2$ *for all* $z \in U$ *then* $f \in K$.

Here $D_1 = \{u+iv \in C\colon u>0 \text{ or } v>0\}$, $D_2 = \{u+iv \in C\colon u>0 \text{ or } v<0\}$.

4. **Particular cases.** In this section we shall point out some important consequences of Theorem 1.

Case I. $\beta = 0$. For $\alpha \neq 0$ we have

$$D(\alpha,0,n) = \{u + iv\colon u > \frac{1 - n - \alpha}{2\alpha^2(1-\alpha)}v^2 + \frac{1-\alpha}{2}(1 + \alpha - n)\}.$$

COROLLARY 1. a) *If* $\alpha < 0$, $f \in A$ *and* $z^2\{f;z\} \in D(\alpha,0,1)$ *for all* $z \in U$ *then* $f \in K(\alpha)$.

b) *If* $\alpha \in (0,1)$, f *has the form* (5) *with* $n \geq 2$ *and* $z^2\{f;z\} \in D(\alpha,0,n)$ *for all* $z \in U$ *then* $f \in K(\alpha)$.

Taking $\alpha = -1/2$ in the above corollary we obtain the following sufficient condition of univalence:

THEOREM 4. *If* $f \in A$ *and* $z^2\{f;z\} \in D(-1/2,0,1)$ *for all* $z \in U$ *then* f *is univalent on* U.

For $\alpha = 0$ the following corollary is valid:

COROLLARY 2. *If* f *has the form* (5) *with* $n \geq 2$ *then*

$$z^2\{f;z\} \prec 2(n-1)\frac{z}{(1-z)^2} \Rightarrow f \in K.$$

Case II. $\beta = \alpha + i\beta_2$, where $\alpha < 1$ and $\beta_2 \neq 0$.

COROLLARY 3. a) *If* $f \in A$ *and* $\beta_2 > 0$ *then*

$$\mathrm{Im}\,[\beta zf''(z)/f'(z) + z^2\{f;z\}] > (\alpha-1)\beta_2,\ z \in U \Rightarrow f \in K(\alpha).$$

b) *If* $f \in A$ *and* $\beta_2 < 0$ *then*

$$\mathrm{Im}\,[\beta zf''(z)/f'(z) + z^2\{f;z\}] < (\alpha-1)\beta_2,\ z \in U \Rightarrow f \in K(\alpha).$$

If $\alpha = 0$ Corollary 3 becomes

COROLLARY 3'. *If* $f \in A$ *and* $x \in \mathbb{R}$ *then*

$$\mathrm{Re}\,[zf''(z)/f'(z) + 1] + x\,\mathrm{Im}\, z^2\{f;z\} > 0,\ z \in U \Rightarrow f \in K.$$

Case III. $\beta = \alpha$.

COROLLARY 4. *If* $\alpha < 1$ *and* f *has the form* (5) *with* $n + \alpha > 1$ *then*

$$\alpha\frac{zf''(z)}{f'(z)} + z^2\{f;z\} \prec 2(1-\alpha)(n+\alpha-1)\frac{z}{(1-z)^2} \Rightarrow f \in K(\alpha).$$

Case IV. $\beta > 0$, $\beta \neq \alpha$.

COROLLARY 5. a) *If* $f \in A$ *and* $\alpha \le 0$ *then*

$$\beta zf''(z)/f'(z) + z^2\{f;z\} \in D(\alpha, \beta, 1),\ z \in U \Rightarrow f \in K(\alpha).$$

b) *If* f *has the form* (5) *with* $n \ge 1$ *and* $\alpha = 0$ *then* $\beta zf''(z)/f'(z) + z^2\{f;z\} \in D(0, \beta, n),\ z \in U \Rightarrow f \in K.$

c) *If* $\alpha \in (0,1)$ *and* f *has the form* (5) *with* $1+\alpha-n-2\beta < 0$ *then*

$$\beta zf''(z)/f'(z) + z^2\{f;z\} \in D(\alpha, \beta, n),\ z \in U \Rightarrow f \in K(\alpha).$$

In this case we have

$$D(\alpha,\beta,n) = \{u+iv:\ u > \frac{1-n-\alpha}{2(1-\alpha)(\beta-\alpha)^2}v^2 + \left(\frac{1+\alpha-n}{2}-\beta\right)(1-\alpha)\}.$$

Remark 1. Taking $\alpha = -1/2$ in Corollary 3 and in Corollary

5, a), we can obtain other sufficient conditions of univalence.

Remark 2. Taking n = 1 in Corollary 5,b), we find the result (1) from [1].

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BOGOLUBOV'S TYPE THEOREM FOR FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH HUKUHARA'S DERIVATIVE

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REZUMAT. - O teoremă de tip Bogolubov pentru incluziuni funţional- diferenţiale cu derivată Hukuhara. Sînt studiate incluziunile funcţional-diferenţiale (1) în care derivata este în sensul lui Hukuhara. Rezultatele obţinute extind rezultatele mai multor autori.

0. **Introduction.** The purpose of the present paper is to give a basic theorem of the method of averaging for functional-differential inclusions with Hukuhara's derivative, i.e. for inclusions of the form

$$D_h X(t) \in F(t, X_t) \qquad (1)$$

Here $D_h X$ denotes the Hukuhara's derivative ([3]) of a multivalued mapping X, $X_t : \theta \to X_t(\theta) = X(t+\theta)$ for $\theta \in [-r, 0]$, $r > 0$ and F is a map from $[0,T] \times C_0$ into $CC(\mathbb{R}^n)$, where $CC(\mathbb{R}^n)$ denotes the collection of all nonempty compact subsets of the compact, convex subsets of Euclidean space $\mathbb{R}^n$ i.e. with conv $\mathbb{R}^n$, and C_0 is a metric space of all continuous mapping $\psi : [-r, 0] \to$ conv $\mathbb{R}^n$.

In Section 1 we shall give some fundamental definitions and conventions. Section 2 contains the proof of the existence theorem for (1). The results obtain in this section generalize the results of Filippov ([5]). Further on, in Section 3 we prove the Bogolubov's type theorem for inclusions (1). The results of this section generalize the results of A.W. Plotnikov ([7]) and of M. Kisielewicz ([6]).

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1. Notations and definitions. Let's denote by conv $\mathbb{R}^n$ the family of all nonempty compact and convex subsets of n-dimensional Euclidian space $\mathbb{R}^n$ endoved with the Hausdorff metric H defined by

$$H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}$$

for $A,B \in$ conv $\mathbb{R}^n$.

It is known ([4]) that (conv $\mathbb{R}^n$,H) is a complete metric space. Let $CC(\mathbb{R}^n)$ denote the space of all nonempty compact subsets of conv $\mathbb{R}^n$. By d we will denote the distance between two collections $A,B \in CC(\mathbb{R}^n)$ i.e.

$$d(A,B) = \max\left\{\max_{a\in A}\min_{b\in B} H(a,b),\ \max_{b\in B}\min_{a\in A} H(a,b)\right\}$$

for $a,b \in$ conv $\mathbb{R}^n$.

Let us denote by ρ a distance between $A \in CC(\mathbb{R}^n)$ and $B \in$ conv $\mathbb{R}^n$ defined by

$$\rho(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B} H(a,b),\ \sup_{b\in B}\inf_{a\in A} H(a,b)\right\}$$

Let $X : [\alpha,\beta] \to$ conv $\mathbb{R}^n$ be a given mapping. Using the definition of the difference in conv $\mathbb{R}^n$ the Hukuhara derivative D_hX ([3]) of X may be introduced in the following way:

$$D_hX(t) = \lim_{h\to 0^+}\frac{1}{h}[X(t+h)-X(t)] = \lim_{h\to 0^+}\frac{1}{h}[X(t)-x(t-x)], \qquad (2)$$

where X is assumed to belong to the class D (clearly not empty) of all functions such that both differences in (2) are possible. The mapping $X : (\alpha,\beta) \to$ conv $\mathbb{R}^n$ will be called Hukuhara diferentiable in (α,β) if D_hX exists for every $t \in (\alpha,\beta)$.

A function $X : [\alpha,\beta] \to \text{conv } \mathbb{R}^n$ is called absolutely continuous ([1]) if for every positive number ε there is a positive number δ such that

$$\sum_{i=1}^{k} H(X(\beta_i), X(\alpha_i)) < \varepsilon$$

whenever $\alpha_1 < \beta_1 \le \alpha_2 < \beta_2 \le \ldots \le \alpha_k < \beta_k$ and

$$\sum_{i=1}^{k} (\beta_i - \alpha_i) < \delta .$$

The Aumann's-Hukuhara integral for multifunction $F:[\alpha,\beta] \to CC(\mathbb{R}^n)$ is the colection $G \in CC(\mathbb{R}^n)$ defined by

$$G = \{g \in \text{conv } \mathbb{R}^n : g = \int_\alpha^\beta f(t)dt \quad \text{for} \quad f(t) \in F(t)\}$$

where $f:[\alpha,\beta] \to \text{conv } \mathbb{R}^n$ and integral of f on a set $[\alpha,\beta]$ is the Hukuhara integral defined in the paper ([3]).

Finally, denote by C_α the metric space of all continuous mapping $\Phi:[-r,\alpha] \to \text{conv } \mathbb{R}^n$, where $\alpha \ge 0$, $r > 0$, with metric ρ_α defined by

$$\rho_\alpha(\Phi,\Psi) = \sup_{-r \le t \le \alpha} H(\Phi(t),\Psi(t)) \quad \text{for} \quad \Phi,\Psi \in C_\alpha .$$

We say that X is a solution of (1) with the initial function $\Phi \in C_0$ if X is a absolutely continuous function from $[-r,T]$ into conv $\mathbb{R}^n$ with the properties:

(a) $X(t) = \Phi(t)$ for $t \in [-r,0]$,

(b) X satisfies (1) for a.e. $t \in [0,T]$.

2. Existence theorem. Let $F:[0,T] \times C_0 \longrightarrow CC(\mathbb{R}^n)$ satisfy the following conditions:

1^0 $F(\cdot,U) : [0,T] \longrightarrow CC(\mathbb{R}^n)$ is measurable for fixed $U \in C_0$;

2^0 $F(t,\cdot) : C_0 \longrightarrow CC(\mathbb{R}^n)$ is Lipschitzean with respect to U, i.e. there exists a Lebesque integrable function $K : [0,T] \longrightarrow \mathbb{R}^+$ such that $d(F(t,U),\ F(t,V)) \le K(t)\rho_0(U,V)$;

3^0 there exists a $M > 0$ such that $d(F(t,U),\{0\}) \le M$ for $(t,U) \in [0,T] \times C_0$.

THEOREM 1. *Let* $\delta:[0,T] \longrightarrow \mathbb{R}$ *be a nonnegative Lebesque integrable function and let* $\Phi \in C_0$ *be absolutely continuous. Suppose* $F : [0,T] \times C_0 \longrightarrow CC(\mathbb{R}^n)$ *satisfies* 1^0 - 3^0 *and let* $Y : [-r,T] \longrightarrow \operatorname{conv} \mathbb{R}^n$ *by an absolutely continuous mapping such that:*

4^0 $Y(t) = \Phi(t)$ *for* $t \in [-r,0]$;

5^0 $\rho(D_hY(t),\ F(t,Y_t)) \le \delta(t)$ *for a.e.* $t \in [0,T]$.

Then there is a solution X *of the initial value problem*

$$(3) \quad \begin{cases} D_hX(t) \in F(t,X_t) & \text{for a.e. } t\in[0,T] \\ X(t) = \Phi(t) & \text{for } t \in [-r,0] \end{cases}$$

such that

$$H(X(t),Y(t)) \le \xi(t) \quad \textit{for} \quad t \in [0,T] \quad \textit{and} \tag{4}$$

$$H(D_hX(t),D_hY(t)) \le \delta(t)+K(t)\xi(t) \ \textit{for a.e.}\ t \in [0,T] \tag{5}$$

where

$$\xi(t) = \int_0^t \delta(s)\exp[m(t) - m(s)]ds \qquad \textit{and} \qquad m(t) = \int_0^t K(t)dt.$$

Proof. We shall define a Cauchy sequence of successive approximations (X^n), such that their derivatives (D_hX^n) form also a Cauchy sequence on $[0,T]$.

Let $Y : [-r,T] \to \text{conv}\,\mathbb{R}^n$ be a given absolutely continuous function satisfying conditions 4^0 and 5^0. Since (in general) $D_hY(t) \notin F(t,Y_t)$ for a.e. $t \in [0,T]$ then there exists a measurable function V^0 such that $V^0(t) \in F(t,Y_t)$ and $H(V^0(t),D_hY(t)) = \rho(D_hY(t),F(t,Y_t)) \le \delta(t)$ for a.e. $t \in [0,T]$. Let us call X^1 the absolutely continuous function defined by

$$\begin{cases} X^1(t) = \Phi(t) & \text{for } t \in [-r,0] \\ X^1(t) = \Phi(0) + \int_0^t V^0(s)\,ds & \text{for } t \in [0,T] \end{cases}$$

In this definition we mean the integral in Hukuhara sense. We have

$$\begin{aligned} H(X^1(t),Y(t)) &= H\Big(\Phi(0)+\int_0^t V^0(s)\,ds,\ Y(0)+\int_0^t D_hY(s)\,ds\Big) \le \\ &\le H(\Phi(0),Y(0)) + H\Big(\int_0^t V^0(s)\,ds, \int_0^t D_hY(s)\,ds\Big) \le \\ &\le \int_0^t H(V^0(s),D_hY(s))\,ds \le \int_0^t \delta(s)\,ds \end{aligned}$$

for $t \in [0,T]$.

We shall define now a sequence of absolutely continuous functions (X^i) in the following way

$$\begin{cases} X^i(t) = \Phi(t) & \text{for } t \in [-r,0] \\ X^i(t) = \Phi(0) + \int_0^t V^{i-1}(s)\,ds & \text{for a.e. } t \in [0,T] \text{ and } i \ge 1 \end{cases}$$

where V^{i-1} is a measurable function such that $V^{i-1}(t) \in F(t,X_t^{i-1})$

and $H(V^{i-1}(t),D_hX^{i-1}(t)) = \rho(D_hX^{i-1}(t),F(t,X_t^{i-1}))$

for a.e. $t \in [0,T]$. Hence for a.e. $t \in [0,T]$ we obtain

$$H(D_hX^i(t),D_hX^{i-1}(t)) = H(V^{i-1}(t),D_hX^{i-1}(t)) = \rho(D_hX^{i-1}(t),F(t,X_t^{i-1})) =$$
$$= \rho(V^{i-2}(t),F(t,X_t^{i-1})) \le d(F(t,X_t^{i-2}),F(t,X_t^{i-1})) \le$$
$$\le K(t)\rho_0(X_t^{i-1},X_t^{i-2}) = K(t)\sup_{-r\le\theta\le 0} H(X^{i-1}(t+\theta),X^{i-2}(t+\theta)) \le$$
$$\le K(t)[\sup_{-r\le\theta\le t}(\sup_{-r\le\tau\le t+\theta} H(X^{i-1}(\tau),X^{i-2}(\tau)))] \le$$
$$\le K(t)\sup_{-r\le\tau\le t} H(X^{i-1}(\tau),X^{i-2}(\tau)) \le K(t)\sup_{0\le\tau\le t} H(X^{i-1}(\tau),X^{i-2}(\tau)).$$

By the definition of X^2 we have

$$\begin{cases} X^2(t) = \Phi(t) & \text{for } \quad t \in [-r,0] \\ X^2(t) = \Phi(0) + \int_0^t V^1(s)ds & \text{for a.e. } \quad t \in [0,T] \end{cases}$$

Therefore

$$D_hX^2(t) = V^1(t) \in F(t,X_t^1) \quad \text{and} \quad H(D_hX^2(t),D_hX^1(t)) =$$
$$= \rho(F(t,X_t^1),D_hX^1(t)) = \rho(F(t,X_t^1),V^0(t)) \le d(F(t,Y_t),F(t,X_t^1)) \le$$
$$\le K(t)\rho_0(Y_t,X_t^1) = K(t)\sup_{-r\le\theta\le 0} H(Y(t+\theta),X^1(t+\theta)) \le$$
$$\le K(t)[\sup_{-r\le\theta\le 0}(\sup_{-r\le\tau\le t+\theta} H(Y(\tau),X^1(\tau)))] \le K(t)[\sup_{-r\le\tau\le t} H(Y(\tau),X^1(\tau))] \le$$
$$\le K(t)[\sup_{0\le\tau\le t} H(Y(\tau),X^1(\tau))] \le K(t)\int_0^t \delta(s)ds.$$

Furtheremore, for $t \in [0,T]$ we have

$$H(X^2(t),X^1(t)) \le H\left(\int_0^t V^1(s)ds, \int_0^t V^0(s)ds\right) \le \int_0^t H(V^1(s),V^0(s))ds \le$$
$$\le \int_0^t d(F(s,X_s^1),F(s,Y_s))ds \le \int_0^t K(s)\rho_0(X_s^1,Y_s)ds \le$$
$$\le \int_0^t K(r)\left[\int_0^t \delta(s)ds\right]dr \le \int_0^t \delta(s)\left[\int_s^t K(r)dr\right]ds \le$$
$$\le \int_0^t \delta(s)[m(t)-m(s)]ds.$$

Using the induction we can show that for every $i \ge 2$

$$H(D_hX^i(t),D_hX^{i-1}(t)) \le K(t)\int_0^t \delta(s)\,\frac{[m(t)-m(s)]^{i-2}}{(i-2)!}ds \qquad (6)$$

for a.e. $t \in [0,T]$ and

$$H(X^i(t),X^{i-1}(t)) \le \int_0^t \frac{[m(t)-m(s)]^{i-1}}{(i-1)!}\delta(s)ds \quad \text{for } t\in[0,T]. \qquad (7)$$

Assume we have defined our functions X^i up to $i=n$. Let us consider (X^n). By measurability of a multivalues mapping $F(\cdot,U)$ there exists a measurable function V^n such that

$$V^n(t) \in F(t,X_t^n) \quad \text{and} \quad H(V^n(t),D_hX^n(t)) = \rho(F(t,X_t^n),D_hX^n(t))$$

for a.e. $t \in [0,T]$.

Define now X^{n+1} by setting

$$\begin{cases} X^{n+1}(t) = \Phi(t) & \text{for } t \in [-r,0] \\ X^{n+1}(t) = \Phi(0) + \int_0^t V^n(s)ds & \text{for a.e. } t \in [0,T]. \end{cases} \qquad (8)$$

We have

$$H(D_hX^{n+1}(t),D_hX^n(t)) = H(V^n(t),V^{n-1}(t)) \le d(F(t,X_t^n),F(t,X_t^{n-1})) \le$$
$$\le K(t)\rho_0(X_t^n,X_t^{n-1}) \le K(t)\int_0^t \frac{[m(t)-m(s)]^{n-1}}{(n-1)!}\delta(s)ds$$

for a.e. $t \in [0,T]$.

We obtain

$$H(X^{n+1}(t),Y(t)) \le H(X^{n+1}(t),X^n(t)) + H(X^n(t),X^{n-1}(t)) + \dots +$$
$$+ H(X^1(t),Y(t)) \le \int_0^t \delta(s)ds + \int_0^t \delta(s)[m(t)-m(s)]ds + \dots +$$
$$+ \int_0^T \delta(s)\frac{[m(t)-m(s)]^n}{n!}ds \le \int_0^t \delta(s)\exp[m(t)-m(s)]ds = \xi(t)$$

for $t \in [0,T]$. Similary for a.e. $t \in [0,T]$ we have

$$H(D_hX^{n+1}(t),D_hY(t)) \le \delta(t) + K(t)\int_0^t \delta(s)\exp[m(t)-m(s)]ds = \delta(t) + K(t)\xi(t).$$

The inequality (6) and (7) imply that (X^n) is a Cauchy sequence of $C_{[0,T]}$, where $C_{[0,T]}$ is the metric space of all continuous mapping of $[0,T]$ into conv $\mathbb{R}^n$.

Let $X = \lim_{n\to\infty} X^n$. Similary from (6) it follows that (V^n) converges pointwise almost everywhere to a measurable function V. Hence, passing to the limit as $n\to\infty$ in (8) we get

$$\begin{cases} X(t) = \Phi(t) & \text{for } t \in [-r,0] \\ X(t) = \Phi(0) + \int_0^t V(s)ds & \text{for a.e. } t \in [0,T] \end{cases}$$

But for a.e. $t \in [0,T]$ $V^n(t) \in F(t,X_t^n)$ and $H(V^n(t),D_hX^n(t)) = \rho(D_hX^n(t),F(t,X_t^n))$.

Therefore for a.e. $t \in [0,T]$ we have

$$H(V^n(t),D_hX^n(t)) = \rho(D_hX^n(t),F(t,X_t^n)) \le \rho(D_hX^n(t),F(t,X_t)) + d(F(t,X_t),F(t,X_t^n)).$$

Hence we obtain

$$\begin{cases} X(t) = \Phi(t) & \text{for } t \in [-r,0] \\ D_hX(t) \in F(t,X_t) & \text{for a.e. } t \in [0,T] \end{cases}$$

which completes the proof.

3. The Bogolubov's type theorem. In this part we will study differential inclusions of the form

$$\begin{cases} D_hX(t) \in \varepsilon F(t,X_t) & \text{for a.e. } t \in [0,\infty) \\ X(t) = \Phi(t) & \text{for } t \in [-r,0] \end{cases} \tag{9}$$

where $F : [0,\infty)\times C_0 \to CC(\mathbb{R}^n)$, $\Phi : [-r,0] \to$ conv $\mathbb{R}^n$ is a given absolutely continuous multifunction, $\varepsilon > 0$ is a small parameter. We shall consider (9) together with the middling inclusions

$$\begin{cases} D_h Y(t) \in \varepsilon F_0(Y_t) & \text{for a.e. } t \geq 0 \\ Y(t) = \Phi(t) & \text{for } t \in [-r,0] \end{cases} \tag{10}$$

where $F_0 : C_0 \longrightarrow CC(\mathbb{R}^n)$ and

6^0 $$\lim_{T\to\infty} d(F_0(U), \frac{1}{T}\int_0^T F(t,U)dt) = 0$$

where the integral is in Aumann's-Hukuhara's sense.

THEOREM 2. *Suppose* $F:[0,\infty)\times C_0 \longrightarrow CC(\mathbb{R}^n)$ *satisfies the conditions* 1^0 - 3^0 *and* 6^0. *Then, for each* $\mu > 0$ *and* $T > 0$ *there exists* ε^0 $(\mu,T) > 0$ *such that for every* $\varepsilon \in (0,\varepsilon^0]$ *the following conditions are satisfied:*

(i) *for each solution* $Y(\cdot)$ *of* (10) *there exists a solution* $X(\cdot)$ *of* (9) *such that:*

$$H(X(t,Y(t)) \leq \mu \quad \textit{for} \quad t \in [-r,T/\varepsilon] \tag{11}$$

(ii) *for each solution* $X(\cdot)$ *of* (9) *there exists a solution* $Y(\cdot)$ *of* (10) *such that* (11) *holds.*

Proof. In the first step of the proof we show the boundary of mapping $F_0:C_0 \longrightarrow \text{conv } \mathbb{R}^n$. Observe that

$$\begin{aligned} d(F_0(U),\{0\}) &\leq d(F_0(U),\frac{1}{T}\int_0^T F(s,U)ds + d(\frac{1}{T}\int_0^T F(s,U)ds,\{0\}) \leq \\ &\leq d(F_0(U),\frac{1}{T}\int_0^T F(s,U)ds) + \frac{1}{T}\int_0^T d(F(s,U),\{0\})ds \leq \\ &\leq d(F_0(U),\frac{1}{T}\int_0^T F(s,U)ds) + M. \end{aligned}$$

Hence, passing to the limit as $T \to \infty$ and by virtue of 6^0 we have $d(F_0(U),\{0\}) \leq M$.

Furthermore the mapping F_0 satisfies the Lipschitz condition with a number $k\geq 0$ because

$$\begin{aligned} d(F_0(U),F_0(\bar U)) &= d(\lim_{T\to\infty}\frac{1}{T}\int_0^T F(t,U)dt,\ \lim_{T\to\infty}\frac{1}{T}\int_0^T F(t,\bar U)dt = \\ &= \lim_{T\to\infty}\frac{1}{T}d(\int_0^T F(t,U)dt,\ \int_0^T F(t,\bar U)dt) \le \\ &\le \lim_{T\to\infty}\frac{1}{T}(\int_0^T d(F(t,U),F(t,\bar U))dt \le \lim_{T\to\infty}\frac{1}{T}\int_0^T k\rho_0(U,\bar U)dt \\ &= k\rho_0(U,\bar U). \end{aligned}$$

Now we can prove the inequality (11).

Let $Y(\cdot)$ be a solution of (10) on $[-r,\infty)$. To prove this theorem we shall consider the solution $X(\cdot)$ in such a way that, for $t\in[-r,0]$, $X(t) = Y(t) = \Phi(t)$, hence $H(X(t),Y(t)) = 0 < \mu$. We will prove inequality (11) on the interval $[0,T/\varepsilon]$. To do this divide the interval $[0,T/\varepsilon]$ on m-subintervals $[t_i,t_{i+1}]$, where $t_i = iT/\varepsilon m$. $i = 0,1,2,\dots,m-1$ and write a solution $Y(\cdot)$ in the form

$$\begin{cases} Y(t) = \Phi(t) & \text{for } t\in[-r,0] \\ Y(t) = Y(t_i) + \varepsilon\int_{t_i}^{t} V(\tau)d\tau & \text{for } t\in[t_i,t_{i-1}] \end{cases} \tag{12}$$

where $V(t)\in F_0(Y_t)$. Let us consider a function $Y^1(\cdot)$ defined by

$$\begin{cases} Y^1(t) = \Phi(t) & \text{for } t\in[-r,0] \\ Y^1(t) = Y(t_i) + \varepsilon U^1(t_i)(t-t_i) & \text{for } t\in[t_i,t_{i+1}] \end{cases} \tag{13}$$

where $U^1(\cdot)$ is measurable multifunction such that $U^1(t)\in F_0(Y^1_{t_i})$ and

$$H\left(\frac{T}{\varepsilon m}U^1(t_i),\int_{t_i}^{t_{i+1}}V(t)dt\right) = \min_{\bar U^1(t)\in F_0(Y^1_{t_i})} H\left(\frac{t}{\varepsilon m}\bar U^1(t),\int_{t_i}^{t_{i+1}}V(t)dt\right).$$

By virtue of (12) for every $t\in[t_i,t_{i+1}]$ we have

$$H(Y(t),Y^1(t_i)) = H\left(Y(t_i)+\varepsilon\int_{t_i}^{t}V(\tau)d\tau,Y^1(t_i)\right) \le H(Y(t_i),Y^1(t_i)) + \\ + \varepsilon M(t-t_i) \le \delta_i+\varepsilon M(t-t_i) \quad \text{where } \delta_i = H(Y(t_i),Y^1(t_i)), i=1,\dots,m-1$$

Furthermore for $t \in [t_i, t_{i+1}]$, $d(F_0(Y_t), F_0(Y^1_{t_i})) \le k\rho_0(Y_t, Y^1_{t_i})$.

But

$$\rho_0(Y_t, Y^1_{t_i}) \le \rho_0(Y_t, Y_{t_i}) + \rho_0(Y_{t_i}, Y^1_{t_i}) = \\ = \sup_{-r\le s\le 0} H(Y(t+s), Y(t_i+s)) + \sup_{-r\le s\le 0} H(Y(t_i+s), Y^1(t_i+s)).$$

By the definition of $Y(\cdot)$ and the properties of multifunction F_0 we have

$$\sup_{-r\le s\le 0} H(Y(t+s), Y(t_i+s)) \le \varepsilon M|t_i - t| \le \frac{MT}{m} \quad \text{for} \quad t\in[t_i, t_{i+1}].$$

Furthermore

$$\sup_{-r\le s\le 0} H(Y(t_i+s), Y^1(t_i+s)) = \sup_{t_i-r\le\tau\le t_i} H(Y(\tau), Y^1(\tau)) = \\ = \sup_{t_i-r\le\tau\le t_i}\left(H(Y(t_i) + \varepsilon\int_{t_i}^{\tau} V(s)ds, Y^1(t_i) + \varepsilon\int_{t_i}^{\tau} U^1(t_i)ds)\right) \le \\ \le \sup_{t_i-r\le\tau\le t_i}\left(H(Y(t_i), Y^1(t_i)) + \varepsilon H(\int_{t_i}^{\tau} V(s)ds, \int_{t_i}^{\tau} U^1(t_i)ds)\right) \le \\ \le \delta_i + \sup_{t_i-r\le\tau\le t_i} \varepsilon\int_{t_i}^{\tau} d(F_0(Y_s), F_0(Y^1_{t_i})) \le \delta_i + \sup_{t_i-r\le\tau\le t_i} \varepsilon(\int_{t_i}^{\tau} d(F_0(Y_s), \{0\}) + \\ + d(F_0(Y^1_{t_i}), \{0\}) \le \delta_i + 2\varepsilon Mr.$$

Then for $t\in[t_i, t_{i+1}]$ we have

$$d(F_0(Y_t), F_0(Y^1_{t_i})) \le k\left(\frac{MT}{m} + \delta_i + 2\varepsilon Mr\right). \tag{14}$$

By virtue (12),(13) and (14) it follows

$$\delta_i = H(Y(t_i), Y^1(t_i)) = \\ = H\left(Y(t_{i-1}) + \varepsilon\int_{t_{i-1}}^{t_i} V(\tau)d\tau\ ,\ Y^1(t_{i-1}) + \varepsilon\int_{t_{i-1}}^{t_i} U^1(t_{i-1})d\tau\right) \le \\ \le H(Y(t_{i-1}), Y^1(t_{i-1})) + \varepsilon\ H\left(\int_{t_{i-1}}^{t_i} V(\tau)d\tau,\ \int_{t_{i-1}}^{t_i} U^1(t_{i-1})d\tau\right) \le \\ \le \delta_{i-1} + \varepsilon\ H\left(\int_{t_{i-1}}^{t_i} F_0(Y_\tau)d\tau\ ,\ \int_{t_{i-1}}^{t_i} F_0(Y^1_{t_{i-1}})d\tau\right) \le \delta_{i-1} +$$

$$+ \varepsilon \int_{t_{i-1}}^{t_i} H(F_0(Y_\tau) \ , F_0(Y^1_{t_{i-1}}) \ )d\tau \le \delta_{i-1} +$$
$$+ \varepsilon\, k(\delta_{i-1} + \frac{MT}{m} + 2e + 2\varepsilon Mr)(t_i - t_{i-1}) =$$
$$= \delta_{i-1} + \frac{kT}{m}(\delta_{i-1} + \frac{MT}{m} + 2\varepsilon Mr) = \delta_{i-1}(1 + \frac{Tk}{m}) + \frac{Tk}{m}(\frac{MT}{m} + 2\varepsilon Mr) =$$
$$= \delta_{i-1}(1 + \frac{a}{m}) + \frac{b}{m},$$

Hence

$$\begin{aligned}\delta_i &\le \delta_{i-1}(1+\frac{a}{m}) + \frac{b}{m} \le (1+\frac{a}{m})[\delta_{i-2}(1+\frac{a}{m}) + \frac{b}{m}] + \frac{b}{m} = \\ &= (1+\frac{a}{m})^2\,\delta_{i-2}+(1+\frac{a}{m})\frac{b}{m} + \frac{b}{m}\le \ldots \le (1+\frac{a}{m})^i\,\delta_0 + \\ &(1+\frac{a}{m})^{i-1}\frac{b}{m} + \ldots + \frac{b}{m} = \frac{b}{m}(1+(1+\frac{a}{m})+\ldots+(1+\frac{a}{m})^{i-1}) = \\ &= \frac{b}{a}((1+\frac{a}{m})^i-1) \le \frac{b}{a}(e^{b}-1) = \frac{M}{m}(T+2\varepsilon mr)(e^{kT}-1)\end{aligned} \tag{15}$$

where $i = 0,1,2,\ldots,m-1$.

For $t \in [t_i,t_{i+1}]$ we have

$$H(Y(t),Y(t_i)) = H\left(Y(t_i)+\varepsilon\int_{t_i}^{t}V(\tau)d\tau, Y_t(t_i)\right) \le \varepsilon H\left(\int_{t_i}^{t}V(\tau)d\tau, \{0\}\right) \le$$
$$\le \varepsilon\int_{t_i}^{t}H(V(\tau),\{0\})d\tau \le \varepsilon M|t-t_i| \le \frac{MT}{m} \quad \text{and} \quad H(Y^1(t),Y^1(t_i)) \le \frac{MT}{m}$$

Hence, we obtain

$$\begin{aligned}H(Y(t),Y^1(t)) &\le H(Y(t),Y(t_i)) + H(Y(t_i),Y^1(t_i)) + \\ &+ H(Y^1(t_i),Y^1(t)) \le \frac{MT}{m} + \frac{MT}{m} + \frac{M}{m}(T+2\varepsilon mr)(e^{kT}-1).\end{aligned} \tag{16}$$

Now we shall consider the function

$$\begin{cases} Y^2(t) = \Phi(t) & \text{for} \quad t \in [-r,0] \\ Y^2(t) = Y^2(t_i) + \varepsilon\int_{t_i}^{t}U^2(\tau)d\tau & \text{for } t \in [t_i,t_{i+1}] \end{cases} \tag{17}$$

where $t_i = \frac{iT}{\varepsilon m}$, $i = 0,1,2,\ldots,m-1$, $U^2(t) \in F(t,Y^1_{t_i})$.

Let us notice that by virtue of condition 6^0 for each $\mu > 0$ there exists a $L_0(\mu)$ such that for every $L > L_0$ we have inequalities:

$$d\left(\frac{1}{L}\int_0^L F(t,Y^1_{t_i})dt, F_0(Y^1_{t_i})\right) < \mu.$$

By virtue of the Hausdorff metric condition (see [2], Lemma 1 (i)) we have

$$d\left(\int_0^L F(t,Y^1_{t_i})dt,\ \int_0^L F_0(Y^1_{t_i})dt\right) \le L\mu.$$

In particular, for $\frac{T}{\varepsilon m} > L_0$ and for every $i \in \{0,1,\dots,m-1\}$

$$d\left(\int_0^{\frac{iT}{\varepsilon m}} f(t,Y^1_{t_i})dt,\ \int_0^{\frac{iT}{\varepsilon m}} F_0(Y^1_{t_i})dt\right) < i\frac{T}{\varepsilon m}\mu \tag{18}$$

and

$$d\left(\int_0^{\frac{(i+1)T}{\varepsilon m}} F(t,Y^1_{t_i})dt,\ \int_0^{\frac{(i+1)T}{\varepsilon m}} F_0(Y^1_{t_i})dt\right) \le (1+i)\frac{T}{\varepsilon m}\mu. \tag{19}$$

Let us observe that $\frac{(1+i)T}{\varepsilon m} = t_{i+1}$ and $\frac{iT}{\varepsilon m} = t_i$.
By virtue of (18),(19) and the Hausdorff matric condition (see Lemma 3 (vi), [2]), we have

$$\begin{aligned}
&d\left(\int_{t_i}^{t_{i+1}} F(t,Y^1_{t_i})dt,\ \int_{t_i}^{t_{i+1}} F_0(Y^1_{t_i})dt\right) \le \\
&\le d\left(\int_0^{t_{i+1}} F(t,Y^1_{t_i})dt,\ \int_0^{t_{i+1}} F_0(Y^1_{t_i})dt\right) + \\
&+ d\left(\int_0^{t_i} F(t,Y^1_{t_i})dt,\ \int_0^{t_i} F_0(Y^1_{t_i})dt\right) \le \\
&\le \frac{(i+i)T}{\varepsilon m}\mu + \frac{iT}{\varepsilon m}\mu = \frac{T}{\varepsilon m}\mu(2i+1) \le (2m+1)\frac{T}{\varepsilon m}\mu.
\end{aligned}$$

Hence $H\left(\frac{m}{T}\varepsilon\int_{t_i}^{t_{i+1}} F(t,Y^1_{t_i})dt,\ \frac{m}{T}\varepsilon\int_{t_i}^{t_{i+1}} F_0(Y^1_{t_i})dt\right) \le (2m+1)\mu$

for $\frac{T}{\varepsilon m} > L_0(\mu)$.

Moreover $d\left(\varepsilon\frac{m}{T}\int_{t_i}^{t_{i+1}}F(t,Y^1_{t_i})dt, F_0(Y^1_{t_i})\right) \le \mu_1 = (2m+1)\mu$

for $\frac{T}{\varepsilon m} > L_0(\frac{\mu_1}{2m+1})$,then for $\varepsilon < \varepsilon_0(\mu,m) = \frac{T}{mL_0(\frac{\mu_1}{2m+1})}$

Hence it follows that

$$d\left(\int_{t_i}^{t_{i+1}}F(t,Y^1_{t_i})dt, \int_{t_i}^{t_{i+1}}F_0(Y^1_{t_i})dt\right) \le \frac{\mu_1 T}{\varepsilon m}.$$

Then $H\left(\int_{t_i}^{t_{i+1}}U^2(\tau)d\tau, \int_{t_i}^{t_{i+1}}U^1(\tau)d\tau\right) \le \frac{\mu_1 T}{\varepsilon m}$ and

$$\begin{aligned} &H(Y^1(t_{i+1}),Y^2(t_{i+1})) \le H(Y^1(t_i),Y^2(t_i)) + \\ &+ \varepsilon\int_{t_i}^{t_{i+1}}H(U^2(\tau),U^1(\tau))d\tau \le H(Y^1(t_i),Y^2(t_i)) + \\ &+ \frac{\mu_1 T}{m} \le \ldots \le m\frac{\mu_1 T}{m} = \mu_1 T. \end{aligned} \tag{20}$$

where $i=0,1,2,\ldots,m-1$

Using the inequality (20) and the fact that for $t \in [t_i,t_{i+1}]$

$H(Y^2(t),Y^2(t_i)) \le \frac{MT}{m}$ and $H(Y^1(t),Y^1(t_i)) \le \frac{MT}{m}$ we have

$$\begin{aligned} &H(Y^1(t),Y^2(t)) \le H(Y^1(t),Y^1(t_i)) + H(Y^1(t_i),Y^2(t_i)) + \\ &+ (Y^2(t_i),Y^2(t)) \le \frac{2MT}{m} + \mu_1 T. \end{aligned} \tag{21}$$

By assumption 2^0 it follows that

$$d(F(t,Y^2_t),F(t,Y^1_{t_i})) \le k\rho_0(Y^2_t,Y^1_{t_i}) \le k(\frac{3MT}{m} + \mu_1 T).$$

By virtue of (17) we have

$$\begin{aligned} &\rho(D_hY^2(t),\varepsilon F(t,Y^2_t)) = \rho(D_hY^2(t),\varepsilon F(t,Y^1_{t_i})) + \\ &+ d(\varepsilon F(t,Y^1_{t_i}),\varepsilon F(t,Y^2_t)) \le k\varepsilon(\frac{3MT}{m} + \mu_1 T). \end{aligned}$$

Now, on the ground of existence theorem (see Theorem 1) there exists the solution $X(\cdot)$ of (1) that for $t \in [0, \frac{T}{\varepsilon}]$

$$H(Y^2(t), X(t)) \le \int_0^t \varepsilon k(\frac{3MT}{m} + \mu_1 T)\exp(\varepsilon k(t-s))ds \le$$
$$\le (\frac{3MT}{m} + \mu_1 T)(\exp(kT) - 1).$$

Using the inequality (16) and (21) it follows

$$H(X(t), Y(t)) \le H(X(t), Y^2(t)) + H(Y^2(t), Y^1(t)) + H(Y^1(t), Y(t)) \le$$
$$\le \frac{4MT}{m} e^{kT} + \mu_1 T e^{kT} + 2\varepsilon M r e^{kT}.$$

Therefore, choosing $m > \frac{12MTe^{kT}}{\mu}$, $\mu_1 = \frac{\mu}{3Te^{kT}}$, and $\varepsilon < \frac{\mu}{6Mre^{kT}}$ we get the inequality

$$H(X(t), Y(t)) \le \mu \quad \text{for} \quad t \in [0, \frac{T}{\varepsilon}].$$

Adopting now the procedure presented above we get condition (ii). In this way the proof is completed for $t \in [-r, \frac{T}{\varepsilon}]$.

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MAXIMUM PRINCIPLES FOR SOME SYSTEMS OF DIFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS AND APPLICATIONS

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REZUMAT. - **Principii de maxim pentru sisteme de ecuaţii diferenţiale cu argument modificat şi aplicaţii.** In lucrare se stabilesc principii de maxim pentru sisteme de ecuaţii diferenţiale cu argument modificat. Apoi, ca aplicaţie a acestora, se demonstrează o teoremă de existenţă şi unicitate pentru o problemă la limită relativă la sisteme de ecuaţii diferenţiale cu argument modificat, şi se stabilesc principii de maxim pentru anumite clase de ecuaţii diferenţiale de ordinul patru, liniare şi neliniare, cu argument modificat.

1. Introduction. Let us consider the following second order system of differential equations with deviating arguments

$$L_k(y)(x) := y_k''(x) + p_k(x) y_k'(x) + q_k(x) y_k(x) + \\ + \sum_{i=1}^{m} \sum_{j=1}^{n} r_{k,j,i}(x) y_j(g_{k,j,i}(x)) = 0 \tag{1}$$

where $x \in [a,b]$, $k=1,\dots,n$, and the following systems of differential inequalities

$$L_k(y) \geq 0 \quad , \quad k=1,\dots,n \tag{2}$$

$$L_k(y) > 0 \quad , \quad k=1,\dots,n \tag{3}$$

$$L_k(y) \leq 0 \quad , \quad k=1,\dots,n \tag{4}$$

$$L_k(y) < 0 \quad , \quad k=1,\dots,n \tag{5}$$

where $p_k, q_k, r_{k,j,i}, g_{k,j,i} \in C[a,b]$, $k=1,\dots,n$, $i=1,\dots,m$, $j=1,\dots,n$ and $a_1 \leq g_{k,j,i}(x) \leq b_1$, $\forall\ x \in [a,b]$, $a_1 \leq a$, $b \leq b_1$, $y=(y_1,\dots,y_n)$.

The aim of this paper is to establish maximum and minimum principles for the solutions of the systems above and to give some applications. Maximum and minimum principles for

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differential equations with deviating arguments are studied in many papers such as the papers of Rus A.Ioan [6], [7], Bellen and Zennaro [1], Zennaro [10], Lasota [4].

By definition a solution of one of these systems of differential inequalities is a function $y\in C([a_1,b_1],R^n)\cap C^2([a,b],R^n)$ which satisfies the inequalities of the system for $k=1,\dots,n$.

2.Maximum and minimum principles.

DEFINITION 1. If $y,z\in C([a,b], R^n)$, $y=(y_1,\dots,y_n)$, $z=(z_1,\dots,z_n)$ then $y\le z$ if and only if $y_k\le z_k$ for $k=1,\dots,n$. If $y\in C([a,b], R^n)$ and $M\in R$, then $y\le M$ if and only if $y_k\le M$ for $k=1,\dots,n$.

DEFINITION 2. A function $y\in C([a_1,b_1],R^n)\cap C^2([a,b],R^n)$ satisfies the maximum principle if

($\max y_k(x) = M>0$ and $y\le M$) implies

$\{x\in[a_1,b_1] \mid y_k(x) = M\} \subset [a_1,a] \cup [b,b_1]$.

DEFINITION 3. A function $y\in C([a_1,b_1],R^n)$ satisfies the minimum principles if

($\min\limits_{x\in[a_1,b_1]} y_k(x) = m<0$ and $y>m$) implies

$\{x\in[a_1,b_1] \mid y_k(x) = m\}\subset[a_1,a]\cup[b,b_1]$.

We have

THEOREM 1. (see[6]). *Let* $y\in C([a_1,b_1],R^n) \cap C^2([a,b],R^n)$ *be a solution of* (1). *If* $r_{k,j,i}(x)\ge 0$, $x\in]a,b[$, $k=1,\dots,n$, $i=1,\dots,m$, $j=1,\dots,n$ *and* $q_k(x) + \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} r_{k,j,i}(x)<0$, $x\in]a,b[$, $k=1,\dots,n$, *then* y *satisfies the maximum principle and the* *minimum principle.*

THEOREM 2. (see[6]). *Let y be a solution of the system* (2). *If* p_k, q_k, $r_{k,j,i}$ *are as in Theorem 1, then y satisfies the maximum principle.*

THEOREM 3. (see[6]). *Let y be a solution of the system* (4). *If* p_k, q_k, $r_{k,j,i}$ *are as in Theorem 1, then y satisfies the minimum principle.*

The proofs of the Theorems 4,5 may be made similarly to the proof of Theorem 1 (see[6]).

THEOREM 4. *Let* $y \in C([a_1,b_1], R^n) \cap C^2([a,b],R^n)$ *be a solution of the following system of differential inequalities*

$$L_k(y)(x)>0 \quad , \quad k=1,\dots,n \tag{3}$$

If $r_{k,j,i}(x)\geq 0$ *for* $x\in]a,b[$, $k=1,\dots,n, i=1,\dots,m, j=1,\dots,n$ (6)

and $q_k(x) + \sum_{i=1}^{m}\sum_{j=1}^{n} r_{k,j,i}(x) \leq 0$, *for* $x\in]a,b[$, $k=1,\dots,n$ (7)

then y satisfies the maximum principle.

THEOREM 5. *Let* $y\in C([a_1,b_1],R^n) \cap C^2([a,b],R^n)$ *be a solution of the system* (5). *If* q_k *and* $r_{k,j,i}$ *satisfy* (6) *and* (7), *then y satisfies the minimum principle.*

THEOREM 6. *Let* $y\in C([a_1,b_1],R^n) \cap C^2([a,b],R^n)$ *be a solution of* (1). *We assume that* q_k *and* $r_{k,j,i}$ *satisfy (6) and (7). If there exists a component* y_k *of y such that* $\max_{x\in[a_1,b_1]} y_k(x)=M\geq 0$, $y\leq M$, *and there exists* $c\in]a,b[$ *with* $y_k(c) = M$, *then* $y_k(x) = M$ *for all* $x\in[a,b]$.

Proof. Suppose the contrary, there exists $d\in]a,b[$ such that $y_k(d) <M$. We shall prove that this assumption leads us to a contradiction.

(i) The case $d>c$. Let us consider the function

$z=(z_1,z_2,\ldots,z_n)$, where

$$z_k(x) = e^{\alpha_k(x-c)} - 1, \quad k=1,\ldots,n$$

with $\alpha_1,\alpha_2,\ldots,\alpha_k > 0$ to be chosen suitable.

We have:

$z_k(x) < 0$, for all $x\in]a,c[$

$z_k(c) = 0$

$z_k(x) > 0$, for all $x\in]c,b[$,

and

$$L_k(z)(x)=[\alpha_k^2+\alpha_k p_k(x)+q_k(x)(1-e^{-\alpha_k(x-c)})]e^{\alpha_k(x-c)}+ \\ +\sum_{i=1}^{m}\sum_{j=1}^{n} r_{k,j,i}(x)(e^{\alpha_j(g_{k,j,i}(x)-c)}-1) > 0 , \quad k=1,\ldots,n$$

for $\alpha_1,\alpha_2,\ldots,\alpha_n > 0$ sufficiently large, from (6) and (7).

Let $w=(w_1,w_2,\ldots,w_n)$, where

$w_i(x) = y_i(x) + z_i(x)$, $i\neq k$

$w_k(x) = y_k(x) + \varepsilon_k z_k(x)$,

$0 < \varepsilon_k < \dfrac{M - y_k(d)}{z_k(d)}$, for we have $y_k(d)<M$.

We have

$w_k(x) < M$, for all $x\in]a,c[$

$w_k(c) = M$

$w_k(d) = y_k(d) + \varepsilon_k z_k(d) < y_k(d) + M - y_k(d) = M$,

because $z_k(d) > 0$.

Therefore w_k has a maximum larger then M in the interior of $]a,d[$. But, for all $i\neq k$, $L_i(w)(x) = L_i(y)(x) + L_i(z)(x) = L_i(z)(x) > 0$, for α_i chosen before, and $L_k(w)(x) = L_k(y)(x) + \varepsilon_k L_k(z)(x) = \varepsilon_k L_k(z)(x) > 0$ for α_k chosen before.

Hence w is a solution of the system

$$L_i(w) > 0, \quad i=1,\ldots,n$$

$q_k, r_{k,j,i}$ satisfy (6) and (7) for $]a,d[\subseteq]a,b[$, and w_k has a maximum larger then M in the interior of $]a,d[$. This represents a contradiction with Theorem 4, so Theorem 6 is proved.

(ii) the case $d < c$. We make a similary argument for the function $z = (z_1, z_2, \ldots, z_n)$, where

$$z_k(x) = e^{-\alpha_k(x-c)} - 1 \ , \ k=1,\ldots,n$$

and we obtain a contradiction with Theorem 5.

The following result generalises the maximum principle and the minimum principle for systems of linear differential equations with deviating arguments (Theorem 1).

THEOREM 7. *Let* $y \in C([a_1,b_1],R^n) \cap C^2([a,b],R^n)$ *be a solution of* (1), *where* $r_{k,j,i}(x) \geq 0$, $x \in]a,b[$, $k=1,\ldots,n$, $i=1,\ldots,m$, $j=1,\ldots,n$. *If there exists a function* $w \in C([a_1,b_1],R^n) \cap C^2([a,b],R^n)$ *with*

$$w > 0 \tag{8}$$

$$L_k(w)(x) < 0 \ , \ \text{for all } x \in]a,b[\tag{9}$$

then the function $z = \left(\frac{y_1}{w_1}, \frac{y_2}{w_2}, \ldots, \frac{y_n}{w_n}\right)$ *satisfies the maximum and the minimum principle.*

Proof. Denote $z_k = \frac{y_k}{w_k}$ We have $y_k = z_k \cdot w_k$, so

$$L_k(y)(x) = w_k(x)\cdot z''_k(x) + [2w'_k(x) + p_k(x)w_k(x)]z'_k(x) +$$
$$+[w''_k(x) + p_k(x)w'_k(x) + q_k(x)w_k(x)]z_k(x) +$$
$$+ \sum_{i=1}^{m}\sum_{j=1}^{n} r_{k,j,i}(x))w_j(g_{k,j,i}(x))\cdot z_j(g_{k,j,i}(x)) = 0$$

for $k = 1,\ldots,n$. Dividing by $w_k > 0$, we obtain

$$z_k''(x)+2\left(\frac{w_k'(x)}{w_k(x)}+p_k(x)\right)z_k'(x)+\left(\frac{w_k''(x)}{w_k(x)}+p_k(x)\frac{w_k'(x)}{w_k(x)}+\right.$$
$$\left.+q_k(x)\right)z_k(x)+\sum_{i=1}^{m}\sum_{j=1}^{n}r_{k,j,i}(x)\frac{w_j(g_{k,j,i}(x))}{w_j(x)}z_j(g_{k,j,i}(x))=0,\; k=1,\dots,n$$

But $r_{k,j,i}(x)\geq 0$ for all $x\in]a,b[$, $k=1,\dots,n$, $i=1,\dots,m$, $j=1,\dots,n$ and because of (8) and (9), we are in the conditions of Theorem 1. Hence, the function $z=(z_1,z_2,\dots,z_n)=\left(\frac{y_1}{w_1},\frac{y_2}{w_2},\dots,\frac{y_n}{w_n}\right)$ satisfies the maximum and the minimum principle.

Remark. This theorem generalises the maximum and the minimum principle. If we consider the function $w\in C([a_1,b_1],R^n)\cap C^2([a,b],R^n)$, $w=(1,1,\dots,1)$, from Theorem 7 we get Theorem 1.

EXAMPLE. Let $[a,b]\subset[a_1,b_1]\subset R_+^*$ and consider $w\in C([a_1,b_1],R^n)\cap C^2([a,b],R^n)$, given by

$w(x)=(x,x,\dots,x)$, for all $x\in[a,b]$.

Then, we have $w(x)>0$ for all $x\in[a,b]$ and the condition $L_k(w)(x)<0$, $x\in]a,b[$ from Theorem 7, gives us the following maximum principle.

THEOREM 7'. *If* $y\in C([a_1,b_1],R^n)\cap C^2([a,b],R^n)$ *is a solution of* (1) and if

$r_{k,j,i}(x)\geq 0$, $x\in]a,b[$, $k=1,\dots,n$, $i=1,\dots,m$, $j=1,\dots,n$

$p_k(x)+xq_k(x)+b_1\sum_{i=1}^{m}\sum_{j=1}^{n}r_{k,j,i}(x)<0$, $x\in]a,b[$

then the function $\left(\frac{y_1}{x},\frac{y_2}{x},\dots,\frac{y_n}{x}\right)$ *satisfies the maximum and the minimum principle.*

3. **Boundary value problem.** We consider the following

boundary value problem for a system of differential equations with deviated arguments

$$L_k(y)(x) := y_k''(x) + q_k(x)y_k(x) + \sum_{i=1}^{m}\sum_{j=1}^{n} r_{k,j,i}(x)y_j(g_{k,j,i}(x)) = f_k(x) \quad ,x \in [a,b]\ ,\ k=1,\ldots,n \tag{10}$$

$$y\Big|_{[a_1,a]} = \varphi, \qquad y\Big|_{[b,b_1]} = \psi \tag{11}$$

where $y = (y_1,\ldots,y_n)$, $f_k, q_k, r_{k,j,i}, g_k \in C[a,b]$, $k = 1,\ldots,n$, $i = 1,\ldots,m$, $j = 1,\ldots,n$, $a_1 \le g_{k,j,i}(x) \le b_1$, $a_1 \le a$, $b \le b_1$ and $\varphi \in C([a_1,a], R^n)$, $\psi \in C([b,b_1],R^n)$.

The object of this paragraph is to establish an existence and uniqueness theorem for the problem (10) + (11). In this purpose we shall use the maximum principle (Theorem 1), and the following surjectivity theorem.

THEOREM 8. (see [9]). *Let X be a Banach and A : X→X a linear and compact operator. The operator* $A - 1_X$ *is surjective if and only if is injective.*

We shall prove the following

THEOREM 9. *If* $r_{k,j,i}(x) \ge 0$, $x \in]a,b[$,

$$k = 1,\ldots,n,\ i = 1,\ldots,m,\ j = 1,\ldots,n \tag{12}$$

$$q_k(x) + \sum_{i=1}^{m}\sum_{j=1}^{n} r_{k,j,i}(x) < 0,\ x \in]a,b[\ ,\ k = 1,\ldots,n \tag{13}$$

then the problem (10) + (11) has exactly one solution.

Proof. Let $\lambda = (\lambda_1,\ldots,\lambda_n) \in C([a_1,b_1],R^n)$ given by

$$\lambda_k(x) = \begin{cases} \varphi_k(x) \ , & x \in [a_1,a] \\ \dfrac{x-a}{b-a}\psi_k(b) + \dfrac{b-x}{b-a}\varphi_k(a) \ , & x \in]a,b[\\ \psi_k(x) \ , & x \in [b,b_1] \end{cases}$$

where $\varphi = (\varphi_1,\ldots,\varphi_n)$, $\psi = (\psi_1,\ldots,\psi_k)$.

Then the problem (10) + (11) is equivalent to the system of integral equations

$$y_k(x) = -\int_a^b \overline{G}(x,s)\left[f_k(s) - q_k(s)y_k(s) - \sum_{i=1}^{m}\sum_{j=1}^{n} r_{k,j,i}(s)y_j(g_{k,j,i}(s))\right]ds + \lambda_k(x) \ , \ x \in [a_1,b_1] \ , \ k=1,\ldots,n \tag{14}$$

$$\text{where } \overline{G}(x,s) = \begin{cases} 0 \ , & x \in [a_1,a] \\ G(x,s) \ , & x \in]a,b[\\ 0 \ , & x \in [b,b_1] \end{cases}$$

and G is Green's function

$$G(x,s) = \begin{cases} \dfrac{(s-a)(b-x)}{b-a} \ , & s \le x \\ \dfrac{(x-a)(b-s)}{b-a} \ , & s > x \end{cases}$$

Let us consider the operator $A: C([a_1,b_1],R^n) \to C([a_1,b_1],R^n)$ where $A = (A_1,\ldots,A_k)$,

$$A_k(y)(x) = \int_a^b \overline{G}(x,s)\left[q_k(s)y_k(s) + \sum_{i=1}^{m}\sum_{j=1}^{n} r_{k,j,i}(s)y_j(g_{k,j,i}(s))\right]ds$$

for $k = 1,\ldots,n$.

Then the system (14) is equivalent to the system

$$y_k(x) = A_k(y)(x) - \int_a^b \overline{G}(x,s)f_k(s)ds + \lambda_k(x) \ , \ k=1,\ldots,n \tag{15}$$

Denote $F = (F_1,\ldots,F_n)$, where

$$F_k(x) = \int_a^b \overline{G}(x,s)f_k(s)ds - \lambda_k(x) \ , \ k=1,\ldots,n$$

Obviously $F \in C([a_1,b_1],R^n)$. Now, the problem (10) + (11) is equivalent to the system

$$y_k = A_k(y) + F_k \qquad , \ k = 1,\ldots,n$$

or

$$\left(A - 1_{C([a_1,b_1],R^n)}\right)(y) = F \tag{16}$$

The operator A is obviously linear. We shall prove that A is compact. In this purpose is sufficient to show that A(U) is a relative compact set in $C([a_1,b_1],R^n)$, where $U = \{y \in C([a_1,b_1],R^n) : \|y\| \leq 1\}$, and the norm in $C([a_1,b_1],R^n)$ is the Chebyshev norm, given by

$\|y\| = \max_{k=1,\dots,n} \max \{|y_k(x)| : x \in [a_1,b_1]\}$, $y = (y_1,\dots,y_n)$.

We shall prove that A(U) is uniormly bounded and equicontinuous in $C([a_1,b_1],R^n)$.

First we show that A(U) is uniformly bounded. We have, for all $x \in [a,b]$ and for all $y \in U$.

$$|A_k(y)(x)| = \left|\int_a^b \bar{G}(x,s)\left[q_k(s)y_k(s) + \sum_{i=1}^{m}\sum_{j=1}^{n} r_{k,j,i}(s)y_j(g_{k,j,i}(s))\right]ds\right|$$

Let L, M_k, $M_{k,j,i}$ be as bellow

$L = \max\{G(x,s) : (x,s) \in [a,b] \times [a,b]\} \geq 0$

$M_k = \max\{|q(s)| : s \in [a,b]\}$, $k = 1,\dots,n$

$M_{k,j,i} = \max\{|r_{k,j,i}(s)| : s \in [a,b]\}$, $k=1,\dots,n$, $i=1,\dots,m$, $j=1,\dots,n$

Then

$$|A_k(y)(x)| \leq L\int_a^b \left[M_k \cdot \|y\| + \|y\| \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} M_{k,j,i}\right]ds \leq$$
$$\leq L(b-a)\left(M_k + \sum_{i=1}^{m}\sum_{j=1}^{n} M_{k,j,i}\right).$$

Therefore

$$\|A(y)(x)\| = \max\{|A_k(y)(x)| : k=1,n\} \le \max\{L(b-a)(M_k+\sum_{i=1}^{m}\sum_{j=1}^{n}M_{k,j,i}),\ k=1,\dots,n\}$$

Hence, $A(U)$ is uniformly bounded.

We prove now that $A(U)$ is equicontinuous, that is $\forall\, \varepsilon > 0$, $\exists\, \delta(\varepsilon) > 0$ such that, if $|x_1 - x_2| < \delta(\varepsilon)$, $x_1, x_2 \in [a,b]$, we have $\|A(y)(x_1) - A(y)(x_2)\| < \varepsilon$, $\forall\, y \in U$.

Let $\varepsilon > 0$. The mapping $G(x,s)\ [q_k(s)y_k(s) + \sum_{i=1}^{m}\sum_{j=1}^{n} r_{k,j,i}(s)y_j(g_{k,j,i}(s))]$ is continuous for all $x \in [a,b]$. Therefore $\exists\, \delta(\varepsilon) > 0$ such that, if $|x_1 - x_2| < \delta(\varepsilon)$

$$|G(x_1,s)[q_k(s)y(s)+\sum_{i=1}^{m}\sum_{j=1}^{n} r_{k,j,i}(s)y_j(g_{k,j,i}(s))] - G(x_2,s)[q_k(s)y(s)+\sum_{i=1}^{m}\sum_{j=1}^{n} r_{k,j,i}(s)y_j(g_{k,j,i}(s))]| < \frac{\varepsilon}{b-a+1}$$

Then, for $x_1, x_2 \in [a,b]$ with $|x_1 - x_2| < \delta(\varepsilon)$ we have

$$\|A(y)(x_1) - A(y)(x_2)\| = \max\{|A_k(y)(x_1) - A_k(y)(x_2)| : k=1,n\} \le \frac{\varepsilon}{b-a+1}\cdot(b-a) < \frac{\varepsilon}{b-a}\cdot(b-a) = \varepsilon\ .$$

Hence $A(U)$ is equicontinuous.

Now, $A(U)$ is uniformly bounded and equicontinuous in $C([a_1,b_1], R^n)$, so $A(U)$ is relative compact in $C([a_1,b_1], R^n)$. That implies that A is a compact operator.

We prove now that the problem (10) + (11) has at most one solution. This thing happens if and only if the following implication holds

$$\left.\begin{cases} L_k(y) = 0\ , & k=1,\dots,n \\ y|_{[a_1,a]}=0\ , & y|_{[b,b_1]} = 0 \end{cases}\right\} \Rightarrow (y=0) \tag{17}$$

Now, we prove (17). If $y \not\equiv 0$, then, by Theorem 1, if there exists a component y_k of y such that $\max_{x\in[a_1,b_1]} y_k = M > 0$,

$y \le M$, then $\max\limits_{x\in[a_1,a]\cup[b,b_1]} y_k = M.$

But $y(x) = 0$ for all $x \in [a_1,a] \cup [b,b_1]$. Thus we have $y \le 0$. By a similary argument we prove that $y \ge 0$. Hence $y = 0$.

Hence the problem (10) + (11) has at most one solution. That means that the equivalent equation (16)

$(A - 1_{C([a_1,b_1],R^n)})(y) = F$ has at most one solution, when $F \in C([a_1,b_1], R^n)$, therefore the operator $A - 1_{C([a_1,b_1],R^n)}$ is injective.

Theorem 8 implies that $A - 1_{C([a_1,b_1],R^n)}$ is also surjective, so $A - 1_{C([a_1,b_1],R^n)}$ is bijective. It results that the equation (16) has exactly one solution in $C([a_1,b_1], R^n)$.

Thus, the problem (10) + (11) has exactly one solution.

4. Fourth order differential equations with deviating arguments. Let us consider the following fourth order linear differential equation with deviating arguments

$$y^{IV}(x)+p_1(x)y'''(x)+p_2(x)y''(x)+p_3(x)y(x)+\sum_{k=1}^{m} q_k(x)y(g_k(x)) = 0 \tag{18}$$

where $p_1\in C^1[a,b]$, $p_2,p_3,q_k,g_k\in C[a,b]$, $a_1\le g_k(x)\le b_1$, $x\in[a,b]$, $k = 1,\dots,n$, $a_1 \le a$, $b \le b_1$.

Let us denote $\varphi(x) = -e^{-\frac{1}{2}\int_a^x p_1(t)dt}$. We shall prove the following

THEOREM 10. *Let* $y \in C[a_1,b_1] \cap C^4[a,b]$ *be a solution of* (18). *We assume that:*

(i) $p_1(x) < 0$, *for all* $x \in]a,b[$
(ii) $q_k(x) \geq 0$, *for all* $x \in]a,b[$, $k=1,\ldots,m$.
(iii) $p_1^2(x)+2p_1'(x)+4p_2(x)+4p_3(x) \leq 0$ *for all* $x \in]a,b[$
(iv) $p_1^2(x)+2p_1'(x)+4p_2(x)+4 > 0$ *and*

$$p_2(x)-p_3(x)+\varphi^2(x)+\sum_{k=1}^{m} q_k(x) < 0 \text{ , for all } x\in]a,b[.$$

In these conditions the vector function

$$x \to (y(x), y''(x) + \varphi(x)y(x))$$

satisfies the maximum principle and the minimum principle.

Proof. By the substitution $y'' + \varphi \cdot y = -u$, with φ chosen before, the equation (18) can be reduced to the following system

$$\begin{cases} u''(x)+p_1(x)u'(x)+[\varphi(x)+p_2(x)]u(x)+[2\varphi'(x)+p_1(x)\varphi(x)]y'(x)+ \\ +[\varphi''(x)+\varphi'(x)p_1(x)+\varphi(x)p_2(x)-p_3(x)+\varphi^2(x)]y(x)+ \\ +\sum_{k=1}^{m} q_k(x)y(g_k)(x))=0 \\ y''(x)+\varphi(x)y(x)+u(x)=0 \end{cases}$$

But φ is a solution of the equation $2\varphi'(x)+p_1(x)\varphi(x) = 0$ so, the system becomes

$$\begin{cases} u''(x)+p_1(x)u'(x)+[\varphi(x)+p_2(x)]u(x)+[\varphi''(x)+\varphi'(x)p_1(x) + \\ \quad +\varphi(x)p_2(x)-p_3(x)+\varphi^2(x)]y(x)+\sum_{k=1}^{m} q_k(x)y(g_k(x)) = 0 \\ y''(x)+\varphi(x)y(x)+u(x) = 0 \end{cases}$$

which is a system of second order linear equations with deviating arguments. We want now to apply the maximum principle from Theorem 1. If

$\varphi''(x)+\varphi'(x)p_1(x)+\varphi(x)p_2(x)-p_3(x)+\varphi^2(x) \geq 0$, $x \in]a,b[$

$q_k(x) \geq 0$, $x \in]a,b[$, $k = 1,\ldots,m$

$\varphi(x) + p_2(x) + \varphi''(x) + \varphi'(x)p_1(x) + \varphi(x)p_2(x) - p_3(x) +$

$$\varphi^2(x)+\sum_{k=1}^{m} q_k(x) < 0 \ , x \in \ ]a,b[$$

$$\varphi(x) + 1 < 0 \ , \ x \in \ ]a,b[$$

then the vector function (y,u) satisfies the maximum and the minimum principle.

We have

$$\varphi''(x) + \varphi'(x)p_1(x) + \varphi(x)p_2(x) - p_3(x) + \varphi^2(x) \geq$$

$$\geq \varphi''(x) + \varphi'(x)p_1(x) + \varphi(x)p_2(x) - p_3(x) =$$

$$= -\frac{1}{4}[p_1^2(x)+2p_1'(x)+4p_2(x)] \quad e^{-\frac{1}{2}\int_a^x p_1(t)dt} \quad -p_3(x)$$

But $p_1(x) < 0$, $x \in \ ]a,b[$, so: $e^{-\frac{1}{2}\int_a^x p_1(t)dt} > 1$

We get

$$\varphi''(x) + \varphi'(x)p_1(x) + \varphi(x)p_2(x) - p_3(x) + \varphi^2(x) >$$

$> -\frac{1}{4}[p_1^2(x)+2p_1'(x) \quad +4p_2(x)+4p_3(x)] \geq 0, \ \forall \ x \in]a,b[$ from (iii).

Then

$$\varphi(x) + p_2(x) + \varphi''(x) + \varphi'(x)p_1(x) + \varphi(x)p_2(x) - p_3(x) + \varphi^2(x) + + \sum_{k=1}^{m} q_k(x) = -\frac{1}{4}[p_1^2(x)+2p_1'(x) \quad + 4p_2(x) + 4]e^{-\frac{1}{2}\int_a^x p_1(t)dt} \quad +p_2(x)- p_3(x)+\varphi^2(x)+\sum_{k=1}^{m} q_k(x) < 0 \ , \ \forall \ x \in \ ]a,b[$$

from (iv).

Obviously $\varphi(x) + 1 = - e^{-\frac{1}{2}\int_a^x p_1(t)dt} + 1 < 0, \ \forall \ x \in]a,b[$.

Hence, by Theorem 1, the vector function (y,u) satisfies the maximum and the minimum principle, so, the function $(y,y'' + \varphi y)$ satisfies the maximum and the minimum principle.

We shall prove, using the above theorem, the following

THEOREM 11. *Let us consider the boundary value problem:*

$$\begin{cases} y^{IV}(x)+p_1(x)y'''(x)+p_2(x)y''(x)+p_3(x)y(x)+\sum_{k=1}^{m} q_k(x)y(g_k(x))=f & (19) \\ y|_{[a_1,a]} = \varphi \ , \ y|_{[b,b_1]} = \psi & (20) \ . \end{cases}$$

where $f\in C[a,b]$, $\varphi\in C[a_1,a]$, $\psi\in C[b,b_1]$. *If the conditions from Theorem 10 hold, then the problem (19) + (20) has at most a solution.*

Proof. The problem (19) + (20) has at most one solution if and only if the following implication holds:

$$\left\{\begin{array}{l} y^{IV}(x)+p_1(x)y'''(x)+p_2(x)y''(x)+p_3(x)y(x)+\sum_{k=1}^{m}q_k(x)y(g_k(x))=0 \\ y|_{[a_1,a]}=0\ ,\ y|_{[b,b_1]}=0 \end{array}\right\}\Rightarrow$$

$$\Rightarrow (y=0) \tag{21}$$

Now, we prove (21). If $y\not\equiv 0$, then, by Theorem 10, if there exists a component Y_k, $k=1,2$, of the vector function $Y=(y,y''+\varphi y)$ such that $\max\limits_{x\in[a_1,b_1]} Y_k = M>0$, $Y\le M$, then $\max\limits_{x\in[a_1,a]\cup[b,b_1]} Y_k = M$. But $Y(x)=0$, for all $x\in[a_1,a]\cup[b,b_1]$. Thus we have $Y\le 0$. By a similary argument we prove $Y\ge 0$, and we obtain $y=0$.

5. **The nonlinear case.** Consider the following nonlinear second order differential operators with deviating arguments:

$$\begin{aligned} L_k(y)(x) := {} & y_k''(x)+f_k(x,y_k'(x),y_k(x),y_1(g_{k,1,1}(x)),\ldots, \\ & y_1(g_{k,1,m}(x)),y_2(g_{k,2,1}(x)),\ldots,y_2(g_{k,2,m}(x)),\ldots, \\ & y_n(g_{k,n,1}(x)),\ldots,y_n(g_{k,n,m}(x))), \end{aligned} \tag{22}$$

where $x\in[a,b]$, $a_1\le g_{k,j,i}(x)\le b_1$, $a_1\le a<b\le b_1$, $k=1,\ldots,n$ and $f_k : [a,b]\times R^{nm+2}\to R$.

We have

THEOREM 12. *(see[7]). Let* $y\in C([a_1,b_1],R^n)\cap C^2([a,b],R^n)$

be a solution of the following system of differential equations

$L_k(y)(x) = 0$, *for all* $x \epsilon [a,b]$ *and* $k = 1,\dots,n$.

Assume that L_k, $k = 1,\dots,n$ *satisfy the conditions:*

$$f_k(x,0,r,\dots,r)<0, \text{ for all } r>0,\ x\epsilon[a,b],\ k = 1,\dots,n \tag{23}$$

$$(t,s\epsilon R^{nm}, t\le s) \text{ implies } f_k(x,0,r,t) \le f_k(x,0,r,s) \tag{24}$$

for all $x \epsilon [a,b]$, $r>0$ *and* $k = 1,\dots,n$

$$f_k(x,0,r,\dots,r)>0, \text{ for all } r>0,\ x\epsilon[a,b],\ k = 1,\dots,n \tag{25}$$

$$(t,s\epsilon R^{nm}, t\le s) \text{ implies } f_k(x,0,r,t) \le f_k(x,0,r,s) \tag{26}$$

for all $x \epsilon [a,b]$, $r<0$, *and* $k = 1,\dots,n$

In these conditions y satisfies the maximum and the minimum principle.

Now, let us consider the following fourth order nonlinear diferential equation with deviating arguments

$$y^{IV}(x)+p_1(x)y'''(x)+p_2(x)y''(x)+p_3(x)y(x)+f(x,y(g_1(x)),\dots, y(g_m(x)))=0 \tag{27}$$

where $p_1\epsilon C^1 [a,b]$, $p_2,p_3,g_k \epsilon C[a,b]$, $a_1\le g_k(x) \le b_1$, $x\epsilon[a,b]$, $k=1,\dots,m$, $a_1\le a<b\le b_1$ and $f:[a,b]xR^{m+1} \to R$.

Let us denote $\varphi(x) = -e^{-\frac{1}{2}\int_a^x p_1(t)dt}$. We shall prove the following

THEOREM 13. *Let* $y\epsilon C[a_1,b_1] \cap C^4[a,b]$ *be a solution of* (27). *We assume that:*

(i) $p_1(x) < 0$, *for all* $x\epsilon[a,b]$

(ii) $f(x,r,\dots,r) < 0$, *for all* $r > 0$, $x\epsilon[a,b]$

(iii) $p_1^2(x)+2p_1'(x)+4p_2(x)+ 4 > 0$ *and*

$p_2(x) - p_3(x) + \varphi^2(x) < 0$ *for all* $x \epsilon [a,b]$

(iv) $(t,s\epsilon R^m, t\le s)$ *implies* $f(x,t) \le f(x,s)$, *for all* $x\epsilon[a,b]$

(v) $p_1^2(x)+2p_1'(x)+4p_2(x)+ 4p_3(x) \le 0$ *for all* $x \epsilon [a,b]$.

In these conditions the vector function $(y, y''+\varphi\cdot y)$ *satisfie* *the maximum and the minimum principle.*

Proof. By the change of function $y''+\varphi\cdot y = -u$, with φ chose before, the equation (27) turns to the following system

$$u''(x)+p_1(x)u'(x)+[\varphi(x)+p_2(x)]u(x)+[2\varphi'(x)+p_1(x)\varphi(x)]y'(x)$$
$$+[\varphi''(x)+\varphi'(x)p_1(x)+\varphi(x)p_2(x)-p_3(x)+\varphi^2(x)]y(x)+f(x,y(g_1(x))$$
$$\ldots,y(g_m(x)))=0$$
$$y''(x)+\varphi(x)y(x)+u(x)=0$$

Taking in consideration that $2\varphi'(x)+p_1(x)\varphi(x)=0$, for φ chosen before, the system becomes

$$u''(x)+p_1(x)u'(x)+[\varphi(x)+p_2(x)]u(x)+[\varphi''(x)+\varphi'(x)p_1(x)+$$
$$+\varphi(x)p_2(x)-p_3(x)+\varphi^2(x)]y(x)+f(x,y(g_1(x)),\ldots,y(g_m(x)))=0$$
$$y''(x)+\varphi(x)y(x)+u(x)=0$$

which is a system of second order nonlinear equations wit deviating arguments. We want to apply the maximum principle fro Theorem 12.

Denoting

$$f_1(x,u'(x),u(x),y(x),y(g_1(x)),\ldots,y(g_m(x)))=p_1(x)u'(x)+$$
$$+[\varphi(x)+p_2(x)]u(x)+[\varphi''(x)+\varphi'(x)p_1(x)+\varphi(x)p_2(x)-p_3(x)+\varphi^2(x)]\cdot$$
$$\cdot y(x)+f(x,y(g_1(x)),\ldots,y(g_m(x))),$$ and

$$f_2(x,y'(x),y(x),u(x))=\varphi(x)\cdot y(x)+u(x),$$

the system may be written:

$$u''(x)+f_1(x,u'(x),u(x),y(x),y(g_1(x)),\ldots,y(g_m(x)))=0$$
$$y''(x)+f_2(x,y'(x),y(x),u(x))=0.$$

The conditions (23) - (25) from Theorem 12 become in ou case

$f_1(x,0,r,\ldots,r)<0$, for all $r>0$, $x\in[a,b]$

$(t,s \in R^{m+1}, t \le s)$ must imply $f_1(x,0,r,t) \le f_1(x,0,r,s)$

for all $r>0$, $x \in [a,b]$

$f_2(x,0,r,r)>0$, for all $r<0, x \in [a,b]$ and

$(t,s \in R, t \le s)$ must imply $f_2(x,0,r,t) \le f_2(x,0,r,s)$

for all $r<0$, $x \in [a,b]$.

We shall prove that all these conditions are fulfilled in our case. Let $r>0$

$$f_1(x,0,r,\dots,r)=[\varphi(x)+p_2(x)]r+[\varphi''(x)+\varphi'(x)p_1(x)+\varphi(x)p_2(x)-$$

$$-p_3(x)+\varphi^2(x)]r+f(x,r,\dots,r)=$$

$$= r\ \{ -\frac{1}{4}[p_1^2(x)+2p_1'(x)+4p_2(x)+4]\ e^{-\frac{1}{2}\int_a^x p_1(t)dt}+p_2(x)-p_3(x)+\varphi^2(x)\}+$$

$+ f(x,r,\dots,r) < 0$, for all $x \in [a,b]$, from (ii) and (iii).

Now, let $t,s \in R^{m+1}$, $t \le s$. We must prove that $f_1(x,0,r,t) \le$ $\le f_1(x,0,r,s)$ for all $r>0$, $x \in [a,b]$.

We have

$$f_1(x,0,r,t)-f_1(x,0,r,s)=[\varphi''(x)+\varphi'(x)p_1(x)+\varphi(x)p_2(x)-$$

$$-p_3(x)+\varphi^2(x)](t_1-s_1)+[f(x,t_2,\dots,t_{m+1})-f(x,s_2,\dots,s_{m+1})].$$

But $\varphi''(x)+\varphi'(x)p_1(x)+\varphi(x)p_2(x)-p_3(x)+\varphi^2(x)\ge 0$, for all $x \in [a,b]$, from (i) and (V), in the same way as in the proof of Theorem 10. Using (iv) and fact that $t_1-s_1 \le 0$ (for $t \le s$), we get $f_1(x,0,r,t)-f_1(x,0,r,s) \le 0$, $x \in [a,b]$, $r>o$, which is $f_1(x,0,r,t) \le f_1(x,0,r,s)$ for all $r>o$, $x \in [a,b]$.

The condition $f_2(x,0,r,r)>0$, for all $r<0$, $x \in [a,b]$ becomes $r[\varphi(x)+1\}>o$, that is $\varphi(x)+1<0$, which is true for φ chosen, from (i).

For the last condition, let $t,s \in R$, $t \le s$. We must prove that $f_2(x,0,r,t) \le f_2(x,0,r,s)$, for all $r<o$, $x \in [a,b]$, that is $\varphi(x)\cdot r+t \le$ $\le \varphi(x)\cdot r+s$ which is obviously true.

Hence, we are in the conditions of Theorem 12, so, the function (y,u) satisfies the maximum and the minimum principles, that is, the vector function $(y, y''+\varphi y)$ satisfies the maximum and the minimum principle.

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BEST APPROXIMATION IN Q-INNER-PRODUCT SPACES

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ABSTRACT. - Caracterizations of best approximation element and cheby - chefian subspaces in Q-inner-product spaces by the use of continuous linear functionals are given.

1. **Introduction.** Q-inner product spaces are real vector spaces X endowed with a Q-inner product, i.e., a positive definite and symmetric mapping q: $X^4 \to \mathbb{R}$ which is linear in the first variable and satisfies an inequality of Cauchy-Schwarz type. Among these spaces we include the usual inner product spaces, the real Lebesque spaces $L^4(\mu)$ and $L^p(\mu)$ with p>4 if $\mu(\Omega)<\infty$, equipped with appropriate Q-inner products generating their inner products and their norms respectively.

DEFINITION([3]). Given a real linear space X, a mapping q : $X^4 \to \mathbb{R}$ is called *quaternary inner-product* or *Q-inner product*, for short, if it satisfies the conditions:

(i) $q(\alpha x_1+\alpha' x'_1, x_2, x_3, x_4)=\alpha q(x_1, x_2, x_3, x_4)+\alpha' q(x_1, x_2, x_3, x_4)$ for all α, α' in $\mathbb{R}$ and all x_1, x'_1, x_2, x_3, x_4 in X;

(ii) $q(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}) = q(x_1, x_2, x_3, x_4)$ for all x_1, x_2, x_3, x_4 in X and any permutation σ of indices (1,2,3,4);

(iii) $q(x_1, x_2, x_3, x_4)>0$ for all nonzero x_1 in X;

(iv) $|q(x_1, x_2, x_3, x_4)|^4 \le \prod_{i=1}^{4} q(x_i, x_i, x_i, x_i)$ for all x_i in X (i=1,...,4)

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A pair (X,q) is said to be a *Q-inner product space* if X is a real vector space and q is a Q-inner product on it. It is easy to see that a Q-inner product space (X,q) will be regarded as a normed space $(X,\|\cdot\|_q)$ with the norm defined by $\|x\|_q := [q(x,x,x,x)]^{1/4}$.

If p is an inner product in a real vector space X, then the function $q : X^4 \to \mathbb{R}$ defined by:
$q(x_1,x_2,x_3,x_4):=3^{-1}[p(x_1,x_2)p(x_3,x_4)+p(x_1,x_3)p(x_2,x_4)+$ $+p(x_1,x_4)p(x_2,x_3)]$ is a Q-inner product on X generating the inner product norm $\|\cdot\|_p$. Let also $(\Omega,\mathbf{A},\mu)$ be a measure space. If x_1,x_2,x_3,x_4 are in $L^4(\mu)$,

$$q(x_1,x_2,x_3,x_4):= \int_\Omega x_1(s)x_2(s)x_3(s)x_4(s)d\mu(s), \quad (1)$$

then this defines a Q-inner product in $L^4(\mu)$ generating the norm $\|\cdot\|_4$ in $L^4(\mu)$. This Q-inner product cannot be recaptured from any usual inner product in $L^4(\mu)$. When $\mu(\Omega)<\infty$, then formula (1) defines a Q-inner product in each space $L^p(\mu)$ with $p>4$.

THEOREM 1.([3]). *Every Q-inner product space (X,q) is uniformly convex and its norm is Gâteaux differentiable. Moreover, the Gâteaux differential $\tau(x,y)$ of $\|\cdot\|_q$ at $x\in X\setminus\{0\}$ in the direction $y\in X$ is given by*

$$\tau(x,y) = \lim_{t\to 0} (\|x+ty\|_q - \|x\|_q)/t = q(x,x,x,y)/ \|x\|_q^3.$$

Given a Q-inner product space (X,q), we say that an element x in X is Q-orthogonal to another element y in X if $q(x,x,x,y)=0$ and we denote this by $x\perp_q y$. By the use of R.C.James result [4] we observe that $x\perp_q y$ iff $x\perp_B y$, i.e., x is orthogonal over y in the sense of Birkhoff [1], [3].

2. The main results. We start to the following characterization theorem.

THEOREM 2. *Let* (X,q) *be a* Q-*inner product space and* E *be its nondense linear subspace. If* $x_0 \epsilon X \setminus \overline{E}$, $x' \epsilon E$, *then the following statements are equivalent:*

(i) $\rho_E(x_0) = \{x'\}$;

(ii) *there exists a unique* $x'' \epsilon E^{\perp q}$ *such that* $x_0 = x' + x''$; *where* $\rho_E(x_0)$ *denotes the set of best approximation elements refering to* x_0 and $E^{\perp q}$ *denotes the* Q-*orthogonal complement of* E (see also [3]).

The proof follows to the fact that $x' \epsilon \rho_E(x_0)$ iff $x - x' \perp_B E$ if $x_0 - x' \perp qE$ and since $(X, \|\cdot\|_q)$ is strictly convex. We omit the details.

THEOREM 3. *Let* (X,q) *be as above,* f *be a nonzero continuous linear functional on it and* $x_0 \epsilon X \setminus Ker(f)$, $g_0 \epsilon Ker(f)$. *Then the following sentences are equivalent:*

(i) $\rho_{Ker(f)}(x_0) = \{g_0\}$;

(ii) g_0 *is the unique element in* Ker(f) *such that*

$$f(x) = f(x_0) q(x, x_0 - g_0, x_0 - g_0, x_0 - g_0) / \|x_0 - g_0\|_q^4 \qquad (2)$$

for all x *in* X.

Proof. "(i)⇒(ii)". If $g_0 \epsilon \rho_{Ker(f)}(x_0)$ then the element $w_0 := x_0 - g_0$ is Q-orthogonal over Ker(f) and since $f(x)w_0 - f(w_0)x$ belongs to Ker(f) for all x in X, hence $q(f(x)w_0 - f(w_0)x, w_0, w_0, w_0) = 0$, what implies the representation (2). If g_0' is another element such that (2) holds, then $x_0 - g_0' \perp_B Ker(f)$, i.e., $g_0' \epsilon \rho_{Ker(f)}(x_0)$

which implies $g_0 = g_0'$ and the implication is proved.

"(ii)⇒(i)". It's obvious.

COROLLARY. *Let* (X,q) *and* E *be as in Theorem 2 and* $x_0 \in X \setminus \bar{E}$, $g_0 \in E$,. *Then the following statements are equivalent:*

(i) $P_E(x_0) = \{g_0\}$;

(ii) g_0 *is the unique element in* E *such that for all continuous linear functional* f *on* $E \oplus Sp(x_0)$ *with* Ker(f)=E *the following representation holds:*

$f(x) = f(x_0)q(x, x_0 - g_0, x_0 - g_0, x_0 - g_0) / \|x_0 - g_0\|_q^4$

for all $x \in E \oplus Sp(x_0)$.

Now, we shall give a characterization of chebychefian linear subspaces E, i.e., the closed linear subspaces E with the property that $P_E(x)$ contains a unique element for all x in X.

THEOREM 4. *Let* (X,q) *be a Q-inner product space and* E *be a nondense linear subspace in it. Then the following assertions are equivalent:*

(i) E *is chebychefian;*

(ii) E *is closed and the following decomposition holds:*

$$X = E \oplus E^{\perp_q} \tag{3}$$

The proof is obvious from Theorem 2 and we omit the details.

Remark 1. If E is finite-dimensional then (3) holds and if $(X, \|\cdot\|_q)$ is complete, then for all closed linear subspace E in X the decomposition (3) is also valid. Note that the last statement improves Theorem 2.4 from [3].

Now, we state the main results of our paper.

THEOREM 5. *Let* (X,q) *be as above and* f *be a nonzero* *continuous linear functional on it. Then the following statements are equivalent:*

(i) Ker(f) *is chebychefian;*

(ii) *there exists a unique element* $u_f \in X$, $\|u_f\| = 1$ *such that:*

$$f(x) = \|f\| q(x,u_f,u_f,u_f) \text{ for all } x \in X. \tag{4}$$

Proof. "(i)⇒(ii)". If Ker(f) is chebychefian, then there exists $w_0 \in X\setminus\{0\}$, $w_0 \perp_q$ Ker(f). As in Theorem 3 we have the representation $f(x) = f(w_0)\, q(x,w_0,w_0,w_0)/\|w_0\|_q^4$ for all x in X. Putting $u_f := w_0/\|w_0\|_q$ if $f(w_0)>0$ or $u_f := -w_0/\|w_0\|_q$ if $f(w_0)<0$ one gets (4).

On the other hand, $f(u_f) = \|f\|$, i.e., u_f is a maximal element of the norm one and since $(X,\|\cdot\|_q)$ is strictly convex, then by Krein's theorem (see for example [5], p.102) we conclude that u_f is the unique element with the property (4).

"(II)⇒(i)". It's also obvious from Krein's theorem and we omit the details.

COROLLARY. *Let* (X,q) *be as above and* E *be its closed linear subspace. Then the following statements are equivalent:*

(i) E *is chebychefian in* X;

(ii) *for every* $x_0 \in X\setminus E$ *and for all* $f \in (E \oplus Sp(x_0))^*$ *such that* Ker(f) = E *there exists a unique element* $u_{x_0,f} \in E \oplus Sp(x_0)$, $\|u_{x_0,f}\| = 1$ *with the property that:*

$$f(x) = \|f\|_{E\oplus Sp(x_0)}\, q(x,u_{x_0,f},u_{x_0,f},u_{x_0,f}) \text{ for all } x \in E\oplus Sp(x_0).$$

Remark 2. If E is a finite-dimensional subspace in (X,q) then for all f a nonzero continuous linear functional on E there exists a unique element $u_{f,E} \in E$, $\|u_{f,E}\| = 1$ such that

$$f(x) = \|f\|_E q(x, u_{f,E}, u_{f,E}, u_{f,E}) \text{ for all } x \in E.$$

If $(X, \|\cdot\|_q)$ is complete, then for all nonzero continuous linear functional f in X there exists a unique element $u_f \in X$, $\|u_f\| = 1$ such that the representation (4) holds. Note this fact improves Theorem 4.3 from [3] and generalizes the clasical theorem of Riesz which works in Hilbert spaces.

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BASIC PROBLEMS OF THE METRIC FIXED POINT THEORY REVISITED (II)

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Dedicated to Professor I. Muntean on his 60^{th} anniversary

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REZUMAT. -Problemele de bază ale teoriei metrice a punctului fix revizitate (II). In anul 1983, în lucrarea [93], am formulat anumite probleme de bază ale teoriei metrice a punctului fix, în cazul operatorilor multivoci. In prezenta lucrare se reanalizează problematica de bază a acestei teorii, din perspectiva rezultatelor obţinute în perioada 1980-1990.

1. Introduction. In 1983, in the paper [93], we formulated some basic problems in the metric fixed point theory for multivalued mappings. The aim of this paper is to analyse these problems from the light of the results given in 1980-1990.

Throughout the paper we follow terminologies and notations in [93] (or see [91]). For the convenience of the reader, we recall some of them.

Let (X,d) be a metric space and $T: X-\circ X$ a m-mapping. Then

$P(X) := \{A \subset X \mid A \neq \phi\}$,

$P_b(X) := \{A \in P(X) \mid A \text{ bounded}\}$,

$P_{cl}(X) := \{A \in P(X) \mid A = \bar{A}\}$,

$P_{cp}(X) := \{A \in P(X) \mid A \text{ a compact set}\}$,

$I(T) := \{A \in P(X) \mid T(A) \subset A\}$,

$I_b(T) := \{A \in I(T) \mid A \text{ a bounded set}\}$,

$\delta(A) := \sup \{d(a,b) \mid a,b \in A\}$,

$\delta(A,B) := \sup \{d(a,b) \mid a \in A,\ b \in B\}$,

$D(A,B) := \inf \{d(a,b) \mid a \in A,\ b \in B\}$,

$H(A,B) := \max (\sup \{D(a,B) \mid a \in A\}, \sup \{D(b,A) \mid b \in B\})$.

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2. Multivalued mappings on metric spaces. Let (X,d) and (Y,ρ) be two metric spaces. A mapping $T : X \to P(Y)$ is

(a) bounded if $A \in P_b(X)$ implies $T(A) \in P_b(Y)$;

(b) compact if $A \in P_b(X)$ implies $\overline{T(A)} \in P_{cp}(Y)$;

(c) upper semicontinuous (u.s.c.) if for each closed subset $A \subset Y$, $T^{-1}(A)$ is a closed subset of X;

(d) lower semicontinuous (l.s.c.), if for each open subset $A \subset Y$, $T^{-1}(A)$ is an open subset of X;

(e) continuous, if it is u.s.c. and l.s.c.;

(f) closed, if for each $x_0 \in X$ we have

$(x_n \to x_0,\ y_n \to y_0,\ y_n \in T(x_n))$ implies $(y_0 \in T(x_0))$.

Remark 2.1. For the basic notions in the theory of multivalued mappings see: [8], [9], [10], [11].

Let (X,d) be a metric and $T : X \to P_{b,cl}(X)$ a m-mapping. In the last thirty years many papers have appeared which establish various fixed point theorems for such type of mappings. In these theorems, the mapping T satisfies various conditions. In what follow we present some of such conditions:

(1) (Markin(1968), Nadler(1969)). There exists $a \in [0,1[$ such that $H(T(x),T(y)) \le a\, d(x,y)$, for all $x,y \in X$;

(2) (Reich(1971)). There exist $a,b,c \in R_+$, $a+b+c<1$, such that

$H(T(x),T(y)) \le a\, d(x,y) + b\, D(x,T(x)) + c\, D(y,T(y))$,

for all $x,y \in X$;

(3) (Iseki(1974)). There exist $a,b,c \in R_+$, $a+2b+2c<1$, such that $H(T(x),T(y)) \le a\, d(x,y)+b[D(x,T(x))+D(y,T(y))] + c[D(x,T(y)) + D(y,T(x))]$, for all $x,y \in X$;

(4) (Ćirić (1972)). There exists $a \in [0,1[$ such that
$H(T(x),T(y)) \le a \max\{d(x,y), D(x,T(x)), D(y,T(y)), 1/2[D(x,T(y))+ + D(y,T(x))]\}$, for all $x,y \in X$;

(5) (Rus(1972)). There exists $a \in [0,1[$ such that
$H(T(x),T(y)) \le a\, d(x,y)$, for all $x \in X$, $y \in T(x)$;

(6) (Rus(1972, 1975, 1979)). There exists $\varphi : R^5_+ \to R_+$, such that
$H(T(x),T(y)) \le \varphi(d(x,y),D(x,T(x)),D(y,T(y)),D(x,T(y)),D(y,T(x)))$ for all $x,y \in X$;

(7) (Reich(1972)). There exist $a,b,c \in R_+$, such that $a+b+c<1$ and $\delta(T(x),T(y)) \le a\, d(x,y)+b\, \delta(x,T(x)) + c\, \delta(y,T(y))$, for all $x,y \in X$;

(8) (Avramescu(1972)). There exist $a,b,c \in R_+$, $a+b+c < 1$, such that
$d(y_1,y_2) \le a\, d(x_1,x_2)+b\, d(x_1,y_1)+c\, d(x_2,y_2)$, for all $y_i \in T(x_i)$;

(9) (Rus(1975,1979)). There exists $\varphi : R^5_+ \to R_+$ such that
$\delta(T(x),T(y)) \le \varphi(d(x,y), \delta(x,T(x)), \delta(y,T(y)), \delta(x,T(y)), \delta(y,T(x))$, $\forall\, x,y \in X$.

(10) (Smithson(1971)).
$H(T(x),T(y)) < d(x,y)$, for all $x,y \in X$, $x \ne y$;

(11) (Rus(1983,1990)). There exists $a \in [0,1[$, such that
$\delta(T(A)) \le a\, \delta(A)$, for all $A \in I_b(T)$;

(12) (Rus(1983,1990)).
$\delta(T(A)) < \delta(A)$, for all $A \in I_b(T)$, $\delta(A) \ne 0$;

(13) (Ćirić (1972)). There exists $a \in [0,1[$, such that
$\delta(T(x)) \cup T(y)) \le a \max\{d(x,y),H(x,T(x)),H(y,T(y)),$

$1/2[D(x,T(y))+D(y,T(x))]$ for all $x,y \in X$.

Remark 2.2. Some metric conditions are degeneracy conditions. For example, in [83] the authors use the following condition:

There exists $a \in [0,1[$, such that

$H(T(x),T(y)) \leq a \{D(x,T(x))\ D(y,T(y))\}^{1/2}$, for all $x,y \in X$.

This condition implies that if $x_0 \in F_T$, then $T(x) = T(x_0)$, for all $x \in X$.

3. **Invariant subsets.** Let $T:X\multimap X$ be a m-mapping. By definition an element $x \in X$ is a fixed point of T if $x \in T(x)$ and a strict fixed point of T if $T(x) = \{x\}$. We denote by F_T the fixed point set of T and by $(SF)_T$ the strict fixed point set of T. By definition a subset $A\subset X$ is an invariant subset under T if $T(A) \subset A$. The following results are well known.

LEMMA 3.1. *Let* $T : X \to P(X)$ *be a m-mapping. Then*

(i) $T^n(X) \in I(T)$, *for all* $n \in N$;

(ii) $(SF)_T$ *is a fixed set of* T.

LEMMA 3.2. (Berge, Martelli; see[97]). *Let* X *be a compact topological space and* $T : X\to P(X)$ *a m-mapping. Then there exists a nonempty closed subset* $Y \subset X$ *such that* $Y = \overline{T(Y)}$. *If* T *is u.s.c. with closed value, then* $Y = T(Y)$.

In general, F_T is not an invariant subset for T but we have $F_T\subset T(F_T)$ (see [94]). The following problem arises:

PROBLEM 1. Let (X,d) be a metric space and $T : X\multimap X$ a m-mapping. Which are metric conditions, on T, which imply that $T(F_T)= F_T$?

References: [2], [5], [34], [61], [62], [66], [94], [97].

We have

THEOREM 3.1. *Let* (X,d) *be a metric space and* $T:X\to P_{b,cl}(X)$ *be a m-mapping. If there exists a function* $\varphi : R^4_+\to R_+$ *such that*

(i) $\varphi(0,r'_2,0,r'_4) \le \varphi(0,r''_2,0,r''_4)$, *for all* r'_2, r''_2, r''_4 $r'_4 \in R_+$, $r'_2 \le r''_2$, $r'_4 \le r''_4$.

(ii) $r-\varphi(0,r,0,r) \le 0$ *implies* $r = 0$.

(iii) $H(T(x),T(y)) \le \varphi(D(x,T(x)),D(y,T(y)),D(y,T(x)),D(x,T(y)))$, *for all* $x,y \in X$.

Then $T(F_T) = F_T$. *Moreover* $x,y \in F_T$ *implies* $T(y) = F_T$.

Proof. First we remark that

$D(x,T(y)) \le H(T(x),T(y))$, for all $x \in F_T$, $y \in X$,

and

$D(y,T(y)) \le H(T(x),T(y))$, for all $x \in X$, $y \in T(x)$.

Now, let $x \in F_T$ and $y \in T(x)$. We have

$H(T(x),T(Y)) \le \varphi(0,H(T(x),T(y)),0,H(T(x),T(y)))$, and from (ii), it follows that $H(T(x),T(y)) = 0$. Thus $T(y) = T(x)$ and $T(F_T) = F_T$.

4. Fixed points.

PROBLEM 2a. Let (X,d) be a complete metric space and $T:X\multimap X$. Which are metric conditions on T which imply that $F_T\neq\phi$?

References : [1], [2], [15], [16], [17], [20], [21], [34], [36], [41], [43], [44], [45], [46], [47], [54], [65], [67], [84], [85], [86], [90], [91], [92], [93], [97], [110], [114].

We have

THEOREM 4.1. *Let* (X,d) *be a complete metric space and*

$T:X \to P_{b,cl}(X)$ *a m-mapping. We suppose that there exists a comparison function* φ $:R_+\to R_+$ *such that*

(i) $H(T(x),T(y)) \le \varphi\,(d(x,y))$, *for all* $x\in X$, $y\in T(x)$.

If

(ii) T *is a closed m-mapping.*

or

(iii) *There exists a function* $\psi:R^5{}_+\to R_+$ *such that*

(a) $\psi(0,0,r,r,0)<r$, *if* $r>0$;

(b) *if* $u_1\le u_2$, $v_1\le v_2$, *then*

$\psi(u,u_1,v,w,v_1) \le \psi(u,u_2,v,w,v_2)$, *for all* $u_i,v_i,u,v,w\in R_+$

(c) $H(T(x),T(y)) \le \psi(d(x,y),\ D(x,T(x)),\ D(y,T(y)),\ D(x,T(y)),\ D(y,T(x)))$, *for all* $x,y\in X$.

then, $F_T \ne \phi$.

Proof. Let us have (i) + (ii). For (i) + (iii) see [92]. Let $q>1$ be such that $q\varphi$ is a comparison function. Let $x_0\in X$ and $x_1\in T(x_0)$. If $H(T(x_0),T(x_1)) = 0$, then $T(x_0) = T(x_1) \ni x_1$. Let $H(T(x_0),T(x_1)) \ne 0$. Then there exists (see[84] or [93]) $x_2\in T(x_1)$ such that $d(x_1,x_2) \le qH(T(x_0),T(x_1))$. This implies that $d(x_1,x_2) \le q\varphi(d(x_0,x_1))$. If $H(T(x_1),T(x_2)) = 0$, then $T(x_1) = T(x_2)\ni x_2$. Let $H(T(x_1),T(x_2)) \ne 0$. Then there exists $x_3\in T(x_2)$ such that $d(x_2,x_3) \le q\varphi(d(x_1,x_2))$. In this way we prove that there exists a convergent sequence $(x_n)_{n\ge0}$ such that $x_{n+1}\in T(x_n)$. Let $x^* := \lim x_n$. From (ii) we have $x^*\in T(x^*)$.

PROBLEM 2b.Let (X,d) be a bounded complete metric space and $T:X\multimap X$. Which are metric conditions, on T, which imply $F_T \ne \phi$?

References: [17], [22], [61], [78], [89], [93], [96], [97], [110].

PROBLEM 2c. Let (X,d) be a compact metric space and $T : X \multimap X$. Which are metric conditions on T which imply that $F_T \neq \phi$?

References: [17], [33], [48], [82], [93], [96], [97], [105], [110].

PROBLEM 3. To extend the theorems of Maia's type to the setting of m-mappings.

References: [78], [93].

We have

THEOREM 4.2. *Let* X *be a nonempty set,* d *and* ρ *two metric on* X *and* $T: X \to P_{b,cl}(X,\rho)$ *a m-mapping. We suppose that*

(i) $d(x,y) \leq \rho(x,y)$, *for all* $x,y \in X$;

(ii) (X,d) *is a complete metric space,*

(iii) $T: (X,d) \to (X,d)$ *is a closed mapping;*

(iv) *there exists a comparison function* $\varphi: R_+ \to R_+$, *such that*

$H_\rho(T(x),T(y)) \leq \varphi(\rho(x,y))$, *for all* $x \in X$, $y \in T(x)$.

Then $F_T \neq \phi$.

Proof. From (iv) there exists a fundamental sequence $(x_n)_{n\geq 0}$, in (X,ρ), such that $x_{n+1} \in T(x_n)$, $n \in N$. By (i) this sequence is fundamental in (X,d). From (ii) it is a convergent sequence. Let x^* be the limit of this sequence. From (iii) we have $x^* \in T(x^*)$.

Remark 4.1. The Theorem 4.2 will remain true if condition (iv) is replaced by

(iv') $H_\rho(T(x),T(y)) \leq a\, d(x,y) + b\, D(y,T(y))$, $x \in X$, $y \in T(x)$;

$a,b \in R_+$, $a+b < 1$.

PROBLEM 4. To extend the theorems of Caristi's type to the setting of m-mappings.

References: [19], [59], [65], [93], [101], [121].

In this connection the following problem may be of interest:

PROBLEM 4a.(J.P.Penot(see[59])). Let (X,d) be a complete metric space, $\varphi: X \to R_+$, be a l.s.c. function and $T: X \to P_{cl}(X)$ satisfyng the following condition

$D(x,T(x)) \leq \varphi(x) - \inf\{\varphi(y) \mid y \in T(x)\}$.

Does T have a fixed point in X?

5.Strict fixed points.

PROBLEM 5a. Let (X,d) be a complete metric space and $T: X \multimap X$. Which are metric conditions, on T, which imply $(SF)_T \neq \phi$?

References: [13], [17], [19], [21], [22], [24], [30], [43], [66], [79], [84], [86], [87], [89], [92], [93], [94].

PROBLEM 5b. Let (X,d) be a bounded complete metric space and $T: X \multimap X$. Which are metric conditions, on T, which imply $(SF)_T \neq \phi$?

References: [17], [33], [54], [93], [94], [96], [97].

One of the main results for this problem is the following:

THEOREM 5.1([97]). *Let* (X,d) *be a bounded complete metric space and* $T: X \to P(X)$ *a* (δ,φ)*-contraction. Then*

(a) $(SF)_T = \{x^*\}$.

(b) $F_T = (SF)_T$.

PROBLEM 5c. Let (X,d) be a compact metric space and $T: X \multimap X$. Which are metric conditions on T which imply $(SF)_T \neq \phi$?

References: [17], [33], [54], [85], [93], [94], [96], [97], [105].

We have

THEOREM 5.2([97]). *Let* (X,d) *be a compact metric space and*

$T: X \to P(X)$ *a δ-condensing m-mapping. Then*

$(SF)_T = \{x^*\}$.

6. **Successive approximations.** Let (X,d) be a metric space and $T: X \to P(X)$. By definition a sequence of successive approximation of T at x_0 is a sequence $(x_n)_{n\geq 0}$ such that $x_{n+1} \in T(x_n)$, for all $n \in Z_+$. The mapping T is asymptotically regular at x_0 with respect to a sequence of successive approximations, $(x_n)_{n\geq 0}$, if $D(x_n, T(x_n)) \to 0$ as $n \to \infty$.

PROBLEM 6. Which metric conditions imply that T is asymptotically regular?

References: [34], [45], [105], [109], [88].

PROBLEM 7. Which metric conditions imply that for all $x_0 \in X$, there exists a sequence of successive approximations for T, what converges to a fixed point of T ?

References: [2], [20], [34], [61], [64], [89], [93], [105], [109], [113], [114], [117].

PROBLEM 8. Which metric conditions imply there exists a successive approximations such that $T(x_n) \to F_T$ as $n \to \infty$?

References: [2], [34], [61], [89], [93].

PROBLEM 9. Which are metric conditions which imply all the following statements:

(i) $F_T \neq \phi$.

(ii) There exists a sequence of successive approximations for T what converges to a fixed point of T.

(iii) $T(y) = F_T$, for all $y \in F_T$.

References: [2], [34], [61], [89], [93], [113], [114].

7. **Stability of fixed point set.** We begin with some remarks on comparison function. By definition (see[93]) a function $\varphi:R_+\to R_+$ is a comparison function if

(a) φ is monoton increasing,

(b) $\varphi^n(t)\to 0$ as $n\to\infty$, for $t\geq 0$.

A comparison function is a strict comparison function if

(c) $t-\varphi(t)\to+\infty$ as $t\to+\infty$.

If φ is strict comparison function, then let

$t_\eta(\varphi) := \sup\{t \mid t-\varphi(t) \leq \eta\}$.

We remark that $t_\eta(\varphi) \to 0$ as $\eta \to 0$.

PROBLEM 10a. Let (X,d) be a complete metric space and $T,T_n:X\to P(X)$, $n\in N$, such that

(i) $(T_n)_n$ converges uniformly to T,

(ii) F_{T_n} and $F_T \neq \varphi$, $n\in N$.

Which are the metric conditions which imply that $H(F_{T_n}, F_T)\to 0$ as $n \to \infty$?

PROBLEM 10b. Let (X,d) be a complete metric space, Y a topological space and $T:X\times Y\to P(X)$ be a continuous m-mapping. Which are metric conditions on $T(\cdot,y)$, which imply that the mapping

$P:Y\to(P(X),H)$, $y\to F_{T(\cdot,y)}$,

is continuous ?

References: [67], [60],[72],[93],[118],[98].

We have:

THEOREM 7.1. *Let* (X,d) *be a compact metric space and* $T,S : X \to P_{cl}(X)$. *We suppose that*

(i) *There exists a strict comparison function, such that*

$H(T(x),T(Y)) \le \varphi(d(x,y))$, *for all* $x,y \in X$,

(ii) $T(F_T) = F_T$,

(iii) $F_S \in P_{cp}(X)$ *and* $S(F_S) = F_S$,

(iv) *there exists* $\eta>0$, *such that* $H(T(x),S(x)) \le \eta$.

Then, $H(F_T,F_S) \le t_\eta(\varphi)$.

Proof. First we remark that

(a) the condition (i) implies, $H(T(A),T(B)) \le \varphi(H(A),B)$, for all $A,B \in P_{cl}(X)$;

(b) the condition (iv) implies, $H(T(A),S(A)) \le \eta$, for all $A,B \in P_{cl}(X)$;

(c) the condition (i) implies that $F_T \neq \varphi$, and $F_T \in P_{cp}(X)$.

We have

$$H(F_T,F_S) = H(T(F_T),S(F_S)) \le H(T(F_T),T(F_S)) + H(T(F_S),S(F_S)) \le$$
$$\le \varphi(H(F_T,F_S)) + \eta \text{ . So, } H(F_T,F_S) \le t_\eta(\varphi).$$

8. Nonself m-mappings.

PROBLEM 11. To analyse the Problems 2a, 2b, 2c, 3, 4, 5a, 5b, and 5c in the case of nonself m-mappings.

References: [6], [7], [17], [48], [103], [116].

Let X be a nonempty set and $Y \in P(X)$. A mapping $\rho: X \to Y$ is called a retraction of X onto Y if $\rho|_Y = 1_Y$. A m-mapping, $T: Y \multimap X$, is retractible onto Y by means of a retraction $\rho: X \to Y$, if $F_{\rho o T} = F_T$.

We have:

LEMMA 8.1. *Let* (X,S,M^o) *be a (strict) fixed point structure* (see [97]). *Let* $y \in S$ *and* $\rho: X \to Y$ *a retraction. Let* $T: Y \multimap X$ *be such that*

(i) $\rho o T \in M^O(Y)$,

(ii) T *is retractible onto* Y *by* ρ.

Then $F_T \neq \phi$ $((SF)_T \neq \phi)$.

Proof. From (i) we have $F_{\rho o T} \neq \phi$ $((SF)_{\rho o T} \neq \phi)$. From (ii) we have $F_T \neq \phi$ $((SF)_T \neq \phi)$.

From Lemma 8.1 it follows:

THEOREM 8.1. *Let* X *be a Hilbert space and* $T:\bar{B}(0;R) \to P_{cp}(X)$. *We suppose that:*

(i) *there exists a comparison function* $\varphi:R_+ \to R_+$ *such that* $H(T(x),T(y)) \leq \varphi(d(x,y))$, *for all* $x,y \in \bar{B}(0;R)$,

(ii) T *is retractible onto* $\bar{B}(0;R)$ *by the radial retraction*.

Then $F_T \neq \phi$.

9.**Fixed point set**. The following result is well known

THEOREM 9.1([3]). *Let* $\varphi:R^5_+ \to R_+$ *a strict comparison function and* $T:R \to P_{cp,cv}(R)$ *a* φ*-contraction. Then* $F_T \in P_{cp,cv}(R)$, *i.e.*, F_T *is nonempty compact convex set.*

The following problem arises:

PROBLEM 12. Which are metric conditions on $T : R \to P_{cp,cv}(R)$ which imply that $F_T \in P_{cp,cv}(R)$.

References: [100], [3].

For other proporties of the fixed point set of a m-mapping $T:(X,d) \to (X,d)$ see: [27], [3], [100], [70], [74].

10.**Common fixed points**. Let (X,d) be a metric space. In the last twenty five years many papers have appeared which contain various common fixed point theorems for a pair, $T,S:X \multimap X$, of m-mappings. Here are some of the metric conditions which appear

(1) (Avram(1975)). There exists $a,b,c\in R_+$, $a+2b+4c<1$, such that

$$\delta(T(x),S(y))\leq a\ d(x,y)+b[\delta(x,T(x))+\delta(y,S(y))]+$$
$$+c[\delta(x,S(y))+\delta(y,T(x))],\ x,y\in X.$$

(2) (Fisher(1980)). There exists $a\in[0,1[$, such that

$$\delta(T(x),S(y))\leq$$
$$\leq a\max\{d(y),\delta(x,T(x)),\delta(y,S(y)),\delta(x,S(x,y)),\delta(y,T(x))\}$$

for all x,y in X.

(3) (Papageorgiou(1983)). There exists $\varphi:R^3_+\to R_+$ such that $H(T(x),S(y))\leq\varphi(d(x,y),D(x,T(x)),D(y,S(y)))$, for all $x,y\in X$.

The following probem arises:

PROBLEM 13. Which are metric conditions wich imply (one of, all of,...)

(i) $F_T\cap F_S\neq\phi$;

(ii) $(SF)_T\cap(SF)_S\neq\phi$;

(iii) $F_T=F_S\neq\phi$;

(iv) $(SF)_T=(SF)_S\neq\phi$;

(v) $(SF)_T=(SF)_S=\{x^*\}$;

(vi) $F_T=(SF)_T=F_S=(SF)_S=\{x^*\}$;

(vii) $F_T\cup F_S\neq\phi$;

References: [12], [16], [18], [25], [28], [29], [31], [32], [35], [38], [50], [51], [52], [53], [57], [69], [71], [73], [76], [77], [78], [80], [99], [104], [106], [107].

11. Other problems.

11.1. Metrical fixed point theory for m-mappings on cartezian product: [17], [23], [14], [93], [110], [115].

11.2. Approximation for fixed point of m-mappings: [7], [17], [64], [93], [102], [110].

11.3. Metrical fixed point theory for generalized metric spaces: [17], [23], [36], [71], [81], [93], [110].

11.4. Nonexpansive m-mappings: [26], [17], [36], [58], [93], [103], [119], [120].

12. **Applications.**

12.1. **Surjectivity theorems.**

PROBLEM 14. Let X be a Banach space and $T : X \to P(X)$ a m-mapping. Which metric conditions imply that $1_X - T : X \multimap X$ is a surjective m-mapping ?

We have

THEOREM 12.1. *Let X be a Banach space and* $T: X \to P_{b,cl(X)}$ *a m-mapping for what there exists a comparison function such that*

$H(T(x),T(y)) \le \varphi(d(x,y))$, *for all* $x,y \in X$.

Then $1_X - T$ *is a surjective m-mapping.*

Proof. Let $y \in Y$. We have, $T(x) + y \ni x \Leftrightarrow x - T(x) \ni y$. Let $S(x) = T(x) + y$. We remark that

$H(S(x_1),S(x_2)) \le \varphi(d(x_1,x_2))$, for all $x_1,x_2 \in X$. This implies that (see &4) S has at least a fixed point.

12.2. **Coincidence points.** Let X and Y be two sets and $T,S: X \to P(Y)$ two m-mappings. By definition $x \in X$ is a coincidence point for the pair T,S if $T(x) \cap S(x) \neq \phi$.

Let $C(T,S) := \{x \in X \mid T(x) \cap S(x) \neq \phi\}$. We remark that

PROBLEM 15. Let (X,d), (Y,ρ) be two metric spaces and

$$C(T,S) \neq \phi \Leftrightarrow F_{T \circ S^{-1}} \neq \phi.$$

$T,S:X \multimap Y$. Which are metric conditions which imply that $C(T,S) \neq \phi$?

References: [93], [37], [75], [88].

12.3. **Optimization theory** (see [22]). Let X be a Banach space. By definition a con $C \subset X$ is a subset with the following property:

(i) $\lambda \in R_+$, $y \in C \Rightarrow \lambda y \in C$;

(ii) $C \cap (-C) = \{0\}$.

Let A be a set and $f : A \to X$. Let $T : X \to P(X)$, $x \multimap \{f(a) \mid a \in A, f(a) \in C+x\}$. We have:

THEOREM 12.2 ([22]). $f(a_0)$ *is a maximal element of* $(f(A), \leq_C)$ *iff* $f(a_0) \in (SF)_T$.

12.4. **Other applications**. For other applications of the fixed point theory see : [8], [17], [39], [40], [68], [98], [111], [112], [113], [114].

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(ε,φ)-LOCALLY CONTRACTIVE MULTIVALUED MAPPINGS AND APPLICATIONS

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Dedicated to Professor I. Muntean on his 60^{th} anniversary

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REZUMAT - (ε,φ) - contractii multivoce locale si aplicatii. Se introduce notiunea de (ε,φ) - contractie locala multivoca si se demonstreaza doua teoreme de punct fix pentru acest tip de multifunctie. Se obtin ca si consecinte citeva rezultate mai generale decit cele date in [1], [2], [8], [10], iar in final o aplicatie la o problema Cauchy multivoca este prezentata.

1. Introduction. In [3] M. Edelstein proved that if X is a complete ε-chainable metric space and $f:X\to X$ is an (ε,λ)-uniformly locally contractive mapping then there is an $x_0\in X$ such that $x_0=f(x_0)$.

S.B.Nadler jr. generalizes this rezult to multivalued mappings. In [4], Nadler defines a multivalued mapping $F:X\to P_{b,cl}(X)$ to be (ε,λ)-uniformly locally contractive (where $\varepsilon>0$ and $\lambda\in[0,1)$) provided that if $x,y\in X$ and $d(x,y)<\varepsilon$ then $H(Fx,Fy)\le\lambda\cdot d(x,y)$. This definition is modeled after Edelstein's definition for singlevalued mappings [3].

One year latter N. Covitz and S.B. Nadler jr. proved (see [2]) that if X is a complete generalized metric space, $F:X\to P_{cl}(X)$ is a (ε,λ)-uniformly locally contractive multivalued mappings (where $P_{cl}(X)$ is endowed with the generalized Hausdorff metric) and $x_0\in X$, then following alternative holds: either:

(i) for each iterative sequence $(x_n)_{n\in N}$ of F at x_0, $d(x_{n-1},x_n)\ge\varepsilon$, for each $n=1,2,...$ or,

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(ii) there exists an iterative sequence $\{x_n\}_{n\in N^*}$ of F at x_0, such that $\{x_n\}_{n\in N^*}$ converges to a fixed point of F.

R. Wegrzyk generalizes Covitz-Nadler's result to multivalued φ-contraction (see [10]).

On the other hand, M. Turinici (see [8] and [9]) using the notion of normal (strong) multivalued contraction proved two fixed point theorems and gives some applications to multivalued Cauchy problems.

The purpose of this paper is to prove some fixed point theorems for a class of multivalued mappings, from which we can obtain some consequences which generalize results given in [1], [2], [8], [10].

2. Basic results. Let (X,d) be a complete generalized metric space, $x\in X$, $Y\subset X$ and $\varepsilon>0$. Throughout this paper we use the following symbols:

$\delta(Y) := \sup\{d(a,b) \mid a,b\in Y\}$

$D(Y,x) := \inf\{d(y,x) \mid y\in Y\}$

$S(Y,\varepsilon)=\{x\in X \mid D(Y,x)<\varepsilon\}$

$P(X)=\{Y\subset X \mid Y\neq\phi\}$

$P_{cl}(X)=\{Y\in P(X) \mid \overline{Y}=Y\}$

$P_{b,cl}(X)=\{Y\in P(X) \mid \overline{Y}=Y,\ \delta(Y)<\infty\}$

Let $H:P_{cl}(X)\times P_{cl}(X)\to \overline{\mathbb{R}}_+$ be a mapping defined by:

$$H(Y,Z):=\begin{cases}\inf\{\varepsilon>0 \mid Y\subset S(Z,\varepsilon), Z\subset S(Y,\varepsilon)\}, & \\ \qquad \text{if } \{\varepsilon>0 \mid Y\subset S(Z,\varepsilon), Z\subset S(Y,\varepsilon)\}\neq\emptyset \\ +\infty & \text{, otherwise}\end{cases}$$

The following lemmas are very useful in the fixed point theory:

LEMMA 2.1.([2]). $(P_{cl}(X),H)$ *is a complete generalized metric space.*

LEMMA 2.2.([8]). *Let* Y,Z *be two nonempty, closed subsets of* X *such that* $H(Y,Z)<\varepsilon$. *Then, for every* $u\in Y$ *(resp* Z*) there is a* $v\in Z$ *(resp* Y*) with* $d(u,v)<\varepsilon$.

LEMMA 2.3.([8]). *Let* Y,Z *be two nonempty, closed subsets of* X *and* $\varepsilon>0$ *such that, for every* $u\in Y$ *(resp* Z*) there is a* $v\in Z$ *(resp* Y*) with* $d(u,v)\le\varepsilon$. *Then, necessarily* $H(Y,Z)\le\varepsilon$.

DEFINITION 2.1.([5]). Let $\varphi,\psi:\mathbb{R}_+\to\mathbb{R}_+$ two mappings. We say that ψ is φ-summable if:

(i) for each $t\in\mathbb{R}_+$

the sequence $\{\varphi^n(t)\}_{n\in N^*}$ converges to zero, as $n\to\infty$ and (2.1)

$$\sum_{n=1}^{\infty}(\psi\circ\varphi)^n(t)<\infty \tag{2.2}$$

(ii) φ is a monoton increasing function on $\mathbb{R}_+$

DEFINITION 2.2.([7]). A function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is a comparison function if it satisfies:

φ is monotone increasing (2.3)

$(\varphi^n(t))_{n\in N}$ converges to 0, for all $t\ge 0$. (2.4)

Remark 2.1.([7]). If $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is a comparison function then $\varphi(0)=0$ and $\varphi(t)<t$, for every $t>0$.

DEFINITION 2.3. A function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ is an expansion function if it satisfies the following conditions:

$\psi(0) = 0$ (2.5)

$\psi(t) > t$, for all $t>0$. (2.6)

DEFINITION 2.4. Let $\varphi,\psi : \mathbb{R}_+\to\mathbb{R}_+$ be two functions. We say that ψ is strong φ-summable if:

ψ is φ-summable (2.7)

ψ is an expansion function (2.8)

$\psi o\varphi$ is a comparison function. (2.9)

DEFINITION 2.5. Let $F:X\to P_{cl}(X)$ be a multivalued mapping (briefly m-mapping)

F is said to be (ε,φ)-locally contractive mapping (where $\varepsilon>0$ and $\varphi:\mathbb{R}_+\to\mathbb{R}_+$) if it satisfies:

$x,y\in X,\ 0<\alpha\leq\varepsilon \qquad d(x,y)<\alpha\Rightarrow\ H(Fx,Fy)\leq\varphi(\alpha)$ (2.10)

The main result for (ε,φ)-locally contractive m-mapping is the following:

THEOREM 2.1. *Let* (X,d) *be a complete generalized metric space,* $F:X\to P_{cl}(X)$ *a m-mapping and* ψ *a strong* φ*-summable function. We suppose that:*

F *is* $a(\varepsilon,\varphi)$*-locally contractive m-mapping* (2.11)

there is an $x_0\in X$ *such that* $D(x_0,Fx_0)<\varepsilon$. (2.12)

Then $F_F \neq \phi$. *(i.e. there is a fixed point of F).*

Proof. Let $x_0\in X$ be such that $D(x_0,Fx_0)<\varepsilon$. If $x_0\in Fx_0$ then $x_0\in F_F$. We suppose $x_0\notin Fx_0$. From (2.12) there is an element $x_1\in Fx_0$ such $d(x_0,x_1)<\varepsilon$.

For $x_0,x_1\in X$ and $\alpha=\varepsilon$: $d(x_0,x_1)<\alpha \Rightarrow H(Fx_0,Fx_1)\leq\varphi(\alpha) = \varphi(\varepsilon)<(\psi o\varphi)(\varepsilon)$.

From Lemma 2.2. there is an $x_2\in Fx_1$ with $d(x_1,x_2)<(\psi o\varphi)(\varepsilon)$.

Using (2.11) for $x_1,x_2\in X$ and $\alpha =(\psi o\varphi)(\varepsilon)<(\varepsilon)$:

$d(x_1,x_2)<(\psi o\varphi)(\varepsilon) \Rightarrow H(Fx_1,Fx_2)\le\varphi(\psi o\varphi)(\varepsilon)<(\psi o\varphi)^2(\varepsilon)$.

Again invoking Lemma 2.2., then is a $x_3\in Fx_2$ with $d(x_2,x_3) <$ $< (\psi o\varphi)^2(\varepsilon)$, etc.

By induction, we get an interative sequence $\{x_n\}_{n\in N}$ satisfying:

$$x_{n+1}\in Fx_n, \text{ for every } n\in N \quad (2.13)$$

$$d(x_n,x_{n+1})<(\psi o\varphi)^n(\varepsilon), \text{ for every } n\in N \quad (2.14)$$

$$H(FX_n,FX_{n+1})<(\psi o\varphi)^{n+1}(\varepsilon), \text{ for every } n\in N \quad (2.15)$$

From (2.14) and definition 2.2. $\{x_n\}_{n\in N}$ is a Cauchy sequence, so $\{x_n\}_{n\in N}$ converges to an element $x^*\in X$, as $n\to\infty$. We now prove that x^* is the required fixed point for F. By (2.11) F is a continuous mapping, so $Fx_n\to Fx^*$, as $n\to\infty$ and since $x_{n+1}\in Fx_n$, for every $n\in N$ conclusion follows if we take the limit as $n\to\infty$. Q.E.D.

Remark 2.2. From theorem 2.1., it follows that $d(x_n,x^*)\le$ $\le\sum_{k=n}^{\infty}(\psi o\varphi)^k(\varepsilon)$,for every $n\in N$.

Remark 2.3. As an important particular case, let the mapping $\varphi:\mathbb{R}\to\mathbb{R}_+$ defined by $\varphi(t)=at$, for every $t\in\mathbb{R}_+$ and some $a\in(0,1)$. Then, the mapping $\psi:\mathbb{R}_+\to\mathbb{R}_+$ defined by $\psi(t)=bt$, for every $t\in\mathbb{R}_+$ and some $b\in(1,1/a)$ is strong φ -summable.

In this way, the above theorem generalizes theorem 1 of /2/.

Remark 2.4. A m-mapping $F:X\to P_{cl}(X)$ is said to be a normal multivalued contraction with respect to a $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ (see /8/) if it satisfies:

$x,y\in X,\ \alpha>0\ d(x,y)\le\alpha \Rightarrow H(Fx,Fy)\ \le\ \varphi(\alpha)$

(or equivalent F is a multivalued φ -contraction, see /7/ and /10/).

If F is a normal multivalued contraction with respect to φ, then F is a (ε,φ) -locally contractive m-mapping. Theorem 2.1 of /8/

is a consequence of the above theorem (see also /9/ and /10/).

DEFINITION 2.6. $F:X\to P_{cl}(X)$ is said to be a strong (ε,φ)-locally contractive m-mapping (where $\varepsilon>0$ and $\varphi:\mathbb{R}_+\to\mathbb{R}_+$) if it satisfies the following condition:

if $x,y\in X$, $0<\alpha\leq\varepsilon$ satisfy $d(x,y)<\alpha$ (2.16)

then, for every $u\in Fx$ (resp. Fy) there is a $v\in Fy$ (resp. Fx) with $d(u,v)<\varphi(\alpha)$. (2.16)

Remark 2.5. From Lemma 2.3. every strong (ε,φ)-locally contractive m-mapping is, necesarily, a (ε,φ)-locally contractive m-mapping.

Now, the second main result of this note is:

THEOREM 2.2. *Let* (X,d) *be a complete generalized metric space,* $F:X\to P_{cl}(X)$ *a m-mapping and* φ *a comparison function.* We *suppose that:*

F is a strong (ε,φ)*-locally contractive m-mapping* (2.17)

there is $x_0\in X$ *such that* $D(x_0,Fx_0)<\varepsilon$. (2.18)

Then, we have $F_F\neq\phi$.

Proof: Let $x_0\in X$ be such that $D(X_0,Fx_0)<\varepsilon$. If $x_0\in Fx_0$ then $x_0\in F_F$.

We suppose $x_0\notin Fx_0$. From (2.18) there is an element $x_1\in Fx_0$ with $d(x_0,x_1)<\varepsilon$. For $x_0,x_1\in X$ and $\alpha=\varepsilon$: $d(x_0,x_1)<\alpha$ implies (taking into account (2.17)) that there is an element $x_2\in Fx_1$ with $d(x_1,x_2)<\varphi(\alpha)=\varphi(\varepsilon)$.

Now, for $x_1,x_2\in X$, $x_2\in Fx_1$ and $\alpha=\varphi(\varepsilon)<\varepsilon$:

$d(x_1,x_2)<\varphi(\varepsilon)$ implies again that there is a $x_3\in Fx_2$ with $d(x_2,x_3)<\varphi^2(\varepsilon)$, etc.

By induction, we get an interative sequence $(x_n)_{n\in\mathbb{N}}$

satisfying $d(x_n,x_{n+1})<\varphi^n(\varepsilon)$, for every $n\in N$.

The last part of the proof is the same as in theorem 2.1. Q.E.D.

Remark 2.6. From Theorem 2.2. it follows that:
$d(x_n,x^*)<\sum_{k=n}^{\infty}\varphi^k(\varepsilon)$, for every $n\in N$, where x^* is a fixed point of F.

THEOREM 2.3. *Let* (X,d) *be a complete* ε*-chainable generalized metric space* ($\varepsilon>0$), $F:X\to P_{cl}(X)$ *a strong* (ε,φ)*-locally contractive m-mapping* (φ *is a comparison function of* $\mathbb{R}_+$ *into itself). Then* $F_F\neq\phi$.

Proof. Conclusion follows from Theorem 2.2. (see also [2]).

Remark 2.7. The above theorems might be compared with those of [1], which contain more restrictive assumptions.

3. **An application.** In this section we use terminologies and notations from [8] or [9].

In what follows $(\mathbb{R}^n,|\cdot|)$ is the euclidean n-dimensional space endowed with a given norm.

We use the following symbols:

$X=\{\ x:\mathbb{R}\to\mathbb{R}^n\ |\ x\text{-continuous}\ \}$

$A=\{\ a:\mathbb{R}_+\to\mathbb{R}_+\ |\ a\text{-continuous}\ \}$

$A_\varepsilon=\{\ a\in A|\ a(t)\le\varepsilon,\ (\forall)t\in\mathbb{R}_+\ \}\quad(\varepsilon>0)$

For every $x\in X$ define $\|x\|\in A$ by $\|x\|(t)=|x(t)|$, for every $t\in\mathbb{R}_+$ and for every $g\in A$, let $\|\cdot\|_g:X\to\mathbb{R}_+$ be defined, for an arbitrary $x\in X$ by:

$$\|x\|_g=\begin{cases}\inf\{\lambda\in\mathbb{R}_+|\ \|x\|\le\lambda g\}, & \text{if } \{\lambda\in\mathbb{R}_+|\ \|x\|\le\lambda g\}\neq\phi\\ +\infty & \text{, otherwise}\end{cases}$$

It is simple to verify that $(X,\|\cdot\|_g)$ is a generalized Banach space (respectively, a complete generalized metric space, by the standard construction of its metric).

For every $g\in A$ denote also

$X_g = \{x\in X \mid \|x\|_g<\infty\}$ and

$C_g(X) = \{Y\subset X \mid Y$ is $\|\cdot\|_g$ -closed$\}$

Now, let $k:X\to P(X)$, $x\to k(x)$ be a m-mapping and $x_0\in\mathbb{R}^n$ a fixed element.

We consider the multivalued Cauchy problem:

$$\text{(C.P.)} \begin{cases} x'(t)\in k(x)(t) & , \text{ for every } t\in\mathbb{R}_+ \\ x(0)=x^0. \end{cases}$$

The following existence result concerning the solutions of (C.P.) may be stated:

THEOREM 3.1. *Suppose that there exists a mapping* $h:A\to A$, *a real number* $\lambda\geq 1$, $g\in A_\lambda$, φ *a comparison map and* $\varepsilon>0$ *such that:*

for every $x\in X$ *the set* $K(x)$ *of all* $\overline{y}\in X$ *with* $\overline{y}(t) = x^0+\int_0^t y(s)ds$, $\forall\, t\in\mathbb{R}_+$ (*for some* $y\in k(x)$) *is a* $\|\cdot\|_g$ *-closed set.* (3.1)

If $x,y\in X$, $\alpha\in A_\varepsilon$ *satisfy* $\|x-y\|<\alpha$, *then for any* $u\in k(x)$ (*resp.* $k(y)$) *there is a* $v\in k(y)$ (*resp.* $k(x)$) *with* $\|u-v\|\leq h(\alpha)$. (3.2)

$$\int_0^t h(g\tau)(s)ds<\varphi(\tau)g(t),\ (\forall)\,\tau>0,\ (\forall)\ t\in\mathbb{R}_+ \qquad (3.3)$$

there is an element $y^0\in X$ *such that*

$$\|y^0(t)-x^0-\int_0^t u(s)ds\|<\varepsilon g(t)\ ,(\forall)\,t\in\mathbb{R}_+ \qquad (3.4)$$

(for some $u\in k(y^0)$*).*

Then, there exists an element $z\in y^0 + X_g$ *solution for (C.P.).*

Proof. Let $F:X\to C_g(X)$ be a m-mapping defined by:

$$F(x) = K(x), \text{ for every } x \in X \qquad (3.5).$$

From (3.4) it follows that there is an element $\bar{u} \in F(y^0)$ such that $\| y^0 - \bar{u} \|_g < \varepsilon$ and so we have $D_g(y^0, Fy^0) < \varepsilon$.

Let $a \in \mathbb{R}$ be such that $0 < a \le \varepsilon/\lambda \le \varepsilon$, and let $x, y \in X$ be such that $\| x-y\|_g < a \le \varepsilon$.

From the definition of $\| \cdot \|_g$ we have $\| x-y\| < a\, g$

Let $\bar{u} \in Fx$ (resp Fy). From (3.5): $\bar{u}(t) = x^0 + \int_0^t u(s)ds$, $(\forall) t \in \mathbb{R}_+$, for some $u \in k(x)$ (resp. $k(y)$).

Let $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ denote a mapping defined by $\alpha(t) = ag(t)$. Because $g \in A_\lambda$, it follows that $\alpha(t) \le \varepsilon/\lambda\ \lambda = \varepsilon$, for every $t \in \mathbb{R}_+$.

So, for $x, y \in X$, $\alpha \in A_\varepsilon$: $\| x-y\| < \alpha$ and $u \in k(x)$ (resp $k(y)$) there is an element $v \in ky$ (resp. $k(x)$) with $\| u-v\| \le h(\alpha) = h(ga)$.

Let $\bar{v} \in X$ be defined by $\bar{v}(t) = x^0 + \int_0^t v(s)ds$, for every $t \in \mathbb{R}_+$

Clearly, $\bar{v} \in Fy$ (resp Fx).

We have:

$$\|\bar{u}(t) - \bar{v}(t)\| < \int_0^t \|u(s) - v(s)\| ds \le \int_0^t h(ga)(s)ds < \varphi(a) g(t),$$

for every $t \in \mathbb{R}_+$, i.e. $\|\bar{u} - \bar{v}\|_g < \varphi(a)$, showing that F is a strong (ε, φ)-locally contractive m-mapping. Thus, theorem 2.2 applied, and the conclusion follows. Q.E.D.

Remark 3.1. A m-mapping $F: X \to P_{cl}(X)$ is said to be weak (ε, φ)-locally contractive if:

$x, y \in X$, $d(x,y) < \varepsilon \Rightarrow H(Fx, Fy) \le \varphi(\varepsilon)$

Open problem: Give some sufficient metric conditions implying existence of fixed points (or ε-fixed points, see [6]) of a weak (ε, φ)-locally contractive m-mapping.

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AFFINE CONNECTIONS WITH GENERALIZED BIRECURRENT TORSION

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Dedicated to Professor I. Muntean on his 60[th] anniversary

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REZUMAT. -Conexiuni afine cu torsiune birecurentă generalizată. În lucrare se studiază spaţii cu conexiune afină cu torsiune birecurentă generalizată, definite prin (2), ce apar ca o generalizare naturală a lui (1) stabilindu-se propoziţiile 1,2, şi 3. Pentru cazul conexiunilor semi-simetrice şi a E-conexiunilor semi-simetrice se stabilesc relaţiile (12), (13), (15), (16) şi (20), pe care le verifică tensorii Q^i_{jkrs} şi φ_{rs} precum şi sistemul (21).

Let A_n be a space with affine connection Γ. In a coordinate system, we denote by Γ^i_{jk}, the components of the affine connection, by T^i_{jk} the components of the torsion tensor of the connection Γ and by $T_k = T^i_{ik}$ the components of the torsion vector (the Vrânceanu's vector).

The space A_n is called space with birecurrent torsion or T-birecurrent space, [4] if there exists a covariant tensor of second order ω_{rs} so that:

$$T^i_{jk,rs} = \omega_{rs} T^i_{jk} \tag{1}$$

where comma denotes the covariant derivation with respect to Γ.

A natural generalization of the relation (1) is obtained in the following way:

DEFINITION 1. The space A_n is called *space with generalized birecurrent torsion,* if we have:

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$$T^i_{jk,rs} = \varphi_{rs} T^i_{jk} + Q^i_{jkrs} \tag{2}$$

where φ_{rs} is a covariant tensor of second order and Q^i_{jkrs} a skew-symmetric tensor in j and k.

Remark 1. Here too, one observes that being given the tensor φ_{rs}, the tensor Q^i_{jkrs} is completely determined by (2).

Remark 2. The fact that (2) is a natural generalization of the relation (1) results easily. Indeed, relation (1) can be written immediately

$$T^i_{jk,rs} = \varphi_{rs} T^i_{jk} + (\omega_{rs} - \varphi_{rs}) T^i_{jk} \tag{3}$$

and the space is with generalized birecurrent torsion, with an arbitrary φ_{rs} and

$$Q^i_{jkrs} = (\omega_{rs} - \varphi_{rs}) T^i_{jk} \tag{4}$$

We have therefore:

PROPOSITION 1. *The* A_n *T-birecurrent spaces are also with generalized birecurrent torsion, with an arbitrary* φ_{rs} *and* Q *given by* (4).

Remark 3. If in (2)

$$Q^i_{jkrs} = a_{rs} T^i_{jk} \tag{5}$$

then, from (5) and (2) it follows (1) and the space is T-birecurrent with $\omega_{rs} = \varphi_{rs} + a_{rs}$.

If in (2) we apply a contraction in i and j we have:

$$T_{k,rs} = \varphi_{rs} T_k + Q_{krs} \tag{6}$$

where $Q_{krs} = Q^i_{ikrs}$ and it follows.

DEFINITION 2. The A_n spaces which satisfy (6) are called *spaces with generalized birecurrent torsion vector.*

From the way (6) was obtained with (2) as it follows:

PROPOSITION 2. *The* A_n *spaces with generalized birecurrent tosion are also with generalized birecurrent torsion vector.*

Remark 4. The converse of the assertion 2 is generally not true. In this paper we will give also a case in which the converse takes place.

The A_n spaces for which exists a convector ω_r and a tensor Q^i_{jkr} so that:

$$T^i_{jk,r} = \omega_r T^i_{jk} + Q^i_{jkr} \tag{7}$$

were called spaces with generalized recurrent torsion [3].

Derivating covariantly (7) with respect to, Γ we have:

$$T^i_{jk,rs} = \omega_{r,s} T^i_{jk} + \omega_r T^i_{jk,s} + Q^i_{jkr,s}$$

and taking count of (7) it follows:

$$T^i_{jk,rs} = (\omega_{r,s} + \omega_r\omega_s) T^i_{jk} + \omega_r Q^i_{jks} + Q^i_{jkr,s} \tag{8}$$

and the space is with generalized birecurrent torsion with

$$\varphi_{rs} = \omega_{r,s} + \omega_r\omega_s \tag{9}$$

and

$$Q^i_{jkrs} = \omega_r Q^i_{jks} + Q^i_{jkr,s} \tag{10}$$

therefore:

PROPOSITION 3. *The* A_n *spaces with generalized recurrent torsion are also with generalized birecurrent torsion with* φ_{rs} *and* Q^i_{jkrs} *given by (9), (10).*

We consider now the space A_n endowed with a semi-symmetri affine connection Γ $(n>1)$ therefore [2]:

$$T^i_{jk} = \frac{1}{n-1}(\delta^i_j T_k - \delta^i_k T_j) \qquad (11)$$

Derivating covariantly (11) twice and taking (2) and (6 into account, we have:

$$Q^i_{jkrs} = \frac{1}{n-1}(\delta^i_j Q_{krs} - \delta^i_k Q_{jrs}) \qquad (12)$$

relation of the same kind as (11), therefore:

PROPOSITION 4. *In a generalized birecurrent torsion* A *space, $n > 1$, with semi-symmetric connection, the tensor* Q^i_{jkrs} *and his contracted* $Q_{krs} = Q^i_{ikrs}$ *satisfy the relation (12).*

For the fixed indexes r,s transvecting (12) by Q_{irs} w have:

$$Q^i_{jkrs}\, Q_{irs} = 0 \qquad (13)$$

and terefore:

PROPOSITION 5. *In the generalized birecurrent torsion* A *spaces with semi-symmetric connection, (13) takes place.*

The relations (11) and (12) give for these spaces, th answer to the remark 4. Indeed, from (11), derivating covariantl twice, and taking (6) and (11) into account, it follows (2) wit Q^i_{jkrs} given by (12).

We have therefore the converse assertion:

PROPOSITION 6. *The* A_n *spaces with semi-symmetri connection and with generalized birecurrent torsion vector a also with generalized recurrent torsion with the same* φ_{rs} *a* *with* Q^i_{jkrs} *given by (12).*

From the relation of S.Golab [5] for semi-symmetric connections:

$$T^i_{sj}\,T^s_{kh} + T^i_{sk}\,T^s_{hj} + T^i_{sh}\,T^s_{jk} = 0 \tag{14}$$

derivating covariantly twice, and taking count of (2) and (14) we have:

$$\begin{aligned} &Q^i_{sjrp}\,T^s_{kh} + Q^s_{khrp}\,T^i_{sj} + Q^i_{skrp}\,T^s_{hj} + \\ &+ Q^s_{hjrp}\,T^i_{sk} + Q^i_{shrp}\,T^s_{jk} + Q^s_{jkrp}\,T^i_{sh} + \\ &+ T^i_{sj,r}T^s_{kh,p} + T^i_{sk,r}T^s_{hj,p} + T^i_{sh,r}T^s_{jk,p} + \\ &+ T^i_{sj,p}T^s_{kh,r} + T^i_{sk,r}T^s_{hj,r} + T^i_{sh,p}T^s_{jk,r} = 0 \end{aligned} \tag{15}$$

Therefore:

PROPOSITION 7. *In the semi-symmetric* A_n *spaces with generalized birecurrent torsion, (15) takes place.*

From (14) by contraction in i and j one gets the well-known [5] relation:

$$T_s\,T^s_{kh} = 0 \tag{16}$$

from which, derivating it covariantly twice with respect to Γ and taking (2), (6) and (16) into account, it follows:

$$Q^i_{jkhs}\,T_i + T^i_{jk}\,Q_{irs} + T^i_{jk,r}\,T_{i,s} + T^i_{jk,s}\,T_{i,r} = 0 \tag{17}$$

PROPOSITION 8. *In the semi-symmetric connection* A_n *spaces with generalized birecurrent torsion, (17) take place between the torsion tensor and the generalized recurrency tensor.*

If the semi-symmetric connection of the A_n space is an E-connection, therefore [2]

$$T_{i,j} - T_{j,i} = 0 \tag{18}$$

from [5]

$$T^i_{jk,r} + T^i_{kr,j} + T^i_{rj,k} = 0 \tag{19}$$

by covariant derivation and taking count of (2) we have:

$$\varphi_{rs}T^i_{jk} + \varphi_{js}T^i_{kr} + \varphi_{ks}T^i_{rj} + Q^i_{jkrs} + Q^i_{krjs} + Q^i_{rjks} = 0 \tag{20}$$

In (20) applying a contraction in i and r and taking count of (6) and (18) it follows:

$$\varphi_{is}\, T^i_{jk} + Q^i_{jkis} = 0 \tag{21}$$

we have therefore:

PROPOSITION 9. *In the semi-symmetric E-connection A_n spaces and with generalized birecurrent torsion, the tensor φ_{rs} is a solution of the n linear systems (21) and verifies (20).*

Remark 5. The systems (21) can also be obtained from the vanishing of the divergence by covariant derivation and taking count of (2).

Remark 6. In remark (1) we emphasized the fact that (2) completely determines the tensor Q^i_{jkrs} if φ_{rs} is given. Now, for the semi-symmetric E-connection we can outline the fact that being given the tensor Q^i_{jkrs}, the problem of the determination of a tensor φ_{rs} that verifies (2) one reduces to the compatibility and solving of the systems (21) with the conditions (20).

Remark 7. From the relations (4) and (9), naturally appears the case in which the tensor Q^i_{jkrs} is degenerate. Therefore one should impose a detailed study of the case in which Q^{i}_{jkr} is degenerate of various kind of degeneration.

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SOME NEW ROOF-SURFACES GENERATED BY BLENDING INTERPOLATION TECHNIQUE

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Dedicated to Profesor I.Muntean on his 60th anniversary.

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Rezumat. Noi suprafeţe acoperiş generate de tehnica interpolării blending. Scopul lucrării este de a prezenta noi suprafeţe acoperiş obţinute cu ajutorul interpolării blending, folosind operatori de interpolare de tip Birkoff.

In some previous papers [3,4,5,6,8] there were studied applications of blending interpolation in generating roof-surfaces for large halls.

The goal of this paper is to construct some new such surfaces using as the start points the blending interpolation on the rectangular respectively the triangular domains.

Let $D=[-a,a]x[-b,b]$ be a rectangular domain in the xOy plane. The problem is to construct a function F, $F:D\to R$ that satisfies the natural conditions: $F|_{\partial D} = 0$ (the roof is staying on its support - the border of D) and $F(0,0)=h$ (the height of the roof in the center of D). To control the position of the tangent planes or the inflaction lines of the surface, can be used some supplementary conditions.

Also, the parabolic points of the surface must be taken into attention, that are the maximum stress points of the surface [6].

1. For the beginning we take as supplementary conditions the

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following:

$$F^{(1,0)}(-\alpha, y) = F^{(1,0)}(\alpha, y) = 0, \quad for\ y \in [-b, b], \quad \alpha \in [0, a],$$

and

$$F^{(0,1)}(x, -\beta) = F^{(0,1)}(x, \beta) = 0, \quad for\ x \in [-a, a], \quad \beta \in [0, b],$$

To interpolate the corresponding data, there are used the Birkhoff's interpolation operators B_4^x and B_4^y defined by

$$(B_4^x F)(x, y) = \varphi_1(x) F(0, y) \quad with\ \varphi_1(x) = \frac{(x^2-a^2)(x^2+a^2-2\alpha^2)}{a^2(2\alpha^2-a^2)}$$

$$(B_4^y F)(x, y) = \psi_1(y) F(x, 0) \quad with\ \psi_1(y) = \frac{(y^2-b^2)(y^2+b^2-2\beta^2)}{b^2(2\beta^2-b^2)} .$$

respectively

As, it is well known, the Boolean sum $F = B_4^x \oplus B_4^y$ is a **blending interpolation operator** that interpolates all the required data. So, we get the family of surfaces

$$F_1(x, y) = \varphi_1(x) F(0, y) + \psi_1(y) F(x, 0) - \varphi_1(x) \psi_1(y) h$$

that depends on the univariate functions $F(0,.)$ and $F(.,0)$. By a suitable selection of these functions we can obtain various kinds of surfaces.

1.1. First, one approximates the functions $F(.,0)$ and $F(0,.)$ by the same Birkhoff's polynomials

$$F(x,0) := (B_4^x F)(x, 0) = \varphi_1(x) F(0, 0)$$

respectively

$$F(0, y) := (B_4^y)(0, y) = \psi_1(y) F(0, 0).$$

One obtains

$$F_{11}(x, y, \alpha, \beta) = \varphi_1(x, \alpha) \psi_1(y, \beta) h.$$

The parabolic points of this surface are given by the solutions of the equation

$$G_{11}(x, y, \alpha, \beta) = 0, \quad (x, y) \in D,$$

with

$$G_{11} = F_{11}^{(2,0)} F_{11}^{(0,2)} - (F_{11}^{(1,1)})^2,$$

or

$$(\varphi_1 \varphi_1'' \psi_1 \psi_1'' - (\varphi_1'' \psi_1 + \varphi_1 \psi_1')^2)(x, y, \alpha, \beta) = 0,$$

where

$$\varphi_1'(x, \alpha) = 4x(x^2 - \alpha^2), \qquad \psi_1'(y, \beta) = \varphi_1'(y, \beta),$$
$$\varphi_1''(x, \alpha) = 4(3x^2 - \alpha), \qquad \psi_1''(y, \beta) = \varphi_1''(y, \beta).$$

Particularly, we have

$$F_{11}(x, y, 0, 0) = \frac{h}{a^4 b^4}(x^4 - a^4)(y^4 - b^4),$$

with the parabolic points given by the equations

$$xy = 0,$$

and/or

$$\frac{x^4}{a^4} + \frac{y^4}{b^4} + \frac{7}{9}\frac{x^4y^4}{a^4b^4} = 1, \qquad (x,y)\in D,$$

$$F_{11}(x,y,\frac{a}{2},\frac{b}{2}) = \frac{h}{a^4b^4}(x^2-a^2)(2x^2+a^2)(y^2-b^2)(2y^2+b^2),$$

and

$$F_{11}(x,y,a,b) = \frac{h}{a^4b^4}(x^2-a^2)^2(y^2-b^2)^2.$$

Remark 1. The last surface was also obtained in [6].

1.2. The second case is obtained for

$$F(x,0) := \frac{(x^2-a^2)(x^2+a^2-6\alpha^2)}{a^2(6\alpha^2-a^2)}h$$

and

$$F(0,y) := \frac{(y^2-b^2)(y^2+b^2-6\beta^2)}{b^2(6\beta^2-b^2)}h.$$

We have

$$F_{12}(x,y,\alpha,\beta) = \frac{(x^2-a^2)(y^2-b^2)}{a^2b^2}\left[\frac{x^2+a^2-2\alpha^2}{2\alpha^2-a^2}\frac{y^2+b^2-6\beta^2}{6\beta^2-b^2} + \frac{x^2+a^2-6\alpha^2}{6\alpha^2-a^2}\frac{y^2+b^2-2\beta^2}{2\beta^2-b^2} - \frac{x^2+a^2-2\alpha^2}{2\alpha^2-a^2}\frac{y^2+b^2-2\beta^2}{2\beta^2-b^2}\right]h.$$

Hence

$$F_{12}(x,y,0,0) = F_{11}(x,y,0,0),$$

$$F_{12}(x,y,\frac{a}{3},\frac{b}{3}) = \frac{(x^2-a^2)(y^2-b^2)}{a^4b^4}\left(\frac{45}{49}x^2y^2+b^2x^2+a^2y^2+a^2b^2\right)h,$$

$$F_{12}(x,y,\frac{a}{2},\frac{b}{2}) = \frac{(x^2-a^2)(y^2-b^2)}{a^4b^4}(a^2b^2-2b^2x^2-2a^2y^2-12x^2y^2)h,$$

and

$$F_{12}(x,y,a,b) = \frac{(x^2-a^2)(y^2-b^2)}{5a^4b^2}(5a^2b^2-b^2x^2-a^2y^2-3x^2y^2)h,$$

1.3. Finally, we take:

$$F(x,0) := \frac{a^2-x^2}{a^2}h; \qquad F(0,y) := \frac{b^2-y^2}{b^2}h.$$

One obtains:

$$F_{13}(x,y,\alpha,\beta) = \frac{(x^2-a^2)(y^2-b^2)}{a^2b^2}\Big[\frac{x^2+a^2-2\alpha^2}{a^2-2\alpha^2}\,\frac{y^2+b^2-6\beta^2}{b^2-2\beta^2} - \frac{x^2+a^2-2\alpha^2}{a^2-2\alpha^2}\,\frac{y^2+b^2-2\beta^2}{b^2-2\beta^2}\Big]h.$$

So,

$$F_{13}(x,y,0,0) = \frac{h}{a^4b^4}(x^2-a^2)(y^2-b^2)(a^2b^2-x^2y^2),$$

$$F_{13}(x,y,\frac{a}{2},\frac{b}{2}) = \frac{h}{a^4b^4}(x^2-a^2)(y^2-b^2)(a^2b^2-4x^2y^2),$$

and

$$F_{13}(x,y,a,b) = F_{13}(x,y,0,0).$$

2. A second starting point is to construct the surface F in the conditions

$$F(0,0)=h,\ F|_{\partial D}=0,\ \frac{\partial^2 F}{\partial x^2}(-\alpha,y) = \frac{\partial^2 F}{\partial x^2}(\alpha,y)=0,\ y\in[-b,b]\ \text{and}\ \alpha\in[0,a],$$

$$\frac{\partial^2 F}{\partial y^2}(x,-\beta) = \frac{\partial^2 F}{\partial y^2}(x,\beta)=0,\quad x\in[-a,a]\quad \text{and}\quad \beta\in[0,b].$$

To satisfy all these conditions is sufficient to take

$F_2 = B_4^x \oplus B_4^y F$, where B_4^x and B_4^y, defined by

$$(B_4^x F(x,y) = \varphi_2(x) F(0,y), \quad \varphi_2(x) = \frac{(x^2-a^2)(x^2+a^2-6\alpha^2)}{a^2(6\alpha^2-a^2)}$$

respectively

$$(B_4^y F(x,y) = \psi_2(x) F(x,0), \quad \psi_2(x) = \frac{(y^2-b^2)(y^2+b^2-6\beta^2)}{b^2(6\beta^2-b^2)},$$

are the Birkhoff's operators that interpolate the data:

$$F(-a,y), \quad \frac{\partial^2 F}{\partial x^2}(-\alpha,y), \quad F(0,y), \quad \frac{\partial^2 F}{\partial x^2}(\alpha,y) \quad \text{and} \quad F(a,y)$$

respectively

$$F(x,-b), \quad \frac{\partial^2 F}{\partial y^2}(x,-\beta), \quad F(x,0), \quad \frac{\partial^2 F}{\partial y^2}(x,\beta) \quad \text{and} \quad F(x,b).$$

So, we have

$$F_2(x,y) = \varphi_2(x) F(0,y) + \psi_2(y) F(x,0) - \varphi_2(x)\psi_2(y) h$$

where

$$F(0,0) = h, \quad F(-a,0) = F(a,0) = 0, \quad F(0,-b) = F(0,b) = 0.$$

2.1. First, we choose

$$F(x,0) = \frac{a^2-x^2}{a^2} h \quad \text{and} \quad F(0,y) = \frac{b^2-y^2}{b^2} h$$

One obtains

$$F_{21}(x,y,\alpha,\beta)=\frac{h}{a^2b^2}(x^2-a^2)(y^2-b^2)\left[\frac{x^2+a^2-6\alpha^2}{a^2-6\alpha^2}+\frac{y^2+b^2-6\beta^2}{b^2-6\beta^2}-\frac{x^2+a^2-6\alpha^2}{a^2-6\alpha^2}\,\frac{y^2+b^2-6\beta^2}{b^2-6\beta^2}\right]$$

So, we have:

$$F_{21}(x,y,0,0) = F_{13}(x,y,0,0),$$

$$F_{21}(x,y,\frac{a}{2},\frac{b}{2})=\frac{h}{a^4b^4}(x^2-a^2)(y^2-b^2)(a^2b^2-4x^2y^2)=F_{13}(x,y,\frac{a}{2},\frac{b}{2}),$$

and

$$F_{21}(x,y,a,b)=\frac{h}{25a^4b^4}(x^2-a^2)(y^2-b^2)(25a^2b^2-x^2y^2).$$

2.2. For $F(.,0)=\varphi_2$ and $F(0,.)=\psi_2$ one obtains:

$$F_{22}(x,y,\alpha,\beta)=\frac{(x^2-a^2)(y^2-b^2)(x^2+a^2-6\alpha^2)(y^2+b^2-6\beta^2)}{a^2b^2(a^2-6\alpha^2)(b^2-6\beta^2)}h.$$

Hence

$$F_{22}(x,y,0,0) = F_{11}(x,y,0,0) = \frac{h}{a^4b^4}(x^4-a^4)(y^4-b^4),$$

$$F_{22}(x,y,\frac{a}{2},\frac{b}{2})=\frac{h}{a^4b^4}(x^2-a^2)(y^2-b^2)(2x^2-a^2)(2y^2-b^2),$$

$$F_{22}(x,y,a,b)=\frac{h}{25a^4b^4}(x^2-a^2)(y^2-b^2)(x^2-5a^2)(y^2-5b^2),$$

and

$$F_{22}(x,y,\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}}) = F_{11}(x,y,a,b).$$

2.3. If F(0,.) and F(.,0) are taken as in the case 1.2, we have:

$$F_{23}(x,y,\alpha,\beta)=\frac{h}{a^2b^2}(x^2-a^2)(y^2-b^2)\left[\frac{x^2+a^2-2\alpha^2}{2\alpha^2-a^2}\,\frac{y^2+b^2-6\beta^2}{6\beta^2-b^2}+\right.$$
$$\left.+\frac{x^2+a^2-6\alpha^2}{6\alpha^2-a^2}\,\frac{y^2+b^2-2\beta^2}{2\beta^2-b^2}-\frac{x^2+a^2-6\alpha^2}{6\alpha^2-a^2}\,\frac{y^2+b^2-6\beta^2}{6\beta^2-b^2}\right].$$

Two particular cases are:

$$F_{23}(x,y,\frac{a}{2},\frac{b}{2})=\frac{h}{a^4b^4}(x^2-a^2)(y^2-b^2)(a^2b^2+2b^2x^2+2a^2y^2-12x^2y^2)$$

and

$$F_{23}(x,y,a,b)=\frac{h}{25a^4b^4}(x^2-a^2)(y^2-b^2)(9x^2y^2-25b^2x^2-25a^2y^2+25a^2b^2)$$

Next, one considers some surfaces over a triangular domain, i.e.

$$T_a=\{(x,y)\in\mathbb{R}^2 \mid x\geq 0,\ y\geq 0,\ x+y\leq a\}.$$

If T_a is taken as a quarter of the support, the roof surface F must satisfy the conditions: F(0,0)=h and F(x,a-x)=0, x∈[0,a]. In [6] there were given some surfaces generated by the blending interpolation procedure on a triangular domain.

Here we give some more such surfaces using also the derivatives in the starting conditions.

3. First, we are going to construct a surface F such that:

$$F(0,0)=h,\ F(x,a-x)=0,\ F'_x(a-y,y)=0,\ F'_y(x,a-x)=0,\ x,y\in[0,a].$$

To this end, we use the Hermite's interpolatory operators H_2^x and H_2^y defined by

$$(H_2^x)(x,y) = \frac{(x+y-a)^2}{(a-y)^2}F(0,y) + \frac{x(2a-2y-x)}{(a-y)^2}F(a-y,y) + \frac{x(x+y-a)}{a-y}F'_x(a-y,y),$$

and

$$(H_2^y)(x,y) = \frac{(x+y-a)^2}{(a-x)^2}F(x,0) + \frac{y(2a-2x-y)}{(a-x)^2}F(x,a-x) + \frac{y(x+y-a)}{a-x}F'_y(x,a-x).$$

Taking into account (1), we have:

$$(H_2^x F)(x,y) = \frac{(x+y-a)^2}{(a-y)^2}F(0,y)$$

$$(H_2^y F)(x,y) = \frac{(x+y-a)^2}{(a-x)^2}F(x,0).$$

Since, $H_2^x \oplus H_2^y F$ satisfies all the conditions (1), we consider $F_3 = H_2^x \oplus H_2^y F$, i.e.

$$F_3(x,y) = \frac{(x+y-a)^2}{(a-y)^2}F(0,y) + \frac{(x+y-a)^2}{(a-x)^2}F(x,0) - \frac{(x+y-a)^2}{a^2}F(0,0),$$

or

$$F_3(x,y) = \frac{(x+y-a)^2}{(a-x)^2}f_1(x) + \frac{(x+y-a)^2}{(a-y)^2}f_2(y) - \frac{(x+y-a)^2}{a^2}h, \qquad (2)$$

where f_1 and f_2 are defined on [0,a] and $f_1(0)=f_2(0)=h$, respectively $f_1(a)=f_2(a)=0$..

This way, we get a family of surfaces $F=F(f_1,f_2,h)$ all of them satisfying the conditions (1). For an fixed h, each selection of the functions f_1 and f_2 gives a surface from the family.

3.1. Let f_1 and f_2 be given by $f_1:=H_2^xF$ and $f_2:=H_2^yF$. From (2), one obtains

$$F_{31}(x,y) = \frac{(x+y-a)^2}{a^2}h.$$

3.2. If $f_1:=L_1^xF$ and $f_2:=L_1^yF$, where L_1^x and L_1^y are the Lagrange's polynomials which interpolate the data $f_1(0)=h$, $f_1(a)=0$, respectively $f_2(0)=h$, $f_2(a)=0$, one obtains:

$$F_{32}(x,y) = \frac{(a-x-y)^2(a^2-xy)}{a^2(a-x)(a-y)}h.$$

3.3. For $\alpha\in[0,a]$, let B_3^x be the Birkhoff's operator that interpolate the data: $f_1(0)=h$, $f_1(a)=f_1'(a)=0$ *and* $f_1''(\alpha)=0$, i.e.

$$(B_3^xf_1)(x)=\frac{(x-a)^2(x+2a-3\alpha)}{a^2(2a-3\alpha)}h,$$

and $(B_3^yf_2)(y) = (B_3^xf_1)(y)$. From (2), one obtains:

$$F_{33}(x,y,\alpha) = \frac{(a-x-y)^2}{a^2}\frac{x+y+2a-3\alpha}{2a-3\alpha}h.$$

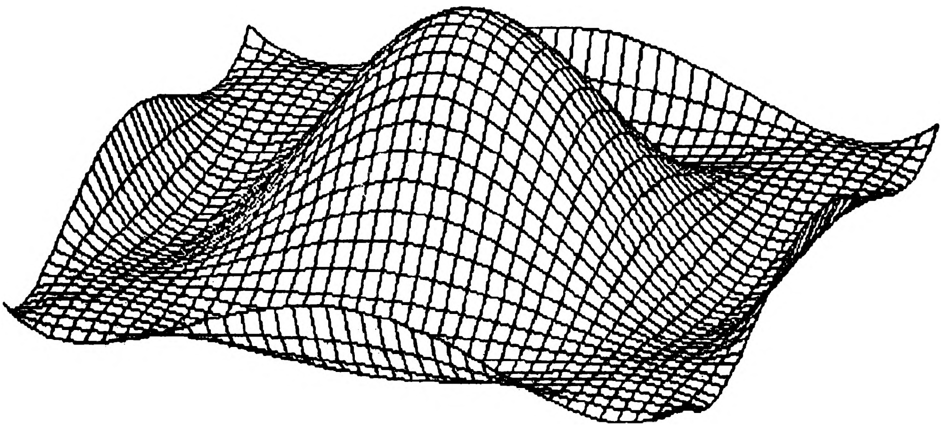
Some particular cases are:

$$F_{33}(x,y,0) = \frac{(a-x-y)^2(x+y+2a)}{2a^3}h,$$

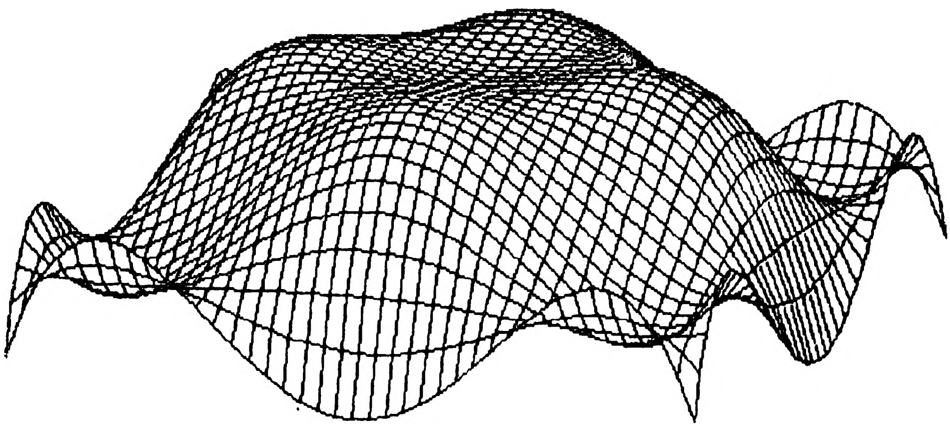
$$F_{33}(x,y,a) = \frac{(a-x-y)^3}{a^3}h,$$

$$F_{33}(x,y,\frac{a}{2}) = \frac{(a-x-y)^2(2x+2y+a)}{a^3}h.$$

Finally, we consider the graphs of two such surfaces for a=20, b=20.



The surface $F_{22}(x,y,\frac{a}{2},\frac{b}{2})$.



The surface $F_{23}(x,y,\frac{a}{2},\frac{b}{2})$.

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ON FEEBLY CONTINUOUS FUNCTIONS II

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REZUMAT. - Asupra funcţiilor slab continue II. În lucrare sînt corectate unele rezultate din lucrarea [6].

The family of all feebly open (resp. semi-open, preopen) sets of a topological space X is denoted by $\alpha(X)$ (resp. SO(X), PO(X)). The affirmation "If $U_1 \in SO(X)$ and $U_2 \in PO(X)$, then $(U_1 \cap U_2) \in \alpha(X)$" didn't result from [3, Lemma 3.1] as is said in the proofs of the Theorems 2 and 5 of [6].

LEMMA 1 [4]. *Let A be a subset of a topological space X. If either $A \in SO(X)$ or $A \in PO(X)$ then $(A \cap V) \in \alpha(A)$ for every $V \in \alpha(X)$.*

LEMMA 2.[1]. *Let $A \subset Y \subset X$, $Y \in \alpha(X)$ and $A \in \alpha(Y)$, then $A \in \alpha(X)$.*

From [5, Ex.5.4] follows that quasicontinuity and weak feebly continuity are independent notions. Also precontinuity and weak feebly continuity are independent notions.

The following Theorems are corrections of the Theorems 2,3,5 and 6 of [6].

THEOREM 1. *A function $f: X \to Y$ is feebly continuous if and only if is weakly feebly continuous and precontinuous.*

Proof. Let G be any open set of Y and $x \in X$ such that $f(x) \in G$. As f is weakly feebly continuous at x there is $U_1 \in \alpha(X)$ containing x such that $f(U_1) \subset Cl(G)$. As f is precontinuous by [2, Theorem 1] there is $U_2 \in PO(X)$ containing x such that $f(U_2) \subset G$. By Lemma 1 $(U_1 \cap U_2) \in \alpha(U_1)$. By Lemma 2 $U = (U_1 \cap U_2) \in \alpha(X)$. Thus $x \in U$, $U \in \alpha(X)$ and

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$f(U) \subset G$ and by [1, Theorem 1.1] f is feebly continuous.

Conversely; if is feebly continuous by [3, Theorem 3.2] f is precontinuous. As f is feebly continuous, f is weakly feebly continuous.

COROLLARY 1. (Noiri [5]). *If* $f:X \to Y$ *is weakly feebly continuous and precontinuous, then* f *is weakly continuous.*

THEOREM 2. *A function* $f:X \to Y$ *is feebly continuous if and only if it is weakly feebly continuous and quasicontinuous.*

Proof. It is similar to the proof of Theorem 1.

THEOREM 3. *A function* $f:X \to Y$ *is feebly continuous if and only if is weakly feebly continuous and* $f^{-1}(Fr(G))$ *is preclosed in* X *for every open set* $G \subset Y$.

Proof. If f is feebly continuous by [3, Theorem 3.2] f is precontinuous and by [2, Theorem 1] the inverse image of each closed set of Y is preclosed in X, thus $f^{-1}(Fr(G))$ is preclosed in X for every open set G of Y. If f is feebly continuous, then f is weakly feebly continuous.

Conversely, let G be any open set of Y and $x \in X$ such that $f(x) \in G$. Then, f being weakly feebly continuous there is $V \in \alpha(X)$ containing x such that $f(V) \subset C\ell(G)$. Let us consider the set $U = V - f^{-1}(Fr(G)) = V \cap (X - f^{-1}(Fr(G)))$. As $f^{-1}(Fr(G))$ is preclosed in $X, X - f^{-1}(Fr(G))$ is preopen. By Lemma 1, $U \in \alpha(V)$ and by Lemma 2, $U \in \alpha(X)$. As $x \in V$ and $f(x) \in G$ it follows that $x \in U$. Let $y \in U$. Then $y \in V$ and $y \overline{\in} f^{-1}(Fr(G))$, thus $f(y) \in C\ell(G)$ and $f(y) \overline{\in} Fr(G)$, thus $f(y) \in G$. As U is feebly open, $x \in U$ and $f(U) \subset G$ it follows by [1, Theorem 1,1] that f is feebly continuous.

THEOREM 4. *A function* $f:X \to Y$ *is feebly continuous if and only* *if f is weakly feebly continuous and* $f^{-1}(Fr(G))$ *is semiclosed in X for every open set* $G\subset Y$.

Proof. It is similar to the proof of Theorem 3.

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A N I V E R S Ă R I

PROFESSOR IOAN MUNTEAN AT HIS 60th ANNIVERSARY

Muntean Ioan was born in Mai 27, 1931, in Sîntimbru, district Alba, Romania. He studied mathematics at the universities of Cluj (1950-1952) and Moscow (1952-1955), and he joined the Faculty of Mathematics of the Cluj University where he became professor in 1976. He gave courses and seminars in classical analysis, qualitative theory of differential equations, optimal control, operational calculus and functional analysis.

Professor Muntean obtained results in the following domains (the numbers in the brackets indicate the works in the enclosed list, where the corresponding results are presented):

Qualitative theory of differential equations: bounded, periodic and almost periodic solutions, limit cycles [1-6, 10-15, 22, 23, 44], stability and exponential convergence [18, 19, 24-26], asymptotic behavior [16, 8].

Optimal control theory: controllability [33, 34, 39, 73, 82], optimal control of thermodynamic systems [38, 40], optimal machine maintenance [53, 57].

Approximation theory and numerical analysis: condensation of singularities for interpolation, Fourier series and quadrature formulas [35, 49, 92, 93, 100], numerical methods for solving equations [41, 43, 104].

Topology, optimization theory and functional analysis: compact mappings and quasiuniform convergence [7-9], Dini convergence theorem [84, 85], fixed point theorems [20, 21, 29, 30, 60], inner product spaces [31, 32, 76], convexity and optimization [17, 37, 42, 62, 66, 80, 91], functional analysis [27, 47, 48, 61, 70, 99, 100, 104].

Real analysis: derivatives and approximate derivatives [45, 50-52, 54, 59, 69, 72, 74, 87, 94], arctangent functional equation [55, 56], elementary functions [81, 102, 103], classification of some sets of real functions [58, 67, 71, 75, 77, 79, 83, 89], teaching calculus [28, 36, 46, 63-65, 68, 78, 86,90, 95, 101].

Some of these results are cited, improved or developed by about 80 mathematicians. Since 1976 Professor Muntean has been a guide of doctorands.

SCIENTIFIC WORKS OF MUNTEAN IOAN

Below we adopt the following abbreviations for name of some publications and expressions:

AIM = Analele Ştiinţifice ale Universităţii "Al. I. Cuza" Iaşi, Secţia I, Matematică,
GMP = Gazeta Matematică, Perfecţionare Metodică şi Metodologică în Matematică şi Informatică,
ISE = Itinerant Seminar on Functional Equations, Approximation and Convexity,
MC = Mathematica (Cluj),
SDM = Lucrările Seminarului de Didactica Matematicii,
SMA = Seminar on Mathematical Analysis,
SUM = Studia Universitatis Babeş-Bolyai, Series Mathematica-Mechanica,
UCM = "Babeş-Bolyai" University of Cluj, Faculty of Mathematics,
H = Hungarian,
R = Romanian.

Mathematical papers:

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2. A boundedness criterion for the solutions of a nonlinear system of differential equations (R). Studii şi Cercet. Matem.(Cluj) 9(1958), 237-243.
3. On an existence theorem of periodic solutions (Russian). MC 1(1959),

287-296.
4. Solutions bornées et solutions périodiques pour certains systèmes d'équations différentielles. Lucrările celui de al IV-lea Congres al Matematicienilor Români (Bucharest, 27 Mai - 4 Juin 1956), pp. 156-157. Editura Academiei, Bucharest, 1960.
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7. Applications complètement continues dans les espaces vectoriels topologiques. MC 7(1965), 297-303.
8. Applications Q-complètement compactes dans les espaces uniformes. MC 8(1966), 309-314.
9. Sur la convergence quasiuniforme. MC 9(1967), 321-324.
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Trans. Automat. Control 19(1974), 459-460.
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RECENSII

Peter West,
Introduction to Supersymmetry and Supergravity,
World Scientific Publ. Co. Pte. Ltd., 1990, 425 p.

This is the extended second edition of the book published by the same Company in 1986. The Contents presents the plane of the work, fragmented in 27 chapters, its origins and motivations.

After the two prefaces, the supersymmetry algebra is introduced going from the "No go Theorem" of Coleman-Mandula. Other approaches of the supersymmetry is related with the existence of the Fermi-Bose symmetry, which is linear. On the other side the supersymmetry is a symmetry mixing the particles of different spin, that is the fermions and bosoms. This kind of symmetry was linearly realized in a four-dimensional model for Wess-Zumino. This construction is typical for a general supersymmetric theory.

An essential step in defining the supersymmetry algebra is given by the generators Q^i_α $(i=1,2,\dots,N)$ carrying out a representation of the Lorentz group and being interpreted as supercharges. The case N=1 is ilustrating by considering an Abelian gauge group, the supersymmetric gauge theory, the Yang-Mill's theory and Noether technique. The irreducible Representations of the supersymmetric group of the states at rest are considered, together with their interpretations. The procedure can be generalized to any semi-direct product $S \otimes T$ with T an Abelian group. The simple supergravity is presented as its invariance. Then are treated also the Theories of extended rigid supergravity (for N=1, N=4), the local tensor calculus concerning the coupling of the supergravity to Matter, the superspaces (for N=1, N=2), the superspace formulation for rigid supersymmetric theory and supergravity.

The supersymmetric theories allow to calculate the quantum effects. Super-Feynman rules are constructed for Wess-Zumino model and N=1. The general formalism is presented as well as some related applications. The finiteness of a large class of extended rigid supersymmetric theories is a significative renormalization property.

Spontaneous breaking of supersymmetry and comments of the Realistic Models are given. The currents in supersymmetric theories are presented in the Wess-Zumino model and in the Super Yang-Mills Theories. A short introduction in the 2-dimensional supersymmetric models and superstring actions are presented. Two-dimensional Supersymmetry Algebras for Minkowski and Euclidean spaces are considered as well as their irreductible representations from a physical point of view, and some models for these spaces are constructed. In a geometrical framework the superspace formulations of two-dimensional supergravities, the superconformal group, the Green function and operator product expansions in some superconformal models are given. The Gauge covariant formulation of strings at different levels concludes the general presentation.

It follows three Appendixes containing (A) "Explanations of the Conventions", (B) "List of Reviews and Books" and (C) "Problems on afferent Chapters". The references includes 260 books, papers and proceedings.

The book is aimed mainly to the theoretical physicists but by its exposure furnishes a good didactical introduction in the subject. For that reason it was also translated in Russian and commented by P.P. Kulysh, being published by the "Mir" Co., Moscow in 1989. By the way, the work excludes the extended supergravity theories, the superstring theories, the extensive disscussions and other phenomenological implications. However it is of great interest even for the mathematicians working in the fields of Algebra, Diferential Geometry and Topology, for the applications of the new techniques in the physical theories.

M. ŢARINĂ

Qualitative Theory of Differential Equations. *Colloquia Mathematica Societatis János Bolyai,* 53. North-Holland Company Amsterdam, 1990, 683 pages, ed. B. Sz.-Nagy and L. Hatvani.

This volume contains expounded versions of 60 lectures given at the Third Colloquium on the Qualitative Theory of Differential Equations held at Szeged (Hungary), August 22-26, 1988. Most papers contain complete proofs, except some survey articles of invited lecturers.

The topics convered by the papers can be summarized as: asymptotic properties of ordinary, functional and partial differential equations, stability problems of the solutions of the ordinary and functional differential equations and stability of the discretization methods, methods of abstract dynamical systems as well as applications in mechanics, physics, biology and control.

In what follows we shall briefly review some papers. The choice of the papers reflects, with no doubt, the reviewer's interests, and this does not mean any value judgement concerning the others.

When an existence problem for solutions of a differential inclusion is approached, it is possible to take into accunt the approximation of the given multifunction by means of a continuous single-valued mapping. Usually, it is assumed that $x \to F(x)$ is upper semicontinuous with compact and convex values. In the last years many attempts have been made to avoid the convexity assumption. In G.Anichini's paper "Approximate Selection and peanian-valued multifunctions" is proved that a necessary condition for an upper semicontinuous and compact valued multifunction to admit a continuous sigle-valued approximation is the connectedness as well as locally connectedness of the sets $F(x)$.

Another paper on differential inclusions is "Boundary value problems for second order nonlinear differential inclusions" by L.Erbe and W.Kracewicz. The authors investigate the existence of solutions of a differential inclusion of the form $y'' \in F(t,y,y')$, $y \in B$, $F: I \times R^n \times R^n \to 2^{R^n}$, $I=[0,1]$, B being boundary condition, which may be nonlinear, periodic or extensions of these. The results apply to continuous functions as well as Caractheodory multifunctions.

The w-limit set is the collection of the cluster points of a trajectory and plays an important role in the study of the local and global behaviour and of the stability of dynamical systems. In Zvi Artstein's paper "On collective limit sets" are shown some advantages of tracking the evolution of a set of points in the state space. The collective sets are analogs of the omega limit sets, when sets, rather than points, are tracked.

Various counterexamples are constructed in order to enlighten some bad features of the structure of the solution set of ordinary differential equations in infinite-dimensional Banach spaces. B.M.Garay in "Sections of solutions, funnels and continuous dependence on initial conditions" shows that the cross-section needs not be closed and it may happen that the closure of an invariant set is no loger invariant in case of solutions which do not depend continuously on initial data. These results together with the failure of Peano property (Godunov's theorem) makes a quite different picture in respect to the ordinary differential equations in finite-dimensional spaces.

A short survey paper "Recent advances in the stability theory of nonlinear systems" by L.Lakshmikanthan, based on Lyapunov function tehniques, presents a systematic account of the recent trends, describes the current state of the theory and provides a unified generall structure applicable to a variety of nonlinear systems.

T.A.Burton in "Lyapunov functionals and periodic solutions", based on Lyapunov's direct method, introduces qualitative results on ordinary differential equations, finite and infinite delay equations and Volterra equations. In a strong connection with the problem of existence of a periodic splution may be considered A.Pelczar's paper "Generalized periodic problems for ordinary and partial differential equations".

MARIAN MUREŞAN

În cel de al XXXVI-lea an (1991). *Studia Universitatis* Babeş—Bolyai apare în următoarele serii:

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STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXVI, 2, 1991

SOME IMPROVED INEQUALITIES

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Dedicated to Professor P. T. Mocanu on his 60[th] anniversary

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REZUMAT. - Cîteva inegalităţi întărite. Cîteva inegalităţi cunoscute sînt întărite folosind o metodă din lucrarea [9].

1. Let $0<a<b$ and let n be an integer, $n\geq 2$. Let $x=(x_1,\ldots,x_n)\in[a,b]^n$. We shall use the following notation:

$A_n=(x_1+\ldots+x_n)/n,\quad G_n=(x_1\ldots x_n)^{1/n},\quad S_n(x)=\sum_{i<j}(x_j-x_i)^2$

$\log(x)=(\log(x_1),\ldots,\log(x_n)),\quad \sqrt{x}=(\sqrt{x_1},\ldots,\sqrt{x_n})$

Then:

$$\frac{1}{2bn^2}S_n(x)\leq A_n-G_n\leq\frac{1}{2an^2}S_n(x) \tag{1}$$

For the long history of (1) see [3]-[5], [8]-[10], [12]. Let us remark that the counterexample to (1), given in [12], is inconclusive.

We have also (see[6]):

$$\frac{1}{n(n-1)}S_n(\sqrt{x})\leq A_n-G_n\leq\frac{1}{n}S_n(\sqrt{x}) \tag{2}$$

2. Let $f\in C^2[a,b]$; let $2m$ and $2M$ be the minimum, respectively the maximum of f'' on $[a,b]$. Then $f(t)-mt^2$ and $Mt^2-f(t)$ are convex functions on $[a,b]$.

This elementary remark, combined with an appropriate choice

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of f, leads to results comparable with (1) and (2). Namely (see [9]):

$$\exp\frac{S_n}{2n^2b^2}\le\frac{A_n}{G_n}\le\exp\frac{S_n}{2n^2a^2} \tag{3}$$

$$\frac{a}{2n^2}S_n(\log(x))\le A_n-G_n\le\frac{b}{2n^2}S_n(\log(x)) \tag{4}$$

$$\frac{2a}{bn^2}S_n(\sqrt{x})\le A_n-G_n\le\frac{2b}{an^2}S_n(\sqrt{x}) \tag{5}$$

Other results obtained by the same method are to be found in [1] and [2].

3. We shall apply the above method in order to improve some results of A.O.Pittenger [8].

Let ϕ be a real-valued function defined on an interval I, possibly unbounded. Let $t_0\in I$. Y will denote a random variable whose range is almost surely in I.

THEOREM ([8]). *Suppose Y has finite mean μ and variance σ^2, and $\phi(Y)$ has finite expectation. If $\mu=t_0$, set $a_0=t_0$ and $p_0=1$; otherwise set $a_0=\mu+\sigma^2/(\mu-t_0)$ and $p_0=(\mu-t_0)^2/(\sigma^2+(\mu-t_0)^2)$. Then if $Y\le t_0$ a.s., and if the function $(\phi(t)-\phi(t_0))/(t-t_0)$ is concave on $(-\infty,t_0)\cap I$, we have*

$$p_0\phi(a_0)\ +\ (1-p_0)\phi(t_0)\le E(\phi(Y)) \tag{6}$$

Equality is attained for the random variable Y_0 which equals a_0 with probability p_0 and t_0 with probability $1-p_0$. If in addition ϕ is convex, the left side of (6) dominates $\phi(\mu)$. All the foregoing hold for $Y\ge t_0$, provided that the function $(\phi(t)-\phi(t_0))/$ $/(t-t_0)$ is convex on $(t_0,\infty)\cap I$.

Remark 1. If I is bounded, the inequality (6) is equivalent to the inequalities given (with different proofs) for n=2 in [11,p.279].

Remark 2. It is easy to verify that if ϕ is 3-convex (in particular, if $\phi^{(3)}\geq 0$), then $(\phi(t)-\phi(t_0))/(t-t_0)$ is convex on $(t_0,\infty)\cap I$.

Using Remark 2 it is easy to check the convexity of $(\phi(t)-\phi(t_0))/(t-t_0)$ in all examples considered in [8]: it suffices to verify that $\phi^{(3)}\geq 0$. Moreover, the inequalities given in those examples can be improved.

For example, let Y be a random variable, $0\leq Y\leq 1$, with mean μ and variance σ^2. Using the above theorem for $t_0 = 0$ and $\phi(t) = t\log(t)$, $\phi(0) = 0$ (note that $\phi^{(3)} \leq 0$), Pittenger obtains in [8]

$$E(Y\log(Y))\leq\mu\log(\mu+\sigma^2/\mu) \tag{7}$$

Using the Jensen inequality for ϕ (note that $\phi^{(2)}\geq 0$) he obtains $\mu\log(\mu) \leq E(Y\log(Y))$ and, finally, the following elegant result

$$0 \leq E(Y\log(Y)) - \mu\log(\mu) \leq \mu\log(1+\sigma^2/\mu^2) \tag{8}$$

Now, in the spirit of Section 2, let us consider the functions $\phi_1(t) = \phi(t) + t^3/6$ and $\phi_2(t) = \phi(t) - t^2/2$. Then $\phi_1^{(3)}\leq 0$ and $\phi_2^{(2)}\geq 0$ for $0<t\leq 1$. By using Pittenger's technique with ϕ_1 and ϕ_2 instead of ϕ, we obtain

$$\sigma^2/2 \leq E(Y\log(Y)) - \mu\log(\mu) \leq \mu\log(1+\sigma^2/\mu^2) - \delta/6 \tag{9}$$

where $\delta = E(Y^3) - \mu^3(1+\sigma^2/\mu^2)^2$ is positive by virtue of the last inequality in [8].

A similar treatment can be applied to the other examples discussed in [8].

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STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXVI, 2, 1991

FUNCTION WITH NEGATIVE COEFFICIENTS n-STARLIKE OF COMPLEX ORDER

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Dedicated to Professor P.T.Mocanu on his 60th anniversary

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REZUMAT. - **Funcţii cu coeficienţi negativi *n*-stelate de un ordin complex.** În lucrare se pun în evidenţă unele relaţii între clasa $T_{n,1}$ a funcţiilor cu coeficienţi negativi n-stelate şi clase $T_{n,b}$ de funcţii cu coeficienţi negativi n-stelate de un ordin complex b.

1. Introduction. Let A denote the class of functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which is analytic in $U = \{z \in C ; |z| < 1\}$. We denote by N the set of nonnegative integers ($N = \{0,1,2,...\}$).

DEFINITION 1 ([3]). We define the operator $D^n : A \to A$, $n \in N$, by : a) $D^0 f(z) = f(z)$; b) $D^1 f(z) = Df(z) = zf'(z)$; c) $D^n f(z) = D(D^{n-1} f(z))$, $z \in U$.

DEFINITION 2 ([3]). A function $f \in A$ is said to be *n-starlike* if $\mathrm{Re}[D^{n+1} f(z) / D^n f(z)] > 0$, $z \in U$, $n \in N$. We denote by S_n the class of n-starlike functions.

We remark that $S_0 = S^*$ is the class of starlike functions and $S_1 = S^c$ is the class of convex functions. In [3] it is proved that all n-starlike functions ($n \in N$) are univalent and $S_n \supset S_{n+1}$.

DEFINITION 3. We say that $f \in A$ is *n-starlike of complex order b* (b is a complex number and $b \neq 0$, $n \in N$) if $D^n f(z) / z \neq 0$,

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($z \in U$) and

$$Re\left[1+\frac{1}{b}\left(\frac{D^{n+1}f(z)}{D^{n}f(z)}-1\right)\right]>0, z\in U.$$

We denote by $S_{n,b}$ the class of n-starlike functions of complex order b.

M.A.Nasr and M.K.Aouf introduced and studied the class $S_{0,b}$ of starlike functions of complex order b ([1]). We also note that $S_{n,1} = S_n$.

DEFINITION 4. Let $n \in N$ and let b be complex and $b \neq 0$; we define the class $T_{n,b}$ by

$$T_{n,b} = \{f\in S_{n,b};\ f(z) = z - \sum_{k=2}^{\infty} a_k z^k,\ a_k\geq 0,\ k=2,3,\dots\}.$$

A function $f \in T_{n,b}$ is said to be a function with negative coefficients n-starlike of complex order b.

The classes $T_{0,1-\alpha}$ and $T_{1,1-\alpha}$, $\alpha \in [0,1)$ were introduced and studied by H.Silverman [4] and the classes $T_{n,1-\alpha}$, $\alpha \in [0,1)$, $n \in N$, were defined in [2].

In this paper we give some relationships between the classes $T_{n,b}$ (b complex) and $T_{n,1}$.

We will use the following lemma

LEMMA A. *Let* $n \in N$ *and let* $\alpha \in [0,1)$; *a function* $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ *is in* $T_{n,1-\alpha}$ *if and only if*

$$\sum_{k=2}^{\infty} k^n (k-\alpha) a_k \leq 1-\alpha.$$

The proof of a more general form of this lemma may be found in

[2].

2. **Main result.** We denote by B the set $\{z \in C, |z - 1/2| \le 1/2$ and $z \neq 0\} = \{z \in C; Re \frac{1}{z} \ge 1\}$.

THEOREM 1. *Let $n \in N$ and let b be in B; then $T_{n,b} \subset T_{n,1}$.*

Proof. Let f be in $T_{n,b}$ and $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ ($a_k \ge 0$, $k=2,3,\dots$). Then, by Definition 3, we have

$$Re\left[1 + \frac{1}{b}\left(\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right)\right] > 0$$

or

$$1 + Re\left[\frac{1}{b}\left(\frac{z - \sum_{k=2}^{\infty} k^{n+1} a_k z^k}{z - \sum_{k=2}^{\infty} k^n a_k z^k} - 1\right)\right] > 0.$$

This last inequality is equivalent to

$$1 + Re\left[\frac{1}{b}\frac{-\sum_{k=2}^{\infty} k^n (k-1) a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} k^n a_k z^{k-1}}\right] > 0.$$

By letting $z \to 1^-$, z real, we obtain

$$1 + Re\,\frac{1}{b}\cdot\frac{-\sum_{k=2}^{\infty} k^n (k-1) a_k}{1 - \sum_{k=2}^{\infty} k^n a_k} \ge 0$$

and this inequality can be rewritten as

$$Re\ \frac{1}{b}\cdot\frac{\sum_{k=2}^{\infty} k^{n}(k-1)a_k}{1-\sum_{k=2}^{\infty} k^{n}a_k} \le 1. \tag{1}$$

By using the condition $b \in B$ which is equivalent to $Re(1/b) \ge 1$, from (1) we deduce

$$\frac{\sum_{k=2}^{\infty} k^{n}(k-1)a_k}{1-\sum_{k=2}^{\infty} k^{n}a_k} \le 1. \tag{2}$$

But we have $1-\sum_{k=2}^{\infty} k^{n}a_k > 0$ because $D^n f(z)\,/\,z = 1-\sum_{k=2}^{\infty} k^{n}a_k z^{k-1} \ne 0$ (Definition 3) and $\lim_{z\to 0}[D^n f(z)\,/\,z] = 1$; from (2) we obtain $\sum_{k=2}^{\infty} k^{n+1}a_k \le 1$ which, by Lemma A, implies $f\in T_{n,1}$.

COROLLARY 1. *If f is a function with negative coefficients starlike of complex order b and $b \in B$, then f is starlike ($f\in S^*$).*

COROLLARY 2. *If f is a function with negative coefficients convex of complex order b and $b\in B$, then f is a convex function ($f\in S^c$).*

3. **Remarks.** 1). If $b \in B$ and $b \ne 1$, then we can find functions f in $T_{n,1}$ such that f are not in $T_{n,b}$ (i.e. $T_{n,1}\not\subset T_{n,b}$). Indeed, let $f(z) = z - z^2/2^n$; then f is in $T_{n,1}$, but for $b \in B$, $1/b = p + iq$, we have

$$1 + \frac{1}{b}\left(\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right) = \frac{(p+1+iq)z-2}{z-2}.$$

and

$$1 + \frac{1}{b}\left(\frac{D^{n+1}f(z_0)}{D^n f(z_0)} - 1\right) = 0$$

for $z_0 = 2/(p+1+iq) \in U$, because $p>1$, and this implies $f \notin T_{n,b}$.

2). Let b be a complex number with $|b| = r > 1$ and for $n \in N$ we consider the functions

$$f_n(z) = z - \frac{r}{2^n(r+1)} z^n;$$

then $f_n \in T_{n,b}$, but $f_n \notin T_{n,1}$ (f_n is not a n-starlike function).

Proof. By using Lemma A we have

$$\sum_{k=2}^{\infty} k^{n+1} a_k = 2^{n+1} a_2 = \frac{2r}{r+1} > 1$$

and this implies $f \notin T_{n,1}$.

Let denote by $U(c;\rho)$ the disc $\{z \in C; |z - c| < \rho\}$. We prove that

$$1 + \frac{1}{b}\left(\frac{D^{n+1}f_n(z)}{D^n f_n(z)} - 1\right) \in U(1;1), \quad z \in U = U(0;1). \tag{3}$$

But (3) is equivalent to $(D^{n+1}f_n(z)/D^n f_n(z) - 1)/b \in U$ and we also have

$$\frac{1}{r}\left(\frac{D^{n+1}f_n(z)}{D^n f_n(z)} - 1\right) = \frac{1}{r}\left(\frac{z - 2rz^2/(r+1)}{z - rz/(r+1)} - 1\right) = \frac{-z}{r+1-rz} \in$$

$$\in U\left(-\frac{r}{2r+1}; \frac{r+1}{2r+1}\right) \subset U(0;1) = U.$$

If we denote by θ the argument of b ($b = r\, e^{i\theta}$), then we also have

$$\frac{1}{b}\left(\frac{D^{n+1}f_n(z)}{D^n f_n(z)} - 1\right) = -\frac{z}{r+1-rz}e^{-i\theta} \in U$$

and we obtain that (3) holds.

From (3) we have $f \in T_{n,b}$.

3). For b real the last result (Remark 2) can be extended. So, by a simple calculation, we also can obtain the next result: if $-1/2 < b < -1/3$ and $1/2 < \beta < -b/(1+b)$ or if $b \leq -1/2$ and $1/2 < \beta < 1$, then $f_\beta(z) = z - \beta z^2 \in T_{0,b}$ and $f_\beta \in T_{0,1}$.

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THE RADIUS OF STARLIKENESS FOR THE ERROR FUNCTION

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Rezumat. - Raza de stelaritate a funcţiei eroare. În lucrare se determină razele de stelaritate pentru funcţiile f_n definite prin relaţia de mai jos. Acestea se exprimă cu ajutorul rădăcinii ecuaţiei (3) din intervalul $(\pi/2,\pi)$.

The purpose of this note is to find the radii of starlikeness for the functions

$$f_n(z)=\int_0^z \exp(-t^n)\,dt\ ,\ z\in\mathbf{C}\ ,\ n\in\mathbf{N}^n\ .$$

Particularly, for $n=1$, the result obtained by P.T.Mocanu in [2] which gives the radius of starlikeness for the exponential function will be refind and, for $n=2$, the radius of starlikeness for the error function $\mathrm{Erf}(z)=f_2(z)$ will be obtained.

Let f be an analytic function around the origin, with $f(0)=0$ and $f'(0)\neq 0$. The radius of starlikeness for f is defined as being the radius of the largest disk centered at 0 in which f is starlike. According to [1], this radius equals $\min|z|$ where z is a root of following system:

$$\mathrm{Re}\ [z\ f'(z)/f(z)]=0$$

$$\mathrm{Re}\ [z\ f''(z)/f'(z)]+1=0$$

For the function f_n this system becomes

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$$\mathrm{Re}\left(\int_0^1 \exp[z^n(1-u^n)]\,du\right)^{-1}=0 \tag{1}$$

$$\mathrm{Re}\ z^n=1/n. \tag{2}$$

Denoting by $r=r^*(f_n)$ the radius of starlikeness of f_n relation (2) gives $(\mathrm{Im}\ z^n)^2=r^{2n}-1/n^2$, so, it follows by (1) that r is the smallest positive root of the equation

$$\int_0^1 \exp[(1-u^n)/n]\cos[(1-u^n)(r^{2n}-1/n^2)^{1/2}]\,du=0$$

Consider the equation

$$F_n(x)=\int_0^1 \exp(-u^n/n)\cos[x(1-u^n)]\,du=0 \tag{3}$$

Then $r^{2n}=(x_n)^2+1/n^2$, where x_n is the smallest positive root of the equation (3).

Let now $n=1$. Then repeated integration by parts in (3) gives the following equation for x_1:

$$x\sin(x)+\cos(x)=1/e\ ,$$

so,as in [2], we obtain $r^*(f_1)=2.83...$

For $n>1$ we have

$$F_n'(x)=-\int_0^1 (1-u^n)\exp(-u^n/n)\sin[x(1-u^n)]\,du=0$$

so F_n is a decreasing function on $[0,\pi]$. It is obvious that $F_n(\pi/2)>0$. We shall show now that $F_n(\pi)<0$.

Let

$$g_n(u)=\exp(-u^n/n)\ \cos[\pi(1-u^n)].$$

Using the sign of g_n and the inequality $\exp(-u^n/n)\le$ $\le\exp[-1/(2n)]$ valid for $u^n\ge 1/2$ we obtain

$$F_n(\pi) < \int_0^{\frac{1}{2}} g_n(u)\,du + \exp[-1/(2n)]\,(1-2^{-\frac{1}{n}}) \;.$$

But for $u \in [0,1/2]$ the following inequalities hold

$$\exp(-u^n/n) \geq \exp(-u),$$

$$\cos[\pi(1-u^n)] \leq \cos[\pi(1-u)] \leq 0,$$

so

$$g_n(u) \leq \exp(-u)\,\cos[\pi(1-u^n)] \leq g_1(u).$$

Integrating by parts it is easy to obtain the next relation

$$\int_0^{\frac{1}{2}} g_1(u)\,du = \frac{-1+\pi\exp(-1/2)}{1+\pi^2} < -\frac{1}{4} \;.$$

Finally we get the following inequality

$$F_n(\pi) < -1/4 + \exp[-1/(2n)]\;(1-2^{-1/n}).$$

If $n \geq 3$ using $\exp[-1/(2n)]<1$ it follows that $4\,F_n(\pi) < 3-4\times 2^{-1/n} < 0$. If $n=2$ we have $\exp(-1/4)\times(1-2^{-1/2}) < 15/64$ so $F_n(\pi) < 0$ for every $n \geq 2$.

We can conclude now that x_n is the unique root of the equation $F_n(x)=0$ situated in $(\pi/2,\pi)$ and

$$r^*(f_n) = [(x_n)^2 + 1/n^2]^{1/(2n)}.$$

By computation the following value is obtained for the radius of starlikeness of the error function:

$$r^*(f_2) = r^*(\mathrm{Erf}) = 1.504\ldots$$

Solving again equation (3) for $n \in \{3,4,5\}$ we obtain

$$r^*(f_3) = 1.268\ldots$$

$$r^*(f_4) = 1.178\ldots$$

$$r^*(f_5) = 1.131\ldots$$

Remark. Since the numbers $r^*(f_n)$ are greater than 1, every function f_n is starlike in the unit disk.

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ON A SUBORDINATION BY CONVEX FUNCTIONS

VALERIA SELINGER*

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REZUMAT. - Asupra unei subordonări prin funcţii convexe. În lucrare sînt determinate condiţii pentru ca $g \prec f$, unde f este o funcţie analitică convexă, iar g este dată de (2).

1. **Introduction.** Let A be class of all analytic functions f in the unit disc U normalized by $f(0)=0$, $f'(0)=1$. A function $f\in A$ is said to be convex in U if

$$\operatorname{Re}\left[\frac{zf''(z)}{f'(z)}+1\right]>0 \quad , \ z\in U \tag{1}$$

Let $f\in A$ be convex in U and let

$$g(z)=\frac{\varphi(z)}{z^{\gamma}}\int_0^z f(\zeta)\,\zeta^{\gamma-1}d\zeta \tag{2}$$

where $\gamma>-1$ and $\varphi(z)$ is analytic in with $\varphi(z)\neq 0$.

In this paper we determine conditions on $\varphi(z)$ so that $g\prec f$. For $\varphi(z)\equiv\lambda$ real or complex number, this problem was solved in [7] for λ real and $\gamma=0$, $\gamma=1$, in [6] for λ real and all $\gamma>-1$ and in [4] for λ complex.

2. **Preliminaries.** We will need the following lemmas to prove our results.

LEMMA 1. ([2]) *Let p be analytic in U, let q be analytic and univalent in $\bar{U}$, with $p(0)=q(0)$. If p is not subordinate to q, then there exists points $z_0\in U$ and $\zeta_0\in\partial U$ and an $m\geq 1$ for wich*

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$p(|z|<|z_0|) \subset q(U)$,

(i) $p(z_0) = q(\zeta_0)$ and

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$

LEMMA 2. ([1] and [8]) *If* $f \in A$ *satisfies* (1), *then*

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > \frac{1}{2} \quad , \ z \in U \tag{3}$$

LEMMA 3. ([3]) *If* $f \in A$ *satisfies* (3), *then the function*

$$g_1(z) = \frac{1}{z^\gamma}\int_0^z f(\zeta)\zeta^{\gamma-1}d\zeta \tag{4}$$

satisfies

$$\mathrm{Re}\frac{zg_1'(z)}{g_1(z)} > \delta(\gamma) \quad , \ z \in U \tag{5}$$

where

$$\delta(\gamma) = \frac{\gamma+1}{2F(1,\gamma+1,\gamma+2,-1)} - \gamma \tag{6}$$

and $F(\alpha,\beta,\gamma,z)$ *is the hypergeometric function:*

$$F(\alpha,\beta,\gamma,z) = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)}{n!\gamma(\gamma+1)\ldots(\gamma+n-1)} z^n$$

3. Main results.

THEOREM. *Let* $f \in A$ *be convex and let* g *be defined by* (2). *If* $\varphi(z)$ *is analytic in* U *with* $\varphi(z) \neq 0$ *and satisfies:*

$$\mathrm{Re}\frac{z\varphi'(z)}{\varphi(z)} \geq 0 \quad , \ z \in U \tag{7}$$

$$(\gamma+2\delta(\gamma))\operatorname{Re}\frac{1}{\varphi(z)}-\operatorname{Re}\frac{z\varphi'(z)}{(\varphi(z))^2}-\left|\frac{\gamma}{\varphi(z)}-\frac{z\varphi'(z)}{(\varphi(z))^2}-1\right|-1\geq 0,\quad z\in U \tag{8}$$

where $\delta(\gamma)$ is given by (6), then $g(z)\prec f(z)$, $z\in U$.

Proof. Without loss of generality we can assume that f and φ satisfies the condition of the theorem of the closed disc $\bar{U}$. If not, then we can replace $f(z)$ by $f_r(z) = f(rz)$, $\varphi(z)$ by $\varphi(z)=\varphi(rz)$ and hence $g(z)$ by $g_r(z) = g(rz)$, where $0<r<1$. $f_r(z)$ is convex on $\bar{U}$. We would then prove $g_r(z)\prec f_r(z)$ for all $0<r<1$. By letting $r\to 1^-$ we obtain $g(z)\prec f(z)$, $z\in U$.

From (2) we deduce:

$$\left(\frac{\gamma}{\varphi(z)}-\frac{z\varphi'(z)}{(\varphi(z))^2}\right)\cdot g(z)+\frac{zg'(z)}{\varphi(z)}=f(z) \tag{9}$$

If g is not subordinate to f, then by Lemma 1. there exists points $z_0\in U$ and $\zeta_0\in\partial U$ and an $m\geq 1$ such that:

$$g(z_0) = f(\zeta_0) \text{ and } z_0 g'(z_0) = m\zeta_0 f'(\zeta_0) \tag{10}$$

From (9) and (10) we obtain:

$$f(z_0)=\left(\frac{\gamma}{\varphi(z_0)}-\frac{z_0\varphi'(z_0)}{(\varphi(z_0))^2}\right)f(\zeta_0)+m\frac{\zeta_0 f'(\zeta_0)}{\varphi(z_0)}$$

hence

$$Q=\frac{f(z_0)-f(\zeta_0)}{\zeta_0 f'(\zeta_0)}=\left(\frac{\gamma}{\varphi(z_0)}-\frac{z_0\varphi'(z_0)}{(\varphi(z_0))^2}-1\right)\frac{f(\zeta_0)}{\zeta_0 f'(\zeta_0)}+\frac{m}{\varphi(z_0)}=$$
$$=m\left[\frac{1}{\varphi(z_0)}+\left(\frac{\gamma}{\varphi(z_0)}-\frac{z_0\varphi'(z_0)}{(\varphi(z_0))^2}-1\right)\frac{g(z_0)}{z_0 g'(z_0)}\right].$$

Since $g(z) = \varphi(z)g_1(z)$, where $g_1(z)$ is defined by (4), if we note $w=\dfrac{g(z_0)}{z_0 g'(z_0)}$ from (5) and (7) we deduce

$$\mathrm{Re}\frac{1}{w}>\delta(\gamma) \quad \text{or} \quad \left|w-\frac{1}{2\delta(\gamma)}\right|\le\frac{1}{2\delta(\gamma)} .$$

Using this result combined with (8) and $m\ge1$, we obtain:

$$\mathrm{Re}\,Q=m\Big(\mathrm{Re}\frac{1}{\varphi(z_0)}+\frac{1}{2\delta(\gamma)}\mathrm{Re}\left(\frac{\gamma}{\varphi(z_0)}-\frac{z_0\varphi'(z_0)}{(\varphi(z_0))^2}-1\right)+$$
$$+\mathrm{Re}\left[\left(\frac{\gamma}{\varphi(z_0)}-\frac{z_0\varphi'(z_0)}{(\varphi(z_0))^2}-1\right)\left(w-\frac{1}{2\delta(\gamma)}\right)\right]\Big)\ge m\,\mathrm{Re}\Big[\frac{1}{\varphi(z_0)}+$$
$$+\frac{1}{2\delta(\gamma)}\mathrm{Re}\left(\frac{\gamma}{\varphi(z_0)}-\frac{z_0\varphi'(z_0)}{(\varphi(z_0))^2}-1\right)-\frac{1}{2\delta(\gamma)}\left|\frac{\gamma}{\varphi(z_0)}-\frac{z_0\varphi'(z_0)}{(\varphi(z_0))^2}-1\right|\Big]\ge0 ,$$

which is equivalent to

$$\left|\arg\frac{f(z_0)-f(\zeta_0)}{\zeta_0 f'(\zeta_0)}\right|\le\frac{\pi}{2} \tag{11}$$

Since $\zeta_0 f'(\zeta_0)$ is the outward normal to the boundary of the convex domain $f(U)$ at $f(\zeta_0)$, (11) implies that $f(z_0)\notin f(U)$. This contradiction shows that $g\prec f$.

If we let $\varphi(z)\equiv\lambda$, complex number, in the Theorem, we obtain the result of [4]:

COROLLARY. *Let $f\in A$ be convex and let* $g(z)=\frac{\lambda}{z^\gamma}\int_0^z f(\zeta)\zeta^{\gamma-1}d\zeta$. *If λ is a complex number which satisfies:*

$$(\gamma+2\delta(\gamma))\mathrm{Re}\frac{1}{\lambda}-\left|\frac{\gamma}{\lambda}-1\right|-1\ge0$$

where $\delta(\gamma)$ is given by (6), then $g\prec f$.

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SUBCLASS OF ANALYTIC FUNCTIONS

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Dedicated to Professor P.T.Mocanu on his 60^{th} anniversary

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REZUMAT. Subclase de funcţii analitice. Scopul acestei lucrări este de a obţine cîteva proprietăţi interesante ale unor subclase de funcţii analitice.

1. **Introduction.** Let A be the class of analytic functions f in the unit disc U, normalized by $f(0) = f'(0) - 1 = 0$.

Definition [2]. Let Ω be a set in C and let q be analytic and univalent on $\bar{U}$ except for those $\zeta\in\partial U$ for which $\lim\limits_{z\to\zeta} q(z)=\infty$. We define $\Psi(\Omega,q)$ to be the class of functions $\psi:C^3 \times U \to U$ for which $\psi(r,s,t;z)\notin\Omega$ when $r = q(\zeta)$ is finite, $s = m\zeta q'(\zeta)$, $\mathrm{Re}\ (1 + \frac{t}{s})\geq m\ \mathrm{Re}\ (1+ \frac{\zeta q''(\zeta)}{q'(\zeta)})$ and $z\in U$, for $m\geq 1$ and $|\zeta| = 1$.

In the special case when Ω is a simply connected domain and h is a conformal mapping of U onto Ω we denote the class by $\Psi(h(U),q)$ or $\Psi(h,q)$.

If $h(z) = q(z) = \frac{1+z}{1-z}$, then

(1) $q(U) = \Omega = h(U) = \{w;\ \mathrm{Re}\ w>0\}$.

LEMMA A[2] *Let the function $\psi\in\Psi(\Omega,q)$, where Ω and q are defined by* (1). *If p is analytic in U, with $p(0)=1$ and if p satisfies*

$$\mathrm{Re}\ \psi(p(z),\ zp'(z),\ z^2p''(z);z)>0,\ z\in U,$$

then Re $p(z)>0$ *for all* $z\in U$.

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2. Main results

THEOREM 1. *Let $M(z)=az^n + a_{n+k}z^{n+k} + \dots$, $N(z) = bz^n + \dots$, be analytic in the unit disc U, $a.b\neq 0$, $n,k\geq 1$. Suppose*

$$\frac{M(z)}{N(z)}\neq 0\ ,\ z\in U\ ,\ \mathrm{Re}\left[\frac{\alpha}{\mu}\frac{N(z)}{zN'(z)}\right]>\delta\ ,\ \text{where } 0\leq\delta<\mathrm{Re}\frac{\alpha}{\mu\cdot n}\ ,$$

$\mu>0$ and $\alpha\in C$ with Re $\alpha>0$. *If*

$$\mathrm{Re}\left[(1-\alpha)\left(\frac{M(z)}{N(z)}\right)^{\mu}+\alpha\frac{M'(z)}{N'(z)}\left(\frac{M(z)}{N(z)}\right)^{\mu-1}\right]>\beta\ ,\ \beta<\left(\frac{a}{b}\right)^{\mu}\ , \tag{2}$$

then

$$\mathrm{Re}\left(\frac{M(z)}{N(z)}\right)^{\mu}>\frac{2\beta+\delta\left(\frac{a}{b}\right)^{\mu}\cdot k}{2+\delta\cdot k},\ \text{for } z\in U\ .$$

Proof. Let $p(z) = \left(\frac{M(z)}{N(z)}\right)^{\mu}$, then p is analytic in U and $p(0)=\left(\frac{a}{b}\right)^{\mu}$.

From (2) we deduce that

$$\mathrm{Re}\left[p(z)+\frac{\alpha}{\mu}\frac{N(z)}{zN'(z)}zp'(z)\right]>\beta \tag{3}$$

We will obtain the real number φ, for which (3) implies Re $p(z)>\varphi$, for $z\in U$.

Let $q(z)=\frac{1}{\left(\frac{a}{b}\right)^{\mu}-\varphi}[p(z)-\varphi]$, then q is analytic in U and $q(z) = 1 + C_k z^k + \dots$

If we define the function $\psi: C^2 \times U \to C$

$$\psi(w_1,w_2;z)=\left[\left(\frac{a}{b}\right)^{\mu}-\varphi\right]w_1+\frac{\alpha}{\mu}\frac{N(z)}{zN'(z)}\left[\left(\frac{a}{b}\right)^{\mu}-\varphi\right]w_2+\varphi-\beta\ ,$$

then from (3) we deduce Re $\psi(q(z), zq'(z); z) > 0$ and

$\mathrm{Re}\,\psi(1,0;\,z) = \left(\frac{a}{b}\right)^{\mu} - \varphi > 0$, for all $z\in U$.

For $s\le -\frac{k}{2}(1+r^2)$, $r\in R$, we obtain

$$\mathrm{Re}\,\psi(ir,s;z) = \mathrm{Re}\left[\frac{\alpha}{\mu}\,\frac{N(z)}{zN'(z)}\right]\left[\left(\frac{a}{b}\right)^{\mu}-\varphi\right]s+\varphi-\beta\le$$
$$\le -\delta\frac{k}{2}(1+r^2)\left[\left(\frac{a}{b}\right)^{\mu}-\varphi\right]+\varphi-\beta\ ,\ if\ \varphi\le\left(\frac{a}{b}\right)^{\mu}.$$

Since $\max\left\{\varphi:\varphi-\beta-\delta\frac{k}{2}(1+r^2)\left[\left(\frac{a}{b}\right)^{\mu}-\varphi\right]\le 0;\ r\in R\right\} =$

$= \dfrac{2\beta-\delta k\left(\frac{a}{b}\right)^{\mu}}{2+\delta k} = \varphi_0$, we have $\varphi_0<\left(\frac{a}{b}\right)^{\mu}$ and

$\mathrm{Re}\,\psi(ir,s;\,z)\le 0$, for $z\in U$, $s\le -\frac{k}{2}(1+r^2)$ and $\varphi\le\varphi_0$.

From Lemma A we deduce that $\mathrm{Re}\,q(z)>0$, for $z\in U$ and $\forall\varphi\le\varphi_0$.

Hence $\mathrm{Re}\,p(z)>\varphi_0$, $z\in U$.

If we let $\mu=1$ in Theorem 1, we obtain

COROLLARY 1. *Let* $M(z) = az^n + a_{n+k}z^{n+k} + \dots$, $N(z) = bz^n +\dots$ *be analytic in* U, $a.b \ne 0$, $n,k\ge 1$.

Suppose that $\mathrm{Re}\left[\alpha\frac{N(z)}{zN'(z)}\right]>\delta$, where $0 \le \delta < \mathrm{Re}\frac{\alpha}{n}$ and

$$\mathrm{Re}\left[(1-\alpha)\frac{M(z)}{N(z)}+\alpha\frac{M'(z)}{N'(z)}\right]>\beta\ ,\ \text{where } \alpha\in C\ ,\ \mathrm{Re}\,\alpha>0\ ,\ \beta<\frac{a}{b}\ ,$$

then

$$\mathrm{Re}\frac{M(z)}{N(z)}>\frac{2\beta+\delta k\frac{a}{b}}{2+\delta k}\ ,\ \text{for } z\in U.$$

This result was recently obtained by *T*. Bulboacă [1].

If we let $a = b = 1$, $\mu = 1/2$, $N(z) = z$, $M(z) = f(z)$, $n = k = 1$ in Theorem 1, then we deduce

COROLLARY 2. *Let* $f\in A$, $\frac{f(z)}{z}\ne 0$ $z\in U$, $\mathrm{Re}\alpha>0$, $\beta<1$ *and suppose that*

$$\operatorname{Re}\left\{\sqrt{\frac{f(z)}{z}}+\alpha z\left(\sqrt{\frac{f(z)}{z}}\right)'\right\}>\beta\ ,$$

then

$$\operatorname{Re}\sqrt{\frac{f(z)}{z}}>\frac{2\beta+\operatorname{Re}\alpha}{2+\operatorname{Re}\alpha}$$

Proof. From (4) we have

$$\operatorname{Re}\left\{\left(1-\frac{\alpha}{2}\right)\left(\frac{f(z)}{z}\right)^{\frac{1}{2}}+\frac{\alpha}{2}f'(z)\left(\frac{f(z)}{z}\right)^{-\frac{1}{2}}\right\}=\operatorname{Re}\left\{\sqrt{\frac{f(z)}{z}}+\alpha z\left(\sqrt{\frac{f(z)}{z}}\right)'\right\}>\beta$$

By using Theorem 1 we obtain $\operatorname{Re}\sqrt{\frac{f(z)}{z}}>u(\delta)$, where $u(\delta)=\frac{2\beta+\delta}{2+\delta}$, $\delta\in[0,\operatorname{Re}\alpha]$.
But $\sup\{u(\delta)\,;\delta\in[0,\operatorname{Re}\alpha]\}=u(\operatorname{Re}\alpha)=\frac{2\beta+\operatorname{Re}\alpha}{2+\operatorname{Re}\alpha}$ hence $\operatorname{Re}\sqrt{\frac{f(z)}{z}}>\frac{2\beta+\operatorname{Re}\alpha}{2+\operatorname{Re}\alpha}$.

The case $\alpha>0$, $0\le\beta<1$ in Corollary 2 improves the result of Shigeyoshi Owa and C.Y.Shen [3].

If we take $M(z)=zf'(z)$, $f\in A$, $N(z)=z$, $\alpha>0$, $\mu=1/2$, $0\le\beta<1$ in Theorem 1, then we deduce the result of Shigeyoshi Owa C.Y.Shen [3].

THEOREM 2. *Let* $f(z)=z+\sum_{n=k+1}^{\infty}a_nz^n$, $k\ge1$ *be analytic in* U, $\frac{f(z)}{z}\ne0$ *in* U *and suppose that*

$$\operatorname{Re}\left[(1-\alpha)\left(\frac{f(z)}{z}\right)^{\mu}+\alpha f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right]>\beta\ ,\ z\in U\ , \tag{5}$$

where $0\le\beta<1$, $\alpha\ge0$, $\mu>0$.

Then $\operatorname{Re}\sqrt{\left(\frac{f(z)}{z}\right)^{\mu}}>\gamma_1$ *for* $z\in U$, *where*

$$\gamma_1 = \frac{\frac{\alpha}{\mu}k + \sqrt{\left(\frac{\alpha}{\mu}k\right)^2 + 4\beta\left(1+\frac{\alpha}{\mu}k\right)}}{2\left(1+\frac{\alpha}{\mu}k\right)} \tag{6}$$

Proof. Let $p(z) = \sqrt{\left(\frac{f(z)}{z}\right)^{\mu}}$, then p is analytic in U and $p(z) = 1 + C_k z^k + \ldots$

We will obtain the real number γ for which $\text{Re}\, p(z) > \gamma$, $z \in U$.

If we set $q(z) = \frac{1}{1-\gamma}[p(z) - \gamma]$, then q is analytic in U and $q(0)=1$.

A simple calculation yields:

$\text{Re}\, \psi(q(z), zq'(z); z) > 0$, where

$\psi(w_1, w_2; z) = (1-\gamma)^2 w_1 + 2\gamma(1-\gamma)w_1 + 2\frac{\alpha}{\mu}(1-\mu)w_2 \cdot$

$((1-\gamma)w_1 + \gamma) + \gamma^2 - \beta$.

We have

$\text{Re}\, \psi(ir, s; z) \le -r^2[(1-\gamma)^2 + \frac{\alpha}{\mu}(1-\gamma)\gamma \cdot k] +$

$+ [\gamma^2 - \beta - \frac{\alpha}{\mu}k\gamma(1-\gamma)] \le 0$, if $0 \le \gamma \le 1$

and $(1-\gamma)^2 + \frac{\alpha}{\mu}(1-\gamma)\gamma k \ge 0$

$\gamma^2(1 + \frac{\alpha}{\mu}k) - \frac{\alpha}{\mu}k\gamma - \beta \le 0$.

These inequalities imply $\gamma \in [0, \gamma_1]$ where γ_1 is deined by (6).

For $\gamma = \gamma_1 < 1$ we deduce $\text{Re}\, \psi(ir, s; z) \le 0$, $z \in U$, $s \le -\frac{k}{2}(1+r^2)$ and by Lemma A we obtain $\text{Re}\, q(z) > 0$, hence $\text{Re}\, p(z) > \gamma_1$, for $z \in U$.

For $\mu = 1/2$, $k=1$ from Theorem 2 we obtain.

COROLLARY 3. *Let $f(z) = z + a_2 z^2 + \ldots$ be analytic in U. Suppose that*

$$\text{Re}\left\{\sqrt{\frac{f(z)}{z}} + \alpha z\left(\sqrt{\frac{f(z)}{z}}\right)'\right\} > \beta$$

where $0 \le \beta < 1$, $\alpha \ge 0$.

Then $\operatorname{Re}\sqrt[4]{\frac{f(z)}{z}} > \frac{\alpha+\sqrt{\alpha^2+4\beta(1+\alpha)}}{2(1+\alpha)}$, for $z\in U$. for $z\in U$.

This result was recently obtained by Shigeoyshi Owa and Zhworen Wu [4].

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SUFFICIENT CONDITIONS FOR UNIVALENCE OBTAINED BY SUBORDINATION METHOD

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REZUMAT. - Condiţii suficiente de univalenţă obţinute cu metoda subordonării. Sînt găsite mai multe condiţii de univalenţă cu ajutorul metodei subordonării.

1. DEFINITION 1. Let $f(z)$, $g(z)$ be two regular functions in $U = \{z: |z| < 1\}$. We say that $f(z)$ is *subordinate* to $g(z)$, written $f(z) \prec g(z)$, if there exists a function $\varphi(z)$ regular in U which satisfies $\varphi(0) = 0$, $|\varphi(z)| < 1$ and

$$f(z) = g(\varphi(z)) \quad |z| < 1 \tag{1}$$

DEFINITION 2. Let $f(z)$ be a regular function in U and $f'(z) \neq 0$ for $z \in U$. The function $f(z)$ is said to be *convex* if

$$Re\frac{zf''(z)}{f'(z)} + 1 > 0, \quad z \in U. \tag{2}$$

Let S denote the class of functions $f(z)$ regular and univalent in the unit disk U, for which $f(0) = 0$, $f'(0) = 1$.

F.G. Avhadiev and L.A. Aksentiev [1] have proved the following theorem:

THEOREM A. *Let $f(z) = z + \ldots$ and $g(z) = z + \ldots$ be two regular functions in U. If*

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$$\left|\frac{zg''(z)}{g'(z)}\right| \le \frac{1}{1-|z|^2} \tag{3}$$

for all $z \in U$ *and* Log $f'(z) \prec$ Log $g'(z)$, Log $f'(0)$ = Log $g'(0)$ = 0, *then the function* $f(z)$ *is univalent in* U.

A generalization of this theorem was obtained in [3]:

THEOREM B. *Let* $f(z)$, $g(z)$ *be regular functions in* U, $f(z) = z + \dots$, $g(z) = z + \dots$, *and let* α *be a complex number,* $0 < Re\alpha \le 1$. *If* Log $f'(z) \prec$ Log $g'(z)$, Log $f'(0)$ = Log $g'(0)$ = 0 *and*

$$(1-|z|^2)\left|\frac{zg''(z)}{g'(z)}\right| \le Re\,\alpha, \tag{4}$$

for any $z \in U$, *then the function*

$$F_\alpha(z) = [\alpha \int_0^z u^{\alpha-1} f'(u)\,du]^{1/\alpha}$$

is regular and univalent in U.

In [2] it is proved the following theorem:

THEOREM C. *Let* β *and* γ *be complex numbers and let* $h(z) = c + h_1 z + \dots$ *be convex (univalent) in* U *with*

$$Re\,[\beta h(z) + \gamma] > 0. \tag{5}$$

If $p(z) = c + p_1 z + \dots$ *is analytic in* U *then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z). \tag{6}$$

In [5] it is proved the next univalence criterion:

THEOREM D. *Let* α *and* c *be complex numbers for which* $|\alpha| < 1$, $|c| \le 1$, $c \ne -1$, $\frac{\alpha-1}{\alpha+1} \notin [1,\infty)$.

If $g(z) = z + \dots$ is a regular function in U, and

(i) $\dfrac{g(z)}{z} \neq 0$ *in* U *when* $\dfrac{1}{\alpha + 1} \in N^* = \{1, 2, \dots\}$

(ii) $\left| c|z|^2 + (1 - |z|^2)\left(\alpha \dfrac{zg'(z)}{g(z)} + \dfrac{zg''(z)}{g'(z)}\right) \right| \le 1$ *for all* z *in* U,

then the function $g(z)$ is univalent in U.

2. In this note we obtain by subordination method new conditions for univalence. First we will prove a consequence of Theorem D.

THEOREM 1. *Let $g(z) = z + \dots$ be a regular function in U with $\dfrac{g(z)g'(z)}{z} \neq 0$ for all $z \in U$, and let α, γ be complex numbers. If the regular function*

$$F(z) = \int_0^z \left[\frac{g(u)}{u}\right]^{1/\gamma} [g'(u)]^{1/\alpha\gamma}\, du \tag{7}$$

is univalent in U,

$|\alpha| < 1$, $\qquad \alpha \dfrac{2\gamma + 1}{\alpha + 1} \notin [1, \infty)$,

and

(i) $\qquad |\alpha| \le \dfrac{1}{4|\gamma|}$ *if* $|\gamma| \ge \dfrac{1}{2}$

(ii) $\qquad |\alpha| \le \dfrac{1}{1 + 4|\gamma|^2}$ *if* $|\gamma| < \dfrac{1}{2}$

then the function $g(z)$ is also univalent.

Proof. We will show that the differential equation

$$\alpha \frac{zg'(z)}{g(z)} + \frac{zg''(z)}{g'(z)} = \alpha + \alpha\gamma \frac{zF''(z)}{F'(z)} \tag{8}$$

has a regular solution $F(z)$, $F(0) = 0$, $F'(0) = 1$ in U.

Integrating (8) from 0 to z we obtain:

$$F'(z) = \left[\frac{g(z)}{z}\right]^{1/\gamma}[g'(z)]^{1/\alpha\gamma} \tag{9}$$

The function $\left(\frac{g(z)}{z}\right)^{1/\gamma}$ is regular because $\frac{g(z)}{z} \neq 0$ (we choose the branch which is equal to 1 at the origin)

We have also $g'(z) \neq 0$. Then $F'(z)$ from (9) is regular and the differential equation (8) has the regular solution

$$F(z) = \int_0^z \left[\frac{g(u)}{u}\right]^{1/\gamma}[g'(z)]^{1/\alpha\gamma}du, \tag{10}$$

for which $F(0) = 0$, $F'(0) = 1$.

For $c = -2\alpha\gamma$ the relation (ii) from Theorem D becomes:

$$\left|\alpha\gamma\ [-2|z|^2 + (1 - |z|^2)\frac{zF''(z)}{F'(z)}] + \alpha(1 - |z|^2)\right| \le 1.$$

Because $F(z) = z + \ldots$ is univalent, we have

$$\left|-2|z|^2 + (1 - |z|^2)\frac{zF''(z)}{F'(z)}\right| \le 4|z|,$$ therefore:

$$\left|\alpha\gamma\ [-2|z|^2 + (1 - |z|^2)\frac{zF''(z)}{F'(z)}] + \alpha(1 - |z|^2)\right| \le$$

$$\le |\alpha\gamma|4|z| + |\alpha|(1 - |z|^2) = |\alpha|[-|z|^2 + 4|\gamma||z| + 1].$$

Calculating the maximum value of expression

$E = |\alpha|[-|z|^2 + 4|\gamma||z| + 1]$ *for* $|z| < 1$ *we obtain*:

$$E \le \begin{cases} 4|\alpha\gamma| & if\ |\gamma| \ge \frac{1}{2} \\ |\alpha|(1 + 4|\gamma|^2) & if\ |\gamma| < \frac{1}{2}. \end{cases}$$

From Theorem D and conditions (i) and, respectively (ii) of Theorem 1 we conclude that $g(z)$ belongs to S.

THEOREM 2. *Let* $g(z) = z + \ldots$, $h(z) = z + \ldots$ *be regular* *functions in U, and let α,γ complex numbers. If* $\frac{h(z)h'(z)}{z} \neq 0$,

$$\left|\frac{zg''(z)}{g'(z)}\right| \le \frac{1}{1-|z|^2} \quad \textit{for all } z \in U, \tag{11}$$

and

$$|\alpha| < 1, \qquad \alpha\,\frac{2\gamma+1}{\alpha+1} \notin [1,\infty), \tag{12}$$

$$\begin{cases} |\alpha| \le \dfrac{1}{4|\gamma|} & \textit{if } |\gamma| \ge \dfrac{1}{2} \\ |\alpha| \le \dfrac{1}{1+4|\gamma|^2} & \textit{if } |\gamma| < \dfrac{1}{2} \end{cases} \tag{13}$$

$$\mathrm{Log}\left\{\left[\frac{h(z)}{z}\right]^{1/\gamma}[h'(z)]^{1/\alpha\gamma}\right\} \prec \mathrm{Log}\, g'(z), \tag{14}$$

then the function h(z) belongs to the class S.

(for $[\frac{h(z)}{z}]^{1/\gamma}[h'(z)]^{1/\alpha\gamma}$ we choose the branch which is equal to 1 at the origin, and for logarithmic functions the branches equal to 0 at the origin).

Proof. Let $f'(z) = [\frac{h(z)}{z}]^{1/\gamma}\,[h'(z)]^{1/\alpha\gamma}$.

From (11), (14), Log $f'(0)$ = Log $g'(0) = 0$, and Theorem A we deduce that $f(z) \in S$. Now, applying Theorem 1 with $F(z) = f(z)$ we have:

$$F'(z) = [\frac{h(z)}{z}]^{1/\gamma}[h'(z)]^{1/\alpha\gamma},$$

that is (9) with $g(z) = h(z)$.

Then Theorem 1 shows that $h(z)$ belongs to the class S.

THEOREM 3. *Let* $g(z) = z + \ldots$, Log $F'(z) = a_1 z + \ldots$ *be regular functions in U, let* Log $G'(z) = b_1 z + \ldots$ *be convex (univalent) in U, and let* α, β, γ, δ *complex numbers.*

If $\frac{g(z)g'(z)}{z} \neq 0$,

$$(1 - |z|^2)\left|\frac{zG''(z)}{G'(z)}\right| \leq \frac{1}{1 - |z|^2} \tag{15}$$

$$[F'(z)]^{\alpha\delta} = [\frac{g(z)}{z}]^{\alpha} g'(z) \qquad for \quad all \quad z \in U, \tag{16}$$

and

$$Re\,\gamma > 0, \quad Re[\mathrm{Log}\, e^{\gamma} G'^{\beta}] > 0, |\alpha| < 1, \alpha\frac{2\delta + 1}{\alpha + 1} \notin [1,\infty),$$

$$\begin{cases} |\alpha\delta| \leq \frac{1}{4} & if \quad |\delta| \geq \frac{1}{2} \\ |\alpha| \leq \frac{1}{1 + 4|\delta|^2} & if \quad |\delta| < \frac{1}{2} \end{cases} \tag{17}$$

$$\mathrm{Log}\, F'(z) + \frac{zF''(z)}{F'(z)\, \mathrm{Log}(e^{\gamma}F'^{\beta})} \prec \mathrm{Log}\, G'(z) \tag{18}$$

(for logarithmic functions we choose the branches equal to 0 at the origin), then the function g(z) belongs to S.

Proof. Let $p(z) = \mathrm{Log}\, F'(z)$, which is regular, and $h(z) = \mathrm{Log}\, G'(z)$ which is convex (univalent) in U. Then

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} = \mathrm{Log} F'(z) + \frac{zF''(z)}{F'(z)\mathrm{Log}(e^{\gamma}F'^{\beta})}$$

By (18) and Theorem C we obtain

$$\mathrm{Log}\, F'(z) \prec \mathrm{Log}\, G'(z) \tag{19}$$

From (15), (19) and Theorem A it follows that $F(z) \in S$. Because all the conditions of Theorem 1 are satisfied, we conclude that $g(z)$ belongs to the class S.

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A NEW GENERALIZATION OF NEHARI'S CRITERION OF UNIVALENCE

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Dedicated to Professor P.T.Mocanu on his 60^{th} anniversary

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REZUMAT: - O nouă generalizare a criteriului de univalenţă al lui Nehari. În această notă se obţine o generalizare a unui bine cunoscut rezultat de univalenţă al lui Nehari.

We denote by U the unit disk $\{ z:|z| < 1 \}$. The aim of this paper is to obtain a generalization of the following well-known result due to Nehari.

THEOREM A [1]. *If* $f(z) = z + a_2z^2 + \dots$ *is a regular function in* U, *and*

$$|\{f;z\}| \le \frac{2}{(1 - |z|^2)^2}, \quad \forall z \in U \tag{1}$$

where

$$\{f;z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \tag{2}$$

then the function $f(z)$ *is univalent in* U.

In the following demonstrations, we shall use the result due to Pommerenke.

THEOREM B [2]. *Let* r_0 *be a real number,* $r_0 \in (0,1]$, $U_{r_0} = \{z:|z| < r_0\}$ *and let* $f(z,t) = a_1(t)z + \dots$, $a_1(t) \neq 0$, *be analytic in* U_{r_0}, *for all* $t \ge 0$ *and locally absolutely continuous in* $I = [0,\infty)$, *locally uniformly with respect to* U_{r_0}

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Supposing that for almost all $t \in I$, $f(z,t)$ *satisfies the equation*

$$z\frac{\partial f(z,t)}{\partial z} = p(z,t)\frac{\partial f(z,t)}{\partial t}, \quad z \in U_{r_0}, \tag{3}$$

where $p(z,t)$ *is analytic in* U *and* $\operatorname{Re} p(z,t) > 0$ *for all* $t \in I$, $z \in U$. *If* $|a_1(t)| \to \infty$ *for* $t \to \infty$ *and if* $\{ f(z,t)/a_1(t) \}$ *forms a normal family in* U_{r_0}, *then for all* $t \in I$, $f(z,t)$ *has an analytic and univalent extension to the whole disk* U.

THEOREM 1. *Let* α *be a real number,* c *be a complex number,* $|c| < 1$ *and* $f(z) = z + a_2z^2 + \ldots$ *a regular function in the unit disk* U. *If*

$$\left|\frac{1-\alpha}{1+\alpha} + \frac{2c}{1+\alpha}\right| < 1 \tag{4}$$

and

$$\left|\frac{1-\alpha}{1+\alpha} + \frac{2}{1+\alpha}ce^{-2t\alpha} + \frac{1}{1+\alpha}\frac{p(e^{-t\alpha}z)}{1+c}z^2(1-e^{-2t\alpha})^2\right| < 1 \tag{5}$$

for all $z \in U$, $t \geq 0$, *where*

$$p(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2, \tag{6}$$

then the function $f(z)$ *is univalent in* U.

Proof. If the function $f(z)$ is regular in U, then

$$f'(z) = \frac{f_1(z)}{f_2(z)} \tag{7}$$

where the functions $f_1(z)$ and $f_2(z)$ verifies the relations

$$f_K''(z) + \frac{p(z)}{2} f_K(z) = 0 \quad (K = 1,2) \tag{8}$$

and

$$\begin{aligned} f_1(0) &= 0, \quad f_1'(0) = 1 \\ f_2(0) &= 1, \quad f_2'(0) = 0. \end{aligned} \tag{9}$$

Let's consider L(z,t) a regular function L: $U_{r_0} \times [0,\infty) \to \mathbf{C}$, $r_0 \in (0,1)$, defined by

$$L(z,t) = \frac{f_1(e^{-t\alpha}z) + \frac{1}{1+c}(e^{t\alpha} - e^{-t\alpha}) z f_1'(e^{-t\alpha}z)}{f_2(e^{-t\alpha}z) + \frac{1}{1+c}(e^{t\alpha} - e^{-t\alpha}) z f_2'(e^{-t\alpha}z)} = \tag{10}$$

$$= a_1(t) z + \ldots$$

where

$$a_1(t) = e^{-t\alpha} + \frac{1}{1+c}(e^{t\alpha} - e^{-t\alpha}). \tag{11}$$

Let's prove that $a_1(t) \neq 0$ for all $t \geq 0$. We observe that if $a_1(t) = 0$, then from (11) results that $c = -e^{-2t\alpha} \in (-\infty,-1]$. Because from hypothesis $c \notin (-\infty,-1]$, it results that $a_1(t) \neq 0$, for all $t \geq 0$.

From (10) we obtain

$$\frac{\partial L(z,t)}{\partial z} = \{[f_1'(e^{-t\alpha}z) f_2(e^{-t\alpha}z) - f_1(e^{-t\alpha}z) f_2'(e^{-t\alpha}z)] \cdot$$

$$\cdot [\frac{e^{t\alpha}+ce^{-t\alpha}}{1+c} + \frac{1}{(1+c)^2} \frac{(1-e^{-2t\alpha})}{(e^{t\alpha}-e^{-t\alpha})^{-1}} z^2 \frac{p(e^{-t\alpha}z)}{2}]\} : \tag{12}$$

$$: \{[f_2(e^{-t\alpha}z) + \frac{1}{1+c}(e^{t\alpha} - e^{-t\alpha}) z f_2'(e^{-t\alpha}z)]^2\};$$

$$\frac{\partial L(z,t)}{\partial t} = \{z[f_1'(e^{-t\alpha}z)f_2(e^{-t\alpha}z) - f_1(e^{-t\alpha}z)f_2'(e^{-t\alpha}z)]\cdot$$

$$\cdot[\frac{\alpha(e^{t\alpha}-e^{-t\alpha})}{1+c} - \frac{1}{(1+c)^2}\frac{(1-e^{-2t\alpha}z)}{(e^{t\alpha}-e^{-t\alpha})^{-1}}z^2\frac{p(e^{-t\alpha}z)}{2}]\}: \tag{13}$$

$$:\{[f_2(e^{-t\alpha}z) + \frac{1}{1+c}(e^{t\alpha} - e^{-t\alpha})zf_2'(e^{-t\alpha}z)]^2\}.$$

Let's prove that $L(z,t)$ is a Loewner chain in U. It is easy to prove that the function $L(z,t)$ is locally absolutely continuous in I and locally uniformly with respect to U_{r_0}. The family of the functions $\{L(z,t)/a_1(t)\}$ forms a normal family of regular functions in $U_{r_1} = \{z:|z| < r_1\}$, $0 < r_1 < r_0$. From (11) we obtain $a_1(t) \to \infty$, for $t \to \infty$. Let we consider the function $Q : U_{r_0} \times I \to \mathbf{C}$, by

$$Q(z,t) = z\frac{\partial L(z,t)}{\partial z} \Big/ \frac{\partial L(z,t)}{\partial t}, \quad z \in U \tag{14}$$

From (12), (13) and (14) it results that

$$Q(z,t) = \frac{(e^{t\alpha}+ce^{-t\alpha}) + \dfrac{p(e^{-t\alpha}z)}{2(1+c)}z^2(1-e^{-2t\alpha})(e^{t\alpha}-e^{-t\alpha})}{(e^{t\alpha}-ce^{-t\alpha}) - \dfrac{p(e^{-t\alpha}z)}{2(1+c)}z^2(1-e^{-2t\alpha})(e^{t\alpha}-e^{-t\alpha})} \tag{15}$$

In order to prove that $L(z,t)$ is a Loewner chain it is sufficient to prove that, there exists a real number $r \in (0,1)$, such that $L(z,t)$ is a regular function in $U_r = \{z:|z| < r\}$, for all $t > 0$, the function $Q(z,t)$ defined from (15) to be regular in U for all $t > 0$ and

$$\operatorname{Re} Q(z,t) > 0, \tag{16}$$

for all $z \in U$ and $t \geq 0$.

Let's consider the function

$$K(z,t) = f_2(e^{-t\alpha}z) + \frac{1}{1+c}(e^{t\alpha} - e^{-t\alpha})zf_2'(e^{-t\alpha}z) \qquad (17)$$

We shall prove that the function $K(z,t) \neq 0$. Because $f_2(0) = 1$ and $f_2(z)$ is regular in U, it results that there exists a number $r\in(0,1)$ such that $K(z,t) \neq 0$ for any $z \in U_r$ and hence the function $L(z,t)$ is regular in U for all $t \geq 0$.

In order to prove that the function $Q(z,t)$ is regular in U and with positive real part in U, for all $t \in I$, it is sufficient to prove that

$$|R(z,t)| < 1 \qquad (18)$$

for all $z \in U$ and $t \geq 0$, where

$$R(z,t) = \frac{Q(z,t) - 1}{Q(z,t) + 1} \qquad (19)$$

From (15) and (19) we obtain

$$R(z,t) = \frac{1-\alpha}{1+\alpha} + \frac{2c}{1+\alpha}e^{-2t\alpha} + \frac{p(e^{-t\alpha}z)}{(1+\alpha)(1+c)}z^2(1-e^{-2t\alpha})^2. \qquad (20)$$

By (5) and (20) it results that the inequality (18) holds true for all $z \in U$ and $t \geq 0$.

Using Theorem B, it results that the function $L(z,t)$ is regular and univalence in U for all $z \in U$ and $t \geq 0$.

It results that $L(z,t)$ is a Loewner chain, and hence the function

$$L(z,0) = f_1(z)/f_2(z) = f(z)$$

is univalent in U.

THEOREM 2. *Let* α *be a real number,* c *a complex number,* $|c| < 1$ *and* $f(z) = z + a_2z^2 + \ldots$ *a regular function in* U.

If

$$\left|\frac{1-\alpha}{1+\alpha}+\frac{2c}{1+\alpha}\right|<1 \tag{21}$$

and

$$\left|\frac{1-\alpha}{1+\alpha}+\frac{2c}{1+\alpha}|z|^2+\frac{1}{1+\alpha}\,\frac{p(z)}{1+\alpha}e^{2i\theta}(1-|z|^2)^2\right|<1 \tag{22}$$

for all $z \in U$ *and* θ *a real number,* $p(z) = \{f;z\}$, *then the function* $f(z)$ *is univalent in* U.

Proof. Using the notations from the Theorem 1 it results that the function $R(z,t)$ defined by the relation (20) is regular in U for all $t > 0$. It results that for all $t > 0$, we have

$$\max_{|z|=1} |R(z,t)| = |R(e^{i\theta},t)| = \left|\frac{1-\alpha}{1+\alpha}+\frac{2c}{1+\alpha}e^{-2t\alpha}+\frac{1}{1+\alpha}\,\frac{p(e^{-t\alpha+i\theta})}{1+c}e^{2i\theta}(1-e^{-2t\alpha})^2\right| \tag{23}$$

where $\theta \in \mathbb{R}$. If $\zeta = e^{-t\alpha+i\theta}$, then $|\zeta| = e^{-t\alpha} < 1$ and hence applying the maximum principle to the function $R(z,t)$ we have

$$|R(z,t)| < \max_{|z|=1} |R(z,t)| = \left|\frac{1-\alpha}{1+\alpha}+\frac{2c}{1+\alpha}|\zeta|^2+\frac{1}{1+\alpha}\,\frac{p(\zeta)}{1+c}e^{2i\theta}(1-|\zeta|^2)^2\right|. \tag{24}$$

Because $\zeta \in U$, from (22) and (24) we obtain

$$|R(z,t)| < 1 \tag{25}$$

for all $z \in U$ and $t > 0$.

From hypothesis we observe that for $t = 0$

$$|R(z,0)| = \left|\frac{1-\alpha}{1+\alpha}+\frac{2c}{1+\alpha}\right|<1 \tag{26}$$

The inequality (25) holds true for all $z \in U$ and for all

$t \geq 0$, and, hence by Theorem 1 it results that the function $f(z)$ is univalent in U.

Remark 1. For $\alpha = 1$ and $c = 0$ we obtain Nehari's criterion of univalence.

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ON CERTAIN ANALYTIC FUNCTIONS WITH POSITIVE REAL PART

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REZUMAT. - Asupra unor funcţii analitice cu partea reală pozitivă. Fie α un număr real şi n un număr întreg pozitiv. Fie P şi Q funcţii analitice în discul unitate U, cu $P(z) \neq 0$, care verifică inegalitatea (3). Se arată că dacă $p(z) = 1 + p_n z^n + \ldots$ este o funcţie analitică în U, care verifică ecuaţia diferenţială (4), atunci $Re\ p(z) > 0$ în U. Acest rezultat este îmbunătăţit în cazul cînd funcţia Q este o constantă reală.

1. Introduction. In this paper we shall show that under certain conditions on α, P and Q the solution $p(z) = 1 + p_n z^n + \ldots$ of the differential equation (4) has positive real part. This result is improved when Q is a real constant and we obtain an extension of the "open door" Theorem in [3].

As a simple application we obtain a sufficient condition of starlikeness. The results are obtained by applying the theory of differential subordination. A survey of this theory and applications may be found in [4].

2. Preliminaries. Let A_n be the class of analytic functions f in the unit disc $U=\{z;\ |z|<1\}$ of the form $f(z)=z+a_{n+1}z^{n+1}+\ldots$, where $n\geq 1$.

Denote $A = A_1$. A function $f \in A$ is said to be starlike if $Re\ [\ z\ f'(z)\ /\ f(z)\ ] > 0$ in U. Denote by S^* the class of the starlike functions.

Let F and G be analytic functions in U. If G is univalent,

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then we say that F is subordinate to G, written $F \prec G$ or $F(z) \prec G(z)$ if $F(0) = G(0)$ and $F(U) \subset G(U)$.

We will need the following lemma to prove our results.

LEMMA A. *Let* Ω *be a set in the complex plane* **C** *and let* n *be a positive integer. Suppose that the function* $\Psi: \mathbf{C}^2 \times U \to \mathbf{C}$ *satisfies the condition*

$$\Psi\,(is,\ t;\ z) \notin \Omega \tag{1}$$

for all real s, $t \leq -(n/2)(1 + s^2)$ *and* $z \in U$.

In the function $p(z) = 1 + p_n z^n + \ldots$ *is analytic in* U *and*

$$\Psi\,[p(z),\ z\,p(z)\ ;\ z] \in \Omega, \tag{2}$$

for $z \in U$, *then* $Re\ \rho(z) > 0$ *in* U.

More general forms of this lemma can be found in [1],[2] and [4].

3. Main results.

THEOREM 1. *Let* α *be real and let* n *be a positive integer.*

Let P *and* Q *be analytic functions in* U, *with* $Re\ P(z) \neq 0$ *and suppose that*

$$(2\alpha + n)\left[\frac{Im\ Q(z)}{Re\ P(z)}\right]^2 - 2\,\frac{Re\ [P(z)\ \overline{Q}(z)]}{Re\ P(z)} + n > 0 \tag{3}$$

for $z \in U$.

If $p(z) = 1 + p_n z^n + \ldots$ *is analytic in* U *and satisfies the differential equation*

$$z\,p'(z) + \alpha p^2(z) + P(z)\ p(z) + Q(z) = 0 \tag{4}$$

then $Re\ p(z) > 0$ *in* U.

Proof. Let $\Psi\,(w_1,\ w_2\ ;\ z) = w_2 + \alpha w_1^2 + P(z) w_1 + Q(z)$

If we let $\Omega = \{0\}$, then equation (4) can be written as

$$\mathcal{F}\,[p(z)\;,\;z\,p'(z)\;;\;z]\in\Omega \tag{5}$$

In order to apply Lemma A we show that $\mathcal{F}$ satisfies condition (1), i.e.

$$t-\alpha s^2+is\,P(z)+Q(z)\neq 0 \tag{6}$$

for all real s, $t\le -(n/2)(1+s^2)$ and $z\in U$.

If for some s,t and z satisfying the above conditions the equality

$$t-\alpha s^2+is\,P(z)+Q(z)=0 \text{ holds, then}$$

$$t-\alpha s^2+s\,Im\,P+Q=0 \tag{7}$$

and

$$s\,Re\,P+Im\,Q=0 \tag{8}$$

From (7) we deduce

$$t=\alpha s^2+s\,Im\,P-Re\,Q\le -(n/2)(1+s^2)$$

hence s satisfies the inequality

$$\frac{2\alpha+n}{2}\,s^2+s\,Im\,P-Re\,Q+\frac{n}{2}\le 0 \tag{9}$$

Since $Re\,P(z)\neq 0$, from (8) we deduce

$$s=-\frac{Im\,Q}{Re\,P}$$

and from (8) we obtain the inequality

$$(2\alpha+n)\left(\frac{Im\,Q}{Re\,P}\right)^2-2\,\frac{Re\,P\,\overline{Q}}{Re\,P}+n\le 0$$

which contradicts (3). Hence condition (6) is satisfied and by Lemma A we deduce $Re\,p(z)>0$ in U.

If the function Q is a real constant then Theorem 1 can be improved by the following result.

THEOREM 2. *Let* n *be a positive integer and let* α *and* β

be real numbers, with $2\alpha + n > 0$ *and* $2\beta + n > 0$. *Let* H *be the function*

$$H(z) = \frac{\beta - \alpha + 2(\alpha+\beta+n)z + (\beta-\alpha)z^2}{1 - z^2}, \quad z \in U \tag{10}$$

Let P *be analytic function in* U *satisfying* $P \prec H$.

If $p(z) = 1 + p_n z^n + \ldots$ *is analytic in* U *and satisfies the differential equation*

$$z\,p'(z) + \alpha p^2(z) + P(z)p(z) = \beta \tag{11}$$

then $Re\ p(z) > 0$ *in* U.

Proof. As in the proof of Theorem 1 we have to check the condition (1) of Lemma A, i.e.

$$t - \alpha s^2 + isP(z) \neq \beta \tag{12}$$

for all real s, $t \leq -(n/2)(1 + s^2)$ and $z \in U$.

If for some s, t and z satisfying the above conditions the equality

$$t - \alpha s^2 + is\,P(z) = \beta$$

holds, then

$$t - \alpha s^2 - s\,Im\,P = \beta \tag{13}$$

and

$$s\,Re\,P = 0 \tag{14}$$

If $Re\ P(z) \neq 0$, then form (14) we deduce $s = 0$ and using (13) we obtain $t = \beta > -n/2$ which contradicts

$$t \leq -(n/2)(1 + s^2) \leq -n/2.$$

Therefore, in this case condition (12) is satisfied.

Suppose now that $Re\ P(z) = 0$.

If $s > 0$, from (13) we deduce

$$Im\, P(z) = \frac{t}{s} - \alpha s - \frac{\beta}{s} \le -\frac{n}{2s}(1 + s^2) - \alpha s - \frac{\beta}{s} =$$

$$= -\frac{1}{2}[(2\alpha+n)s + (2\beta+n)\frac{1}{s}] = \varphi(s)$$

It is easy to show that the maximum value of $\varphi(s)$ is given by

$$-\sqrt{(2\alpha + n)(2\beta + n)}.$$

Hence $Im\, P(z) \le -\sqrt{(2\alpha + n)(2\beta + n)}$.

Similarly, for $s < 0$ we deduce

$$Im\, P(z) \ge \sqrt{(2\alpha + n)(2\beta + n)}.$$

Therefore condition (12) holds if either

$Re\, P(z) \ne 0$ or $Re\, P(z) = 0$ and $|Im\, P(z)| < \sqrt{(2\alpha + n)(2\beta + n)}$.

If we let

$$C = \sqrt{(2\alpha + n)(2\beta + n)} \quad \text{and}$$

$$G(z) = 2\,C\,\frac{z}{1 - z^2}$$

then $H(z) = G\left(\frac{z+a}{1+az}\right)$, where $2\,C\,\frac{a}{1-a^2} = \beta-\alpha$.

We deduce that $H(U) = G(U)$ is the complex plane slit along the half-lines $Re\, w = 0$ and $|Im\, w| \ge C$ and $H(0) = \beta-\alpha$.

From the above analyisis we deduce that condition (12) holds if $P \prec H$. By applying Lemma A we deduce $Re\, p(z) > 0$.

4. A starlikeness condition

THEOREM 3. *Let* $f \in A_n$, *with* $\frac{f(z)\, f'(z)}{z} \ne 0$ *in* U *and suppose that*

$$1 + \frac{z\, f''(z)}{f'(z)} - \frac{f(z)}{z\, f'(z)} \prec \frac{2(2 + n)z}{1 - z^2}. \tag{15}$$

Then $f \in S^*$.

Proof. If we let $\alpha = \beta = 1$ then in (10) we have

$$H(z) = \frac{2(2+n)z}{1-z^2}.$$

If we denote $p(z) = \frac{z f'(z)}{f(z)}$ then (15) becomes

$$p(z) + \frac{z p'(z)}{p(z)} - \frac{1}{p(z)} \prec H(z)$$

and if we take $P(z) = \frac{1}{p(z)} - p(z) - \frac{z p'(z)}{p(z)}$

then

$$P(z) \prec H(z) = H(-z)$$

and from Theorem 2 we deduce $Re\ p(z) > 0$, which shows that $f \in S^*$.

COROLLARY. *If* $f \in A_n$, $\frac{f(z) f'(z)}{z} \neq 0$

and

$$\left| Im \left[1 + \frac{z f''(z)}{f'(z)} - \frac{f(z)}{z f'(z)} \right] \right| < 2 + n$$

then $f \in S^*$.

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DISTORSION OF LEVEL LINES OF THE CAPACITY FUNCTIONS UNDER K-qc MAPPINGS

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Dedicated to Professor P. T. Mocanu on his 60^{th} anniversary

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REZUMAT. - Deformarea liniilor de nivel ale funcţiilor capacitate prin transformări K-qc. Fie R şi R' două suprafeţe Riemann deschise cu frontierele ideale Γ şi Γ', iar p_Γ şi $p_{\Gamma'}$, funcţiile capacitate ale celor două frontiere. În lucrare se studiază imaginea printr-o funcţie f a liniilor de nivel ale lui $p_\Gamma(\cdot, z_0)$ în raport cu cele ale lui $p_{\Gamma'}(\cdot, z_0')$ unde $f:R\to R'$, $f(z_0)=z_0'$ este o transformare K-qc (omeomorfism K-cvasiconform).

0. **Introduction.** The capacity functions have introduced by L.Sario [7]. Let R be an open Riemann surface, Γ its ideal boundary, z_0 a point in R and D an arbitrary but fixed parametric disc containing z_0. The **capacity function** of the ideal boundary Γ of R with respect to z_0 and D [7], [8,p.179] is a function $p_\Gamma(\cdot\ ,z_0) = p_R(\cdot\ ,z_0)$ with the following properties:

1) p_Γ is harmonic on $R\setminus z_0$,

2) $p_\Gamma(z,z_0) = \log|z-z_0|+h(z)$, $z\in D$, where $h(z)$ is a harmonic function with $h(z_0)=0$, and

3) p_Γ minimizes the integral

$$\frac{1}{2\pi}\int_\Gamma \varphi * d\varphi := \lim_{n\to\infty}\frac{1}{2\pi}\int_{\Gamma_n}\varphi * d\varphi$$

in the family of the functions $\varphi:R\setminus z_0 \to \mathbf{R}$ which verify 1) and 2), where Γ_n is the boundary of a regular region [1, p.26] π_n from a countable exhaustion of R with $z_0\in\pi_0$.

One knows that

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$$k_\Gamma=\frac{1}{2\pi}\int_\Gamma p_\Gamma * dp_\Gamma=\sup_{z\in R} p_\Gamma(z,z_0)\le +\infty$$

is the Robin constant of R with respect to z_0 and D, while $c_\Gamma=e^{-k_\Gamma}$ is the capacity of Γ with respect to z_0 and D, the Riemann surface R being hyperbolic or parabolic according as $k_\Gamma<+\infty$ or $k_\Gamma=+\infty$.

In the hyperbolic case,

$$p_\Gamma(\cdot\ ,z_0) + G_\Gamma(\cdot\ ,z_0) = k_\Gamma, \qquad (0.1)$$

where $G_\Gamma(\cdot\ ,z_0) = G_R(\cdot\ ,z_0)$ is the **Green function** on R with the logarithmic pole z_0, [8, pp.180-181], [10,IX,I], which is characterized by the following properties:

1) $G_R(\cdot\ ,z_0)$ is harmonic on $R\backslash z_0$,

2) $G_R(z,z_0) = \log\frac{1}{|z-z_0|} + v(z)$, $z\in D$, where v is a harmonic function, and

3) $G_R(\cdot\ ,z_0) = \inf\{P:P$ is positive and satisfies to 1) and 2)$\}$.

In what follows we consider two open Riemann surfaces R and R' with the ideal boundaries Γ and Γ' respectively, two arbitrarily fixed points $z_0\in R$ and $z_0'\in R'$, two parametric discs $D\ni z_0$ and $D'\ni z_0'$ and the corresponding capacity functions $p_\Gamma(\cdot\ ,z_0)$ and $p_{\Gamma'}(\cdot,z_0')$. We denote by z and z' points in R and R' as well as parameters on these surfaces.

Suppose that there are K-qc mappings (K-quasiconformal homeomorphisms) $f:R\to R'$ with $f(z_0) = z_0'$ and denote by $\mathcal{F}$ their family and $z'=f(z)$.

If f would be a conformal mapping $p_{\Gamma'}(z',z_0')=p_\Gamma(z,z_0)$ (by a convenient choice of the parameter of D'), hence level lines of

$p_\Gamma(\cdot\ ,z_0)$ will be mapped by f in level lines of $p_{\Gamma'}(\cdot,z_0')$. Generally this property does not hold for K-qc mappings.

Our aim is to study the image under f of the level lines of $p_\Gamma(\cdot\ ,z_0)$ by means of the level lines of $p_{\Gamma'}(\cdot,z_0')$. For hyperbolic surfaces we first treat this problem by working with the Green functions $G_R(\cdot\ ,z_0)$ and $G_{R'}(\cdot,z_0')$ and taking into account (0.1). This way enlarges the possibilities of application in as much as the form of the results for the Green function is more adequate.

In the proofs we use the following

LEMMA [3]. *Let R and R' be arbitrary Riemman surfaces which are not conformally equivalent to C and $\hat{C}$. If $z_0\epsilon R$ and $z_0'\epsilon R'$, the family $\mathscr{F}$ of the K-qc mappings $f:R\to R'$ with $f(z_0)= z_0'$ is normal and closed.*

Compacity property will play a main role in the paper. Thus we consider only Riemann surfaces of **class** R_p, i.e. Riemann surfaces on which there exists a capacity function with compact level lines [8, p.30], [6, p.231]. As it was proved by M.Nakai, this class contains all parabolic Riemann surfaces [8,IV,§1]. The interior of a compact bordered Riemann surface gives an example of a hyperbolic Riemann surface of class R_p.

1. Level lines of the Green function.

1.1. Let be R and R' two hyperbolic Riemann surfaces of class R_p, z_0 and z_0' two points in R and R', $G_R(\cdot\ ,z_0)$ and $G_{R'}(\cdot,z_0')$ the corresponding Green functions and $\mathscr{F}$ the family defined above.

We designate by C_λ the level line $G_R(z,z_0) = \lambda$ where $\lambda\epsilon(0,+\infty)$, by Π_λ the regular region $\{z\epsilon R:G_R(z,z_0)>\lambda\}$ and for

$\lambda_2<\lambda_1$ by $C_{\lambda_1\lambda_2}$ the curve family $\{C_\lambda : \lambda\in[\lambda_2,\lambda_1]\}$ and $\Pi_{\lambda_1\lambda_2}=\Pi_{\lambda_2}\setminus\overline{\Pi}_{\lambda_1}$. Further we introduce on R' the notations $C'_{\lambda'}$ - the level line $G_{R'}(z',z'_0)=\lambda'$, $\lambda'\in(0,+\infty)$, and similary $\Pi'_{\lambda'}$, $C'_{\lambda'_1\lambda'_2}$ and $\Pi'_{\lambda'_1\lambda'_2}$, $\lambda'_2<\lambda'_1$.

The modulus of $\overline{\Pi}_{\lambda_1\lambda_2}$ defined as the modulus of the curve family separating C_{λ_1} from C_{λ_2} in $\Pi_{\lambda_1\lambda_2}$ is given by the modulus of $C_{\lambda_1\lambda_2}$ [2], namely

$$\operatorname{Mod} C_{\lambda_1\lambda_2}=\frac{\lambda_1-\lambda_2}{2\pi}. \tag{1.1}$$

Since $R\in R_p$, we can define for every $f\in\mathcal{F}$ the functions:

$$\lambda'_0(\lambda,f)=\min\{\lambda'=G_{R'}(z',z'_0) : z'\in fC_\lambda\}$$

and

$$\Lambda'_0(\lambda,f)=\min\{\lambda'=G_{R'}(z',z'_0) : z'\in fC_\lambda\}.$$

PROPOSITION 1.1. *The functions* $\lambda'_0(\lambda,f)$ *and* $\Lambda'_0(\lambda,f)$ *are strictly increasing with respect to* λ, *and verify the inequalities:*

$$\lambda'_0(\lambda,f) \le \lambda' \le \Lambda'_0(\lambda,f) \tag{1.2}$$

and

$$K^{-1}\lambda'_0(\lambda,f) \le \lambda \le K\,\Lambda'_0(\lambda,f). \tag{1.3}$$

1) *Proof that* $\Lambda'_0(\lambda,f)$ *is a strictly increasing function of* λ. We remark that fC_λ descomposes R' in $f\Pi_\lambda$ and $R'\setminus f\overline{\Pi}_\lambda=:\Omega'_\lambda$, that $\max\{G_{R'}(z',z'_0) : z'\in\overline{\Omega}'_\lambda\}=\Lambda'_0(\lambda,f)$ and $C'_{\Lambda'_0(\lambda,f)}\subset f\overline{\Pi}_\lambda$. Further if $\lambda_1>\lambda_2$, then $fC_{\lambda_2}\subset\Omega'_{\lambda_1}$. Suppose that $C'_{\Lambda'_0(\lambda_2,f)}$ does not intersect fC_{λ_1}; since it has at least a common point with fC_{λ_2}, hence with Ω'_{λ_1}, it follows that $\Lambda'_0(\lambda_2,f) < \max\{G_{R'}(z',z'_0) : z'\in\overline{\Omega}'_{\lambda_1}\}=\Lambda'_0(\lambda_1,f)$. If $C'_{\Lambda'_0(\lambda_2,f)}$ intersects fC_{λ_1} then $\Lambda'_0(\lambda_2,f) \le \Lambda'_0(\lambda_1,f)$; however,

equality cannot occur, since otherwise there would exist a point in $fC_{\lambda_2}\cap C'_{\Lambda'_0(\lambda_1,f)}$, hence in $\Omega'_{\lambda_1}\cap f\overline{\Pi}_{\lambda_1}=\emptyset$.

2) *Proof of (1.3)*. According to (1.1) and to the Grötzsch inequalities

$(2\pi)^{-1}\Lambda'_0(\lambda,f)=\mathrm{Mod}\,C'_{\Lambda'_0(\lambda,f)0}=$ the modulus of the curve family separating $C'_{\Lambda'_0(\lambda,f)}$ from Γ' on $R'\setminus\pi'_{\Lambda'_0(\lambda,f)}\geq\mathrm{Mod}\,fC_{\lambda_0}\geq K^{-1}\mathrm{Mod}C_{\lambda_0}=(2\pi K)^{-1}\lambda$.

Similary,

$(2\pi)^{-1}\lambda'_0(\lambda,f)=\mathrm{Mod}C'_{\lambda'_0(\lambda,f)0}\leq K\mathrm{Mod}f^{-1}C'_{\lambda'_0(\lambda,f)0}\leq K$ the modulus of the curve family separating C_λ from Γ on $R\setminus\Pi_\lambda$ = K Mod $C_{\lambda_0}=(2\pi)^{-1}K\lambda$.

PROPOSITION 1.2. *If* $\lambda_1>\lambda_2$, *then*

$$K^{-1}[\lambda'_0(\lambda_1,f)-\Lambda'_0(\lambda_2,f)]\leq\lambda_1-\lambda_2\leq$$
$$\leq K[\Lambda'_0(\lambda_1,f)-\lambda'_0(\lambda_2,f)]. \tag{1.4}$$

The inequalities (1.3) are a particular case of the inequalities (1.4). They can be obtained from (1.4) by taking $\lambda_1=\lambda$ and $\lambda_2=0$, since $\Lambda'_0(0,f)=\lambda'_0(0,f)=0$. However the proof of (1.4) is also similar to that of (1.3):

$$(2\pi)^{-1}[\Lambda'_0(\lambda_1,f)-\lambda'_0(\lambda_2,f)]=\mathrm{Mod}C'_{\Lambda'_0(\lambda_1,f)\lambda'_0(\lambda_2,f)}\geq$$
$$\geq\mathrm{Mod}fC_{\lambda_1\lambda_2}\geq K^{-1}\mathrm{Mod}C_{\lambda_1\lambda_2}=(2\pi K)^{-1}(\lambda_1-\lambda_2)$$

and, if $\lambda'_0(\lambda_1,f)>\Lambda'_0(\lambda_2,f)$,

$$(2\pi)^{-1}[\lambda'_0(\lambda_1,f)-\Lambda'_0(\lambda_2,f)]=\mathrm{Mod}C'_{\lambda'_0(\lambda_1,f)\Lambda'_0(\lambda_2,f)}\leq$$
$$\leq K\mathrm{Mod}f^{-1}C'_{\lambda'_0(\lambda_1,f)\Lambda'_0(\lambda_2,f)}\leq K\mathrm{Mod}C_{\lambda_1\lambda_2}=(2\pi)^{-1}K(\lambda_1-\lambda_2).$$

Remark 1.1 The image fC_λ of a level line of $G_R(\ ,z_0)$ is included in $\overline{\Pi}_{\Lambda'_0(\lambda,f)\lambda'_0(\lambda,f)}$ so that its distorsion from the level lines of $G_{R'}(\ ,z'_0)$ could be measured by

$\mathrm{Mod}C'_{\Lambda'_0(\lambda,f)\lambda'_0(\lambda,f)}=(2\pi)^{-1}[\Lambda'_0(\lambda,f)-\lambda'_0(\lambda,f)]$.

In the family $\mathcal{F}$ there are K-*qc* mappings with the property:

$fC_\lambda = C_{\lambda'}$, for $\lambda' = \lambda'(\lambda, f)$. For such a function, if we write $\lambda'_j = \lambda'(\lambda_j, f)$, j=1,2, the inequalities (1.3) and (1.4) become

$$K^{-1}\lambda' \le \lambda \le K\lambda' \text{and} \tag{1.3'}$$

$$K^{-1}\ (\lambda'_1-\lambda'_2) \le \lambda_1-\lambda_2 \le K\ (\lambda'_1-\lambda'_2)\ . \tag{1.4'}$$

This case implies the equality in some of the inequalities used to prove (1.3) or (1.4). The results in [2] show that equality in the right- (left-) hand side of (1.3) and (1.4) is assured if we add to this property of f the conditions: the dilatation quotient of f is the constant K and the major axes of the characteristic ellipses are orthogonal (or respectively tangent) to the curves C_λ a.e. in R. Then we have e.g.

$$\lambda' = K^{-1}\lambda \qquad \text{(or } \lambda' = K\lambda, \text{ respectively)}. \tag{1.3"}$$

If $K=1$, the equality holds in both sides, $\lambda'=\lambda$, and expresses the conformal invariance of the Green function as in the Lindelöf principle.

Remark 1.2. If we denote by $\lambda_0(\lambda', f^{-1}) = \min\{G_R(z,z_0) : z \in f^{-1}C'_{\lambda'}\}$ and by $\Lambda_0(\lambda'_0, f^{-1}) = \max\{G_R(z,z_0) : z \in f^{-1}C'_{\lambda'}\}$,
then we obtain

$$\lambda_0[\ \Lambda'_0(\lambda,f), f^{-1}] = \lambda = \Lambda_0[\ \lambda'_0(\lambda)].$$

1.2. Till now we studied the functions $\lambda'_0(\lambda,f)$ and $\Lambda'_0(\lambda,f)$ which correspond to a K-qc mapping $f \in \mathcal{F}$. We now introduce two functions which delimit the distorsion of the Green level lines with respect to the whole family of mappings $\mathcal{F}$. Namely we define

$$\lambda'_0(\lambda) = \inf\{\ \lambda'_0(\lambda,f)\ :\ f \in \mathcal{F}\}$$

and

$$\Lambda'_0(\lambda) = \sup\{\ \Lambda'_0(\lambda,f)\ :\ f \in \mathcal{F}\}.$$

PROPOSITION 1.3. *If R and R' are hyperbolic Riemann surfaces* *of class R_p, there exist extremal mappings $f_{0,\lambda}$ and $F_{0,\lambda}\in\mathcal{F}$ such that $\lambda_0'(\lambda)=\lambda_0'(\lambda,f_{0,\lambda})$ and $\Lambda_0'(\lambda)=\Lambda_0'(\lambda,F_{0,\lambda})$.*

Proof. Let λ be an arbitrary but fixed positive number and $\{f_n\}$ be a sequence in $\mathcal{F}$ such that $\lambda_0'(\lambda,f_n)\to\lambda_0'(\lambda)$. According to the Lemma quoted in Introduction the family $\mathcal{F}$ is normal and closed, such that $\{f_n\}$ contains a subsequence again denoted by $\{f_n\}$ which uniformly converges in the compact subsets of R to a mapping $f_{0,\lambda}\in\mathcal{F}$. Let us choose for any n a point $z_n\in C_\lambda$ such that $G_{R'}(f_n(z_n),z_0')=\lambda_0'(\lambda,f_n)$. As $R\in R_p$, the sequence $\{z_n\}$ contains a convergent subsequence with a limit $z^*\in C_\lambda$. By a new change of notations we may suppose that $\{f_n\}\subset\mathcal{F}$ uniformly converges on the compact subsets of R to $f_{0,\lambda}$, that $G_{R'}(f_n(z_n),z_0')=\lambda_0'(\lambda,f_n)$ and that $z_n\to z^*$. Since $f_n(z_n)\to f_{0,\lambda}(z^*)$, $\lambda_0'(\lambda)=\lim_{n\to\infty}G_{R'}(f_n(z_n),z_0')=$ $=G_{R'}(f_{0,\lambda}(z^*),z_0')\geq\lambda_0'(\lambda,f_{0,\lambda})$. It follows thus by the definition that $\lambda_0'(\lambda)=\lambda_0'(\lambda,f_{0,\lambda})$. The proof for $\Lambda_0'(\lambda)$ is similar.

PROPOSITION 1.4. *The functions $\lambda_0'(\lambda)$ and $\Lambda_0'(\lambda)$ are strictly increasing. They verify the inequalities*

$$\lambda_0'(\lambda)\leq\lambda'\leq\Lambda_0'(\lambda), \tag{1.5}$$

where $\lambda'=G_{R'}(z',z_0')$for $z'=f(z)$ and $z\in C_\lambda$,

$$K^{-1}\lambda_0'(\lambda)\leq\lambda\leq K\Lambda_0'(\lambda),\ \text{and if}\ \lambda_1>\lambda_2 \tag{1.6}$$

$$K^{-1}[\lambda_0'(\lambda_1)-\Lambda_0'(\lambda_2)]\leq\lambda_1-\lambda_2\leq K[\Lambda_0'(\lambda_1)-\lambda_0'(\lambda_2)]. \tag{1.7}$$

Proof that $\Lambda_0'(\lambda)$ is strictly increasing. Suppose that $\lambda_1>\lambda_2$. From the definition of the function $\Lambda_0'(\lambda)$ and since $\Lambda_0'(\lambda,f)$, $f\in\mathcal{F}$, is strictly increasing, we deduce:

$$\Lambda_0'(\lambda_1)\geq\Lambda_0'(\lambda_1,F_{0,\lambda_2})>\Lambda_0'(\lambda_2,F_{0,\lambda_2})=\Lambda_0'(\lambda_2).$$

The proof for $\lambda_0'(\lambda)$ is similar.

Proof of the inequalities. From (1.2) and (1.3) it follows directly (1.5) and (1.6) respectively, by passing to $\inf_{f\in\mathcal{F}}(\sup_{f\in\mathcal{F}})$ in the left-(right-) hand side. Starting from (1.4) one obtains (1.7) by means of the inequalities

$$\Lambda_0'(\lambda_1,f)-\lambda_0'(\lambda_2,f)\le\Lambda_0'(\lambda_1)-\lambda_0'(\lambda_2) \quad \text{and}$$
$$\lambda_0'(\lambda_1,f)-\Lambda_0'(\lambda_2,f)\ge\lambda_0'(\lambda_1)-\Lambda_0'(\lambda_2) \quad \text{respectively.}$$

Remark 1.3. Proposition 1.3 shows that the functions $\lambda_0'(\lambda)$ and $\Lambda_0'(\lambda)$ are finite and Proposition 1.4. permits us to obtain a uniform majorant. If λ_1, $\lambda_2 \in [m,M]$, $\lambda_1 > \lambda_2$, then

$$\Lambda_0'(\lambda_1)-\lambda_0'(\lambda_2)\le\Lambda_0'(M)-\lambda_0'(m) .$$

2. Level lines of the capacity function.

2.1. We now consider two arbitrary Riemann surfaces R and R' of class R_p, and - as in Introduction - the capacity functions $p_\Gamma(\ ,z_0)$ and $p_{\Gamma'}(\ ,z_0')$ of the ideal boundaries Γ of R and Γ' of R' with respect to $z_0\in R$, the parametric disc D and $z_0'\in R',D'$ respectively.

We denote by c_τ the level line $p_\Gamma(z,z_0)=\tau$, where $\tau\in(-\infty,k_\Gamma)$ and by Π_τ the regular region $\{z\in R:\ p_\Gamma(z,z_0)<\tau\}$. For $\tau_1<\tau_2$ let $c_{\tau_1\tau_2}=\{c_\tau\ :\ \tau\in[\tau_1,\tau_2]\}$ and $\Pi_{\tau_1\tau_2}=\Pi_{\tau_2}\setminus\overline{\Pi}_{\tau_1}$. The modulus of $\overline{\Pi}_{\tau_1\tau_2}$ is now given by

$$\operatorname{Mod} c_{\tau_1\tau_2} = \frac{\tau_2-\tau_1}{2\pi}. \tag{2.1}$$

Further we introduce similar notations $c'_{\tau'}, c'_{\tau'_1\tau'_2}, \Pi_{\tau'}, \Pi_{\tau'_1\tau'_2}$ on R', we consider the family $\mathcal{F}$ and we define as in 1.1. the functions

$$\tau'_0(\tau,f)=\min\{\tau'=p_{\Gamma'}(z',z'_0)\ :\ z'\in fc_\tau\}$$

and

$$T'_0(\tau,f)=\max\{\tau'=p_{\Gamma'}(z',z'_0)\ :\ z'\in fc_\tau\}.$$

By the same device as in 1.1 which is now applied to the capacity function instead of the Green function (in the hyperbolic case by using (0.1)) we prove the following results.

PROPOSITION 2.1. *The functions* $\tau'_0(\tau,f)$ *and* $T'_0(\tau,f)$ *are strictly increasing with respect to* τ. *They verify the inequalities*

$$\tau'_0(\tau,f)\le\tau'\le T'_0(\tau,f), \tag{2.2}$$

and in the hyperbolic case

$$K^{-1}[k_{\Gamma'} - T'_0(\tau,f)] \le k_\Gamma - \tau \le K[k_{\Gamma'} - \tau'_0(\tau,f)]. \tag{2.3}$$

PROPOSITION 2.2. *If* $\tau_1 < \tau_2$, *then*

$$K^{-1}[\tau'_0(\tau_2,f)-T'_0(\tau_1,f)]\le\tau_2-\tau_1\le K[T'_0(\tau_2,f)-\tau'_0(\tau_1,f)]. \tag{2.4}$$

Remark 2.1. Once again equality in the right-(left-) hand side of (2.4) takes place for a mapping $f\in\mathcal{F}$ with the properties: $fc_\tau=c'_{\tau'}$, for a function $\tau'=\tau'(\tau,f)$ (then $\tau'_0(\tau,f)=T'_0(\tau,f)=\tau'$), the dilatation quotient of f is the constant K and the major axes of the characteristic ellipses are orthogonal (tangent) to the curves c_τ a.e. in R. Then (2.4) becomes $\tau'_2-\tau'_1 = K^{-1}(\tau_2-\tau_1)$, (or $=K(\tau_2-\tau_1)$ respectively). Inequalities similar to (1.3') and (1.4'), and equalities as in Remark 1.2. are valid.

2.2. As in 1.2. if we introduce the functions

$$\tau_0'(\tau) = \inf\{\tau_0'(\tau, f) \ : \ f \in \mathcal{F}\}$$

and

$$T_0'(\tau) = \sup\{T_0'(\tau, f) \ : \ f \in \mathcal{F}\}$$

we obtain

PROPOSITION 2.3. *Let R and R' be Riemann surfaces of class R_p not conformally equivalent to C. There exist mappings $f_{0,\tau}$ and $F_{0,\tau} \in \mathcal{F}$ such that* $\tau_0'(\tau) = \tau_0'(\tau, f_{0,\tau})$ and $T_0'(\tau) = T_0'(\tau, F_{0,\tau})$.

PROPOSITION 2.4. *The functions $\tau_0'(\tau)$ and $T_0'(\tau)$ are strictly increasing and verify the inequalities*

$$\tau_0'(\tau) \le \tau' \le T_0'(\tau) \tag{2.5}$$

where $\tau' = p_{\Gamma'}(z', z_0')$, $z' = f(z)$ *and* $z \in c_\tau$,

$$K^{-1}[\tau_0'(\tau_2) - T_0'(\tau_1)] \le \tau_2 - \tau_1 \le K[T_0'(\tau_2) - \tau_0'(\tau_1)] \tag{2.6}$$

for $\tau_1 < \tau_2$, and

$$K^{-1}[k_{\Gamma'} - T_0'(\tau)] \le k_\Gamma - \tau \le K[k_{\Gamma'} - \tau_0'(\tau)] \tag{2.7}$$

in the hyperbolic case.

Remark 2.2. If $\tau_1, \tau_2 \in [m, M]$, $\tau_1 < \tau_2$, then

$$T_0'(\tau_2) - \tau_0'(\tau_1) \le T_0'(M) - \tau_0'(m)$$

so that we have again a uniform majorant.

Remark 2.3. The compact Riemann surfaces can be also studied with this method - as parabolic surfaces hence surfaces of class R_p - namely if S and S' are two such surfaces (for Propositions 2.3 and 2.4 not conformally equivalent to C), one deals with $R = S \setminus z_\infty$ and $R' = S' \setminus z_\infty'$ for two arbitrary points $z_\infty \in S$, $z_\infty' \in S'$. The family $\mathcal{F}$ consists now of all the K-qc mappings $f: S \to S'$ with

$f(z_h) = z_h'$,

$h=0,\infty, z_0 \in R$ and $z_0' \in R'$.

Remark 2.4. These results have been applied in [4] in order to generalize a Gehring's theorem. The paper [4] contains also proofs of Propositions 2.1.-2.4. Let us mention that our main tools - the functions $\lambda_0'(\lambda, f)$, $\Lambda_0'(\lambda, f)$ and $\tau_0'(\tau, f)$, $T_0'(\tau, f)$ - generalize classical functions considered in the plane by different authors and which have various applications. As an example we quote [9] where for the level lines of the capacity function in the plane (the Evans-Selberg potential) with respect to 0, $c_r: |z| = r$, the function $M(r,f) = \max\{|f(z)| : z \in c_r\}$ is used to thoroughly study the growth of the entire quasiregular functions.

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LOCALLY BILIPSCHITZ MAPPINGS AS A SUBCLASS OF QUASICONFORMAL HOMEOMORPHISM IN A NORMED SPACE

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REZUMAT. - Aplicaţii local biLipschitziene ca o subclasă de homeomorfisme cvasiconforme în spaţii normate. În lucrare se dau două caracterizări ale clasei aplicaţiilor local biLipschitziene.

In this paper, I show in a normed space X, the locally biLipschitz mappings $f:D\to D'$ (D,D' domains in X), the local quasi-isometries and a certain subclass of quasiconformal mappings (considered in my paper [2]) and characterized by the quasi-invariance of a certain kind of module $\mathrm{mod}_{\infty}^{D}\Gamma$ of an arc family Γ, coincide. I show also that the distance $d_D(E_0,E_1)$ between two sets E_0,E_1 relatively to a domain D coincide to the extremal length $\lambda_{\infty}^{D}(E_0,E_1,D)$ of the family $\Gamma(E_0,E_1,D)$ of the arcs γ joining E_0 and E_1 in D and defined as the inverse of the module $\mathrm{mod}_{\infty}^{D}\Gamma(E_0,E_1,D)$. This allows us to give another characterization of the subclass from above by means of the quasi-invariance of the relative distance.

Now let Γ be a family of arcs $\gamma\subset D$ (by abuse and for simplicity sake, I shall denote it by $\Gamma\subset D$) and let

$F^D(\Gamma) = \{\rho;\rho\geq 0, \rho|_{CD}=0$, bounded and continuous in D and such that

$$\int_{\gamma}\rho\, ds\geq 1\ \forall\gamma\in\Gamma\} \quad (\ \forall = \text{"for every"})$$

be the corresponding class of admissible functions. Then, we define the module of Γ as

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$$\operatorname{mod}_\infty^D\Gamma = \inf_{\rho\in F^D(\Gamma)} \sup_X \rho(x) = \inf_{\rho\in F^D(\Gamma)} \|\rho\|_\infty$$

We recall that the origin of this concept of module (cf our paper [1]) is the n-dimensional module

$$\operatorname{mod}\Gamma = \inf_{\rho\in F^D(\Gamma)} \left(\int_{\mathbb{R}^n}\rho^n dm\right)^{\frac{1}{n}},$$

taking into account that, for ρ continuous,

$$\lim_{n\to\infty}\left(\int_X\rho^n dm\right)^{\frac{1}{n}} = \sup_X\rho(x) = \|\rho\|_\infty .$$

In this paper, a map f means a homeomorphism $f:D\to D'$, where D,D' domains of the normed space X.

Let us denote by $\Gamma(E_0,E_1,D)$ the family of open arcs $\gamma\subset D$ - an open arc being the homeomorphic image of the linear open interval (0,1) - such that the closure $\bar\gamma$ of γ is a homeomorphic image of the closed linear interval [0,1], the endpoints of $\bar\gamma$ belonging to E_0 and E_1, respectively.

A homeomorphism f is K-**quasiconformal** if $\forall$ $E_0,E_1\subset D$, the double inequality

$$\frac{\operatorname{mod}_\infty^D(E_0,E_1,D)}{K} \le \operatorname{mod}_\infty^{D'}\Gamma(E_0',E_1',D') \le K\operatorname{mod}_\infty^D\Gamma(E_0,E_1,D) \tag{1}$$

holds, where $E_k'= f(E_k)$ $(k=0,1)$. A **quasiconformal mapping** is a K-quasiconformal one with non specified K. In this paper, by K-quasiconformal mapping, we understand only the mappings of the subclass characterized by (1).

We recall that the relative distance $d_E(E_0,E_1)$ (with respect to a set E) between two sets E_0,E_1 is

$$d_E(E_0,E_1) = \inf_{\gamma\in\Gamma(E_0,E_1,E)} H^1(\gamma),$$

where H^1 is linear Hausdorff measure. If $\Gamma(E_0,E_1,E)=\emptyset$, then, we consider $d_E(E_0,E_1)=\infty$.

PROPOSITION 1. $E_0,E_1\subset\bar{D}$ →

$$\operatorname{mod}_{\infty}^{D}\Gamma(E_0,E_1,D) = \frac{1}{d_D(E_0,E_1)}$$

(P.Caraman [2], lemma 5).

Taking into account that the extremal length $\lambda_{\infty}^{D}\Gamma=\frac{1}{\operatorname{mod}_{\infty}^{D}\Gamma}$, the preceding proposition yields the following

COROLLARY. $d_D(E_0,E_1)= \lambda_{\infty}^{D}\Gamma(E_0,E_1,D)$.

PROPOSITION 2. *f is K-quasiconformal iff* $\forall\ E_0,E_1\subset D$,

$$\frac{d_D(E_0,E_1)}{K} \le d_{D'}(E_0',E_1') \le K d_D(E_0,E_1)\,. \tag{2}$$

(P.Caraman [2], lemma 7).

COROLLARY. *f is K-quasiconformal iff* $\forall\ E_0,E_1\subset D$,

$$\frac{\lambda_{\infty}^{D}\Gamma(E_0,E_1,D)}{K} \le \lambda_{\infty}^{D'}\Gamma(E_0',E_1',D') \le K\lambda_{\infty}^{D}\Gamma(E_0,E_1,D)\,.$$

A mapping f is said to be a **local** C-**isometry** with $0<C<\infty$ if $\forall\, x\in D$, there exists a neighbourhood $U_x\subset D$ such that

$$\frac{\|y-z\|}{C} \le \|f(y)-f(z)\| \le C\|y-z\|$$

$\forall\, y,z\in U_x$.

THEOREM 1. *f is K-quasiconformal iff it is a local K-isometry.*

Proof. Suppose f is K-quasiconformal and consider an arbitrary point $x\in D$. Next, let $U_x = B(x,r)\subset D$. Then, on account of the preceding proposition in the particular case $E_0=\{y\}$,

$E_1=\{z\}$,

$$\|f(y)-f(z)\| \leq d_{D'}[f(y),f(z)] \leq Kd_D(y,z) = K|y-z| \quad (3)$$

$\forall\ y,z \in U_x$. But also the converse is true. Indeed, let $x' \in D'$ be an arbitrary point and $V_{x'} = B(x',r') \subset D'$. Then, on account of the preceding proposition, in the particular case $E_0=\{y\}, E_1=\{z\}$,

$$\|y-z\| \leq d_D(y,z) \leq Kd_{D'}(y',z')=K\|y'-z'\|=K\|f(y)-f(z)\|$$

$\forall\ y,z \in V_x = f^{-1}(V_{x'})$. This relation, together with (3), yields

$$\frac{\|y-z\|}{K} \leq \|f(y)-f(z)\| \leq K\|y-z\|$$

$\forall\ y,z \in U_x \cap f^{-1}(V_{x'})$.

Now, let us prove also the opposite implication. Assume f is a local K-isometry $(1 \leq K < \infty)$, $x_0 \in D$, $U_x=B(x_0,r_0) \subset D$. Then,

$$d_D(x,y)=\|x-y\| \leq K\|f(x)-f(y)\| \leq Kd_{D'}(x',y') \quad (4)$$

$\forall\ x,y \in U_0$. But, also conversely, if $x_0 \in D$ and V_0 is a neighbourhood of x_0 such that $f(V_0) \subset B[f(x_0), r_0'] \subset D'$, then

$$d_{D'}(x',y')=\|f(x)-f(y)\| \leq K\|x-y\| \leq Kd_D(x,y)$$

$\forall\ x,y \in V_0$, hence and on account of (4), we obtain

$$\frac{d_D(x,y)}{K} \leq d_{D'}(x',y') \leq Kd_D(x,y) \quad (5)$$

$\forall\ x,y \in W_0 \subset U_0 \cap V_0$.

Next, we observe that there exist two sequences $\{x_n'\} \subset E_0'$, $\{y_n'\} \subset E_1'$ such that

$$d_{D'}(E_0',E_1') = \inf_{\substack{x' \in E_0' \\ y' \in E_1'}} d_{D'}(x',y') = \lim_{n\to\infty} d_{D'}(x_n',y_n') = \lim_{n\to\infty} \inf_{\gamma_n' \in \Gamma(x_n',y_n',D')} H^1(\gamma_n'). \quad (6)$$

And now, $\forall\ \varepsilon>0$ and $\forall\ n \in N$, there is an arc $\gamma_n' \in \Gamma(y_n', z_n', D')$ such that

$$d_D(x'_n, y'_n) > H^1(\gamma'_n) - \varepsilon \ ; \tag{7}$$

but $\overline{\gamma_n} = f^{-1}(\overline{\gamma'_n})$ is compact, hence $d(\overline{\gamma_n}, \partial D) = d_n > 0$. Then, let $x_n^k \in \gamma_n$ such that $x_n^0 = x_n = f^{-1}(x'_n)$, $x_n^p = y_n = f^{-1}(y'_n)$ and $d(x_n^k, x_n^{k+1}) < d_n$ $(k = \overline{0, m-1})$. But, on account of (5) and (7),

$$d_D(x_n, y_n) \le \sum_{k=0}^{p} d(x_n^k, x_n^{k+1}) \le K \sum_{k=0}^{p} d_{D'}(x_n'^k, x_n'^{k+1}) \le$$
$$\le K \sum_{k=0}^{p} H^1(\gamma_n'^k) = K H^1(\gamma'_n) < K d_{D'}(x', y') + K\varepsilon,$$

where $\gamma_n'^k$ is the subarc of γ'_n joining $x_n'^k$ and $x_n'^{k+1}$, hence, letting $\varepsilon \to 0$, it follows that

$$d_D(x_n, y_n) \le K d_{D'}(x'_n, y'_n) \ ,$$

whence and since $x_n \in E_0, y_n \in E_1$, we obtain

$$d_D(E_0, E_1) \le d_D(x_n, y_n) \le K d_{D'}(x'_n, y'_n) \qquad \forall\ n \in N,$$

so that, taking into account (6), we obtain

$$d_D(E_0, E_1) \le K \lim_{n\to\infty} d_{D'}(x'_n, y'_n) = K d_{D'}(E'_0, E'_1) \ . \tag{8}$$

In order to establish the opposite inequality, we use a similar argument. We observe first that there exist two sequences $\{x_n\}, \{y_n\}$ such that

$$d_D(E_0, E_1) = \lim_{n\to\infty} d_D(x_n, y_n) = \lim_{n\to\infty} \inf_{\gamma_n \in \Gamma(x_n, y_n, D)} H^1(\gamma_n) \ , \tag{9}$$

hence, $\forall\ n \in N$, and $\varepsilon > 0$, there exist $\gamma_n \in \Gamma(x_n, y_n, D)$ such that $d_D(x_n, y_n) > H^1(\gamma_n) - \varepsilon$. Since $\overline{\gamma'_n} = \overline{f(\gamma_n)}$ is compact, $d(\overline{\gamma'_n}, \partial D') = d'_n > 0$ so that we may choose $x_n'^k \in \gamma'_n$ $(k = 0, 1, \dots, p)$ so that $x_n'^0 = x'_n, x_n'^p = y'_n$ and $d(x_n'^k, x_n'^{k+1}) < d'_n$. But then, taking into account (5), we get

$$d_{D'}(x_n', y_n') \le \sum_{k=0}^{p} d(x_n'^k, x_n'^{k+1}) \le K\sum_{k=0}^{p} d_D(x_n^k, x_n^{k+1}) =$$
$$= K\sum_{k=0}^{p} H^1(\gamma_n^k) = KH^1(\gamma_n) < Kd_D(x_n, y_n) + \varepsilon K$$

and, letting $\varepsilon \to 0$, we deduce that

$$d_{D'}(x_n', y_n') \le Kd_D(x_n, y_n),$$

which, taking into account (9), yields

$$d_{D'}(E_0', E_1') \le \lim_{n\to\infty} d_{D'}(x_n', y_n') \le K \lim_{n\to\infty} d_D(x_n, y_n) = Kd_D(E_0, E_1),$$

which, together with (8), yields (2), implying (by the preceding proposition) the K-quasiconformality of f, as desired.

Arguing as in the preceding theorem, we obtain the

COROLLARRY. *f is K-quasiconformal iff* $\forall\, x, y \in D$,

$$\frac{d_D(x,y)}{K} \le d_{D'}(x', y') \le Kd_D(x,y).$$

A mapping f is said to be **uniformly locally Lipschitz with the constant** $M>0$ if $\forall\, x\in D$, there exists a neighbourhood $U_x \subset D$ such that $\forall\, y, z \in U_x$, $\|f(y)-f(z)\| \le M\|y-z\|$ f is said **uniormly locally biLipschitz with the constant** $M>0$ if f and f^{-1} are uniformly locally Lipschitz with the constant M.

THEOREM 2. *f is K-quasiconformal iff it is uniformly locally bilipschitz with the constant K.*

Proof. If f is K-quasiconformal, then, according to the preceding theorem, f is K-isometry, hence f and f^{-1} are uniformly locally Lipschitz with the constant K. The converse follows by a similar argument.

COROLLARY. *A K-quasiconformal mapping* $f: B(x_0, R) \to D'$ *is*

Lipschitz with constant K.

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ON A CONJECTURE OF HORN IN COINCIDENCE THEORY

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Dedicated to Professor P.T.Mocanu on his 60^{th} anniversary

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REZUMAT. - Asupra unei conjecturi a lui Horn în teoria coincidenţei. Conjectura lui Horn afirmă că doi operatori continui şi comutativi ce invariază un compact convex dintr-un spaţiu Banach, au cel puţin un punct de coincidenţă. In prezenta lucrare se dau mai multe propoziţii echivalente cu conjectura lui Horn. In finalul lucrării se introduce noţiunea de structură de coincidenţă şi se stabileşte o teoremă generală de coincidenţă.

1. Introduction. Horn's conjecture ([1]) states that if two commutative mappings, onto a compact convex subset of a Banach space into it self are continuous, then this pair of mappings has at least a coincidence point. In this paper we present some equivalent statements with the Horn's conjecture.

2. Measures of noncompactness. Let X be a Banach space. By a weak measure of noncompactness on X we mean a mapping, $\alpha: P_b(X)\to R_+$, which satisfies the following conditions:

(i) $\alpha(A)=0$ implies $\bar{A}\in P_{cp}(X)$,

(ii) $\alpha(\overline{co}A)=\alpha(A)$, for all $A\in P_b(X)$.

By definition a weak measure of noncompactness is a measure of noncompactness if satisfies the condition

$\bar{A}\in P_{cp}(X)$ implies $\alpha(A)=0$.

For example, α_k (Kuratowski's measure of noncompactness) and α_H (Hausdorff's measure of noncompactness) are measure of noncompactness and δ is a weak measure of noncompactness.

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3. **Invariant subsets.** Let X be a nonempty set and let $f,g:X\to X$ be two mappings. We denote

$I(f):=\{A\subset X \mid A\neq\phi,\ f(A)\subset A\}$,

$I(f,g):=I(f)\cap I(g)$,

$F_f:=\{x\in X \mid f(x)=x\}$,

$C(f,g):=\{x\in X \mid f(x)=g(x)\}$.

We have

LEMMA 1. *Let X be a nonempty set, $\mu:\mathcal{P}(X)\to\mathcal{P}(X)$ a clasure operator, $Y\in F_\mu$ and $f,g:Y\to Y$ such that $f\circ g=g\circ f$. Let $A_1\subset Y$, $A_1\neq\phi$. Then there exists $A_0\subset Y$ such that*

(i) $A_0\supset A_1$,

(ii) $A_0\in F_\mu$,

(iii) $A_0\in I(f,g)$,

(iv) $\mu(f(A_0)\cup g(A_0)\cup A_1)=A_0$.

Proof. Let $\mathcal{B}:=\{B\subset Y \mid B$ satisfies (i)+(ii)+(iii)$\}$. We have $\cap\mathcal{B}\in\mathcal{B}$. Let $A_0:=\cap\mathcal{B}$. We remark that $\mu(f(A_0)\cup g(A_0)\cup A_1)\in\mathcal{B}$ and $\mu(f(A_0)\cup g(A_0)\cup A_1)\subset A_0$. This implies (iv).

4. **α-condensing pair.** Let X be a Banach space, $Y\subset X$ and $f,g:Y\to Y$. Let $\theta:P_b(x)\to R_+$. The pair (f,g) is θ-condensing if

(i) $A\in P_b(Y)$ implies $f(A)$, $g(A)\in P_b(Y)$

(ii) $\theta(f(A)\cup g(A)<\theta(A)$, $\forall\ A\in I_b(f,g)$, $\theta(A)\neq 0$.

Example 1. Let $Y\in P_b(X)$ and let $f,g:Y\to Y$ be two compact mapping. Then the pair (f,g) is α_k-condensing.

Example 2. Let $Y\in P_b(X)$ and let $f,g:Y\to Y$ be two δ-condensing mapping. In general, the pair (f,g) is not δ-condensing.

Now we consider

Statement $S(\theta)$. Let Y be a bounded closed convex subset of a Banach space X and let $f,g:Y \to Y$ be commuting continuous mappings. If the pair is θ-condensing, then $C(f,g) \neq \phi$.

The main results of this paper is the following:

THEOREM 1. *The following statements are equivalent:*

(i) *(Horn) Let Y be a compact convex subset of X and let $f,g:Y \to Y$ be commuting continuous mappings. Then $C(f,g) \neq \phi$.*

(ii) *Statement $S(\alpha_k)$.*

(iii) *Statement $S(\alpha)$, for an α - a measure of noncompactness on X.*

(iv) *Statement $S(\alpha)$ for all α - measures of noncompactness on X (i.e., $\{S(\alpha) \mid \alpha \in$ the set of all measure of noncompactness on $X\}$*

(v) *Statement $S(\alpha)$ for all α - weak measures of noncompactness on X (i.e., $\{S(\alpha) \mid \alpha \in$ the set of all weak measures of noncompactness on $X\}$.*

Proof. The proof follows from the following implications:

$$(v) \to (iv) \begin{matrix} \nearrow (iii) \searrow \\ \searrow (ii) \nearrow \end{matrix} (i) \Rightarrow (v).$$

We will prove (i) $\to$ (v). Let $A_1 = F_f$ and $\mu(A) = \overline{co}\ A$. By Schauder's fixed point theorem, $F_f \neq \phi$. We have $f(F_f) = F_f$ and $g(F_f) \subset F_f$. By Lemma 1, there exists $A_0 \subset Y$ such that

$\overline{co}(f(A_0) \cup g(A_0) \cup F_f \supset = A_0.$

Since, $F_f \in f(A_0) \cup g(A_0)$, hence $\overline{co}(f(A_0) \cup g(A_0)) = A_0$. We have

$\alpha(\overline{co}(f(A_0) \cup g(A_0)) = \alpha(f(A_0) \cup g(A_0)) = \alpha(A_0)$

Thus implies that $A_0 \in P_{cp,cv}(X)$.

From (i), we have that, $C(f,g) \neq \phi$.

5. **Coincidence property.** Let X be a nonempty set and $Y \in P(X)$. We denote by $M(Y)$ the set of all mappings, $f:Y \to Y$. A triple (X,S,M) is a coincidence structure if

(i) $S \subset P(X)$, $S \neq \phi$,

(ii) $M: P(X) \multimap \bigcup_{Y \in P(X)} M(Y)$, $Y \mapsto M(Y) \subset M(Y)$, is a mapping such that, if $Z \subset Y$, $Z \neq \phi$, then $M(Z) \supset \{f|_Z : f \in M(Y) \text{ and } f(Z) \subset Z\}$,

(iii) ($Y \in S$, $f,g \in M(Y)$, $f \circ g = g \circ f$) imply $C(f,g) \neq \phi$.

For example (see [1]), if $X=R$, $S=\{[a,b] \mid a,b \in R\}$ and $M(Y)=C(Y):=\{f:Y \to Y \mid f\text{-continuous}\}$, then the triple (X,S,M) is a coincidence structure.

Let (X,S,M) be a coincidence structure. A pair (θ,μ) is compatible with (X,S,M) if

(i) $\theta: Z \to R_+$, $S \subset Z \subset P(X)$,

(ii) $\mu: P(X) \to P(X)$ is a clasure operator, $S \subset \mu(Z) \subset Z$, and $\theta(\mu(Y)) = \theta(Y)$, for all $Y \in Z$,

(iii) $F_\mu \cap Z_\theta \subset S$.

The Theorem 1 suggests us the following very general results

THEOREM 2. *Let (X,S,M) be a coincidence structure and (θ,μ) a compatible pair with (X,S,M). Let $Y \in \mu(Z)$ and $f,g \in M(Y)$ such that $f \circ g = g \circ f$.*

We suppose that

(i) $\theta(f(A) \cup g(A)) < \theta(A)$, *for all* $A \in I(f,g)$, $\alpha(A) \neq 0$;

(ii) $F_f \neq \phi$.

Then

$C(f,g)\neq\phi$.

Proof. Let $A_1=F_f$. From the Lemma 1 there exists $A_0\subset Y$ such that

$\mu(f(A_0)\cup g(A_0)\cup F_f)=A_0$.

since (θ,μ) is a compatible pair with (X,S,M), it follows

$\theta(\mu(f(A_0)\cup g(A_0)\cup F_f))=\theta(A_0)$.

This implies $\theta(A_0)=0$. Thus, $A_0\in F_\mu\cap Z_\theta$.

$S_0,A_0\in S$, i.e., $C(f,g)\neq\phi$.

Remark 1. In the Theorem 2, insted of the condition (ii), we can take the following

(ii') $x\in Y$ $A\in Z$ implies $A\cup\{x\}\in Z$ and $\theta(A\cup\{x\})=\theta(A)$.

Remark 2. For the θ-condensing mappings see: [2] [3].

Remark 3. For the coincidence theory, see [4].

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BIVARIATE BIRKHOFF INTERPOLATION OF SCATTERED DATA

GH. COMAN* and L. ŢÂMBULEA*

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REZUMAT. Interpolare Birkhoff bidimensională pentru date arbitrare. Se studiază formule de interpolare de tip Birkhoff pentru funcţii de două variabile definite pe un domeniu plan oarecare, obţinute prin generalizarea cazului rectangular.

0. In a previous paper [1] there was considered the following general scattered data interpolation problem (*SDIP*): let f be a real valued function defined on a given domain $D\subset\mathbb{R}^2$, $\Pi=\{D_k\subset D\,|\,k=1,\ldots,N\}$ a given partition of D and L_kf some given informations on the function f at D_k, $k=1,\ldots,N$. Find a function g, from a given set of functions, say A, such that $L_kg=L_kf$, $k=1,\ldots,N$.

Remark 1. The usual informations are the values or some medium values of the function f and of certain of its derivatives $f^{(\mu,\nu)}$, $(\mu,\nu)\in\mathbb{N}^2$.

Remark 2. If $\Pi=\{D_1,\ldots,D_N\}$ is a set of discret points then the (*SDIP*) is a punctual interpolation problem and it is a transfinite interpolation problem otherwise.

Particularly, if L_kf are the Lagrange informations ($L_kf = f(x_k,y_k)$) then the (*SDIP*) take the clasical fashion (the scattered data-fitting problem).

Remark 3. The (*SDIP*) can be also a deterministic or a non-deterministic problem if L_kf are deterministic or non-

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deterministic informations.

DEFINITION 1. The degree of exactness of the interpolation formula defined by the informations $L_k f$, $k=1,\dots,N$, will be called the exactness degree of these informations.

Remark 4. For the bivariate case we cane have the total degree of exactnees and the degree of exactness on regard with each variable.

We remark two ways for solving a (*SDIP*):

1) to generalize the tensor product or the Boolean sum techniques from a regular domain D (rectangle or triangle) to a unusual shape on.

2) to generalize or to modify the Shepard's method.

The goal of this note is to derive some scattered data interpolation formulas using first way and the Birkhoff informations of the function f.

1. For the begining, one supposes that $L_k f = f(x_k, y_k)$, $k=1,\dots,N$.

Now, if the partition Π is

$$\Pi=\{(x_i,y_i)\in D \mid i=0,1,\dots,m;\ j=0,1,\dots,n\}$$

then the solution of the corresponding (*SDIP*) is given by the tensor product of the univariate Lagrange operators L_m^x and L_n^y corresponding to the nodes x_i, $i=0,1,\dots,m$ respectively $y_j, j=0,1,\dots,n$, i.e.

$$(L_m^x \otimes L_n^y)(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{u(x)}{(x-x_i)\,u'(x_i)} \; \frac{v(y)}{(y-y_j)\,v'(y_j)} \; f(x_i,y_j)$$

where $u(x) = (x-x_0)\ldots(x-x_m)$; $v(y) = (y-y_0)\ldots(y-y_n)$.

In [5], J.F.Steffensen had given a first generalization of the Lagrange interpolation problem for the partition

$$\Pi = \{(x_i,y_j) \in D \mid i=0,1,\ldots,m;\; j=0,1,\ldots,n_i \text{ and } n_i \in \mathrm{N}\}$$

One obtains

$$(P_1 f)(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n_i} \frac{u(x)}{(x-x_i)\,u'(x_i)} \; \frac{v_i(y)}{(y-y_j)\,v_i'(y_j)} \; f(x_i,y_j)$$

where $v_i(y) = (y-y_0)\ldots(y-y_{n_i})$.

In 1957, D.D.Stancu [3] had given a new extension of the Lagrange interpolation problem, that is also a generalization of the Steffensen problem, taking

$$\Pi = \{(x_i,y_{ij}) \in D \mid i=0,1,\ldots,m;\; j=0,1,\ldots,n_i \text{ with } n_i \in \mathrm{N}\}$$

i.e.

$$(P_2 f)(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n_i} \frac{u(x)}{(x-x_i)\,u'(x_i)} \; \frac{v_i(y)}{(y-y_{ij})\,v_i'(y_{ij})} \; f(x_i,y_{ij})$$

with $v_i(y) = (y-y_{i,0})\ldots(y-y_{i,n_i})$.

We note that in both generalization are given expressions for the error functions $(f-P_i f, i=1,2)$ in terms of divided differences.

We, also, remark that $P_2 f$ is a solution for the clasical (*SDIP*), i.e. a Lagrange's scattered data interpolation polynomial.

Next, we consider the (*SDIP*) with the punctual Birkhoff's type informations on f.

2. Let $M=\{(x_k,y_k),\ k=1,\dots,N\}$ be a given set of points in D.

Following [1], on considers the partition M_i $i=0,1,\dots,p$ of the set M, where M_i is the set of all points $(x_k,y_k)\in M$ with $x_k=x_i$, $k=0,1,\dots,q_i$ and $x_i\neq x_j$ for $i\neq j$, i.e. $M_i=\{(x_i,y_{ij})\mid j=0,1,\dots,q_i\}$ for $i=0,1,\dots,p$.

Let $L^{\mu,\nu}_{i,i_j}=f^{(\mu,\nu)}(x_i,y_{ij}),\ j=0,1,\dots,q_i;\ i=0,1,\dots,p$ and $(\mu,\nu)\in I_i\times J_{ij}$ with $I_i,J_{ij}\subset\mathbb{N}$, be informations of the Birkhoff type of the function f, while $L^{\mu}_i f=f^{(\mu,0)}(x_i,.)$ respectively $L^{\nu}_j f=f^{(0,\nu)}(.,y_j)$ will be considered as partial informations of the function f on regard to x and y.

2.1. For the begining one considers the rectangular case, i.e. $D=[x_0,x_p]\times[y_0,y_q]$, $M=\{x_0,\dots,x_p\}\times\{y_0,\dots,y_q\}$ and $I_i,J_j\subset\mathbb{N}$, with $|I_0|+\dots+|I_p|=m+1$, $|J_0|+\dots+|J_q|=n+1$. If B^x_m and B^y_n are the Birkhoff's interpolation operators corresponding to the partial informations $L^{\mu}_i f=f^{(\mu,0)}(x_i,.)$, $i=0,1,\dots,p$; $\mu\in I_i$ respectively $L^{\nu}_j f=f^{(0,\nu)}(.,y_j),\ j=0,1,\dots,q$; $\nu\in J_j$ then the well known bivariate interpolation formula is

$$f=B^x_m\otimes B^y_n f+R^x_m\oplus R^y_n f\ , \tag{1}$$

where R^x_m and R^y_n are the corresponding remainder operators. More precisely

$$(B^x_m\otimes B^y_n f)(x,y)=\sum_{i=0}^{p}\sum_{j=0}^{q}\sum_{\mu\in I_i}\sum_{\nu\in J_j}b_{i\mu}(x)\,b_{j\nu}(y)\,f^{(\mu,\nu)}(x_i,y_j)$$

and for $f\in C^{m+1,n+1}(D)$,

$$(R_m^x \oplus R_n^y f)(x,y) = \int_{x_0}^{x_p} \phi_m(x,s) f^{(m+1,0)}(s,y)\,ds +$$

$$+\int_{y_0}^{y_q} \psi_n(y,t) f^{(0,n+1)}(x,t)\,dt - \iint_D \phi_m(x,s)\psi_n(y,t) f^{m+1,n+1}(s,t)\,dsdt$$

where $b_{i\mu}$ and $b_{j\nu}$ are the fundamental interpolation polinomials, while ϕ_m and ψ_n are the Peano's kernels.

Remarks 5. It is obviously that the degree of exactness of the formula (1) is (m,n) (m on regard to x and n on regard to y).

2.2. One considers now the general case.

So, $M=M_0\cup\ldots\cup M_p$ with $M_i=\{(x_i,y_{ij})\,|\,j=0,1,\ldots,q_i\}$. Let B_m^x be the same operator that interpolate the data $f^{(\mu,0)}(x_i,.)$ for $i=0,1,\ldots,p$ and $\mu\in I_i$ with $|I_0|+\ldots+|I_p|=m+1$. Using this operator we obtain, in a first level of interpolation, the formula

$$f = B_m^x f + R_m^x f \tag{2}$$

where

$$(B_m^x f)(x,y) = \sum_{i=0}^{p}\sum_{\mu\in I_i} b_{i\mu}(x) f^{(\mu,0)}(x_i,y)$$

Now, let $B_{n_i}^y$ be the Birkhoff's operators which interpolate, respectively the data $f^{(\mu,\nu)}(x_i,y_{ij}), j=0,1,\ldots,q_i$ and $\nu\in J_{ij}$ with $|J_{i,0}|+\ldots+|J_{i,q_i}|=n_i+1$ and $R_{n_i}^y$ the corresponding remainder operators, for all $i=0,1,\ldots,p$ and $\mu\in I_i$. Applying these operator, from (2) one obtains, in a second level of interpolation, the final scattered data interpolation formula

$$f(x,y)=\sum_{i=0}^{p}\sum_{j=0}^{q_i}\sum_{\mu\in I_i}\sum_{\nu\in J_{ij}} b_{i\mu}(x)\,b_{ij\nu}(y)\,f^{(\mu,\nu)}(x_i,y_{ij})+(Rf)(x,y) \tag{3}$$

with

$$(Rf)(x,y)=(R_m^x f)(x,y)\ +\sum_{i=0}^{p}\sum_{\mu\in I_i} b_{i\mu}(x)\,(R_{n_i}^y f)(x_i,y)\,.$$

PROPOSITION 1. *The degree of exactness of the formula (3) is (m,r), where $r=\min\{n_0,\ldots,n_p\}$.*

The proof is a consequence of the theorem 1 from [1].

From the Peano's kernel theorem we also have:

PROPOSITION 2. *If $f(.,y)\in H^{m+1}[x_0,x_p]$ and $f^{(\mu,n_i+1)}(x_i,.)\in H^{n_i+1}[y_{i_0},y_{i,n_i}]$ for all $i=0,1,\ldots,p$, then*

$$(Rf)(x,y)=\int_{x_0}^{x_p}\Phi_m(x,s)\,f^{(m+1,0)}(s,y)\,ds+$$
$$\sum_{i=0}^{p}\sum_{\mu\in I_i} b_{i\mu}(x)\int_{y_{i_0}}^{y_{i,n_i}}\Phi_{n_i}(y,t)\,f^{(\mu,n_i+1)}(x_i,t)\,dt$$

where

$$\Phi_m(x,s)=\frac{(x-s)_+^m}{m!}\ -\sum_{i=0}^{p}\sum_{\mu\in I_i} b_{i\mu}(x)\,\frac{(x_i-s)_+^{m-\mu}}{(m-\mu)!}$$

and

$$\Phi_{n_i}(y,t)=\frac{(y-t)_+^{n_i}}{n_i!}-\sum_{j=0}^{q_i}\sum_{\nu\in J_{ij}} b_{ij\nu}(y)\,\frac{(y_{ij}-t)_+^{n_i-\nu}}{(n_i-\nu)!}$$

Remark 6. From the first proposition it follows that the best case, from the degree of exactness point of view, is obtained for $n_0=n_1=\ldots=n_p$. In this case (3) is a homogeneous interpolation formula on regard to the variable y [1]. But, the structure of the interpolation formula depend on the given informations. So, if the initial informations do not permit to construct a homogeneous formula (there exists $i,j\in\{0,1,\ldots,p\}$, $i\neq j$ such that $n_i\neq n_j$) then there exist two posibilities: to generate new informations on f or to try to interpolate the function f first on regard with the variable y and than on regard with x. Anyhow, an interpolation formula as closed as possible of a homogeneous one is recomandable.

Summarizing the given procedure we have:

1. Input data:

$$M_i=\{(x_i,y_{ij})\mid j=0,1,\ldots,q_i\}\ ,i=0,1,\ldots,p;$$
$$I_i,J_{ij},;\ j=0,1,\ldots,q_i;i=0,1,\ldots,p;$$
$$f^{(\mu,\nu)}(x_i,y_{ij}),\ \mu\in I_i,\nu\in J_{ij},\ j=0,1,\ldots,q_i;i=0,1,\ldots,p.$$

2. One determines the fundamental interpolation polynomials $b_{i\mu}$ and b_{ij}, solving the linear algebraic systems:

$$b_{kj}^{(r)}(x_\nu)=0,\ \nu\neq k,\ r\in I_\nu,$$
$$b_{kj}^{(r)}(x_k)=\delta_{jr},\ r\in I_k,$$
$$\text{for } j\in I_k \text{ and } k,\mu=0,1,\ldots,p,$$

respectively

$$b_{ik\mu}^{(s)}(y_{ir})=0,\ r\neq k,\ s\in I_{ir},$$
$$b_{ik\mu}^{(s)}(y_{ik})=\delta_{\mu s},\ s\in I_{ik},$$
$$\text{for } \mu\in I_k;\ k,r=0,1,\ldots,q_i;$$
$$\text{for all } i=0,1,\ldots,p.$$

3. Compute $F(x,y)$;

$$F(x,y) = \sum_{i=0}^{p} \sum_{j=0}^{q_i} \sum_{\mu \in I_i} \sum_{\nu \in J_{ij}} b_{i\mu}(x)\, b_{ij\nu}(y)\, f^{(\mu,\nu)}(x_i, y_{ij}) \tag{4}$$

EXAMPLE. The test function is

$$f(x,y) = \frac{1}{x^2+y^2+1},$$

with the graph in fig.1. The input data are:

$$M_0=\{(-1,-1);(-1,0);(-1,1)\};$$
$$M_1=\{(-1/2,0)\};$$
$$M_2=\{(0,-1);(0,0);(0,1)\};$$
$$M_3=\{(1/2,0)\};$$
$$M_4=\{(1,-1);(1,0);(1,1)\}.$$

$$I_0=I_1=I_2=I_3=I_4=\{0\};$$
$$J_{00}=\{1\};\ J_{01}=\{0\};\ J_{02}=\{1\};$$
$$J_{10}=\{0,1,2\};$$
$$J_{20}=\{1\};\ J_{21}=\{0\};\ J_{22}=\{1\};$$
$$J_{30}=\{0,1,2\};$$
$$J_{40}=\{1\};\ J_{41}=\{0\};\ J_{42}=\{1\}.$$

So, it is used a Lagrange's interpolation with regard to x and a Birkhoff's interpolation with regard to y.

The graph of the interpolating surfaces computed by (4) is in fig.2.

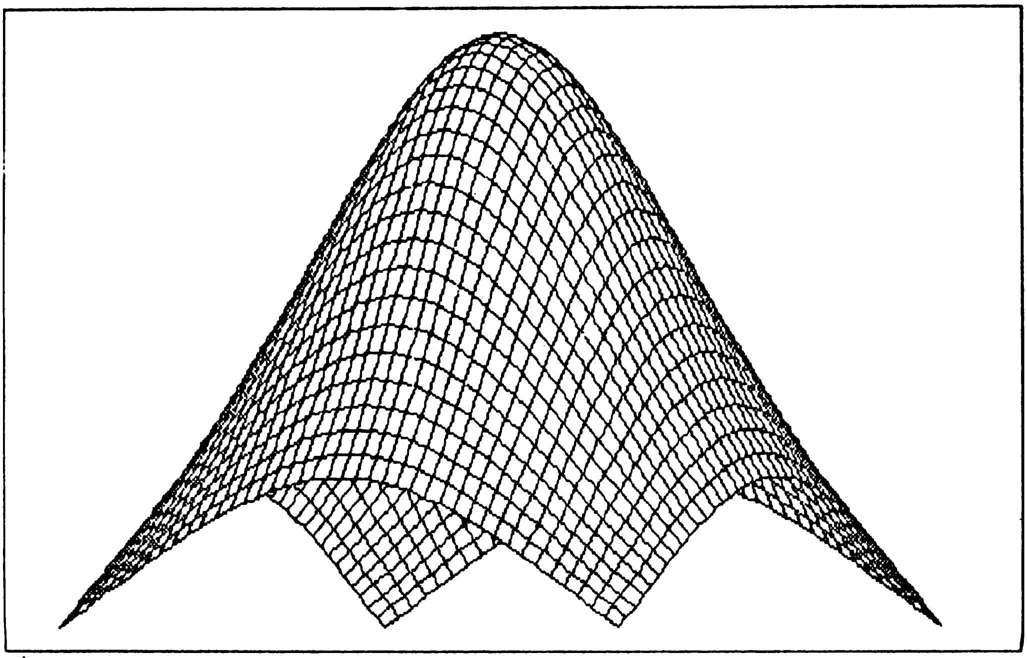

Fig.1.

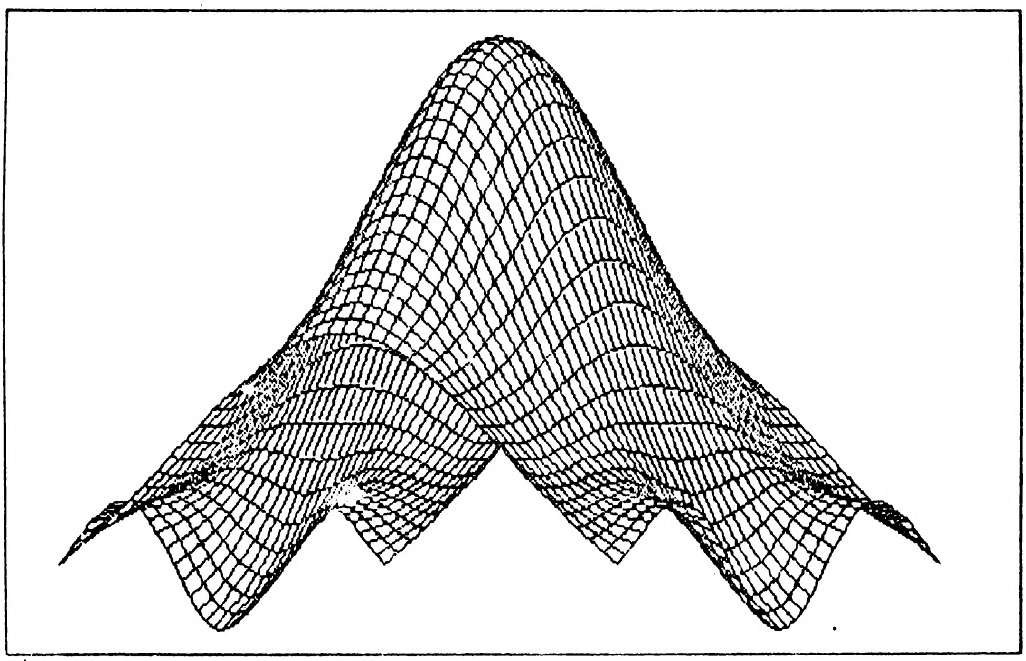

Fig.2.

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ON THE INITIALLY CIRCULAR MOTION AROUND A ROTATION ELLIPSOID

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Dedicated to Professor P.T.Mocanu at his 60^{th} anniversary

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REZUMAT. - Asupra mişcării circulare în jurul unui elipsoid de rotaţie. Se studiază mişcarea iniţial circulară a unei particule test în cîmpul gravitaţional necentral al unui elipsoid de rotaţie. Se stabileşte o formulă analitică pentru perioada mişcării, cu o precizie de ordinul al doilea în raport cu parametrul caracterizînd turtire elipsoidului, generalizîndu-se astfel rezultate anterioare (ale altor autori şi proprii).

1. **Introduction.** Consider a point mass orbiting an attracting body (under the only gravitational influence of this one) at a distance r. We shall describe the relative motion of the point mass with respect to a Cartesian right-handed frame originated in the mass centre of the attracting body by means of the Keplerian orbital elements $\{ y \in Y \; ; \; u \}$, all time-dependent, where:

$$Y = \{p, \; q = e \cos\omega, \; k = e \sin\omega, \; \Omega, \; i\}, \tag{1}$$

and p = semilatus rectum, e = eccentricity, ω = argument of pericentre, Ω = longitude of the ascending node, i = inclination, u = argument of latitude.

Many authors studied such a motion (for a brief survey see e.g. [2]) with very various hypotheses. First and (sometimes) second order perturbations of the orbital parameters were analytically estimated, as well as first order perturbations of the nodal or anomalistic period [1,2,6,7]. We must emphasize the

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fact that the anomalistic period cannot be used to the case of very low eccentric (and especially circular) orbits; that is why we use in this paper the nodal period. Also, as far as we know, nobody determined second order perturbations of the nodal period for a specified perturbing factor.

We shall estimate analytically the nodal period, with a second order accuracy in respect of a small parameter σ on which the perturbing factor is depending, in the following hypotheses:

(i) The attracting body is a rotation ellipsoid with a corresponding mass distribution.

(ii) The initial orbit of the point mass is circular.

(iii) The initial orbital elements are considered in the ascending node of the orbit.

2. **Equations of motion.** Considering hypothesis (i), let us choose the Cartesian right-handed frame mentioned in Section 1 as follows: The basic plane is the equatorial plane of the ellipsoid, while the third axis (normal to this plane) is the rotation axis. Since we study the nodal period, we describe the perturbed motion with respect to this frame by means of the Newton- Euler system written in the form (e.g. [3,5]):

$$\begin{aligned}
dp/du &= 2(Z/\mu)r^3T, \\
dq/du &= (Z/\mu)(r^3kBCW/(pD) + r^2T(r(q + A)/p + A) + r^2BS), \\
dk/du &= (Z/\mu)(-r^3qBCW/(pD) + r^2T(r(k + B)/p + B) - r^2AS), \\
d\Omega/du &= (Z/\mu)r^3BW/(pD), \qquad (2) \\
di/du &= (Z/\mu)r^3AW/p, \\
dt/du &= (Zr^2(\mu p)^{-1/2},
\end{aligned}$$

where μ=gravitational parameter of the dynamic system, $A=\cos u$, $B = \sin u$, $C = \cos i$, $D = \sin i$, $Z = (1 - r^2C\dot{\Omega}/(\mu p)^{1/2})^{-1}$, while S, T, W stand respectively for the radial, transverse, and binormal components of the perturbing acceleration.

For the needs of Section 4, it is to be specified that we consider, as usually, that the elements (1) have small variations over one revolution, such that they may be taken as constant and equal to $y_0 = y(u_0) = y(u(t_0))$, $y \in Y$, in the right-hand side of equations (2), and these ones can be separately considered. So, we can write $y = y_0 + \Delta y$, where, according to hypothesis (iii):

$$\Delta y = \int_0^u (dy/du)\,du, \quad y \in Y. \tag{3}$$

These integrals are estimated from (2) by successive approximations, with $Z \approx 1$, limiting the process to the first order approximation.

In what follows, for simplicity, we shall no longer use the subscript "0" to mark the initial values of elements (1) and of functions of them. In fact, every quantity which does not depend on u (explicitly or through A,B) will be considered constant over one revolution.

3. **Perturbing acceleration.** Since the gravitational field generated by the attracting body is not Newtonian, the point mass will undergo a perturbing acceleration. Having in view the hypothesis (i), the components of this acceleration are [2,7]:

$$S = -(3/2)c_{20}\mu R^2r^{-4}(3D^2B^2 - 1),$$
$$T = 3c_{20}\mu R^2r^{-4}D^2AB, \tag{4}$$

$$W = 3c_{20}\mu R^2 r^{-4} CDB,$$

where R = equatorial radius of the ellipsoid, while c_{20} is a small parameter featuring the oblateness.

4. Variations of orbital elements. Firstly remind the orbit equation in polar coordinates: $r = p / (1 + e \cos v)$, where v = true anomaly, or:

$$r = p / (1 + qA + kB). \tag{5}$$

Replacing (4) and (5) in (2), taking into account hypothesis (ii), in other words $q(0) = k(0) = 0$, then performing integrals (3) as we showed in Section 2, we obtain:

$$\begin{aligned}
\Delta p &= 3c_{20}\ (R/p)^2 pD^2B^2, \\
\Delta q &= (c_{20}/2)(R/p)^2(7D^2AB^2 + (2C^2 + 1)(1 - A)), \\
\Delta k &= (c_{20}/2)(R/p)^2(7D^2B^3 - 3B), \\
\Delta\Omega &= (3c_{20}/2)(R/p)^2C(u - AB), \\
\Delta i &= (3c_{20}/2)(R/p)^2CDB^2.
\end{aligned} \tag{6}$$

5. Nodal period. As we showed in [4], the nodal period can be written as:

$$T_N = T_0 + \Delta_1 T_N + \Delta_2 T_N, \tag{7}$$

where T_0 is the Keplerian period for $u = 0$; with hypothesis (ii):

$$T_0 = 2\pi p^{3/2}\mu^{-1/2}. \tag{8}$$

The first order (in σ) perturbation is [4]:

$$\Delta_1 T_N = p^{3/2}\mu^{-1/2}\int_0^{2\pi} [\ -2(J_q + J_k) + (3/2)p^{-1}J_p + P^2\mu^{-1}J_e]\ du, \tag{9}$$

where, with hypothesis (ii):

$$J_p = \Delta p, \quad J_q = A\Delta q, \quad J_k = B\Delta k, \quad J_\sigma = B\sigma(CW/D)_\sigma. \tag{10}$$

As to the second order (in σ) perturbation, this one has the expression, according to [4]:

$$\Delta_2 T_N = p^{3/2}\mu^{-1/2}\int_0^{2\pi}[3(J_{qq} + J_{kk} + 2J_{qk}) - 3p^{-1}(J_{pq} + J_{pk}) + \\ + (3/8)p^{-2}J_{pp} + (7/2)p\mu^{-1}J_{p\sigma} + \\ + p^2\mu^{-1}\cdot(-5J_{q\sigma} - 5J_{k\sigma} + J_{\Omega\sigma} + J_{i\sigma}) + \\ + (1/2)p^4\mu^{-2}J_{\sigma\sigma}]\,du, \tag{11}$$

where, with hypothesis (ii):

$$J_{xy} = J_x J_y, \quad x \in \{p,q,k\}, \quad y \in \{p,q,k,\sigma\}, \tag{12}$$

$$J_{\Omega\sigma} = \Delta\Omega(J_\sigma)_\Omega, \quad J_{i\sigma} = \Delta i(J_\sigma)_i, \quad J_{\sigma\sigma} = B^2\sigma^2(C^2W^2/D^2)_{\sigma\sigma}. \tag{13}$$

We must emphasize the fact that the subscript σ in the right-hand side of the last formula (10), and the subscripts Ω, i, and $\sigma\sigma$ in the right-hand sides of (13) mark the respective partial derivatives. As to the subscripts added to J in (9) - (13), they are simple identifying notations.

6. Results. Substituting W from (4) in the last formula (10) and calculating the required partial derivative (the part of σ is played by c_{20}), then substituting (6) in (10) and the results in (9), and finally performing the integral (9), we obtain:

$$\Delta_1 T_N = 3\pi c_{20} R^2 p^{-1/2}\mu^{-1/2}(3 - 5D^2/2). \tag{14}$$

Analogously, replacing W in the last formula (13) and calculating the partial derivative ($\sigma = c_{20}$, too), then introducing (6) and the previously calculated (10) in (12) -

(13), substituting the results in (11) and performing the integral, we obtain:

$$\Delta_2 T_N = (\pi/32)\, c_{20}^2 R^4 p^{-5/2} \mu^{-1/2} (1527D^4 - 3180D^2 + 1620). \qquad (15)$$

We must mention that (14) confirms the results of [2,7], while the result (15) is entirely new. Moreover, this result constitutes a first application of our formulae given in [4] to the case of a concrete perturbation.

With (8), (14), and (15), the nodal period (7) can be written as:

$$T_N = T_0(1 + Kf_1(D) + K^2 f_2(D)), \qquad (16)$$

where $K = c_{20}(R/p)^2$, and:

$$f_1(D) = (18-15D^2)/4, \quad f_2(D) = (1527D^4-3180D^2+1620)/64. \qquad (17)$$

This new, better approximation for the real (perturbed) nodal period could be very useful in the case in which the ellipsoid is strongly oblate and the point mass orbits in its immediate neighbourhood. According to K and to the orbital inclination, the contribution of f_2 (which can act as f_1 or inversely) in altering the period could be sensible.

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ON A COMPLEX BOUNDARY ELEMENT METHOD FOR THE "WALL EFFECT"

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REZUMAT. - Asupra unei metode de element pe frontieră cu valori compexe pentru "efectul de perete". Prima parte a lucrării conţine o trecere în revistă a unor consideraţii matematice legate de mişcaroa fluidă generată de deplasarea unui profil în prezenţa unui perete nelimitat, rezultate ale autorului care au fost deja prezentate pe larg în [2]. Partea a doua dezvoltă o metodă de element pe frontieră cu valori complexe (CVBEM), pentru care se stabilesc o schemă de utilizare în problema propusă ca şi un rezultat final de convergenţă.

1. Let us consider as given a plane incompressible, potential, inviscid fluid "basic" flow of complex velocity $w_B(z)$. This fluid flow could have some singularities, too, takes place in the presence of an unlimited fixed wall δ.

Let now the plane fluid flow, produced by a general displacement (rototranslation) in the mass of an arbitrary profile (C), in the presence of the same wall δ and which superposes over the basic flow. We assume that during its displacement the profile (C) doesn't cross the singularities of the given basic flow.

A general method to determine the fluid flow which results from the mentioned superposition, method establishing also the existence and the uniqueness of the solution of the joined mathematical model, has been already developed by us [2]. In what follows we intend to make a sketch of a complex variable boundary element method (CVBEM) which could easily be used for the studied

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problem and whose convergence has been already established in the case of the unbounded flow [3, 4].

Concerning the unlimited wall δ and the contour C, we suppose that their parametrical equations $z=\alpha(\varphi)$ and respectively $z=\beta(\psi)$, defined for $\varphi,\psi\in E_1$ versus a fixed, rectangular, Cartesian system of axes, are 2π periodical functions on the interval $(0,2\pi)$, with $\alpha(0) = \infty$ and $\beta(0)$ taking a finite value, which define Jordan positively oriented curves with continuous curvature[1].

In what concerns the given function $w_B(z)$, it belongs to a class (a) of functions having the properties [2]:

1a) They are holomorphic functions in the domain D, bounded by the wall δ, except a finite number of points $\{z_k\}_{k=\overline{1,q}}$ placed at a finite distance, and which represent singular points of these functions; let $D_1^*=D_1\setminus\{z_k\}_{k=\overline{1,q}}$;

2a) They are continuous bounded functions in $V(\infty)$; let $\lim_{|z|\to\infty} w_B(z) = w_B(\infty)$;

3a) They are höllderian functions in the points of $\delta\setminus\{\infty\}$ satisfying also the following boundary condition:

$\exists\ V_B:(0,2\pi) \to \mathbf{R}$ so that $\overline{w_B(\alpha(\varphi))}=V_B(\varphi)\dot\alpha(\varphi)/|\dot\alpha(\varphi)|$, $\forall\varphi\in(0,2\pi)$

With regard to the unknown function $w(z)$, the complex velocity of the fluid resulting by the considered superposition, must be determined among the functions of class (b), i.e.:

1b) They are holomorphic functions in the domain $D=D_1\setminus\{\overline{\mathrm{Int}\ C}\}$

[1] This last condition is equivalent to the assumption that the functions $1/(\alpha(\varphi)-z_0)$ (where z_0 is a point placed on the "right" side of δ) and $\beta(\psi)$ are from $C^2[0,2\pi)$ having also a nonvanishing first derivative.

except the same points $\{z_k\}_{k=\overline{1,q}}$ which are singular points of the same nature as for $w_B(z)$;

2b) They are continuous bounded functions in $V(\infty)$ where they have an identical behaviour with $w_B(z)$ and consequently

$$\lim_{|z|\to\infty} w(z) = w_\infty \equiv w_B(\infty);$$

3b) They are holomorphic functions in the points of $C \cup \delta \setminus \{\infty\}$ where these functions also satisfy the following boundary conditions:

$$\exists\, V_1 : (0,2\pi) \to \mathbb{R} \text{ so that } \overline{w(\alpha(\varphi))} = V_1(\varphi)\dot{\alpha}(\varphi)/|\dot{\alpha}(\varphi)|, \quad \forall\varphi\in(0,2\pi);$$

$$\exists\, V_2 : [0,2\pi) \to \mathbb{R} \text{ so that } \overline{w(\beta(\psi))} = V_2(\psi)\dot{\beta}(\psi)/|\dot{\beta}(\varphi)| + 1 + im + i\omega(\beta(\psi) - z_A)$$
$$\forall\psi\in[0,2\pi),$$

where $(1,m,\omega)$ are the given functions of time corresponding to the components of the rototranslation of the profile (C) evaluated in the point $z_A \in \{\text{Int } C\}$;

4b) They satisfy the equality:

$$\int_C w(z)\, dz = \Gamma,$$

where Γ is an "a priori" given constant (circulation).

2. As a consequence of the requirements imposed on the functions $w_B(z)$ and $w(z)$ we remark that the function $g(z) \equiv w(z) - w_B(z)$, known together with $w(z)$, is:

- holomorphic in the fluid flow domain D which also contains the points $\{z_k\}_{k=\overline{1,q}}$;

- continuous and bounded in $\bar{D}$ (the point of infinity included, where $\lim_{|z|\to\infty} g(z) = 0$);

- hölderian on $C \cup \delta\setminus\{\infty\}$;

- satisfying the condition $|z^{\tau}g(z)|_{\delta} < A$, where (A,τ) is a suitable pair of real numbers.

The last condition will ensure the existence (in the Cauchy sense) of the integral taken on the unlimited curve δ [2].

Let us consider now Cauchy's formula for the function $g(z)$ and the domain D. According to the behaviour at far field of this function, we can write [1]:

$$g(z) = -\frac{1}{2\pi i}\int_{C}\frac{g(\zeta)}{\zeta - z}d\zeta + \frac{1}{2\pi i}\int_{\delta}\frac{g(\tau)}{\tau - z}d\tau \ .$$

This formula, which is in fact the integral representation associated with the proposed boundary problem, allows us to determine $g(z)$ (i.e. $w(z)$) once found out its values on the boundary $C \cup \delta$. But the determination of these values through a classical BEM requires the construction and, obviously, the solution of an integral equation on boundary, which could be obtained, for instance, making $z \to \zeta_0 \in C$ and $z \to \tau_0 \in \delta$, respectively.

In what follows we shall succeed to avoid the construction and solution of the boundary integral equations mentioned above, which means a serious and essential step in simplifying all algorithms. The technique used and described by us in the case of the unbounded fluid [3, 4] represents a so-called "improved" CVBEM.

Let d and d' be two divisions of the curves C and δ, consisting of the nodes $z_0, z_1, \ldots, z_n$ $(z_0 = z_n)$ on C (counterclockwise oriented) and, respectively, $z_1', z_2', \ldots, z_n'$ on δ (clockwise oriented). We denote by C_j $(j=1,\ldots,n)$ the corresponding boundary elements (arcs) on C, and by $C_{-\bullet}'$, C_j'

$(j=1,\dots,n)$, C'_{*} the boundary elements on δ. Let us consider now the approximations $\mathfrak{g}_d(\zeta)$ in the points of C, and, respectively, $\mathfrak{g}'_{d'}(\tau)$ in the points of $\delta\backslash\{C'_{*}\cup C'_{*}\}$ of the function g(z), where $\mathfrak{g}_d(\zeta)$ and $\mathfrak{g}'_{d'}(\tau)$ are suitable interpolating spline functions related to the divisions d and d' accordingly.

At once, for every $z \in D$, we have the approximation $g^*(z)$ of $g(z)$, i.e.

$$g^*(z) = -\frac{1}{2\pi i}\int_C \frac{\mathfrak{g}_d(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\int_{[z'_1,z'_n]} \frac{\mathfrak{g}'_{d'}(\tau)}{\tau - z}\,d\tau \ .$$

where the right side could be calculated explicitly sometimes even precisely [3, 4]. Accepting then the existence of

$\lim_{z\to z_k} g^*(z)\,(=g^*(z_k))$ and $\lim_{z\to z'_k} g^*(z)\,(=g^*(z'_k))$,

by separating the real and imaginary parts of the approximate equalities

$g_k = g(z_k) = g^*(z_k)$ and $g'_k = g(z'_k) = g^*(z'_k)$,

we finally get an agebraic system in the unknowns u_k, v_k, u'_k, v'_k, i.e. the real and imaginary parts of g_k and g'_k. Obviously, while solving this system we should take into account the data connected with the values of $g(z)$ on $C \cup \delta$ (in fact it is a boundary value problem of Hilbert type for $g(z)$) and, of course, the circulation given "a priori"[2]. Once solved this system, via the already written Cauchy formula, one gets the approximate solution $g^*(z)$ valid in all the points of the flow domain.

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[2] In the particular case of a "piecewise" Lagrange interpolating system, which is essentially an approximation by spline functions of first order, the algebraic system becomes linear and it has a unique solution, due to the "a priori" given circulation.

In respect of the convergence of the method, if we admit the "acceptability" of the divisions d and d' (i.e. for every $t \in C_j$ or $t \in C_j'$ there exists $\max\{|t-z_j|, |t-z_{j-1}|\} < |z_j - z_{j-1}|, j=1,\ldots,n$), the uniform continuousness of the approximation $\tilde{g}$ in the points of $C \cup [z_1', z_n']$ allows to prove, like in the case of the unbounded flow [3, 4], the following final result:

THEOREM. *For every point $z \in D$,*

$$\lim_{\substack{n \to \infty \\ (\delta, \delta' \to 0)}} g^*(z) = g(z) ,$$

where δ and δ' are the norms of the acceptable divisions d and d', respectively.

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A N I V E R S Ă R I

PROFESSOR PETRU T. MOCANU AT HIS 60th ANNIVERSARY

Professor Petru T. Mocanu was born in June 1st 1931 in Brăila, Romania. He attended primary and secondary school in Brăila, then universitary studies (1950-1953) and higher studies (1953-1957) at the Faculty of Mathematics, University of Cluj. In 1959 he defended his doctoral thesis "Variational methods in the theory of univalent functions" (under the supervision of the great romanian mathematician G. Călugăreanu). He worked at the University of Cluj ("Babeş-Bolyai University) as assistant professor (1953-1957), lecturer (1957-1962), associate professor (1962-1966 and 1967-1970), full professor (since 1970) and he was visiting professor at Conakry (1966-1967). He has tought the basis course of Complex Analysis and many special courses (Univalent Functions, Measure Theory, Hardy Spaces etc.).

Professor P.T.Mocanu obtained scientific results in the following domains (see "List of publications, Scientific papers"): **extremal problems in the theory of univalent functions** [1-5, 8, 9, 12, 13, 15, 17, 18, 20-22, 24, 25, 27, 37, 42, 47, 72, 91], **new classes of univalent functions** [7, 10, 14-16, 19, 23, 26-32, 34-36, 38, 40, 41, 43, 46, 65, 70, 95, 99, 102, 103, 109], **integral operators on classes of univalent functions** [44, 45, 48, 55-58, 62-64, 66, 67, 69-71, 73, 76, 78, 80-83, 88-90, 92-94, 96-98, 100, 102, 103, 106, 110, 111], **differential subordinations** [49, 50, 52, 53, 61, 64, 68, 74, 77, 79, 85-87, 101, 104, 105, 107], **extentions of certain geometric conditions for injectivity to the case of nonanalytic functions** [51-54, 59, 60, 75, 95]. Some of this results are cited by about 150 mathematicians in more then 300 papers.

Professor P.T.Mocanu was appointed dean of the Faculty of Mathematics (1968-1976 and 1984-1987), head of the chair of Functions Theory (1976-1984 and since 1990) and he is vice-rector of Babeş-Bolyai University since 1990. He is also editor of Mathematica (Cluj) and member of the editorial board of Studia Univ. Babeş-Bolyai, Bulletin de Mathématiques and Gazeta Matematică.

Since 1972 Professor P.T.Mocanu has been a guide of doctorands (twelve students had taken Ph.D. degrees and other ten are prepering their dissertations). He is the chairman of the Seminar on Geometric Function Theory at Babeş-Bolyai University and the head of the Romanian School of Univalent Functions.

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S U M A R - C O N T E N T S - S O M M A I R E

A BINARY TREE CLASIFIER BASED ON FUZZY SETS

IOANA MARIA BOIER*

Dedicated to Professor P. T. Mocanu on his 60-th aniversary

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Rezumat. Arbore binar de clasificare bazat pe mulţimi fuzzy. În această lucrare este descris un algoritm de proiectare şi implementare a unui clasificator binar. Acest algoritm îşi propune îmbunătăţirea algoritmului propus de Fu şi Mui [3]. O mulţime de date de test este utilizată în construcţia clasificatorului. În abordarea acestei probleme, Fu şi Mui folosesc proiecţia datelor în plan şi inspecţia vizuală ca metode de separare a clusterilor. Abordarea de faţă propune o separare automată, bazată pe mulţimi fuzzy.

The design of the binary tree classifier.

A method to design a binary tree classifier has been proposed in [3]. According to Fu and Mui, there are three major tasks to be implemented, to design a binary tree classifier:

a) a tree skeleton or hierarchical ordering of class labels

b) the choice of features at each nonterminal node

c) the decision rule to be used at each nonterminal node.

These tasks involve the specification of the following parameters:

a) the number of descendant nodes at each nonterminal node

b) the number of features used at each nonterminal node

c) an appropriate decision rule to be considered at each nonterminal node.

Since any conventional single stage classification scheme can be represented by a binary tree classifier which has exactly two immediat descendant nodes for each nonterminal node [3], we

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consider the number of descendant nodes at each nonterminal node to be two. The next parameter to be specified is the maximum number of features used at each nonterminal node. This number depends on the specific classification problem and it is a constant for the problem. Let us denote it by K. To determine K, the number of all features, the size of test sample and the average number of samples per class are to be considered. The decision rule chosen at each nonterminal node is:

if $d(X,L^1) \le d(X,L^2)$ then X is classified into calss A_1 (1)

otherwise X is classified into class A_2,

where X is the feature vector of the unknown sample to be classified, L^i is the prototype of the class A_i (i=1,2) [1] and d is a norm induced by distance in $\mathbb{R}^p$:

$$d(x,y)=\|x-y\|.$$

The next steps we have to perform are to design the tree skeleton or hierarchical ordering of class labels and to establish the actual features used at each nonterminal node. The fundamental problem which appears when the tree skeleton is built is the separation of the two groups of classes in each nonterminal node and the choice of features which are effective in separating these groups of classes. But, generally, the choise of the most effective features depends on the classes to be separated and the separation of the classes depends on what features are used. A method to break this deadlock is proposed in what follows. Using General Fuzzy Isodata algorithm [1] a fuzzy class is divided into two groups. Then, a method similar to the one presented by Fu and Mui [3] is used to chose the features which are "most effective" in separating the two groups of classes.

Let us assume that the predetermined number of classes is n and that the classes are labeled 1, 2,...,n. We also assume that the dimension of the features space is p. Suppose we have reached with the construction of the tree skeleton to a nonterminal node. Let C be the fuzzy set describing the membership degrees of class label i to this node, for all i from 1 to n. For example, at the beginning, when the nonterminal node is the root, the membership degrees are $C(i)=1.0$ for all i from 1 to n. Further, using the General Fuzzy Isodata algorithm, a fuzzy partition [1] $P = \{A_1, A_2\}$ of C is detected. According to the definition of a fuzzy partition, we have:

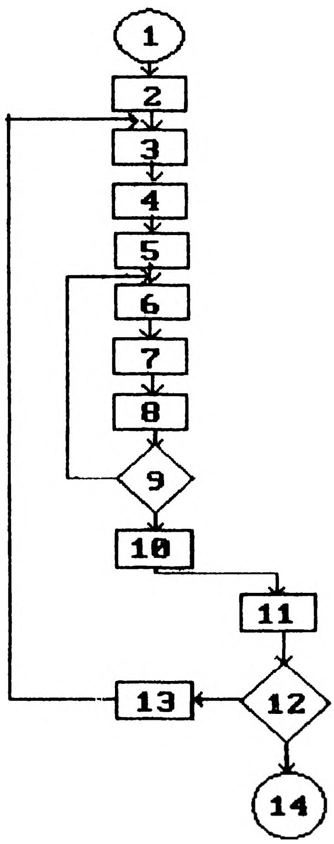
$$C(i) = A_1(i) + A_2(i) \ , \ i=\overline{1,n}$$

For the classification accuracy, the following correction rule is used:

$$\textit{if } A_j(i) < 0.1 \text{ then } A_{2-j+1}(i)=C(i) \text{ and } A_j(i)=0.0 \ , \ i=\overline{1,n}, \ j=1,2$$

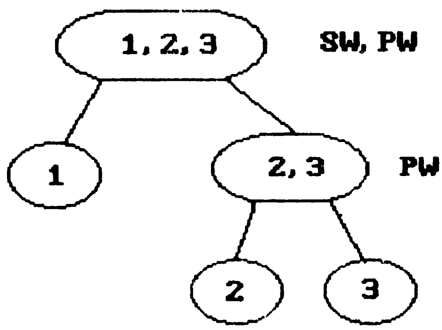
In determining the partition P, we use n feature vectors, representing the mean values of the features for each of the n classes. However, it is possible that not all the p features are needed to split the class C into A_1 and A_2. Using the set of test samples, we shall find the "best" up to K features in separating the two groups of classes. First, the best single feature is selected an this feature is used to perform classification based on the decision rule [1]. The result of the classification is computed and represents the number of test samples well classified. The "best 2" up to the "best K" feature subsets are obtained. The feature subset which give the best classification result of the K "best" feature subsets is chosen as the feature subset for the node considered. When an unknown sample to be classified reaches this node and we use the decision rule to go further, only those features from the feature vector of the unknown sample which correspond to the feature subset associated with the current node will be considered in order to compute the distances to the prototypes of the descendants.

The flowchart which describes the binary tree classifier design process is given below:



1. Start
2. Find the mean values of the features for each of the n classes
3. Obtain separable clusters using General Fuzzy Isodata algorithm
4. If needed, use the correction rule
5. $l = 1$
6. Find the "best l" features
7. Perform classification using these l features
8. $l = l + 1$
9. Is $l > K$?
10. Find the best classification result from the result corresponding to each of the K "best" feature subsets
11. Use the best result obtained to build up the decision tree
12. No new nonterminal node?
13. Get a new nonterminal node
14. Stop

Results. The method described above has been used to design a binary tree classifier for the classification of 147 samples of Iris spread over 3 classes [2]: Iris Setosa, Iris Virginica and Iris Versicolor. There are 4 characteristics taken into consideration: petal width (*PW*), petal length (*PL*), sepal width (*SW*), sepal length (*SL*). Considering for each of the 3 classes the mean values of the 4 characteristics listed below and the set of test samples as consisting of the first 20 samples from each class listed in Anex *A*, the following tree classifier is obtained ($K=2$):



Although all the lass labels (1,2,3) appear in each node, only those which have the membership degree to the node i ($i=1,\ldots,5$) greater than have been represented for the node i in the figure above. Beside each nonterminal node is the set of features used.

	Setosa	Versicolor	Virginica
PW	0.2	1.4	2.5
PL	1.4	4.7	6.0
SW	3.5	3.2	3.3
SL	5.1	7.0	6.3

mean values of the 4 characteristics

The classification results are as follows:

	samples no.	well classified	percent
Setosa	49	49	100%
Virginica	49	26	51.02%
Versicolor	49	49	100%
Total	147	127	85.03%

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Anexa A

Iris Setosa				Iris Versicolor				Iris Virginica			
SL	SW	PL	PW	SL	SW	PL	PW	SL	SW	PL	PW
5.1	3.5	1.4	0.2	7.0	3.2	4.7	1.4	6.3	3.3	6.0	2.5
4.9	3.0	1.4	0.2	6.4	3.2	4.5	1.5	5.8	2.7	5.1	1.9
4.7	3.2	1.3	0.2	6.9	3.1	4.9	1.5	7.1	3.0	5.9	2.1
4.6	3.1	1.5	0.2	5.5	2.3	4.0	1.3	6.3	2.9	5.6	1.8
5.0	3.6	1.4	0.2	6.5	2.8	4.6	1.5	6.5	3.0	5.8	2.2
5.4	3.9	1.7	0.4	5.7	2.8	4.5	1.3	7.6	3.0	6.6	2.1
4.6	3.4	1.4	0.3	6.3	3.3	4.7	1.6	4.9	2.5	4.5	1.7
5.0	3.4	1.5	0.2	4.9	2.4	3.3	1.0	7.3	2.9	6.3	1.8
4.4	2.9	1.4	0.2	6.6	2.9	4.6	1.3	6.7	2.5	5.8	1.8
4.9	3.1	1.5	0.1	5.2	2.7	3.9	1.4	7.2	3.6	6.1	2.5
5.4	3.7	1.5	0.2	5.0	2.0	3.5	1.0	6.5	3.2	5.1	2.0
4.8	3.4	1.6	0.2	5.9	3.0	4.2	1.5	6.4	2.7	5.3	1.9
4.8	3.0	1.4	0.1	6.0	2.2	4.0	1.0	6.8	3.0	5.5	2.1
4.3	3.0	1.1	0.1	6.1	2.9	4.7	1.4	5.7	2.5	5.0	2.0
5.8	4.0	1.2	0.2	5.6	2.9	3.6	1.3	5.8	2.8	5.1	2.4
5.7	4.4	1.5	0.4	6.7	3.1	4.4	1.4	6.4	3.2	5.3	2.3
5.4	3.9	1.3	0.4	5.6	3.0	4.5	1.5	6.5	3.0	5.5	1.8
5.1	3.5	1.4	0.3	5.8	2.7	4.1	1.0	7.7	3.8	6.7	2.2
5.7	3.8	1.7	0.3	6.2	2.2	4.5	1.5	7.7	2.6	6.9	2.3
5.1	3.8	1.5	0.3	5.6	2.5	3.9	1.1	6.0	2.2	5.0	1.5
5.4	3.4	1.7	0.2	5.9	3.2	4.8	1.8	6.9	3.2	5.7	2.3
5.1	3.7	1.5	0.4	6.1	2.8	4.0	1.3	5.6	2.8	4.9	2.0
4.6	3.6	1.0	0.2	6.3	2.5	4.9	1.5	7.7	2.8	6.7	2.0
5.1	3.3	1.7	0.5	6.1	2.3	4.7	1.2	6.3	2.7	4.9	1.8
4.8	3.4	1.9	0.2	6.4	2.9	4.3	1.3	6.7	3.3	5.7	2.1
5.0	3.0	1.6	0.2	6.6	3.0	4.4	1.4	7.2	3.2	6.0	1.8
5.0	3.4	1.6	0.4	6.8	2.8	4.8	1.4	6.2	2.8	4.8	1.8
5.2	3.5	1.5	0.2	6.7	3.0	5.0	1.7	6.1	3.0	4.9	1.8
5.2	3.4	1.4	0.2	6.0	2.9	4.5	1.5	6.4	2.8	5.6	2.1
4.7	3.2	1.6	0.2	5.7	2.6	3.5	1.0	7.2	3.0	5.8	1.6
4.8	3.1	1.6	0.2	5.5	2.4	3.8	1.1	7.4	2.8	6.1	1.9
5.4	3.4	1.5	0.4	5.5	2.4	3.7	1.0	7.9	3.8	6.4	2.0
5.2	4.1	1.5	0.1	5.8	2.7	3.9	1.2	6.4	2.8	5.6	2.2
5.5	4.2	1.4	0.2	6.0	2.7	5.1	1.6	6.3	2.8	5.1	1.5
4.9	3.1	1.5	0.2	5.4	3.0	4.5	1.5	6.1	2.6	5.6	1.4
5.0	3.2	1.2	0.2	6.0	3.4	4.5	1.6	7.7	3.0	6.1	2.3
5.5	3.5	1.3	0.2	6.7	3.1	4.7	1.5	6.3	3.4	5.6	2.4
4.9	3.6	1.4	0.1	6.3	2.3	4.4	1.3	6.4	3.1	5.5	1.8
4.4	3.0	1.3	0.2	5.6	3.0	4.1	1.3	6.0	3.0	4.8	1.8
5.1	3.4	1.5	0.2	5.5	2.5	4.0	1.3	6.9	3.1	5.4	2.1
5.0	3.5	1.3	0.3	5.5	2.6	4.4	1.2	6.7	3.1	5.6	2.4
4.5	2.3	1.3	0.3	6.1	3.0	4.6	1.4	6.9	3.1	5.1	2.3
4.4	3.2	1.3	0.2	5.8	2.6	4.0	1.2	5.8	2.7	5.1	1.9
5.0	3.5	1.6	0.6	5.0	2.3	3.3	1.0	6.8	3.2	5.9	2.3
5.1	3.8	1.9	0.4	5.6	2.7	4.2	1.3	6.7	3.3	5.7	2.5
4.8	3.0	1.4	0.3	5.7	3.0	4.2	1.2	6.7	3.0	5.2	2.3
5.1	3.8	1.6	0.2	5.7	2.9	4.2	1.3	6.3	2.3	5.0	1.9
4.6	3.2	1.4	0.2	6.2	2.9	4.3	1.3	6.5	3.0	5.2	2.0
5.3	3.7	1.5	0.2	5.1	2.5	3.0	1.1	6.2	3.4	5.4	2.3

ON INDEPENDENT SETS OF GRAPHS

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Rezumat. - Asupra seturilor independente de grafe. Lucrarea trece în revistă unii algoritmi de determinare a mulţimilor independente (mulţimi interior stabile) referitoare la un graf. În prima parte se prezintă cîţiva algoritmi care au la bază expresii şi/sau ecuaţii booleene precum şi un algoritm recursiv şi anume algoritmul dat de Taulbee şi Bednarek. În final autorii dau un algoritm recursiv inspirat din acest ultim algoritm.

1. **Definition, properties.** Let $G = (V,T)$ be an undirected graph where:

- V is the set of vertices and $|V| = n$;

- $T : V \to V$ is the aplication which defines the graph.

DEFINITION : Let $S \subset V$. S is an independent set (IS) iff $\forall v \in S$, $T_v \cap S = \emptyset$.

(where we denote $T(v)$ by T_v).

In other words the vertices of S don't have any edges between each other.

Observations:

1^0 We may define $G = (V,E)$ where E is the set of edges, $E \subset \subset V \times V$, an edge is $[x,y]$, $x,y \in V$ and $[x,x] \notin E$.

2^0 Let S be an IS. S is called maximal if S is maximal by sets inclusion.

3^0 We denote by $\mathbf{S}$ the set of all maximal IS of G.

We remember that:

a) $\alpha(G)$ is the number of internal stability:

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$$\alpha(G) = \max_{S \in \mathcal{S}} |S|$$

b) $\gamma(G)$ is the chromatic number of G, $\gamma(G)$ is the smallest number of IS, disjoints, which cover G.

4^0 Moon and Moser have proved that:

$$\alpha(G) \le \begin{cases} 3^{\frac{n}{3}} & ,\text{if } n=3k \\ 4*3^{\frac{n-1}{3}-1} & ,\text{if } n=3k+1 \\ 2*3^{\frac{n-2}{3}} & ,\text{if } n=3k+2 \end{cases}$$

2. **Algorithms for determining *IS*.** In many problems it is important to find the *IS* family.

There are some algebraic or combinatorial algorithms to find *IS*.

2.1. Maghout and Weissman'S algorithm based on boolean expression.

2.2. Malgrange algorithm's which finds every squared matrix containing only 0 (zero) of the adjacent matrix where:

$$A = (a_{i,j}) \; ; \; i=\overline{1,n} \; ; \; j=\overline{1,n} \text{ with}$$

$$a_{ij} = \begin{cases} 1 & ,\text{if } [v_i, v_j] \in E \\ 0 & ,\text{otherwise} \end{cases}$$

2.3. The Rudeanu's method, using the boolean equations which caracterize the *IS* family.

Let $G = (V,E)$, where $V = \{v_1,\dots,v_n\}$.

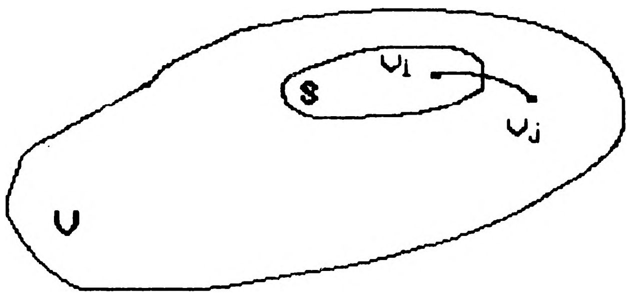
If $S \subset V$ is an *IS* then we associate to each $v_i \in V$ an boolean variable b_i define by :

$$b_i = \begin{cases} 1 & ,\text{if } v_i \in S \\ 0 & ,\text{if } v_i \notin S \end{cases}$$

We have the following result:

If $a_{ij} = 1$ then it results $a_{ij}.b_i.b_j = 0$ (1)

(because $v_i \in S$ and $v_j \notin S$, see the diagram).



So from (1) it results that $\bigvee_{a_{ij}=1} b_i \cdot b_j = 0$ iff $\bigwedge_{a_{ij}=1} \overline{b_i \cdot b_j} = 1$ iff $\bigwedge_{a_{ij}=1} (\overline{b_i} \vee \overline{b_j}) = 1$ iff $\bigvee \overline{b_{k_1}} \cdot \overline{b_{k_2}} \ldots \overline{b_{k_p}} = 1$

So, for each factor $\overline{b_{k_1}} \cdot \overline{b_{k_2}} \ldots \overline{b_{k_p}} = 1$ we have an *IS* :

$$S = \{x_{k_{p+1}}, x_{k_{p+2}}, \ldots, x_{k_n}\}.$$

2.4. Bednarek and Taulbee's recursiv algorithm

Let $G = (V,E)$ be an undirected graph:

- $\forall\, k = 1,\ldots,n$ we denoted by $V_k = \{v_1,\ldots,v_k\}$;
- for each subgraph with $V_k = \{v_1,\ldots,v_k\}$; we denoted by L_k the maximal *IS* family;
- we also denote $Y_k = \{y \in V_k \,/\, [x_k, y] \in E\}$.

The steps of the algorithm are:

S1. Let $Y_1 = \{v_1\}$, $L_1 = \{v_1\}$, $k=1$.

S2. One finds the next family: $I_k=\{S/S=M \cap Y_{k+1},\ M \in L_k\}$.

S3. One finds $I'_k = \{I\ /\ I \subset I_k,\ I$ maximal with respect to sets inclusion}.

S4. One finds L^*_{k+1} family, for each $M \in L_k$:

a) if $M \subset Y_{k+1} \rightarrow M \cup \{v_{k+1}\} \in L^*_{k+1}$

b) if $M \not\subset Y_{k+1} \rightarrow M \in L^*_{k+1}$ and $\{v_{k+1}\} \cup (M \cap Y_{k+1}) \in L^*_{k+1}$ iff $M \cap Y_{k+1} \in I'_k$

The L^*_{k+1} family contains only these sets of S4.

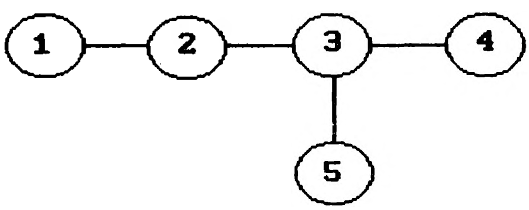
S5. One finds the maximal family L_{k+1} from L^*_{k+1} with respect to sets inclusion.

S6. Repeat S2,S3,S4,S5 for $k=2,\dots,n-1$. Finally we have L_n which contains the maximal *IS* of *G*.

Example :

Let $G = (V,E)$ be an undirected graph with

$V = \{1,2,3,4,5\}$, $E = \{[1,2],[2,3],[3,4],[3,5]\}$



In the next table we have:

k	Y_{k+1}	I_k	I'_k	L^*_{k+1}	L_{k+1}
1	{2}	∅	∅	{1},{2}	{1},{2}

2	{1,3}	∅,{1}	{1}	{1,3},{2}, {3}	{2}, {1,3}
3	{1,2,4}	∅,{1},{2}	{1},{2}	{1,3},{1,4}, {2,4},{4}	{1,3}, {1,4}, {2,4}
4	{1,2,4,5}	∅,{1},{1,4}	{1,4}	{1,3}, {1,4,5}, {2,4,5},{5}	{1,3}, {1,4,5} {2,4,5}

$\rightarrow L = \{\{1,3\},\ \{1,4,5\},\ \{2,4,5\}\}$

2.5. In what follows we suggest the next algorithm:

The notations used:

Let $G = \{V,E\}$ be an undirected graphs and:

$V_k = \{v_1,\ldots,v_k\}$, $|V| = n, 1 \le k \le n;$

L_k = the sets family of *IS* associated with V_k, $1 \le k \le n$.

The steps of the algorithm are:

S1. $L_1 = \{v_1\}$, $k=2$.

S2. One finds L_k :

a) if $M \in L_{k-1} \rightarrow M \in L_k$.

b) if $M \in L_{k-1}$ and $\forall\ y \in M$ with $[y,x_k] \notin E \rightarrow M \cup \{x_k\} \in L_k$.

c) $\{v_k\} \in L_k$.

Repeat S2 for $k=2,3,\ldots,n$.

S3. Reducing L_n with respect to sets inclusion:

$\forall\ M,N \in L_n$ and $M \subset N \rightarrow L_n = L_n \setminus M$.

For the previous graph we have:

L_1 : {1}.

L_2 : {1}, {2}.

L_3 : {1}, {2}, {1,3}, {3}.

L_4 : {1}, {2}, {1,3}, {3}, {1,4}, {2,4}, {4}.

L_5 : {1}, {2}, {1,3}, {3}, {1,4}, {2,4}, {4}, {1,5}, {2,5}
{1,4,5}, {2,4,5}, {4,5}, {5}.

Applying S3 we obtain:

L : { {1,3}, {1,4,5}, {2,4,5} }.

The algorithm is very simple and it works only with a single sets family.

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ON SOME PARALLEL METHODS IN LINEAR ALGEBRA

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REZUMAT. - Asupra unor metode paralele în algebra liniară. Sînt studiate din punct de vedere al complexităţii mai multe metode numerice de inversare a matricelor şi de rezolvare a sistemelor algebrice liniare.

The parallel computation had became an actual problem in many application fields.

Of course, not each mathematical method can be efficientely projected in a parallel version.

To characterize the depth of the parallelism of a given method there exists specifically criterions. Such criterions are the speed and the efficiency. The goal of this paper is to discus some methods in linear algebra from the parallelism point of view.

Let X be a linear space, X_0 a subset of X, $(Y, \|\cdot\|)$ a normed linear space and $S, S: X_0 \to Y$, a given operator. The problem: for given $\varepsilon > 0$ and $x \in X_0$ find an $y \in Y$ such that $\|S(x) - y\| \leq \varepsilon$ is called a S - problem, x is the problem element, S is the solution operator and $s = S(x)$ is the solution element. $\tilde{s} \in Y$ for which $\|\tilde{s} - s\| \leq \varepsilon$ is called an ε - approximation of the solution s.

In order to solve a S - problem there are necessary some informations on the problem element x. So, let Z be a set (the set of informations). The operator $\mathfrak{I}: X \to Z$ is called the informational operator and $\mathfrak{I}(x)$, $x \in X_0$, is the information on

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x. To compute a solution of a S - problem for a given information $\mathfrak{I}(x)$ we need an algorithm, which is defined as an application α: $\mathfrak{I}(X_0)\to Y$. So, for a given $x \in X_0$, $\alpha(\mathfrak{I}(x))$ is the approximation of the solution $S(x)$ given by the algorithm α with the information $\mathfrak{I}(x)$ as the input data. If α $(\mathfrak{I}(x))$ is an ε - approximation of $S(x)$ then $\mathfrak{I}$ and α are called ε - admissible. So, to solve a S - problem means to find an ε - admissible informational operator and an ε - admissible algorithm for it.

DEFINITION 1. A couple $(\mathfrak{I},\alpha)$ with $\mathfrak{I}:X\to Z$ and α : $\mathfrak{I}$ $(X_0)\to Y$ is called a method associated to a S - problem.

If $\mathfrak{I}$ and α are ε - admissible then the corresponding method is called also ε - admissible.

Next, one denotes by $M(S)$ the set of all admissible methods for the problem S. A method $\mu\in M(S)$, $\mu = (\mathfrak{I},\alpha)$, is called a serial method if all the computations are described as a single instructions stream (α is a serial algorithm). If the computations are described as a multiple instructions streams then μ is called a parallel method (α is a parallel algorithm).

To distinguish the two kind of methods one denotes by $M_s(S)$ the set of all serial methods for the problem S and by $M_p(S)$ the set of all parallel methods for S.

For a method $\mu\in M(S)$ one denotes by $CP(\mu;x)$, $x \in X_0$, its computational complexity for the element x or the local complexity, while

$$CP(\mu) = \sup_{x\in X_0} CP(\mu;x)$$

is the complexity of the method μ for the problem S(global complexity) [3].

DEFINITION 2. The method $\overline{\mu} \in M_s$ (S) for which

$$CP(\overline{\mu}) = \inf_{\mu \in M(S)} CP(\mu)$$

is called the optimal method with regard to the complexity.

Now, let μ be a serial method, $\mu \in M_s$ (S).

Generally speaking, by a parallel method $\mu_p \in M_p$ (S), associated to μ we understand a method in which all the operations, independent to each others, are performed in parallel (in the same time). So, we can image the serial method divided in many parts (segments - streams of instructions) independently or partial independently from the computation point of view, say $\mu_1,\ldots,\mu_r$. Then

$$CP(\mu_p) = \max_{1 \le i \le r} CP(\mu_i)$$

is the complexity of the corresponding parallel method μ_p.

DEFINITION 3. Let S be a given problem, $\mu_p \in M_p(S)$ a parallel method and $\overline{\mu_s} \in M_s$ (S) the optimal serial method with regard to the complexity.

Then

$$S(\mu_p) = \frac{CP(\overline{\mu}_s)}{CP(\mu_p)}$$

is called the speed of the parallel method μ_p.

Remark 1. The speed is also denoted by $S(\mu_p;r)$, where r is the number of the instructions streams of the method μ_p.

Obviously, $S(\mu_p; r) \le r$.

Remark 2. A more practical value to judge the parallel version μ_p of a serial method μ_s is

$$s(\mu_p;r) = \frac{CP(\mu_s)}{CP(\mu_p)}.$$

We also have $s(\mu_p;r) \geq S(\mu_p;r)$.

DEFINITION 4. The value

$$E(\mu_p) = \frac{S(\mu_p;r)}{r}$$

is called the efficiency of the parallel method μ_p.

As $0 \leq S(\mu_p;r) \leq r$ it follows that $0 \leq E(\mu_p) \leq 1$.

Next, we consider first some examples.

E.1. Let $\mathscr{E}$ be the following expression :

$\mathscr{E} = t_1 \;\rho\; t_2 \;\rho\; \dots \;\rho\; t_n$

where ρ is an associative operation.

The serial computational complexity of $\mathscr{E}$ is

$CP(\mathscr{E}) = (n-1)\; CP(\rho)$,

where $CP(\rho)$, is the complexity of the operation ρ.

A parallel version $\mathscr{E}_p$ of the expression $\mathscr{E}$ is obtained as follows: in the first step we compute, say $t_i^1 := t_{2i-1}\rho\; t_{2i}$, for all possible i. To do it more clear, let $m \in \mathrm{N}$ be such that $2^{m-1} < n \leq 2^m$. If $n < 2^m$ then we supplement the expression $\mathscr{E}$ by

$t_{n+1} = \dots = t_{2m} = 0$, i.e.

$\mathscr{E} = t_1 \;\rho\; t_2 \;\rho\; \dots \;\rho t_n \;\rho\; t_{n+1} \;\rho\; \dots \;\rho\; t_{2^m}$

so,

$t_i^1 := t_{2i-1} \;\rho\; t_{2i}\,,\quad i = 1,\dots,2^{m-1}.$

In the second step we have

$t_i^2 := t_{2i-1}^1 \;\rho\; t_{2i}^1\,,\qquad i = 1,\dots,2^{m-2}$

and so on

$$t_i^k := t_{2i-1}^{k-1} \, \rho \, t_{2i}^{k-1} , \qquad i = 1,\dots,2^{m-k}$$

for $k = 3,\dots,m$. Finally, we have $\mathscr{E} = t_1^m$ .Hence, the necessary steps to compute $\mathscr{E}$ is m. Taking into account that $2^{m-1} < n \leq 2^m$, one obtains $m = [\log_2 n]$, where $[x]$, $x \in \mathbb{R}$ is the integer with the property $x \leq [x] < x + 1$.

It follows that

$$CP\ (\mathscr{E}_p) = [\log_2 n]\ CP\ (\rho) \ .$$

So, we have

$$s(\mathscr{E}_p; [n/2]\ ) = \frac{n - 1}{[log_2 n]}$$

and

$$E(\mathscr{E}_p) = \frac{n - 1}{[n/2]\ [log_2 n]} \approx \frac{2}{[log_2 n]}$$

where $[x]$ is the integer part of x.

Remark 3. If we consider the binary tree associated to the expression $\mathscr{E}$ then the complexity of the parallel computation of $\mathscr{E}$ is the depth of the tree [5].

I.2. Let be $X = M_n(\mathbb{R})$, $X_0 = X$, $Y = \mathbb{R}$ and $S : X \to Y$, $A \to \det A$. Hence, S is the problem to compute the determinant $\det A$ of the matrix A. The method used consists in the transformation of the determinant

$$\det A = \begin{vmatrix} a_{11} & a_{12}\dots & a_{1n} \\ a_{21} & a_{22}\dots & a_{2n} \\ \dots & \dots & \dots \\ a_{n1} & a_{n2}\dots & a_{nn} \end{vmatrix} .$$

in the form

$$\det A := a_{11}^1 * \dots * a_{nn}^n * \begin{vmatrix} 1 & a_{11}^2 & \dots & a_{1n}^2 \\ 0 & 1 & \dots & a_{2n}^3 \\ \dots & & & \dots \\ 0 & 0 & . & 1 \end{vmatrix}$$

using the operations :

$$a_{ij}^1 := a_{ij}, \quad i,j = 1,\dots,n$$

$$a_{ip}^{p+1} := \frac{a_{ip}^p}{a_{pp}^p}, \quad i = p+1,\dots,n$$

$$a_{ij}^{p+1} := a_{ij}^p - a_{pj}^p * a_{ip}^{p+1}, \quad i,j = p+1,\dots,n$$

for $p = 1,\dots,n-1$.

So, we have

$$\det A = a_{11}^1 * a_{22}^2 * \dots * a_{nn}^n .$$

Remark 4. Next we suppose that $CP(+) = 1$ (a unit time) and $CP(*) = CP(/) = 3$.

If one denotes by μ_s the serial method to compute det A, one obtains

$$CP(\mu_s) = \frac{1}{6}(n-1)(8n^2+5n+18) .$$

A parallel version of the considered method using n parallel instructions strems (n processors) is :

begin

det A: = 1;

for p: = 1 step 1 until n -1 do

begin

(det A: = det A * a_{pp}^p; $(p + 1 \le j \le n)$ $a_{pj}^{p+1} := \frac{a_{pj}^p}{a_{pp}^p}$) ;

(($p+1 \le j \le n$) for i:=p+1 step 1 until n do

$$a_{ij}^{p+1} := a_{ij}^{p} - a_{ip}^{p} * a_{pj}^{p+1})$$

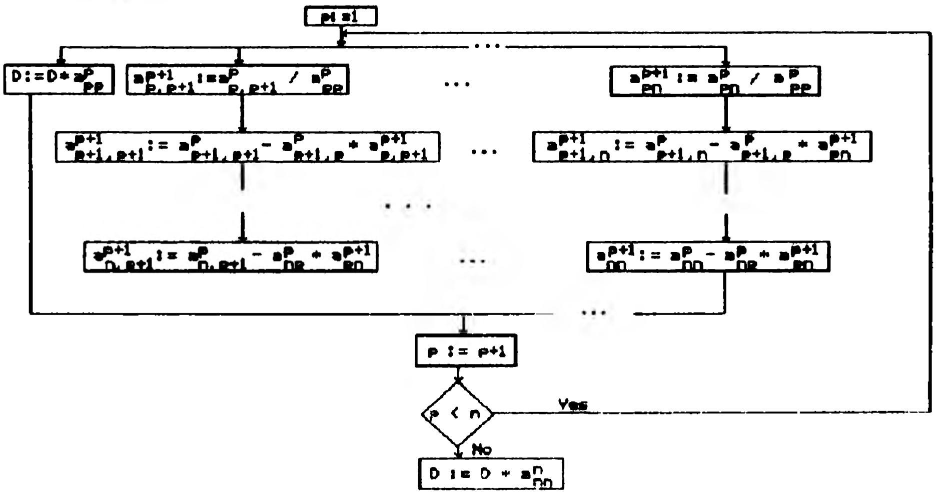
end

det A := det A * a_{nn}^{n}

end

Remark 5. $(I, (1 \le k \le m)\ I_k)$ means that the instructions I, I_1, ..., I_m are performed in parallel.

For a better illustration of the parallel method, say μ_p, we give the next diagram (D:= det A) :



The complexity of the parallel method μ_p, as it can be easy seen, is:

$CP(\mu_p;n) = n(2n - 1)$.

So,

$$s(\mu_p;n) = \frac{(n-1)(8n^2 + 5n + 18)}{/6n(2n - 1)} \approx \frac{2}{3}\, n + \frac{1}{12}$$

and

$$E(\mu_p) \approx \frac{2}{3} .$$

E.3. For $X = M_n$ ($\mathbb{R}$), $X_0 = \{A \mid \det A \neq 0, A \in X\}$, $Y = M_n$ ($\mathbb{R}$) and $S(A) = A^{-1}$, S is the problem to compute the inverse of a matrix.

We use the method based on the succesive transformations of the matrix $[A \mid I_n]$ in the matrix $[I_n \mid A]$, where I_n is the unit matrix of order n. The transformations are : first one denotes the elements of the matrix $[A \mid I_n]$ by t^1_{ij}, $i=1,\ldots,n$; $j=1,\ldots,2n$. Now,

$$t^{p+1}_{pj} := \frac{t^p_{pj}}{t^p_{pp}} \; , \; j=p+1,\ldots,2n$$

$$t^{p+1}_{ij} := t^p_{ij} - t^p_{ip} * t^{p+1}_{pj} \; , \; i=1,\ldots,n, \; i \neq p \; ; \; j=p+1,\ldots,2n$$

$$t^n_{nj} := \frac{t^n_{nj}}{t^n_{nn}} \; , \; j=n+1,\ldots,2n \; ,$$

for all $p = 1,\ldots,n-1$.

So,

$$A^{-1} = (t^n_{ij}) \; i = \overline{1,n} \; ; \quad j = \overline{n+1 \, , \, 2n}$$

If μ_s is the corresponding serial method then

$$CP(\mu_s) = \frac{3}{2} n \, (4n^2 - 5n + 3) .$$

A parallel method, μ_p, can be projected as follows :

begin

$t^2_{12} := \dfrac{t^1_{12}}{t^1_{11}}$;

for p:=1 step 1 until $n-1$ do

begin for j:=p+1 step 1 until $2n$ do

$$\left(t_{p,j+1}^{p+1} := \frac{t_{p,j+1}^{p}}{t_{pp}^{p}} \ ;\ (1 \le i \le n,\ i \ne p)\ t_{ij}^{p+1} := t_{ij}^{p} - t_{ip}^{p} * t_{pj}^{p+1} \right)$$

end;

$$\left((n+1 \le j \le 2n) \quad t_{nj}^{n} := \frac{t_{nj}^{n}}{t_{nn}^{n}} \right)$$

end

We have

$$CP(\mu_p;n) = 6(n^2 - n + 1)$$

and

$$s(\mu_p;n) \approx n - \frac{1}{4}$$

respectively

$$E(\mu_p) \approx 1.$$

Remark 6. From these three examples we can see that the matrix inversion permites a very good parallelism ($E(\mu_p) \approx 1$), while for the determinant computation $E(\mu_p) \approx 2/3$ and in the first example

$$E(\mathscr{E}_p) \approx 2/[\log_2 n] \ .$$

Linear algebraic systems.

If $X = \{[A|b] \mid A \in M_n(\mathbb{R}),\ b \in M_{n,1}(\mathbb{R})\}$, $X_0 = \{[A|b] \in X \mid \det A \ne 0\}$ $S([A|b]) = A^{-1}b$ then S is the problem to solve the system $As=b$.

Next, there are discused serial and parallel versions for some well known numerical methods for the solution of linear algebraic systems.

I. **Cramer's method.** Taking into account that the solution is given by $s_i = D_i/D, i=1,\dots,n$, where D=det A and D_i is the determinant obtained by D changing the i-th column vector by b.

So, we have to compute $n + 1$ determinants of order n, with the complexity $CP(\mu_s)$ from the example E_2, and n divisions. It follows that the serial complexity of Cramer's method μ_s^c is $CP(\mu_s^c) = (n+1)CP(\mu_s) + nCP(/)$, i.e.

$$CP(\mu_s^c) = \frac{1}{6}(8n^4 + 5n^3 + 1on^2 + 13n - 18). \qquad (1)$$

A natural parallel method here is to compute in parallel the $(n+1)$ determinants and than to perform the n divisions. So,

$$CP(\mu_p^c) = \frac{1}{6}(8n^3 - 3n^2 + 13n) \qquad (2)$$

where μ_p^c is the mentioned parallel method.

Hence, one obtains

$$s(\mu_p^c; n+1) = (n+1) - \frac{18}{8n^3-3n^2+13n} \approx n+1$$

and

$$E(\mu_p^c) \approx 1. \qquad (3)$$

As a conclusion we can remark the very good parallelism of Cramer's method $\left(E(\mu_p^c) \approx 1\right)$.

II. **Gaussian elimination method.** As, it is well known first the given matrix $[A|b] \in X_0$ is transformed in the matrix $[T_n \mid \tilde{b}]$, where T_n is an upper triungular matrix ($T_n = (a_{ij}^i)$ $i=1,\ldots,n;\ j=i+1,\ldots,n;\ a_{ii}^i=1$) using the relations

$$a_{pj}^p := a_{pj}^p \;/\; a_{pp}^p,\ j=p+1,\ldots,n;\ b_p^p \;/\; a_{pp}^p$$

$$a_{ij}^{p+1} := a_{ij}^p - a_{ip}^p * a_{pj}^p\ ,\ i,j=p+1,\ldots,n$$

$$b_i^{p+1} := b_i^p - a_{ip}^p * b_p^p\quad ,\ i = p+1,\ldots,n$$

for $p = 1,\ldots,n-1$, and $b_n^n := \dfrac{b_n^n}{a_{nn}^n}$, where for the begining $a_{ij}^1 := a_{ij},\ b_i^1 := b_i,\ i,j=1,\ldots,n.$

The complexity of this computation is $n(n^2-1)/3*[CP(+)+CP(*)]$ + + $n(n+1)/2 * CP(/)$. Now the triangular system $T_n s = b$ is solved by back substitution method:

$$s_n := b_n^n$$

$$s_i := b_i^i - \sum_{j=i+1}^{n} a_{ij}^i * x_j,\quad i=n-1,\ldots,1,$$

with the computational complexity $n(n-1)/2 * [CP(+) + CP(*)]$. It follows that

$$CP(\mu_s^G) = \frac{1}{6}\ (8n^3 + 21n^2 - 11n)\,. \qquad (4)$$

A parallel version μ_p^G of the Gauss method is :

begin

for p:=1 step 1 until n-1 do

begin

$$\left((p+1 \le j \le n+1)\quad a_{ij}^p := \frac{a_{pj}^p}{a_{pp}^p}\right);$$

for i:=p+1 step 1 until n do

begin $\left((p+1 \le j \le n+1)\quad a_{ij}^{p+1} := a_{ij}^p - a_{ip}^p * a_{pj}^p\right)$; $b_i^{p+1} := b_i^p - a_{ip}^p * b_p^p$ end

end;

$a_{n,n+1}^{n}:=\frac{a_{n,n+1}^{n}}{a_{nn}^{n}}$

for k:=1 step 1 until $n-1$ do

$\left((k\le i\le n-1)\quad a_{n-i,n+1}^{n}:=a_{n-i,n+1}^{n}-a_{n-i,n-k+1}*a_{n-k+1,n+1}^{n}\right)$

end

where $a_{p,n+1}^{p}=b_p^p$.

So, $s_i:=a_{i,n+1}^{n}$, $i=1,\ldots,n$.

It follows that

$$CP(\mu_p^G;n) = 2n^2 + 5n - 11 \tag{5}$$

and

$$s(\mu_p^G;n) \approx \frac{2}{3}n + \frac{1}{2}$$

respectively

$$E(\mu_p^G) \approx \frac{2}{3}. \tag{6}$$

III. **Total elimination method.** The matrix $[A|b] \in X_0$ is transformed in the matrix $[\ I_n \mid b^n]$.

First,

$a_{ij}^1:=a_{ij},\quad a_{i,n+1}^1=b_i,\quad i,j=1,\ldots,n.$

Now, one applies the succesive transformations

$a_{pj}^{p+1}:=\frac{a_{pj}^{p}}{a_{pp}^{p}},\quad j=p+1,\ldots,n+1;$

$a_{ij}^{p+1}:=a_{ij}^{p}-a_{ip}^{p}*a_{pj}^{p+1};\quad i=1,\ldots,n;\quad i\neq p;\quad j=p+1,\ldots,n+1$

for all $p=1,\ldots,n$.

So, the solution is $s_i:=a_{i,n+1}^{n+1},\quad i=1,\ldots,n$.

The computational complexity of this method in the serial version (μ_s^T) is

$$CP(\mu_s^T) = \frac{1}{2}(4n^3 + 3n^2 - n). \tag{7}$$

As a parallel version (μ_p^T) of the total elimination method is the following :

begin

$a_{12}^2 := \frac{a_{12}^1}{a_{11}^1}$;

for $p:=1$ step 1 until $n-1$ do

begin

for $j:=p+1$ step 1 until n do

$$\left(a_{p,j+1}^{p+1} := \frac{a_{p,j+1}^p}{a_{pp}^p};\quad (1\le i\le n\quad i\ne p)\quad a_{ij}^{p+1} : a_{ij}^p - a_{ip}^p * a_{pj}^{p+1}\right)$$

$$\left(a_{p+1,p+2}^{p+2} := \frac{a_{p+1,p+2}^{p+1}}{a_{p+1,p+2}^{p+1}};\quad (1\le i\le n,\ i\ne p)\quad a_{i,n+1}^{p+1} := a_{i,n+1}^p - a_{ip}^p * a_{p,n+1}^{p+1}\right)$$

end

$$\left(a_{n,n+1}^{n+1} := \frac{a_{n,n+1}^n}{a_{nn}^n};\quad (1\le i\le n-1)\quad a_{i,n+1}^{n+1} := a_{i,n+1}^n - a_{in}^m * a_{n,n+1}^{n+1}\right)$$

end

We have

$$CP(\mu_p^T) = 2n^2 + 2n + 3. \tag{8}$$

So,

$$s(\mu_p^T ; n) = n - \frac{1}{4}$$

and

$$E(\mu_p^T) = 1. \tag{9}$$

IV. **Iterative methods.** One considers two iterative methods.

IV.1. **Jacobi iteration.** For a given $x^{(0)} = (x_1{}^{(0)}, \ldots, x_n{}^{(0)})^T$, the sequence of the succesive approximation $x^{(m+1)}$ is given by

$$x_i^{(m+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{\substack{j=1 \\ j\neq i}} a_{ij}x_j^{(m)}\right), \quad i = 1,\ldots,n.$$

If $CPI(\mu_s^T)$ is the computational complexity of one iteration then the serial complexity of the Jacobi method is

$$CP(\mu_s^T) = m_J(\varepsilon) \quad CPI(\mu_s^J),$$

where $m_J\ (\varepsilon)$ is the iterations number for which $x^{(m_J(\varepsilon))}$ is an ε-approximation of the solution. So, we have

$$CP(\mu_s^J) = (4n^2 - n)\ m_J(\varepsilon)\ . \tag{10}$$

A parallel version of the method μ_s^J is to compute, in parallel, each $x_i^{(m+1)}$, $i=1,\ldots,n$.

Hence,

$$CP(\mu_p^J\ ;n) = (4n-1)\ m_J\ (\varepsilon)\ . \tag{11}$$

It follows that

$$S(\mu_p^J; n) = n$$

and

$$E\left(\mu_p^J\right) = 1$$

IV.2. **Gauss - Siedel iteration.** Starting with $x^{(0)}$, the iterations are given by

$$x_i^{(m+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij}x^{(m+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(m)}\right), i=1,\ldots,n.$$

The serial complexity of the Gauss-Siedel method is

$$CP(\mu_s^{GS}) = (4n^2 - n)\ m_{GS}(\varepsilon)\ , \tag{13}$$

where $m_{GS}(\varepsilon)$ is the iterations number.

It is obviously that the parallelism of the Gauss-Siedel method is more less than of the Jacobi iteration. Certainly we solve for $x_2^{(m+1)}$ using already the "new" value $x_1^{(m+1)}$, for $x_3^{(m+1)}$ it is used the "new" values $x_1^{(m+1)}$, $x_2^{(m+1)}$ and so on. Hence, $x_2^{(m+1)}$ can be computed only when the computation of $x_1^{(m+1)}$ is finished and the computation of $x_3^{(m+1)}$ must wait for $x_1^{(m+1)}$ and $x_2^{(m+1)}$ and so on. It follows that a parallel version μ_p^{GS} is to do the computation begining with the first line ($x_1^{(m+1)}$) than the second one ($x_2^{(m+1)}$) and so on. One obtains

$$CP(\mu_p^{GS}) = n([\log_2 n]+6)\, m_{GS}(\varepsilon)$$

and

$$E(\mu_p^{GS}) = \frac{4(1-1/n)}{[\log_2 n]+6}\ .$$

Conclusions. Taking into account the serial and parallel complexity of the above methods for linear algebraic systems it follows:

PROPOSITION 1. $CP(\mu_s^G) < CP(\mu_s^T) < CP(\mu_s^C)$, $\forall n > 2$.

The proof follows directly by (1), (4) and (7).

Remark 7. Of the Gauss - Siedel procedure may be viewed as an acceleration of Jacobi method, so we generally have $m_{GS}(\varepsilon) \leq m_J(\varepsilon)$ i.e.

$$CP(\mu_s^{GS}) < CP(\mu_s^J)\ .$$

Now, from (2) and (10), it follows :

PROPOSITION 2. *If* $m_{GS}(\varepsilon) \leq [n/3]$ *then*

$$CP(\mu_s^{GS}) < CP(\mu_s^G)\ .$$

Remark 8. For the systems with a lagre number of equation (such that $[(n/3)+1]$ iterations are sufficient to get a good approximation the Gauss - Siedel iteration is better than of the Gauss elimination method.

The following two propositions give some informations regarding with the parallel methods.

PROPOSITION 3. $CP(\mu_p^T) < CP(\mu_p^G) < CP(\mu_p^C) \quad \forall n > 2$.

The proof is based on the relations (2), (5) and (8).

Remark 9. For the parallel version μ_p^G and μ_p^T we have $CP(\mu_p^G) > CP(\mu_p^T)$ just if in the serial case the relation is $CP(\mu_s^G) < CP(\mu_s^T)$. So, generally a good serial method does not conduct to a good parallel version.

PROPOSITION 4. *If* $m_T(\epsilon) < [n/2]$ *then* $CP(\mu_p^T) < CP(\mu_p^T)$.

Remark 10. In the parallel case it can be done just $[n/2]$ iterations without passing the complexity of the best parallel method μ_p^T

Finally, from (3), (6), (9) and (12) it follows that the best parallelism is possessed by the Jacobi iteration method ($E(\mu_p^T)=1$). Also, a good parallelism has the total elimination method $(E(\mu_p^T) \approx 1 - \frac{1}{4n})$ and the Cramer's method $(E(\mu_p^C) \approx 1)$. But the complexity of the Cramer method is, in both serial and parallel versions, a polynomial function on degree with a unity greater than the other ones. So, the Cramer's method is never recommended from the computational complexity point of view.

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ON THE CONVERGENCE OF THE THREE-ORDER METHODS IN FRECHET SPACES

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REZUMAT. - **Asupra convergenţei metodelor de ordinul trei în spaţii Fréchet.** În lucrare se demonstrează existenţa şi unicitatea existenţei ecuaţiei (1) precum şi convergenţa metodei iterative (2), renunţînd la uniform mărginirea operatorului $\Lambda = [x',x'';P]^{-1}$.

1. It is known that the rapidity of convergence for the sequence of approximates (x_n) of solution of the operatorial equation

$$P(x) = \theta \tag{1}$$

given by an iterative method, can be improved if the first and the second order divided differences, which enter in the algorithm exprimation, are taken on special nodes.

In the case of operatorial equation

$$P(x) \equiv x-F(x) = \theta \tag{2}$$

using the metod

$$x_{n+1} = x_n-\Lambda_n(I-[x_n,u_n,v_n;P]\ \Lambda_n\ P(u_n)\ \bar{\Lambda}_n)^{-1}\ P(x_n) \tag{3}$$

where

$$\Lambda_n = [x_n,u_n;P]^{-1};\ \ \bar{\Lambda}_n = [u_n,v_n;P]^{-1}$$

and

$$u_n = F(x_n);\ v_n=F(u_n) = (F(x_n))$$

this property is proved in the paper [1]. The following theorem

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are proved:

THEOREM A. *If for $x_0 \in X$, there exist μ_0, B, $M>1$ and N so that the following conditions:*

1) $\| P(x_0) \| < \mu_0$;

2) *For any $x', x'', x''', x^{IV} \in S(x_0, R)$, R remaining to be defined, we have*

a. $\Lambda = [x', x''; P]^{-1}$ exists and $\| \Lambda \| < B$;

b. $\| [x', x''; F] \| < M$;

c. $\| [x', x'', x'''; P] \| < K$;

d. $\| [x', x'', x^{IV}; P] - [x', x'', x'''; P] \| <$
$< N \| x^{IV} - x''' \|$

3) $G_0 h_0 < 1$ *where* $h_0 := B^2 M K \mu_0 < 1/2$ *and*

$$G_0^2 := \frac{M(1+BK\mu_0)\,[1+BK\mu_0(1+M)]}{(1+h_0)^2(1-2h_0)}\left(1+\frac{N}{BK^2}\right)$$

hold, then the equation (2) has the solution $x^* \in S(x_0, R)$, where

$$R = (1+M)\mu_0 + M^2 Q \quad \text{and} \quad Q = \frac{B\mu_0}{1-h_0}\sum_{m=0}^{n}(G_0 h_0)^{3^m-1}$$

solution which is the limit of the sequence genereted by (3), the rapidity of convergence being given by

$$\|x^* - x_m\| \le (G_0 h_0)^{3^m-1} Q.$$

THEOREM B. *In the conditions of Theorem A, the solution of equation (2) is unique.*

In the following, we will change the condition 2a of Theorem A, removing the uniform bounded of the operator Λ.

2. Let us consider the equation

$$P(x) = x - F(x) = \theta$$

where $P:X\to X$ is a continuous operator considered with its generalized divided difference [2] up to the second order, inclusively, X is a Fréchet space with a quasinorm induced by a distance invariant to translation, i.e $)|\ x\ |(= d(x,\theta)$, $x,\theta \in \in X$ [3].

To solve the equation (2) we consider the algoritm (3).

Concerning the convergence of the method (3), we prove

THEOREM. *If for* $x_0 \in X$ *exists* $\overline{\mu_o}$, $\overline{M} > 1$, $\overline{K}$ *and* $\overline{N}$ *such that the following conditions:*

1^0 *For any* $x', x'', x''', x^{IV} \subset S$, *where* $S = \{x\ |)\ |x - x_0|(\ < R\}$,

$$R = (1+\overline{M})\,\mu_0 + \overline{M}^2\,\overline{D}\ ; \quad \overline{D} = \frac{\overline{\mu}_0}{1-h_0}\sum_{m=0}^{n} (\overline{G}_0 \overline{h}_0)^{3^m-1}$$

we have:

a. $\Lambda = [x', x''; P]^{-1}$ *exists;*

b. $)|\ \Lambda[x', x''; F]\ |(< \overline{M}$;

c. $)|\ \Lambda[x', x'', x'''; P]\ |(< \overline{K}$;

d. $)|\ \Lambda\ ([x', x'', x^{IV}; P] - [x', x'', x'''; P])\ |(<$
$< \overline{N}\ )|x^{IV} - x'''\ |($

2^0 $)|\ \Lambda P(x_0)\ |(< \overline{\mu_0}$;

3^0 $\overline{G_0}\,\overline{h_o} < 1$ *where* $\overline{h_o} := \overline{M^2}\,\overline{K}\,\overline{\mu_o} < 1/2$ *and*

$$\overline{G}_0^2 = \frac{\overline{N}(1+\overline{K}\overline{\mu}_0)\,[1+\overline{K}\overline{\mu}_0(1+\overline{M}]}{(1+h_0)^2(1-2h_0)}\left(1+\frac{\overline{N}}{K^2}\right)$$

hold, then the equation (2) has the unique $x^* \subset S(x_0, R)$, *which is* *the limit of the sequence genereted by* (3)*, the rapidity of convergence being given by*

$$\| x^*-x_n \| < (\overline{C}_0 \overline{h}_0)^{3^n-1}\cdot \overline{\varrho}.$$

Proof. We consider the equation

$$\overline{P}(x) = \theta \qquad (6)$$

where

$$\overline{P}(x) = \Lambda P(x) = \Lambda(x-F(x)), \quad \Lambda=[x',x'';P]^{-1}$$

equation which is equivalent to (2).

Indeed, if x^* is a solution of equation (2), i.e. $P(x^*)=\theta$, due to liniarity of Λ, it results

$$\Lambda P(x^*) = \overline{P}(x^*) = \theta. \qquad (7)$$

Reciprocal, if x^* is a solution of equation (6), i.e.

$$\overline{P}(x^*) = \Lambda P(x^*) = \theta$$

from the existence of operator Λ, it results $\Lambda^{-1}=[x',x'';P]$ which, applied to the left of the equation (6), leads to $P(x^*)=\theta$.

For solving this equation, we have the iterative method

$$\overline{x}_{n+1}=\overline{x}_n-\overline{\Lambda}_n(I-[\overline{x}_n,\overline{u}_n,\overline{v}_n;\overline{P}]\overline{\Lambda}_n\overline{P}(\overline{u}_n)\ \tilde{\overline{\Lambda}}_n)^{-1}\overline{P}(\overline{x}_n)\ . \qquad (8)$$

Using the induction, one can prove that for $x_0=\overline{x_0}$, $u_0=\overline{u_0}$, $v_0=\overline{v_0}$

the sequence given by (8) is identical with the sequence (3).

For the operator $\bar{P}$, the conditions of Theorem A and B are true. Indeed

1^0 $\|\bar{P}(x_0)\| = \|\Lambda P(x_0)\| < \bar{\mu}_0$;

2^0 For any $x', x'', x''', x^{IV} \in S(x_0, R)$, we have

a) $\bar{\Lambda} = [x', x''; \bar{P}]^{-1} = (\Lambda[x', x''; P]^{-1} = I$, then

$\bar{\Lambda}$ exists and $\|\bar{\Lambda}\| = 1 = \bar{B}$;

b) $\|[x', x''; \bar{P}]\| = \|\Lambda[x', x''; F]\| < \bar{M}$;

c) $\|[x', x'', x'''; \bar{P}]\| = \|\Lambda[x', x'', x'''; P]\| < \bar{K}$

d) $\|[x', x'', x^{IV}; \bar{P}] - [x', x'', x'''; \bar{P}]\| =$

$= \|\Lambda([x', x'', x^{IV}; P] - [x'', x'', x'''; P])\| <$

$< \bar{N} \|x^{IV} - x'''\|$;

3^0 $\bar{G}_0\bar{h}_0 < 1$, where $\bar{h}_0 := \bar{B}^2 \bar{M} \bar{K} \bar{\mu}_0 < \frac{1}{2}$ and

$$\bar{G}^2 := \frac{\bar{M}(1+\bar{B}\bar{K}\bar{\mu}_0)[1+\bar{B}\bar{K}\bar{\mu}_0(1+\bar{M})]}{(1+h_0)^2(1-2h_0)}\left(1+\frac{\bar{N}}{BK^2}\right)$$

It results that the hypotesis of Theorem A are satisfied by P, hence the equation (6) has a solution $x^* \in S$, which is the limit of sequence generated by the algoritm (3) or (8), the rapidity of convergente being given by (4).

Because (6) is equivalente to (2), the statement results.

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SOFTWARE FOR CLASSIFICATION

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Rezumat. Software pentru clasificare. Articolul prezintă un sistem de programe destinat clasificării automate a unei colecţii de obiecte caracterizate prin valorile mai multor parametri. Programele au fost elaborate de autor şi se bazează pe o serie de algoritmi din literatură, precum şi pe unii originali. Principalele componente ale sistemului sînt: extractorul de caracteristici, clasificatorul ierarhic diviziv, clasificatorul neierarhic, clasificatorul bazat pe arborescenţa de acoperire minimală şi componenta destinată interpretării calitative a partiţiilor obţinute. Pentru fiecare componentă se prezintă structura şi funcţiile ei, algoritmii implementaţi, datele de intrare şi ieşire.

0. Introduction. The aim of this paper is to describe a program system designed for pattern preprocessing, classification and interpretation of data sampled from a non-homogeneous population. The programs, which belong to the author of the paper, implement classical algorithms as well as some original ones. The main components of the system are: the pattern preprocessor, the divisive hierarchical classifier, the single-level classifier, the minimum forest classifier and the component which enables a qualitative interpretation of the obtained partitions. The input of the system is the collection of objects to be classified, characterized by the values of d variables recorded within a usual text file.

1. The pattern preprocessor. This component performs the transformation of data from the original pattern space to the feature space. The output is also a text file in which s features

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($s \le d$) for each object are recorded.

This program has several processing options: normalization of the original patterns, Mahalanobis distance, principal component analysis and combinations of the above options. The first processing type is a simple scaling of the original variables by the overall mean and standard deviation such that they become comparable. The second option implements the Mahalanobis distance by an appropriate coordinate transformation. The principal component analysis projects the original patterns onto the eigenvectors of the covariance matrix of parameters corresponding to the first s eigenvalues in their decreasing order. A given threshold indicates what percentage of the original information should be preserved after data compression.

2. **The divisive hierarchical classifier.** This component implements the fuzzy divisive hierarchical algorithm [2], and its corresponding hard version. These algorithms perform a hierarchical descendant classification, iteratively splitting the current fuzzy (hard) cluster into two fuzzy (hard) subclusters until the clustering degree of this binary partition [3] becomes less than a given threshold; in this case, the cluster is considered homogeneous.

The input of this classifier is the file containing the features of the objects to be classified. There are three output files: one containing the hard partition (in the fuzzy case, it is obtained by defuzzification), another containing the fuzzy partition (in the fuzzy case) and the last one containing the prototypes of the partition clusters.

The structure of the classifier is presented in fig. 1. Module HIER performs the hierarchical classification. Module SPLIT implements the generalized Fuzzy c-Means algorithm with two clusters, which is used to split the current cluster. It consists of the reiteration of modules: CENTRE (computation of

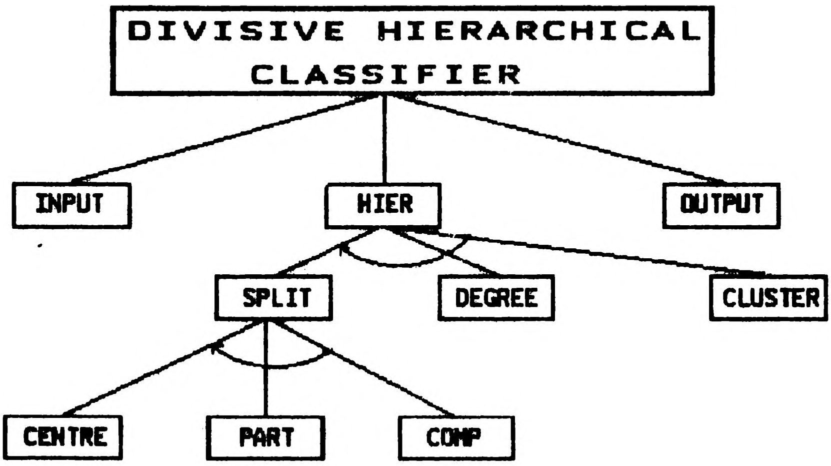

Fig.1

the sub-clusters centres), PART (computation of a new partition - fuzzy or hard) and COMP (comparison of the last two partitions). Module DEGREE computes the clustering degree of the current cluster binary partition. Module CLUSTER displays the clusters from the hierarchy and records within an output file final clusters (which are no longer splitted).

3. **The single-level classifier.** This classifier implements the following algorithms: Fuzzy c-Means, Fuzzy c-Lines, fuzzy clustering algorithms with linear manifold prototypes,

combinations of them [1] and hyperellipsoid prototypes [8], as well ae their hard versions. This classifier performs a non-hierarchical classification, hence the number of clusters must be given by the user.

The input files of the classifier are: the file containing the features of the objects to be classified and a file containing an initial partition (fuzzy or hard) or an initial set of prototypes. Thus, the output of the divisive hierarchical classifier may become the input of this classifier, in order to obtain an improved partition. Moreover, if the input is a set of prototypes, this classifier may be used as a trainable classifier: the prototypes are computed from a training set and then unknown samples are classified according to the dissimilarities with respect to these prototypes. The output files are the same as those of the divisive hierarchical classifier.

The structure of this classifier is presented in Fig.2.

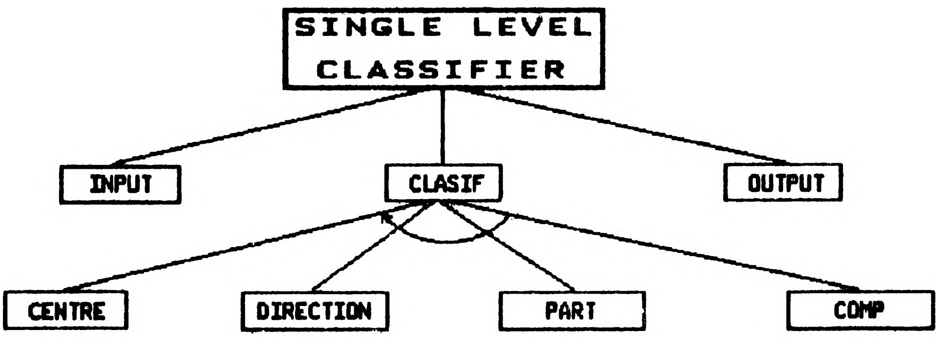

Fig.2

Module CLASIF performs the single-level classification. It consists of the reiteration of modules: CENTRE (computation of the centres of the clusters), DIRECTION (computation of the directions of prototypes), PART (computation of a new partition - fuzzy or hard) and COMP (comparison of the last two partitions) until the last two partitions coincide (in the hard case) or the maximum difference of corresponding membership degrees does not exceed a given error level (in the fuzzy case).

4. **The minimum spanning forest (MSF) classifier.** From numerical experiments we noticed that Fuzzy *c*-Means and other related algorithms often misclassify samples situated at the border of the clusters. One way to prevent this situation, if the distribution of the cluster samples is close to a normal one, is to use hyperellipsoid prototypes. If the distribution is arbitrary, we propose Prim's MSF clustering algorithm [9] which is based on a graph-theoretical approach.

The MSF classifier first detects the subclusters containing samples which were surely correctly classified by a Fuzzy *c*-Means type algorithm and states them as "centres"; this operation, implemented in module SELECT (fig. 3), is done by selecting those samples which have the membership degree in the corresponding fuzzy cluster higher than a given threshold. The remaining samples represent the "objects" (according to the terminology used in [9]) and will be reclassified. Module DISSIM computes object-to-centre dissimilarities as point-to-set dissimilarities, i.e. the least dissimilarities between the objects and the samples in the centres. Module MSF, implementing Prim's MSF clustering algorithm, associates the objects one by one with the centres. Thus, misclassified samples are reclassified and will probably get into the correct cluster.

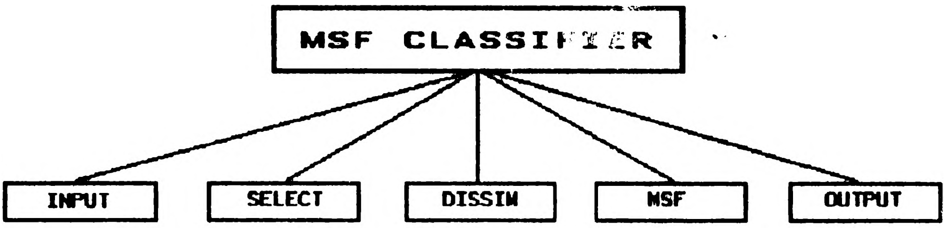

Fig.3

As input data we need the features of the samples, the fuzzy and the defuzzified partitions. The output is a hard partition.

5. **Interpretation of the obtained partitions.** We first give a theoretical model of partition qualitative interpretation. Let X be the set of samples partitioned into the clusters $A_1,\dots,A_c$. Consider a qualitative feature (or a combination of qualitative features) F defined on X and taking a finite number of values $\{f_1, \dots, f_k\}$ such that $k \geq c$. We are lookink for the onto function

$$\varphi : \{1,\dots,k\} \to \{1,\dots,c\}$$

which maximizes the cardinality of overlapping clusters from the initial partition of X and the one induced by the feature F:

$$S_\varphi = \sum_{i=1}^{k} | F^{-1}(f_i) \cap A_{\varphi(i)} |$$

This problem can be formulated as a maximum matching problem for which we apply the Hungarian algorithm. Thus the cluster A_j may be interpreted as a mixture of the qualitative values from the set

$$\{f_i | \varphi(i) = j\}$$

We also define the matching degree as

$$\frac{S_\varphi}{|X|}$$

which is a sub-unitary value and characterizes the proportion in which the qualitative feature can explain the partition of X.

The component of our program system which enables the qualitative interpretation of obtained partitions enters quantitative and qualitative features of the classified samples fulfilling certain criteria concerning their features. Then a second module classifies the selected samples into groups as it was done by previous clustering procedures or according to the values of grouping features. Groups are identified as codes (for qualitative features) or intervals (for quantitative features). Quantitative features transformations can be performed. If two partitions are thus obtained, they can be compared as it was shown above. Selections, groups and transformed features can be stored in output files. Scatter plots of one quantitative feature against another can also be obtained and are useful in examining the performances of the clustering algorithms we used, the discrimination power of the two features and the regions delimited in the plane by the clusters.

6. **Facilities of the software.** This software gathers in a

unitary conception various aspects of classification: hierarchical and single-level classification, fuzzy and hard partitioning, pattern preprocessing and postprocessing, graph-theoretical methods and methods based on the minimization of a certain functional, supervised and unsupervised classification. Here are now some of the implementation facilities of this system:

- portability, as being written in Pascal language;
- simple text file structure of input and output data, which enables its use in a sequence of processing stages;
- independence of the system components, which permits interchanging their order during processing, omitting or reiterating them in order to obtain improved partitions;
- possibility of modifying the memory limits for data according to the capacity of the computer (this is done simply by modifying some constants and recompiling the programs);
- listing file option for obtaining a list with intermediate or final results;
- supplementary possibilities to limit the execution of iterative procedures by setting time, number of steps and maximum level in the classification hierarchy limits;
- other options which enable a flexible execution of programs.

This software was applied in geology, to the determination of certain types of mineralizations and to the parallelization of tuff horizons [5], in geography, to the regionalization of hydroenergetical potentials [6] and water resources [7] and in biology, to the determination of plants associations specific to certain environment conditions [4].

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TOPOGRAPHICAL DATA MANAGEMENT SYSTEM

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Rezumat. - Sistem de gestiune a datelor topografice. Articolul prezintă un sistem original de gestiune a datelor topografice implementat sub sistemele de operare RSX şi PC DOS. Informaţia preluată de pe hărţi o constituie coordonatele punctelor de observaţie (puncte în care s-au efectuat anumite determinări calitative sau cantitative), precum şi entităţile grafice curbe, regiuni, semne convenţionale, texte). Culegerea datelor se realizează prin digitizare, sub controlul unui editor grafic. Exploatarea bazei de date presupune extragerea şi reprezentarea datelor situate în fereastra de lucru, definirea de noi entităţi grafice, calcule simple (arii, medii ale unor funcţii de parametri cantitativi).

0. Introduction. The aim of this paper is to present an original topographical data management system for the acquisition and processing of data taken from maps. We may also consider, instead of maps, any kind of drawing consisting of curves, regions, conventional signs and texts. This system was implemented under the operating systems RSX and PC DOS.

1. Map entities. An item (data element) on a map will be called topographical entity. Two kinds of topographical entities are considered: observation points and graphical entities. Observation points are those points on a map where certain qualitative or quantitative parameters were determined. For instance, on a geological map, mineral resources and petrographical types are qualitative parameters while percentages of certain chemical elements are quantitative parameters. Graphical entities are curves (opened or closed), regions (areas

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delimited by a closed curve and filled with a certain colour), conventional signs (circles, triangles, cross-marks, etc.) and texts. Graphical entities are characterized by three features: their type, the plotting mode (for instance the type and the colour of line for curves, the filling colour for regions, the character set for texts) and the user code. This last feature is an integer associated by the user with each graphical entity in order to handle it easier. The position on the map of a graphical entity is defined by a variable number of points in a given order in the case of curves and regions and by a fixed number of points for the other two entities; thus, one point is needed to indicate the position of a standard conventional sign or text, while two points are needed for variable radius circles or inclined texts.

Observation points are numbered; their qualitative and quantitative parameters as well as the coordinates of their positions on the map are stored in separate files. A group of graphical entities is formed by a single entity with variable number of points or by several entities with fixed number of points and the same features. Three files are used to store graphical entities: an entity file, a coordinate file and a text file. The first one contains a record for each group of graphical entities; this record consists of three features of the graphical entities from the group (entity type *ET*, plotting mode *PM*, user's code *UC*) and two pointers *FP*, *LP* to the first and the last point in the coordinate file which define the position of entities. The coordinate file contains sequences of (x, y, z) - coordinates corresponding to the groups or graphical entities from the entity file. Points indicating the position of a text have instead of the *z* - coordinate a pointer to the corresponding text in the text file. This file contains all the texts from the map; the end of each text and the letter of which position was indicated in the coordinate file (if not the first one) are marked. The described structure of graphical entities is given in fig. 1.

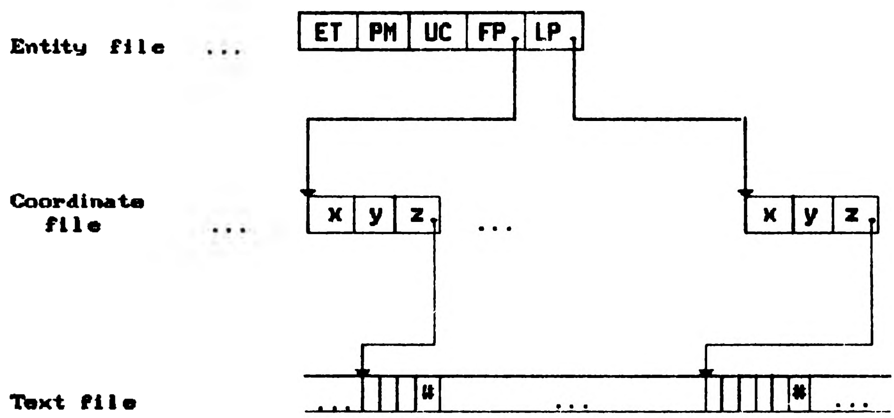


Fig. 1

2. **Data acquisition**. We are now concerned only with the acquisition of coordinates which indicate the position of topographical entities. This is done by means of a digitizer. The map may be fastened to its plane table in any position since the acquisition programs make a corrective rotation. The absolute coordinates are then computed according to the map scale and the coordinates of its origin. Thus, maps of adjacent zones can be assembled to form the general map.

Observation points can be digitized consecutively, in the order specified by the user, examining only not yet digitized points from a given interval or a certains group of points at a time. In this latter case, we have to provide a file with the number of group to which each observation point belongs.

The acquisition of graphical entities is performed by a graphical editor which plots the entities on the display as they are stored into computer. A graphical cursor may be moved in the current window represented on the display (corresponding to a rectangular area from the map) by means of the arrow keys of the keyboard or a mouse and its coordinates are indicated. The position of the digitizer cursor on the display can also be indicated.

Graphical entities which are edited at a time are stored in the computer memory hierarchically, on four levels. This structure enables a quick performing of editing operations. Graphical entities are divided into fragments which are portions of curves or groups of conventional signs or texts placed or not in the current window. Fragment points are divided into sequences which are stored in certain memory locations called pages. We are now able to describe the tree structure of edited data (fig. 2). The first level is a two-way list of data groups referring to groups of graphical entities. Curves which do not intersect the current window at all do not appear in this list. A data group consists of the three features of the corresponding graphical entities (entity type *ET*, plotting mode, user's code *UC*) and two pointers *FF, LF* to their first and last fragments. On the second level are placed the two-way lists of data groups referring to fragments. Chaining of curve fragments observes the order in which they are placed on the curve. A data group contains the fragment type *FT* (inside or outside the current window), a pointer *GE* to the group of graphical entities to which it belongs and two pointers *FP, LP* to the first and last point of the fragment. Only points of fragments inside the current window are stored in the internal memory; thus, for fragments outside the window, pointers *FP* and *LP* point directly to the coordinate file. On the third level are placed the two-way lists of pages, each one containing a pointer *PF* to the fragment to which it belongs and the sequence of data groups referring to points. Such a data group consists of the (x, y, z) - coordinates and two pointers *P1, P2*, which chain in a two-way list the points placed in the same square on the map. The number of squares into which the current window is divided is given by the user. This chaining enables a quick retrieval of points given a certain neighbourhood of them. For instance, when digitizing a point which has already been digitized, we can search for this point in a neighbourhood of the curently digitized point and replace the coordinates of the latter by those of the former. Connection of curves can thus be carried out without errors.

The following editing operations can be performed:

- acquisition of a graphical entity (coordinates from the digitizer, features and texts from the keyboard);

- prolongation of an open curve (the corresponding extremity of the curves is indicated using the graphical cursor);

- connection of two open curves or of the extremities of an open curve to form a closed one (the extremities of the curves/curve are indicated using the graphical cursor and then

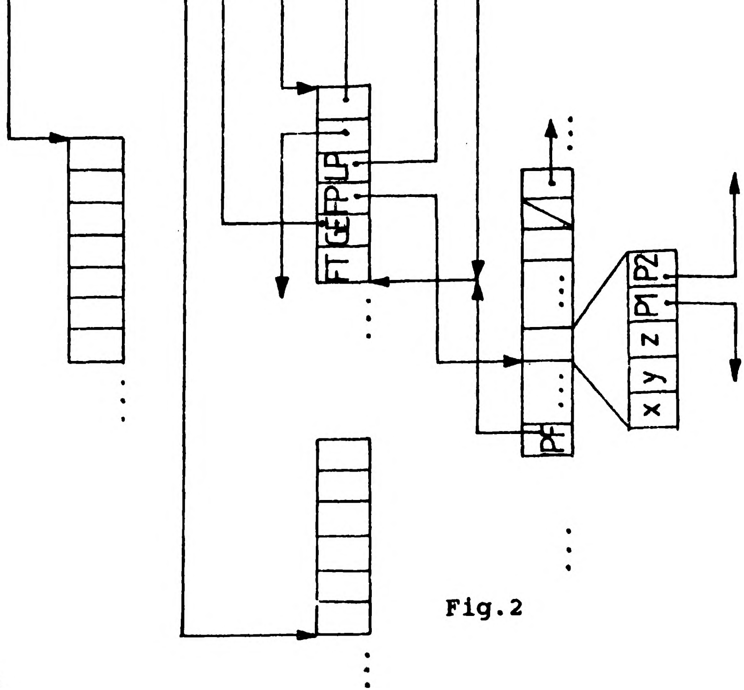
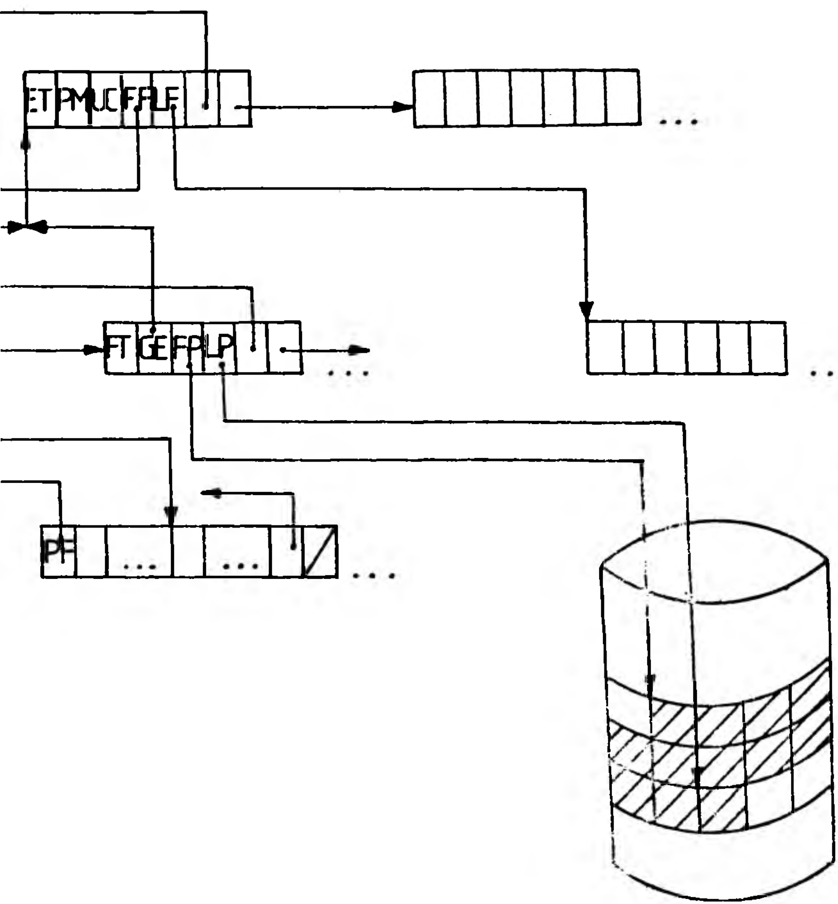


Fig.2


ET
PM
UC
FF
LF
FT
GE
FP
LP
PF

the new curve fragment is digitized);

- modification of a curve fragment (a certain curve fragment is deleted and then its extremities are connected by a new fragment);
- deletion of a curve fragment, of a curve entirely, of a conventional sign or text (the corresponding graphical entities are indicated using the graphical cursor);
- change of the features of a graphical entity;
- hardcopy of the current window;
- asking for help on the editing menu;

Deletion and feature change operations can be performed on several graphical entities at a time without viewing their effects. These entities are also selected by indicating their features (wild cards are permitted for some of the features).

3. **Data base enquiry.** Observation points retrieval is carried out in terms of the following criteria: point number, qualitative and quantitative parameters. Qualitative parameters are specified as codes while quantitative ones as intervals. Selected observation points can be classified according to the value of a grouping parameter; hence cluster identifier is a supplementary retrieval criterion. Retrieval criteria for graphical entities are their three features. Accessing of a map entity is followed by plotting it on the display. A curve can be plotted by joining its points with straight-line segments or by smoothing it. Smoothing can be carried out using a cubic spline interpolation or an original method which iteratively halves the

angle between two neighbour straight-line segments until these segments become sufficiently small. Thus, the user can construct a map of an area he desires and containing only the information he indicates. Moreover, he can add graphical entities, perform some elementary computations and blow up a rectangular area.

Once the map is thus constructed, certain entities on it can be identified using the graphical cursor. A temporary selection of observation points is considered. We now explain how entities are identified and what kind of operations can we perform on identified entities.

a) Operations involving a single observation point:

- viewing the observation point with a given number;

- finding the number of an observation point;

- inserting/removing an observation point in/from the temporary selection;

- displaying the quantitative parameters of an observation point or elementary functions of them.

b) Operations on groups of observation points. A group of observation points is formed by the points of a cluster, by the temporary selected points or by the points placed in a rectangular or arbitrary region indicated by the user. The following operations can be performed on such groups:

- displaying the numbers of observation points from the group;

- inserting/removing the points of the group from the temporary selection;

- means computation of the group points quantitative parameters or functions of them.

c) Operations on graphical entities.

- adding/deleting a graphical entity on/from the display (we indicate its position using the graphical cursor);

- storing/removing a graphical entity in/from the data base;

- modification of graphical entity features;

- displaying the area of a region and the number of observation points placed in it (these parameters can be used together with the quantitative parameters of the observation points to compute the value of certain elementary functions depending on them);

- displaying the coordinates of the graphical cursor.

We conclude by mentioning that the system developed in this paper is a useful tool for the management of a data base containing topographical data which can be then used by other software to construct 3*D* plottings. Referring to the parameters of observation points, the system is also useful for a primary data analysis and for the interpretation of statistical processing or clustering results.

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THEORETICAL SUPPORT FOR OBJECT-ORIENTED AND PARALLEL PROGRAMMING

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Rezumat. Un model teoretic privind programarea paralelă şi orientată-obiect. Lucrarea prezintă unele aspecte principale ale unui model algebric pentru specificarea unor concepte de bază din limbajele de programare. Legătura cu alte lucrări din domeniul specificării algebrice este prezentată în partea introductivă. Secţiunea 2 prezintă pe scurt conceptul algebric de ierarhie *HAS*. Conceptele de obiect, metoda şi clasa, specifice programării orientate-obiect sînt expuse în secţiunea 3. Secţiunea 4 este destinată conceptelor de algoritm secvenţial, algoritm paralel, proces secvenţial şi proces paralel.

1. **Introduction.** Generally, the specification of a programming language has two purposes. The first purpose is the specification of the data types proper to the language to be specified, which include the primitive (predefined) data types and the composed data types. The second purpose involves the specification of the operations (statements) which act on the data types. The algebraic approach of a language specification constitutes the topics of a great number of papers. We shall mention further down some such works directly related to the present paper.

In [3] and [4] T. Rus presents a hierarchical and algebraic specification model. A context-free algebra is associated to a context-free grammar. The specification aiming at such a model involves a cascade of heterogeneous algebras. The last algebra from the cascade (hierarchy) corresponds to the complete specification of the language. In this model, the hierarchical

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order (hierarchy depth) depends on the context-free grammar which specifies the syntax of the language. My paper, which uses an algebraic model [2], starts from the algebraic definition of certain primitive data types (defined in the form of homogeneous algebras). These will be organized into a specification base (signature) for the whole hierarchy of heterogeneous algebras which will constitute the specification levels of the programming language to be specified. This time the hierarchical order (the number of levels of the hierarchy) depends on the complexity of the elements of the language to be specified. The proposed model defines in a personal manner the concepts of object, method and class, proper to the object-oriented programming.

An algebraic approach of a language specification for abstract data types specification can be found in [6]. Unlike this paper, in my paper the concepts of object and method are defined on complexity levels (corresponding to the hierarchical levels), allowing in this way a relatively easy implementation of the model. The final result of a language specification will look like a multi-level tree, unlike [1] where the specification appears as a tree with a single level. At each level the data types and their corresponding operations are defined.

Lastly, on the basis of the algebraic definition of the concepts of parallel algorithm and parallel process [5] implanted on the specification levels of the proposed model, the algebraic specification becomes concurrent algebraic specification.

2. ***HAS* hierarchy concept.** The concept of *HAS* hierarchy was presented in detail by T. Rus in [3] and [4]. This concept is based on the following two principles:

p1. Every homogeneous algebraic structure is a *HAS* of zero hierarchical level in a *HAS* hierarchy;

p2. Every i-level *HAS* can be chosen as a base for an $(i+1)$-level *HAS*.

Let HAS^i be the i-level *HAS* given in the form of the pair:

$$HAS^i = < A^i \ , \ \Omega^i >,$$

where A^i is the support, while Ω^i is the collection of operations defined on the support A^i. The support of the HAS^i is used as index set for the specification of the HAS^{i+1}. Consider the n-ary operation $\omega\epsilon\Omega^i$, and $a = \omega\ (a_1,a_2,\ldots,a_n)$, where $a,a_1,a_2,\ldots,a_n\ \epsilon$ $\epsilon\ A^i$. The set of operation schemes Σ_ω is defined as follows:

$$\Sigma_\omega = \{\sigma = <n,\omega,a_1a_2\ldots a_na> \ / \ a=\omega(a_1,a_2,\ldots,a_n> \}.$$

Since the operations specified by means of the operation schemes σ are heterogeneous operations, we consider the symbol of the operation ω to be distributed upon its operands. In other words, the operation scheme σ becomes $\sigma=<n,s_0s_1\ldots s_n,a_1a_2\ldots a_na>$. For a complete specification of the HAS^{i+1} by the HAS^i , consider a function F which associates to each operation scheme $\sigma\epsilon\Sigma_\omega$, $\omega\epsilon\Omega^i$, a heterogeneous operation specific to HAS^{i+1}. If $\sigma=<n,s_0\ldots s_n,$ $a_1\ldots a_na>$, then F_σ is a specific operation in HAS_{i+1}, that is , the function:

$$F_\sigma \ : \ A^i_{a_1}\times A^i_{a_2}\times\ldots\times A^i_{a_n} \dashrightarrow A^i_a \ .$$

In these conditions HAS^{i+1} is specified on the basis of HAS^i

and has the form:

$$HAS^{i+1} = \langle A^{i+1} = (A_a^{i+1})_{a \in A^i}, \Sigma = (\Sigma_\omega)_{\omega \in \Omega^i}, F \rangle .$$

On the basis of the concept of HAS hierarchy, in Section 3 we shall present schematically a model for the specification of a programming language. The terms of object, method, and class, specific to the object-oriented languages, are found again in our model. Section 4 presents briefly the algebraic formalization of the concepts of parallel algorithm and parallel process [5] adapted to our model.

3. **Object-oriented hierarchical specification.** An abstract object is defined as heterogeneus algebra as follows:

Object name : NAME;

Supports : $NAME_1, NAME_2, \ldots, NAME_n$;

Operation schemes : $\sigma_1, \sigma_2, \ldots, \sigma_m$;

Variables : $LIST_1$: $NAMEi_1$

$LIST_2$: $NAMEi_2$

....

$LIST_k$: $NAMEi_k$

Axioms : $w_{11}=w_{12}$.

$w_{21}=w_{22}$,

......

$w_{p1}=w_{p2}$;

end NAME.

We choose as zero level of the *HAS* hierarchy (HAS^0) the following partial homogeneous algebra:

$HAS^0 = \langle A^0, \Omega^0, H^0 : A^0 \rightarrow I \rangle,$

where:

A^0 = support of the algebra, consisting of a set of predefined abstract objects $(A_j^0)_{j \in J}$ specified as homogeneous algebras;

Ω^0 = the set of operations defined on the objects (A_j^0), $j \in J$;

H^0 = function which associates a measure to every $a \in A^0$;

I = the set of the measures printed out by the function H^0.

The level one of the HAS hierarchy (HAS^1) is defined on the basis of HAS^0 and has the form:

$$HAS^1 = \langle A^1 = (A_i^1)_{i \in I},\ \Sigma = (\Sigma_\omega)_{\omega \in \Omega^0},\ H^1 : A^1 \dashrightarrow I, F^1 \rangle ,$$

where:

A^1 = a family of subsets with the property that the same measure $H^0(a)$, $a \in A_i^1$, $i \in I$, is associated to all elements of such a subset;

Σ = the set of operation schemes specified by the operations corresponding to the zero level;

H^1 = the same definition as in the case of H^0;

F^1 = symbol of a function $\left(F^1 : (\Sigma_\omega)_{\omega \in \Omega^0} \dashrightarrow OP(A^1)\right)$, where $OP(a^1)$ is the set of all operations defined on A^1.

For $\omega \in \Omega^0$, $a_i \in A^0$, $i=1,2,\dots,n$, $a=\omega(a_1,a_2,\dots,a_n)$, an operation scheme

$$\sigma=<n,s_0s_1\ldots s_n,H^0(a_1)H^0(a_2)\ldots H^0(a_n)H^0(a)>$$

is associated.

If $\sigma\in\Sigma$ is an operation scheme, then F_σ^1 , , defined as follows:

$$F_\sigma^1 : A^1_{H^0(a_1)}xA^1_{H^0(a_2)}x\ldots xA^1_{H^0(a_n)}\rightarrow A^1_{H^0(a)}$$

is a heterogeneous operation in HAS^1.

The function H^0 associates to every $a\epsilon A^1$ a characteristic called measure. Another characteristic associated to an $a\epsilon A^1$ is the interpretation mode. For a given measure $H^0(a)$ associated to an element $a\epsilon A^1$ there can exist several interpretation modes (integer, real, boolean, character, etc.). Hence the interpretation mode for an $a\epsilon A^1$ is the significance (integer, real, boolean, character, etc.) assigned to the representation (encoding) of a. We particularize the function H^0 as follows: for every $a\epsilon A^1$, $H^0(a)=1(a)$, where $1(a)$ is the representation length, representing a in the computer storage. The conection between the interpretation mode and the measure (representation length) of the unstructured type data can be performed by a bi-dimensional matrix. One considers $a_{ij}=1$ if for the representation length i there exists the interpretation mode j, and $a_{ij}=0$ in the opposite case. The row index of the matrix ($i=1,2,\ldots,n$) signifies the possible representation length of the data from A^1, while the column index ($j=1,2,\ldots,m$) is the index associated to the elements of the set of the interpretation modes. Having these elements, we define the level two of the *HAS* hierarchy (HAS^2) on the basis of the preceding level:

$$HAS^2=\langle A^2=(A^2_{ij})_{i\in N, j\in M}, \Sigma=(\Sigma_\omega)_{\omega\in OP(A^1)}, H^2:A^1xM\to NxM, F^2\rangle$$

where:

A^2_{ij} = the support of the data type i interpreted in the mode j;

$\Sigma\omega$ = the set of the operation schemes generated by the operation $\omega\epsilon OP(A^1)$;

N = a set containing all possible representation length;

M = a set containing all possible interpretation modes;

H^2 = function which establishes by its values the index set for the family of sets A^2;

$\forall\ (a,j)\epsilon A^1xM, H^2(a,j)=(H^1(a),j)a_{H^1(a)j}$

F^2 = the symbol of a function which associates to an operation scheme

$$\sigma=\langle n, s_0s_1\ldots s_n, H^2(a_1,j_1)H^2(a_2,j_2)\ldots H^2(a_n,j_n)H^2(b,j_{n+1})\rangle$$

a heterogeneous operation on the data set A^2.

The zero level HAS^0 points out the primitive (predefined) data types together with the operations defined on them. The level one HAS^1 points out the data division in subsets, the criterion of differentiation being the representation length. The level two HAS^2 differentiates the data according to a characteristic supplementary to the preceding level, that is, the interpretation mode. This allows the specification of data types (short integer, long integer, real on different length, character, etc.). The composed data types are specified at this level, too.

Let us denote by $OP(HAS^{i})_{i=0,1,2}$ the set of the heterogeneous operations defined on the three hierarchical levels. An equivalence relation, denoted HAS^{e}, is defined on this set. Two operations ω_1, ω_2 $\in(HAS^{i})_{i=0,1,2}$ are, by definition, called equivalent if they have the same definition domain. Consider ω_1: $A_1 x A_2 x \ldots A_n \rightarrow A_{n+1}, \omega_2 : B_1 x B_2 x \ldots x B_n \rightarrow B_{n+1}$, $\omega_1 HAS^{e} \omega_2$, where

$$A_i = \begin{cases} A_i^0, & \text{if } \omega_i \in \Omega^0 \\ A^1_{H^0(a_i)}, & \text{if } \omega_1 = F^1_\sigma, \sigma \in \sum, \sigma = \langle n, s_0 s_1 \ldots s_n, \\ & \quad H^0(a_1) \ldots H^0(a_n) H^0(a) \rangle \\ A^2_{H^2(a_i, j_i)}, & \text{if } \omega_1 = F^2_\sigma, \sigma \in \sum, \sigma = \langle n, s_0 s_1 \ldots s_n, \\ & \quad H^2(a_1, j_1) \ldots H^2(a_n, j_n)\, H^2(a, j_{n+1}) \rangle \end{cases}$$

$i=1,2,\ldots n+1$, and B_i is obtained in a similar manner. Follows that $A_1=B_1, A_2=B_2, \ldots, A_n=B_n$ (equality of sets). Let $E = OP(HAS^{i})_{i=0,1,2}/HAS^{e}$ be the set of the equivalence classes. An equivalence class $e \in E$ consists of all operations with the same definition domain. Let $C_1 x C_2 x \ldots x C_n$ be the common definition domain for the operations belonging to the equivalence class e. We call object an n-uple $(c_1, c_2, \ldots, c_n) \in C_1 x C_2 x \ldots x C_n$, while the set of all objects of this form will be called the class of objects associated to the equivalence class e. Let us denote by K the set of all classes of objects. We add to each class of objects $k \in K$ the object nil_k which constitutes the nil object of the class k. An object of the class k is either the object nil_k or an instance of the class k. The specification of a class k consists firstly of the specification of the form of the objects belonging to the class k. The form of a class k specifies the n-

uples which can appear as values of the instances of the objects belonging to the class *k*. On the other hand, the specification of the class *k* consists of the specification of a colection of methods belonging to the class *k*, too. An operation $\omega\epsilon\epsilon$ acts on the class of objects *k* according to a law well defined during the stage of hierarchical level construction. Such an operation on a class of objects *k* will be called method. The set of methods is classified on hierarchical levels, but there also exists hybrid methods whose definition domains originate in several hierarchical levels.

The number of levels in the *HAS* hierarchy is arbitrary. It depends on the needs of defining certain objects of high complexity degree. We stopped at the level two of the hierarchy, considering it to be sufficient for a concise exposition of the model.

4. Concurrent hierarchical specification. Let *<A,OP>* a homogeneous algebra, where *A* is the support and *OP* is the set of the operations defined on *A*. If the set *OP* also contains relations, then *<A,OP>* will be called algebraic system. The concept of heterogeneous algebraic system is obtained analogously.

Let us consider the heterogeneous algebra *HA=<KL,ME>*, where *KL* is the set of the classes of objects, while *ME* is the set of the methods specified in Section 3. We define the concept of algorithm over the given heterogeneous algebra HA as being the heterogeneous algebraic system *AL=<K*1*,Me,R>*, where:

$K1$ = a finite set of classes belonging to the support HA;

Me = a finite set of methods belonging to the set Me;

R = a relation indicating the order of execution of the methods from Me on the objects of $K1$.

If the ordering relation R is linear (or total) on Me, then the algorithm is called sequential algorithm.

If the ordering relation R is partial on Me, then the algorithm is called parallel (or concurrent) algorithm.

Since the set Me of the methods specifying an algorithm is finite, this one can always be decomposed into the subsets Me_1, $Me_2,\ldots,Me_k$, such that every Me_i, $i=1,2,\ldots,k$, is linearly ordered by the execution of the methods. Let us denote by R_i the linear relation defined by the order of execution of the methods in Me_i. If for every $i,j,i\neq j,i,j=1,2,\ldots,k$, the subset $K1_i\subseteq K1$ on which the methods from Me_i are acting and the subset $K1_j\subseteq K1$ on which the methods from Me_j are acting are disjoint each other, then the algorithm $<K1,Me,R>$ gives rise to a family of sequential algorithms $<K1_i,Me_i,R_i>$, $i=1,2,\ldots,k$. These sequential algorithms can be parallelly executed and keep the consistence of the computations specified by the original algorithm.

In this context a processor is identified with an abstract agent able to execute any method featuring the HA specification. The concept of process over the HA specification is defined through the couple Process (HA)=<Processor,AL>.

A process P=<Processor,AL> will be called sequential if the defined algorithm AL is sequential.

A process P=<Processor,AL> will be called parallel (or concurrent) if the defined algorithm *AL* is a parallel algorithm.

5. **Conclusions**. The basic theoretical concepts (belonging to the programming languages) specified by means of the proposed model constitutes a theoretical nucleus for the simultaneous approach of parallel programming and object-oriented programming. The nucleus, semantically and syntactically defined, could constitute a reference basis for any other semantic construction reductible to one of the semantic forms from the nucleus by established transformation rules.

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FORMAL INTEGRATION OF CERTAIN CLASSES OF FUNCTIONS

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REZUMAT. - Integrarea formală a unor clase de funcţii. Lucrarea prezintă o metodă de determinare analitică a primitivei unei funcţii raţionale. Legat de aceasta, sînt expuşi şi algoritmi de manipulare simbolică a polinoamelor precum şi de factorizare a polinoamelor peste Z[X]. Este descrisă de asemenea determinarea substituţiilor prin care problema integrării funcţiilor din anumite clase se poate reduce la cazul raţional.

1. Introduction. The symbolic computation represents the entrance in a new computer usage era, in which the computer becomes smarter and powerful enough to do complex scientific computation, for example the formal integration. We can notice here the software packages for scientific computation MACSYMA, REDUCE, MATHCAD and MATHEMATICA.

In this paper we present the formal integration of rational functions with integer coefficients ($R(x)$) and related to this, the formal integration of functions from the classes R(exp) and R(sin, cos, tan) where the arguments of the exp, sin, cos and tan functions have the form kx with $k \in \mathbb{Z}$.

With these algorithms I realized a Pascal program for IBM PC compatible computers running MS-DOS, which can be easily extended for larger classes of functions.

2. Substitutions. Since the problem of the formal integration of rational functions is simpler than the same problem for another function types, we try to reduce the given

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function to a rational one by using suitable substitutions. For this reason the determination and the effectuation of the suitable substitution represents one of the most important part of a formal integration program.

In our case, we can apply the classical substitutions.

If the function belongs to the $R(\exp)$ class, the suitable substitution is $\exp(x)\to t$ and all the terms $\exp^m(nx)$ become t^{mn}.

If the function belongs to the $R(\sin, \cos, \tan)$ class we can transform the function to a equivalent function f from the $R(\sin, \cos)$ class. We have three cases:

$$f(-\sin, -\cos) = f(\sin, \cos)$$

$$f(-\sin, \cos) = -f(\sin, \cos)$$

$$f(\sin, -\cos) = -f(\sin, \cos)$$

The corresponding substitutions are $\tan(x)\to t$, $\cos(t)\to t$ and $\sin(x)\to t$. If our function doesn't verify any of these conditions, the suitable substitution is $\tan(x/2)\to t$.

Through these substitutions we transform our function in a $R(x)$ class function.

3. The formal integration of a R(x) class function. Suppose we have to integrate the function $f(x)=p(x)/q(x)$ where $p,q\in Z[x]$ are primitive polynomials, $\deg p(x)<\deg q(x)$ and $\gcd(p(x), ,q(x))=1$.

Obviously, every polynomial $q\in Z[x]$ has a unique squarefree decomposition:

$$q(x)=q_1(x)(q_2(x))^2\dots(q_k(x))^k$$

where $q_i\in Z[x]$ are squarefree polynomials (some of them can be constants 1) and $\gcd(q_i(x), q_j(x))=1$ for $1\le i, j\le k$ and $i\ne j$.

This decomposition can be obtained with Yun's algorithm described in section 4.

Using the simple fraction decomposition method, described in section 5, we obtain the polynomials $p_i(x)$ so that

$$\int \frac{p(x)}{q(x)}dx = \sum_{i=1}^{k}\int \frac{p_i(x)}{(q_i(x))^i}dx$$

Certainly, if $q_i(x)=1$ then $p_i(x)=0$.

In order to reduce the numerator's degree and to extract the rational part of the result we use the Hermite-Ostrogradsky method (described in section 6) and we determine the polynomials $s_i(x)$ and $r_i(x)$ for which:

$$\int \frac{p_i(x)}{(q_i(x))^i}dx = \frac{s_i(x)}{(q_i(x))^{i-1}} + \int \frac{r_i(x)}{q_i(x)}dx \quad 1<i\le k$$

In this moment, $\sum_{i=2}^{k} \frac{s_i(x)}{(q_i(x))^{i-1}}$ represents the rational part of the result. The remainder integrals will give us logarithmic or arctangent terms.

We need now the factorization of the polynomials q_i over $Z[x]$.

$$q_i(x)=q_{i1}(x)\ldots q_{in_i}(x)$$

where $q_{ij}(x)$ are irreducible polynomials over $Z[x]$.

This problem can be solved by using the Berlekamp-Hensel algorithm described in section 7.

Using again the simple fraction decomposition algorithm, we determine the polynomials $r_{ij}(x)$ for which:

$$\frac{r_i(x)}{q_i(x)} = \sum_{j=1}^{n_i} \frac{r_{ij}(x)}{q_{ij}(x)}$$

Now we have to compute $\int \frac{r_{ij}(x)}{q_{ij}(x)} dx \quad 1 \le i \le k, \quad 1 \le j \le n_i$.
If $r_{ij}(x) = a * g'_{ij}(x)$ $(a \in Q)$ then the result is the logarithmic term $a \ln(q_{ij}(x))$. However, if deg $r_{ij}(x)$=deg $q_{ij}(x)-1$ we can extract a logarithmic term $\ln(q_{ij}(x))$ in order to reduce the degree of the numerator at the highest deg $s_{ij}(x)-2$.

If deg $q_{ij}(x)=2$ then we have an arctangent or a logarithmic term depending on the sign of the discriminant.

If deg $q_{ij}(x) \in \{3,4\}$ the equation $q_{ij}(x)$ can be solved through radicals and therefore we can factorize $q_{ij}(x)$ in a product of two polynomials of degree 1 or 2, over a radical extension of $Q[x]$.

If deg $q_{ij}(x) > 4$ we shall search for a substitution in order to reduce the denominator's degree. Let's suppose we have to determine:

$$\int \frac{u(x)}{v(x)} dx$$

with $v \in Z[x]$ a irreducible polynomial over $Z[x]$, deg $v(x) > 4$ and that we can effectuate the substitution $g(x) \rightarrow t$. In this situation there exist the polynomials $f, h \in Q[x]$ so that:

$$\frac{u(x)}{v(x)} = \frac{g'(x) f(g(x))}{h(g(x))}$$

If deg $g(x)$ = a then follows:

deg $u(x)$ = a-1+a deg $f(x)$

deg $v(x)$ = a deg $h(x)$

$$u(x) = g'(x)\ f(g(x))$$

$$v'(x) = g'(x)\ h'(g(x))$$

This relations shows that we can search $g'(x)$ (the derivative of the possible substitution $g(x)$) among the divisors of gcd($u(x)$, $v'(x)$) with the property that 1+deg $g'(x)$ = deg $g(x)$ divides gcd(1+deg $u(x)$, deg $v(x)$).

4. **The squarefree decomposition Yun's algorithm.** It is fairly easy to show that if $q \in Z[x]$ and $q_i(x)$ is a polynomial such that it's roots are the i order roots of q, then $q_i \in Z[x]$, all the roots of $q_i(x)$ have the order 1 and $(q_i(x))^i$ divides $q(x)$.

Let's suppose that all the roots of $q(x)$ have the order less or equal to $k \in$ N. In this case:

$$q(x)=q_1(x)\ (q_2(x))^2 \ldots (q_k(x))^k.$$

Furthermore, since for $i \neq j$ $q_i(x)$ and $q_j(x)$ haven't common roots

$$\gcd(q_i(x),\ q_j(x)) = 1.$$

We can now see that:

$$q'(x) = q_1'(x) \ldots (q_k(x))^k + \ldots + k q_1(x) \ldots q_k'(x)\,(q_k(x))^{k-1}$$

$$c(x) = \gcd(q(x), q'(x)) = q_2(x)\,(q_3(x))^2 \ldots (q_k(x))^{k-1}$$

$$r(x) = \frac{q(x)}{c(x)} = q_1(x)\,q_2(x) \ldots q_k(x)$$

$$s(x) = \gcd(c(x), r(x)) = q_2(x) \ldots q_k(x)$$

In this moment $q_1(x) = \frac{r(x)}{s(x)}$ and we see that making $q(x) \leftarrow c(x)$ and repeating the above operations until $q(x)$ become constant, we obtain the polynomials $q_1(x), \ldots, q_k(x)$. We also remark that $r, c, z \in Z[x]$.

The above relations represent the mathematical basis of the

Yun's algorithm. The complete description can be found in [2].

5. Simple fraction decomposition algorithm. Assume $p,u,t \in Z[x]$ and $\gcd(u(x),t(x)) = 1$. This algorithm will compute the polynomial $r \in Q[X]$ so that:

$$\frac{p(x)}{u(x)\,t(x)} = \frac{r(x)}{u(x)} + \frac{s(x)}{t(x)}$$ and $\deg r(x) < \deg u(x)$, where $s \in Q[x]$ can be computed analogously.

From the above relation we obtain that:

$$p(x) = r(x)t(x) + u(x)s(x)$$

and

$$r(x) \equiv r(x) \bmod u(x).$$

This implies that:

$$\begin{aligned} p(x) \bmod u(x) &\equiv r(x)t(x) \bmod u(x) \\ &\equiv (r(x) \bmod u(x))\ (t(x) \bmod u(x)) \bmod u(x) \\ &\equiv r(x)\ (t(x) \bmod u(x)) \bmod u(x). \end{aligned}$$

Since $\gcd(u(x),\ t(x)) = 1$, there exist the polynomials $v,w \in Q[x]$ such that:

$$u(x)v(x) + w(x)t(x) = 1.$$

(The polynomials v and w can be computed using the Extended GCD Algorithm).

By dividing this relation by $u(x)$ we can see that:

$$w(x) \equiv t(x)^{-1} \bmod u(x)$$

and this tells us that

$$r(x) \equiv (p(x) \bmod u(x))\ w(x) \bmod u(x).$$

6. The Hermite-Ostrogradski algorithm. This algorithm computes the polynomials $a,b \in Q[x]$ so that:

$$\int \frac{p(x)}{(q(x))^n}dx = \frac{a(x)}{(q(x))^{n-1}} + \int \frac{b(x)}{q(x)}dx$$

where $p,q \in Z[x]$ and q is squarefree.

It is easy to show that $\gcd(q(x),q'(x)) = 1$ since q is squarefree. Therefore we can use the Extended GCD algorithm in order to determine the polynomials $v,w \in Q[x]$ so that:

$v(x)q'(x) + w(x)q(x) = 1.$

If we multiply this relation with $-p(x)/(n-1)$ and:

$s(x) = -\frac{p(x)\,v(x)}{n-1}, \quad t(x) = -p(x)\,w(x)$

we obtain that $s(x)\,q'(x) + \frac{t(x)\,q(x)}{n-1} = -\frac{p(x)}{n-1}$ and

$$-(n-1)s(x)q'(x) = p(x) + t(x)q(x).$$

Consequently,

$$\left[\frac{s(x)}{(q(x))^{n-1}}\right]' = \frac{s'(x)}{(q(x))^{n-1}} - \frac{(n-1)\,s(x)\,q'(x)}{(q(x))^n} =$$

$$= \frac{s'(x)}{(q(x))^{n-1}} + \frac{p(x)+t(x)\,q(x)}{(q(x))^n} = \frac{p(x)}{(q(x))^n} + \frac{s'(x)+t(x)}{(q(x))^{n-1}}$$

This means that if $r(x) = s'(x) + t(x)$ then

$$\int \frac{p(x)}{(q(x))^n}dx = \frac{s(x)}{(q(x))^{n-1}} - \int \frac{r(x)}{(q(x))^{n-1}}dx$$

It is now clear that using this algorithm for $n-1$ times, we will obtain

$$\int \frac{p(x)}{(q(x))^n}dx = \frac{s_1(x)}{(q(x))^{n-1}} + \dots + \frac{S_{n-1}(x)}{(q(x))} + \int \frac{b(x)}{q(x)}dx$$

and thus $a(x) = s_1(x) + s_2(x)q(x) + \dots + s_{n-1}(x)(q(x))^{n-2}$.

7. The Berlekamp-Hensel algorithm. Let $f(x) = a_n x^n + \dots + a_1 x +$

$+ a_0$ be a squarefree and primitive polynomial with integer coefficients.

Also let

$S = a_0^2 + \ldots + a_n^2$

$M(f) = 2^n S$ (1)

$q \geq M(f)$, $q \in Z$

The algorithm presented here computes $r \in \mathrm{N}$ and the polynomials $u_1, \ldots, u_r \in Z[x]$ irreducible over $Z[x]$, such that

$f(x) = u_1(x) \ldots u_r(x)$.

It can be prove that if $b \in Z[x]$, $b(x) = b_0 + b_1 x + \ldots + b_s x^s$ and b divides f then $|b_i| < M(f)$ $i=0,s$. (see [4])

This means that if $b_i > 0$ then

$$b_i = b_i \mod q \in \left(0, \frac{q}{2}\right).$$

and if $b_i < 0$ then

$$b_i \mod q = q - b_i \in \left(\frac{q}{2}, q\right) \quad (2)$$

These observations lead us to the idea that the factorization of f over $Z_q[x]$ could be fairly closed to the factorization of f over $Z[x]$, since if

$f(x) = p(x)t(x)$ with $p, t \in Z[x]$

then

$f(x) \equiv p(x)t(x) \mod q$

and according to (2) we can determine the coefficients of $p(x)$ mod q which correspond to negative coefficients of $p(x)$.

The Berlekamp-Hensel algorithm is based on these conclusions and it has the following steps:

S1) Determine a prime number p, the least possible, for which $\deg f(x) = n$ (q doesn't divide the leading coefficient of f) and f remain squarefree in $Z_p[x]$.

S2) Use the Berlekamp's algorithm (see [3]) for the factorization of $f(x)$ over $Z_p[x]$

$$f(x) \equiv u_1(x)\dots u_s(x) \mod p$$

S3) Compute $M(f)$ given by (1).

S4) Pass from the factorization of f over $Z_p[x]$ to the factorization of f over $Z_{p^2}[x], \dots, Z_q[x]$ using the formula given by the Hensel's lema (see [3]), until $q=p^k \geq 2M(f)$. This step computes the polynomials $u_{1k}, \dots, u_{sk} \in Z_q[x]$ such that

$$f(x) \equiv u_{1k}(x)\dots u_{sk}(x) \mod q$$

$$u_{ik}(x) \equiv u_i(x) \mod p, \quad i=1,s.$$

S5) Compute the product of each possible combination of $1,2,\dots,s$ $u_{ik}(x)$ polynomials in $Z_q[x]$.
Normalise the coeficients of the product according to (2) by subtracting q from the coefficients greater than $\frac{q}{2}$.
If this normalised product divides f then it represents a factor of f and the $u_{ik}(x)$ polynomials which compose the product will be excluded from further combinations since f is squarefree.

Note that this is a polynomial time algorithm. There also exists the Kronecker's algorithm which is simpler and more intuitive but it requires exponential time and it become very inefficient for polynomials of degree greater than 5.

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A NEW METHOD FOR THE PROOF OF THEOREMS

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Resumat. În lucrare se prezintă un sistem formal de demonstrare prin respingere a teoremelor. Condiţia necesară şi suficientă impusă acestui sistem se bazează pe metoda lui J.Hsiang de demonstrare a teoremelor cu ajutorul sistemelor de rescriere a termenilor.

1. Introduction. Let T be a set of linguistic, algebraic or symbolic objects (as, for instance, first-order terms, programs) and let ~ be an equivalence relation on T.

DEFINITION [2]. A computable function $S:T \to T$ is called a canonical simplifier for the equivalence relation ~ on T iff for all $s, t \in T$:

$$S(t) \sim t$$

$$S(t) \leq t$$

(for some ordering $\leq$ on T)

$$t \sim s \to S(t) = S(s)$$

For computer algebra, the problem of constructing canonical simplifiers is basic, because of the following theorem:

THEOREM [2]. *Let T be a set of linguistic objects and ~ an equivalence relation on T. Then ~ is decidable iff there exists a canonical simplifier S for ~ .*

Let $T = T(F,V)$ be the algebra free generated by the set of variables V with the set of functions F; that is T is the minimal set of words on the alphabet $F \cup V \cup \{(,)\}$ such that:

1. $V \subseteq T$

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2. If $f \in F$, $\alpha(f)$ is its arity, and if $t_1,\dots,t_{\alpha(f)} \in T$, then

$f(t_1,\dots,t_{\alpha(f)}) \in T$

Let $E \subseteq T(F,V) \times T(F,V)$ be a set of equations. By the Birkhoff theorem (1935) s and t are semantically equal in the equational theory $E(E \models s = t)$ iff s and t are provably equal in the theory $E(E \vdash s = t)$.

Let $s \sim t$ be the equivalence relation defined by $E \vdash s = t$. Then $\sim$ is decidable iff there exists a canonical simplifier S for $\sim$.

2. Associated term rewriting system and the completion. Let E be a set of equations $E \subseteq T \times T$ and let R_E a term rewriting system (TRS) obtained such that

$\ell \to r \in R_E \Leftrightarrow \ell = r \in E$ and

$v(r) \subseteq v(\ell)$, where $v(t)$ is the set of variables in the term (object) $t \in T$. This system will be called TRS associated with E. The rewriting relation $\vec{R}_E$ has the inverse relation, transitive closure, the reflexive-symmetric-transitive closure denoted by $\overleftarrow{R}_E$, $\overset{+}{\vec{R}}_E$ and $\overset{*}{\leftrightarrow}_{R_E}$ respectively. Also, we have:

$$\tilde{E} = \overset{*}{\leftrightarrow}_{R_E}$$

For a TRS denoted R let be the following definition [3], [7], [8]:

DEFINITION. R is noetherian (R has the finite termination property) iff there is no infinite chain

$$t_1 \;\vec{R}_E\; t_2 \;\vec{R}_E\; t_3 \;\vec{R}_E \dots$$

DEFINITION. R is confluent iff $\forall\, x, y, z \in T \;\; \exists\, u \in T$ such that if $x \overset{*}{\vec{R}}_E z$ and $x \overset{*}{\vec{R}}_E y$ then $z \overset{*}{\vec{R}}_E u$, $y \overset{*}{\vec{R}}_E u$.

DEFINITION. If $x \in T$, $x\downarrow \in T$, $x \overset{*}{\underset{R_E}{\rightarrow}} x\downarrow$ and it does not exist t such that $x\downarrow \underset{R}{\rightarrow} t$ then $x\downarrow$ is normal form for x in TRS R (denoted $x \downarrow R$).

If R_E which is associated with a system of equation E is noetherian and confluent (i.e. complete) then, for $\forall x \in T$, the application $S(x) = x \downarrow R_E$ is a canonical simplifier. Then ~ is decidable, and we have :

$$s \sim t \quad \text{iff} \quad s \downarrow R_E = t \downarrow R_E$$

Stated in the context of confluence, the idea of completion is straightforward:

Given a set of equations E we try to find a set of equations F such that: $\underset{E}{=} \, = \, \underset{F}{=}$ and the relation $\vec{R}_F$ is confluent.

If this set of equations do not exists, then the completion must terminate with failure or the completion is impossible.

The first completion algorithm for rewrite rules is that of Knuth-Bendix (1967). For a general formulation of this algorithm some additional notion for describing the replacement of terms in terms are needed.

DEFINITION [1],[2],[5]. Let O(t) be the set of occurrences of a term t. If s, $t \in T(F,V)$ and $u \in O(t)$ then $t[u \leftarrow a]$ is the term that derives from t if the term occurring at u in t is replaced by the term s (t/u becomes s).

DEFINITION. $s \rightarrow t$ iff there is a rule $a \rightarrow b \in R_E$ (or an equation $((a,b) \in E)$, a substitution τ and an occurrence $u \in O(s)$ such that

$$s/u = \tau\ (a) \text{ and } t = s\ [u \leftarrow \tau\ (b)]$$

DEFINITION. The terms p and q form a critical pair in E iff there are equations $(a_1,b_1) \in E$ and $(a_2,b_2) \in E$, an occurrence u in $O(a_1)$ and the substitution τ_1, τ_2 such that:

1. a_1/u is not a variable
2. $\tau_1(a_1/u) = \tau_2(a_2)$
3. $p = \tau_1(a_1)\ [u \leftarrow \tau_2(b_2)]$

   $q = \tau_1(b_1)$

The algorithm Knuth-Bendix is based on the

THEOREM: *A TRS noetherian R_E is confluent iff for all critical pairs (p,q) of E: $p \downarrow R_E = q \downarrow R_E$.*

Then it suggests to augment R_E by the rule $p \downarrow R_E \rightarrow q \downarrow R_E$ or $q \downarrow R_E \rightarrow p \downarrow R_E$. This process may be iterated until, hopefully, all critical pairs have a unique normal form or it may never stops: the algorithm is at least a semidecision procedure for $\sim$.

The completion algorithm for rewrite rules (Knuth-Bendix, 1967) is therefore [2]:

Input: A finite set of equations E such that $\vec{R}_E$ is noetherian.

Output: 1. A finite set of equations F such that

$$\overset{*}{\underset{R_E}{\longleftrightarrow}} = \overset{*}{\underset{R_F}{\longleftrightarrow}}$$

and relation $\vec{R}_F$ (therefore system R_F) is confluent (therefore is decidable) or

2. the procedure stops with failure or
3. the procedure never stops

Algorithm [2]:

1. $F: = E$;

2. C: = set of critical pairs of F;

3. while $C \neq 0$ do

3.1. if $(p,q) \in C$ and $(p \downarrow R_F \neq q \downarrow R_F)$ then

3.1.1. if $p{\downarrow}R_F \rightarrow q{\downarrow}R_F$ leaves R_F noetherian then $R_F := R_F \cup \{p{\downarrow}R_F \rightarrow q{\downarrow}R_F\}$ else if $q{\downarrow}R_F \rightarrow p{\downarrow}R_F$ leaves R_F noetherian then $R_F := R_F \cup \{ q \downarrow R_F \rightarrow p \downarrow R_F \}$
else STOP (FAILURE)

3.1.2. $C=C \cup \{$ critical pairs in $F \cup \{(p \downarrow R_F , q \downarrow R_F)\}\}$

3.1.3. $F=F \cup \{(p \downarrow R_F = q \downarrow R_F)\}$

3.2. $C := C \setminus \{(p, q)\}$

4. STOP(R_F).

The above crude form of the algorithm can be refined in many ways. The sequence of critical pairs chosen by the procedure in 3.1. may have a crucial influence on the efficiency of the algorithm.

3. The J. Hsiang's completion procedure. It is well known that a formula in first-order predicate calculus is valid, iff the closed Skolemized version of its negation is false under Herbrand interpretation. Equivalentely, a formula is valid if the set of the clauses in its clausal form is insatisfiable. Hsiang [7] first suggested using a complete rewrite system in a resolution-like theorem-proving strategy.

Let $C = \{C_1,\ldots,C_n\}$ the set of clauses of a formula in first-order predicate calculus.

Let $C_1 = L_1 \vee L_2 \vee \ldots \vee L_k$ be a clause where L_j is a literal, and let H be a mapping transforming terms of a Boolean algebra into terms of a Boolean ring:

$$H(C_i) = \begin{cases} 1 & \text{if } C_i \text{ is empty clause} \\ x+1 & \text{if } C_i \text{ is } x \\ x & \text{if } C_i \text{ is } \overline{x} \\ H(L_1)*H(L_2V\ldots VL_k) & \text{otherwise} \end{cases}$$

THEOREM (Hsiang[7]: *Given a set of clauses $\mathfrak{C}$ in first-order predicate Calculus, $\mathfrak{C}$ is inconsistent iff the system*

$$H(C_i) = 0,\ C_i \in \mathfrak{C},\ i = 1,n$$

has not a solution.

Now, let BR be the complete TRS [7]:

$x + 0 \to 0$

$x + x \to 0$

$x * 1 \to x$

$x * 0 \to 0$

$x * x \to x$

$x *(y+z) \to x * y + x * z$

For each equation $H(C_i) = 0$ let us consider the equation $a_i=b_i$, where a_i is the biggest monomial of boolean polynomial $H(C_i)$ and let E be the system corresponding in this fashion to the system of equations:

$$H(C_i) = 0, i = \overline{1,n}$$

The TRS R_E having all the rules of the form $a_i \to b_i$ is noetherian [7]. In the TRS formed by $R_E \cup BR$ we have:

$$s \underset{H(C_i)=0}{\sim} t \Rightarrow s \underset{E}{\sim} t \Rightarrow s \overset{*}{\underset{R_E \cup BR}{\longleftrightarrow}} t$$

because $a_i = b_i$ is equivalent with $a_i + b_i = H(C_i) = 0$

A critical pair (p,q) may be added to system R_E not only in the form $p{\downarrow}R_E \rightarrow q{\downarrow}R_E$ or in the form $q{\downarrow}R_E \rightarrow p{\downarrow}R_E$, but also in the form $p'{\downarrow}R_E \rightarrow q'{\downarrow}R_E$ where p' is the biggest monomial of Boolean polynomial $P + q$. Hence, the polynomial $p + q$ is an intermediate form to study for critical pair.

Then, the previous theorem becomes:

THEOREM [7]. *A set of clauses $\mathfrak{C}$,in first-order predicate calculus is inconsistent iff by Knuth-Bendix completion algorithm applied to the TRS formed by $R_E \cup BR$, where E is the set of equations $a_i{=}b_i$, $i = 1,...,n$ (a_i is the biggest monomial of $H(C_i)$), the critical pair $1{\rightarrow}0$ is obtained.* Let us observe that KB algorithm of completion is allways terminating by STOP.

4. A new method for proving a formula. Let $S = (\Sigma, F, A, R)$ be a formal system, where Σ is the alphabet for the term in a boolean ring (including + and *), F is the set of boolean polynomials, $A = \emptyset$ and R is the single deductive rule denoted "res" or $\vdash$:

$f_i,\ f_j \vdash f_k$ iff

$f_i,\ f_j,\ f_k \in F$ and there exist the monomials $\alpha,\ \beta \in F$ and the substitution τ_1 and τ_2 such that:

$$(\alpha * \tau_1(f_i)) \downarrow BR = (\ \beta * \tau_2(f_j) + f_k) \downarrow BR$$

where the equality is modulo associativity and commutativity.

For this formal system the following theorem is true:

THEOREM : *Given a set of clauses $\mathfrak{C} = \{C_1,...,C_n\}$ in first-order predicate calculus, $\mathfrak{C}$ is inconsistent if in formal system S:*

$H(C_1),\dots,H(C_n) \vdash 1.$

The proof of theorem in propositional calculus consists of the following three propositions (the proof of theorem in predicate calculus is analogous).

PROPOSITION 1. *If f_i, $f_j \vdash f_k$ and f_i, f_j, f_k are the clause polynomials then $H^{-1}(f_i) \wedge H^{-1}(f_j) \rightarrow H^{-1}(f_k)$.*

Proof. By the assumption:

$f_i = \tilde{a}_{i_1} * \dots * \tilde{a}_{i_k}$, where

$$\tilde{a}_{i_s} = \begin{cases} a_{i_s}+1 \\ \text{or} \\ a_{i_s} \end{cases}, \quad s=\overline{1,k}$$

and $f_j = \tilde{b}_{j_1} * \dots * \tilde{b}_{j_l}$, where

$$\tilde{b}_{j_t} = \begin{cases} b_{j_t}+1 \\ \text{or} \\ b_{j_t} \end{cases}, \quad t=\overline{1,e}$$

If $\tilde{a}_u = a + 1$, $\tilde{b}_v = a$, $u \in \{i_1,\dots,i_k\}$, $v \in \{j_1,\dots,j_e\}$: by the commutativity of operation $*$ we can write:

$$f_i = (a + 1) * \gamma$$

$$f_j = a * \gamma$$

In boolean ring the following identity is obvious:

$$\delta * (a+1) * \gamma = \delta * a * \gamma + \delta * \gamma$$

By the comparison with the relation:

$$\alpha * f_i = \beta * f_j + f_k$$

(because $\tau_1 = \tau_2 =$ the identic substitution in propositional calculus), we observe that $f_k = \delta * \gamma$, and that $H^{-1}(f_k) = H^{-1}(\delta) \vee H^{-1}(\gamma)$.

In the propositional calculus the following implications are true:

$$(a \vee a_{i_1}^{\alpha_{i_1}} \vee \ldots \vee a_{i_k}^{\alpha_{i_k}}) \wedge (\bar{a} \vee b_{j_1}^{\alpha_{j_1}} \vee \ldots \vee b_{j_e}^{\alpha_{i_e}}) \to (a_{i_1}^{\alpha_{i_1}} \vee \ldots \vee b_{j_1}^{\alpha_{j_1}})$$

where $i_s \neq u$, $j_t \neq v$,

$$\alpha_{i_s}, \alpha_{j_t} \in \{0,1\}, s = \overline{1,k} \quad , \quad t = \overline{1,e}$$

and

$$a_{i_s}^{\alpha_{i_s}} = \begin{cases} a_{i_s} & \text{if } \alpha_{i_s} = 1 \\ \overline{a_{i_s}} & \text{if } \alpha_{i_s} = 0 \end{cases}$$

and analogously for $b_{j_t}^{\alpha_t}$.

The above implication is therefore:

$H^{-1}(f_i) \wedge H^{-1}(f_j) \to H^{-1}(f_k)$

PROPOSITION 2. *If* $\mathfrak{C} = \{C_1, \ldots, C_n\}$ *is a set of clauses, and if:*

$H(C_1), \ldots, H(C_n) \vdash U$

U is clause polinomial, then

$$C_1 \wedge \ldots \wedge C_n \to H^{-1}(U)$$

Proof: To prove this proposition we proceed by induction after the length i of the deduction of U from $H(C_1), \ldots, H(C_n)$ in formal system S.

If $i = 0$, then exists j such that $U = H(C_j)$ and

$$H^{-1}(H(C_j)) = C_j$$

The following implication is true:

$$C_1 \wedge \ldots \wedge C_n \to C_j \ , \ j = 1, \ldots, n$$

We suppose that the proposition 2 is true for the length $\leq i - 1$ of deduction, and let $f_0, \ldots, f_m = U$ a deduction of U with the length i.

For the three last polynomials f_{m-2}, f_{m-1}, f_m in the system S there is the relation:

$\alpha * f_{m-2} = \beta * f_{m-1} + f_m$

Moreover, if f_m is a clause polynomial, f_{m-2} and f_{m-1} are too, and f_{m-2} and f_{m-1} are obtained by the deduction of length $\leq i - 1$.

From the induction hypothesis we have:

$C_1 \wedge \ldots \wedge C_n \rightarrow H^{-1}(f_{m-2})$

$C_1 \wedge \ldots \wedge C_n \rightarrow H^{-1}(f_{m-1})$

By the formula:

$\vdash (A \rightarrow B) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow B \wedge C))$

results by modus poneus:

$\vdash C_1 \wedge \ldots \wedge C_n \rightarrow H_{-1}(f_{m-2}) \wedge H_{-1}(f_{m-1})$

From proposition 1 we have:

$\vdash H^{-1}(f_{m-2}) \wedge H^{-1}(f_{m-1}) \rightarrow H^{-1}(f_m)$ and by the rule of syllogism

$\vdash C_1 \wedge \ldots \wedge C_n \rightarrow H^{-1}(f_m)$

or

$\vdash C_1 \wedge \ldots \wedge C_1 \rightarrow H^{-1}(U)$ q.e.d.

PROPOSITION 3. *If* $H(C_1),\ldots,H(C_n) \vdash 1$ *then* $\mathfrak{C} = \{C_1,\ldots,C_n\}$ *is inconsistent.*

Proof. From the proposition 2 we have:

$\vdash C_1 \wedge \ldots \wedge C_n \rightarrow H^{-1}(1)$

but $H^{-1}(1)$ is the empty clause. q.e.d.

But the condition (x) "$H(C_1),\ldots,H(C_n) \vdash 1$ iff $\mathfrak{C} = \{C_1,\ldots,C_n\}$ is inconsistent" is also true hence the implication

"$H(C_1),\dots,H(C_n) \vdash 1 \rightarrow \mathfrak{C} = \{C_1,\dots,C_n\}$ is inconsistent" is true even through not all the polynomials f_i, f_j, f_k in the propositions are the clause polynomials.

Exemple: (In propositional calculul $\tau_1 = \tau_2$ = identic substitution) $\mathfrak{C} = \{P \vee \bar{Q} \vee R, \bar{P} \vee Q \vee \bar{R}, \bar{P} \vee \bar{Q}, Q \vee P, P \vee \bar{R}\}$

$H(C_1) = PQR + QR + PQ + Q$

$H(C_2) = PQR + PR$

$H(C_3) = PQ$

$H(C_4) = QR + Q + R + 1$

$H(C_5) = PR + R$

$H(C_1),\ H(C_2) \vdash PR + PQ + RQ + Q$

(due to the fact that $PQR + PQ + RQ + Q = (PQR + PR) + (PR + PQ + QR + Q)$)

$PQ + PR + RQ + Q,\ H(C_3) \vdash PR + RQ + Q$

$PR + RQ + Q\ ,\qquad H(C_5) \vdash RQ + Q + R$

$(PR + RQ + Q = H(C_5) + QR + Q + R)$

$H(C_4),\ RQ + Q + R \vdash 1$

This set of clauses is inconsistent, and the triplet f_i, f_j, f_k is not in each step the clause polynomials (like in proposition 1).

In fact the following observation is true: if A_i is the set of all the clauses with i positive variables (nonnegative): $C_1 \in A_i$ and $C_2 \in A_j$ are two clauses, $|i-j| \geq 2$, and $H(C_1)$, $H(C_2) \vdash f_k$ then f_k is not a clause polynomial. Moreover, if $C_1 \in A_i$ and $C_2 \in A_{i+1}$ differ by a number n of variables, with $n \geq 2$, and $H(C_1)$, $H(C_2) \vdash f_k$ then f_k is not a clause polynomial.

The condition (*) results from Hsiang's theorem (§ 3) by following observations:

Let us observe that the deductive rule "res": $f_i, f_j \vdash f_k \Leftrightarrow \exists\ \alpha, \beta$ (monomials) such that $(\alpha * \tau(f_i))\downarrow BR = (\beta*\tau_2(f_j) + f_k)\downarrow BR$ is a special fashion to calculate a critical pair. Indeed, the biggest monomial in $\alpha*\tau_1(f_i)$ (i.e. $MP\ f_i$) and the biggest monomial in $\beta*\tau_2(f_j)$ (i.e. $MP\ f_j$) are equal and:

$(f_k)\downarrow BR = (\alpha*\tau_1(f_i) + \beta*\tau_2(f_j))\downarrow BR = (MP\ f_i + MP\ f_j + \text{REST}\ f_i + \text{REST}\ f_j\ )\downarrow BR = (\text{REST}\ f_i + \text{REST}\ f_j)\downarrow \text{BR}$

This is the case $\tau_1(a_1) = \tau_2(a_2)$ and $(p,q) = (\tau_1(b_1), \tau_2(b_2)$ is a critical pair. The intermediate form $p + q$ of critical pair (in our case f_k) is studied.

THEOREM: *The set of clauses* $\mathfrak{C} = \{C_1,\ldots,C_n\}$ *is inconsistent iff*

$$H(C_1),\ldots,H(C_n)\vdash 1$$

Proof: If $\mathfrak{C} = \{C_1,\ldots,C_n\}$ is inconsistent, by Hsiang's theorem the system $H(C_i) = 0,\ i = 1,\ldots,n$ has not a solution, or, equivalentaly, by completion in R_E the rule $1 \to 0$ is obtained. Therefore, a critical pair (1,0) or $(f_k,\ 0)$ is obtained. We have:

$$(f_k)\downarrow BR = 1 = (1 + P + P)\downarrow BR$$

In formal system S we can write $1 + P,\ P \vdash f_k(= 1)$ where P is a boolean polynomial.

Conversely, if $H(C_1),\ldots,H(C_n) \vdash 1$ then there exists a deduction $f_0,\ldots,f_k = 1$ from $H(C_1),\ldots,H(C_n)$.

Therefore, there exists f_i and f_j such that $f_i, f_j \vdash f_k(=1)$. But f_k is a critical pair corresponding to a rule $1\to 0$, and by Hsiang's theorem $\mathfrak{C}$ is inconsistent.

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METRIC CONVEXITY IN GRAPHS

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REZUMAT. - Convexitatea metrică în grafuri. În această lucrare se prezintă o sinteză a unor rezultate recente în domeniul convexităţii metrice în grafuri. Sînt analizate diferite proprietăţi ale mulţimilor şi funcţiilor convexe în grafuri, caracterizările unor clase de grafuri cu ajutorul convexităţii.

Contents

1. Introduction. It is well-known that the ideas and results of convex analysis are of high importance for many mathematical disciplines. Convex analysis has shown itself as a powerful instrument usefut for applications. Therefore the development of mathematical structures and the enlargement of their applications lead to the creation of distinct analogies and generalizations of the notions of convex sets and convex functions (see, for instance, [31], [47], [73]).

Among them the notion of metric convexity introduced by K.Menger [52] is one of the most developed. Recall that a set A

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in a metric space (X,d) is called *convex* provided for every pair of points $x,y \in A$, the metric interval

$$[x,y] = \{z \in X : d(x,z) + d(z,y) = d(x,y)\}$$

is contained in A. For any set $B \subset X$, its *convex hull* conv$B$ is defined in a standard way to be the intersection of all convex sets in X containing B. Since the intersection of any family of convex sets is again a convex set, conv$B$ is the least convex set in X containing B.

The notions of metric convex set and covex hull became fruitful in general topology, differential geometry and functional analysis. (A sufficiently complete list of results and references on metric convexity in metric spaces and linear normed spaces can be found in [13], [73].)

Later the notion of a convex function on a metric space (X,d) was defined (see [70], [71]): a real-valued function f on X is called *convex* provided

$$f(z) \leq \frac{d(z,y)}{d(x,y)}\cdot f(x) + \frac{d(x,z)}{d(x,y)}\cdot f(y)$$

for all points $x,y \in X$ $(x \neq y)$ and $z \in [x,y]$.

The actual period in the development of metric convexity is connected with investigations of discrete structures and of some extreme problems on them (see, for instance, [61], [62]). At the same time, a considerable part of the results on convexity in discrete spaces is concentrated around metric convexity in graphs. It is interesting to mention that the notions of convex set and convex function in graphs appeared previously in connection with some location problems (see [25], [66], [68],

[69], [82]). And only later, due to the development of generalized convexity theory, some properties of metric convexity and related metric behaviour of graphs where studied by distinct authors.

In this article, we deal with metric convexity in ordinary (may be, infinite) graphs. Since this topic became too wide to be described compact, we will be concentrated below on some results closely connected with the author's interests in this field. Some additional information on metric convexity in graphs can be found in the literature placed at the end of the paper.

For the convenience, we mention here some necessary definitions connected with graphs.

Everywhere below $G = (X,U)$ denotes a graph with vertex-set X and edge-set U. A graph G is called *finite* if the set X is finite. If card$X = n$, then G will be denoted by G_n. By a *subgraph* H of G we mean the induced one, i.e., two vertices x,y are adjacent in H if and only if they are adjacent in G. For any set $Y \subset X$, the subgraph in G induced by Y is denoted by $G(Y)$.

A graph G is *connected* if for any two vertices u, v in G, there exists a finite chain containing u, v. *We assume that all the considered below graphs are connected.*

In order to consider metric convexity in G, we assume that G is equipped with standard metric: for any vertices $x, y \in X$, denote by $d(x,y)$ the least number of edges in a chain connecting x, y. Is easily seen that d indeed is a metric on X, i.e., $d(x,y)$ satisfies the following conditions:

1) $d(x,y) \geq 0$, with $d(x,y) = 0$ if and only if $x = y$,

2) $d(x,y) = d(y,x)$,

3) $d(x,y) \le d(x,z) + d(z,y)$.

If vertices x, y belong to a connected subgraph H of G, then $d_H(x,y)$ denotes the distance between x, y in the graph H. A connected subgraph H is called an *isometric subgraph* of G if $d_H(x,y) = d_G(x,y)$ for every pair of vertices x, y in H.

A *clique* in G is a vertex-set having every two distinct vertices adjacent. If X is a clique, then G is called a *complete* graph. K_n denotes a complete graph with n vertices. The supremum of the cardinality of a clique in G is called the *density* of G, and is denoted by φ.

A vertex z in G is called *simplicial* provided the set $O(z)$ of all vertices in G adjacent with z form a clique. The *degree* $\deg(z)$ of z is the number of all vertices neighbor to z. Put $\Sigma(z) = O(z) \cup \{z\}$.

A sequence $l = (\dots, v_{i-1}, v_i, v_{i+1}, \dots)$ of vertices in G such that every two consecutive vertices are adjacent is called a *chain*. A chain l is *finite* if it is of the form $l = (v_1, \dots, v_n)$; it is *one-side infinite* provided it has one of the forms $l = (v_1, v_2, \dots)$, $l = (\dots, v_2, v_1)$; l is *infinite* if it has no end-vertex. A chain is *simple* if all its vertices are distinct. A *circuit* of length n in G is a chain of the form $(v_1, v_2, \dots, v_n, v_1)$. A circuit is *simple* if all its vertices $v_1, \dots, v_n$ are distinct. Let C_n denote the simple circuit of length n.

A simple chain $l = (\dots, v_{i-1}, v_i, v_{i+1}, \dots)$ of vertices in G is called *geodesic* if any two vertices of the form v_{i-1}, v_{i+1}

are not adjacent in G. A *segment* (a *ray*, a *line*) is a finite (respectively, one-side infinite, both-side infinite) geodesic chain in G.

A *disconnecting* vertex-set in a graph G is a set $Y \subset X$ such that the induced graph $G(X \setminus Y)$ is disconnected. A graph without disconnecting vertices is called a *block*. A *tree* is a connected graph without circuits.

A *bipartite* graph is a graph containing no circuit of odd length. The vertex-set of a bipartite graph X can be partitioned into two disjoint sets Y, Z such that every edge in G joins a vertex in Y and a vertex in Z.

Also recall that G is named a *chord* graph provided it contains no simple circuit of the length greater than three as an induced subgraph. A *Husimi* tree is a graph such that each its block is a complete subgraph.

A graph G is called *planar* if it can be placed in the plane such that every vertex of G is a point and every edge of G is a rectifiable arc with end-points in X satisfying the properties: 1) every vertex x of G is an end of each arc incident with x, 2) a common point of two arcs is a vertex for both of them.

2. **Extremal structure of convex sets.** In this section some analogies of Krein-Mil'man's theorem about extremal points of convex sets in linear space are studied. Recall that Krein-Mil'man's theorem [50] states that every compact convex set in Hausdorff linear topological space is the closed convex hull of its extremal points.

Since every vertex-set in G is closed, the closed convex hull in G is identical with the convex hull, and a set of vertices in G is compact if and only if it is finite. Therefore we will discuss below the following problem. To determine necessary and sufficient conditions for the implementation of the assertion: for every finite set A of vertices in G, its convex hull coincides with the convex hull of extremal vertices of A.

By analogy with the linear space, we introduce the following definition. A vertex z of a set $A \subset X$ is called *extremal* in A if $z \notin [x,y]$ for all $x, y \in A \setminus \{z\}$, where $[x,y]$ is the metric interval with the ends x, y. By ext$A$ the set of all extremal vertices in A will be denoted.

It will be shown below that extremal vertices are closely related with simplicial vertices. The following well-known result (see [27], [51]) gives a sufficient condition for the existence of simplicial vertices in a graph.

LEMMA 2.1. *Any nonempty finite chord graph G contains at least one simplicial vertex; if G is not complete, then it contains at least two nonadjacent simplicial vertices.*

The following theorem strengths this assertion.

THEOREM 2.2. [74]. *For a graph $G = (X,U)$ the following conditions are equivalent:*

1) *every nonempty finite set in X contains at least one extremal vertex,*

2) *every nonempty finite subgraph in G contains at least one simplicial vertex,*

3) *G is a chord graph.*

The relation between extremal and simplicial vertices is shown in the following lemma.

LEMMA 2.3. *Every extremal vertex of a set $A \subset X$ is simplicial in the subgraph $G(A)$. If A is convex, then every simplicial vertex of $G(A)$ is extremal in A.*

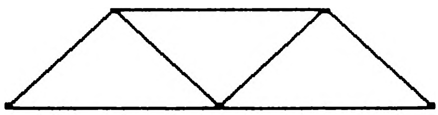
Now we can formulate an assertion analogous to Krein-Mil'man's theorem.

THEOREM 2.4. [74]. *For a graph $G = (X,U)$ the following conditions are equivalent:*

1) $\mathrm{conv}A = \mathrm{conv}(\mathrm{ext}A)$ *for every finite set* $A \subset X$,

2) $\mathrm{conv}A = \cup \{[x,y]: x, y \in \mathrm{ext}A\}$ *for every finite set* $A \subset X$,

3) *G is a chord graph containing no subgraph*



(1)

Note that for finite graphs, the equivalence of items 1) and 3) in Theorem 2.4 was established independently in [37], [47], and [72], [73].

In connection with Theorem 2.4, we mention two interesting lemmas. We say that a segment $(v_1,\dots,v_n)$ is a shortest path provided $d(v_1,v_n) = n - 1$.

LEMMA 2.5. [47]. *Let G be a finite chord graph. Then every its vertex belongs to a segment whose ends are simplicial vertices in G.*

LEMMA 2.6. [45]. *For a graph G the following conditions are* *equivalent:*

1) *every segment in G is a shortest path,*

2) *G is a chord containing no subgraph* (1).

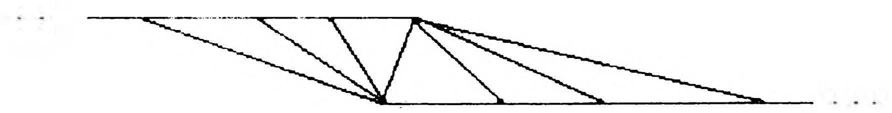
The following result shows some conditions for a graph G to satisfy conditions 2) and 3) of Theorem 2.4 for subgraphs and sets of any (may be, infinite) cardinality. These conditions are sufficiently cumbersome in the general case. Therefore, for the compactness of the description, we will restrict our attention on the class of graphs which contain no infinite complete subgraphs. Denote this class of graphs by K.

THEOREM 2.7. [74]. *For a graph* $G = (X,U) \in K$ *the following conditions are equivalent:*

1) *every nonempty set in* X *contains at least one extremal vertex,*

2) *every nonempty subgraph of* G *contains at least one simplicial vertex,*

3) *G is a chord graph containing no line and no subgraph*



THEOREM 2.8. [74]. *For a graph* $G = (X,U) \in K$ *the following conditions are equivalent:*

1) $\mathrm{conv}A = \mathrm{conv}(\mathrm{ext}A)$ *for every set* $A \subset X$,

2) $\mathrm{conv}A = \cup \{[x,y]: x, y \in \mathrm{ext}A\}$ *for every set* $A \subset X$,

3) $A = \mathrm{conv}(\mathrm{ext}A)$ *for every convex set* $A \subset X$,

4) $A = \cup \{[x,y]: x, y \in \text{ext}A\}$ *for every convex set* $A \subset X$,

5) *G is a chord graph containing no ray and no subgraph* (1).

Another well-known result on extremal structure of convex sets in linear space belongs to S.Straszewicz [81]: every compact convex set in finite-dimensional linear topological space is the closed convex hull of its exposed points. Recall that a boundary point x of a convex set A in a linear space is called exposed if there exists a hyperplane H such that $A \cap H = \{x\}$.

In order to formulate the respective analogous result for graphs, we need some definitions. A vertex-set H in $G = (X,U)$ is called a half-space provided both H and $X \setminus H$ are convex. A vertex z of a set $A \subset X$ is called *exposed* in A provided $\{z\} = A \cap H$ for some half-space $H \subset X$. Denote by exp$A$ the set of all exposed vertices of A. It is easily seen that any exposed vertex of a set is also extremal for the set, i.e., $\text{exp}A \subset \text{ext}A$ for every set $A \subset X$.

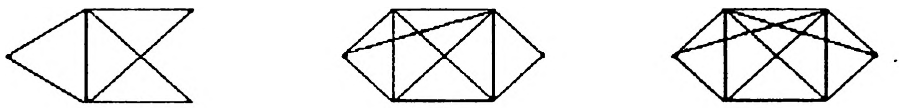
The following result is analogous to Straszewicz's theorem.

THEOREM 2.9. *For a graph* $G = (X,U)$ *the following conditions are equivalent:*

1) $\text{conv}A = \text{conv}(\text{exp}A)$ *for every finite set* $A \subset X$,

2) $\text{conv}A = \cup\{[x,y]: x,y \in \text{exp}A\}$ *for every finite set* $A \subset X$,

3) *G is a chord graph containing no subgraph* (1) *and none of*



(2)

THEOREM 2.10. *For a graph* $G = (X,U) \in K$ *the following conditions are equivalent:*

1) $\text{conv}A = \text{conv}(\text{exp}A)$ *for every set* $A \subset X$,

2) $\text{conv}A = \cup \{[x,y]: x, y \in \text{exp}A\}$ *for every set* $A \subset X$,

3) $A = \text{conv}(\text{exp}A)$ *for every convex set* $A \subset X$,

4) $A = \cup \{[x,y]: x, y \in \text{exp}A\}$ *for every convex set* $A \subset X$,

5) *G is a chord graph containing no ray and none of* (1) *or* (2).

At the end of this section we formulate two open problems.

PROBLEMS 2.11. *To describe the family of graphs* $G = (X,U)$ *satisfying at least one of the conditions:*

1) $\text{exp}A \neq \phi$ *for every nonempty convex set* $A \subset X$,

2) $\text{ext}A = \text{exp}A$ *for every convex set* $A \subset X$.

3. **Convexity of balls, ball neighborhoods, and diametrally maximal sets.** It is well-know that some classes of convex sets are of special interest in the convexity theory. These are balls, ball neighborhoods, diametrally maximal sets, etc. Below we establish conditions under which these sets are convex in a graph. Recall that a set of the form

$$\Sigma_r(z) = \{ x \in X : d(x,z) \leq r\}$$

is called the ball with center *x* and radius *r*. A set of the form

$$\Sigma_r(A) = \{ x \in X : d(x,A) \leq r\}$$

is called the r-neighborhood of a set $A \subset X$. A set A in X is called diametrally maximal if diam$(z \cup A)$ > diam$A$ for every vertex $z \in X \setminus A$, where diam$K$ denotes the diameter of a set $K \subset X$.

Let M be a connected set in X. (M is called connected if the subgraph $G(M)$ is a connected component in G.) For any vertex $x \in M$, put

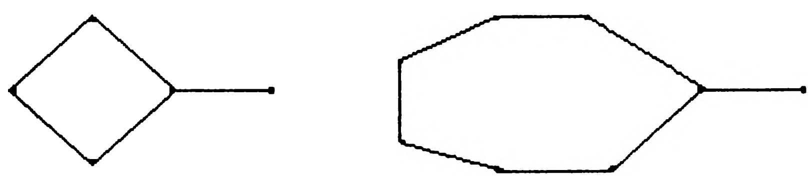
$$Q_M(x) = \{y \in M : \text{diam}M = d(x,y) = d_M(x,y)\}.$$

A pair $\{x,y\}$ is called *diametral* in M provided $y \in Q_M(x)$ (or $x \in Q_M(y)$, which is the same).

THEOREM 3.1. [76]. *Every diametrally maximal set in G is convex if and only if the following conditions are fulfilled:*

1) *G contains no simple circuit isometric to* C_6 *or* C_n, $n \geq 8$,

2) *G contains no subgraph isometric to one of*



3) *if* $Q_T(y) = \{x\}$ *for some vertices x, y in a simple circuit* $T \neq C_4$, *then x is simplicial in the subgraph* $G(T)$.

THEOREM 3.2. [76]. *Every ball in G is convex if and only if the following conditions are fulfilled:*

1) *G contains no simple circuit isometric to* C_4 *or* C_n, $n \geq 6$,

2) *if* $Q_T(y) = \{x\}$ *for some vertices x, y in a simple circuit*

T in G, then x is simplicial in the subgraph G(T).

THEOREM 3.3. [76]. *For a graph G the following conditions are equivalent:*

1) *for any convex set $A \subset X$ and $r \geq 0$, the r-neighborhood $\Sigma_r(A)$ is convex,*

2) *for any vertices $a, b \in X$, the 1-neighborhood $\Sigma_1(\text{conv}\{a,b\})$ is convex,*

3) *G contains no simple circuit isometric to C_n, $n \geq 4$.*

Note that Theorems 3.2 and 3.3 are repeated in [37] in an equivalent form.

COROLLARY 3.4. *If a graph G contains no simple circuit isometric to C_n, $n \geq 4$, then the following conditions are equivalent:*

1) *every diametrally maximal set in G is convex,*

2) *every ball in G is convex,*

3) *every neighborhood of a convex set in G is convex,*

4) *G is a tree.*

4. **Convex functions.** Recall that a real-valued function f on X is called *convex* provided

$$f(z) \leq \frac{d(z,y)}{d(x,y)} \cdot f(x) + \frac{d(x,z)}{d(x,y)} \cdot f(y)$$

for all vertices $x, y \in X$ $(x \neq y)$ and $z \in [x,y]$. One can state the following simple properties of convex functions on X.

THEOREM 4.1. 1) *For any convex functions f, g and real number $\lambda \geq 0$, the functions $f + g$ and λf are convex,*

2) *the least upper bound of any family of convex functions*

is a convex function,

3) *the limit of any pointwise convergent sequence of convex functions is a convex function,*

4) *for any convex function f and real number λ, the sets*

$$\{z \in X : f(z) \le \lambda\},\ \{z \in X : f(z) < \lambda\}$$

are convex.

Similarly to the case of linear space, we can define an *affine* function f on X as a real-valued function such that both functions f and $-f$ are convex. In other words, f is affine if

$$f(z) = \frac{d(z,y)}{d(x,y)} \cdot f(x) + \frac{d(x,z)}{d(x,y)} \cdot f(y)$$

for all vertices x, $y \in X$ $(x \ne y)$ and $z \in [x,y]$. From this definition follows immediately

COROLLARY 4.2. 1) *For any affine functions f_1, f_2 and real numbers λ_1, λ_2, the function $\lambda_1 f_1 + \lambda_2 f_2$ is affine,*

2) *the limit of any pointwise convergent sequence of affine functions is an affine function,*

3) *for any affine function f and real number λ, the sets*

$$\{z \in X : f(z) \le \lambda\},\ \{z \in X : f(z) < \lambda\}$$

are half-spaces.

A function $f : X \to R$ is called *quasiconvex* if for every real number λ, the set $\{z \in X : f(z) \le \lambda\}$ is convex. Equivalently, f is quasiconvex if

$$f(z) \le \max\{f(x), f(y)\}$$

for all vertices x, $y \in X$ and $z \in [x,y]$.

THEOREM 4.3. 1) *For any quasiconvex function f and real numbers $\lambda \ge 0$, $\mu \in R$, the function $\lambda f + \mu$ is quasiconvex,*

2) *the least upper bound of any family of quasiconvex functions is a quasiconvex function,*

3) *the limit of any pointwise convergent sequence of quasiconvex functions is a quasiconvex function.*

Similarly, a function $f : X \to R$ is called *quasiaffine* if both functions f and $-f$ are quasiconvex, i.e., f is quasiaffine if

$$\min\{f(x),f(y)\} \le f(z) \le \max\{f(x),f(y)\}$$

for all vertices $x, y \in X$ and $z \in [x,y]$.

COROLLARY 4.4. 1) *For any quasiaffine functions f_1, f_2 and any real numbers λ_1, λ_2, the function $\lambda_1 f_1 + \lambda_2 f_2$ is quasiaffine,*

2) *the limit of every pointwise convergent sequence of quasiaffine functions is a quasiaffine function,*

3) *a function f is quasiaffine if and only if for every real number λ, the sets*

$$\{z \in X : f(z) \le \lambda\}, \{z \in X : f(z) < \lambda\}$$

are half-spaces.

Below we study some properties of the classes of convex, affine, quasiconvex, and quasiaffine functions on X. Let $\mathbf{A}$, $\mathbf{D}$, $\mathbf{CA}$, and $\mathbf{CD}$ denote, respectively, the collection of all affine, convex, quasiaffine, and quasiconvex functions on X, and let $\mathbf{F}$ (respectively, $\mathbf{I}$) denote the family of all real (constant) functions on X. Trivially,

$$\begin{array}{ccccc} & & \mathbf{CA} & \subset & \mathbf{CD} \subset \mathbf{F} \\ & & \cup & & \cup \\ \mathbf{I} & \subset & \mathbf{A} & \subset & \mathbf{D} \end{array}$$

THEOREM 4.5. [65], [75]. 1) *The following conditions are equivalent:* $\mathbf{A} = \mathbf{F}$, $\mathbf{D} = \mathbf{F}$, $\mathbf{CA} = \mathbf{F}$, $\mathbf{CA} = \mathbf{CD}$, $\mathbf{CD} = \mathbf{F}$, $\mathbf{A} = \mathbf{CD}$,

$D = CD$, G is a complete graph,

2) any two of the classes A, D, CA coincide if and only if the two classes are trivial, i.e., are equal to I or to F.

THEOREM 4.6. [65], [75]. 1) *$A \neq I$ if and only if the graph $G = (X,U)$ can be decomposed into at most countable family of pairwise disjoint complete subgraphs G_i such that every vertex z in G_i is adjacent to all the vertices in $G_{i-1} \cup G_{i+1}$ and only to them,*

2) for a finite graph G, one has $D \neq I$ if and only if X contains a convex set Y with connected complement $X \setminus Y$ such that every vertex $z \in Y$ adjacent in $X \setminus Y$ is adjacent to all the vertices in $X \setminus Y$,

3) $CA \neq I$ if and only if X contains at least one half-space,

4) $CD \neq I$ (provided card$X > 1$).

For any family H of functions on X, let H_+ denote the collection of all functions which are the sums of finite subfamilies of H. We have the relations

$$\begin{array}{ccccccccc} & & D & = & D_+ & \subset & CD & \subset & CD_+ = F \\ & & \cup & & \cup & & \cup & & \cup \\ I & \subset & A & = & A_+ & \subset & CA & \subset & CA_+ \end{array}$$

THEOREM 4.7. [65], [75]. *The following implications hold:*

1) $CA_+ = A \leftrightarrow CA = A$,

2) $CA_+ = D \leftrightarrow CA = D$,

3) $CD_+ = CD$ holds if and only if G is a complete graph,

4) $CA_+ = CA$ holds if and only if the intersection of every collection of half-spaces in G is either empty or a half-space.

The supremum properties of convex functions play an important role in convex analysis. For example, at the base of the finite dimensional theory of duality of convex functions lies the famous theorem by Minkowski: every convex function is a pointwise supremum of affine functions. Below we investigate an analogous assertion for convex functions on a graph.

For any family H of functions on X, let H_s denote the collection of all finite functions which are pointwise supreme of subfamilies of H. It is easily seen that the following relations are valid:

$$\begin{array}{ccccccccc} & & CA & \subset & CA_s & \subset & CD & = & CD_s & \subset F \\ & & \cup & & \cup & & \cup & & \cup & \\ I & \subset & A & \subset & A_s & \subset & D & = & D_s & \end{array}$$

In our notations, the analogous assertion to Minkowski's theorem for convex functions on graphs looks as in item 4) of Theorem 4.8.

THEOREM 4.8. [65], [75]. 1) *The following conditions are equivalent:* $A = CA$, $A = CA_s$, $A_s = CA$, $A_s = CA_s$,

2) $A = A_s \leftrightarrow$ *either* $A = I$ *or* $A = F$,

3) $CA = CA_s$ *holds if and only if the intersection of every collection of half-spaces in* G *is either empty or a half-space,*

4) $A_s = D$ *holds if and only if* G *is either a complete graph or a simple chain,*

5) $A_s = CD$ *holds if and only if* G *is a complete graph,*

6) $CA_s = D \leftrightarrow CA = D$,

7) *if* G *is a finite graph, then* $CA_s = CD$ *holds if and only if the intersection of every collection of half-spaces in* G *is either empty or a half-space.*

As a logical consequence of this circle of questions, we will consider the family H_* which is the smallest collection of functions on X containing a family H of functions and is closed with respect to taking finite sums and finite supreme. We have the relations

$$\begin{array}{ccccccc} & D = D_* & \subset & CD & \subset & CD_* & \subset F \\ & \cup \quad \cup & & \cup & & \cup & \\ I \subset & A \subset A_* & & CA & \subset & CA_* & \end{array}$$

THEOREM 4.9. [75]. *The following implications hold:*

1) $A_* = A \leftrightarrow A_s = A$,

2) $A_* = D \leftrightarrow A_s = D$,

3) $CA_* = A \leftrightarrow CA = A$,

4) $CA_* = CA \leftrightarrow CA_s = CA$,

5) $A_* = CA \leftrightarrow A_* = CA_* \leftrightarrow A = CA$,

6) $A_* = CD$ *holds if and only if* G *is a complete graph,*

7) *the following conditions are equivalent:* $A = CD_*$, $D = CD_*$ $CA = CD_*$, *G is a complete graph.*

In connection with the above results, we formulate some open problems.

PROBLEMS 4.10. 1) *To determine conditions for the feasibility of any of the relations:*

a) $D \neq I$, $CA_+ = CA$, $CA_+ = CD$, $CA_+ = CD_+$, $CA_+ = F$, $CD_+ = F$,

b) $A_* = CD_*$, $CA_* = CD$, $CA_* = CD_*$, $CD_* = CD$, $CD_* = F$,

2) *to determine conditions for the feasibility of the following property: the intersection of every collection of half-spaces in G is either empty or is a half-space.*

The remained part of this section is devoted to the study of separation properties of convex functions on X. Below we consider a graph G to be finite. A family H of functions on X will be said to have *separation property* if for any disjoint convex sets $Y, Z \subset X$ there exists a function $f \in H$ such that

$$\inf \{f(x): x \in Y\} > \sup \{f(x): x \in Z\}.$$

Il the set Y (respectively, Z) is a singleton, then we will speak about *upper (lower) separation property.* If both sets Y and Z are singletons, we will say that *H separates vertices.*

THEOREM 4.11. [75]. 1) *For the family* **A**, *separation property, upper separation property, lower separation property, and separation property for vertices are equivalent and hold if and only if the graph G is either complete or a simple chain,*

2) *the following properties are equivalent:*

a) *D separates vertices,*

b) *D has lower separation property,*

c) *G is a chord graph,*

3) *the following conditions are equivalent:*

a) *D has separation property,*

b) *D has upper separation property,*

c) *G is a chord graph containing no subgraph* (1).

4) a) ***CA*** *separates vertices if and only if any two vertices of G can be separated by some complementary half-spaces,*

b) ***CA*** *has lower separation property* $\leftrightarrow$ ***CA*** *has upper separation property* $\leftrightarrow$ *convexity in G is regular,*

c) ***CA*** *has separation property if and only if convexity in G is normal,*

5) ***CD*** *has separation property.*

Sometimes it is necessary to know about the existence in a given class of a function satisfying the respective separation condition. We say that a real-valued function f on X satisfies *separation condition* if for any disjoint convex sets Y, Z in X one of the inequalities

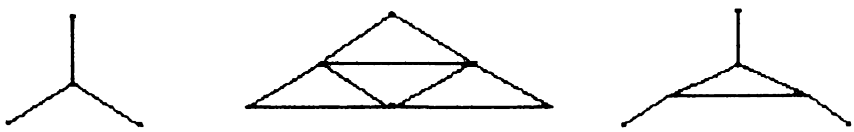
$$\inf \{f(x): x \in Y\} > \sup \{f(x): x \in Z\},$$

$$\inf \{f(x): x \in Z\} > \sup \{f(x): x \in Y\}$$

holds. If one of the sets Y, Z is a singleton, we speak about *weak separation condition.*

THEOREM 4.12. [75]. 1) *There exists a function* $f \in A$ *separating vertices in* X *if and only if* G *is either a complete graph or a simple chain,*

2) *there exists a function* $f \in CA$ *separating vertices in* X *if and only if* G *is a chord graph containing none of the subgraphs*



3) *the following conditions are equivalent:*

a) *there is a function* $f \in D$ *separating vertices in* X*,*

b) *there is a function* $f \in CD$ *separating vertices in* X*,*

c) G *is a chord graph,*

4) *the following conditions are equivalent:*

a) *each of the classes* A, D, CA *and* CD *contains a function satisfying separation condition,*

b) *each of the classes* A, D, CA *and* CD *contains a function satisfying weak separation condition,*

c) *graph* G *is a simple chain.*

5. Convexity of Steiner functions. As we know, Steiner's problem (or Weber's problem, in a different terminology) on a graph consists in finding a minimum of a function

$$f(z) = \Sigma\mu(x)\cdot d(z,x), \tag{3}$$

where $\mu(x) \geq 0$ and the sum is taken over the set of all verces $x \in X$. Unlike to the case of Euclidean space, functions (3) have no "good" properties like convexity, which guarantee the absence of local minima different from the global one. Therefore it is reasonable to find the class of all graphs for which Steiner's problem is confined to the scheme of convex analysis. An analogous problem will be studied below for functions

$$F(z) = \Sigma\mu(A)\cdot d(z,A), \tag{4}$$

where $\mu(A) \geq 0$ and the sum is taken over the family of all convex sets A in X.

THEOREM 5.1. [72], [79]. *For a graph $G = (X,U)$ the following conditions are equivalent:*

1) *every function (3) is convex,*

2) *for every vertex $x \in X$, the function $p(z) = d(z,x)$ is convex,*

3) *every function (3) is quasiconvex,*

4) *for any vertices x_1, $x_2 \in X$, the function*

$$p(z) = \mu_1 d(z,x_1) + \mu_2 d(z,x_2), \quad \mu_1,\mu_2 \geq 0$$

is quasiconvex,

5) *G is a chord graph containing no subgraph of the form (1).*

From Theorem 3.2 follows

COROLLARY 5.2. *Every function* $p(z) = d(z,x)$, $x \in X$ *is quasiconvex if and only if the following conditions are fulfilled:*

1) *G contains no simple circuit isometric to* C_4 *or* C_n, $n \geq 6$,

2) *if* $Q_T(y) = \{x\}$ *for some vertices* x,y *in a simple circuit* $T \subset G$, *then x is simplicial in the subgraph* $G(T)$.

THEOREM 5.3. [79]. *For a graph* $G = (X,U)$ *the following conditions are equivalent:*

1) *every function* (4) *is convex,*

2) *for every convex set* $A \subset X$ *with* card$A \leq 2$, *the function* $p(z) = d(z,A)$ *is convex,*

3) *every function* (4) *is quasiconvex,*

4) *for any convex set* A_1, $A_2 \subset X$, *with* card$A_1 \leq 2$ *and* card$A_2 \leq 2$, *the function* $p(z) = d(z,A_1) + d(z,A_2)$ *is quasiconvex,*

5) *G is a Husimi tree.*

From Theorem 3.3 follows

COROLLARY 5.4. *For a graph* $G = (X,U)$ *the following conditions are equivalent:*

1) *for every convex set* $A \subset X$, *the function* $p(z) = d(z,A)$ *is quasiconvex,*

2) *for every convex set* $A \subset X$ *with* card$A \leq 2$, *the function* $p(z) = d(z,A)$ *is quasiconvex,*

3) *G contains no simple circuit isometric to* C_n, $n \geq 4$.

A function $f : X \to R$ is called *strictly convex* (respectively, *strictly quasiconvex*) provided it is convex (respectively, quasiconvex) and

$$f(z) < \frac{d(z,y)}{d(x,y)}\cdot f(x) + \frac{d(x,z)}{d(x,y)}\cdot f(y)$$

respectively, $f(z) < \max\{f(x), f(y)\}$

for all vertices $x, y \in X$ $(x \neq y)$ and $z \in [x,y] \setminus \{x,y\}$ in case $f(x) \neq f(y)$.

THEOREM 5.5. *For a graph $G = (X,U)$ the following conditions are equivalent:*

1) *for every vertex $x \in X$, the function $p(z) = d(z,x)$ is strictly convex,*

2) *for any vertices x_1, $x_2 \in X$, the function $p(z) = d(z,x_1) + d(z,x_2)$ is strictly convex,*

3) *for every convex set $A \subset X$ with* card$A \le 3$, *the function $p(z) = d(z,A)$ is strictly quasiconvex,*

4) *G is a complete graph.*

THEOREM 5.6. *For a graph $G = (X,U)$ the following conditions are equivalent:*

1) *for every vertex $x \in X$, the function $p(z) = d(z,x)$ is strictly quasiconvex,*

3) *for every convex set $A \subset X$ with* card$A \le 2$, *the function $p(z) = d(z,A)$ is strictly quasiconvex,*

4) *G is a Husimi tree.*

THEOREM 5.7. [79]. *For a graph G with at most countable number of vertices, the folloving conditions are equivalent:*

1) *every finite function* (3) *with $\mu(x) > 0$ for all $x \in X$ is strictly convex,*

2) *G is a chord graph containing no subgraph* (1).

THEOREM 5.8. *The following conditions are equivalent:*

1) *every finite function* (4) *with* $\mu(A) > 0$ *for all convex sets* A *in* X *strictly convex,*

2) G *is a Husimi tree.*

At the end of this section we put the following problem.

PROBLEM 5.9. *For a graph* $G = (X,U)$, *to determine conditions for the feasibility of the following property: the function* $f(z) = \Sigma\ \{d(z,x) : x \in Y\}$ *is convex for every finite set* $Y \subset X$.

6. **Convex sets in chord graphs.** It was shown above that chord graphs play a special role for metric convexity. In this connection, we collect here different properties of convex sets in chord graphs.

We say that convexity in a graph $G = (X,U)$ has *join property* provided

$$\mathrm{conv}(A \cup B) = \cup\ \{[a,b] : a \in A,\ b \in B\}$$

for any convex sets A,B in X, and that it has *cone property* if

$$\mathrm{conv}(a \cup B) = \cup\ \{[a,b] : b \in B\}$$

for every vertex a and every convex set B in X.

For any set $A \subset X$, put $P(A) = \cup\ \{[x,y] : x,\ y \in A\}$.

THEOREM 6.1. [77]. *For a chord graph* $G = (X,U)$ *the following conditions are equivalent:*

1) *convexity in* G *has join property,*

2) *convexity in* G *has cone property,*

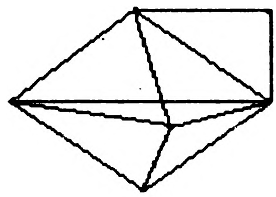
3) $\mathrm{conv}\{x,y,z\} = \cup\ \{[x,v] : v \in [y,z]\}$ *for any vertices* $x,\ y,\ z \in X$ *such that* $\mathrm{diam}\{x,y,z\} \le 2$,

4) $\mathrm{conv}A = P(A)$ *for every set* $A \subset X$,

5) $\mathrm{conv}A = P(A)$ *for every set* $A \subset X$ *with* $\mathrm{card}A \le 3$ *and*

diam$A \le 2$,

6) *G contains none of the subgraphs*



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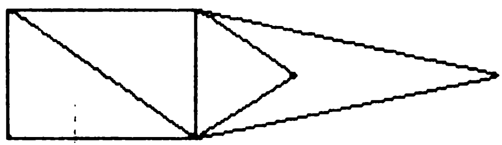

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The following results complete Theorem 6.1.

THEOREM 6.2. [64]. *For a chord graph $G = (X,U)$ the following conditions are equivalent:*

1) $\text{conv}(a \cup B) = \cup \{[a,b] : b \in B\}$ *for every vertex $a \in X$ and every set $B \subset X$ of diameter one,*

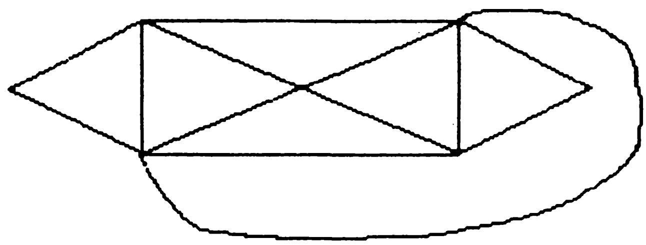
2) $\text{conv}\{a,b,c\} = [a,b] \cup [a,c]$ *for every vertex $a \in X$ and every edge $(b,c) \in U$ such that* $\text{diam}\{a,b,c\} \le 2$,

3) *G contains no subgraph* (5).

THEOREM 6.3. [77]. *For a chord graph $G = (X,U)$ the following conditions are equivalent:*

1) *for any vertices $x, y \in X$, the interval $[x,y]$ is convex,*

2) *G contains no isometric subgraph*



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LEMMA 6.4. [64]. *Let* $G = (X,U)$ *be a chord graph. For any pair of vertices* $x, y \in X$ *such that* $d(x,y) \le 2$, *the interval* $[x,y]$ *is convex.*

For any sets A, B in X, the sets

$$A/B = \{z \in X : B \cap [\ \cup [z,a] : a \in A] \neq \phi\},$$

$$A//B = \{z \in X : B \cap \mathrm{conv}(z \cup A) \neq \phi\}$$

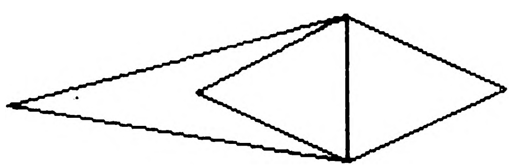
are called, respectively, weak and strong shadows of A relative to B.

THEOREM 6.5. [64]. *For a chord graph* $G = (X,U)$ *the following conditions are equivalent:*

1) *for any vertices* $a, b \in X$, *the set* a/b *is convex,*

2) *for any adjacent vertices* $a, b \in X$, *the set* a/b *is convex,*

3) *G contains no isometric subgraph* (7) *and none*



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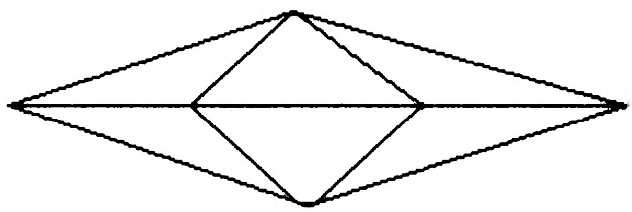
THEOREM 6.6. [64]. *For a chord graph $G = (X,U)$ the following conditions are equivalent:*

1) *for every convex set $B \subset X$ and every vertex $a \in X \setminus B$, the set a/B is convex,*

2) *for any pairwise adjacent vertices $a, b, c \in X$, the set $a/\{b,c\}$ is convex,*

3) *G contains no isometric subgraphs (7), (8), and no subgraph*



THEOREM 6.7 [64]. *For a chord graph $G = (X,U)$ the following conditions are equivalent:*

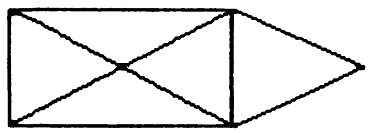
1) *for any disjoint convex sets $A, B \subset X$, the set A/B is convex,*

2) *for every convex set $A \subset X$ and every vertex $b \in X \setminus A$, the set A/b is convex,*

3) *for every edge $(a,c) \in U$ and every vertex $b \in X \setminus \{a,c\}$ such that $\max\{d(a,b), d(b,c)\} \le 2$, the set $\{a,c\}/b$ is convex,*

4) *for any pairwise adjacent vertices $a,b,c \in X$, the sets a/b and $\{a,c\}/b$ are convex,*

5) *G contains none of the subgraph (8) and*



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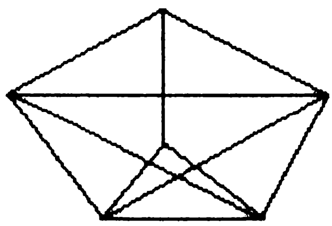
THEOREM 6.8. [64]. *For a chord graph G = (X,U) the following conditions are equivalent:*

1) *for any pairwise adjacent vertices a, b, c ∈ X, the set {a,c}/b is convex,*

2) *G contains none of the subgraph (6), (9), and*



THEOREM 6.9. [64]. *For a chord graph G = (X,U) the following conditions are equivalent:*

1) *for any convex sets A, B ⊂ X, the set A/B is convex,*

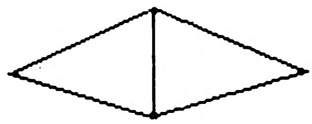
2) *for every convex set A ⊂ X and every vertex b ∈ X, the set A/b is convex,*

3) *for every convex set A ⊂ X of diameter two and every vertex b ∈ X, the set A/b is convex,*

4) *for every vertex a ∈ X and every convex set B ⊂ X, the set a/B is convex,*

5) *for every edge (a,b) ∈ U, the set a/{a,b} is convex,*

6) *G contains no subgraph*



Now we will discuss some separation properties of convex sets. A *half-space* in X is any convex set $A \subset X$ with convex complement $X \setminus A$. We say that two complementary half-spaces P, Q *separate* sets A, B if $A \subset P$ and $B \subset Q$. Convexity in X is called:

i) *separating*,

ii) *regular*,

iii) *normal*,

provided it is possible to separate by complemantary half-spaces, respectively:

i) any two distinct points,

ii) any convex set and any its exterior point,

iii) any two disjoint convex sets.

THEOREM 6.10. [64]. *For a chord graph G the following conditions are equivalent:*

1) *every semispace in G is a half-space,*

2) *G contains none of the subgraphs* (2).

THEOREM 6.11. [77]. *For a finite chord graph G the following conditions are equivalent:*

1) *every half-space in G is a semispace,*

2) *G is a tree.*

COROLLARY 6.12. *For a chord graph G the following conditions are equivalent:*

1) *convexity in G is regular,*

2) *for every set $B \subset X$ of diameter one and every vertex $a \in B$, the sets $\{a\}$ and $B \setminus \{a\}$ are separated by complementary half-spaces,*

3) *for every set $B \subset X$ of diameter one with at most four vertices and for every vertex $a \in B$, the sets $\{a\}$ and $B \setminus \{a\}$ are separated by complementary half-spaces,*

4) *G contains no subgraph* (2).

THEOREM 6.13. [77]. *For a chord graph $G = (X,U)$ the following conditions are equivalent:*

1) *convexity in G is normal,*

2) *any two disjoint edges $(a,b),(c,d) \in U$ such that the set $\{a,b,c,d\}$ has at most one pair of nonadjacent vertices, are separated by complementary half-spaces,*

3) *any two disjoint parts of a set with at most four vertices in X are separated by complementary half-spaces,*

4) *G contains none of the subgraphs* (8), (9).

Note that some sufficient conditions for the separability of vertices in a chord graph by complementary half-spaces are studied in [63].

We continue with some combinatorial problems on convex sets in chord graphs. Further $\mathcal{S}$ denotes the family of all convex sets in G. Put

$$\mathcal{S}_k = \{A \in \mathcal{S} : \text{card}A = k\},\ k = 0,1,\ldots$$

THEOREM 6.14. [77]. *For a chord graph G_n with n vertices, one has* $\text{card}\mathcal{S}_k \geq n - k + 1$.

1) $\text{card}\mathcal{S}_2 = n - 1$ *if and only if G_n is a tree,*

2) *for* $3 \leq k \leq n - 1$, *the equality* $\text{card}\mathcal{S}_k = n - k + 1$ *holds*

if and only if G_n *is a simple chain.*

For any vertex $z \in X$, call by a *semispace* corresponding to z any convex set in $X \setminus \{z\}$ maximal with respect to inclusion. It is known that the family of sets consisting of X and of all semispaces in X forms *the least base* B of convexity; i.e., every convex set in X can be represented as the intersection of some elements from B, and every proper subfamily of B does not satisfy this property.

THEOREM 6.15. [77]. *If* B *is the least base of convexity in a chord graph* G_n*, then* card$B \geq n + 1$. *The equality* card$B = n + 1$ *holds if and only if* G_n *is a complete graph.*

Denote by P the family of all half-spaces in a graph G.

THEOREM 6.16. *For a chord graph* G_n*,* $n \geq 4$*, one has* card$P \geq 6$. *For* $n = 4$*, the equality* card$P = 6$ *holds if and only if* G_n *is either a chain or a star, and for* $n \geq 5$*, one has* card$P = 6$ *if and only if* G_n *contains a complete subgraph* K_{n-3} *such that every vertex in* $G_n - K_{n-3}$ *is adjacent to all vertices in* K_{n-3} *and only to them.*

For a set $A \subset X$, put

$$P_0(A) = P(A), \quad P_{k+1}(A) = P(P_k(A)), \quad k = 0,1,\ldots$$

It is easy to prove that

$$A \subset P_1(A) \subset P_2(A) \subset \ldots \subset \operatorname{conv}A = \cup \{P_k(A) : k \geq 0\}.$$

This method of convex hull construction gives us the following characteristic number for convex hulls: for any set $A \subset X$, denote by $\beta(A)$ the least natural number k such that $\operatorname{conv}A = P_k(A)$.

THEOREM 6.17. [77]. *For any vertex-set* A *in a chord graph*

G_n, $n \geq 5$, *one has* $\beta(A) \leq n - 4$, and $\beta(A) \leq 1$ *in case* $n \leq 5$.

Classical Helly [43], Caratheodory [15], and Radon [59] theorems about convex sets in linear space became a starting point for the following definitions. The *Helly number* of X is the least natural number h satisfying the property: any finite family of convex sets in X has a common point if and only if each its h-membered subfamily has a common point. The *Radon number* of X is the least natural number r such that every set $A \subset X$ containing at least r vertices can be divided in two disjoint subsets whose convex hulls have a common point. The *Caratheodory number* in X is the least natural number c such that for every set $A \subset X$

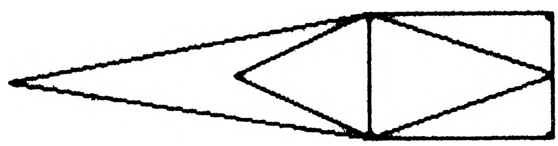
$$\text{conv}A = \cup \{\text{conv}B : B \subset A, \text{card}B \leq c\}.$$

THEOREM 6.18. [21]. *The Helly number of convexity in a chord graph G equals the density of G.*

THEOREM 6.19. [21], [72]. *If G is a chord graph with density φ, then for the Radon number r in G, one has:*

1) $3 \leq r \leq 4$ *if* $\varphi = 2$, *and* $r = 3$ *if and only if G is a simple chain,*

2) $4 \leq r \leq 5$ *if* $\varphi = 3$, *and* $r = 4$ *if and only if G contains no subgraph*



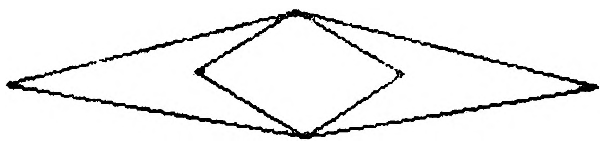
THEOREM 6.20. [77]. *For a chord graph G the following*

conditions are equivalent:

1) *the Caratheodory number of convexity in G is at most two,*

2) *G contains none of the subgraphs* (5), (6).

7. **Convex simple and quasisimple planar graphs.** From the point of view of generalized convexity theory, graphs with the most poor collection of convex sets are of certain interest. Since any vertex, any pair of end vertices of an edge, and the whole vertex-set X in a graph are convex, it is interesting to study those graphs which contain no other convex set. In [59] the following definition is introduced. A graph $G = (X,U)$ is called convex simple if every proper convex set in G (i.e., a set different from empty set and the whole X) has at most two vertices. For example, the graph shown below is convex simple.



The class of all convex simple graphs is too large to have a suitable description. Therefore we concentrate our attention on planar convex simple graphs. The following theorem was first proved in [60] for finite graphs.

THEOREM 7.1. [20]. *A planar graph G different from the graph of cube Q_3 is convex simple if and only if it contains no convex set of three vertices.*

The following theorem gives an interesting characterization of convex simple planar graphs. Recall that S (S_k) means the family of all (respectively, all k-membered) convex sets in G.

THEOREM 7.2. [20], [72], [78]. *For a finite graph G_n,*

$$\mathrm{card}S \geq 3n - 2, \quad \mathrm{card}S_2 + \mathrm{card}S_3 \geq 2n - 4.$$

The following conditions are equivalent:

1) $\mathrm{card}S = 3n - 2$,

2) *G_n is different from the graph of cube Q_3 and* $\mathrm{card}S_2 +$ $+ \mathrm{card}S_3 = 2n - 4$,

3) *G_n is planar and convex simple.*

THEOREM 7.3. [19]. *A planar graph $G = (X,U)$ with* $\mathrm{card}X \geq 5$ *different from the graph of octahedron F_3 is convex simple if and only if it contains at least one vertex of* degree ≥ 2, *and every such a vertex has a unique dual vertex in G (a vertex z is dual for x provided $O(z) = O(x)$).*

Now we are going to describe convex simple planar graphs. Denote by T any tree with at least three vertices, and let T_0 be a copy of a subtree formed by all the interior vertices in T. Denote by $L(T,T_0)$ the graph containing $T \cup T_0$ with the following additional edges: any vertex z in T_0 is adjacent to all the vertices in $O(\overline{z})$ and only to them (here $\overline{z} \in T$ is a copy of z and $O(z) = \{v \in T : v \text{ is adjacent to } \overline{z}\}$).

THEOREM 7.4. [18]. *For any planar convex simple graph G there is a tree T such that $G = L(T,T_0)$.*

In connection with the previous theorem, there appears the problem to describe those trees T for which the graph $L(T,T_0)$ is planar and convex simple.

THEOREM 7.5. [19]. *For any tree T with at least three vertices, the graph $L(T,T_0)$ is convex simple.*

The class of all trees T for which the graph $L(T,T_0)$ is planar is not described. We know only one particular result.

THEOREM 7.6. [19]. *If a tree T has at most countable number of vertices, then the graph $L(T,T_0)$ is planar.*

We are interested to know about the uniqueness of the representation of a planar convex simple graph in the form $L(T,T_0)$ for a suitable tree T.

THEOREM 7.7. [19]. *For any trees S and T, the graphs $L(S,S_0)$ and $L(T,T_0)$ are isomorphic if and only if S and T are isomorphic.*

The obtained results permit a description of a more wide class of graphs. By definition (see [11]), a graph G is called *convex quasisimple* if every proper convex set in G generates a complete subgraph. In other words, a graph is convex quasisimple if the diameter of every proper convex vertex-set in G is at most one.

THEOREM 7.8. [19]. *A planar graph G is convex quasisimple if and only if it contains no convex vertex-set inducing one of the following subgraphs:*

THEOREM 7.9. [17]. *Any planar convex quasisimple graph G contains a convexly simple subgraph.*

Let T be a tree with at least three vertices. Denote by $R(T)$ the family of graphs R obtained from T by the addition of some new edges in correspondence with the following rules:

1) the distance (in T) between the ends of any new edge (x,y) is equal to two,

2) any new edge is incident to at least one end-vertex of T,

3) for any end-vertex of T, its degree in R is at most three,

4) if one of the vertices of a new edge (x,y) is interior for T and a vertex z lies between x and y in T, then $\deg_T z = 2$,

5) if T is not a star, then R contains no simple circuit containing the end-vertices of T only,

6) if T is a star and R contains a simple circuit containing the end-vertices of T only, then this circuit contains all the end vertices of T.

Let T_0 be a subtree, consisting of all the interior vertices of a tree T. For any graph $R \in \mathcal{R}(T)$, denote by $L(R,T_0)$ the graph containing R, T_0 and the following edges: every vertex $z \in T_0$ is adjacent to all the vertices in $O(\overline{z})$, where $\overline{z} \in T$ means the copy of z and $O(\overline{z}) = \{v \in T : v \text{ is adjacent to } \overline{z}\})$.

THEOREM 7.10. [19]. *For any planar convex quasisimple graph G with* card$X \geq 4$ *different from complete graph K_4, there is a tree T and a graph $R \in \mathcal{R}(T)$ such that $G = L(R,T_0)$.*

THEOREM 7.11. [19]. *For any tree T with at least three vertices and for any graph $R \in \mathcal{R}(T)$, the graph $L(R,T_0)$ is convex quasisimple.*

THEOREM 7.12. [19]. *If a tree T has at most countable number of vertices, then for any graph $R \in \mathcal{R}(T)$ the graph $L(R,T_0)$ is* *planar.*

THEOREM 7.13. [19]. *Let S and T be some trees, $Q \in R(S)$ and $R \in R(T)$. The graphs $L(Q,S_0)$ and $L(R,T_0)$ are isomorphic if and only if S and T are isomorphic.*

8. Characterization of hypercubes and Hamming graphs by means of convexity. Let S be any set. The graph of *hypercube* $H(S)$ is defined as follows (see [28]): the vertex-set of $H(S)$ consists of all finite subsets in S (the empty set inclusively); two vertices A, B in $H(S)$ are adjacent if and only if the symmetric difference $(A \setminus B) \cup (B \setminus A)$ of the sets A, B is a one-point set.

Below we assume that any graph isomorphic to a graph of hypercube also is called a graph of hypercube. Observe, that for a finite set S with card$S = k$, $H(S)$ is the graph of k-dimensional cube.

It is not hard to prove that the function

$$d(A,B) = \text{card}[(A \setminus B) \cup (B \setminus A)]$$

is the induced metric on $H(S)$; i.e., $d(A,B)$ is equal to the number of edges in a shortest path in $H(S)$ with the ends A,B.

A graph G is called *median* if for any vertices x, y, $z \in X$ their "median" $[x,y] \cap [y,z] \cap [x,z]$ consists of a vertex.

THEOREM 8.1. [80]. *For a graph G the following conditions are equivalent:*

1) *G is a hypercube,*

2) *G contains no three-vertex convex set, and any two disjoint convex sets in G are separated by complementary half-spaces,*

3) *G contains no three-vertex convex set, and any two vertices in G are separated by complementary half-spaces,*

4) *G contains no three-vertex convex set and convexity in G satisfies cone condition,*

5) *G is a median graph and contains no three-vertex convex set.*

For proof of Theorem 8.1 we use the following lemmas.

LEMMA 8.2. [4]. *For a bipartite graph G the following conditions are equivalent:*

1) *G is a hypercube,*

2) *every interval $[x,y]$ in G generates a hypercube.*

LEMMA 8.3. [54]. *Any median graph G is bipartite. Every interval $[x,y]$ in a median graph G is a convex set.*

The relation between hypercubes and median graphs is shown in the following lemma.

LEMMA 8.4. [4]. *A graph G is a hypercube if and only if it is median and any two vertices in G have either two common adjacent vertices or have no common adjacent vertices.*

Let $\{S_\omega\}$, $\omega \in I$ be a family of pairwise disjoint sets. The *Hamming* graph H is defined as follows: the family of vertices in H consists of all finite subsets $A \subset \cup S_\omega$ such that card $(A \cap S_\omega) \le 1$ for each $\omega \in I$; two distinct vertices A, B in H are adjacent if and only if the symmetric difference $(A \setminus B) \cup (B \setminus A)$ of the sets A, B is contained in one of the sets S_ω, $\omega \in I$. If $\{S_\omega\}$ is a finite family of finite sets, then H is the Cartesian product of complete graphs.

It is easily seen that for any vertices A, B in the Hamming graph H, the induced distance $d(A,B)$ looks as

$$d(A,B) = \sum_{\omega} \text{sign card}([(A \setminus B) \cup (B \setminus A)] \cap S_{\omega}).$$

Recall that for any vertex $x \in X$ and for any set $M \subset X$, the value $d(x,M) = \min\{d(x,u) : u \in M\}$ is called the distance from x to M, and the set

$$N_x(M) = \{z \in M : d(x,z) = d(x,M)\}$$

is named the *metric projection* of x on M. For $z \in M$, put

$$W_z(M) = \{x \in X : N_x(M) = \{z\}\}.$$

A set $M \subset X$ is named *Chebishev* provided $N_x(M)$ is a one-vertex set for every $x \in X$.

THEOREM 8.5. [80]. *A graph G is a Hamming graph if and only if the following conditions are fulfilled:*

1) *every three-vertex set in G induced a complete subgraph,*

2) *every clique in G is a Chebishev set,*

3) *for every clique C in G and for every vertex $z \in C$, the set $W_z(C)$ is convex.*

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MEANS AND CONVEXITY

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REZUMAT. - Medii şi convexitate. În lucrare se consideră o noţiune de convexitate în raport cu o medie de puteri, numită r-convexitate. Se generalizează inegalitatea lui Hermite-Hadamard pentru funcţii cu inversă r-convexă aşa cum în [3] s-a procedat pentru funcţii cu inversă logaritmic convexă.

1. Introduction. In this paper we consider a notion of convexity with respect to a power mean called r-convexity. We generalize Hermite-Hadamard's inequality for functions with r-convex inverse. Then we apply it for the study of the monotony of the "relative growth" of generalized logarithmic means. We try to analyse so the position of the mean values of two numbers between those numbers.

As moust of the definitions and results which we need may be found in the book of P. S. Bullen, D. S. Mitrinović and P. M. Vasić [1] we content ourself to refer mainly at it.

2. Means. We shall use in what follows some means of two positive numbers $0<a<b$. They all belong to the familly of extended mean values defined by K. B. Stolarsky (see [1], p.345) for $r\neq s$, $rs\neq 0$ by:

$$E_{rs}(a,b)=((r/s)(b^s-a^s)/(b^r-a^r))^{1/(s-r)}$$

the definition for other values being obtained by taking limits. As special cases we have the power means:

$$P_r=E_{r,2r} \quad \text{for } r\neq 0$$

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$$P_r = E_{r,2r} \quad \text{for } r \neq 0$$

and

$$P_0(a,b) = G(a,b) = (a \cdot b)^{1/2}$$

then the generalized logarithmic means defined by:

$$L_r = E_{1,r+1}, \quad \text{for } r \neq -1,\ r \neq 0$$

but

$$L_{-1}(a,b) = L(a,b) = (b-a)/(\log b - \log a)$$

and

$$L_0(a,b) = I(a,b) = (1/e)(b^b/a^a)^{1/(b-a)}.$$

Also we use weighted power means defined for $0 \le t \le 1$ by:

$$P_{rt}(a,b) = (ta^r + (1-t)b^r)^{1/r} \text{ if } r \neq 0$$

and

$$P_{0t}(a,b) = G_t(a,b) = a^t b^{1-t}.$$

For $t=1/2$ we get the usual power means and for $r=1$ the weighted arithmetic mean $P_{rt} = A_t$.

Among the properties of these means we are interested in their monotony with respect to the parameter. So we have (see [1], p.159) for $r<s$:

$$P_{rt}(a,b) < P_{st}(a,b), \quad 0<t<1 \tag{1}$$

and also (see [1] p. 347):

$$L_r(a,b) < L_s(a,b). \tag{2}$$

3. **r-Convexity.** Let us consider the following notion: we said that the positive function $f:[a,b] \to \mathbb{R}$ is r-convex if:

$$f(A_t(x,y)) \le P_{rt}(f(x), f(y)), \quad \forall x,y \in [a,b],\ t \in [0,1].$$

As we can remark, this notion differs from a similar one given in [1] called r-mean convexity.

From (1) we deduce that if f is r-convex then it is also s-convex for every $s>r$. Also from the definition we deduce that f is r-convex if and only if: a) f^r is convex, for $r>0$; b) $\log f$ is convex, for $r=0$ and c) f^r is concave for $r<0$. Thus 0-convexity is in fact logarithmic convexity.

The paper [3] deals with functions which have logarithmic convex inverse. We consider also functions with r-convex inverse. Let us denote by $K_r^-[a,b]$ the set of positive, strictly increasing functions with r-convex inverse defined on [a,b]. We have:

$$K_r^-[a,b] \subset K_s^-[a,b] \ , \quad \text{for} \quad r < s \ . \tag{3}$$

It is also easy to check the following:

LEMMA 1. *If the positive function f is twice differentiable then it belongs to* $K_r^-[a,b]$ *if and only if:*

$$f'(x)>0 \text{ and } 1+xf''(x)/f'(x)\le r,\ \forall\ x\in[a,b] \tag{4}$$

Integrating the differential equation obtained from (4) we get functions which can be considered to be r-linear. As a special case we have:

LEMMA 2. *The function* f_r *defined by:*

$$f_r(x) = \begin{cases} x^r - a^r\ , & r > 0 \\ \log x - \log a\ , & r = 0 \\ a^r - x^r\ , & r < 0 \end{cases} \tag{5}$$

has the properties:

$$f_r(x) \ge 0\ ,\ f_r'(x) > 0\ ,\ 1+xf_r''(x)/f_r'(x) = r\ ,\ \forall\ x \ge a\ .$$

4. Hermite-Hadamard's inequality. For a function $f:[a,b]\to\mathbb{R}$ consider the integral arithmetic mean defined by:

$$A(f;a,b) = \int_a^b f(x)\,dx/(b-a) \;.$$

Hermite-Hadamard's inequality (see[1], p.30) gives for a concave function f the evaluation:

$$(f(a)+f(b))/2 \le A(f;a,b) \le f((a+b)/2). \tag{6}$$

Also H.-J.S e i f f e r t proved in [3] that for a function f from $K_0^-[a,b]$ holds:

$$A(f;a,b) \le f(I(a,b)). \tag{7}$$

We remark that from (2) it follows:

$$I(a,b)=L_0(a,b)<L_1(a,b)=(a+b)/2$$

thus (7) improves the right side of (6) for this special case. We can do the same thing for functions from $K_r^-[a,b]$ with $r\neq 0$.

In the proof of the relation (7) it is used the following result, proposed as a problem by R. E u l e r in [2]:

$$\lim_{n\to\infty}\left(\prod_{i=1}^{n}(c+(i-1)/n)\right)^{1/n} = I(c,c+1)\;,\;\forall\, c>0\;. \tag{8}$$

The expression from the first member of (8) is a geometric mean (of n numbers). We can prove a similar relation to (8) for an arbitrary power mean.

LEMMA 3. *If $r\neq 0$ and $c>0$ then:*

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{\left(c+\frac{i-1}{n}\right)^r}{n}\right)^{1/r} = L_r(c,c+1)\;. \tag{9}$$

Proof. If $r>0$, the mean value theorem of the differential calculus applied to the function $f(x)=(x+1)^{r+1}$, $x>0$, gives:

$$((x+1)^{r+1}-x^{r+1})/(r+1)<(x+1)^r<((x+2)^{r+1}-(x+1)^{r+1})/(r+1). \tag{10}$$

For $n>1/c$, we get by addition:

$$L_r\left(c-\frac{1}{n}\,,\,c+1-\frac{1}{n}\right) < \left(\sum_{i=1}^{n}\left(c+\frac{\left(\frac{i-1}{n}\right)^r}{n}\right)\right)^{\frac{1}{r}} < L_r(c\,,\,c+1)$$

hence (9). For $r<0$, $r\neq-1$, we have to do minor changes in the proof, while for $r=-1$ we must replace (10) by:

$$\log(x+2)-\log(x+1)<(x+1)^{-1}<\log(x+1)-\log x.$$

Finally we remark that the case $r=0$, excepted from (9), is contained in (8).

Replacing (8) by (9) in the proof of (7) given in [3] we get:

THEOREM 1. *If the function f belongs to* $K_r^-[a,b]$ *then:*

$$A(f;a,b)\le f(L_r(a,b)). \tag{11}$$

Let us remark that the function f_r *defined by (5) verifies:*

$$A(f_r;a,b)=f_r(L_r(a,b)). \tag{12}$$

We can improve also the left inequality from (6) for the same class of functions.

THEOREM 2. *If the function f belongs to* $K_r^-[a,b]$ *then:*

$$A(f;a,b)\ge$$
$$\ge(f(a)(b^r-L_r^r(a,b))+f(b)(L_r^r(a,b)-a^r))/(b^r-a^r), \tag{13}$$

if $r\neq0$ *and*

$$A(f;a,b)\ge(f(a)(L(a,b)-a)+f(b)(b-L(a,b)))/(b-a) \tag{14}$$

if $r=0$.

Proof. For $t\in[a,b]$ we have:

$$f(t)=\frac{f(b)-f(t)}{f(b)-f(a)}f(a)+\frac{f(t)-f(a)}{f(b)-f(a)}f(b)\ . \tag{15}$$

So, if $r>0$, $(f^{-1})^r$ being convex:

$$t^r \le \frac{f(b)-f(t)}{f(b)-f(a)}a^r+\frac{f(t)-f(a)}{f(b)-f(a)}b^r$$

or

$$f(t) \ge \frac{f(b)-f(a)}{b^r-a^r}t^r+\frac{b^r f(a)-a^r f(b)}{b^r-a^r}\ .$$

It is also valid for $r<0$. By integration we get (13). For $r=0$; $\log(f^{-1})$ is convex and (15) gives:

$$\log t \le \frac{f(b)-f(t)}{f(b)-f(a)}\log a+\frac{f(t)-f(a)}{f(b)-f(a)}\log b\ .$$

Isolating $f(t)$ and integrating we get (14).

5. **The relative growth.** We consider the following expression:

$$D_r(a,b)=\begin{cases}\dfrac{L_r^r(a,b)-a^r}{b^r-a^r}\ , & r\neq 0\\ \dfrac{b-L(a,b)}{b-a}\ , & r=0\ .\end{cases}$$

which we call relative growth of L_r. It is easy to see that:

$0\le D_r(a,b)\le 1,\ \forall r;\ D_1(a,b)=1/2.$

THEOREM 3. *If $r<s$ and $0<a<b$ then:*

$$D_r(a,b)\ge D_s(a,b)\ . \tag{16}$$

Proof. As the function f_r given by (5) belongs to $K_r^-[a,b]$ and $r<s$, from (3) it follows that it is also in $K_s^-[a,b]$ and so (12), (13) and (14) implies:

$A(f_r;a,b)=f_r(L_r(a,b))\ge f_r(b)D_s(a,b)$

which gives (16). In fact we must consider separately the cases: $0<r<s$, $0=r<s$, $r<s=0$ and $r<s<0$.

Remark 1. From (2) it follows that the evaluation given by (11) is improved by decreasing the value of the parameter r. The same conclusion is valid for (13) and (14) if we take into account (16). On the other hand, from (4) we deduce that for a strictly increasing and continuously twice differentiable function, there is a sufficiently large r for which (11) and (13) be valid.

Remark 2. An inequality similar to (16) for power means was proved by A.J.Goldman (see [1], p.203). On the other hand we remark that (16) contains many inequalities between means. For example, for $r>1$ it is equivalent with $L_r(a,b)\geq P_r(a,b)$ and for $0<r<1$ it gives $L_r(a,b)\leq P_r(a,b)$. For $r<0<s$ we get:

$$E_{r,r+1}(a,b)\leq L(a,b)\leq E_{s,s+1}(a,b).$$

All these relations may be found in [1]. We also have:

$L_{r,r+1}(a,b)L(a,b)\leq G^2(a,b)$, for $r<-1$ but the converse inequality for $-1<r<0$.

Remark 3. From $0\leq D_r(a,b)\leq 1$ we deduce that it may be preferable to use instead D_r the differences $D_r-1/2$, that is:

$$\frac{L_r^r(a,b)-P_r^r(a,b)}{b^r-a^r} \text{ for } r\neq 0\ ;\ \frac{A(a,b)-L(a,b)}{b-a}\ ,\ r=0$$

where $A=A_{1/2}$. These are between $-1/2$ and $1/2$ and are decreasing upon r, as D_r is.

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THÉORÈMES DE POINT FIXE DANS LES ESPACES AVEC MÉTRIQUE VECTORIELLE

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REZUMAT. - Teoreme de punct fix în spaţii cu metrică vectorială. În această lucrare se stabilesc trei teoreme de punct fix în spaţii cu metrică vectorială analoge teoremelor de punct fix pe spaţii metrice demonstrate în lucrarea [3].

1. Notions préliminaires

DEFINITION 1.1. Soit X un ensemble ordonné. Une suite $\{x_n\}_{n\in N}$ d'élèments de X *(o)-converge* vers un élèment $x\in X$ s'il existe deux suites $\{a_n\}_{n\in N}$ et $\{b_n\}_{n\in N}$ d'élèments de X, telles que $a_n\le x_n\le b_n$ $(\forall)$ $n\in N$ et $a_n\uparrow x$, $b_n\downarrow x$.

Nous désignons par $x = (0) - \lim x_n$ ou $x_n \xrightarrow{o} x$.

DEFINITION 1.2. Un ensemble ordonné X s'appele *ensemble réticulé* si pour tous $x,y\in X$ (donc aussi pour tout nombre fini d'élèments) il existe $x\vee y$ et $x\wedge y$.

DEFINITION 1.3. Un ensemble réticulé X s'appelle *ensemble réticulé relativement complet* si pour tout sous-ensemble denombrable borné de X il existe la borne supérieure et la borne inférieure.

DEFINITION 1.4. On appelle *espace linéaire complétement réticulé* tout espace linéaire ordonné qui est un ensemble réticulé relativement complet.

DEFINITION 1.5. Un espace linéaire ordonné X est appellé *espace archimédien* si $\bigwedge_{n\in N} \frac{1}{n}x = 0$ pour tout $x>0$, $x\in X$.

DEFINITION 1.6. Dans un espace linéaire réticulé archimédien

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X une suite $\{x_n\}_{n\in N}$ d'élèments de X *ρ-converge* (ou converge avec régulateur) vers un élèment x, s'il existe $v>0$ (appellé régulateur de convergeance) tel que: pour tout nombre $\varepsilon>0$ il existe $n_\varepsilon \in N$ de maniére que:

$$|x_n - x| \leq \varepsilon v \quad \text{si} \quad n \geq n_\varepsilon$$

On note $x = (\rho) - \lim_n x_n$ (ou $x_n \xrightarrow{\rho} x$)

Si $x = (\rho) - \lim_n x_n$ alors $x = (0) - \lim_n x_n$

DEFINITION 1.7. On appelle *espace régulier* tout espace linéaire réticulé archimedien tel que: tout suite (o)-convergente est (ρ)-convergeante.

2. **Définitions et notations.** Soit X un espace linéaire complétement réticulé et $Z \neq \emptyset$ un ensemble. On définit une métrique vectorielle d: $Z \times Z \to X$ et pour $A \subset Z$ on note le diamètre de A par $\delta(A) = \sup\{d(z_1, z_2) / z_1, z_2 \in A\}$.

DEFINITION 2.1. On dit que l'ensemble $B \subset Z$ est *d-fermé* si tout $\{z_n\}_{n\in N}$, $z_n \in B$, $z_n \xrightarrow{d} z$ implique $z \in B$.

$$\left(\text{!!} \quad z_n \xrightarrow{d} z \;\rightarrow\; d(z_n, z) \xrightarrow{o} 0 \right)$$

DEFINITION 2.2. Soit $B \subset Z$. Définissons par:

$$\overline{B} = \{ z \in Z \;/\; z = d\text{-}\lim_n z_n \;,\; z_n \in B \}$$

LEMME 2.1. *Si Z est d-complet la suite $\{B_n\}\downarrow$, $B_n \subset Z$, B_n d-fermé et $\delta(B_n)\downarrow 0$ alors il existe $z_0 \in B$ unique tel que*

$$\bigcap_{n\in N} B_n = \{z_0\} \; .$$

DEFINITION 2.3. Soit l'ensemble $Z \neq \emptyset$ d-complet. Une application $f: Z \to Z$ s'appelle *application de Picard* s'il existe $z^* \in Z$ telle que $\mathrm{Fix}(f) = \{z^*\}$ et la suite $\{f^n(z_0)\}_{n\in N}$ d-converge vers z^* pour tout $z_0 \in Z$.

DEFINITION 2.4. Soit $Z \neq \emptyset$ un certain ensemble. Une application $f: Z \to Z$ est une *application de Janos* si

$$\bigcap_{n\in N} f^n(Z) = \{z^*\} \quad \text{où} \quad \{z^*\} = \text{Fix}(f)$$

DEFINITION 2.5. Soit $Z\neq\emptyset$ un certain ensemble, $f, f_n: Z\to Z$, $n\in N$. La suite est *assimptotiquement uniformément convergente* (designons par $f_n \overset{a}{\rightrightarrows} f$) s'il existe $v>0$, $v\in X$ tel que pour tout $\varepsilon>0$ il existe $n_0(\varepsilon)$, $m_0(\varepsilon)\in N$ tels que $d(f_n^m(z), f^m(z)) < \varepsilon v$ pour tout $n>n_0$, $m>m_0$ et $z\in Z$.

3. Théorèmes de point fixe dans les espaces avec métrique vectorielle.

THÉORÈME 3.1. *Soit X un espace linéaire complétement réticulé, $Z\neq\emptyset$ d-complet, f: $Z\to Z$ et $\varphi: X_+\to X_+$. Nous supposons que:*

(i) φ est $\downarrow$ et $\varphi^n(t) \xrightarrow{o} 0$ pour $t > 0$ et $n\to\infty$

(c'est - à - dire: φ est function de comparaison)

(ii) *$\delta(f(A))\le\varphi(\delta(A))$ pour tout $A\subset Z$ tel que $f(A)\subset A$*

(c'est - à - dire: f est (δ,φ) - contraction generalisée)

Alors:

(a) f est application de Picard

(b) f est application de Janos

Démonstration a) Soit $A_1 = \overline{f(Z)}$, $A_2 = \overline{f(A_1)}, \dots, A_{n+1} = \overline{f(A_n)}$ Alors nous avons: $A_{n+1}\subset A_n$, $A_n=\bar{A}_n$ et $f(A_n)\subset A_n$ pour tout $n\in$N. D'autre part:

$$\delta(A_{n+1}) = \delta(\overline{f(A_n)}) = \delta(f(A_n)) \le \varphi(\delta(A_n)) \le \varphi^2(\delta(A_{n-1})) \le \dots \le \varphi^n(\delta(Z)) \xrightarrow{o} 0$$

Donc $\delta(A_{n+1})\downarrow 0$. Alors d'après le LEMME 2.1 il existe $z^*\in Z$ unique tel que $\bigcap A_n = \{z^*\}$ et $f\left(\bigcap_{n=1} A_n\right) \subset \bigcap_{n=1} A_n$, donc $\text{Fix}(f)=\{z^*\}$. Soit $z_0\in Z$ et $B_n = \{f_2^n(z_0), f^{n+1}(z_0), \dots, z^*\}$. Comme $f(B_n) = \{f^{n+1}(z_0), f^{n+2}(z_0), \dots, z^*\} = B_{n+1}\subset B_n$ et $\delta(B_n) = \delta(f(B_n)) \le \varphi(\delta(B_n))$ il résulte que $\delta(B_n)\downarrow 0$ pour $n\to\infty$, c'est - à - dire $f^n(z_0)\to z^*$, $n\to\infty$

b) $z^*\in\bigcap_{n\in N} f^n(Z) \subset \bigcap_{n=1} A_n = \{z^*\}$ et donc $\bigcap_{n\in N} f^n(Z) = \{z^*\}$ q.e.d.

THÉORÈME 3.2. *Soit X un espace linéaire complétement réticulé $Z\neq\emptyset$ d - complet, f: $Z\to Z$ une application ayant la proprieté suivante: il existe $n_k\in$N* tel que f^{n_k} soit une (δ,φ) contraction generalisée.*

Alors:

a) f est une application de Picard

b) f est une application de Janos

Démonstration (a) + (b). Dans le théorème 3.1. nous avons Fix $(f^{n_k}) = \{z^*\}$ et $\delta(f^{n_k}(Z)) \downarrow 0$ pour $k \to \infty$ D'autre part:

$$Z \supset f(Z) \supset f^2(Z) \supset \ldots \supset f^{n_k}(Z) \supset \ldots \quad \text{donc} \quad \bigcap_{n=1}^{\infty} f^{n_k}(Z) = \{z^*\}$$ q.e.d.

THEOREM 3.3. *Soit X un espace linéaire complétement réticulé et régulier, $Z \neq \emptyset$ d - complet et f, f_n: $Z \to Z$ $n \in$N. Supposons que:*

i) f est une application de Picard (On note Fix (f) = $\{z^*\}$)

ii) $f_n \xrightarrow{a} f$

iii) Fix $(f_n) \neq \emptyset$ *pour tout* $n \in N$ *(On note Fix* $(f_n) = \{z_n^*\}$)

Alors: $z_n^* \xrightarrow{d} z^*$

Démonstration. Nous avons:

$$d(z_n^*, z^*) = d(f_n(z_n^*), z^*) = d(f_n^m(z_n^*), z^*) \le$$
$$\le d(f_n^m(z^*), f^m(z_n^*)) + d(f^m(z_n^*), z^*)$$

D'après (ii) il en résulte qu'il existe $v>0$, $v \in X$ tell que pour tout $\varepsilon>0$ ils existent $n_0(\varepsilon)$, $m_0(\varepsilon) \in$N:

$d(f_n^m(z_n^*), f^m(z_n^*)) < \frac{\varepsilon}{2} v$ quel que soit $n > n_0(\varepsilon)$; $m > m_0(\varepsilon)$.

D'après (i) il en résulte que pour tout $n \in$N nous avons:

$d(f^m(z_n^*), z^*) \xrightarrow{o} 0$, pour $m \to \infty$.

L'espace X etant régulier il existe un régulateur de convergeance $w \ge v$ tel que: $(\forall) \varepsilon > 0$ il existe $m(\varepsilon, n) \ge m_0(\varepsilon)$ tel que:

$d(f^m(z_n^*), z^*) \le \frac{\varepsilon}{2} w$ $(\forall)$ $m \ge m(\varepsilon, n)$

Donc on obtient:

$d(z_n^*, z^*) \le \frac{\varepsilon}{2} v + \frac{\varepsilon}{2} v \le \varepsilon w$ pour tout $n \ge n_0(\varepsilon)$

Donc: $z_n^* \xrightarrow{d} z^*$

q.e.d.

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A NEW DOUBLE FOURIER EXPONENTIAL-LAGUERRE SERIES FOR FOX'S H-FUNCTION

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REZUMAT. - O nouă serie Fourier exponenţial-Laguerre pentru H-funcţiile lui Fox. În această notă este prezentată o nouă serie Fourier exponenţial-Laguerre pentru H-funcţiile lui Fox, în două variabile.

1. Introduction. The object of this paper is to introduce a new class of double Fourier Exponential-Laguerre series for Fox's H-function [4] and present one double Fourier series of this class.

In what follows for sake brevity:

$$\sum_{j=1}^{p} e_j - \sum_{j=1}^{q} f_j = A, \quad \sum_{j=1}^{n} e_j - \sum_{j=n+1}^{p} e_j + \sum_{j=1}^{m} f_j - \sum_{j=n+1}^{q} f_j = B.$$

The following formulas are required in the proof:

The integral [1,p.704,(2.2)]:

$$\int_0^{\pi} \cos 2ux \left(\sin\frac{x}{2}\right)^{-2w_1} H_{p,q}^{m,n}\left[z\left(\sin\frac{x}{2}\right)^{-2h} \middle| \begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right] dx =$$

$$= \sqrt{\pi}\, H_{p+2,q+2}^{m+1,n+1}\left[z \middle| \begin{matrix}(1-w_1-2u,h),(a_p,e_p),(1-w_1+2u,h)\\(\frac{1}{2}-w_1,h),(b_q,f_q),(1-w_1,h)\end{matrix}\right], \qquad (1.1)$$

where $h>0$, $A\le 0$, $B>0$, $|\arg z|< 1/2B\pi$, $\mathrm{Re}[1-2w_1+2h(1-a_j)/e_j]>0$, $(j=1,\dots,n)$.

The integral [2,p.711,(2.3)]:

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$$\int_0^\infty y^{w_2+a} e^{-y} L_v^a(y)\, H_{p,q}^{m,n}\left[zy^k \,\middle|\, \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] dy =$$

$$= \frac{(-1)^v}{v!} H_{p+2,q+1}^{m,n+2}\left[z \,\middle|\, \begin{matrix} (-w_2-a,k),\ (-w_2,k),\ (a_p,e_p) \\ (b_q,f_q),\ (v-w_2,k), \end{matrix} \right], \tag{1.2}$$

where $h>0$, $A\le 0$, $B>0$, $|\arg z|<1/2B\pi$, $\mathrm{Re}[w_2+a+kb_j]>-1$ $(j=1,2,\ldots,m)$, $\mathrm{Re}\,a>-1$.

The orthogonality property of Laguerre polynomials [3,p.292-293, (2) & (3)]:

$$\int_0^\infty x^a e^{-x} L_m^a(x)\, L_n^a(x)\, dx = \begin{cases} 0, & m\neq n,\ \mathrm{Re}\,a>-1; \\ \dfrac{\Gamma(a+n+1)}{n!}, & m=n,\ \mathrm{Re}\,a>0. \end{cases} \tag{1.3}$$

The following orthogonality property:

$$\int_0^\pi e^{2imx} \cos 2nx\, dx = \begin{cases} 0, & m\neq n \\ \dfrac{\pi}{2}, & m=n\neq 0 \\ \pi, & m=n=0. \end{cases} \tag{1.4}$$

2. Double Fourier Exponential-Laguerre series. The double Fourier Exponential-Laguerre series to be established is

$$(\sin\frac{x}{2})^{-2w_1} y^{w_2} H_{p,q}^{m,n}\left[z(\sin\frac{x}{2})^{-2h} y^k \,\middle|\, \begin{matrix} (a_p,e_p) \\ (b_q,f_q) \end{matrix} \right] =$$

$$= \frac{2}{\sqrt{(\pi)}} \sum_{r=-\infty}^{\infty} \sum_{t=0}^{\infty} \frac{(-1)^t}{\Gamma(a+t+1)} e^{2irx} L_t^a(y) \ . \tag{2.1}$$

$$\times H_{p+4,q+3}^{m+1,n+3}\left[z \,\middle|\, \begin{matrix} (1-w_1-2r,h),\ (-w_2-a,k),\ (-w_2,k),\ (a_p,e_p),\ (1-w_1+2r,h) \\ (\frac{1}{2}-W_1,h),\ (b_q,f_q),\ (1-w_1,h),\ (t-w_2,k) \end{matrix} \right],$$

valid under the conditions of (1.1), (1.2), and (1.3).

Proof. To establish (2.1), let

$$f(x,y)=(\sin\frac{x}{2})^{-2w_1}y^{w_2}h_{p,q}^{m,n}\left[z(\sin\frac{x}{2})^{-2h}y^{k}\left|\begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right.\right]=$$

$$=\sum_{r=-\infty}^{\infty}\sum_{t=0}^{\infty}A_{r,t}\,e^{2irx}L_t^{a}(y)\ . \qquad (2.2)$$

Equation (2.2) is valid, since f(x,y) is continuous and of bounded variation in the region $0 < x < \pi$, $0 < y < \infty$.

Multiplying both sides of (2.2) by $y^{a}e^{-y}L_v^{a}(y)$, integrating with respect to y from 0 to ∞, and using (1.2) and (1.3), we obtain

$$(-1)^{v}(\sin\frac{x}{2})^{-2w_1}H_{p+2,q+2}^{m,n+2}\left[z(\sin\frac{x}{2})^{-2h}\left|\begin{matrix}(-w_2-a,k),(w_2,k),(a_p,e_p)\\(b_q,f_q),(v-w_2,k)\end{matrix}\right.\right]=$$

$$=\sum_{r=-\infty}^{\infty}A_{r,v}\Gamma(a+v+1)\,e^{2irx}\ . \qquad (2.3)$$

Multiplying both sides of (2.2) by $\cos 2ux$, integrating with respect to x from 0 to π and using (1.1) and (1.4), we get

$$A_{u,v}=\frac{2(-1)^{v}}{\sqrt{(\pi)}\,\Gamma(a+v+1)} \qquad (2.4)$$

$$\times H_{p+4,q+3}^{m+1,n+3}\left[z\left|\begin{matrix}(1-w_1-2u,h),(-w_2-a,k),(-w_2,k),(a_p,e_p),(1-w_1+2u,h)\\(\frac{1}{2}-w_1,h),(b_q,f_q),(1-w_1,h),(v-w_2,k)\end{matrix}\right.\right],$$

except that $A_{0,v}$ is one-half of the above value.

From (2.2) and (2.4), the formula (2.1) is obtained.

Since on specializing the parameters the H-function yields almost all special functions appearing in applied mathematics and physical sciences. Therefore, the result presented in this paper is of a general character and hence may encompass several casses of interest.

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THE CLOSE-TO-CONVEXITY RADIUS OF SOME FUNCTIONS

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REZUMAT. - Razele de aproape convexitate ale unor funcţii. În lucrare sînt determinate razele de aproape convexitate ale funcţiilor sinus integral şi sinus hiperbolic integral, folosindu-se condiţia de aproape convexitate a lui Kaplan.

1. **Preliminaries.** Let f be an analytic function in the unit disk U. The function f is said to be convex if it is univalent in U and if $f(U)$ is a convex domain. The function f is said to be close-to-convex if there is a convex function φ on U such that $\mathrm{Re}(f'(z)/\varphi'(z)) > 0$ for $z\in U$. Using a well known criterion of univalence, due to Ozaki and Kaplan, it follows from definition that every close-to-convex function is univalent in U. The following theorem is also due to Kaplan and gives a necessary and sufficient analytic condition for close-to-convexity.

THEOREM 1 [1]. *An analytic function f in U is close-to--convex if and only if f' is a nonvanishing function in U and*

$$\int_{t_1}^{t_2}\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}+1\right)dt > -\pi\ , \quad z=re^{it}\ ,$$

for every $r\in(0,1)$ and $0\le t_1<t_2<2\pi$.

For a function f which is analytic around the origin we define its close-to-convexity radius as being the radius of the largest disk centered at 0 in which f is close-to-convex. It is obvious that the problem of finding the close-to-convexity radius of f is the same with that of determining the maximum value of

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the real and positive parameter λ for which the function $g(z)=f(\lambda z)$ is close-to-convex in U.

Finding the close-to-convexity radius of a function is important as an independent problem and also because in this way is obtained a lower bound for the radius of univalence.

2. **Main problem.** We deal in this note with the problem of finding the close-to-convexity radii for the functions

$$\mathrm{Si}(z) = \int_0^z \frac{\sin t}{t} dt , \quad z \in \mathbf{C}$$

$$\mathrm{Shi}(z) = \int_0^z \frac{\mathrm{sh}\, t}{t} dt , \quad z \in \mathbf{C}$$

Note first that these functions have the same close-to-convexity radius, denoted by r_0. This becomes clear from the relation $\mathrm{Shi}(z)=\mathrm{Si}(iz)/i$. The nonvanishing condition for the derivative implies that $r_0 \leq \pi$.

Letting $\Delta = \{(t_1,t_2):0\leq t_1<t_2<2\pi\}$ Theorem 1 applied to these functions now gives

$$I_1(t_1,t_2) = \int_{t_1}^{t_2} \mathrm{Re}(z \operatorname{ctg} z)\, dt > -\pi \tag{1}$$

$$I_2(t_1,t_2) = \int_{t_1}^{t_2} \mathrm{Re}(z \operatorname{cth} z)\, dt > -\pi \tag{2}$$

where $z=re^{it}$, for every $r \in (0,r_0)$ and $(t_1,t_2) \in \Delta$.

If we put

$x=x(t)=r\cos(t)$, $y=y(t)=r\sin(t)$

$g_1(t)=y\,\mathrm{sh}(2y)+x\sin(2x)$, $g_2(t)=x\,\mathrm{sh}(2x)+y\sin(2y)$

$h_1(t)=\mathrm{Re}(z \operatorname{ctg} z)=g_1(t)/(\mathrm{ch}(2y)-\cos(2x))$

$h_2(t)=\mathrm{Re}(z \text{ cth } z)=g_2(t)/(\mathrm{ch}(2x)-\cos(2y))$

then the functions g_1,h_1,g_2,h_2 are even, periodical of period π and verify the relations

$$g_j(t)=g_j(\pi-t),\ h_j(t)=h_j(\pi-t),\ \mathrm{sgn}\ h_j=\mathrm{sgn}\ g_j,\ j=1,2 \qquad (3)$$

$$g_2(t)=g_1(t-\pi/2),\ h_2(t)=h_1(t-\pi/2).$$

Using the well-known inequalitis $\sin(a)/a \leq 1 \leq \mathrm{sh}(b)/b$, $\cos(a) \leq 1 \leq \mathrm{ch}(b)$, $a,b\in\mathbb{R}^*$ and the sign of g_1' if follows that g_1 increases on $[0,\pi/2]$, decreases on $[\pi/2,\pi]$, and $r\sin(2r) \leq$ $\leq g_1(t) \leq r\mathrm{sh}(2r)$. Consequently relations (1), (2) are fulfiled for $r \leq \pi/2$ because g_j and h_j are positive, so $r_0 \in (\pi/2,\pi]$.

Using the sign of h_1 it follows that the minimum points (t_1,t_2) of I_1 with respect to $\overline{\Delta}$ verify $t_1 \in \{0,\pi-t_0,2\pi-t_0\}$, $t_2 \in \{t_0,\pi+t_0,2\pi\}$ where $t_0 = t_0(r)$ is the unique root of the equation $g_1(t) = 0$ situated in $[0,\pi/2]$. Aplying relations (3) we find

$$I_1(0,t_0)=I_1(2\pi-t_0,2\pi)=I_1(\pi-t_0,\pi+t_0)/2<0$$

$$I_1(\pi-t_0,2\pi)=I_1(0,\pi+t_0).$$

So the minimum points (t_1,t_2) of I_1 with respect to $\overline{\Delta}$ satisfy the relation $(t_1,t_2)\in\{(0,\pi+t_0),(\pi-t_0,\pi+t_0),(0,2\pi)\}$. We distinguish two cases:

a) If $I_1(0,\pi-t_0)\geq 0$ then $I_1(\pi-t_0,\pi+t_0)\leq I_1(0,\pi+t_0)\leq I_1(0,2\pi)$ so $\min\{I_1(t_1,t_2):(t_1,t_2)\in\Delta\}=\min\{I_1(t_1,t_2):(t_1,t_2)\in\overline{\Delta}\}=I_1(\pi-t_0,\pi+t_0)$.

b) If $I_1(0,\pi-t_0)<0$ then $I_1(0,2\pi)<I_1(0,\pi+t_0)<I_1(\pi-t_0,\pi+t_0)$ so $\min\{I_1(t_1,t_2):(t_1,t_2)\in\overline{\Delta}\}=\inf\{I_1(t_1,t_2):(t_1,t_2)\in\Delta\}=I_1(0,2\pi)$.

Consequently, with a previous use of relations (3), the close-to-convexity condition (1) for the function Si and for a fixed r becomes

$I_1(0,t_0)>-\pi/2$ and $I_1(0,\pi/2)\geq-\pi/4$. (4)

Consider now the case of the function Shi. Denoting by $t_0'=t_0'(r)$ the root of the equation $g_2(t)=0$ situated in $[0,\pi/2]$ we have by (3) that $t_0+t_0'=\pi/2$. Using again (3) and proceeding in an analogous way as before it follows that the minimum value of I_2 with respect to Δ may be

$$I_2(t_0',\pi-t_0') = 2I_2(t_0',\pi/2) = 2I_1(0,t_0)$$

or

$$I_2(t_0',2\pi-t_0') = I_2(0,\pi)+2I_2(t_0',\pi/2) = 2[I_1(0,t_0)+I_1(0,\pi/2)].$$

So, the close-to-convexity condition (2) for the function Shi and for a fixed r becomes

$I_1(0,t_0)>-\pi/2$ and $I_1(0,t_0) + I_1(0,\pi/2)>-\pi/2$. (5)

It follows now, from (4) and (5), that the following conditions are fulfiled when r equals r_0:

$I_1(0,t_0)=-\pi/2$ or $I_1(0,\pi/2)=-\pi/4$ (6)

$I_1(0,t_0)=-\pi/2$ or $I_1(0,t_0) + I_1(0,\pi/2)=-\pi/2$. (7)

Presuming that $I_1(0,t_0)>-\pi/2$ for $r=r_0$ we obtain from (6) and (7) that $I_1(0,\pi/2)=-\pi/4$ and $I_1(0,t_0) + I_1(0,\pi/2)=-\pi/2$, so $I_1(0,t_0)=I_1(0,\pi/2)=-\pi/4$ which is impossible because h_1 is negative on $[o,t_0]$ and positive on $[t_0,\pi/2]$.

Finally, the close-to-convexity radius r_0 of the functions Si and Shi is the smallest root, situated in $(\pi/2,\pi]$, of the equation

$$I_1(0,t_0(r)) = \int_0^{t_0(r)} \frac{y\operatorname{sh}(2y)+x\sin(2x)}{\operatorname{ch}(2y)-\cos(2x)}dt = -\frac{\pi}{2},$$

where $x=r\cos(t)$, $y=r\sin(t)$ and $t_0(r)$ is the unique root of the equation $y\text{sh}(2y) + x\sin(2x) = 0$ situated in $(0,\pi/2)$.

An approximative value obtained for r_0 is $r_0 \approx 3.1411...$

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GENERALIZED PRE-STARLIKE FUNCTIONS

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REZUMAT. - Funcţii prestelare generalizate. Lucrarea se ocupă cu funcţii prestelare cu mai mulţi parametri, de ordinul α şi tipul β. Sînt stabilite unele inegalităţi privind coeficienţii acestor funcţii.

Introduction. A function $f(z)$ normalised by $f(0)=f'(0)-1=0$ is said to be in the class S if it is analytic and univalent in the unit disc $U=\{z:|z|<1\}$. A function $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ is said to be in the class of functions starlike of order α, $0\le\alpha<1$, denoted by $S^*(\alpha)$, if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}\ge\alpha \qquad (z\in U)$$

Further we say that f in S belongs to the class $S(\alpha,\beta)$ if f satisfies

$$\left|\frac{\dfrac{zf'(z)}{f(z)}-1}{\dfrac{zf'(z)}{f(z)}+1-2\alpha}\right|<\beta$$

where $\beta\in(0,1]$, $0\le\alpha<1$.

The convolution or Hadamard product of two power series $f(z)=\sum_{n=0}^{\infty} a_n z^n$ and $g(z)=\sum_{n=0}^{\infty} b_n z^n$ is defined as the power series $f*g(z)=\sum_{n=0}^{\infty} a_n b_n z^n$. A normalised analytic function is said to be in the class of functions prestarlike of order α and type β,

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$0 \le \alpha < 1$, $\beta \in (0,1]$, denoted by R_α^β, if $f * S_{\alpha,\beta} \in S(\alpha,\beta)$ where $S_{\alpha,\beta} = z(1-\beta z)^{-2(1-\alpha)}$. For $\beta=1$ we get the class R_α introduced by Ruscheweyh [2].

Main Results. We need the following lemma due to Kulkarni S.R. [1]:

LEMMA: *Let f be in $S(\alpha,\beta)$, then for z in $\overline{U}$*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1-\beta(1-2\alpha)}{1+\beta} \quad . \tag{1}$$

We also need,

LEMMA: *Let*

$$S_{\alpha,\beta} = z(1-\beta z)^{-2(1-\alpha)} = z + \sum_{n=2}^{\infty} \gamma(n,\alpha,\beta) z^n, \quad \textit{then}$$

$$\gamma(n,\alpha,\beta) = \frac{\prod_{k=2}^{n} [\beta(k-2\alpha)]}{(n-1)!} \quad \textit{for } n=2,3,4,\ldots \tag{2}$$

Proof: We have

$$S = z(1-\beta z)^{-2(1-\alpha)} =$$

$$= z\left\{1 + \sum_{n=2}^{\infty}\left(\frac{\prod_{k=2}^{n}[\beta(k-2\alpha)]}{(n-1)!}\right) z^{n-1}\right\} =$$

$$= z + \sum_{n=2}^{\infty}\left(\frac{\prod_{k=2}^{n}[\beta(k-2\alpha)]}{(n-1)!}\right) z^n$$

Hence the result follows.

THEOREM 1. *Let f be in* R_α^β *, then*

$$\operatorname{Re}\{G(z)\} > \frac{1}{1+\beta} \tag{3}$$

where

$$G(z) = \frac{f*\left(\dfrac{z}{(1-\beta z)^{3-2\alpha}}\right)}{f*\left(\dfrac{z}{(1-\beta z)^{2-2\alpha}}\right)}$$

Proof : Since f is in R_α^β , $F=f*S_{\alpha,\beta}$ belongs to $S(\alpha,\beta)$

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1-\beta(1-2\alpha)}{1+\beta}$$

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}+1-2\alpha\right\} > \frac{2(1-\alpha)}{1+\beta}$$

We have

$$F(z) = f*S_{\alpha,\beta}(z)$$

$$zF'(z) = f*z(S_{\alpha,\beta}(z))'$$

$$\frac{zF'(z)}{F(z)}+1-2\alpha = 2(1-\alpha)\frac{f*(z(1-\beta z)^{-(3-2\alpha)})}{f*(z(1-\beta z)^{-(2-2\alpha)})}$$

Hence the result follows.

THEOREM 2. *If* $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ *be in* R_α^β *and* $S_{\alpha,\beta} = z(1-\beta z)^{-2(1-\alpha)}$ *then*

$$|a_n| \le \frac{1+\sum_{k=2}^{n-1} |a_k|\gamma(k,\alpha,\beta)}{\beta^{n-1}+\sum_{k=2}^{n} \beta^{n-k}\gamma(k,\alpha,\beta)} .$$

Proof : In view of Theorem 1, we can write

with $|b_n| \le 1$.

$$f*(z(1-\beta z)^{-(3-2\alpha)}) = (f*S_{\alpha,\beta})\left(1+\sum_{n=1}^{\infty} b_n z^n\right) \qquad (4)$$

Equating the coefficients of z^n in the power series expansion of (4), we have,

$$a_n\left(\beta^{n-1}+\sum_{k=2}^{n} \beta^{n-k}\gamma(k,\alpha,\beta)\right)=$$
$$=b_{n-1}+\sum_{k=2}^{n-1} a_k\gamma(k,\alpha,\beta)b_{n-k}+a_n\gamma(n,\alpha,\beta)$$

Whence the result.

Note : For $\beta=1$, we get the result of Silverman and Silvia [3].

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ON A PARTICULAR n-α-CLOSE-TO-CONVEX FUNCTION

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REZUMAT. - Asupra unor funcţii n-α-aproape convexe. În lucrare sînt stabilite cîteva proprietăţi ale unor funcţii n-α-convexe.

1. Introduction. Let A be the class of functions $f(z)$ which are analytic in the unit disc $U=\{z\in C : |z|<1\}$, with $f(0)=f'(0)-1=0$. In [2] the author defined the class $K_{n,\alpha}(\delta)$, the class of functions $f\in A$ which satisfy

$$\text{Re}\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^nf(z)}+\alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right]>\delta\ ,\ z\in U$$

where $\alpha\geq 0$, $\delta<1$ and $D^nf(z)=\frac{z}{(1-z)^{n-1}}*f(z)=\frac{z(z^{n-1}f(z))^{(n)}}{n!}$,

where (*) stands for the Hadamard product (convolution) of power series, i.e. if $r(z)=\sum_{j=0}^{\infty}r_jz^j$ and $s(z)=\sum_{j=0}^{\infty}s_jz^j$, then $(r*s)(z)=\sum_{j=0}^{\infty}r_js_jz^j$.

Note that the classes $K_{n,\alpha}(\delta)$ and $Z_n(\delta)\equiv K_{n,0}(\delta)$ were studied in [2] and the classes $K_{n,\alpha}(1/2)$ and $Z_n(1/2)$ were introduced by H.S.Al-Amiri [1] and S.Ruscheweyh [7] respectively.

We denote by $AC_n(\delta)$ (the class of n-close-to-convex functions of order δ) the class of functions $f\in A$ which satisfy

$$\text{Re}\frac{D^{n+1}f(z)}{D^{n+1}g(z)}>\delta\ ,\ z\in U$$

where $g\in Z_{n+1}(\delta)$, $\delta<1$ and let $C_{n,\alpha}(\delta)$ (the class of n-α-close-to-

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convex functions of order δ) the class of functions $f \in A$ which satisfy

$$\mathrm{Re}\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^{n+1}g(z)}+\alpha\frac{D^{n+2}f(z)}{D^{n+2}g(z)}\right]>\delta\ ,\ z\in U$$

where $g \in Z_{n+2}(\delta)$, $\delta<1$. These classes were introduced in [3] and we have presented in [4] some properties by using sharp subordination results from [5] and [6], and the classes $C_{n,\alpha}(1/2)$, $AC_n(1/2)$ were studied in [1].

Let $\gamma \in C$ with Re $\gamma>-1$ and $b_\gamma(z)=\sum_{j=1}^{\infty}\frac{\gamma+1}{\gamma+j}z^j$. In [7], S.Ruscheweyh showed that if Re $\gamma \geq (n-1)/2$ and $f \in Z_n(1/2)$, then $f*b_\gamma \in Z_n(1/2)$.

In [4] we presented some new results concerning this function and in this paper we will give other new properties of the function $b_\gamma(z)$.

2. **Preliminaries.** We will need the next lemmas to prove our results.

LEMMA A. [4, Theorem 3]. *Let* $\gamma>-1$ *and*

$$\delta_0=\max\left\{\frac{n-\gamma}{n+1}\ ,\ \frac{2n-\gamma}{2(n+1)}\right\}\leq\delta<1\ .$$

If $f \in Z_n(\delta)$ *then* $f*b_\gamma \in Z_n(\tilde{\delta}(n,\gamma,\delta))$ *where*

$$\tilde{\delta}(n,\gamma,\delta)=\frac{1}{n+1}\left[\frac{\gamma+1}{F(1,2(n+1)(1-\delta),\gamma+2;1/2)}-\gamma+n\right]$$

and this result is sharp.

LEMMA B. [4, Theorem 4]. *Let* $\gamma>-1$ *and*

$$\max\left\{\frac{n-\gamma+1}{n+2}\ ,\ \frac{2n-\gamma+2}{2(n+2)}\right\} \le \delta \le \frac{2n-\gamma+3}{2(n+2)}\ .$$

If $f\epsilon AC_n(\delta)$ related to $g\epsilon Z_{n+1}(\delta)$ then $f*b_\gamma\epsilon AC_n(\delta)$ related to $g*b_\gamma\epsilon Z_{n+1}(\delta)$.

3. Main results.

THEOREM 1. *If $-1 < \gamma \le 0$, then $f\epsilon Z_n\left(\frac{n-\gamma}{n+1}\right)$ implies that*

$$f*b_\gamma\epsilon Z_n\left(\tilde{\delta}\left(n,\gamma,\frac{n-\gamma}{n+1}\right)\right)\ ,\ where$$

$$\tilde{\delta}\left(n,\gamma,\frac{n-\gamma}{n+1}\right) = \frac{1}{(n+1)\sqrt{\pi}}\ \frac{\Gamma(\gamma+3/2)}{\Gamma(\gamma+1)} + \frac{n-\gamma}{n+1}$$

and the result is sharp.

Proof. If $-1 < \gamma \le 0$, then $\max\left\{\frac{n-\gamma}{n+1}\ ,\ \frac{2n-\gamma}{2(n+1)}\right\} = \frac{n-\gamma}{n+1}$ and by using Lemma A for $\delta = (n-\gamma)/(n+1)$ and a simple calculus we obtain our result.

THEOREM 2. *If $\gamma \ge 0$, then $f\epsilon Z_n\left(\frac{2n-\gamma}{2(n+1)}\right)$ implies that*

$$f*b_\gamma\epsilon Z_n\left(\frac{2n-\gamma+1}{2(n+1)}\right)$$

and this result is sharp.

Proof. If $\gamma \ge 0$ then $\max\left\{\frac{n-\gamma}{n+1}\ ,\ \frac{2n-\gamma}{2(n+1)}\right\} = \frac{2n-\gamma}{2(n+1)}$; taking $\delta = \frac{2n-\gamma}{2(n+1)}$ in Lemma A and using the well-known relation $F(1,a,a,;z)=1/(1-z)$ we have $\tilde{\delta}\left(n,\gamma,\frac{2n-\gamma}{2(n+1)}\right) = \frac{2n-\gamma+1}{2(n+1)}$ and we obtain our result.

Taking $\gamma = 0$ in Lemma A we obtain the next result.

COROLLARY 1. *Let $\frac{n}{n+1}\le\delta<1$ and $f\epsilon Z_n(\delta)$; then*

$$f*b_0\in Z_n(\tilde{\delta}(n,0,\delta))\ ,\ where$$

$$\tilde{\delta}(n,0,\delta)=\begin{cases}\frac{1}{n+1}\left[\frac{1-2(n+1)(1-\delta)}{2-2^{2(n+1)(1-\delta)}}+n\right],\ for\ \delta\neq\frac{2n+1}{2(n+1)}\\ \frac{1}{n+1}\left[\frac{1}{2\ln 2}+n\right],\ for\ \delta=\frac{2n+1}{2(n+1)}\end{cases}$$

and this result is sharp.

Taking $n=0$ in the above corollary we obtain:

COROLLARY 2. *Let* $0\le\delta<1$ *and* $f\in A$ *with* $\mathrm{Re}\frac{zf'(z)}{f(z)}>\delta\ ,\ z\in U$. *Then* $\mathrm{Re}\frac{zF'(z)}{F(z)}>\tilde{\delta}\ ,z\in U$ *where*

$$\tilde{\delta}=\begin{cases}\frac{2\delta-1}{2-2^{2(1-\delta)}}\ ,\ for\ \delta\neq\frac{1}{2}\\ \frac{1}{2\ln 2}\ ,\ for\ \delta=\frac{1}{2}\end{cases}$$

and $F(z)=f(z)*b_0(z)$ *, and this result is sharp.*

Considering $n=0$ and $\delta=0$ in Lemma B we obtain the following result:

COROLLARY 3. *Let* $2\le\gamma\le3$ *and* $f,g\in A$. *Then* $\mathrm{Re}\frac{f'(z)}{g'(z)}>0,\ z\in U$, *where* $\mathrm{Re}\left(1+\frac{zg''(z)}{g'(z)}\right)>-1\ ,\ z\in U$ *implies* $\mathrm{Re}\frac{F'(z)}{G'(z)}>0,\ z\in U$

where $\mathrm{Re}\left(1+\frac{zG''(z)}{G'(z)}\right)>-1\ ,\ z\in U$ *and*

$F(z)=f(z)*b_\gamma(z)\ ,\ G(z)=g(z)*b_\gamma(z)$.

Taking $n=0$ and $\delta=1/2$ in Lemma B we obtain the next result:

COROLLARY 4. *Let* $0\le\gamma\le1$ *and* $f,g\in A$. *Then*

$\mathrm{Re}\frac{f'(z)}{g'(z)}>\frac{1}{2}\ ,\ z\in U$ *where* $\mathrm{Re}\left(1+\frac{zg''(z)}{g'(z)}\right)>0\ ,\ z\in U$ *implies*

$\mathrm{Re}\frac{F'(z)}{G'(z)}>\frac{1}{2}\ ,\ z\in U$ *where* $\mathrm{Re}\left(1+\frac{zG''(z)}{G'(z)}\right)>0\ ,\ z\in U$ *and*

$F(z)=f(z)*b_\gamma(z)$, $G(z)=g(z)*b_\gamma(z)$.

Considering $n=0$, $\gamma=0$ in Lemma B and $\gamma=0$ in Theorem 2 we obtain the next two results concerning $f*b_0$ respectively.

COROLLARY 5. *Let* $1/2 \le \delta \le 3/4$ *and* $f,g\in A$. *Then* $\mathrm{Re}\frac{f'(z)}{g'(z)}>\delta$, $z\in U$ *where* $\mathrm{Re}\left(1+\frac{zg''(z)}{g'(z)}\right)>2\delta-1$, $z\in U$ *implies* $\mathrm{Re}\frac{F'(z)}{G'(z)}>\delta$, $z\in U$ *where* $\mathrm{Re}\left(1+\frac{zG''(z)}{G'(z)}\right)>2\delta-1$, $z\in U$ *and* $F(z)=f(z)*b_0(z)$, $G(z)=g(z)*b_0(z)$.

COROLLARY 6. *If* $f\in Z_n\left(\frac{n}{n+1}\right)$ *then* $f*b_0\in Z_n\left(\frac{2n+1}{2(n+1)}\right)$ *and this result is sharp.*

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ON A MARX-STROHHÄCKER DIFFERENTIAL SUBORDINATION

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REZUMAT. - Asupra unei subordonări diferenţiale Marx-Strohhäcker. Fie A clasa funcţiilor f, $f(0) = f'(0) - 1 = 0$, analitice în discul unitate U. Fie q o funcţie univalentă în U, cu $q(0) = 1$. Presupunem că sînt verificate condiţiile (5) şi (6), unde $h(z) = q(z) + zq'(z)/q(z)$. Rezultatul principal al lucrării afirmă că dacă $f \in A$ şi $zf''(z)/f'(z) \prec zk''(z)/k'(z)$ atunci $zf'(z)/f(z) \prec zk'(z)/k(z)$, unde k este definită de (9). Se consideră cazul particular $k(z) = (e^{\lambda z} - 1)/\lambda$ unde $|\lambda| \le 4$.

1. Introduction. If the function f with $f'(0) \neq 0$ is analytic in the unit disc U, then f is convex in U (i.e. f is univalent and $f(U)$ is a convex domain) if and only if $\text{Re}[zf''(z)/f'(z)+1]>0$ in U. Let A denote the class of analytic functions f in U, which are normalized by $f(0) = 0$ and $f'(0)=1$. A function f in A is starlike in U (i.e. f is univalent and $f(U)$ is starlike with respect to the origin) if and only if $\text{Re}[zf'(z)/f(z)] > 0$. If $\text{Re}[zf'(z)/f(z)] > \alpha$, $0 \le \alpha < 1$, then f is called starlike of order α. A classic result due to Marx [2] and Strohhäcker [7] asserts that a convex function f in A is starlike of order 1/2, i.e.

$$f \in A,\ \text{Re}\frac{zf''(z)}{f'(z)}+1 > 0\ (z \in U) \Rightarrow \text{Re}\frac{zf'(z)}{f(z)} > \frac{1}{2}\ (z \in U). \qquad (1)$$

If F and G are analytic functions in U and G is univalent then we say that F is subordinate to G, written $F \prec G$, or $F(z) \prec G(z)$, if $F(0) = G(0)$ and $F(U) \subset G(U)$.

If we let $k(z) = z/(1-z)$, then the implication (1) can be

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rewritten as

$$f\in A,\ \frac{zf''(z)}{f'(z)} \prec \frac{zf''(z)}{k'(z)} \Rightarrow \frac{zf'(z)}{f(z)} \prec \frac{zk'(z)}{k(z)}. \tag{2}$$

In [4] S.S.Miller and the present author determined certain general sufficient conditions on the function k in A, for which the implication (2) holds.

In this paper we determine other sufficient conditions on k in A, for which (2) holds. For example these new conditions are satisfied if $k(z) = (e^{\lambda z} - 1)/\lambda$, where $|\lambda| \le 4$. This example is an improvement of a recent result of V.Anisiu and the author [1]. In particular we offer a new and more simple proof of the starlikeness condition obtained in [1].

2. **Preliminaries.** We shall use the following lemmas to prove our results.

LEMMA 1. *Let G be an analytic and univalent function on $\bar{U}$, with $G'(\zeta) \neq 0$, for $\zeta \in \partial U$. Let F be analytic in U, with $F(0) = G(0)$. If F is not subordinate to G, then there exist points $z_0 \in U$ and $\zeta \in \partial U$, and an $m \ge 1$, for which*

(i) $F(z_0) = G(\zeta)$ *and*

(ii) $z_0F'(z_0) = m\zeta G'(\zeta)$.

More general forms of this lemma may be found in [3]. A recent survey on the theory and applications of differential subordinations is given in [5].

LEMMA 2.[1]. *The radius of univalence of the function $f(z) = (e^z - 1)/z$ is given by $r = 4.83\ldots$, where r satisfies the system*

$$\begin{cases} e^{r\cos t}\sin(r\sin t - t) + \sin t = 0 \\ e^{r\cos t}[r\cos(r\sin t) - \cos(r\sin t - t)] + \cos t = 0 \end{cases}.$$

LEMMA 3. *If* $|z| < r_0 = 4.046\ldots$ *, where* r_0 *is the root in the interval* $(0,2\pi)$ *of the equation* $r[1 + \operatorname{ctg}(r/2)] = 2$, *then*

$$\operatorname{Re}\left(1+\frac{1}{e^z-1}-\frac{1}{z}\right) > 0.$$

Proof. It is well known that

$$\frac{z}{e^z-1} = 1-\frac{z}{2}+\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}z^{2n}, \quad |z| < 2\pi,$$

where B_{2n} are the Bernoulli numbers. Therefore we have

$$1+\frac{1}{e^z-1}-\frac{1}{z} = \frac{1}{2}+\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}z^{2n-1}$$

and we deduce

$$\operatorname{Re}\left(1+\frac{1}{e^z-1}-\frac{1}{z}\right) \geq \frac{1}{2}-\left|\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}z^{2n-1}\right|.$$

Using the well known formula

$$\frac{z}{2}\operatorname{ctg}\frac{z}{2} = 1-\sum_{n=1}^{\infty}\frac{|B_{2n}|}{(2n)!}z^{2n}, \quad |z| = r < 2\pi,$$

we easily obtain

$$\operatorname{Re}\left(1+\frac{1}{e^z-1}-\frac{1}{z}\right) \geq \frac{1}{2}-\frac{1}{r}+\frac{1}{2}\operatorname{ctg}\frac{r}{2}$$

and from this last inequality we deduce the desired result.

LEMMA 4.[6]. *The inequality*

$$\mathrm{Re}\frac{\lambda z}{e^{\lambda z}-1} > 0$$

holds for all $z \in U$, *if and only if* $|\lambda| \le r^*$, *where* $r^* = 2.832\ldots$ *is given by*

$$r^* = \sqrt{1+y_0^2} \tag{3}$$

and y_0 *is the smallest positive root of the equation*

$$y\sin y + \cos y = \frac{1}{e}\ . \tag{4}$$

We note that r^* is the radius of starlikeness of the function $f(z) = e^z - 1$.

3. Main results.

THEOREM 1. *Let* q *be univalent in* U, *with* $q(0) = 1$. *Let*

$$h(z) = q(z) + \frac{zq'(z)}{q(z)}$$

and suppose that

$$h \text{ is convex in } U \tag{5}$$

$$\mathrm{Re}\left[\frac{h'(z)}{q'(z)}q(z)\right] > 0\ ,\ z\in U\ . \tag{6}$$

If P *is an analytic function in* U, *such that*

$$P(z) \prec h(z), \tag{7}$$

then the analytic solution p *of the differential equation*

$$zp'(z) + P(z)p(z) = 1 \tag{8}$$

satisfies $p \prec 1/q$.

Proof. Condition (6) implies $q(z) \neq 0$ in U, hence the function $1/q$ is analytic and univalent.

Without loss of generality we can assume q is univalent, with $q(z) \neq 0$ on $\bar{U}$ and $q'(z) \neq 0$ for $z \in \partial U$. If not, then we can replace q, h, P, and p by $q_r(z) = q(rz)$, $h_r(z) = h(rz)$, $P_r(z) = P(rz)$ and $p_r(z) = p(rz)$ respectively, where $0 < r < 1$. These new functions satisfy the conditions of the Theorem on $\bar{U}$. We would then prove $p_r \prec 1/q_r$ and by letting $r \to 1$ we obtain $p \prec 1/q$.

Now assume that $p \not\prec 1/q$. From Lemma 1 there exist points $z_0 \in U$ and $\zeta \in \partial U$ and $m \geq 1$ such that $p(z_0) = 1/q(\zeta)$ and $z_0 p'(z_0) = -m\,\zeta\, q'(\zeta)/q^2(\zeta)$. Therefore from (8) we obtain

$$P(z_0) = \frac{1}{p(z_0)} - \frac{z_0 p'(z_0)}{p(z_0)} = q(\zeta) + \frac{m\zeta q'(\zeta)}{q(\zeta)}$$
$$= h(\zeta) + \frac{(m-1)\zeta q'(\zeta)}{q(\zeta)} .$$

If we let

$$\delta = \frac{P(z_0) - h(\zeta)}{\zeta h'(\zeta)} = \frac{(m-1)\, q'(\zeta)}{q(\zeta)\, h'(\zeta)} ,$$

then from (6) and the fact that $m \geq 1$ we deduce $\operatorname{Re}\delta > 0$, or equivalenty $|\arg \delta| < \pi/2$. Since $\zeta\, h'(\zeta)$ is the outward normal to the boundary of the convex domain $h(U)$ we deduce that $P(z_0) \notin h(U)$, which contradicts (5). Hence we have $p \prec 1/q$.

THEOREM 2. *Let q satisfy the conditions* (5) *and* (6) *of Theorem* 1 *and let*

$$k(z) = z \exp\int_0^z \frac{q(t)-1}{t}\,dt . \tag{9}$$

If $f \in A$ and

$$\frac{zf''(z)}{f'(z)} \prec \frac{zk''(z)}{k'(z)} , \tag{10}$$

then $zf'(z)/f(z)$ is analytic in U and

$$\frac{zf'(z)}{f(z)} \prec \frac{zk'(z)}{k(z)} .$$

Proof. From (9) we obtain

$$q(z) = \frac{zk'(z)}{k(z)} \text{ and } h(z) = q(z) + \frac{zq'(z)}{q(z)} = 1+\frac{zk''(z)}{k'(z)} .$$

Since condition (10) implies $f'(z) \neq 0$, the function $P(z) = 1 + zf''(z)/f'(z)$ is analytic in U and satisfies (7). For this particular P equation (8) has the analytic solution $p(z) = f(z)/[zf'(z)]$. Thus all conditions of Theorem 1 are satisfied and we deduce $p \prec 1/q$. Since $1/q(z) \neq 0$, this implies $p(z) \neq 0$ and so $1/p(z) = zf'(z)/f(z)$ is analytic in U. In addition from $p \prec 1/q$ and $q(z) \neq 0$ we obtain $1/p \prec q$, i.e.
$zf'(z)/f(z) \prec zk'(z)/k(z)$.

4. **A particular case.** If we let

$$q(z) = \frac{\lambda z}{e^{\lambda z}-1} ,$$

then from Lemma 2 we deduce that q is univalent in U if $|\lambda| \leq 4.83...$ and in this particular case we have

$$k(z) = \frac{1-e^{-\lambda z}}{\lambda} \text{ and } h(z) = 1-\lambda z .$$

On the other hand we have

$$\frac{q'(z)}{h'(z)q(z)} = 1+\frac{1}{e^{\lambda z}-1}-\frac{1}{\lambda z}$$

and by using Lemma 3 we deduce that

$$\mathrm{Re}\frac{q'(z)}{h'(z)q(z)} > 0\ ,\quad \text{if}\quad |\lambda| \le r_0 = 4.046\ldots$$

Thus if $|\lambda| \le r_0$ all conditions of Theorem 2 are satisfied and we obtain the following result.

THEOREM 3. *Let* $r_0 = 4.046\ldots$ *be the root in the interval* $(0,2\pi)$ *of the equation* $r[1 + \mathrm{ctg}(r/2)] = 2$. *If* $f \in A$ *and*

$$\left|\frac{f''(z)}{f'(z)}\right| \le M \le r_0\ ,\quad \text{for}\quad z \in U\ ,$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{\lambda z}{e^{\lambda z}-1}\ ,\quad \text{for}\quad |\lambda| = M\ .$$

This theorem is an improvement of a result in [1].

By using Lemma 4, from Theorem 3 we deduce the following sufficient condition of starlikeness, which was obtained in [1].

THEOREM 4. *Let* $r^* = 2.83\ldots$ *be given by* (3) *and* (4). *If* $f \in A$ *and*

$$\left|\frac{f''(z)}{f'(z)}\right| \le r^*\ ,\quad \text{for}\quad z \in U\ ,$$

then f is starlike in U and this result is sharp.

Example. Let $f \in A$ be defined by

$$f(z) = \int_0^z e^{\lambda t^2} dt\ .$$

From Theorem 4 we deduce that f is starlike if $|\lambda| \le r^*/2 = 1.41\ldots$

In particular, if we denote by ρ the radius of starlikeness of the error function

$$\operatorname{erf}(z) = \int_0^z e^{-t^2} dt \ ,$$

then $\rho \geq \sqrt{\frac{r^*}{2}} = 1.19\ldots$ We note that the inequality $\rho \geq r^*/2$ in [1] has to be corrected by $\rho \geq \sqrt{\frac{r^*}{2}}$.

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CONVOLUTION OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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REZUMAT. - Convoluţii de funcţii univalente cu coeficienţi negativi. În lucrare sînt stabilite unele proprietăţi ale convoluţiilor de funcţii stelate de ordin α şi tip β cu coeficienţi negativi.

1. Let A denote the class of functions of the form

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad n=2,3,\ldots$$

that are analytic in the unit disc $U = \{z\in\mathbf{C};\ |z|<1\}$. The function $f\in A$ is said to be starlike of order α, $\alpha\in[0,1)$, with negative coefficients, if

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > \alpha, \quad z\in U.$$

We denote this class by $S^*(\alpha)$. Let $\alpha\in[0,1)$ and $\beta\in(0,1]$; we define the class $S^*(\alpha,\beta)$ of starlike functions of order α and type β with negative coefficients by

$$S^*(\alpha,\beta) = \{f\in A;\ J(f(z);\alpha)<\beta,\ z\in U\},$$

where

$$J(f(z);\alpha) = \left|\frac{\frac{zf'(z)}{f(z)}-1}{\frac{zf'(z)}{f(z)}+1-2\alpha}\right|.$$

Remark 1. Let D be the disc with the center at $a = (1-2\alpha\beta^2+\beta^2)/(1-\beta^2)$ and the radius $r = 2\beta(1-\alpha)/(1-\beta^2)$ when $\beta\in(0,1)$ and

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$\alpha\in[0,1)$, and let $D = \{w\in\mathbb{C};\ \mathrm{Re}\ w>\alpha\}$, when $\beta=1$ and $\alpha\in[0,1)$. Then for $z\in U$ we have

$$J(f(z);\alpha) \prec \beta \Leftrightarrow \frac{zf'(z)}{f(z)} \in D \tag{1}$$

and we deduce that if $f\in S^*(\alpha,\beta)$, then

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > \sigma\ ,\quad z \in U\ ,$$

where $\sigma=\sigma(\alpha,\beta)$ and

$$\sigma(\alpha,\beta) = \frac{1+2\alpha\beta-\beta}{1+\beta}\ .$$

We obtain $S^*(\alpha,1) = S^*(\alpha)$ and $S^*(\alpha,\beta)\subset S^*(\sigma)$, where $\sigma=\sigma(\alpha,\beta)$.

Remark 2. By using (1) we also obtain

a) if $0\le\alpha_1<\alpha_2<1$, then $S^*(\alpha_2,\beta)\subset S^*(\alpha_1,\beta)$;

b) if $0<\beta_1<\beta_2\le 1$, then $S^*(\alpha,\beta_1)\subset S^*(\alpha,\beta_2)$.

Let f and g be two functions in A,

$$f(z) = z-\sum_{n=2}^{\infty} a_n z^n \ \text{ and }\ g(z) = z-\sum_{n=2}^{\infty} b_n z^n\ .$$

Then we define the (modified) Hadamard product or convolution of f and g by

$$(f*g)(z) = z-\sum_{n=2}^{\infty} a_n b_n z^n\ .$$

In this paper we show that if f, $g\in S^*(\alpha,\beta)$, then $f*g\in S^*(\alpha,\gamma)\cap S^*(\delta,\beta)$, where $0<\gamma<\beta$ and $\alpha<\delta<1$.

We will use the following result due to V.P.Gupta and P.K.Jain [1].

THEOREM A. *A function f,*

$$f(z) = z-\sum_{n=2}^{\infty} a_n z^n , \quad a_n \geq 0 , \quad n=2,3,\ldots$$

is in $S^*(\alpha,\beta)$ *if and only if*

$$\sum_{n=2}^{\infty} \frac{n-1+\beta(n+1-2\alpha)}{2\beta(1-\alpha)} a_n \leq 1 .$$

The result is sharp.

2. THEOREM 1. *Let* $f,g\in S^*(\alpha,\beta)$, $\alpha\in[0,1)$, $\beta\in(0,1]$. *Then* $f*g\in S^*(\alpha,\gamma(\alpha,\beta))$, *where*

$$\gamma(\alpha,\beta)=\frac{2\beta^2(1-\alpha)}{(3-2\alpha)(\beta+1)^2-2(1-\alpha)}$$

and $0<\gamma(\alpha,\beta)<\beta$. *The result is sharp.*

Proof. From Theorem A we know that if $f,g\in S^*(\alpha,\beta)$ and

$$f(z) = z-\sum_{n=2}^{\infty} a_n z^n , \quad g(z) = z-\sum_{n=2}^{\infty} b_n z^n ,$$

then

$$\sum_{n=2}^{\infty} \frac{n-1+\beta(n+1-2\alpha)}{2\beta(1-\alpha)} a_n \leq 1 \tag{2}$$

and

$$\sum_{n=2}^{\infty} \frac{n-1+\beta(n+1-2\alpha)}{2\beta(1-\alpha)} b_n \leq 1 . \tag{3}$$

From Theorem A we also know that $f*g\in S^*(\alpha,\gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{n-1+\gamma(n+1-2\alpha)}{2\gamma(1-\alpha)} a_n b_n \le 1 \tag{4}$$

and we wish to find the smallest $\gamma=\gamma(\alpha,\beta)$ such that (4) holds.

From (2) and (3) we get by means of the Cauchy-Schwarz inequality

$$\sum_{n=2}^{\infty} \frac{n-1+\beta(n+1-2\alpha)}{2\beta(1-\alpha)} \sqrt{a_n b_n} \le 1 \tag{5}$$

which implies

$$\sqrt{a_n b_n} \le \frac{2\beta(1-\alpha)}{n-1+\beta(n+1-2\alpha)}, \quad n=2,3,\ldots \tag{6}$$

We observe that the inequalities

$$\frac{n-1+\gamma(n+1-2\alpha)}{2\gamma(1-\alpha)} a_n b_n \le \frac{n-1+\beta(n+1-2\alpha)}{2\beta(1-\alpha)} \sqrt{a_n b_n}, \quad n=2,3,\ldots \tag{7}$$

imply (4). We also observe that (7) is equivalent to

$$\frac{n-1+\gamma(n+1-2\alpha)}{\gamma} \sqrt{a_n b_n} \le \frac{n-1+\beta(n+1-2\alpha)}{\beta}, \quad n=2,3,\ldots \tag{8}$$

By using (6) we obtain

$$\frac{n-1+\gamma(n+1-2\alpha)}{\gamma} \sqrt{a_n b_n} \le \frac{2\beta(1-\alpha)(n-1+\gamma(n+1-2\alpha))}{\gamma(n-1+\beta(n+1-2\alpha))}, \quad n=2,3,\ldots$$

In order to obtain (8) it will be sufficient to show that

$$\frac{2\beta(1-\alpha)(n-1+\gamma(n+1-2\alpha))}{(n-1+\beta(n+1-2\alpha))\gamma} \le \frac{n-1+\beta(n+1-2\alpha)}{\beta}, \quad n=2,3,\ldots$$

These last inequalities are equivalent to

$$\frac{n-1}{\gamma}+n+1-2\alpha \le \frac{(n-1+\beta(n+1-2\alpha))^2}{2\beta^2(1-\alpha)}, \quad n=2,3,\ldots$$

or

$$\gamma \geq \gamma(n) = \frac{2\beta^2(1-\alpha)(n-1)}{(n-1+\beta(n+1-2\alpha))^2-2\beta^2(1-\alpha)(n+1-2\alpha)}, \quad n=2,3,\ldots$$

We note that $\gamma(n) \leq \gamma(2)$ for all $n = 2,3,\ldots$ and then we choose

$$\gamma(\alpha,\beta) = \gamma(2) = \frac{2\beta^2(1-\alpha)}{(3-2\alpha)(\beta+1)^2-2(1-\alpha)}.$$

We have $\gamma(2)>0$ because $2\beta^2(1-\alpha)>0$ and

$(3-2\alpha)(\beta+1)^2 - 2(1-\alpha) = 2(1-\alpha)(\beta+1)^2 - 2(1-\alpha) +$

$+ (\beta+1)^2 = 2(1-\alpha)(\beta^2+2\beta) + (\beta+1)^2 > 0.$

We also have

$$\beta-\gamma(\alpha,\beta) = \frac{(1-\beta)^2+8\beta(1-\alpha)+4\alpha\beta(1-\beta)+2\alpha\beta^2}{(3-2\alpha)(\beta+1)^2-2(1-\alpha)} > 0 .$$

By Theorem A the function

$$f_2(z) = z-\frac{2\beta(1-\alpha)}{1+\beta(3-2\alpha)}z^2 \tag{9}$$

is an element of $S^*(\alpha,\beta)$ and

$$(f_2*f_2)(z) = z-\frac{4\beta^2(1-\alpha)^2}{(1+\beta(3-2\alpha))^2}\cdot z^2 \in S^*(\alpha,\gamma(\alpha,\beta)),$$

because

$$\frac{4\beta^2(1-\alpha)^2}{(1+\beta(3-2\alpha))^2} = \frac{2\gamma(1-\alpha)}{1+\gamma(3-2\alpha)}, \quad \text{when } \gamma = \gamma(\alpha,\beta).$$

Then the functions $f = g = f_2$ are extremal functions for this theorem.

COROLLARY 1.1. *If* $f,g\in S^*(\alpha,\beta)$, *then* $f*g\in S^*(\alpha,\beta)$.

Proof. We use Theorem 1 and Remark 2 b).

COROLLARY 1.2. *If* $f,g\in S^*(\alpha,\beta)$, *then* $f*g\in S^*(\rho(\alpha,\beta))$, *where*

$$\rho(\alpha,\beta) = 1-\frac{4\beta^2(1-\alpha)^2}{(\beta+1)(5\beta+1-4\alpha\beta)} . \tag{10}$$

Proof. If $f,g\in S^*(\alpha,\beta)$, then $f*g\in S^*(\alpha,\gamma(\alpha,\beta))\subset S^*(\sigma(\alpha,\gamma(\alpha,\beta)))$, where

$$\sigma(\alpha,\gamma(\alpha,\beta)) = \frac{1+2\alpha\gamma(\alpha,\beta)-\gamma(\alpha,\beta)}{1+\gamma(\alpha,\beta)} = \rho(\alpha,\beta)$$

and $\rho(\alpha,\beta)$ is given by (10).

COROLLARY 1.3. *If $f,g\in S^*(\alpha)$, then $f*g\in S^*\left(\frac{2-\alpha^2}{3-2\alpha}\right)$.*

Proof. We know that $S^*(\alpha) = S^*(\alpha,1)$ (see Remark 1) and by using Corollary 1.2 we obtain $f*g\in S^*(\rho(\alpha,1))$ and $\rho(\alpha,1)=\frac{2-\alpha^2}{3-2\alpha}$.

The preceding result (Corollary 1.3) are due to A. Schild and H.Silverman [2].

3. THEOREM 2. *Let $\alpha\in[0,1)$ and $\beta\in(0,1]$. If $f,g\in S^*(\alpha,\beta)$, then $f*g\in S^*(\delta(\alpha,\beta),\beta)$, where*

$$\delta(\alpha,\beta) = 1 \quad \frac{2\beta(1-\alpha)^2}{5\beta+1-4\alpha\beta} ,$$

and $\alpha<\delta(\alpha,\beta)<1$. The result is sharp.

Proof. If $f,g\in S^*(\alpha,\beta)$, then (6) holds. By Theorem A we know that $f*g\in S^*(\delta,\beta)$ if and only if

$$\sum_{n=2}^{\infty}\frac{n-1+\beta(n+1-2\delta)}{2\beta(1-\delta)}a_nb_n \le 1 \tag{11}$$

and we wish to find the largest $\delta=\delta(\alpha,\beta)$ such that (11) be satisfied.

We note that the next inequalities

$$\frac{n-1+\beta(n+1-2\delta)}{1-\delta}\sqrt{a_nb_n} \le \frac{n-1+\beta(n+1-2\alpha)}{1-\alpha} , \quad n=2,3,... \tag{12}$$

implies (11).

By using (6) we have

$$\frac{n-1+\beta(n+1-2\delta)}{1-\delta}\sqrt{a_n b_n} \le \frac{2\beta(1-\alpha)}{1-\delta}\cdot\frac{n-1+\beta(n+1-2\delta)}{n-1+\beta(n+1-2\alpha)}, \quad n=2,3,\ldots$$

and we deduce that

$$\frac{2\beta(1-\alpha)}{1-\delta}\cdot\frac{n-1+\beta(n+1-2\delta)}{n-1+\beta(n+1-2\alpha)} \le \frac{n-1+\beta(n+1-2\alpha)}{1-\alpha}, \quad n=2,3,\ldots$$

or

$$2\beta(n-1+\beta(n+1-2\delta))(1-\alpha)^2 \le (1-\delta)(n-1+\beta(n+1-2\alpha))^2, \; n=2,3,\ldots \qquad (13)$$

implies (12).

The inequalities (13) are equivalent to

$A\delta \le B$,

where

$$A = -4\beta^2(1-\alpha)^2 + (n-1)^2 + 2\beta(n-1)(n+1-2\alpha) +$$
$$+ \beta^2(n+1-2\alpha)^2 = (n-1)(\beta+1)((n-1)(\beta+1) + 4\beta(1-\alpha)) > 0$$

and

$$B = (n-1)^2 + 2\beta(n-1)(n+1-2\alpha) + \beta^2(n+1-2\alpha)^2 -$$
$$- 2\beta(1-\alpha)^2(n-1) - 2\beta^2(1-\alpha)^2(n+1) =$$
$$= (n-1)(\beta+1)((n-1)(\beta+1) + 4\beta(1-\alpha) - 2\beta(1-\alpha)^2).$$

We obtain

$$\delta \le \frac{B}{A} = \frac{(n-1)(\beta+1)+4\beta(1-\alpha)-2\beta(1-\alpha)^2}{(n-1)(\beta+1)+4\beta(1-\alpha)} =$$
$$=1-\frac{2\beta(1-\alpha)^2}{(n-1)(\beta+1)+4\beta(1-\alpha)} = \delta(n)$$

We have

$$\delta \le \delta(2) \le \delta(n), \quad n = 2,3,\ldots,$$

because $\delta(n)$ is an icreasing function of n.

Now we choose

$$\delta(\alpha,\beta) = \delta(2) = 1 - 2\beta(1-\alpha)^2/(5\beta+1-4\alpha\beta).$$

We have $\delta(\alpha,\beta) > \alpha$ because

$$\delta(\alpha,\beta)-\alpha = \frac{(3\beta+1)(1-\alpha)+4\alpha^2\beta}{4\beta(1-\alpha)+\beta+1} > 0$$

and $\delta(\alpha,\beta) < 1$, because

$$1-\delta(\alpha,\beta) = \frac{2\beta(1-\alpha)^2}{4\beta(1-\alpha)+\beta+1} > 0 .$$

The extremal functions are $f = g = f_2$ given by (9).

Remark 3. By using Theorem 2 and Remark 2.a) we obtain again Corollary 1.1.

Remark 4. Since $\sigma(\delta(\alpha,\beta),\beta) = \rho(\alpha,\beta)$, where $\rho(\alpha,\beta)$ is given by (10), we obtain $S^*(\delta(\alpha,\beta),\beta) \subset S^*(\rho(\alpha,\beta))$. So we can prove Corollary 1.2 by using 2 and Remark 1.

Remark 5. We have $\delta(\alpha,1) = (2-\alpha^2)/(3-2\alpha)$, hence we can obtain Corollary 1.3 by using Theorem 2 and Remark 1.

Remark 6. For given α and β, $\alpha \in [0,1)$, $\beta \in (0,1]$, the classes $S^*(\alpha,\gamma(\alpha,\beta))$ and $S^*(\delta(\alpha,\beta),\beta)$ are inclused in $S^*(\rho(\alpha,\beta))$, but they are generally distinct.

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UNIVALENCY CRITERIA OF KUDRIASOV'S TYPE

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REZUMAT. - Un criteriu de univalenţă de tip Kudriasov. În lucrare se obţin condiţii de univalenţă similare cu cele date de Kudriasov, condiţii care folosesc şi coeficientul a_2.

Let A be the class of regular functions in $U = \{z:|z|<1\}$, $f(z) = z + a_2z^2 + \ldots$ and $f(z)/z \neq 0$ for all $z \in U$.

THEOREM A [3]. *Let $f(z)$ be a regular function in U, $f(z) = z + a_2z^2 + \ldots$*

If

$$\left|\frac{f''(z)}{f'(z)}\right| \leq M \tag{1}$$

for all $z \in U$, where $M = 3,05$, then the function $f(z)$ is univalent in U.

In Kudriasov's results the constant M doesn't depend from a_2. The result could be improved for valours of $|a_2|$ approaching to 0.

In this paper we obtain the conditions of univalence similar to the result of Kudriasov's type, conditions which use coefficient a_2 too.

THEOREM 1. *If $f(z)$ is a regular function in U, $f(z) = a + a_3z^3 + \ldots$, and*

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$$\left|\frac{f''(z)}{f'(z)}\right| \le 4 \tag{2}$$

for all $z \in U$, then the function $f(z)$ is univalent in U.

Proof. Let's consider the function $g(z) = \frac{1}{4}\frac{f''(z)}{f'(z)}$. Using Schwarz's lemma [2] and Becker's univalence criterion [1], for the function $g(z)$, we obtain

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| = (1-|z|^2)\,|z|\cdot 4\,|g(z)| \le 4\,(1-|z|^2)\,|z|^2 \le 1,$$

and, hence, it results that the function $f(z)$ is univalent in U.

THEOREM 2. *Let α be a complex number,* Re $\alpha>0$ *and the function $f(z)$ belongs to the class A.*

If

$$\left|\frac{f''(z)}{f'(z)}\right| \le M \tag{3}$$

for all $z \in U$, where the constant M verifies the condition

$$M \le \frac{1}{\max\limits_{|z|\le 1}\left[\dfrac{(1-|z|^{2\,\mathrm{Re}\,\alpha})}{\mathrm{Re}\,\alpha}\,|z|\,\dfrac{|z|+\dfrac{2|a_2|}{M}}{1+\dfrac{2|a_2|}{M}|z|}\right]} \tag{4}$$

then, for every complex number β, Re $\beta \ge$ Re α *the function*

$$F_\beta(z) = \left[\beta\int_0^z u^{\beta-1}f'(u)\,du\right]^{\frac{1}{\beta}} \tag{5}$$

is regular and univalent in U.

Proof. Let's consider the function $F:[0,1] \to \mathbb{R}$,

$$F(x) = \frac{(1-x^{2\mathrm{Re}\,\alpha})}{\mathrm{Re}\,\alpha}\, x \, \frac{x+\dfrac{2|a_2|}{M}}{1+\dfrac{2|a_2|}{M}x} \quad ; \; x = |z|$$

Because $F\left(\frac{1}{2}\right) \neq 0$ it results that $\max\limits_{x\in[0,1]} F(x)>0$. Let's consider the function $g(z) = \frac{1}{M}\frac{f''(z)}{f'(z)}$. Using the generalization of Schwarz's lemma [2] for the function $g(z) = \frac{1}{M}\frac{f''(z)}{f'(z)}$, where M is a real positive constant which verifies the inequality (4), we obtain

$$\left|\frac{1}{M}\frac{f''(z)}{f'(z)}\right| \le \frac{|z|+\dfrac{2|a_2|}{M}}{1+\dfrac{2|a_2|}{M}|z|} \tag{6}$$

for all $z \in U$, and, hence we have

$$\begin{aligned} &\frac{(1-|z|^{2\mathrm{Re}\,\alpha})}{\mathrm{Re}\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le \\ &\le M \cdot \max_{|z|\le 1}\left|\frac{(1-|z|^{2\mathrm{Re}\,\alpha})}{\mathrm{Re}\,\alpha}\,|z|\,\frac{|z|+\dfrac{2|a_2|}{M}}{1+\dfrac{2|a_2|}{M}|z|}\right| \end{aligned} \tag{7}$$

From (4) and (7) we obtain

$$\frac{(1-|z|^{2\mathrm{Re}\,\alpha})}{\mathrm{Re}\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \tag{8}$$

for all $z \in U$ and from Pascu's univalence criterion [4], it results that the function $F_\beta(z)$ is regular and univalent in U.

COROLLARY 1. *If the function $f(z)$ belongs to the class A and*

$$\left|\frac{f''(z)}{f'(z)}\right| \le M \tag{9}$$

for all $z \in U$, where the constant M verifies the inequality

$$M \le \frac{1}{\max\limits_{|z|\le 1}\left[(1-|z|^2)\,|z|\,\dfrac{|z|+\dfrac{2|a_2|}{M}}{1+\dfrac{2|a_2|}{M}|z|}\right]}; \tag{10}$$

then the function $f(z)$ is univalent in U.

Proof. From the THEOREM 2, for $\alpha = 1$ and $\beta = 1$, we obtain the COROLLARY 1.

Observation. From Kudriasov's result it doesn't result the THEOREM 1,but from COROLLARY 1 for $a_2=0$ we obtain the THEOREM 1.

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TEST SETS IN QUANTITATIVE KOROVKIN APPROXIMATION

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REZUMAT. - Mulţimi test în aproximarea Korovkin cantitativă. Lucrarea conţine rezultate cantitative de tip Korovkin în care - în afară de funcţiile liniare - este utilizată ca funcţie test o singură funcţie convexă.

1. Let (X,d) be a compact metric space and let $B(X)$ denote the space of all real-valued bounded functions on X.

Let $C(X)$ be the subspace of $B(X)$ consisting of all continuous functions on X. For $f \in B(X)$ and $\delta > 0$ let

$$\omega(f,\delta) = \sup\{|f(x) - f(y)| : x,y \in X,\ d(x,y) \le \delta\}.$$

Suppose that there exists a constant $\mu > 0$ such that

$$\omega(f,t\delta) \le (1 + \mu t)\omega(f,\delta) \tag{1}$$

for all $f \in B(X)$ and all $t,\delta > 0$.

Let F be a nonnegative function in $B(X^2)$ such that

$$F(\cdot,y) \in C(X) \text{ for each } y \in X. \tag{2}$$

Suppose that there exist constants $q \ge 1$ and $k > 0$ such that

$$d^q(x,y) \le kF(x,y) \text{ for all } x,y \in X. \tag{3}$$

T.Nishishiraho [2] has proved

THEOREM 1. *Let* $T:C(X) \to B(X)$ *be a positive linear operator such that* $T1 = 1$. *Then*

$$|Tf(x) - f(x)| \le (1 + \mu k\delta^{-q}\, TF(\cdot,x)(x))\omega(f,\delta)$$

for all $f \in C(X)$, $x \in X$ *and* $\delta > 0$.

2. Let E be a normed real space and E' the dual of E endowed

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with the usual norm. Let X be a compact convex subset of E.

For $f \in C(X)$, $h_1,\dots,h_m \in E'$ and $\delta > 0$ let us denote

$$\omega(f;h_1,\dots,h_m) = \sup\{|f(x) - f(y)|: x,y \in X,$$

$$\sum_{i=1}^{m} (h_i(x) - h_i(y))^2 \le 1\},$$

$$\Omega(f,\delta) = \inf\{\omega(f;h_1,\dots,h_m) : m \ge 1,\ h_1,\dots,h_m \in E',$$
$$\sum_{i=1}^{m} \|h_i\|^2 = \delta^{-2}\}.$$

In what follows let $L:C(X) \to C(X)$ be a positive linear operator such that $L1 = 1$ and $Lh = h$ for all $h \in E'$.

For $x \in X$ and $\delta > 0$ let us denote

$$\tau(\delta,x) = \sup\{\sum_{i=1}^{m} (Lh_i^2(x) - h_i^2(x)) : m\ge1,\ h_1,\dots,h_m \in E',\ \sum_{i=1}^{m} \|h_i\|^2 = \delta^{-2}\}.$$

Then we have (see [1], [5, Th.1.4]):

THEOREM 2. *Let $f \in C(X)$, $x \in X$, $\delta > 0$. Then*

(i) $0 < \delta_1 \le \delta_2$ *implies* $\Omega(f,\delta_1) \le \Omega(f,\delta_2)$ *and*

$$\tau(\delta_1,x) \ge \tau(\delta_2,x)$$

(ii) $\lim_{\delta \to 0} \Omega(f,\delta) = 0$

(iii) $|Lf(x) - f(x)| \le (1 + \tau(\delta,x))\Omega(f,\delta)$.

In Theorem 1 the test set is $\{1\} \cup \{F(\cdot,x) : x \in X\}$.

In Theorem 2 it is $\{1\} \cup E' \cup \{h^2 : h \in E'\}$.

Suppose now that there exists a constant $c > 0$ such that

$$2(\|x\|^2 + \|y\|^2) - \|x + y\|^2 \ge c\|x - y\|^2 \tag{4}$$

for all $x,y \in E$ (See[6,p.86], [7]). In this case we shall obtain quantitative results in which- besides the linear functions - - only one convex function is involved as test function.

Let us remark that $c \le 1$; moreover, $c = 1$ if and only if E is an inner - product space. Condition (4) implies that E is uniformly convex (see[5]).

THEOREM 3. *Let* $f \in C(X)$, $x \in X$, $\delta > 0$. *Then:*

$$|Lf(x) - f(x)| \le (1 + LF(\cdot,x)(x)/c\delta^2)\omega(f,\delta) \tag{5}$$

$$|Lf(x) - f(x)| \le (1 + (Le - e)(x)/c\delta^2)\Omega(f,\delta) \tag{6}$$

where $e(x) = \|x\|^2$ *and* $F(x,y) = 2(e(x) + e(y)) - e(x+y)$, $x,y \in X$.

Proof. In this case (1) holds with $\mu = 1$ (see [2,Lemma 3]). By virtue of (4) we can choose $q = 2$ and $k = 1/c$; so (2) and (3) are also satisfied. Now (5) is a consequence of Th.1.

For $x,y \in X$, $a \in [0,1]$ and $f \in C(X)$ let us denote

$$(x,a,y;\ f) = (1 - a)f(x) + af(y) - f((1 - a)x + ay)$$

From (4) it follows (see[4]) that $(x,a,y;\ e) \ge ca(1-a)e(x-y)$ for all $x,y \in X$. Let $x \in X$. Then $f \in C(X) \to Lf(x)$ defines a probability Radon measure on X with barycenter x. It has been proved in [3] that for all $f \in C(X)$ there exist $u,v \in X$, $u \ne v$ and $a \in (0,1)$ such that

$$Lf(x) - f(x) = (Le(x) - e(x))(u,a,v;\ f)/(u,a,v;\ e).$$

Let $h \in E'$. Then we have $Lh^2(x) - h^2(x) = (Le(x) -$

$- e(x))(u,a,v;h^2)/(u,a,v;e) \le (Le(x) - e(x))(h(u) - h(v))^2/$

$/ce(u-v) \le (Le - e)(x)\ \|h\|^2/c$

It follows that $\tau(\delta,x) \le (Le - e)(x)/c\delta^2$ and thus (6) is a consequence of Th.2.

Let us remark that in (6) the test functions are the constant function 1, the linear functions and the convex function e. On the other hand it is easy to verify that $\omega \le \Omega$.

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