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A NOTE ON THE FUNCTIONS $\sigma_k(n)$ AND $\varphi_k(n)$

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REZUMAT. — Notă despre funcţiile $\sigma_k(n)$ şi $\varphi_k(n)$. În această notă studiem printre altele cîteva proprietăţi ale funcţiilor compuse $\sigma_k \circ \varphi_s$ şi $\varphi_k \circ \varphi_s$, unde k şi s sînt numere naturale nenule.

1. Let $\sigma_k(n)$ and $\varphi_k(n)$ denote the sum of kth powers of divisors of the natural number n and Jordan's arithmetical function, respectively. (See, e. g. [2], [4], [9]). Clearly, $\sigma_1(n) = \sigma(n)$ — the sum of divisors of n, and $\varphi_1(n) = \varphi(n)$ — the Euler arithmetical function ([9], [4], [7]). Our aime is to study certain new properties of these arithmetical functions, and especially some results for the composite functions $\sigma_k \circ \varphi_s$ and $\varphi_k \circ \varphi_s$ with $k, s \geqslant 1$ positive integer numbers. We will use also the generalized Dedekind function, $\psi_k(n)$ ([9], [4]).

2. Let $n = \prod p^a$ be the canonical representation of $n > 1$ where $p \mid n$ are the prime divisors of n. Recall that $d \mid n$ denotes the fact that d is a divisor of n. Then it is well-known that σ_k, φ_k, ψ_k are multiplicative functions and

$$\sigma_k(n) = \prod_{p|n} \frac{p^{k(a+1)} - 1}{p^k - 1}, \quad \varphi_k(n) = n^k \cdot \prod_{p|n} (1 - p^{-k}), \tag{1}$$
$$\psi_k(n) = n^k \cdot \prod_{p|n} (1 + p^{-k})$$

As in [5], [6], let us introduce the following notation: Denote by $n \wedge m$ the property that there exists at least a prime t with $t \mid n$ and $t \nmid m$. Let $k \geqslant 1$ be a fixed natural number. First we prove

LEMMA 1. $\sigma_k(mn) \geqslant n^k \sigma_k(m)$ *for all* $m, n = 1, 2, 3, \ldots$; (2)
$\sigma_k(mn) \leqslant \sigma_k(m)\sigma_k(n)$ *for all* $m, n = 1, 2, 3, \ldots$; (3)
$\sigma_k(mn) \geqslant (n^k + 1)\sigma_k(m)$ *for* $n \wedge m$. (4)

Proof. We shall prove only (4) and note that (2) and (3) follow by the same lines. (See also [6], [7]). Let $m = \prod p^a \cdot \prod q^b$, $n = \prod p^{a'} \prod t^c$ be the prime factorizations of m and n, where $(p, q) = (p, t) = (q, t) = 1$ and $c \geqslant 1$ (since $n \wedge m$). Using (1), one has at once

$$\sigma_k(mn) / \sigma_k(m) = \prod (p^{k(a+a'+1)} - 1)/(p^{k(a+1)} - 1) \cdot \prod (t^{k(c+1)} - 1)/(t^k - 1)$$

The simple algebraic inequalities

$$(x^{a+a'+1} - 1)/(x^{a+1} - 1) \geqslant x^{a'} (a\,; a' \geqslant 0,\ x > 1) \text{ and}$$

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$$(x^{c+1}-1)/(x-1) \geqslant x^c+1 \quad (c \geqslant 1)$$

imply $\sigma_k(mn)/\sigma_k(m) \geqslant \prod p^{ka'} \cdot \prod (t^{kc}+1) \geqslant \prod p^{ka'}(\prod t^{kc}+1) \geqslant n^k+1$, yielding (4).

The same results are valid for the Dedekind function ψ_k with a slightly different but analogous proof:

LEMMA 2. $\psi_k(mn) \geqslant n^k\psi_k(m)$ *for all* $m, n = 1, 2, 3, \ldots$; (5)

$\psi_k(mn) \leqslant \psi_k(m)\psi_k(n)$ *for all* $m, n = 1, 2, 3 \ldots$; (6)

$\psi_k(mn) \geqslant (n^k+1)\psi_k(m)$ *for* $n \wedge m$. (7)

For the function φ_k, the above inequalities are reversed:

LEMMA 3. $\varphi_k(mn) \leqslant n^k\varphi_k(m)$ *for all* $m, n = 1, 2, 3 \ldots$; (8)

$\varphi_k(mn) \geqslant \varphi_k(m)\varphi_k(n)$ *for all* $m, n = 1, 2, 3 \ldots$; (9)

$\varphi_k(mn) \leqslant (n^k-1)\varphi_k(m)$ *for* $n \wedge m$. (10)

Proof. For $n \wedge m$ we have

$\varphi_k(mn)/\varphi_k(m) = n^k \cdot \prod (1-t^{-k})$. Here $1-n^{-k} = 1-\prod p^{-ka} \cdot \prod t^{-kc} \geqslant$ $\geqslant 1-\prod t^{-kc} \geqslant 1-\prod t^{-k}$ for all $c \geqslant 1, a \geqslant 0$. This finishes the proof of (10). Relations (8) and (9) are almost obvious and we omit the details.

We note that the above arithmetical functions are connected in the following manner:

LEMMA 4. $\varphi_k(n) \leqslant \psi_k(n) \leqslant \sigma_k(n)$, $n = 1, 2, 3, \ldots$ (11)

Lastly, we need the following elementary inequalities:

LEMMA 5. $x^k+(1-x)^k \geqslant 2^{-k+1}$ *for all* $x \in [0, 1]$; (12)

$$\frac{x^{a+1}-1}{x^2-1} \leqslant \frac{a+1}{2} \cdot x^{a-1} \leqslant ax^{a-1} \text{ for all } x > 1,\ a \geqslant 1 \quad (13)$$

Proof. For (12) we may study the function $f(x) = x^k+(1-x)^k$, $x \in [0,1]$; while for (13) we apply the Cauchy mean-value theorem $\frac{f(u)-f(v)}{g(u)-g(v)} = \frac{f'(\xi)}{g'(\xi)}$, $\xi \in (u, v)$ by choosing $f(t) = t^{a+1}$, $g(t) = t^2$, $[u, v] = [1, x]$. Obviously, (13) is a consequence of $1 \leqslant \xi \leqslant x$ and $(a+1)/2 \leqslant a$.

3. We now are in a position to state and prove the main results of this paper. In what follows, $\omega(n)$ will be the number of all distinct prime factors of n.

THEOREM 1. *Let A denote the set of all numbers $n > 1$ with the property*

$$\sigma_k(\varphi_s(n)) \geqslant n^{ks} \cdot 2^{-(k-1)\omega(n)} \quad (14)$$

(with k, s fixed natural numbers). Let p be a prime number. Then:

a) *If $n \in A$ and $p \mid n$, then $np \in A$ is valid, too.*

b) *If $n \in A$, $p \nmid n$ and $p^s-1 \wedge \varphi_s(n)$, then $np \in A$, too.*

Proof. a) For $p \mid n$ one has $\varphi_s(np) = p^s\varphi_s(n)$, so by (2) we get $\sigma_k(\varphi_s(np)) \geqslant$ $\geqslant p^{sk} \cdot n^{sk} \cdot 2^{-(k-1)\omega(n)} = (np)^{sk} \cdot 2^{-(k-1)\omega(np)}$, i.e. $np \in A$

b) In the second case we apply (3) and (12):

$$\sigma_k(\varphi_s(np)) = \sigma_k(\varphi_s(n) \cdot (p^s - 1)) \geqslant \frac{(p^s - 1)^k + 1}{p^{ks}} (np)^{ks} \cdot 2^{-(k-1)\omega(n)} \geqslant (np)^{ks} \cdot 2^{-(k-1)\omega(np)}$$

because of $\omega(np) = \omega(n) + 1$ for $p \nmid n$.

Remark. The same result is valid also for the function $\psi_k(\varphi_s(n))$, with application of (5) and (7). In this case, on view of (11), the obtained result is a slightly stronger one.

THEOREM 2. *Let B denote the set of all odd numbers $n > 1$ whose prime factors $p_1, p_2, \ldots, p_r$ satisfy the following conditions:*

(C) $p_3^s - 1 \wedge (p_1^s - 1)(p_2^s - 1), \ldots, p_r^s - 1 \wedge (p_1^s - 1)(p_2^s - 1) \ldots (p_{r-1}^s - 1)$.

Then $B \subset A$, i.e. for $n \in B$, inequality (14) *is true.*
If $n \geqslant 2$ is an even number with $m \in B$, then

$$\sigma_k(\varphi_s(n)) \geqslant \frac{n^{ks}}{2^{ks} - k + 1} \cdot (2^s - 1)^k \cdot 2^{-(k-1)\omega(n)} \tag{15}$$

where m denotes the greatest odd divisor of n.

Proof. The proof follows by induction with respect to $r = \omega(n)$. Indeed, let $n = \prod p^a \in B$ be the prime factorization of $n > 1$. Then

$$\sigma_k(\varphi_s(n)) = \sigma_k(\prod p^{s(a-1)} \cdot \prod (p^s - 1)) \geqslant \prod p^{ks(a-1)} \cdot \sigma_k(\prod (p^s - 1)), \text{ by } \quad (2)$$

Thus it will be sufficient to prove that

$$\sigma_k(\prod (p^s - 1)) \geqslant \prod [(p^s - 1)^k + 1] \tag{16}$$

For $r = 1$, i.e. when $\prod (p^s - 1) = p^s - 1$, this is trivial; for $r = 2$, (16) is true for all odd primes p_1, p_2, since $1, p_1^s - 1, p_2^s - 1, (p_1^s - 1)(p_2^s - 1)$ are distinct divisors of $(p_1^s - 1)(p_2^s - 1)$.
Now, using condition (C), via (4) we obtain $\sigma_k \left(\prod\limits_{\omega(n)=3} (p^s - 1)\right) \geqslant [(p_3^s - 1)^k + 1]$ $[(p_1^s - 1)^k + 1][(p_2^s - 1)^k + 1]$, and so on, by induction we conclude with (16). Relation (14) follows by repeated application of inequality (12). In order to prove (15), let $n = 2^a \cdot m$ be an even number with $(2, m) = 1$. Then $\varphi_s(n) = \varphi_s(2^a)\varphi_s(m) = 2^{s(a-1)}(2^s - 1)\varphi_s(m)$, so by (14), applied this time for m, by taking into account of (2), we can derive (15).

THEOREM 3. *For all $n \in B$ we have*

$$\varphi_k(\varphi_s(n)) \leqslant n^{ks} \cdot \prod_{p \mid n} \frac{(p^s - 1)^k - 1}{p^{ks}} \tag{17}$$

If $n \geqslant 2$ is an even number with $m \in B$, then

$$\varphi_k(\varphi_s(n)) \leqslant n^{ks} \cdot \left(\frac{2^s - 1}{2^s}\right)^k \cdot \prod_{p \mid m} \frac{(p^s - 1)^k - 1}{p^{ks}} \tag{18}$$

where m denotes the greatest odd divisor of n.

Proof. The proof is similar with the proof of Theorem 2, but we now consider (10) and the simple inequality $\varphi_k(a) \leqslant a^k - 1$. We shall omit the details.

The last result involves also the arithmetical function $f(n) = \prod a$, where $n = \prod p^a$, p prime.

THEOREM 4. *Let S denote a set of natural numbers which have the same prime factors. Then*

$$\max\left\{\frac{\sigma_k(n)}{n^k \cdot f(n)} : n \in S\right\}$$

is taken for squarefree numbers $n \in S$.

Proof. Inequality (13) applied for $n = p^k$ gives $\frac{p^{k(a+1)} - 1}{(p^k - 1)p^{ka} \cdot a} \leqslant \frac{p^k + 1}{p^k}$, so by a simple multiplication it results $\sigma_k(n)/n^k \cdot f(n) \leqslant \sigma_k(m)/m^k \cdot f(m)$, where $m = \prod p \in S$ is a squarefree number.

Remarks. 1) For $k = 1$ we reobtain, with a new proof, a result from [1]. 2) If we apply the stronger inequality from (13), we get

$$\psi_k(n) \geqslant 2^{\omega(n)} \cdot \sigma_k(n)/d(n),$$

where $d(n)$ is the number of distinct divisors of n. This is stronger than a result of R.P. Sahu [3]. For other proofs and consequences of this inequality see [5], [8].

Finally, we conjecture that relation (14) is valid for all odd natural numbers $n > 1$ (with k, $s \geqslant 1$ positive integers).

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INEQUALITIES FOR DIVIDED DIFFERENCES OF n-CONVEX FUNCTIONS

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REZUMAT — Inegalităţi pentru diferenţe divizate ale funcţiilor n-convexe
Se demonstrează cîteva inegalităţi referitoare la diferenţe divizate; rezultatele obţinute le completează pe cele din lucrarea [2].

1. Let $a \leqslant x_o \leqslant x_1 \leqslant \ldots \leqslant x_n \leqslant b$ be fixed real numbers. Define $\mu : C[a, b] \to \mathbf{R}$, $\mu(f) = n![x_o, x_1 \ldots, x_n]f_n$, where $f_n \in C^n [a, b]$ $f_n^{(n)} = f$ and $[x_o, x_1, \ldots, x_n]f_n$ is the divided difference of f_n at the points $x_0, x_1, \ldots, x_n$. Clearly μ is a positive linear functional on $C[a, b]$ and $\mu(1) = 1$; therefore μ can be identified with a probability Radon measure on $[a, b]$.

Let $p_i(t) = t^i$, $t \in [a, b]$, $i = 1, 2, \ldots$. The barycenter of μ is

$$b(\mu) = \mu(p_1) = n![x_0, x_1, \ldots, x_n][p_{n+1}/(n+1)!] = (x_o + x_1 + \ldots + x_n)/(n+1)$$

If $f \in C[a, b]$ is convex, we have $f(b(\mu)) \leqslant \mu(f)$, i.e.,

$$f((x_c + \ldots + x_n)/(n+1)) \leqslant n![x_o, \ldots, x_n]f_n \tag{1}$$

This inequality was proved in [2]; moreover, it was shown there that if f is convex, then

$$f((x_o + \ldots + x_n)/(n+1)) \leqslant n![x_o, \ldots, x_n]f_n \leqslant$$
$$\leqslant (f(x_o) + \ldots + f(x_n))/(n+1) \tag{2}$$

Generalizations of these inequalities are given in [3].

From (2) we obtain in particular

$$f(1/2) \leqslant n!\left[0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right]f_n \leqslant$$
$$\leqslant \left(f(0) + f\left(\frac{1}{n}\right) + \ldots + f\left(\frac{n-1}{n}\right) + f(1)\right)/(n+1) \tag{3}$$

for every convex function $f \in C[0, 1]$ and every $n \geqslant 1$.

The present note contains an improved version of the inequalities (3) and some results related to the inequalities (2).

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2. For $f \in C[0, 1]$ and $n \geqslant 1$ let us denote

$$d_n(f) = n!\left[0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right]f,$$

$$s_n(f) = \left(f(0) + f\left(\frac{1}{n}\right) + \ldots + f\left(\frac{n-1}{n}\right) + f(1)\right)/(n+1)$$

Clearly $(s_n(f))$ converges to $\int_0^1 f(t)dt$ (see also (8) below).

THEOREM. (i) *For all* $f \in C[0, 1]$, $(d_n(f))$ *converges to* $f(1/2)$ (see also (below).

(ii) *If* $f \in C[0, 1]$ *is convex, then*

$$f(1/2) \leqslant d_n(f) \leqslant d_{n-1}(f) \leqslant \int_0^1 f(t)\mathrm{d}t \leqslant s_n(f) \leqslant s_{n-1}(f) \leqslant (f(0) + f(1))/2 \ ($$

for all $n \geqslant 2$.

Proof. For $f \in C[0,1]$ let $\mu(f) = \int_0^1 f(t)\mathrm{d}t$. Then $b(\mu) = 1/2$. Using notatio from [6], let

$$B_n f(1/2) = \int_{[0,1]^n} f((t_1 + \ldots + t_n)/n)\mathrm{d}(\mu \otimes \ldots \otimes \mu)(t_1, \ldots, t_n) =$$

$$= \int_0^1 \ldots \int_0^1 f((t_1 + \ldots + t_n)/n)\mathrm{d}t_1 \ldots \mathrm{d}t_n.$$

By a well-known result,

$$\left[0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right]f_n = (1/n!)\int_0^1 \ldots \int_0^1 f((t_1 + \ldots + t_n)/n)\mathrm{d}t_1 \ldots \mathrm{d}t_n$$

and hence $d_n(f) = B_n f(1/2)$. For every $f \in C[0, 1]$ we have $\lim B_n f(1/2) = f(1/2)$ (see [6] and the references given there). We conclude that $(d_n(f))$ cc verges to $f(1/2)$.

Let $f \in C[0, 1]$ be a convex function. Then $B_n f(1/2) \geqslant B_{n+1} f(1/2)$, $n = 1, 2, \ldots$ (see [4], [6]), hence $(d_n(f))$ is a decreasing sequence. It follo that $f(1/2) \leqslant d_n(f) \leqslant d_{n-1}(f) \leqslant d_1(f) = \int_0^1 f(t)\,\mathrm{d}t$.

We have also $f\left(\frac{k}{n}\right) \leqslant \frac{k}{n} f\left(\frac{k-1}{n-1}\right) + \frac{n-k}{n} f\left(\frac{k}{n-1}\right)$, $k = 1, \ldots, n -$

therefore $\frac{1}{n+1}\sum_{k=0}^{n} f\left(\frac{k}{n}\right) \leqslant \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n-1}\right)$, i.e., $s_n(f) \leqslant s_{n-1}(f)$, $n \geqslant 2$, and the theorem is proved.

Let us remark that (4) is an interpolating inequality for the well-known Hermite-Hadamard inequalities. Also, note that the inequality $\int_0^1 f(t)\mathrm{d}t \leqslant \leqslant s_n(f)$ can be proved by using the same Hermite-Hadamard inequality. Indeed,

$$\int_0^1 f(t)\mathrm{d}t + n\int_0^{1/n} f(t)\mathrm{d}t + \ldots + n\int_{(n-1)/n}^1 f(t)\mathrm{d}t \leqslant (1/2)\left[(f(0) + f(1)) + \left(f(0) + f\left(\frac{1}{n}\right)\right) + \right.$$

$$\left. + \ldots + \left(f\left(\frac{n-1}{n}\right) + f(1)\right)\right], \text{ i.e., } (n+1)\int_0^1 f(t)\mathrm{d}t \leqslant \sum_{i=0}^{n} f(i/n).$$

3. In the context of section 1 let $f \in C^2[a, b]$. Denote $m_f = \min\{f''(t) : : t \in [a, b]\}$, $M_f = \max\{f''(t) : t \in [a, b]\}$. Using a method from [5] and [1] let us consider the functions $g(t) = f(t) - (m_f/2)t^2$ and $h(t) = (M_f/2)t^2 - f(t)$. Clearly they are convex on $[a, b]$. Let $K_n = \sum_{i<j} (x_i - x_j)^2/2(n+1)^2(n+2)$.

Let us apply (2) for the convex functions g and h; we obtain

$$m_f K_n \leqslant n![x_0, \ldots, x_n]f_n - f((x_9 + \ldots + x_n)/(n+1)) \leqslant M_f K_n \tag{5}$$

$$(n+1)m_f K_n \leqslant (f(x_0) + \ldots + f(x_n))/(n+1) - n![x_0, \ldots, x_n]f_n \leqslant \leqslant (n+1)M_f K_n. \tag{6}$$

In particular, if $f \in C^2[0,1]$ we have

$$m_f/24n \leqslant d_n(f) - f(1/2) \leqslant M_f/24n. \tag{7}$$

Similarly,

$$m_f/12n \leqslant s_n(f) - \int_0^1 f(t)\mathrm{d}t \leqslant M_f/12n. \tag{8}$$

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A GENERALIZATION OF BECKER'S UNIVALENCE CRITERION

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REZUMAT — O generalizare a criteriului de univalenţă al lui Becker. Fie A mulţimea funcţiilor olomorfe în discul unitate U, normate cu ajutorul condiţiilor $f'(0) = 1, f(0) = 0$. În lucrare se prezintă condiţii suficiente pentru univalenţa funcţiilor de forma (5). Pentru $\alpha = -\beta$ condiţia obţinută devine criteriul de univalenţă al lui Becker.

1. **Introduction.** Let A be the class of functions f, which are analytic in the unit disc $U = \{z \in C : |z| < 1\}$, with $f(0) = 0$, $f'(0) = 1$.

In order to prove our main results we shall need the theory of Loewner.

DEFINITION. A function $L(z, t)$, $z \in U, t \geqslant 0$ is called a *Loewner chain* or a *subordination chain* if $L(z, t)$ is analytic and univalent in U for all positive t and, for all s, t with $0 \leqslant s < t$, $L(z, s) \prec L(z, t)$. In addition, $L(z, t)$ must be continuously differentiable on $[0, \infty)$, $z \in U$. [by $\prec$ we denote the relation of subordination].

2. **Preliminaries.** Denote by U_r, $0 < r \leqslant 1$ the disc of the z — plane: $\{z \in C : |z| < r\}$.

THEOREM A (*Pommerenke*) ([8], [9]). *Let* $r_0 \in (0, 1]$ *and let* $L(z, t) = a_1(t)z + a_2(t)z^2 + a_3(t)z^3 + \dots$, $a_1(t) \neq 0$ *be analytic in* U_{r_0} *for all* $t \geqslant 0$, *locally absolutely continuous in* $[0, \infty)$ *locally uniform with respect to* U_{r_0}. *For almost all* $t \geqslant 0$ *suppose*

$$z\frac{\partial L(z, t)}{\partial z} = p(z, t) \cdot \frac{\partial L(z, t)}{\partial t} \tag{1}$$

where $p(z, t)$ *is analytic in* U *and* $\operatorname{Re} p(z, t) > 0$, $z \in U$, $t \geqslant 0$. *If* $|a_1(t)| \to \infty$ *for* $t \to \infty$ *and* $\{L(z, t)/a_1(t)\}$ *forms a normal family in* U_{r_0}, *then, for each* $t \in [0, \infty)$. $L(z, t)$ *has an analytic and univalent extension to the disc* [*and is, consequently, a Loewner chain*].

THEOREM B ([1]). *If* $f(z) = z + a_2z^2 + \dots$ *is an analytic function in* U *and*

$$(1 - |z|^2) \cdot \left|\frac{z \cdot f''(z)}{f'(z)}\right| \leqslant 1 \text{ for all } z \in U \tag{2}$$

then, the function $f(z)$ *is univalent in* U.

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Theorem B was proved by J. Becker an is an useful criterion of univalence. Many generalizations of this criterion were obtained in [2], [4], [5], [6], [7], [9].

In this paper we shall give a sufficient condition for univalence of a class of functions which generalize the results obtained in [3] and is also a new generalization of Becker's univalence criterion.

3. Main results.

THEOREM 1. *Let $f \in A$ and let α and β be complex numbers If*:

$$|\alpha + \beta| < 1, \ \alpha \neq 0 \tag{3}$$

$$\left|\alpha(1 - |z|^2) \cdot \left(\frac{zf'(z)}{f(z)} - 1\right) + \alpha + \beta\right| \leqslant 1 \text{ for all } z \in U \tag{4}$$

then, the function

$$g(z) = \left[(\alpha + \beta + 1) \int_0^z f^\alpha(u) \cdot u^\beta du\right]^{\frac{1}{\alpha+\beta+1}} \tag{5}$$

is analytic and univalent in U.

Proof. The function

$$h(u) = \frac{f(u)}{u} = 1 + a_1 u + a_2 u^2 + \ldots \tag{6}$$

is analytic in U and $h(0) = 1$.
Then, we can choose r_0, $0 < r_0 \leqslant 1$, so that h does not vanish in Ur_0. In this case, denote by $h_1(u)$ the uniform branch of $[h(u)]^\alpha$ which is analytic in Ur_0 and $h_1(0) = 1$. Let

$$h_2(z, t) = (\alpha + \beta + 1) \int_0^{e^{-t} \cdot z} h_1(u) u^{\alpha+\beta} \cdot du = (e^{-t} \cdot z)^{\alpha+\beta+1} + \ldots \tag{7}$$

It is clear that, if $z \in U_{r_0}$ then $e^{-t} \cdot z \in U_{r_0}$ and, from the analyticity of h in U_{r_0}, we have that $h_2(z, t)$ is also analytic in U_{r_0} for all $t \geqslant 0$ and

$$h_2(z, t) = (e^{-t} \cdot z)^{\alpha+\beta+1} \cdot h_3(z, t) \tag{8}$$

where

$$h_3(z, t) = 1 + \ldots \text{ If we put} \tag{9}$$

$$h_4(z, t) = h_3(z, t) + (e^{2t} - 1) \cdot h_1(e^{-t} \cdot z) \tag{10}$$

we have that
$h_4(0, t) = e^{2t} \neq 0$ for all positive t. Then, we can choose r_1, $0 < r_1 \leqslant r_0$ so that h_4 does not vanish in U_{r_1}, $t \geqslant 0$. Now, denote by $h_5(z, t)$ the uniform branch of $[h_4(z, t)]^{1/(\alpha+\beta+1)}$ which is analytic in U_{r_1}, and $h_5(0, t) = e^{2t/(\alpha+\beta+1)}$. It follows that the function

$$L(z, t) = e^{-t} \cdot z \cdot h_5(z, t) \tag{11}$$

is analytic U_{r_1} and $L(0, t) = 0$ for all $t \geqslant 0$.

It is also clear that $e^{-t} \cdot h_5(0, t) = e^{\frac{1-(\alpha+\beta)}{1+(\alpha+\beta)} \cdot t}$. Now, we can formally. write (using (7), (8), (9), (10) and (11))

$$L(z, t) = \left[(\alpha + \beta + 1) \int_0^{e^{-t} z} \cdot f^\alpha(u) \cdot u^\beta \, du + (e^{2t} - 1) \cdot f^\alpha(e^{-t} z) \cdot (e^{-t} z)^{\beta+1}\right]^{\frac{1}{\alpha+\beta+1}}$$

$$= z \cdot e^{\frac{1-(\alpha+\beta)}{1+(\alpha+\beta)} \cdot t} + \ldots = z \cdot a_1(t) + \ldots$$

From (3) we have that $\mathrm{Re} \dfrac{1 - (\alpha + \beta)}{1 + (\alpha + \beta)} > 0$ and then

$$\lim_{t \to \infty} |a_1(t)| = \lim_{t \to \infty} \left| e^{\frac{1-(\alpha+\beta)}{1+((\alpha+\beta)} \cdot t} \right| = \lim_{t \to \infty} e^{t \cdot \mathrm{Re} \frac{1-(\alpha+\beta)}{1+(\alpha+\beta)}} = \infty.$$

$L(z, t)/a_1(t)$ is analytic in U_{r_1} for all $t \geqslant 0$ and then, it follows that $\{L(z, t)/a_1(t)\}$ is uniformly bounded in $U_{r_1|_2}$. Applying Montel's theorem, we have that $\{L(z,t)/ /a_1(t)\}$ is a normal family in $U_{r_1|_2}$. Using (10) and (11) we have.

$$\frac{\partial L(z, t)}{\partial t} = (e^{-t}z) \cdot \left[\frac{1}{\alpha + \beta + 1} (h_4(z, t))^{\frac{-\alpha-\beta}{\alpha+\beta+1}} \cdot \frac{\partial h_4(z, t)}{\partial t} + (h_4(z, t))^{\frac{1}{\alpha+\beta+1}}\right] \tag{13}$$

Because $h_4(0, t = e^{2t} \neq 0$ we can define an uniform branch for $[h_4(z, t)]^{\frac{-\alpha-\beta}{\alpha+\beta+1}}$ which is analytic in U_{r_2}, where $0 < r_2 \leqslant r_{1/2}$ is choosen so that the above -mentioned uniform branch, which takes in $(0, t)$ the value $e^{2t \frac{-\alpha-\beta}{\alpha+\beta+1}}$ and does not vanish in U_{r_2}.
It is also clear that $\partial h_4(z, t)/\partial t$ is analytic in U_{r_2} and then $\partial L(z, t)/\partial t$ is also. It follows that $L(z, t)$ is locally absolutely continuous. Let

$$p(z, t) = \frac{z \partial L(z, t)}{\partial z} \Big/ \frac{\partial L(z, t)}{\partial t} \tag{14}$$

In order to prove that $p(z, t)$ has an analytic extension with positive real part in U, for all $t \geqslant 0$, it is sufficient to prove that the function:

$$w(z, t) = \frac{p(z, t) - 1}{p(z, t) + 1} \tag{15}$$

is analytic in U for $t \geqslant 0$ and

$$|w(z, t)| < 1 \text{ for all } z \in U \text{ and } t \geqslant 0. \tag{16}$$

Using (6), after simple calculations we obtain

$$w(z, t) = \frac{(\alpha + \beta e^{2t}) \cdot h(e^{-t} \cdot z) + \alpha(e^{2t} - 1) f'(e^{-t} z)}{e^{2t} \cdot h(e^{-t} z)} \tag{17}$$

Because $h(e^{-t} \cdot z)$ does not vanish in U_{r_2}, and is analytic, it follows that $w(z, t)$ is also analytic in the same disc, for all $t \geqslant 0$. Then, $w(z, t)$ has an analytic extension in U, denoted also by $w(z, t)$. For $t = 0$, $|w(z, t)| = |\alpha + \beta| < 1$ from (3).

Let now $t > 0$. In this case $w(z, t)$ is analytic in $\bar{U}$ because: $|e^{-t} \cdot z| \leqslant e^{-t} < 1$ for all $z \in \bar{U}$. Then

$$|w(z, t)| < \max_{|z|=1} |w(z, t)| = |w(e^{i\theta}, t)| \tag{18}$$

with θ real.

For proving (16) it is sufficient that:

$$|w(e^{i\theta}, t)| \leqslant 1 \text{ for all } t > 0. \tag{19}$$

Note $u = e^{-t} \cdot e^{i\theta}$, $u \in U$. Then $|u| = e^{-t}$ and, from (17), after calculations, we obtain:

$$|w(e^{i\theta}, t)| = \left|\alpha(1 - |u|^2) \cdot \left(\frac{u \cdot f'(u)}{f(u)} - 1\right) + \alpha + \beta\right| \tag{20}$$

and inequality (19) becomes:

$$\left|\alpha(1 - |u|^2) \cdot \left(\frac{uf'(u)}{f(u)} - 1\right) + \beta + \alpha\right| \leqslant 1. \tag{21}$$

Because $u \in U$, relation (4) implies (21). Combining (18), (19), (20) and (21), it follows that $|w(z, t)| < 1$ for all $z \in U$, $t \geqslant 0$ Applying Theorem A, we have that $L(z, t)$ is a subordination chain and then, the function $L(z, 0) = g_{\alpha,\beta}(z)$ defined by (5), is analytic and univalent in U.

COROLLARY 1. *If* $f \in A$, α, β *are complex numbers with* $|\alpha+\beta| < 1$ *and* $\alpha \neq 0$ *and*

$$\left|\frac{zf'(z)}{f(z)} - 1\right|(1 - |z|^2) \leqslant \frac{1 - |\alpha + \beta|}{|\alpha|}, \tag{22}$$

then the function:

$$g_{\alpha,\beta}(z) = \left[(\alpha + \beta + 1)\int_0^z f^{\alpha}(u)u^{\beta}du\right]^{\frac{1}{\alpha+\beta+1}} \text{ is}$$

analytic and univalent in U.

Proof. $\left|\alpha(1 - |z|^2)\left(\frac{zf'(z)}{f(z)} - 1\right) + \alpha + \beta\right| \leqslant |\alpha| \cdot (1 - |z|^2) \cdot \left|\frac{zf'(z)}{f(z)} - 1\right| + |\alpha + \beta|$. From (22) it follows that (4) holds and then, the assertion follows easyly.

Remark 1. From Theorem 1 with $\beta = -\alpha$ we have:

If $f \in A$ and α ,is a complex number, $\alpha \neq 0$ and

$$\left|(1 - |z|^2) \cdot \left(\frac{\alpha zf'(z)}{f(z)} - \alpha\right)\right| \leqslant 1 \text{ for all } z \in U. \tag{23}$$

then the function

$$h(z) = \int_0^z \left[\frac{f(u)}{u}\right]^{\alpha} du \tag{24}$$

is analytic and univalent in U.

After simple calculations, we have that condition (23) is equivalent to:

$$\left|(1 - |z|^2) \cdot \frac{zh''(z)}{h(z)}\right| \leqslant 1. \tag{25}$$

Then, Theorem 1 with $\beta = -\alpha$ is equivalent with Becker's criterion of univalence (see Theorem B). Then, Theorem 1 is a generalization of Becker's criterion of univalence.

Remark 2. If, in Theorem 1, $\beta = -1$ we find the results obtained in [3].

4. So [illegible] **cases.**

The following particular cases have been studied with other methods by prof. P.T. Mocanu in his works related to hypergeometric functions. The reason of taking again these examples is to show that the subordination chains method is available too for such kind of problems and is liable to improvements.

Example 1. If $a \in C$ and b is real, $b > 1$ and

(a) $\alpha, \beta \in C, \dfrac{|\alpha|}{1 - |\alpha + \beta|} \leqslant b, \ \alpha \neq 0, \ |\alpha, + \beta| < 1$

(b) $m \in (-\infty, -b-1] \cup [b + 1, \infty)$ where

$m = \max \{\mathrm{Re}a, \ \mathrm{Im}a\}$.

then the function

(c) $F(z) = \left[(\alpha + \beta + 1) \displaystyle\int_0^z \frac{a^{\alpha} \cdot u^{\alpha+\beta}}{(a + u)^{\alpha}} du\right]^{\frac{1}{\alpha+\beta+1}}$ is analytic and **univalent** in U.

Proof. Let $f(z) = \dfrac{az}{a + z}$. It is clear that $f \in A$.

It is simple to show that:

$|zf'(z)/f(z) - 1| \leqslant \dfrac{1}{b} \leqslant \dfrac{1 - |\alpha + \beta|}{|\alpha|}$ and because

$1 - |z|^2 \leqslant 1, \ z \in U$, it follows that

$(1 - |z|^2) \cdot |zf'(z)/f(z) - 1| \leqslant \dfrac{1 - |\alpha + \beta|}{|\alpha|}$. Now we can apply Corollary 1 for f, α and β and we obtain the result.

Example 2. If α, β are complex, $\alpha \neq 0$, $|\alpha + \beta| < 1$ and

(d) $\dfrac{1 - |\alpha + \beta|}{|\alpha|} \geqslant 1$ then, the function

(e) $F(z) = \left[(\alpha + \beta + 1)\int_0^z \dfrac{2^\alpha \cdot u^{\alpha+\beta}}{(2+u)^\beta}\, du\right]^{\frac{1}{\alpha+\beta+1}}$ is analytic and univalent in U

Proof. Let $f(z) = \dfrac{2z}{2+z}$. It is clear that $f \in A$. After simple calculati

we have that:

$$(1 - |z|^2) \cdot |zf'(z)/f(z) - 1| \leqslant |zf'(z)/f(z) - 1| \leqslant 1 \leqslant \frac{1 - |\alpha + \beta|}{|\alpha|}, z \in$$

If we apply Corollary 1 for f, we obtain the result.

Remark 3. Example 2 is not a particular case of Example 1 because Exemple 1, condition $b > 1$ was necessary and in Example 2 we have $b =$

Remark 4. If α, β are real, $\beta \in [-1, 0]$, $\alpha \neq 0$, $-\beta \leqslant \alpha \leqslant \dfrac{1-\beta}{2}$. Th the function F defined by (e) is analytic and univalent in U.

Proof. $-\beta \leqslant \alpha \leqslant \dfrac{1-\beta}{2}$ is equivalent with $\dfrac{1 - |\alpha+\beta|}{|\alpha|} \geqslant 1$ and then, applying Example 2 we obtain the result.

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ON SOME PARTICULAR CLASSES OF INTEGRAL OPERATORS

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REZUMAT — Asupra unor clase de operatori integrali. În lucrare sînt demonstrate mai multe teoreme de subordonare diferenţială, care generalizează o serie de rezultate cunoscute.

1. **Introduction.** Let A be the class of analytic functions f in the disc U normalized by $f(0)=0, f'(0)=1$, and let α, c be complex numbers, with $Re(\alpha + c) > 0$, $\alpha \neq 0$.

We define the integral operators I and J on A, by

$$F(z) = I(f)(z) = \left[\frac{\alpha + c}{z^c}\int_0^z f^\alpha(t)\varphi_{(t)}^{c-1} h'(t)\mathrm{d}t\right]^{1/\alpha} \tag{1}$$

and

$$G(z) = J(g)(z) = \left[\frac{\alpha + c}{z^c}\int_0^z g^\alpha(t)\varphi_{(t)}^{c-1} h'(t)\mathrm{d}t\right]^{1/\alpha} \tag{2}$$

respectively.

DEFINITION 1 [4]. Let c be a complex number such that $\mathrm{Re}\, c > 0$ and let

$$N = N(c) = [|c|(1 + 2\mathrm{Re}\, c)^{1/2} + Im\, c]/\mathrm{Re}\, c.$$

If h is the univalent function $h(z) = 2Nz/(1 - z^2)$ and $b = h^{-1}(c)$, then we define the *open door* function Q as

$$Q_c(z) = h\left(\frac{z + b}{1 + \bar{b}z}\right), \; z \in U. \tag{3}$$

We will need the following four lemmas to prove our results.

LEMMA 2. *Let* α, $c \in \mathbf{C}$, *with* $\mathrm{Re}(\alpha + c) > 0$, $\alpha \neq 0$ *and let* f, g, $h \in A$ *satisfy*

$$\alpha\frac{zf'(z)}{f(z)} + (c - 1)\frac{zg'(z)}{g(z)} + \frac{zh''(z)}{h'(z)} + 1 \prec Q_{\alpha+c}(z),$$

where $Q_{\alpha+c}$ *is given by* (3). *Then* $F \in A$ *and* $F(z)/z \neq 0$ *in* U, $\mathrm{Re}\left[\alpha\frac{zF'(z)}{F(z)} + c\right] > 0.$

This lemma is a slight extention of Theorem 1 in [5].

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LEMMA 3 [2]. *Let* $\lambda(z)$ *be a complex function defined in* U *with* $\text{Re}\lambda(z) \geq 0$ *for* $z \in U$. *If* $p(z)$ *is analytic in* U *and* $\text{Re}[p(z) + \lambda(z)zp'(z)] > 0$ *for* $z \in U$ *then* $\text{Re}\, p(z) > 0$ *for* $z \in U$.

LEMMA 4 [1]. *Let* β *and* γ *be complex numbers and let* $h(z) = c + h_1 z + \ldots$ *be convex (univalent) in* U *with* $\text{Re}[\beta h(z) + \gamma] > 0, z \in U$. *If* $p(z) = c + p_1 c +$ *is analytic in* U *then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z), \quad z \in U. \tag{4}$$

LEMMA 5 [3]. *Let* β *and* γ *be complex numbers with* $\beta \neq 0$, *and let* $h(z)$ *be convex (univalent) in* U. *Set* $P(z) = \beta h(z) + \gamma$ *and suppose* $\text{Re}\, P(z) > 0$, $z \in U$. *If* $1/P$ *is convex in* U *then the solution of the equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \;\; (q(0) = h(0)),$$

is univalent and is best dominant of (4).

2. Main results.

THEOREM 1. *Let* $f, g, h, \varphi \in A$ *and let* q *be a convex (univalent) in* U *with* $q(0) = 1$. *Let* $\alpha, c \in \mathbb{C}$, $\alpha \neq 0$ *with* $\text{Re}(\alpha + c) > 0$ *and suppose* $g, h,$ *satisfy the conditions*

$$\alpha \frac{zg'(z)}{g(z)} + (c-1)\frac{z\varphi'(z)}{\varphi(z)} + \frac{zh''(z)}{h'(z)} + 1 \prec Q_{\alpha+c}(z), \; z \in U. \tag{5}$$

Suppose that $F \in A$ *and* $F(z)/z \neq 0$ *in* U. *If* $\frac{F(z)}{G(z)} \neq 0, \frac{f(z)}{g(z)} \neq 0,$

$$z\frac{f'(z)}{g^{\alpha}(z)f^{1-\alpha}(z)} + \left(\frac{f(z)}{g(z)}\right)^{\alpha} \cdot \left[\frac{c-1}{\alpha} \cdot \frac{z\varphi'(z)}{\varphi(z)} + \frac{1}{\alpha} \cdot \frac{zh''(z)}{h'(z)} + \frac{1-c}{\alpha}\right] \prec q(z)$$

then $\frac{zF'(z)}{G^{\alpha}(z)F^{1-\alpha}(z)} \prec q(z)$ *where* F *and* G *are defined by* (1) *and* (2) *respectively.*

Remark. The condition $F \in A$ and $F(z)/z \neq 0$, $z \in U$ is satisfied if (for example):

$$\alpha\frac{zf'(z)}{f(z)} + (c-1)\frac{z\varphi'(z)}{h'(z)} + \frac{zh''(z)}{h'(z)} + 1 \prec Q_{\alpha+c}(z), \; z \in U$$

Proof. By Lemma 2 the function G defined by (2) is analytic in U, $G(z)/z \neq 0$ and $\text{Re}[\alpha zG'(z)/G(z) + c] > 0$ in U. If we set $p(z) = zF'(z)/G^{\alpha}(z)F^{1-\alpha}(z)$, then p is analytic in U and $p(0) = 1$. A simple calculation yields

$$\frac{zf'(z)}{g^{\alpha}(z)f^{1-\alpha}(z)} + \left(\frac{f(z)}{g(z)}\right)^{\alpha} \cdot \left[\frac{c-1}{\alpha} \cdot \frac{z\varphi'(z)}{\varphi(z)} + \frac{1}{\alpha} \cdot \frac{zh''(z)}{h'(z)} + \frac{1-c}{\alpha}\right] =$$

$$= p(z) + \lambda(z) \cdot zp'(z)$$

where $\lambda(z) = 1/[\alpha zG'(z)/G(z) + c]$ and so $\text{Re}\,\lambda(z) > 0$ in U;

Since f, g, h, φ satisfy (5) we deduce that:

$$p(z) + \lambda(z) \cdot zp'(z) \prec q(z), \; z \in U$$

It is clear that all the conditions of Lemma 5 are satisfied and we obtain

$$p(z) \prec q(z) \text{ in } U.$$

If we let $\varphi(z) = z$, $h(z) = z$, $\alpha \in \mathbf{R}$ in Theorem 1, then we obtain:

COROLLARY 2. *Let $f \in A$ and let q be convex (univalent) in U with $q(0) = 1$. Let α be a real number with $\alpha > 0$ and c be a complex number with* $\mathrm{Re}\,(\alpha + c) > 0$ *and suppose $g \in A$ satisfies the condition*

$$\alpha \frac{zg'(z)}{g(z)} + c \prec Q_{\alpha+c}(z), \; z \in U.$$

Suppose that $F \in A$ and $F(z)/z \neq 0$ in U.

$$\text{If } \frac{zf'(z)}{g^{\alpha}(z)\, f^{1-\alpha}_{(z)}} \prec q(z) \text{ then } \frac{zF'(z)}{G^{\alpha}(z)\, F^{1-\alpha}_{(z)}} \prec q(z), \; z \in U, \text{ where}$$

$$F(z) = \left[\frac{\alpha + c}{z^c} \int_0^z f^{\alpha}(t) \cdot t^{c-1}\, dt\right]^{1/\alpha} \quad (6)$$

$$G(z) = \left[\frac{\alpha + c}{z^c} \int_0^z g^{\alpha}(t) \cdot t^{c-1}\, dt\right]^{1/\alpha}, \quad (7)$$

This result was recently obtained by S. Ponnusamy [6, 105—106].

If we take $g = f$ then, from Theorem 1, we obtain:

COROLLARY 3. *If* $\alpha \frac{zf'(z)}{f(z)} + (c-1)\frac{z\varphi'(z)}{\varphi(z)} + \frac{zh''(z)}{h'(z)} + 1 \prec Q_{\alpha+c}(z), \; z \in U,$

then $\frac{zf'(z)}{f(z)} + \frac{c-1}{\alpha} \cdot \frac{z\varphi'(z)}{\varphi(z)} + \frac{1}{\alpha} \cdot \frac{zh''(z)}{h'(z)} + \frac{1-c}{\alpha} \prec q(z)$, *implies that* $\frac{zF'(z)}{F(z)} \prec q(z)$, *where F is defined by* (1).

If we take $\varphi(z) = z$, $h(z) = z$, in Corollary 3, then we have

COROLLARY 4. *If q is a convex (univalent) in U with $q(0) = 1$, if $f \in A$ satisfies* $\alpha \frac{zf'(z)}{f(z)} + c \prec Q_{\alpha+c}(z)$ *then the function F defined by* (6) *also satisfies* $\frac{zF'(z)}{F(z)} \prec q(z), \; z \in U$.

For $q(z) = (1+z)/(1-z)$, from Corollary 4, we deduce:

COROLLARY 5. *If $f \in A$ satisfies* $\alpha zf'(z)/f(z) + c < Q_{\alpha+c}(z) \;\; z \in U$, *then $f \in S^*$ implies $F \in S^*$.*

If we take $c = 0$, $\alpha \in \mathbf{R}$ in Corollary 5, we have:

COROLLARY 6. *If* $\alpha > 0$ *and if* $f \in A$ *satisfies* $\alpha z f'(z)/f(z) < Q_\alpha(z)$ *then function* F *given by* $F(z) = \left(\int_0^z f^\alpha(t)^{-1} dt\right)^{1/\alpha}$ *belongs to* S^*.

This result was obtained by P. T. M o c a n u [4, Corollary 2.1].

THEOREM 7. *If* $f, g, h \in A$ *satisfy*

$$\frac{zf'(z)}{f(z)} + \frac{c-1}{\alpha} \cdot \frac{z\varphi'(z)}{\varphi(z)} + \frac{1}{\alpha} \frac{zh''(z)}{h'(z)} + \frac{1-c}{\alpha} \prec q(z), \quad (8)$$

where q *is convex (univalent) in* U, *and* $\operatorname{Re}(\alpha q(z) + c) > 0$, $z \in U$, $q(0) =$ *then* $zF'(z)/F(z) < q(z)$, *where* F *is given by* (1). $q(0)=1$

Proof. If we let $p(z) = zF'(z)/F(z)$, then p is analytic in U, with $p(0) =$
Since $p(0)=1$

$$F(z) = \left[\frac{\alpha + c}{z^c} \int_0^z f^\alpha(t) \varphi_{(z)}^{c-1} h'(t) dt\right]^{1/\alpha}$$

we deduce

$$f(z) = F(z) \cdot \left[\frac{c + \alpha zF'(z)/F(z)}{c + \alpha}\right]^{1/\alpha} \cdot \left[\frac{z}{\varphi_{(z)}^{c-1} h'(z)}\right]^{1/\alpha} \quad (9)$$

Hence p satisfies the differential equation

$$p(z) + \frac{zp'(z)}{c + \alpha p(z)} = \frac{zf'(z)}{f(z)} + \frac{c-1}{\alpha} \cdot \frac{z\varphi'(z)}{\varphi(z)} + \frac{1}{\alpha} \cdot \frac{zh''(z)}{h'(z)} + \frac{1-c}{\alpha}$$

and according to (8) we have

$$p(z) + \frac{zp'(z)}{c + \alpha p(z)} < q(z), \quad z \in U$$

Hence by Lemma 4 we obtain that $p < q$.

If we take $\varphi(z) = z$, $h(z) = z$, $\alpha \to 0$, from Theorem 7 we deduce

THEOREM 8 [6, 107—108]. *Let* $f \in A$, *let* c *be a complex number with* $\operatorname{Re} c >$ $c \neq 0$, *let* q *be a convex (univalent) in* U *with* $q(0) = 1$ *and let* F *defined by* (6). $\operatorname{Re} c \geq 0$

Then $zf'(z)/f(z) < q(z)$, $z \in U$, *implies that*

$$\frac{zF'(z)}{F(z)} < c \cdot z^{-c} \int_0^z t^{c-1} \cdot q(t) dt, \ \textit{for } z \in U. \quad (10)$$

The result is sharp.

Proof. If we put $p(z) = zF'(z)/F(z)$ then we have

$$\frac{zf'(z)}{f(z)} = c^{-1}zp'(z) + p(z), \quad z \in U.$$

Sience $f \in A$ satisfies (9) $\alpha \mapsto 0$, the conclusion of the theorem follows from ma 3. For $-\frac{\pi}{2} < \lambda < \frac{\pi}{2}$ and

$$q(z) = \frac{1 + e^{-i\lambda}\,(2\rho\cos\lambda - e^{-i\lambda})\cdot z}{1+z}, \quad \rho < 1, \tag{11}$$

obtain that

$$\mathrm{Re}\left[e^{i\lambda}\cdot\frac{zf'(z)}{f(z)}\right] > \rho\cos\lambda \text{ implies} \tag{11'}$$

$$e^{i\lambda}\cdot\frac{zF'(z)}{F(z)} < e^{i\lambda}\cdot\left[cz^{-c}\int_0^z t^{c-1}\cdot q(t)dt\right],$$

re $q(z)$ is given by (11).

The case $\rho = 0$ of (11′) improves the result of Yoshikai [7]

THEOREM 9. *Let $f, g, h \in A$ satisfy the condition* (8) *of* Theorem 7. *If* log P 1/P *are convex in* U, *where* $P(z) = \alpha q(z) + c$, $z \in U$, *then*

$$\frac{zF'(z)}{F(z)} \prec q_1(z),$$

re

$$q_1(z) = \frac{1}{\alpha g_1(z)} - \frac{c}{\alpha}, \tag{12}$$

$$g_1(z) = \frac{1}{z^c\cdot(k(z))^\alpha}\cdot\int_0^z k^\alpha(t)\cdot t^{c-1}dt, \text{ and}$$

$$k(z) = z\cdot\exp\int_0^z\frac{q(t)-1}{t}dt$$

result *is sharp.*

Proof. If we set $p(z) = zF'(z)/F(z)$, then by Lemma 5, we obtain that the tion q_1 defined by (12) is univalent and $p \prec q_1$. This result is the best possible.

If we take $\varphi(z) = z$, $h(z) = z$, then from Theorem 9, we obtain:

COROLLARY 10. *Let* $f \in A$, $c \in C$, $\mathrm{Re}(\alpha + c) > 0$. *If* $\frac{zf'(z)}{f(z)} < q(z)$.

Re $(\alpha q(z) + c) > 0$, $\log P$ *and* $1/P$ *are convex in* U, *with* $P(z)$:
$z \in U$, *then* $zF'(z)/F(z) < q_1(z)$, *where* F *is given by* (6) *and* q_1 *is gi*
The result is sharp.

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EXISTENCE AND CONTINUATION OF SOLUTIONS FOR NCTIONAL-DIFFERENTIAL INCLUSIONS OF NEUTRAL TYPE[1]

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REZUMAT — Existenţa şi continuarea soluţiilor incluziunilor funcţional- diferenţiale de tip neutral. În lucrare se studiază existenţa şi continuarea soluţiilor incluziunii funcţional-diferenţiale (1) generalizîndu-se unele rezultate ale lui J. K. Hale.

1. **Introduction.** In this paper we study the functional-differential inclu-
:

$$\frac{d}{dt} D(t,\ x_t) \in F(t,\ x_t) \tag{1}$$

re F is a multivalued mapping taking as its values nonempty compact sets of $\mathbf{R}^n$ and D is a single-valued mapping with values in $\mathbf{R}^n$. We will sider the equation (1) by the assumption that F satisfies the Carathéodory conditions. This paper is related to the previous paper ([5]) of this author, re the existence of solutions of (1), in the case when F has a Carathéodory ctor, has been considered.

The aim of this paper is to present the existence theorem and continuation olutions for functional — differential inclusion (1). The results of this paper eralize some results of J. K. Hale ([2]).

2. **Notations and definitions.** Suppose $r \geqslant 0$ is a given real number, $\mathbf{R} = (-\infty, \infty)$, $\mathbf{R}^n$ is a real n-dimensional linear vector space with norm $|\cdot|$, $C([a, b], \mathbf{R}^n)$ is the Banach space of continuous functions mapping the interval $[a, b]$ into $\mathbf{R}^n$ with the topology of uniform convergence. If $[a,\ b] = [-r,\ 0]$, let $C_{or} = C([-r,\ 0],\ \mathbf{R}^n)$ and designate the norm of an element Φ in C_{or} $\|\Phi\| = \sup_{-r \leqslant \theta \leqslant 0} |\Phi(\theta)|$. For every fixed $t \in [\sigma,\ \sigma + a]$, $\sigma \in \mathbf{R}$, $a \geqslant 0$ by we denote a mapping of $[-r,\ 0]$ into $\mathbf{R}^n$ defined by $x_t(\theta) = x(t + \theta)$ for $[-r, 0]$ where $x \in C([\sigma\ - r,\ \sigma + a],\ \mathbf{R}^n)$ is given, $r \geqslant 0$.

By $L([a,\ b],\ \mathbf{R}^n)$ we denote the Banach space of all Lebesque integrable tions of $[a,\ b]$ into $\mathbf{R}^n$ with the norm $|\cdot|$ defined by $|f| = \int_a^b |f(t)|\,dt$ for $L([a, b],\ \mathbf{R}^n)$. For Banach spaces X and Y, by $\mathfrak{L}(X,\ Y)$ we denote the ach space of bounded linear mappings from X into Y with the operator

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topology. If $K \in \mathcal{L}(C_{or}, \mathbf{R}^n)$, then the Riesz representation theore that there is an $n \times n$ matrix function η on $[-r, 0]$ of bounded vari

th at $K\Psi = \int_{-r}^{0} [d\eta(\theta)]\Psi(\theta)$, $\Psi \in C_{or}$.

Let Comp $(\mathbf{R}^n)$ denote the space of all nonempty compact $\mathbf{R}^n$. We will consider Comp $(\mathbf{R}^n)$ together with the Hausdorff metr

Assume that Ω is an open set in $\mathbf{R} \times C_{or}$ and that $F : \Omega \to ($ satisfies the following conditions:

(i) $F(\cdot, \Phi)$ is measurable for fixed $\Phi \in \Pi_{C_{or}}(\Omega)$,

(ii) $F(t, \cdot)$ is continuous for fixed $t \in \Pi_{\mathbf{R}}(\Omega)$,

(iii) there is a Lebesgue integrable function $m : \mathbf{R} \to \mathbf{R}^+$ such th

$$h(F(t, \Phi), \{0\}) \leqslant m(t) \text{ for } (t, \Phi) \in \Omega$$

where $\Pi_{\mathbf{R}}(\Omega)$ and $\Pi_{C_{or}}(\Omega)$ denote the projections of Ω on the real li respectively.

DEFINITION 1. Suppose Ω is an open set in $\mathbf{R} \times C_{or}$, $D : \Omega \to \mathbf{R}$ nuous, $D(t, \Phi)$ has a continuous Fréchet derivative $D'_{\Phi}(t, \Phi)$ with Φ on Ω and

$$D'_{\Phi}(t, \Phi)\Psi = \int_{-r}^{0} [d_{\theta}\eta\,(t, \Phi, \theta)]\Psi(\theta)$$

for $(t, \Phi) \in \Omega$, $\Psi \in C_{or}$, where $\eta\,(t, \Phi, \theta)$ is an $n \times n$ matrix wit of bounded variation in $\theta \in [-r, 0]$. For any β in $[-r, 0]$ we say D at β on Ω if

$$\eta(t, \Phi, \beta^+) - \eta(t, \Phi, \beta^-) = A(t, \Phi, \beta)$$

$$\det A(t, \Phi, \beta) \neq 0$$

where $A(t, \Phi, \beta)$ is continuous in (t, Φ) and there is a scalar $\gamma(t, \Phi, s, \beta)$ continuous for $(t, \Phi) \in \Omega$ $s \geqslant 0$, $\gamma(t, \Phi, 0, \beta) = 0$ suc

$$(*) \left| \int_{\beta-s}^{\beta+s} [d_{\theta}\eta(t, \Phi, \theta)]\Psi(\theta) - A(t, \Phi, \beta)\Psi(\beta) \right| \leqslant \gamma(t, \Phi, s, \beta) \, \|\Psi\| \text{ for}$$

$s \geqslant 0$, $\Psi \in C_{or}$.

DEFINITION 2. Suppose $\Omega \subseteq \mathbf{R} \times C_{or}$ is open, $D : \Omega \to \mathbf{R}^n$ is a tinuous function, D is atomic at zero and $F : \Omega \to$ Comp $(\mathbf{R}^n)$ satisfies (i) — (iii). The relation

$$\frac{d}{dt} D(t, x_t) \in F(t, x_t)$$

is called the *neutral functional — differential inclusion* (NFDI).

DEFINITION 3. For a given NFDI a function x is said to be a solution (1) if there are $\sigma \in \mathbf{R}$, $a > 0$ such that $x \in C([\sigma - r, \sigma + a), \mathbf{R}^n)$, $(t, x_t) \in \Omega$, $[\sigma, \sigma + a)$ and x satisfies (1) for a.e. $t \in (\sigma, \sigma + a)$.

Notice that this definition implies that $D(t, x_t)$ and not $x(t)$ is continuously differentiable on $(\sigma, \sigma + a)$.

DEFINITION 4. For a given $(\sigma, \Phi) \in \Omega$ we say $x(\sigma, \Phi)$ is a *solution* of NFDI $[\sigma - r, \sigma + a)$ through (σ, Φ) if $x(\sigma, \Phi)$ is a solution of (1) and $x_\sigma(\sigma, \Phi) = \Phi$.

3. Existence theorem.

THEOREM 1. *If $\Omega \subset \mathbf{R} \times C_{0r}$ is an open set and (1) is NFDI then for any $(\sigma, \Phi) \in \Omega$ there exists a solution of (1) through (σ, Φ).*

Proof. If the derivative $D'_\Phi(t, \Phi)$ of $D(t, \Phi)$ with respect to Φ is represented as

$$D'_\Phi(t, \Phi)\Psi = A(t, \Phi, 0)\Psi(0) - \int_{-r}^{0-} [d_\theta \eta(t, \Phi, \theta)]\Psi(\theta) \tag{2}$$

then the definition of atomic at zero implies $\det A(t, \Phi, 0) \neq 0$, $A(t, \Phi, 0)$ is continuous in (t, Φ) and there is a scalar function $\gamma(t, \Phi, s, 0)$ continuous for $(t, \Phi) \in \Omega$, $s \geqslant 0$, $\gamma(t, \Phi, 0, 0) = 0$ such that condition $(\ast)$ for $\beta = 0$ is true.

A function x is a solution of (1) through (σ, Φ) if there is an $a > 0$ such that $x \in C([\sigma - r, \sigma + a), \mathbf{R}^n)$ and x satisfies the inclusion

$$\begin{cases} \dfrac{d}{dt} D(t, x_t) \in F(t, x_t) \text{ for a.e. } t \in [\sigma, \sigma + a] \\ x_\sigma = \Phi \end{cases} \tag{3}$$

Let $C^\circ([\sigma, \sigma + a], \mathbf{R}^n)$ denote a Banach space of all continuous function $z : [\sigma, \sigma + a] \to \mathbf{R}^n$ such that $z(\sigma) = 0$. For every $z \in C^\circ([\sigma, \sigma + a], \mathbf{R}^n)$, we define $\tilde{z} \in C([\sigma - r, \sigma + a], \mathbf{R}^n)$ by $\tilde{z}(t) = \chi_{[\sigma - r, \sigma]}(t) \cdot 0 + \chi_{[\sigma, \sigma + a]}(t) \cdot z(t)$. Now, we can define, for every $\Phi \in C_{0r}$ and for each fixed $\sigma \in \mathbf{R}$ and $a > 0$, a mapping $(\Phi \oplus \tilde{z}) \in C([\sigma - r, \sigma + a], \mathbf{R}^n)$ by setting

$$(\Phi \oplus \tilde{z})(t) = \begin{cases} \Phi(t - \sigma) \text{ for } t \in [\sigma - r, \sigma) \\ \Phi(0) + \tilde{z}(t) \text{ for } t \in [\sigma, \sigma + a]. \end{cases}$$

In what follows, we shall denote $(\Phi \oplus \tilde{z})$ by $(\Phi \oplus z)$. If x is a solution of (1) through (σ, Φ) on $[\sigma - r, \sigma + a]$ and $x(t) = (\Phi \oplus z)(t)$, $t \in [\sigma - r, \sigma + a]$, then z satisfies the equation

$$\begin{cases} \dfrac{d}{dt} D(t, (\Phi \oplus z)_t) \in F(t, (\Phi \oplus z)_t) \text{ for a.e. } t \in [\sigma, \sigma + a] \\ z_\sigma = 0 \end{cases} \tag{4}$$

Since $F : \Omega \to \text{Comp}(\mathbf{R}^n)$ satisfies conditions (i) — (iii), then in virtue of paper ([3]) there exists a continuous mapping $f : \Lambda \to L([\sigma, \sigma + a], \mathbf{R}^n)$, where Λ is compact subset of $C([\sigma, \sigma + a], \mathbf{R}^n)$, such that $f(z)(t) \in F(t, (\Phi \oplus z$ a.e. $t \in [\sigma, \sigma + a]$. Then x is a solution of (1) through (σ, Φ) if and only if there is an $a > 0$ such that z satisfies the equation:

$$\begin{cases} D(t, (\Phi \oplus z)_t) = D(\sigma, \Phi) + \int_\sigma^t f(z)(\tau)d\tau \text{ for a.e. } t \in [\sigma, \sigma + a] \\ z_\sigma = 0 \end{cases} \quad (5)$$

In virtue of (2) and the definition of the mapping $(\Phi \oplus z)$ we have

$$A(t,(\Phi \oplus 0)_t, 0)z_t(0) = D'_\Phi(t,(\Phi \oplus 0)_t)z_t + \int_{-r}^{0-} [d_\theta \eta(t,(\Phi \oplus 0)_t,\theta)]z_t(\theta). \quad (6)$$

Therefore, using the formula (5) and the property of mapping D we get

$$A(t,(\Phi \oplus 0)_t,0)z_t(0) = \quad (7)$$

$$= D'_\Phi (t,(\Phi \oplus 0)_t)z_t + \int_{-r}^{0-} [d_\theta \eta(t,(\Phi \oplus 0)_t,\theta)]z_t(\theta) +$$

$$+ D(\sigma,\Phi) + \int_\sigma^t f(z)(\tau)d\tau - D(t,(\Phi \oplus z)_t) =$$

$$= D'_\Phi (t,(\Phi \oplus 0)_t)z_t + \int_{-r}^{0-} [d_\theta \eta(t,(\Phi \oplus 0)_t,\theta)]z_t(\theta) +$$

$$+ D(\sigma,\Phi) + \int_\sigma^t f(z)(\tau)d\tau - D(t,(\Phi \oplus 0)_t) - [D(t,(\Phi \oplus z)_t) -$$

$$- D(t,(\Phi \oplus 0)_t)] = D'_\Phi (t,(\Phi \oplus 0)_t)z_t + D(\tau,\Phi) +$$

$$+ \int_{-r}^{0-} [d_\theta \eta(t,(\Phi \oplus 0)_t, \theta)]z_t(\theta) + \int_\sigma^t f(z)(\tau)d\tau +$$

$$- D(t,(\Phi \oplus 0)_t) - D'_\Phi (t,(\Phi \oplus 0)_t)z_t - g(t,(\Phi \oplus 0)_t,z_t) =$$

$$= \int_{-r}^{0-} d_\theta \eta(t,(\Phi \oplus 0)_t,\theta)\, z_t(\theta) + D(\sigma, \Phi) +$$

$$- D(t,(\Phi \oplus 0)_t) + \int_\sigma^t f(z)(\tau)d\tau - g(t,(\Phi \oplus 0)_t,z_t)$$

re $g(t, \varphi, 0) = 0$, $|g(t, \varphi, \Psi) - g(t, \varphi, \xi)| \leqslant \mathcal{E}(t, \varphi, \delta) \| \Phi - \xi \|$ for $(t, \varphi) \in \Omega$, $\|\Psi\| \leqslant \delta$, $\|\xi\| \leqslant \delta$, $\mathcal{E}(t, \varphi, \delta)$ is continuous in t, φ, δ for $(t, \varphi) \in \Omega$, $\geqslant 0$ and $\mathcal{E}(t, \varphi, 0) = 0$. Then, because $z_t(0) = z(t)$, it follows

$$\begin{cases} z(t) = A^{-1}(t, (\Phi \oplus 0)_t, 0) \left\{ \int_{-r}^{0-} [d_\theta \eta(t, (\Phi \oplus 0)_t, \theta)] z_t(\theta) + \right. \\ \left. \quad + D(\sigma, \Phi) - D(t, (\Phi \oplus 0)_t) - g(t, (\Phi \oplus 0)_t, z_t) + \int_\sigma^t f(z)(\tau) d\tau \right\} \\ \qquad \text{for a.e. } t \in [\sigma, \sigma + a] \\ z_\sigma = 0 \end{cases} \tag{8}$$

we let

$$(Tz)(t) = \begin{cases} 0 & \text{for } t \in [\sigma - r, \sigma] \\ A^{-1}(t, (\Phi \oplus 0)_t, 0) \left\{ \int_{-r}^{0-} [d_\theta \eta(t, (\Phi \oplus 0)_t, \theta)] z_t(\theta) + \right. \\ \left. + D(\sigma, \Phi) - D(t, (\Phi \oplus 0)_t) - g(t, (\Phi \oplus 0)_t, z_t) \right\} & \text{for a.e. } t \in [\sigma, \sigma + a] \end{cases}$$

nd

$$(Sz)(t) = \begin{cases} 0 & \text{for } t \in [\sigma - r, \sigma] \\ A^{-1}(t, (\Phi \oplus 0)_t, 0) \int_\sigma^t f(z)(\tau) d\tau & \text{for a.e } t \in [\sigma, \sigma + a] \end{cases}$$

hen (8) is equivalent to the equation:

$$z(t) = (Tz)(t) + (Sz)(t) \text{ where } z \in C([\sigma - r, \sigma + a], \mathbf{R}^n), \; z_\sigma = 0.$$

One now proceeds as in [5], (Lemma 1), to show that there are positive $\bar{a}$, $\bar{b}$ so that if

$$\mathcal{A}(\bar{a}, \bar{b}) = \{\xi \in C([\sigma - r, \sigma + \bar{a}], \mathbf{R}^n) : \xi_\sigma = 0, \; \|\xi_t\| \leqslant \bar{b} \text{ for } t \in [\sigma, \sigma + \bar{a}]\}$$

then $T : \mathcal{A}(\bar{a}, \bar{b}) \to C([\sigma - r, \sigma + \bar{a}], \mathbf{R}^n)$ is a contraction, $S : \mathcal{A}(\bar{a}, \bar{b}) \to C([\sigma - r, \sigma + \bar{a}], \mathbf{R}^n)$ is completely continuous and $T + S : \mathcal{A}(\bar{a}, \bar{b}) \to \mathcal{A}(\bar{a}, \bar{b})$. It implies the existence of a fixed point of $T + S$ in $\mathcal{A}(\bar{a}, \bar{b})$ (see [1], Lemma 2.1) and thus a solution of (I) through (σ, Φ). The proof is complete.

4. **Continuation of solutions.** Let Ω be a nonempty open subset of $\mathbf{R} \times C_{or}$ and let F and D be such that for every $(\sigma, \Phi) \leqslant \Omega$ NFDI has at least one local solution on $[\sigma - r, a)$, $a > \sigma$, through (σ, Φ). We say $\hat{x}$ is a con-

tinuation of x if there is a $b > a$ such that $\hat{x}$ is defined on $[\sigma - r, b)$, cides with x on $[\sigma - r, a)$ and satisfies (1) on (a, b).

A solution x of (1) on $[\sigma - r, a)$ through (σ, Φ) is said to be *nonnuable* if no such continuation exist, that is the interval $[\sigma - r, a)$ is the ximal interval of existence of the solution x.

The existence of noncontinuable solutions of NFDI follows from the ratowski—Zorn's lemma. The proof of the following theorem is based on thesis of M. Kisielewicz ([4]).

THEOREM 2. *Let D and F are such that for every $(\sigma, \Phi) \in \Omega$, $\Omega \subset \mathbf{R} \times$ is open, NFDI has at least one local solution through (σ, Φ). Then for $(\sigma, \Phi) \in \Omega$ there exists a noncontinuable solution of NFDI through (σ, Φ).*

Proof. Denote by $\chi(\sigma, \Phi, F)$ the set of all functions $x\colon [\sigma - r, a_x)$ that each restriction $x|_{[\sigma-r,\mu]} = x^{\mu}$ with $\mu \in (\sigma, a_x)$ is a solution of NFDI $[\sigma - r, \mu]$ through (σ, Φ). Let us introduce in $\chi(\sigma, \Phi, F)$ an ordered rel $\prec$ by setting $x \prec y$ if and only if x is a restriction of y to any subint $[\sigma - r, a_x)$ contained in the domain $[\sigma - r, a_y)$ of y, for $x, y \leqslant \chi(\sigma,$ $(\chi(\sigma, \Phi, F), \prec)$ is a partially ordered system such that only $(X_x, \prec)$ with containing all restrictions x^{μ} of $x \in \chi(\sigma, \Phi, F)$ are totally ordered subsy of $\chi(\sigma, \Phi, F)$. Since for every $z \in X_x$ we have $z \prec x$, then every totally red subsystem of $\chi(\sigma, \Phi, F)$ has an upper bound in $\chi(\sigma, \Phi, F)$. Thus, by tue of Kuratowski—Zorn's lemma, there exists in $\chi(\sigma, \Phi, F)$ a maximal ment $x_{\max}$ defined on any interval $[\sigma - r, a)$ that is a noncontinuable sol of *NFDI* through (σ, Φ).

DEFINITION 4. If D is atomic at β on Ω and W is a subset of Ω, we s is uniformly atomic at β on W if there is an $N > 0$ such that $|A^{-1}(t,$ $\beta)| \leqslant N$, $|D'_{\Phi}(t, \Phi)| \leqslant N$ for all $(t, \Phi) \in W$ and $\gamma(t, \Phi, s, \beta) \to 0$ as $s \to 0$ unifo for $(t, \Phi) \in W$.

Now, similarly as in the paper of W. Melvin ([6]) we can prove following general continuation theorem.

THEOREM 3. *Suppose Ω is an open set in $\mathbf{R} \times C_{or}$, (1) is a NFDI for any closed bounded set W in with a δ — neighborhood also in Ω, $F\colon$ $\to$ Comp $(\mathbf{R}^n)$, $D(t, \Phi)$, $D'_{\Phi}(t, \Phi)$ are uniformly continuous on W and D is formly atomic at zero on W. If x is a noncontinuable solution of (1) on $[\sigma -$ $b)$, then there is a $t' \in [\sigma, b)$ such that $(t', x_{t'}) \in W$.*

Proof. Suppose $r > 0$ and b finite.

I. If there is a sequence $t_k \to b^-$ and $\Psi \in C_{or}$ such that $x_{t_k} \to \Psi$, then the that $r > 0$ implies that $x(t)$ is uniformly continuous on $[\sigma - r, b)$ and $x($ $\to \Psi(0)$ as $t \to b$. If we define $x(b) = \Psi(0)$, then (b, x_b) must belong to boundary of Ω or x would be continuable beyond b. Now, the fact that is continuous and the distance of (b, x_b) from any closed bounded set W positive imply the existence of a t_w such that $(t, x_t) \notin W$ for $t \in [t_w, b)$.

II. If no such sequence exists, there are two cases to consider:

1) the case where the set $V = \{(t, x_t) : t \in [\sigma, b)\}$ is unbounded

2) the case where the set V, defined above, is bounded. In the first case have that for any closed bounded set $W \subset \Omega$, there is a constant k_W such $||\Phi|| < k_W$ for $(t, \Phi) \in W$. Let $k_{W'} = \max\ \{||x_\sigma||, k_W\}$. From hypothe

there is a sequence $t_k \to b^-$ monotonically such that $||x_{t_k}|| > k_{W'}$. Using the property of the norm in C_{or} and definition of the mapping x_t we get the existence of a t_W such that $(t, x_t) \notin W$ for $t \in [t_W, b)$.
In the second case, if the set V is bounded and has a δ — neighborhood in Ω then this set is also closed since there are no subsequences $t_k \to b^-$ such that x_{t_k} converges. We want to show there is an $\alpha > 0$ such that x is uniformly continuous on $[b - \alpha, b)$ and therefore $\{(t, x_t), t \in [\sigma, b)\}$ belongs to a compact set in Ω. This will obviously be a contradiction.

If $x(t)$ is not uniformly continuous for $t \in [\sigma - r, b)$, there are an $\varepsilon > 0$, a monotone decreasing sequence of positive numbers Δ_k, $\Delta_k \to 0$ as $k \to \infty$ and a sequence of real numbers, $t_k \in [\sigma, b)$, $t_k - \Delta_k \in [\sigma, b)$ such that $|x(t_k) - x(t_k - \Delta_k)| \geqslant \varepsilon$ for all k. For any $p > 0$ the fact that x is uniformly continuous on $[\sigma - r, b - p]$ implies for any $\varepsilon_1 > 0$ the existence of a $\Delta > 0$ such that $|x(t) - x(t_1)| \leqslant \varepsilon_1$ for $|t - t_1| < \Delta$ and $t, t_1 \in [\sigma - r, b - p]$. In virtue of assumptions $D(t, \Phi)$ is uniformly continuous on V and we can choose Δ so that

$$|D(t, \Phi) - D(t_1, \Phi)| \leqslant \varepsilon_1 \tag{9}$$

for $|t - t_1| < \Delta$ and $(t, \Phi) \in V$, $(t_1, \Phi) \in V$.

From the hypotheses on V, D and Definitions 1 and 4, there are a $\beta_0 > 0$ and continuous functions $\gamma(s)$, $s \geqslant 0$, $\mathscr{E}(\beta)$, $0 \leqslant \beta \leqslant \beta_0$, $\gamma(0) = \mathscr{E}(0) = 0$ and a constant N such that

$$|A^{-1}(\sigma, \Phi, 0)| \leqslant N, \ |D'_\Phi(\sigma, \Phi)| \leqslant N,$$

$$D(\sigma, \Phi + \Psi) = D(\sigma, \Phi) + D'_\Phi(\sigma, \Phi)\Psi + g(\sigma, \Phi, \Psi)$$

where $|g(\sigma, \Phi, \Psi)| \leqslant \mathscr{E}(\beta)||\Psi||$ and

$$\left|\int_{-s}^{0} [d_\theta \eta(\sigma, \Phi, \theta)]\Psi(\theta)\right| \leqslant \gamma(s) \sup_{-s \leqslant \theta \leqslant 0} |\Psi(\theta)|$$

for $(\sigma, \Phi) \in V$, $||\Psi|| \leqslant \beta$, $0 \leqslant \beta \leqslant \beta_0$, $s \geqslant 0$.

Hence,

$$\begin{aligned} |D(\sigma, \Phi + \Psi) - D(\sigma, \Phi)| &= \\ = |D'_\Phi(\sigma, \Phi)\Psi + g(\sigma, \Phi, \Psi)| &\geqslant |A(\sigma, \Phi, 0)\Psi(0)| - \\ - \left|\int_{-r}^{0-} [d_\theta \eta(\sigma, \Phi, \theta)]\Psi(\theta)\right| &- |g(\sigma, \Phi, \Psi)| \geqslant \\ \geqslant \frac{|\Psi(0)|}{N} - \gamma(s)||\Psi|| - N &\sup_{-r \leqslant \theta \leqslant -s} |\Psi(\theta)| - \mathscr{E}(\beta)||\Psi|| \end{aligned} \tag{10}$$

for $(\sigma, \Phi) \in V$, $||\Psi|| \leqslant \beta$, $0 \leqslant \beta \leqslant \beta_0$, $s \geqslant 0$.

Suppose $0 < \beta \leqslant \beta_0$ is given. Choose $\varepsilon_1 < \min(\beta, \varepsilon)$ and K sufficiently that $|\Delta_k| < \Delta$, $k \geqslant K$. For every $k \geqslant K$, let

$p_k = \inf\{t \in [\sigma, b) : |x(t) - x(t - \Delta_k)| \geqslant \min(\beta, \varepsilon)$.

From (9) and (10) we have

$$|D(p_k, x_{p_k}) - D(p_k - \Delta_k, x_{p_k - \Delta_k})| \geqslant$$

$$\geqslant |D(p_k, x_{p_k}) - D(p_k, x_{p_k - \Delta_k})| - |D(p_k, x_{p_k - \Delta_k}) - D(p_k - \Delta_k, x_{p_k - \Delta_k})|$$

$$\geqslant |D(p_k, x_{p_k}) - D(p_k, x_{p_k - \Delta_k})| - \varepsilon_1 \geqslant$$

$$\geqslant \frac{\min(\beta, \varepsilon)}{N} - \gamma(s)\beta - N\varepsilon_1 - \mathcal{E}(\beta)\min(\beta, \varepsilon) - \varepsilon_1 = \bar{\varepsilon}.$$

Now, one can obviously choose β_0, s, ε_1 so that $\bar{\varepsilon} > 0$. Consequently the h thesis that $x(t)$ is not uniformly continuous on $[\sigma - r, b)$ implies that D is not uniformly continuous on $[\sigma, b)$.

On the other hand for a.e. $t, t + \tau \in [\sigma, b)$ we have

$$D(t + \tau, x_{t+\tau}) = D(t, x_t) + \int_t^{t+\tau} f(x)(s)ds,$$

where f is a continuous mapping, such that $f(x)(t) \in F(t, x_t)$.
Since

$$\left|\int_t^{t+\tau} f(x)(s)ds\right| \leqslant \int_t^{t+\tau} m(s)ds$$

we have

$$|D(t + \tau, x_{t+\tau}) - D(t, x_t)| \leqslant M$$

for $(s, x_s) \in W$ and some constant M.
Then, the function $D(t, x_t)$ is uniformly continuous on $[\sigma, b)$. This contrad completes the proof of the theorem.

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PERIODIC SOLUTIONS OF CERTAIN SIXTH ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. — In this paper, we shall give sufficient conditions for the nonexistence of nontrivial periodic solutions of the autonomous equation (2.1) and for the existence of periodic solutions of the nonautonomous equation (2.2).

1. **Introduction.** Consider the sixth order constant-coefficient differential equation:

$$x^{(6)} + a_1 x^{(5)} + a_2 x^{(4)} + a_3 \dddot{x} + a_4 \ddot{x} + a_5 \dot{x} + a_6 x = 0. \tag{1.1}$$

As it was shown in [1] that if

$$a_1 \neq 0, \; a_6 \neq 0 \text{ and} \left(a_5 - \frac{1}{4} a_3^2 \; a_1^{-1}\right) \operatorname{sgn} a_1 > 0, \tag{1.2}$$

then the auxiliary equation corresponding to (1.1) has no purely imaginary roots whatever. By the general theory; this, in turn, implies first of all that (1.1) has no periodic solution whatever other than $x = 0$, and secondly that the perturbed equation

$$x^{(6)} + a_1 x^{(5)} + a_2 x^{(4)} + a_3 \dddot{x} + a_4 \ddot{x} + a_5 \dot{x} + a_6 x = p(t) \tag{1.3}$$

in which $p(\neq 0)$ is any continuous T-periodic function of t, has an w-periodic solution subject to (1.2).

The argument in [4] shows clearly that similar cases are valid for the qeuations (1.1) and (1.3) under conditons:

$$a_4 > \frac{1}{4} a_2^2 \; , \; a_6 < 0. \tag{1.4}$$

The object of the present paper is to extend the result (1.4) for (1.1) and (1.3) to certain equations in which a_2, a_3, a_4, a_5 and a_6 are not all constants.

Note that this problem was pointed out in [1].

2. **Statement of the results.** We shall be concerned with the two equations:

$$x^{(6)} + a_1 x^{(5)} + f_1(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}, x^{(5)}) x^{(4)} + f_2(\ddot{x}) \dddot{x} \tag{2.1}$$

$$+ f_3(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}, x^{(5)}) \ddot{x} + f_4(\dot{x}) + f_5(x) = 0 \quad (f_4(0) = 0, \; f_5(0) = 0),$$

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$$x^{(6)} + a_1 x^{(5)} + g_1(\dddot{x})x^{(4)} + g_2(\ddot{x})\dddot{x} + g_3(\dot{x})\ddot{x} + g_4(\dot{x}) \qquad (2.2)$$
$$+ g_5(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}, x^{(5)})$$

in which a_1 is a constant. The functions $f_1, f_2, f_3, f_4, g_1, g_2, g_3, g_4$ and p are continuous functions depending only on the arguments shown with p w-periodic in t, that is $p(t + w, x_1, x_2, \ldots, x_6) = p(t, x_1, x_2, \ldots, x_6)$ for some $w >0$ and arbitrary $t, x_1, x_2, \ldots, x_6$. We shall however require here that $f_5'(x)$ and $g_5'(x)$ exist and are continuous for all x.

We shall establish here the following theorems:

THEOREM 1. *If*

$$f_5'(x_1) < 0 \textit{ for all } x_1, \qquad (2.3)$$

and

$$f_3(x_1, x_2, x_3, x_4, x_5, x_6) > \frac{1}{4} f_1^2 (x_1, x_2, x_3, x_4, x_5, x_6) \qquad (2.4)$$

for arbitrary $x_1, x_2, \ldots, x_6$, *then the equation (2.1) has no periodic solution whatever other than* $x = 0$.

The conditions here can be seen to be a generalization of (1.4). Note that there are no restrictions on a_1, f_2 and f_4.

THEOREM 2. *Suppose that*

(i) *there exist constants* $a_2 \geqslant 0$, $c > 0$ *such that*

$$|g_1(x_4)| \leqslant a_2 \textit{ for all } x_4, \qquad (2.5)$$

$$a_4 = \inf_{x_2} g_3(x_2) > \frac{1}{4} a_2^2 \qquad (2.6)$$

$$g_5'(x_1) < -c \textit{ for all } x_1, \qquad (2.7)$$

(ii) *there are constants* $A_0 \geqslant 0$, $A_1 \geqslant 0$ *such that*

$$|p(t, x_1, x_2, x_3, x_4, x_5, x_6)| \leqslant A_0 + A_1(|x_3| + |x_4|) \qquad (2.8)$$

for all $t, x_1, x_2, x_3, x_4, x_5$ *and* x_6;

Then there exists a constant $\zeta_0 > 0$ *such that* (2.2) *has at least one* w*-periodic solution if* $A_1 < \zeta_0$.

Note again the absence of any restriction on a_1, g_2 and g_4.

3. Proof of Theorem 1. The procedure here is exactly as in [1] and [3]. We consider the equivalent differential system for (2.1):

$$\dot{x}_i = x_{i+1} \qquad (i = 1, 2, \ldots, 5)$$
$$\dot{x}_i = -a_1 x_6 - f_1(x_1, x_2, \ldots, x_6)x_5 - f_2(x_3)x_4 - f_3(x_1, x_2, \ldots, x_6)x_3 - \qquad (3.1)$$
$$- f_4(x_2) - f_5(x_1)$$

tained by setting $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \ddot{x}$, $x_4 = \dddot{x}$, $x_5 = x^{(4)}$ and $x_6 = x^{(5)}$ (2.1). Let $(\zeta_1, \ldots, \zeta_6) = (\zeta_1(t), \ldots, \zeta_6(t))$ be an arbitrary α-periodic solution (3.1), that is

$$(\zeta_1(t), \ldots, \zeta_6(t)) = (\zeta_1(t+\alpha), \ldots, \zeta_6(t+\alpha)) \tag{3.2}$$

some $\alpha > 0$. It will be shown that, subject to the conditions in Theorem **1**,

$$\zeta_1 = 0 = \zeta_2 = \ldots = \zeta_6.$$

Our main tool here is the function $V(x_1, \ldots, x_6)$, introduced in [4], which defined by

$$V = x_3(x_6 + a_1x_5) + x_4(x_5 + \frac{1}{2}a_1x_4) - \int_0^{x_3} sf_2(s)ds - \int_0^{x_2} f_4(u)du - f_5(x_1)x_2 \tag{3.3}$$

Consider the function

$$\theta(t) = V(\zeta_1(t), \ldots, \zeta_6(t)).$$

Since V is continuous and $\zeta_1, \ldots, \zeta_6$ are periodic in t, $\theta(t)$ is clearly bounded. Also it can be verified by an elementary differentation along solutions paths of (3.1) that

$$\dot{\theta}(t) = \frac{d}{dt} V(\zeta_1, \ldots, \zeta_6)$$

$$= \left\{\zeta_5 + \frac{1}{2} f_1(\zeta_1, \ldots, \zeta_6)\zeta_3\right\}^2 + \left\{f_3(\zeta_1, \ldots, \zeta_6) - \frac{1}{4}f_1^2(\zeta_1, \ldots, \zeta_6)\right\}\zeta_3^2$$

$$- f_5'(\zeta_1)\zeta_2^2.$$

Hence $\dot{\theta}(t) \geqslant 0$, by (2.3) and (2.4), so that $\theta(t)$ is monotone in t, and therefore the rest of the proof, can be shown in the same way as in [1]. which gives

$$\zeta_1 = 0 = \zeta_2 = \ldots = \zeta_6$$

subject to the conditions in Theorem 1.

4. **Proof of Theorem 2: preliminaries.** The proof will be by the Leray-Schauder technique, with the equation (2.2) embedded in the parameter-dependent equation :

$$x^{(6)} + a_1x^{(5)} + \{(1-\mu)a_2 + \mu g_1(\dddot{x})\}x^{(4)} + \mu g_2(\ddot{x})\dddot{x} + \{(1-\mu)a_4 + \mu g_3(\dot{x})\}\ddot{x}$$

$$+ \mu g_4(\dot{x}) + \{\mu g_5(x) - (1-\mu)cx\} = \mu p(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}, x^{(5)}) \quad 0 \leqslant \mu \leqslant 1. \tag{4.1}$$

Note that when $\mu = 1$ (4.1) reduces to (2.2). Also, when $\mu = 0$ it reduces to the linear equation

$$x^{(6)} + a_1x^{(5)} + a_2x^{(4)} + a_4\ddot{x} - cx = 0$$

which, in view of the conditions (1.4), has no non-trivial w-periodic solution.

The equation (4.1) thus has the base features expected of parameter-depe-dent equations for application of the usual fixed point considerations in [2] and hence, in order to establish the theorem 2, it remains only for us to sho[w] that there is a constant $D > 0$ independent of $\mu(0 \leqslant \mu \leqslant 1)$ such that

$$|x(t)| \leqslant D,\ |\dot{x}(t)| \leqslant D,\ |\ddot{x}(t)| \leqslant D,\ |\dddot{x}(t)| \leqslant D,\ |x^{(4)}(t)| \leqslant D,\ |x^{(5)}(t)| \leqslant D \quad (0 \leqslant t \leqslant w), \qquad (4.2)$$

for any w-periodic solution $x(t)$ of (4.1).

Before proceeding to the actual verification of (4.2) we shall introduce so[me] notations. Throughout what follows, the D, D_0, D_1, ... whenever, they occ[ur] are positive constants whose magnitudes depend only on a_1, a_2, a_4, c, A_0, [...] g_2, g_3, g_4 and g_5 but not on μ. The numbered D's : D_0, D_1 ,... retain the sa[me] identity throughout while the unnumbered D's are not necessarily the sa[me] each time they occur.

We shall take (4.1) in the more compact form :

$$x^{(6)} + a_1 x^{(5)} + g_1^*(\dddot{x})x^{(4)} + \mu g_2(\ddot{x})\dddot{x} + g_3^*(\dot{x})\ddot{x} + \mu g_4(\dot{x}) + g_5^*(x) = \mu p(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}, x^{(5)}), \qquad (4.3)$$

by setting

$$\begin{aligned} g_1^*(\dddot{x}) &= (1-\mu)a_2 + \mu g_1(\dddot{x}) \\ g_3^*(\dot{x}) &= (1-\mu)a_4 + \mu g_3(\dot{x}) \\ g_5^*(x) &= -(1-\mu)cx + \mu g_5(x). \end{aligned} \qquad (4.4)$$

Note here that

$$|g_1^*(x_4)| \leqslant a_2,\ g_3^*(x_2) \geqslant a_4,\ g_5^{*\prime}(x_1) < -c \qquad (4.5)$$

by (2.5), (2.6) and (2.7).

Throughout what follows in this paper let $x = x(t)$ be an arbitrary w-periodic solution of (4.3) and the function $W = W(t)$,analogous to the V of sectio[n] 3, defined by

$$W = -x_3(x_6 + a_1 x_5) + x_4\left(x_5 + \frac{1}{2} a_1 x_4\right) - \mu \int_0^{x_3} s g_2(s) ds - \mu \int_0^{x_2} g_4(u) du - g_5^*(x_1)x_2$$

An elementary differentiation gives that

$$\dot{W} = U - \mu x p(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}, x^{(5)}) \qquad (4.6)$$

where

$$\begin{aligned} U &= x^{(4)2} + g_1^*(\dddot{x})\ddot{x}x^{(4)} + g_3^*(\dot{x})\ddot{x}^2 - g_5^*(x)\dot{x}^2 \\ &\geqslant x^{(4)2} - a_2|\ddot{x}||x^{(4)}| + a_4\ddot{x}^2 + c\dot{x}^2 \end{aligned} \qquad (4.7)$$

by (4.5). Subject to the condition (2.6) it is possible to obtain the following more refined estimate for U:

$$U \geqslant D_0(x^{(4)^2} + \dddot{x}^2 + \dot{x}^2) \tag{4.8}$$

for some suitable fixed D_0. Indeed by (4.7),

$$U - \{D_0(x^{(4)^2} + x^2 + x^2)\} \geqslant (1 - D_0)x^{(4)^2} - a_2|\dddot{x}||x^{(4)}| + (a_4 - D_0)\dddot{x}^2 +$$

$$+ (c - D_0)\dot{x}^2 = (1 - D_0)\Big[|x^{(4)}| - \frac{1}{2}a_2(1 - D_0)^{-1}|\dddot{x}|\Big]^2 + \frac{1}{4}(1 - D_0)^{-1}$$

$$[(4a_4 - a_2^2) - 4D_0(1 + a_4) + 4D_0^2]\dddot{x}^2 + (c - D_0)\dot{x}^2$$

in D_0' is fixed such that

$$D_0 < \min\left\{1, \frac{1}{4}(4a_4 - a_2^2)(1 + a_4)^{-1}, c\right\}. \tag{4.9}$$

The term $(4a_4 - a_2^2)$ and c here are positive by (2.6) and (2.7), so that the choise of a positive D_0 satisfying (4.9) is possible. We can therefore assume (4.8) subject to (4.9) on D_0. Hence, by (4.6) and (2.8),

$$\dot{W} \geqslant D_0(x^{(4)} + \dddot{x}^2 + \dot{x}^2) - \{A_0|\dddot{x}| + A_1(\dddot{x}^2 + |\dddot{x}||\ddot{x}|)\}$$

$$\geqslant D_0(x^{(4)^2} + \dot{x}^2 + (D_0 - \frac{3}{2}A_1)\dddot{x}^2 - \frac{A_1}{2}\ddot{x}^2 - A_0|\dddot{x}|$$

$$\geqslant D_1(x^{(4)^2} + \dddot{x}^2 + \dot{x}^2) - \frac{A_1}{2}\ddot{x}^2 - D_2 \tag{4.10}$$

for some D_1, D_2, if A_1 is fixed sufficiently small.

Because of the (assumed) w-periodicity of x, we have, on integrating (4.10), that

$$0 \geqslant D_1\int_2^w (x^{(4)^2} + \dddot{x}^2 + \dot{x}^2)dt - \frac{1}{2}A_1\int_0^w \dddot{x}^2 dt - D_2 w. \tag{4.11}$$

Combined with the inequality

$$\int_0^w \ddot{x}^2 dt \leqslant \frac{1}{4}w^2\pi^{-2}\int_0^w x^{(4)^2}\, dt, \tag{4.12}$$

which can be verified by substituting the Fourier expansions of $\ddot{x}$ and $x^{(4)}$ in (4.12), (4.11) leads to the estimate

$$D_1 - \frac{1}{8}w^2\pi^{-2}A_1)\int_0^w x^{(4)^2}\, dt + D_1\int_0^w (\dddot{x}^2 + \dot{x}^2)dt \leqslant D_2 w.$$

Hence, if A_1 is further fixed such that

$$A_1 w^2 \pi^{-2} \leqslant 4D_1$$

as we assume henceforth, then

$$\frac{1}{2} D_1 \int_0^w (x^{(4)^2} + \ddot{x}^2 + \dot{x}^2)dt \leqslant D_2 w.$$

In particular

$$\int_0^w x^{(4)^2}\, dt \leqslant D_3. \tag{4.13}$$

Considering now the identity:

$$\dddot{x}(t) = \dddot{x}(T_1) + \int_{T_1} x^{(4)}(s)ds$$

with T_1 fixed (as is possible in view of the periodicity condition $\dddot{x}(0) = \dddot{x}(w)$) such that $\dddot{x}(T_1) = 0$, we have that

$$\max_{0 \leqslant t \leqslant w} |\dddot{x}(t)| \leqslant \int_0^w |x^{(4)}(s)|ds \leqslant w^{1/2}\left(\int_0^w x^{(4)^2}(s)ds\right)^{1/2}$$

by Schwarz's inequality. Thus (4.13) implies that

$$\max_{0 \leqslant t \leqslant w} |\dddot{x}(t)| \leqslant w^{1/2} D_3^{1/2}. \tag{4.14}$$

Likewise the fact that $\ddot{x}(T_2) = 0$ at some $T_2 \in [0, w]$ combines with (4.14) to yield the remaining estimate:

$$\max_{0 \leqslant t \leqslant w} |\ddot{x}(t)| \leqslant w^{3/2} D_3^{1/2}., \tag{4.15}$$

and similarly from the fact that $\dot{x}(T_3) = 0$ at some $T_3 \in [0, w]$ we have that

$$\max_{0 \leqslant t \leqslant w} |\dot{x}(t)| \leqslant w^{5/2} D_3^{1/2}. \tag{4.16}$$

To obtain an estimate for $x(t)$ first note that, because of the w-periodicity of x, integration of both sides of (4.3) yields the result

$$\int_0^w -\{g_1^*(x) - \mu p(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}, x^{(5)})\}dt = \mu \int_0^w g_4(\dot{x})dt$$

or indeed, in view of (4.16), that

$$\left|\int_0^w \{g_1^*(x) - \mu p(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}, x^{(5)})\}dt\right| \leqslant D_4. \tag{4.17}$$

But, by (2.8), (4.14) and (4.15)

$$|\mu p(t, x, \dot{x}, \ddot{x}, \dddot{x}\ x^{(4)}, x^{(5)})| \leqslant D_3^{1/2} w^{1/2}$$

for some D_3. Since g_5 is subject to (2.7) and $c > 0$, it is thus evident from (4.17) that $x(t)$ must satisfy $|x(T_4)| < D_5$ at some $T_4 \in [0, w]$, for D_5 independent of μ. The result that $|x(T_4)| < D_5$ combined with (4.16) to yield the required boundedness estimate for x:

$$\max_{0 \leqslant t \leqslant w} |x(t)| \leqslant D_5 + D_3^{1/2} w^{7/2}. \tag{4.18}$$

It remains now to obtain estimates for $|x^{(4)}(t)|$ and $|x^{(5)}(t)|$ in order to complete our verification of (4.2). For this, note that if (4.3) is written as:

$$x^{(6)} + a_1 x^{(5)} = Q \tag{4.19}$$

the function Q, by virtue of (2.5), (2.8), (4.14), (4.15), (4.16) and (4.18) would satisfy

$$|Q| \leqslant D_6\,(|x^{(4)}| + 1).$$

Thus, if we multiply both sides of (4.19) by $x^{(6)}$ and integrate from $t = 0$ to $t = w$, we shall have, x being w-periodic, that

$$\int_0^w x^{(6)2} dt \leqslant D_6 \left(\int_0^w |x^{(4)}|\,|x^{(6)}|\,dt + \int_0^w |x^{(6)}|\,dt \right)$$

or, on applying Schwarz's inequality, that

$$\int_0^w x^{(6)2} dt \leqslant D_6 \left[\int_0^w x^{(6)2} dt \right]^{1/2} \left[w^{1/2} + \left(\int_0^w x^{(4)2} dt \right)^{1/2} \right]$$

$$\leqslant D_7 \left(\int_0^w x^{(6)2}\, dt \right)^{1/2}$$

by (4.13). Hence

$$\int_0^w x^{(6)2}\, dt \leqslant D_8. \tag{4.20}$$

Since $x^{(5)}(T_5) = 0$ for some T_5, it follows from the identity

$$x^{(5)}(t) = x^{(5)}(T_5) + \int_{T_5} x^{(6)}(s) ds$$

and the result (4.20), in the usual manner, that

$$\max_{0 \leqslant t \leqslant w} |x^{(5)}(t)| \leqslant w^{1/2} D_8^{1/2}. \qquad (4$$

In turn (4.21), combined with the identity

$$x^{(4)}(t) = x^{(4)}(T_6) + \int_{T_6}^{t} x^{(5)}(s)ds$$

with T_6 chosen such that $x^{(4)}(T_6) = 0$, implies that

$$\max_{0 \leqslant t \leqslant w} |x^{(4)}(t)| \leqslant w^{3/2} D_8^{1/2}. \qquad (4$$

The result (4.14), (4.15), (4.16), (4.18), (4.21) and (4.22) fully verify (for the arbitrarily chosen w-periodic solution $x(t)$ of (4.1) if the A_1 in (is sufficiently small. This now completes the proof of Theorem 2.

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CONTINUOUS DEPENDENCE OF THE SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS ON INITIAL DATAS

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REZUMAT — Dependenţa continuă de date a soluţiilor ecuaţiilor diferenţiale cu argument modificat Rezultatele stabilite în această lucrare extind o serie de rezultate cunoscute cu privire la dependenţa continuă de date a soluţiilor unor clase de ecuaţii diferenţiale cu argument modificat.

1. **Introduction.** Let us consider the following Cauchy problem for a first order differential equation with deviating argument:

$$\begin{cases} y'(x) = f(x, y(x), y(g(x))), & x \in [a, b] \\ y|_{[a_1, a]} = \varphi & , a_1 < a \end{cases} \tag{1}$$

where $f \in C([a, b] x \mathbf{R}^2)$, $g \in C([a, b], [a_1, b])$ and $\varphi \in C[a_1, a]$.

We search the solutions of this problem in $C[a_1, b] \cap C^1[a. b]$.

The problem (1) is equivalent to the following integral equation

$$y(x) = \begin{cases} \varphi(a) + \int_a^x f(s, y(s), y(g(s)))ds, & , x \in [a, b] \\ \varphi(x) & , x \in [a_1, a] \end{cases}$$

We have

THEOREM 1. (Existence and Uniqueness Theorem; see [3]). *Let $f \in C([a, b] \times \mathbf{R}^2)$, $g \in C([a, b], [a_1, b])$, $\varphi \in C[a_1, a]$.*
We suppose that:
(i) *f satisfies the following Lipschitz condition: there is a number $L_f > 0$ such that*
$|f(x, y, u) - f(x, \bar{y}, \bar{u})| \leqslant L_f(|y - \bar{y}| + |u - \bar{u}|)$, *for all* $x \in [a, b]$ *and for all* $y, u, \bar{y}, \bar{u} \in \mathbf{R}$,
(ii) *g satisfies the condition:*

$$\left\| \int_a^{(\cdot)} e^{\tau(g(s)-(\cdot))} ds \right\|_{C[a,b]} \to 0, \text{ when } \tau \to \infty.$$

Then problem (1) has a unique solution in $C[a_1, b]$, which can be obtained by using the iteration method, starting with any element from $C[a_1, b]$.

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2. **Preliminaries.** We will need the following theorem to prove our main result.

THEOREM 2. (see [4]). *Let (X, d) be a complete metric space. Let $A, B : X \to x$ be two mappings. We suppose that:*
(i) *the mapping A satisfies the conditions from the contraction principle and $F_A = \{x_A^*\}$,*
(ii) $x_B^* \in F_B$,
(iii) *there is a number $\eta > 0$ such that*

$$d(A(x),\ B(x)) \leqslant \eta,\ \textit{for all}\ x \in X.$$

Then $d(x_A^*, x_B^*) \leqslant \dfrac{\eta}{1-a}$, *where a is the contraction constant of mapping A.*

We denote by F_f, the fixed points set of mapping f.

Let $[\gamma, b] \cap \mathbf{R}$ and $\alpha, \delta \in [\gamma, b]$. We consider the following Cauchy problem:

$$\begin{cases} y'(x) = f(x, y(x), y(g(x))), & x \in [a, b] \\ y\,|\gamma_a] = \varphi\,|\gamma_a] \end{cases} \tag{2}$$

We assume the following conditions:

(i) $f \in CLM([\alpha, b] \times \mathbf{R}^2)$, i.e.,

— there is a number $L_f > 0$ such that $|f(x, y, u) - f(x, y, v)| \leqslant L_f|u - v|$, for all $x \in [\alpha, b]$ and for all $y, u, v \in \mathbf{R}$,

— there is a number $M_f > 0$ such that

$|f(x, y, u)| \leqslant M_f$, for all $x \in [\alpha, b]$ and for all $y, u \in \mathbf{R}$

(ii) $g \in C([\alpha, b], [\gamma, \delta])$

(iii) $\varphi \in CL[\gamma, b]$, i.e.,

— there is a number $L_\varphi > 0$ such that

$|\varphi(x) - \varphi(y)| \leqslant L_\varphi |x - y|$, for all $x, y \in [\gamma, b]$.

Now, we define the following mapping,
$\mathcal{C}: CLM([\alpha, b] \times \mathbf{R}) \times C([\alpha, b], [\gamma, \delta]) \times CL[\gamma, b] \times [\alpha, b] \to C_T[\gamma, b]$, which maps any (f, g, φ, a) into the unique solution of problem (2), corresponding to f, g, φ and a.

We denote by $C_T[\gamma, b]$ the set of continuous functions $y \in C[\gamma, b]$, which satisfy the Lipschitz clondition:

$$|y(x_1) - y(x_2)| \leqslant T\,|x_1 - x_2|,\ \text{for all}\ x_1,\ x_2\ \text{from}\ [\gamma, b].$$

We suppose that $M_f + L_\varphi \leqslant T$.

3. **Main results.** Now, we can establish the main result of this paper.

THEOREM 3. *Under the above conditions, the mapping $\mathcal{C}$ is continuous with respect to all its arguments.*

Proof. We consider the problem

$$\begin{cases} y'(x) = f(x, y(x), y(g_1(x))), & x \in [a, b] \\ y|_{[\gamma, a]} = \varphi|_{[\gamma, a]} \end{cases} \tag{1'}$$

and, also, the same problem with perturbed initial datas:

$$\begin{cases} z'(x) = h(x, z(x), z(g_2(x))), & x \in [\tilde{a}, b] \\ z|_{[\gamma, \tilde{a}]} = \psi|_{[\gamma, \tilde{a}]} \end{cases} \tag{1''}$$

where $f, h \in CLM([\alpha, b] \times \mathbf{R}^2)$, $g_1, g_2 \in C([\alpha, b], [\gamma, \delta])$, $\varphi, \psi \in CL[\gamma, b]$. We suppose that

$$||f - h|| < \varepsilon_f$$

$$||g_1 - g_2|| < \varepsilon_g$$

$$||\varphi - \psi|| < \varepsilon_\varphi$$

$a, \tilde{a} \in [\alpha, b]$ such that $|a - \tilde{a}| < \varepsilon_a$

We have the integral equations equivalent to (1') and (1''):

$$y(x) = \begin{cases} \varphi(a) + \int_a^x f(s, y(s), y(g_1(s)))ds, & x \in [a, b] \\ \varphi(x), & x \in [\gamma, a] \end{cases} \tag{2'}$$

$$y(x) = \begin{cases} \psi(\tilde{a}) + \int_{\tilde{a}}^x h(s, y(s), y(g_2(s)))ds, & x \in [\tilde{a}, b] \\ \psi(x), & x \in [\gamma, \tilde{a}] \end{cases} \tag{2''}$$

We denote by $(A(y))(x)$, respectively $(B(y))(x)$ the second parts of (2') and (2'').

We shall use Theorem 2.

Of course, $(C_T[\gamma, b], d)$ is a complete metric space where d is Cebişev metric, i.e.,

$$d(f, g) = ||f - g|| = \max_{x \in [\gamma, b]} |f(x) - g(x)|.$$

Mappings A and B, as we defined them, map $C_T[\gamma, b]$ into itself, because

$$|(A(y))(x_1) - (A(y))(x_2)| \leqslant (L_\varphi + M_f)|x_1 - x_2| \leqslant T|x_1 - x_2|$$

for all x_1, x_2 from $[\gamma, b]$.

So, we have $A, B : C_T[\gamma, b] \to C_T[\gamma, b]$

The conditions (i) and (ii) from Theorem 2 are satisfied (see Theorem 1). So we still have to find a number $\eta > 0$ such that $||A(y) - B(y)|| \leqslant \eta$, for all $y \in C_T[\gamma, b]$.

We suppose that $a < \tilde{a}$. There are three cases:

I. $x \in [\gamma, a] \Rightarrow (x \in [\gamma, \tilde{a}])$

II. $x \in [\tilde{a}, b] \Rightarrow (x \in [a, b])$

III. $x \in [a, \tilde{a}]$

Case I is very clear. We have

$||A(y) - B(y)|| < \varepsilon$ so, here $\eta = \varepsilon_\varphi$

Case II. We have the following deliminations:

$$|A(y))(x) - B(y))(x)| \leqslant |\varphi(a) - \psi(\tilde{a})| +$$

$$+ \int_{\tilde{a}}^{x} |f(s, y(s), y(g_1(s))) - h(s, y(s), y(g_2(s)))| ds +$$

$$+ \int_{a}^{\tilde{a}} |f(s, y(s), y(g_1(s)))| ds \leqslant |\varphi(a) - \psi(a)| + |\psi(a) - \psi(\tilde{a})| +$$

$$+ \int_{\tilde{a}}^{x} |f(s, y(s), y(g_1(s))) - h(s, y(s), y(g_1(s)))| ds +$$

$$+ \int_{\tilde{a}}^{x} |h(s, y(s), y(g_1(s))) - h(s, y(s), y(g_2(s)))| ds + M_f|a - \tilde{a}| \leqslant$$

$$\leqslant ||\varphi - \psi|| + L_\psi|a - \tilde{a}| + ||f - h||(b - \tilde{a}) + L_h T \int_{\tilde{a}}^{x} |g_1(s) - g_2(s)| ds +$$

$$+ M_f|a - \tilde{a}| < \varepsilon_\varphi + L_\psi \varepsilon_a + \varepsilon_f(b - \alpha) + L_h T \varepsilon_g(b - \alpha) + M_f \varepsilon_a =$$

$$= \varepsilon_\varphi + (L_\psi + M_f)\varepsilon_a + (b - \alpha)\varepsilon_f + L_h T(b - \alpha)\varepsilon_g$$

Hence $||A(y) - B(y)|| < \eta$, where

$\eta = (b - \alpha)\varepsilon_f + L_h T(b - \alpha)\varepsilon_g + \varepsilon_\varphi + (L_\psi + M_f)\varepsilon_a$

Case III. Using the same technique, we find:

$||A(y) - B(y)|| < \varepsilon_\varphi + (L_\psi + M_f)\varepsilon_a$

So, here, we have $\eta = \varepsilon_\varphi + (L_\psi + M_f)\varepsilon_a$

In each of these three cases we have:

$\eta \to 0$, if $\varepsilon_f \to 0$, $\varepsilon_g \to 0$ $\varepsilon_\varphi \to 0$, $\varepsilon_a \to 0$.

Since the Bielecki metric is equivalent to Cebîşev metric, we have that, the mapping $\mathcal{C}$ is continuous with respect to all its arguments.

4. Remarks.

a). A similar theorem of continuous dependence, using the same arguments, can be established for a Cauchy problem of the form

$$\begin{cases} y'(x) = f(x, y(x), y(g_1(x)), y(g_2(x)), \ldots, y(g_m(x))), & x \in [a, b] \\ y|_{[a_1, a]} = \varphi \end{cases}$$

b). The results established in this paper extend the theorem given by R.D. Driver in [1], concerning the continuous dependence on φ, of the solution of the equation

$$\begin{cases} y'(t) = \mathcal{F}(t, y(s(t))) & , \alpha \leqslant s(t) \leqslant t), \text{ for } t > t_0 \\ y(t) = \varphi(t) & , t \in [\alpha, t_0] \end{cases}$$

and also the theorem given by L. E. El'sgol'c, S. B. Norkin in [2], regarding the continuous dependence on φ, of the solution of the equation

$$\begin{cases} y'(t) = f(t, x(t), x(t - \tau_1(t)), \ldots, x(t - \tau_m(t))), & , t \geqslant t_0 \\ y(t) = \varphi(t) & , t \in [\alpha, t_0] \end{cases}$$

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GENERALIZED TOPOLOGICAL TRANSVERSALITY AND MAPPINGS OF MONOTONE TYPE

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REZUMAT. — Transversalitate topologică generalizată şi aplicaţii de tip monoton. În lucrare se demonstrează o teoremă de existenţă de tip Browder [2]. Noutatea constă în faptul că în locul condiţiei de coercivitate se impune o condiţie de semn, mai generală. Demonstraţia se bazează pe teorema de transversalitate topologică generalizată, obţinută în [4]. Această notă constituie un addendum la lucrările [4] şi [5].

*In this paper a Browder's type result [2] is proved by using our g*eneralized topological transversality theorem given in [4] (see also [5]). W show that the coercivity condition assumed by Browder can be replaced by more general sign condition. This note is an addendum to our previous paper [4] and [5].

1. **The generalized topological transversality principle.** Let K be a norm topological space, X and A two proper closed subsets of K, $A \subset X$, $A \neq X$ an consider a nonvoid class of mappings.

$$\mathcal{A}_A(X, K) \subset \{f: X \to K\,;\ \mathrm{Fix}\,(f) \cap A = \emptyset\}, \tag{1}$$

where Fix(f) stands for the set of all fixed points of f. The mappings in $\mathcal{A}_A(X, K)$ are said to be *admissible*.

An admissible mapping f is said to be *essential* if

$$f' \in \mathcal{A}_A(X, K),\ f|_A = f'|_A \text{ imply } \mathrm{Fix}(f') \neq \emptyset. \tag{2}$$

Otherwise, f is said to be *inessential*.

Also consider an equivalence relation $\sim$ on $\mathcal{A}_A(X, K)$ and assume that the following conditions are satisfied for f and f' in $\mathcal{A}_A(X, K)$:

(a) *if* $f|_A = f'|_A$ *then* $f \sim f'$;

(h) *if* $f \sim f'$ *then there is* $h: [0, 1] \times X \to K$

such that $h(0, .) = f'$, $h(1, .) = f$,
$\mathrm{cl}(\bigcup\{\mathrm{Fix}(h(t, .))\,;\ t \in [0, 1]\}) \bigcap A = \emptyset$ *and* $h(\eta(.), .)$ *is admissible for any* $\eta \in C(X\,;\ [0, 1])$ *satisfying* $\eta(x) = 1$ *for all* $x \in A$.

We now state the generalized topological transversality theorem.

PROPOSITION 1. *If f and f' are admissible mappings and $f \sim f'$, then f and f' are both essential or both inessential.*

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The next proposition is useful in order to establish the essentiality of certain admissible mappings. It is formulated in terms of fixed point structures.

By a *fixed point structure* on a certain space K we mean a pair (S, M) where S is a class of nonempty subsets of K and M is a mapping attaching to each $D \in S$ a family $M(D)$ of mappings from D into D having, each of them, at least one fixed point.

PROPOSITION 2. *Let (S, M) be a fixed point structure on the normal topological space K and let $f_0 \in \mathcal{A}_A(X, K)$. If for every $f \in \mathcal{A}_A(X, K)$ satisfying $f|_A = f_0|_A$, there exist $D_f \in S$ and $\tilde{f} \in M(D_f)$ such that*

$$f|_{X \cap D_f} = \tilde{f}|_{X \cap D_f}$$

and

$$\mathrm{Fix}\,(\tilde{f}) \setminus X = \emptyset,$$

then f_0 is essential.

The proofs of Proposition 1 and Proposition 2 and some applications can be found in the papers 4] and [5].

The aim of this paper is to give another application of Proposition 1.

2. **The fixed point structure.** Now we describe the fixed point structure which will be used in the next section.

Let E be a real reflexive Banach space which is normed so that E and its dual E^* are locally uniformly convex and let $J : E \to E^*$ be the duality mapping. Set

$$S = \{D\,;\ S \text{ is a nonvoid bounded closed convex subset of } E\} \tag{3}$$

and for each $D \in S$,

$$M(D) = \{(J + T)^{-1}(J - N) : D \to D(T)\,;\ T \subset D \times E^* \text{ is maximal monotone in } E \times E^* \text{ and } N : D \to E^* \text{ is pseudomonotone, bounded and demicontinuous}\}. \tag{4}$$

Recall that a mapping $N : D \to E^*$ is said to be *pseudomonotone* if, for any sequence (x_n) in D for which $x_n \rightharpoonup x$ and $\limsup \langle N(x_n), x_n - x\rangle \leqslant 0$, we have $\langle N(x), x - y\rangle \leqslant \liminf \langle N(x_n), x_n - y\rangle$ for all $y \in D$. Also, N is said to be of *type* (S_+) if for any sequence (x_n) in D for which $x_n \rightharpoonup x$ and $\limsup \langle N(x_n), x_n - x\rangle \leqslant 0$, it follows $x_n \to x$.

LEMMA 1. *The pair (S, M) given by (3) and (4) is a fixed point structure on E.*

This statement is just a Browder's result [2] (see also [6, Theorem 32 A]). Nevertheless, we will insert here its proof.

Proof of Lemma 1. We have to show that each mapping in $M(D)$ has at least one fixed point, i.e., there exists at least one solution to

$$x_0 \in D(T),\ 0 \in N(x_0) + T(x_0). \tag{5}$$

We will first solve (5) under the assumption that N is of type (S_+): In view of the maximal monotonicity of T in $E \times E^*$, (6) is equivalent to:

$$x_0 \in D, \quad \langle x^* + N(x_0),\ x - x_0 \rangle \geqslant 0 \tag{6}$$

for all $(x, x^*) \in T$.
For any finite — dimensional subspace Y of E with $Y \cap D \neq \emptyset$, we look for a solution y to

$$y \in Y \cap D, \quad \langle x^* + N(y),\ x - y \rangle \geqslant 0 \tag{7}$$

for all $(x, x^*) \in T$ with $x \in Y$.
Since N is demicontinuous, a solution to (7) exists in view of Debrunner-Flor's lemma (see [6, Proposition 2.17]). Thus, the set

$$V_Y = \{(y, -N(y)) \in D \times E^*;\ \langle x^* + N(y),\ x - y \rangle \geqslant 0 \ \text{for all}\ (x, x^*) \in T \ \text{with}\ x \in Y\}$$

is nonempty. Clearly, the family $\{V_Y\}$ has the finite intersection property; thus the family of weak-compact sets $\{\text{w}-\text{cl}(V_Y)\}$, $Y \cap D \neq \emptyset$, has a nonvoid intersection. Let (x_0, x_0^*) an element of its intersection.

Note that, due to the maximal monotonicity of T, there exists $(z_0, z_0^*) \in T$ such that

$$\langle z_0^* - x_0^*,\ z_0 - x_0 \rangle \leqslant 0. \tag{8}$$

Now, for an arbitrary $(x, x^*) \in T$, we choose Y such that x, x_0 and z_0 belong to Y and we take a sequence $(y_n, -N(y_n))$ in V_Y such that $y_n \rightharpoonup x_0$ and $-N(y_n) \rightharpoonup x_0^*$. We have

$$\langle z^* + N(y_n),\ z - y_n \rangle \geqslant 0, \tag{9}$$

for all $(z, z^*) \in T$ with $z \in Y$.
From (9) we get

$$\begin{aligned}\langle N(y_n),\ y_n - w \rangle &= \langle N(y_n),\ y_n - z \rangle + \langle N(y_n),\ z - w \rangle \leqslant \\ &\leqslant \langle z^*,\ z - y_n \rangle + \langle N(y_n),\ z - w \rangle\end{aligned} \tag{10}$$

for all $(z, z^*) \in T$ with $z \in Y$ and $w \in E$.
Taking $w = x_0$, $z = z_0$, $z^* = z_0^*$ we obtain

$$\langle N(y_n),\ y_n - x_0) \rangle \leqslant \langle z_0^*,\ z_0 - y_n \rangle + \langle N(y_n),\ z_0 - x_0 \rangle,$$

whence, letting $n \to \infty$ and taking into account (8), we get

$$\limsup \langle N(y_n),\ y_n - x_0 \rangle \leqslant 0.$$

This, since N is supposed of type (S_+), implies that $y_n \to x_0$. Consequently, $x_0^* = -N(x_0)$ and passing to limit in (10) with $w = z = x$ and $z^* = x^*$ we obtain just (6). This proves the solvability of (5) in case N is of type (S_+).

Finally, for N pseudomonotone, use the fact that $N + \varepsilon J$ is of type (S_+) for each $\varepsilon > 0$, in order to deduce the existence of an y_ε, solution to

$$0 \in N(x_\varepsilon) + \varepsilon J(x_\varepsilon) + T(x_\varepsilon)$$

and letting $\varepsilon \to 0$, find a solution to (5). This step is well known and we omit the details. The lemma is thus proved.

The existence of a solution to (5) is known even if D is unbounded, but under the additional hypothesis that N is coercive with respect to 0 (see [2, p. 92] or [6, Theorem 32. A]). In what follows we shall prove, via Proposition 1, that the coercivity of N may be replaced by a more general sign condition.

3. **Application of the generalized transversality theorem.** The main result of this note is the following proposition.

THEOREM 1. *Let E be a real reflexive Banach space, K an unbounded closed convex subset of E, $T \subset K \times E^*$ maximal monotone in $E \times E^*$ with $(0, 0) \in T$ and let $N : K \to E^*$ be a bounded demicontinuous pseudomonotone mapping such that there exists $r > 0$ so that*

$$\langle N(x), x \rangle \geqslant 0 \text{ for all } x \in K \text{ with } ||x|| = r. \tag{11}$$

Then there exists $x \in D(T)$ a solution to

$$0 \in N(x) + T(x).$$

Remark. Condition (11) is less restrictive than the coercivity condition:

$$\langle N(x), x \rangle > 0 \text{ for all } x \in K \text{ with } ||x|| \geqslant r.$$

Under the coercivity condition on N, Theorem 1 was proved in [2, p. 92].

Proof of Theorem 1. The same argument as in the proof of Lemma 1, allows us, setting $N + \varepsilon J (\varepsilon > 0)$ in place of N, to assume that N is of type (S_+) and in addition, that the inequality in (11) is strict.

We shall succed two steps:

1) *Application of Proposition* 1. Consider the class

$$\mathcal{C}_A(K_r, K) = \{(J + T)^{-1} \circ \eta_\lambda (J - N) : K_r \to D(T) \ ; \eta_\lambda \in C(K_r ; [0,1]), \\ \eta_\lambda(x) = \lambda \text{ for } x \in A\}$$

where $A = \{x \in K ; \ ||x|| = r\}$ and for each $R > 0$ we denote

$$K_R = \{x \in K ; \ ||x|| \leqslant R\}.$$

Note that the mappings in $\mathcal{C}_A(K_r, K)$ can not have fixed points in A because, in view of (11), the inclusion

$$(\lambda - 1) J(x) - \lambda N(x) \in T(x)$$

is false for all $x \in A$.

Also define an equivalence relation on $\mathcal{C}_A(K_r, K)$ by setting

$$(J + T)^{-1} \circ \eta_\lambda (J - N) \sim (J + T)^{-1} \circ \eta_{\lambda'} (J - N)$$

if and only if

$$\begin{aligned} \lambda = \lambda' \text{ or } \{\lambda, \lambda'\} = \{0, 1\}, &\text{ in case } J - N \not\equiv 0 \text{ on } A \\ \text{always,} \quad &\text{ in case } J - N \equiv 0 \text{ on } A. \end{aligned}$$

Since $(J + T)^{-1}$ is one-to-one, condition (a) is satisfied. In order to ver condition (h), set

$$h(t, .) = (J + T)^{-1} \circ [(1 - t)\eta_{\lambda'} + t\eta_\lambda](J - N).$$

Clearly, $h(\eta(.), .) \in \mathcal{A}_A(K_r, K)$ for each $\eta \in C(K_r; [0, 1])$ satisfying $\eta(x) =$ for all $x \in A$. Also, by (11), the sets A and $Z = \cup \{\text{Fix}\,(h(t, \cdot))\,;\ t \in [0, 1$ are disjoint. It remains only to show that Z is closed. For this, let (x_n) be sequence in Z such that $x_n \to x_0$. We have $h(t_n, x_n) = x_n$ for some $t_n \in [0, 1$ We may assume $t_n \to t_0$. Setting

$$\mu_n = (1 - t_n)\eta_{\lambda'}(x_n) + t_n\eta_\lambda(x_n) \text{ and } \mu_0 = (1 - t_0)\eta_{\lambda'}(x_0) + t_0\eta_\lambda(x_0),$$

we thus have

$$\langle -(1 - \mu_n)J(x_n) - \mu_n N(x_n) - x^*,\ x_n - x\rangle \geqslant 0$$

for all $(x, x^*) \in T$. Letting $n \to \infty$ and using the demicontinuity of J and we get

$$\langle -(1 - \mu_0)J(x_0) - \mu_0 N(x_0) - x^*,\ x_0 - x\rangle \geqslant 0$$

for all $(x, x^*) \in T$, i.e., $h(t_0, x_0) = x_0$, as desired.

Therefore, Proposition 1 can be applied. But

$$(J + T)^{-1}(J - N) \sim (J + T)^{-1} \circ 0(J - N) \equiv 0.$$

Hence, in order that the mapping $(J + T)^{-1}(J - N)$ have a fixed point, it i sufficient to prove that the null operator is essential in $\mathcal{A}_A(K_r, K)$.

2) *Use of the fixed point structure.* We shall now prove that the null operator is essential, i.e., each mapping

$$f = (J + T)^{-1} \circ \eta_\lambda(J - N)$$

satisfying $f \equiv 0$ on A, has at least one fixed point. Remark that if $J - N \not\equiv 0$ on A, then λ must be zero, while if $J - N \equiv 0$ on A, then λ is any number in $[0, 1]$. To do this, for any fixed $R \geqslant r$ we consider the mapping

$$f_R = (J + T_R)^{-1} \circ \tilde{\eta}(J - N)\ : K_R \to K_R,$$

where $T_R \subset K_R \times E^*$ is maximal monotone in $E \times E^*$ and $T|_{K_R} \subset T_R$ (see [1, Theorem 1.4])
and

$$\tilde{\eta}(x) = \begin{cases} \eta_\lambda(x), & \text{if } x \in K_r \\ 0, & \text{if } x \in K_R \setminus K_r. \end{cases} \tag{12}$$

Clearly, $K_R \in S$. We shall prove that $f_R \in M(K_R)$, i.e., the mapping

$$\tilde{N} : K_R \to E^*, \tilde{N} = J + \tilde{\eta}(N - J) \tag{13}$$

is pseudomonotone, bounded and demicontinuous. The last two properties are immediate. To prove its pseudomonotonicity, consider any sequence (x_n) in K_R such that $x_n \rightharpoonup x$ and

$$\limsup \langle N(x_n), x_n - x\rangle \leqslant 0. \tag{14}$$

According to (13), we have

$$\min\{\langle J(x_n), x_n - x\rangle, \langle N(x_n), x_n - x\rangle\} \leqslant \langle \tilde{N}(x_n), x_n - x\rangle. \tag{15}$$

Now, from (14) and (15) and since J and N are both of type (S_+), it easily follows that $x_n \to x$. Hence, $\tilde{N}$ is of type (S_+) and since $\tilde{N}$ is also demicontinuous, it follows that $\tilde{N}$ is pseudomonotone (see [6, Proposition 27.6]). Therefore, $f_R \in M(K_R)$ and according to Lemma 1, there exists a fixed point $x_R \in K_R$ for f_R. Moreover, by (12), $x_R \in K_r$. Since $f_R(x_R) = x_R$,

$$\langle z^* + \tilde{N}(x_R), z - x_R\rangle \geqslant 0 \tag{16}$$

for all $(z, z^*) \in T_R$ and in particular, for all $(z, z^*) \in T$, with $z \in K_R$. Now let (R_n) be an increasing sequence such that $R_n \to \infty$ and denote $x_n = x_{R_n}$. We may assume

$$x_n \rightharpoonup x_0 \in K \text{ and } \tilde{N}(x_n) \rightharpoonup x_0^* \in E^*.$$

Choose a pair $(z_0, z_0^*) \in T$ such that

$$\langle z_0^* + x_0^*, z_0 - x_0\rangle \leqslant 0. \tag{17}$$

Now for an arbitrary pair $(x, x^*) \in T$, there is n_0 such that $x_0, z_0, x \in K_{Rn}$ for all $n \geqslant n_0$. From (16), we get

$$\langle \tilde{N}(x_n), x_n - w\rangle \leqslant \langle z^*, z - x_n\rangle + \langle \tilde{N}(x_n), z - w\rangle \tag{18}$$

for all $(z, z^*) \in T$ with $z \in K_{Rn}$ and $w \in E$.
Taking $w = x_0$, $z = z_0$, $z^* = z_0^*$, letting $n \to \infty$ and using (17), we get

$$\limsup \langle \tilde{N}(x_n), x_n - x_0\rangle \leqslant 0$$

whence, since $\tilde{N}$ is of type (S_+), $x_n \to x_0$ and
$x_0^* = \tilde{N}(x_0)$. Clearly, $x_0 \in K_r$ and
$\tilde{N}(x_0) = J(x_0) + \eta_\lambda(x_0)(N(x_0) - J(x_0))$. Finally, passing to limite in (18) with $w = z = x$ and $z^* = x^*$, we obtain

$$\langle x^* + J(x_0) + \eta_\lambda(x_0)(N(x_0) - J(x_0)), x - x_0\rangle \geqslant 0.$$

Consequently, x_0 is a fixed point of f and the proof is complete.

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ON THE STRONG BOUNDEDNESS OF INFINITE SERIES

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ABSTRACT. — In this paper we have established a relation between the $[\bar{N}, p_n]_k$ and $[\bar{N}, q_n]_k$ boundedness. Also some results have been obtained.

1. Let Σa_n be a given infinite series with the sequence of partial sums (s_n). Let (p_n) be a sequence of positive real constants such that

$$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty \text{ as } n \to \infty, \ (P_{-i} = p_{-i} = 0, \ i \geqslant 1) \tag{1}$$

The series $\sum a_n$ is said to be *bounded* $[C, 1]$, if (see [4])

$$\sum_{\nu=1}^{n} |s_\nu| = O(n) \text{ as } n \to \infty. \tag{2}$$

We say that the series $\sum a_n$ is *bounded* $[C, 1]_k$, $k \geqslant 1$, if

$$\sum_{\nu=1}^{n} |s_\nu|^k = O(n) \text{ as } n \to \infty. \tag{3}$$

In the special case $k = 1$, $[C, 1]_k$ boundedness is the same as $[C, 1]$ boundedness. The series $\sum a_n$ is said to be *bounded* $[R, \log n, 1]_k$, $k \geqslant 1$, if (see [3])

$$\sum_{\nu=1}^{n} \nu^{-1} |s_\nu|^k = O(\log n) \text{ as } n \to \infty. \tag{4}$$

The series $\sum a_n$ is said to be *bounded* $[\bar{N}, p_n]$, if (see [2])

$$\sum_{\nu=1}^{n} p_\nu |s_\nu| = O(P_n) \text{ as } n \to \infty, \tag{5}$$

and it is said to be *bounded* $[\bar{N}, p_n]_k$, $k \geqslant 1$, if (see [1])

$$\sum_{\nu=1}^{n} p_\nu |s_\nu|^k = O(P_n) \text{ as } n \to \infty. \tag{6}$$

If we take $k = 1$ $\left(\text{resp. } p_n = \frac{1}{n}\right)$, then $[\bar{N}, p_n]_k$ boundedness is the same as $[\bar{N}, p_n]$ (resp. $[R, \log n, 1]_k$) boundedness.

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Also in the special case $p_n = 1$ for all values of n, $[\bar{N}, p_n]_k$ boundedne the same as $[C, 1]_k$ boundedness.

2. The object of this paper is to establish a relation between the $[\bar{N},$ and $[\bar{N}, q_n]_k$ boundedness. Now we shall prove the following theorems.

THEOREM 1. *Let $k \geqslant 1$. If Σa_n is $[\bar{N}, p_n]_k$ bounded, then it is also $[\bar{N},$ bounded provided that (p_n) and (q_n) are positive sequences such that as n*

$$\text{a) } p_n Q_n = O(q_n P_n), \quad b) \; q_n P_n = O(p_n Q_n). \tag{7}$$

3. *Proof of Theorem* 1. To prove the theorem we have to show that

$$\sum_{\nu=1}^{n} q_\nu |s_\nu|^k = O(Q_n) \text{ as } n \to \infty. \tag{8}$$

Since $q_n = O\left(\frac{p_n Q_n}{P_n}\right)$, by (7. b), we have

$$\sum_{\nu=1}^{n} q_\nu |s_\nu|^k = O(1) \sum_{\nu=1}^{n} \frac{Q_v}{P_v} p_\nu |s_\nu|^k \tag{9}$$

Applying Abel's transformation to the right hand side of (9). That is to we have

$$\sum_{\nu=1}^{n} \frac{Q_\nu}{P_\nu} p_\nu |s_\nu|^k = \sum_{\nu=1}^{n-1} \Delta\left(\frac{Q_\nu}{P_\nu}\right) \sum_{r=1}^{\nu} p_r |s_r|^k + \frac{Q_n}{P_n} \sum_{\nu=1}^{n} p_\nu |s_\nu|^k.$$

Since $\sum_{\nu=1}^{n} p_\nu |s_\nu|^k = O(P_n)$, by hypothesis, we have that

$$\sum_{\nu=1}^{n} \frac{Q_\nu}{P_\nu} p_\nu |s_\nu|^k = O(1) \sum_{\nu=1}^{n-1} \Delta\left(\frac{Q_\nu}{P_\nu}\right) P_v + O(Q_n)$$

$$= O(1) \sum_{\nu=1}^{n-1} \left\{ \frac{Q_\nu}{P_\nu} - \frac{Q_{\nu+1}}{P_{\nu+1}} - \frac{Q_{\nu+1}}{P_\nu} + \frac{Q_{\nu+1}}{P_\nu} \right\} P_v + O(Q_n)$$

$$= O(1) \sum_{\nu=1}^{n-1} (Q_v - Q_{\nu+1}) + O(1) \sum_{\nu=1}^{n-1} Q_{\nu+1} \left(\frac{1}{P_\nu} - \frac{1}{P_{\nu+1}} \right) P_v + O(Q_n)$$

$$= O(1) \sum_{\nu=1}^{n-1} (-q_{\nu+1}) + O(1) \sum_{\nu=1}^{n-1} Q_{\nu+1} \frac{p_{\nu+1}}{P_{\nu+1}} + O(Q_n)$$

$$= O(1) \sum_{=2}^{n} q_\nu + O(1) \sum_{\nu=2}^{n} Q_\nu \frac{p_\nu}{P_\nu} + O(Q_n)$$

$$= O(1) \sum_{\nu=0}^{n} q_\nu + O(1) \sum_{\nu=0}^{n} Q_\nu \frac{p_\nu}{P_\nu} + O(Q_n).$$

ince $Q_n \frac{p_n}{P_n} = O(q_n)$, by (7. a), we have

$$\sum_{\nu=1}^{n} \frac{Q_\nu}{P_\nu} p_\nu |s_\nu|^k = O(1) \sum_{\nu=0}^{n} q_\nu + O(1) \sum_{\nu=0}^{n} q_\nu + O(Q_n) = O(Q_n).$$

nce ,

$$\sum_{\nu=1}^{n} q_\nu |s_\nu|^k = O(1) \sum_{\nu=1}^{n} \frac{Q_\nu}{P_\nu} p_\nu |s_\nu|^k = O(Q_n) \text{ as } n \to \infty,$$

ich completes the proof of Theorem 1.

If we interchange the roles of p_n and q_n in this theorem, then we obtain following corollary.

COROLLARY 1. *If* $q_n P_n = O(p_n Q_n)$, *and* $p_n Q_n = O(q_n P_n)$, (10)

$[\bar{N}, q_n]_k$ *implies* $[\bar{N}, p_n]_k$, $k \geqslant 1$.

If we put two results together we have the following theorem.

THEOREM 2. *Suppose* (q_n) *and* (p_n) *are positive sequences such that satisfy the* *dition* (7). *Then boundedness* $[\bar{N}, p_n]_k$ *is equivalent to boundedness* $[\bar{N}, q_n]$, $k \geqslant 1$.

It should be noted that if we take $q_n = 1$ for all values of n in the Theo- 1, then we have the following corollary.

COROLLARY 3. *If* $np_n = O(P_n)$, *and* $P_n = O(np_n)$, (11)

$[\bar{N}, p_n]_k$ *implies* $[C, 1]_k$, $k \geqslant 1$.

Furthermore if we take $q_n = 1/n$ in the Theorem 1, then we obtain the wing corollary.

COROLLARY 2. *If* $np_n \log n = O(P_n)$ *and* $P_n = O(np_n \log n)$, (12)

$[\bar{N}, p_n]_k$ *implies* $[R, \log n, 1]_k$, $k \geqslant 1$.

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SOME REMARKS ON THE CHARACTERIZATION OF NEAREST

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REZUMAT. — Observaţii asupra caracterizării elementelor de cea mai bu aproximare. Se consideră problema celei mai bune aproximări prin elemen ale unei submulţimi p-convexe a unui spaţiu normat X. Se demonstrează teoremă de caracterizare a elementelor de cea mai bună aproximare în termen unor funcţionale extremale ale bilei unitate din spaţiul dual X^*.

For a normed space X (over **R** or **C**), a nonvoid subset Y of point $x \in X$, denote by $d(x, Y) = \inf\{||x - y|| : y \in Y\}$ — the *dist* x to Y and by $P_Y(x) = \{y \in Y : ||x - y|| = d(x, Y)\}$ — the (possi set of *nearest points* to x in Y (the elements of $P_Y(x)$ are called also pr of x onto Y or elements of best approximation for x by element

We shall consider characterization theorems for the elements of bes ximation in terms of some functionals belonging to a prescribed sub the unit sphere S_{X^*} of the conjugate space X^* of X. There are two type theorems:

— Theorems in which Γ is a fundamental subset of S_{X^*} (a sub S_{X^*} is called *fundamental* if it is w^* — closed and for every $x \in X$ there exists $f \in \Gamma$ such that $|f(x)| = ||x||$) on the line of V. N. Ni [15], [16] and S. A. Azizov [2]; and

— Theorems in which $\Gamma = \text{ext } B_{X^*}$ (the set of the extreme point unit ball B_{X^*} of X^*). Such type theorems were first proved by I. S [19] (see also [20]) in the case of the approximation by elements of a s and by A. L. Garkavi [8], in the case of approximation by eleme convex subset.

In a previous paper [4], we have proved a characterization the the first kind in the case of approximation of a p-convex set. For number p, $0 < p < 1$, a subset Y of a linear space is called p-c $pY + (1 - p)Y \subset Y$. The p-convex sets are special cases of so calle convex sets ([10]). See [17] for other generalizations of the notion of co

The aim of this paper is to extend Garkavi's theorem [8] to of p-convex sets. Some duality and characterization theorem, for approximation by elements of p-convex sets were proved in [5], e other results of A. L. Garkavi [7].

The main facts about p-convex sets we shall need in the nex tained in the following theorem:

THEOREM 1. a) *If Y is a p-convex subset of a topological vector s the closure and the interior of Y are convex sets,* [1];

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b) *If Y is a p-convex subset of a real locally convex space X and $x \in X \setminus \bar{Y}$, there exists $f \in X^*$ such that* $\inf \{f(y) : y \in Y\} > f(x)$, [13];

c) *If Y is a p-convex subset with nonvoid interior of a real locally convex space, every boundary point of Y is contained in a closed hyperplane supporting* s*et Y at x,* [13].

Remark. Taking into account the representation of a complex linear ctional f as function of its real part, the above results (the assertions a) l b) from Theorem 1) can be authomatically extended to the complex case, lacing in all the statements the functional f by $\mathrm{Re} f$.

The following extension theorem was proved by I. Singer [19] (see also], Lemma II. 12) and A. L. Garkavi [8]:

THEOREM 2. *Let X be a normed space and Z a subspace of X. If $\varphi \in Z^*$* an *extremal point of the unit ball B_{Z^*} of Z^* then φ admits an extension f* ich *is an extremal point of the unit ball of X^*.*

Now we can state the characterization theorem, which was proved by Singer [19] in the case of a subspace Y of X and by A. L. Garkavi] for a convex subset Y of X.

THEOREM 3. *Let X be a normed space, Y a nonvoid p-convex subset of X* *$x \in X \setminus Y$. An element $y_0 \in Y$ is a nearest point to x in Y if and only* *every $y \in Y$, there exists an extremal point $f = f_y$ of the unit ball B_{X^*} of* *, such that*

i) $f(x - y_0) = ||x - y_0||$

ii) $\mathrm{Re} f(y_0 - y) \geqslant 0.$

Proof. Sufficiency. Suppose $x \in X \setminus Y$ and that $y_0 \in Y$ fulfills the hypothe- the theorem. To show that y_0 is a nearest point to x in Y, consider an ary element $y \in Y$ and let $f \in \mathrm{ext}\, B_{X^*}$, satisfying the condition i) and Then

$$||x - y_0|| = f(x - y_0) = \mathrm{Re} f(x - y_0) = \mathrm{Re} f(x - y) + \mathrm{Re} f(y - y_0) \leqslant$$
$$\leqslant \mathrm{Re} f(x - y) \leqslant |f(x - y)| \leqslant ||x - y||,$$

nce $y_0 \in P_Y(x)$.

Necessity. If $x \in X \setminus Y$ and y_0 is a nearest point to x in Y, then $y_0 - x$ nearest point to 0 in $Y - x$. For $y \in Y$ denote by Z the linear space ned by $y - x$ and $y_0 - x$ and let $W = (Y - x) \cap Z$. The set W is p-con- and $y_0 - x$ is a nearest point to 0 in W. By [13], Corollary 2.5, there s a real linear functional $\varphi \in Z^*$ such that $||\varphi|| = 1$, $\varphi(x - y_0) = ||x - y_0||$ $\varphi(y_0) = \sup \{\varphi(y') : y' \in Y\}$. Putting $\psi(z) = \varphi(z) - i \cdot \varphi(iz)$, it follows the complex linear functional ψ verifies the conditions:

$||\psi|| = 1$, $\mathrm{Re}\, \psi(y_0 - y) \geqslant 0$ and $\psi(x - y_0) = ||x - y_0||$.

Indeed

$||x - y_0|| \geqslant |\psi(x - y_0)| = [||x - y_0||^2 + (\varphi(i(x - y_0))^2]^{1/2}$, implying

$\varphi(i(x - y_0)) = 0$ and $\psi(x - y_0) = \varphi(x - y_0) = ||x - y_0||$.

Now, considering the conjugate space $Z_{\mathbf{R}}^*$ over the real scalars th dimension at most four and φ is an element of its unit ball. By the w Carathéodory's theorem, the functional φ can be written in the form $\varphi=$ where $1 \leqslant r \leqslant 5$, $\alpha_k > 0$, $\sum_{k=1}^{r} \alpha_k = 1$ and φ_k are extremal points of ball of $Z_{\mathbf{R}}^*$. The functionals $\psi_k(z) = \varphi_k(z) - i\varphi_k(iz)$ will be extremal poi unit ball of the complex Banach space Z^* and $\psi = \sum_{k=1}^{r} \alpha_k \psi_k$. The $\psi(x - y_0) = ||x - y_0||$ implies $\psi_k(x - y_0) = ||x - y_0||$, $k = 1, ..$ Re $\psi(y_0 - y) \geqslant 0$ implies that there exists $k_0 \in \{1, \ldots, r\}$ such that Re $\geqslant 0$.

Now, by Theorem 2, ψ_{k_0} can be extended to an extremal functi of the unit ball of X^*, which ends the proof of the theorem.

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ON AN APPROXIMATION PROPERTY FOR CONTINUOUS LINEAR FUNCTIONALS IN BANACH SPACES

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REZUMAT. — Asupra unei proprietăţi de aproximare pentru funcţionalele liniare şi continue în spaţii Banach. În această lucrare se pune în evidenţă o clasă de spaţii Banach care are proprietatea (A) şi se dă o aplicaţie pentru soluţiile aproximative ale unei ecuaţii operatoriale.

1. We propose the following:

PROBLEM. Find the classes of Banach spaces X with the property that: (A) for every nonzero continuous linear functional f and for every $\varepsilon > 0$ there exists a nonzero closed linear subspace X_ε in X which is not included in Ker (f) and such that:

$$|f(x)| \leqslant \varepsilon ||x|| \text{ for all } x \in X_\varepsilon, \tag{1}$$

i.e.,

$$||f||_{X_\varepsilon} \leqslant \varepsilon,$$

where $||f||_{X_\varepsilon} := \sup \{|f(x)| : x \in X_\varepsilon, \; ||x|| \leqslant 1\}$.

2. Further on, we shall give an example of such spaces.

THEOREM. *Every infinite-dimensional Banach space which is isomorphic (top-linear) to a Hilbert space has the (A)-property.*

To prove this fact, we need the following lemmas.

LEMMA 1. *Let $(X; (,))$ be a Hilbert space and $e \in X$, $||e|| = 1$. Then for every $u, v \in X$ such that $u \perp v$, we have the inequality*

$$||u|| \, ||v|| \geqslant 2 \; |(u, e)(e, v)|. \tag{2}$$

Proof. In paper [1] (see also [2] or [3]) we proved teh following refinement of Schwarz' inequality in prehilbertian spaces:

$$||x|| \; ||y|| \geqslant |(x, y) - (x, e)(e, y)| + |(x, e)(e, y)| \geqslant |(x, y)|$$

for all x, y in X and $e \in X$, $||e|| = 1$.

In this inequality if we put: $x = u$, $y = v$ and $(u, v) = 0$ we conclude (2) and the lemma is proved.

The following statement is also valid.

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LEMMA 2. *Let $(X; (\ ,\ ))$ be an infinite-dimensional Hilbert space a nonzero continuous linear functional on it. Then there exists a closed linea space Y in X such that*

(i) *Y and $Y^\perp$ are infinite-dimensionals*

and

(ii) $\|f\|_Y, \|f\|_{Y^\perp} > 0$.

Proof. Denote $N = \mathrm{Ker}\,(f)$. Then there exists a closed linear sub $X_0 \subset N$ such that X_0 is infinite-dimensional and $X_0^\perp$ is also infinite-di sional.

Let $e \in X$ such that $e \notin N$ and $N^\perp$ is not included in $Y := X \oplus E$, $E := \mathrm{Sp}\,\{e\}$. Then Y is closed in X, infinite-dimensional and $\|f\|_Y > 0$ $f(e) \neq 0$.

On the other hand $Y^\perp$ is closed, infinite-dimensional and $Y^\perp$ is not inc in N since if we assume that $Y^\perp \subset N$ we obtain $Y = (Y^\perp)^\perp \supset N^\perp$ what pro a contradiction. Consequently, $\|f\|_{Y^\perp} > 0$ and the statement is proved.

By the use of the previous lemmas, we have

LEMMA 3. *Let $(X; (\ ,\ ))$ and f be as above. Then for all decomposition $X = Y \oplus Y^\perp$ as in Lemma 2 we have the inequality*

$$0 < \|f\|_Y \|f\|_{Y^\perp} \leqslant 1/2\, \|f\|^2. \qquad (3)$$

Proof. Let x_f be the representation element (by Riesz' theorem) of fun nal f. Then for all $u \in Y$, $v \in Y^\perp$ we have (see Lemma 1):

$$2\,|(u, x_f)(x_f, v)| \leqslant \|u\|\,\|v\|\,\|x_f\|^2 = \|u\|\,\|v\|\,\|f\|^2,$$

what implies the desired inequality.

The following result is important in itself too.

LEMMA 4. *Let $(X; (\ ,\ ))$ be an infinite-dimensional Hilbert space. Then every $f \in X^* \setminus \{0\}$ there exists a sequence of nonzero closed linear subspaces (X_n) such that:*

(i) $X = X_0 \supset X_1 \supset X_2 \supset \dots \supset X_n \supset X_{n+1} \supset \dots$

(ii) *X_n is not included in* Ker (f) *for all $n \in \mathbf{N}$*

(iii) *we have the inequality*:

$$0 < \|f\|_{X_n} \leqslant 1/2^{n/2}\, \|f\| \text{ for all } n \in \mathbf{N}. \qquad (4)$$

Proof. Let $f \in X^* \setminus \{0\}$. Then by Lemma 2 there exists a closed li subspace Y in X such that $X = Y \oplus Y^\perp$, Y, $Y^\perp$ are infinite-dimensional $\|f\|_Y, \|f\|_{Y^\perp} > 0$. We can suppose that $0 < \|f_Y\| \leqslant \|f\|_{Y^\perp}$ Then by Lemm we obtain:

$$0 < \|f\|_Y^2 < \|f\|_Y \|f\|_{Y^\perp} \leqslant 1/2\, \|f\|^2,$$

from where results

$$0 < ||f||_Y \leqslant 1/2^{1/2} \; ||f||.$$

Denote $X_1 = Y$ and consider the restriction $f_{/X_1}$ of f to closed linear subspace X_1. Then $f_{/X_1}$ is nonzero on X_1 and by an argument similar to that presented above we can find a closed linear subspace X_2 in X_1 such that:

$$0 < ||f||_{X_2} \leqslant 1/2^{1/2} ||f||_{X_1} \leqslant (1/2^{1/2})^2 \; ||f||.$$

By induction, we have a sequence of closed linear subspaces $(X_n)_{n \in N}$ such that (i), (ii) and (iii) hold and the lemma is thus proved.

The proof of theorem. If $(X, \; ||\cdot||)$ is a Banach space isomorphic top-linear to a Hilbert space, then there exists an euclidian norm $||\cdot||_H$ on X such that:

$$m \; ||x|| \leqslant ||x||_H \leqslant M ||x|| \text{ for all } x \in X \; (m > 0).$$

Now, let f be a nonzero continuous linear functional on X and $\varepsilon > 0$. Then there exists $n_\varepsilon \in \mathbf{N}$ such that

$$||f||^H/(m \; 2^{n_\varepsilon/2}) \leqslant \varepsilon, \text{ where } ||f||^H := \sup_{x \neq 0} |f(x)|/||x||_H \tag{5}$$

By Lemma 4 we can find a nonzero closed linear subspace X_{n_ε} so that:

$$||f||^H_{X_{n_\varepsilon}} \leqslant 1/2^{n/2} \; ||f||^H. \tag{6}$$

Consequently, from (5) and (6) there exists $X_\varepsilon := X_{n_\varepsilon}$ so that:

$$||f||_{X_\varepsilon} \leqslant 1/m \; ||f||^H_{X_\varepsilon} \leqslant ||f||^H/(m \; 2^{n_\varepsilon/2}) \leqslant \varepsilon,$$

and the theorem is proved.

3. An application. Let $(X; (\,,\,))$ be an infinite-dimensional Hilbert space, $A: D \subset X \to X$ be a mapping on X and $y \in X$. The element $x_0 \in D$ is called an ε-solution ($\varepsilon > 0$) for the equation $(A; y)$

$$Ax = y,$$

relative to the closed linear subspace X_0 in X if

$$|(x, Ax_0 - y)| \leqslant \varepsilon \; ||x|| \text{ for all } x \in X.$$

PROPOSITION. *For every $x_0 \in D$ which is not a solution of $(A; y)$ and for any $\varepsilon > 0$ there exists a nonzero closed linear subspace X_0 in X with $Ax_0 - y$ is not orthogonal on it and such that x_0 is an ε-solution of $(A; y)$ relative to X_0.*

Proof. Let consider the functional $f_0: X \to K$, $f_0(x) := (x, Ax_0 - y)$. Since x_0 is not a solution of $(A; y)$ it follows that f is not zero and by the above considerations, for every $\varepsilon > 0$ there exists a nonzero closed linear subspace X_0 in X such that X_0 is not included in Ker (f) and with the property : $|f_0(x)| \leqslant \varepsilon ||x||$ for all $x \in X_0$.

The proof is finished.

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RECENT ADVANCES IN TRIANGLE INEQUALITIES

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ABSTRACT. — In this paper we prove some inequalities related to the elements of a triangle. We shall follow the terminology of [1]. The results are improvements of the results from [1] and some are of a new nature.

Notation. a, b, c-sides BC, CA, AB of a triangle ABC; α, β, γ-its angles; h_a, h_b, h_c-altitudes; m_a, m_b, m_c-medians; w_a, w_b, w_c — angle bisectors; R - radius of circumcircle; r — radius of incircle; s — semiperimeter; F - area of triangle ABC; r_a, r_b, r_c — radii of excircles.

THEOREM 1. *In every tiangle*:

$$\sum a \cdot w_a^2 \leqslant 2rs(4R + r) \tag{A}$$

Equality in (A) *holds if and only if the triangle is equilateral.*

Proof 1. Since (see [1], p. 76)

$$a \cdot w_a^2 = abc(1 - a^2/(b + c)^2), \quad a = 4R \sin \alpha/2 \cdot \cos \alpha/2$$

$$b + c = 4R \cos \alpha/2 \cdot \cos(\beta - \gamma)/2, \quad abc = 4Rrs$$

$$\cos(\beta - \gamma)/2 \leqslant 1,$$

we have

$$a \cdot w_a^2 \leqslant 4 \cdot R \cdot r \cdot s\,(1 - \sin^2 \alpha/2),$$

that is

$$a \cdot w_a^2 \leqslant (3 - \sum \sin^2 \alpha/2)\, 4Rrs. \tag{1}$$

Then, since

$$\sin^2 \alpha/2 = 1 - r/2R,$$

(1) implies

$$a \cdot w_a^2 \leqslant 2rs(4R + r)$$

i.e., (A) holds.

Proof 2. It is known ([1], 8.8) that

$$w_a \leqslant \sqrt{s(s - a)}, \quad w_b \leqslant \sqrt{s(s - b)}, \quad w_c \leqslant \sqrt{s(s - c)}.$$

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We have

$$a \cdot w_a^2 \leqslant s\left(\sum a(s-a)\right)$$

By aplying the formulas

$$abc = 4Rrs, \ \sum a = 2s \ \text{and} \ \sum a^2 = 2(s^2 - 4Rr - r^2)$$

we obtain the required inequality. Equality in (A) occurs only if the tria
is equilateral.

Remark 1. The inequality (A) is more precise than 8.13 shown in [1
$(\lambda = 1)$

THEOREM 2. *In every triangle*:

$$(h_a + 2r)/(r + r_a) \geqslant 27/4 \quad (B)$$

Equality occurs if and only if the triangle is equilateral.

Proof. As

$$h_a = 2F/a, \ r_a = F/(s-a) \ \text{and} \ r = F/s$$

we have

$$(h_a + 2r_a)/(r + r_a) = 2s^2(1/a(2s-a))$$

that is

$$\sum(h_a + 2r_a)/(r + r_a) = 2s^2\left(\sum 1/a(2s-a)\right) \quad (3)$$

We define the function

$$f(x) = 1/x(2s-x) \ \ (0 < x < 2s) \quad (4)$$

Since

$$f''(x) = 2(3x^2 - 6sx + 4s^2)/x^3(2s-x)^3 < 0$$

function f, given by (4) is convex, so that

$$\sum_{i=1}^{3} 1/x_i(2s - x_i) \geqslant 27/\left(\sum_{i=1}^{3} x_i\right)\left(6s - \sum_{i=1}^{3} x_i\right) \quad (5)$$

If in the inequality (5) we put $x_1 = a$, $x_2 = b$ and $x_3 = c$, then we obta

$$\sum 1/a(2s-a) \geqslant 27/8s^2 \quad (6)$$

Now (6) and (3) imply that

$$\sum(h_a + 2r_a)/(r + r_a) \geqslant 27/4$$

i.e., (B) holds.

THEOREM 3. *In every triangle*:

$$\sum(b + c)/(r_b + r_c) \geqslant 4s/3r \quad (C)$$

and the equality is true if and only if the triangle is equilateral.

Proof. Let

$$S = \sum(b + c)/(r_b + r_c)$$

Since

$$r_a = F/(s - a),\ r_b = F/(s - b),\ r_c = F/(s - c),$$
$$F = rs,\ (s - a)(s - b)(s - c) = r^2 \cdot s$$
$$\text{and } b + c = 2s - a$$

We have

$$S = r\sum(2s - a)/a(s - a) \tag{7}$$

By the formulas

$$\sum bc(2s - a)(2s - b)(2s - c) = 2s(\sum b^2c^2) + 7abs - (2s^2 + abc)(\sum bc),$$
$$abc = 4Rrs,\ \sum bc = r^2 + s^2 + 4Rr$$
$$\text{and } \sum b^2c^2 = s^4 - 8Rrs^2 + 2r^2s^2 + 16R^2r^2 + 8Rr^3 + r^4$$

We obtain

$$S = (2Rs)^{-1}(8R + 6Rr + r^2 + s^2). \tag{8}$$

It is known that ([1], 5.8, 5.1)

$$s^2 \leqslant 4R^2 + 4Rr + 3r^2 \text{ and } 2r \leqslant R$$

where from implies

$$5s^2 \leqslant 5(4R^2 + 4Rr + 3r^2) + 2(R - 2r)(2R + 3r) \tag{9}$$

The inequality (9) is equivalent to

$$(2Rs)^{-1}(8R + 6Rr + r^2 + s^2) \geqslant 4s/3R \tag{10}$$

On the basis of (10) and (8) we obtain the required inequality.

QUESTION. Is the inequality

$$\sum(b + c)/r_b + r_c) \leqslant 2 \cdot \sqrt{3}$$

true for acute-angled triangles?

THEOREM 4. ([3]). *In every triangle*:

$$\sum w_a^2 \leqslant s^2 - r(R/2 - r) \tag{D}$$

Equality in (D) *occurs if and only if the triangle is equilateral.*

Proof. It is known (see [1], p. 76), that

$$w_a^2 = bc - a^2bc/(b + c)^2$$

that is

$$\sum w_a^2 = \sum bc - abc\,(\sum a/(b+c)^2) \quad (12)$$

By applying Cauchy's inequality we have

$$(\sum \sqrt{a}/(b+c)\cdot\sqrt{a})^2 \leqslant (\sum a/(b+c)^2)(\sum a), \text{ i.e.}$$

$$\sum a/(b+c)^2 \geqslant 1/2s(\sum a/(b+c))^2 \quad (13)$$

Using the inequality 1.16 in [1], i.e.

$$\sum a/(b+c) \geqslant 3/2$$

we get

$$\sum a/(b+c)^2 \geqslant 9/8s \quad (14)$$

On the basis $bc = r^2 + s^2 + 4Rr$ and $abc = 4Rrs$, from the relations (12) a (14) we obtain the required inequality.

THEOREM 5. *In every triangle*:

$$\sum 1/r_a(h_b + h_c) \geqslant 1/6r^2 \quad (E)$$

with the equality only if the triangle is equilateral.

Proof. Since the function $g(x) = x^{-1}$ $(x < 0)$ is convex the famous varia on the Jensen's inequality for a convex function may be applid:

$$g\left(\frac{\sum_{i=1}^{3} x_i}{\sum_{i=1}^{3} p_i}\right) \leqslant \frac{\sum_{i=1}^{3} p_i g\left(\frac{x_i}{p_i}\right)}{\sum_{i=1}^{3} p_i},\ x_i>0,\ \frac{x_i}{p_i} > 0 \quad (15)$$

Putting

$$x_1 = (h_b + h_c)/r_a,\ x_2 = (h_c + h_a)/r_b,\ x_3 = (h_a + h_b)/r_c$$

$$p_1 = 1/r_a,\ p_2 = 1/r_b,\ p_3 = 1/r_c$$

On (15), because $\sum 1/r_a = 1/r$ and

$$(h_b + h_c)/r_a = 6 \text{ (see [2], theorem 1.)}$$

we obtain the inequality (E).

THEOREM 6. *In every triangle*:

$$3r\cdot\sqrt{6} \leqslant \sum a\cdot\sqrt{\sin\alpha/2} \leqslant 3\cdot\sqrt{R(2R-r)} \quad (F)$$

Equality holds if and only if the trinagle is equilateral.

Proof. Using the equalities

$$\Pi\sin\alpha/2 = r/4R,\ \Pi a = 4Rrs$$

and the connection between the arithmetic and geometric means, we have

$$\sum a \cdot \sqrt{\sin\alpha/2} \geqslant 3 \sqrt[6]{(abc)^2 \quad \sin\alpha/2 \cdot \sin\beta/2 \cdot \sin\gamma/2} = \\ = 3 \cdot \sqrt[6]{4R^3s^2} \tag{16}$$

Then, since ([1], 5.3, 5.11)

$$R \geqslant 2s/(3 \cdot \sqrt{3}),\ s \geqslant 3r\sqrt{3}$$

(16) implies

$$\sum a \cdot \sqrt{\sin\alpha/2} \geqslant 3r \cdot \sqrt{6}$$

i.e. the left part of the inequality (F).
Using the equality

$$\sum a \cdot \mathrm{tg}\alpha/2 = 4R - 2r \tag{17}$$

(see [4], Theorem 1.)

and the inequality

$$\sum a \cdot \cos\alpha/2 \leqslant 9/2R \text{ ([4], Remark 4)} \tag{18}$$

We have by virtue of Cauchy's inequality

$$(\sum a \cdot \cos\alpha/2)(\sum a \cdot \mathrm{tg}\alpha/2) \geqslant (\sum a \cdot \sqrt{\sin\alpha/2})^2$$

and (17), (18) implies the right part of (F).

THEOREM 7. *In every triangle*:

$$\sum a \cdot \sin\alpha/2 \leqslant \frac{3 \cdot \sqrt{2}}{2} \cdot \sqrt{R(2R - r)} \tag{G}$$

Equality holds if and only if the triangle is equilateral.

Proof. Using Cauchy's inequality we have:

$$\sum\sqrt{2} \cdot a \cdot \sin\alpha/2 = \sum\sqrt{2 \cdot a \cdot \sin\alpha/2 \cdot \cos\alpha/2} \cdot \sqrt{a \cdot \mathrm{tg}\alpha/2} \leqslant \\ \leqslant \sqrt{(\sum a \cdot \sin\alpha)(\sum a \cdot \mathrm{tg}\alpha/2)}$$

But (17) and

$$\sum a \cdot \sin\alpha \leqslant 9/2R \qquad \text{(see [1], 3.14)}$$

we have

$$\sum a \cdot \sin\alpha/2 \leqslant 3\sqrt{2}/2 \cdot \sqrt{R(2R - r)}$$

i.e. the right part of (G).

THEOREM 8. *In every triangle*:

$$3^{1+n/2} \cdot 2^{n-1} \cdot r^n \leqslant \sum a^n \cdot \sin\alpha/2 \leqslant (s(1 - r/R)(\sum a^{2n-1}))^{1/2}\ (n \geqslant 1) \tag{H}$$

Equality holds if and only if the triangle is equilateral.

Proof. By the inequality of arithmetic and geometric means we have:

$$\sum a^n \sin\alpha/2 \leqslant 3[(\Pi\sin\alpha/2)(\Pi a^n)]^{1/3} =$$
$$= 3\left(\frac{s(s-a)(s-b)(s-c)}{sabc} a^n b^n c^n\right)^{1/3} =$$
$$= 3 \cdot ((F^2/s)(abc)^{n-1})^{1/3} = 3 \cdot (r^2 s \cdot (abc)^{n-1})^{1/3}$$

Then since $abc = 4Rrs$, $R \geqslant 2r$, $s \geqslant 3\sqrt{3}/2R$ we have

$$abc \geqslant 24\sqrt{3} \cdot r^3$$

and

$$\sum a^n \cdot \sin\alpha/2 \geqslant 3^{1+n/2} \cdot 2^{n-1} \cdot r^n$$

i.e. the left part of the inequality (H).

Using Cauchy's inequality, we have

$$(\sum a^{(2n-1)/2} \cdot a^{1/2} \cdot \sin\alpha/2)^2 \leqslant (\sum a^{2n-1})(\sum a \cdot \sin^2\alpha/2) \quad (19)$$

Then since

$$\sum a \cdot \sin^2\alpha/2 = s \cdot (1 - r/R)$$

19) implies the right part of (H).

THEOREM 9. *In every triangle*:

$$\sum a^n \tan\beta/2 \cdot \tan\gamma/2 \geqslant 2^n \cdot (r(R+r))^{n/2}, \; n \geqslant 2 \quad (I)$$

(*Equality holds if and only if the triangle is equilateral and* $n = 2$.

Proof. If in Jensen's inequality (15) for a convex function

$g(x) = x^{n/2} (x > 0,\; n \geqslant 2)$ we put

$x_1 = a^2 \cdot \tan\beta/2 \cdot \tan\gamma/2$, $x_2 = b^2 \cdot \tan\gamma/2 \cdot \tan\alpha/2$,

$x_3 = c^2 \cdot \tan\alpha/2 \cdot \tan\beta/2$

$p_1 = \tan\beta/2 \cdot \tan\gamma/2$, $p_2 = \tan\gamma/2 \cdot \tan\alpha/2$, $p_3 = \tan\alpha/2 \cdot \tan\beta/2$

Then from

$$\sum \tan\beta/2 \cdot \tan\gamma/2 = 1$$

and

$$\sum a^2 \cdot \tan\beta/2 \cdot \tan\gamma/2 = 4r(R+r)$$

we obtain the inequality (H).

Remark 2. This method was shown in [5].

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A MOST DIRECT PROOF OF COMPACTNESS OF THE PRODUCT OF COMPACT SPACES

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REZUMAT. — O demonstraţie directă a compactităţii produsului de spaţii compacte. În lucrare este prezentată o demonstraţie directă a teoremei lui Tychonoff asupra compactităţii produsului unor spaţii topologice compacte.

In the existing literature there are two quite polished proofs of Tychonoff's product theorem which states:

THEOREM. *Let $A = \{a, b, c, d, e, \ldots\}$ be a (finite or infinite) index set and for every $t \in A$ let C_t be a compact topological space. Then the cartesian product*

$$P = C_a \cdot C_b \cdot C_c \cdot C_d \cdot C_e \cdot \ldots \quad (1)$$

is compact with respect to the product topology.

Both of the abovementioned proofs can be found in [1, p. 143]. However both proofs are rather indirect. The first proof uses Alexander's subbase theorem [1, p. 139] and therefore does not start the proof directly with an arbitrary open cover. The second proof (Bourbaki) considers the dual definition of compactness in terms of the closed subsets and then digresses by passing to the closures of the projections of closed subsets.

Our proof will reflect some sensitive points (such as the distributivity of $\cup$ w.r.t. $\cap$) of the proof of Alexander's subbase theorem and (such as the use of Zorn's lemma) of Bourbaki's proof. However, in our proof these points are invoked directly at the most natural places.

Our proof is intentionally leisurely, self-contained and special effort is made to keep the proof as clear as possible.

For the sake of simplicity, we introduce a definition and prove two easy (but essential) lemmas.

Using the notations introduced in the above Theorem, for $t \in A$, let u_t be an open subset of the topological space C_t. Then the cartesian product $E(u_t)$ given by

$$E(u_t) = C_a \cdot C_b \cdot C_c \cdot \ldots \cdot u_t \cdot \ldots \cdot C_s \cdots \quad (2)$$

is called an *elementary strip of type t determined by u_t*.

Clearly, an elementary strip is an open subset of P.

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LEMMA 1. *Let P be given as in* (1) *of the Theorem. If P is covered by a set K of elementary strips then P is already covered by some elementary strips belonging to K and which are all of the same type.*

Proof. Let us assume to the contrary.

Thus, there exists (one or possibly more) $x_a \in C_a$ such that the subset $\{x_a\} \cdot C_b \cdot C_c \cdot C_d \cdot C_e \ldots$ of P is not covered by any elementary strip of type a belonging to K.

Similarly, there exists (one or possibly more) $x_b \in C_b$ such that the subset $C_a \cdot \{x_b\} \cdot C_c \cdot C_d \cdot C_e \ldots$ of P is not covered by any elementary strip of type b belonging to K.

In general, for every $t \in A$ there exists (one or possibly more) $x_t \in C_t$ such that the subset $C_a \cdot C_b \cdot C_c \cdot \ldots \cdot \{x_t\} \cdot \ldots \cdot C_z \ldots$ of P is not covered by any elementary strip of type t belonging to K.

But then (using the axiom of Choise) there exists a point, say, $(x_a, x_b, x_c, \ldots, x_z, \ldots)$ of P which is not covered by K.

Thus, our assumption lead to a contradiction and Lemma 1 is proved.

Remark. We observe that neither the definition of an elementary strip nor Lemma 1 depended on the compactness of C_t's. Indeed, Lemma 1 is valid for the product of any topological space C_t (compact or noncompact). However, in Lemma 2 below compactness of every C_t is essential.

LEMMA 2. *Let P be given as in* (1) *of the Theorem. If P is covered by a set K of elementary strips then P is already covered by a finite number of elementary strips belonging to K and which are all of the same type.*

Proof. By Lemma 1, P is already covered by a set K of elementary strips, say, of the same type t given by

$$K = \{E(u_t), E(v_t), \ldots, E(m_t), \ldots, E(w_t), \ldots\}$$

Clearly, $\{u_t, v_t, \ldots, m_t, \ldots, w_t, \ldots\}$ is a cover of the compact space C_t and therefore it has a finite subcover, say,
$\{u_t, v_t, \ldots, m_t\}$. But then obviously, the finite set
$\{E(u_t), E(v_t), \ldots, E(m_t)\}$ of elementary strips covers P, as claimed by the Lemma.

Proof of the Theorem. Let us assume to the contrary that P is not a compact space (relative to the product topology). Thus, there exists an open cover V of P such that V has no finite subcover. By Zorn's lemma there exists an open cover M of P such that $V \subset M$ and M is maximal with respect to the property of having no finite subcover. Because of this it is clear that M has the following properties:

if $H \in M$ and B is an open set of P such that $B \subset H$ then $B \in M$. (3)

the union of any finite number of elements of M is an element of M. (4)

if E is an open set of P such that $E \notin M$ then the union of E with some element of M is equal to P. (5)

Now, let $x \in P$. Then x is covered by an element H of M. Thus, there exist (by the definition of the product topology) a finite number of elements, say, a, b, c of A which define a basic open set B of P given by

$$B = u_a \cdot u_b \cdot u_c \cdot C_d \cdot C_e \ldots \tag{6}$$

where u_a, u_b, u_c are open subsets respectively of the topological spaces C_a, C_c and where

$$x \in B \text{ and } B \subset H \text{ and } H \in M \quad (7)$$

In connection with (6), let us consider the following elementary strip

$$E(u_a) = u_a \cdot C_b \cdot C_c \cdot C_d \cdot \ldots \quad (8)$$

$$E(u_b) = C_a \cdot u_b \cdot C_c \cdot C_d \cdot \ldots \quad (9)$$

$$E(u_c) = C_a \cdot C_b \cdot u_c \cdot C_d \cdot \ldots \quad (10)$$

From (6), (8), (9), (10) it follows that

$$B = E(u_a) \cap E(u_b) \cap E(u_c) \quad (11)$$

We claim that one of $E(u_a)$, $E(u_b)$, $E(u_c)$ is an element of M. If not, (5) there exist elements m_1, m_2, m_3 of M such that

$$m_1 \cup E(u_a) = m_2 \cup E(u_b) = m_3 \cup E(u_c) = P \quad (12)$$

Since P is the entire topological space, by (12), we have

$$m_1 \cup m_2 \cup m_3 \cup E(a_a) = m_1 \cup m_2 \cup m_3 \cup E(u_b) = m_1 \cup m_2 \cup m_3 \cup E(u_c) = P \quad (13)$$

Since m_1, m_2, m_3 are elements of M, from (4) it follows that

$$m = m_1 \cup m_2 \cup m_3 \text{ is an element of } M \quad (14)$$

From (13) and (14), we obtain

$$m \cup E(u_a) = m \cup E(u_b) = m \cup E(u_c) = P$$

and therefore,

$$(m \cup E(u_a)) \cap (m \cup E(u_b)) \cap (m \cup E(u_c)) = P$$

which, by the distributivity of $\cup$ with repect to $\cap$ and (11) implies

$$m \cup (E(u_a) \cap E(u_b) \cap E(u_c)) = m \cup B = P \quad (15)$$

However, from (3) and (7) it follows that $B \in M$. By (14) we also ha $m \in M$. Thus, by (4) we must have $(m \cup B) \in M$, which by (15) contradic our assumption that M has no finite subcover.

Thus, one of $E(u_a)$, $E(u_b)$, $E(u_c)$, say, $E(u_c)$, is an element of M.

But then from (7) and (11) it follows that $x \in E(u_a)$. Thus, every element P is covered by an elementary strip belonging to M. Consequently, by Lemma we see that P is already covered by a finite number of elements of M, contr dicting our assumption mentioned above.

Hence, our assumption is false and the Theorem is proved.

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FIXED POINT THEOREMS FOR θ—CONDENSING MAPPINGS

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REZUMAT. — Teoreme de punct fix pentru aplicaţii θ-condensatoare. În lucrările [30], [31], [32] şi [35] am folosit tehnica structurilor de punct fix pentru a stabili teoreme noi de punct fix. În prezenta lucrare dăm o generalizare a Teoremei 6.1. din [32] şi prezentăm o gamă largă de consecinţe ale acestui rezultat. Printre altele obţinem rezultate date de Darbo (a se vedea [1] şi [5]), Sadovskii ([1], [5]), Amann [3], Bae [4], de Blasi [6], Iseki [15] şi [16], Jones [17], Kassay [18], Reich [25], şi Reinermann [26]. În finalul lucrării se formulează cîteva conjecturi.

1. **Introduction.** In the papers [30], [31], [32] and [35] we use the technique of the fixed point structures to give same new fixed point theorems. In this paper we improve the Theorem 6.1. in [32] and we give same consequences of this result. Thus we obtain some results given by Darbo (see [1] and [5]), Sadovskii (see [1] and [5]), Amann [3], Bae [4], de Blasi [6], Danes [9], Iseki ([15] and [16]), Jones [17], Kassay [18], Pasicki [24], Reich [25(, Reibermann [26] and Tineo [39]. Some conjectures are formulated.

The plan of the paper is the following:

2. Fixed point structures
3. Compatible pair with the fixed point structures
4. θ—condensing mappings
5. Invariant subsets and fixed subsets
6. A general fixed point principle
7. Banach spaces
8. Metric spaces
9. Convex metric space
10. Locally convex topological vector space
11. Nonself mappings
12. Asymptotic fixed point theorems.

Through this paper we follow terminologies and notations in [28] and [42].

2. **Fixed point structures.** We begin with

DEFINITION 2.1. (see [31] and [32]). Let X be a nonempty set and $Y \in P(X)$. We denote by $\mathbf{M}(Y)$ the set of all mappings $f: Y \to Y$. A triple (X, S, M) is said to be a *fixed point structure* if:

(i) $S \subset P(X)$, $S \neq \emptyset$

(ii) $M: P(X) \to \cup\ \mathbf{M}(Y)$, $Y \to M(Y) \subset \mathbf{M}(Y)$ is

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a mapping such that, if $Z \subset Y$, $Z \neq \emptyset$, then $M(Z) \supset \{f|_Z \mid f \in M(Y)$ and $f(Z) \subset$ $\subset Z\}$.

(iii) every $Y \in S$ has the fixed point property with respect to $M(Y)$.

Example 2.1. X is a nonempty set, $S = \{\{x\} \mid x \in X\}$ and $M(Y) = \mathbf{M}(Y)$.

Exemple 2.2. (X, d) is a metric space $S = P_{cp}(X)$ and $M(Y) = \{f: Y \to Y \mid f$ is continuous and δ—condensing$\}$.

Example 2.3. X is a Banach space, $S = P_{cp,cv}(X)$ and $M(Y) = C(Y, Y)$.

Example 2.4. X is a locally convex space, $S = P_{cp,cv}(X)$ and $M(Y) =$ $= C(Y, Y)$.

Example 2.5. X is a Hilbert space, $S = P_{b,\, cl,\, cv}(X)$ and $M(Y) = \{f: Y \to Y \mid f$ is nonexpansive$\}$.

Remark 2.1. The notion „fixed point structure" is a generalization of some notions as „topological space with fixed point property", „ordered set with fixed point property", „mapping with fixed point property on a family of sets" (Jones [17]), „object with fixed point property" (Rus [26]), ...

Remark 2.2. For other examples, of fixed point structure see [30], [31] and [32].

3. **Compatible pair with fixed point structure.** The following notion is useful in what follows.

DEFINITION 3.1. (see [31] and [32]). Let (X, S, M) be a fixed point structure, $(O, \leqslant)$ an ordered set with a minimal element, which we will denote by 0. The pair $(\theta, \eta)(\theta : Z \to (O, \leqslant)$ and $\eta : P(X) \to P(X))$ is said to be *compatible with* (X, S, M) if

(i) η is a closure operator,

(ii) $S \subset \eta(Z) \subset Z \subset P(X)$ and $\theta(\eta(Y)) = \theta(Y)$, for all $Y \in Z$,

(iii) $F_\eta \cap Z_\theta \subset S$, where $F_\eta := \{A \subset X \mid \eta(A) = A\}$ and $Z_\theta = \{Y \in Z \mid \theta(Y) = 0\}$.

Example 3.1. Let (X, d) be a metric space. Let (X, S, M) be as in Example 2.1, $Z = P_b(X)$, $\theta = \delta$ and $\eta(A) = \bar{A}$. Then (δ, η) is a compatible pair with (X, S, M).

Example 3.2. Let (X, S, M) be as in Example 2.2, $Z = P_b(X)$, $\theta = \alpha_K$ or α_H and $\eta(A) = \bar{A}$. Then (θ, η) is a compatible pair with (X, S, M).

Example 3.3. Let (X, S, M) be as in Example 2.3, $Z = P_b(X)$, $\theta = \alpha_K$ or α_H and $\eta(A) = \overline{\text{co}}\, A$. Then (θ, η) is a compatible pair with (X, S, M).

Example 3.4. Let (X, S, M) be as in Example 2.4. Let V be the family of all closed balanced convex neighbourhoods of zero. For $Y \in P_b(X)$ let $\theta(Y) :=$ $= \{A \in V \mid$ there exists a totally bounded subset $T \subset X$ with $Y \subset T + A\}$. If $U_1, U_2 \in P(V)$, then $U_1 \leqslant U_2$ if and only if $U_1 \supset U_2$. Thus we have the partial ordered set $(P(V), \leqslant)$ with the minimal element V. The pair $(\theta, \overline{\text{co}})$ is a compatible pair with (X, S, M).

Remark 3.1. For other examples of compatible pair see [30], [31] and [32].

4. **θ-condensing mappings.** Let X be a nonempty set, $Y \subset X$, $Z \subset P(X)$ ($Y \neq \emptyset$, $Z \neq \emptyset$) and $\theta : Z \to (O, \leqslant)$.

DEFINITION. 4.1. The mapping $f : Y \to Y$ is said to be θ-*condensing* if

(i) $A \in P(Y) \cap Z$ implies $f(A) \in Z$,

(ii) $A \in P(Y) \cap Z$, $f(A) \subset A$, $\theta(f(A)) \geqslant \theta(A)$ imply that $\theta(A) = 0$.

DEFINITION. 4.2. The mapping $f : Y \to Y$ is said to be θ-*condensing* if

(i) $A \in P(Y) \cap Z$ implies $f(A) \in Z$,

(ii) $A \in P(Y) \cap Z$, $f(A) \subset A$, $\theta(A) \neq 0$ imply $\theta(f(A)) < \theta(A)$.

DEFINITION. 4.3. The mapping $f : Y \to Y$ is said to be *strong* θ-*condensing* if

(i) $A \in P(Y) \cap Z$ implies $f(A) \in Z$,

(ii) $A \in P(Y) \cap Z$, $\theta(f(A)) \geqslant \theta(A)$ imply that $\theta(A) = 0$.

DEFINITION. 4.4. The mapping $f : Y \to Y$ is said to be *strong* θ-*condensing* if

(i) $A \in P(Y) \cap Z$ implies $f(A) \in Z$,

(ii) $A \in P(Y) \cap Z$, $\theta(A) \neq 0$, imply $\theta(f(A)) < \theta(A)$.

Remark 4.1. For some examples of θ-condensing mappings see [1], [5], [6], [8], [10], [17], [26], [32] and [38].

5. **Invariant subsets and fixed subsets.** The following notions are very tied to the fixed point theory.

DEFINITION. 5.1. Let X be a set and $f : X \to X$, a mapping. A subset $Y \subset X$ is said to be *invariant subset* for f if $f(Y) \subset Y$ and a *fixed subset* if $f(Y) = Y$.

Let $I(f) := \{Y \in P(X) \mid f(Y) \subset Y\}$ the family of all nonempty invariant subset of f.

The following results are well known:

LEMMA 5.1. (see [4], [11], [19], [23]). *Let (X, τ) be a compact topological space and $f : X \to X$ a continuous mapping. Then the subset $X_\infty := \bigcap_{n=1}^{\infty} f^n(X)$ is a fixed subset for $f(X_\infty \in P_{cp}(X))$.*

LEMMA 5.2. (M a r t e l l i [20]). *Let (X, τ) be a compact topological space and $f : X \to X$ a mapping. Then there exists a nonempty subset $Y \subset X$ such that $Y = \overline{f(Y)}$. If f is continuous then $Y = f(Y)$.*

Remark 5.1. In the conditons of the Martelli's Lemma there exists a minimal $Y \in I_{cl}(f)$ such that $Y = \overline{f(Y)}$ (or $Y = f(Y)$ if f is continuous).

LEMMA 5.3. (R u s [30] and [32]). *Let X be a nonempty set, $\eta : P(X) \to P(X)$ a closure operator, $Y \in F_\eta$ and $f : Y \to Y$ a mapping. Let $A \subset Y$ be a nonempty subset of Y. Then there exists $A_0 \subset Y$ such that*: (i) $A_0 \supset A$, (ii) $A \in F_\eta$, (iii) $A_0 \in I(f)$, (iv) $\eta(f(A_0) \cup A) = A_0$.

DEFINITION 5.2. (J o n e s [17]; see also de B l a s i [6]). Let X be a set, $U \subset P(X)$, $U \neq \emptyset$. A mapping $f : X \to X$ is said to be *reducible on* U if for any $A \in$

$\in U$, such that $f(A) \subset A$ and card $A > 1$, there exists a proper subset $B \subset A$, invariant for f.

Example 5.1. Let (X, d) be a complete metric space and $f: X \to X$ a (δ, φ)--contraction. Then f is reducible on $P_b(X)$.

DEFINITION. 5.3. Let X be a nonempty set. A family $U \subset P(X)$, $U \neq \emptyset$ has the *intersection property* if for any totally ordered subset $V \subset U$ (U is partial ordered by the set inclusion) we have $\cap V \in U$.

Example 5.2. Let (X, τ) be a topological space. Then the family $P_{cp}(X)$ has the intersection property.

Example 5.3. Let $(X, \|\cdot\|)$ be a reflexive Banach space. Then $P_{cl,cv,b}(X)$ has the intersection property.

We have

THEOREM 5.1. *Let X be a set, $U \subset P(X)$, $U \neq \emptyset$, a family with the intersection property and $f: X \to X$ such that*

(i) $I(f) \cap U \neq \emptyset$,

(ii) *f is reducible on U.*

Then there exists $x_0 \in X$ such that $\{x_0\} \in I(f)$, i.e., $x_0 \in F_f$.

Proof. Let $U_1 := U \cap I(f)$. From (i), $U_1 \neq \emptyset$. We consider the partial ordered set $(U_1, \subset)$. Let C be a totally ordered subset of U_1. We have $\cap C \in U_1$, and $\cap C$ is a lower bound of C. By Zorn's lemma there exists at least a minimal element, A_0 of U_1. We have $A_0 \in U \cap I(f)$. Since f is reducible on U it follows $A = \{x_0\}$.

From the Theorem 5.1. we have

THEOREM 5.2. ([3], [12]) *Let (X, d) be a compact metric space and $f: X \to X$ a continuous δ—condensing mapping. Then $F_f = \{x^*\}$.*

Proof. We remark that f is reducible on $P_{cp}(X)$. Let $A \in P_{cp}(X) \cap I(f)$. Then if $\delta(A) \neq 0$, it follows $\delta(f(A)) < \delta(A)$. This implies that $f(A)$ is a proper invariant subset of f. The theorem follows from the Theorem 5.1.

Remark 5.2. The Theorem 5.2. follows also from the Martelli's lemma.

6. **A general fixed point principle.** We have

THEOREM 6.1. (see Rus [30], for $O = \mathbf{R}^+$). *Let (X, S, M) be a fixed point structure, and (θ, η) $(\theta: Z \to (O, \leqslant))$ a compatible pair with (X, S, M). Let $Y \in \eta(Z)$ and $f \in M(Y)$. We suppose that*

(i) *$A \in Z$, $x \in Y$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$,*

(ii) *f is θ—condensing (as in Definition 4.1. or in Definition 4.2).*

Then

(a) $F_f \neq \emptyset$,

(b) *if $F_f \in Z$, then $\theta(F_f) = 0$.*

Proof. (a) Let $y_0 \in Y$ and $A = \{y_0\}$. By Lemma 5.3. there exists $A_0 \in F_\eta \cap I(f)$ such that $\eta(f(A_0) \cup \{y_0\}) = A_0$.

We have
$\theta(\eta(f(A_0) \cup \{y_0\}) = \theta(f(A_0) \cup \{y_0\}) = \theta(f(A_0)) = \theta(A_0)$.
This implies $A_0 \in Z_0$. Thus $A_0 \in F_\eta \cap Z_0$ and $f|_{A_0} \in M(A_0)$. Since (X, S, M) is a fixed point structure, we have $F_f \neq \emptyset$.

(b) From $f(F_f) = F_f$, it follows $\theta(F_f) = 0$.

From the Theorem 6.1. we have

THEOREM 6.2. *Let (X, S, M) be a fixed point structure and (θ, η) a compatible pair with (X, S, M). Let $Y \in F_\eta$ and $f \in M(Y)$ such that $f(Y) \in Z$. We suppose that*

(i) *$A \in Z$, $x \in Y$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$,*

(ii) *f is $\theta-$condensing (as in Definition 4.1. or in Definition 4.2.).*

Then

(a) *$F_f \neq \emptyset$,*

(b) *if $F_f \in Z$, then $\theta(F_f) = 0$.*

Proof. We remark that $\eta(f(Y)) \subset Y$ and $\eta(f(Y)) \in I(f)$.

In what follow we give some consequences of the Theorem 6.2.

7. **Banach spaces.** Let X be a Banach space. Let $\theta : P_b(X) \to (O, \leqslant)$ be such that

(i) $\theta(A) = 0$, implies $\bar{A} \in P_{cp}(X)$,

(ii) $\theta(\overline{\text{co}}\, A) = \theta(A)$, for all $A \in P_b(X)$,

(iii) for all bounded sequences in X we have

$\theta\{x_n | n \geqslant 1\} = \theta\{x_n \mid n \geqslant 2\}$.

Let (X, S, M) be as in Example 2.3. Then the pair $(\theta, \overline{\text{co}})$ is compatible with (X, S, M). From the Theorem 6.1. and 6.2. we have

THEOREM 7.1. (S a d o v s k i i, see [1]). *Let X be a Banach space, $Y \in P_{b,cl,cv}(X)$ and $f : Y \to Y$ a continuous $\theta-$condensing mapping. Then $F_f \neq \emptyset$ and $\theta(F_f) = 0$.*

THEOREM 7.2. *Let X be a Banach space, $Y \in P_{cl,cv}(X)$ and $f : Y \to Y$ a continuous $\theta-$condensing mapping such that $f(Y) \in P_b(X)$. Then $F_f \neq \emptyset$ and $\theta(F_f) = 0$.*

Remark 7.1. In the Theorem 7.1. and 7.2. we can take $\theta = \alpha_K$ or $\theta = \alpha_H$ (see [1], [5], [33]). Thus we have

THEOREM 7.3. (D a ţ b o). *Let X be a Banach space, $Y \in P_{b,cl,cv}(X)$ and $f : Y \to Y$ a continuous $(\alpha_K, a)-$contraction. Then $F_f \neq \emptyset$ and F_f is a compact set.*

THEOREM 7.4. *Let X be a Banach space, $Y \in P_{cl,cv}(X)$ and $f : Y \to Y$ a continuous $(\alpha_K, a)-$contraction such that $f(Y) \in P_b(X)$. Then $F_f \neq \emptyset$ and F_f is a compact set.*

8. **Metric spaces.** Let (X, d) be a bounded complete metric space. $\alpha_{DP}:P(X)\to\mathbf{R}^+$ be the Danes—Pasicki measure of noncompactness (see D a [9], P a s i c k i [24]; see also de B l a s i [6] and R u s [32]), i.e.,

(i) $\alpha_{DP}(A) = 0$ implies $\bar{A} \in P_{cp}(X)$,

(ii) $\alpha_{DP}(\bar{A}) = \alpha_{DP}(A)$, for all $A \in P(X)$,

(iii) $A \subset B$ implies $\alpha_{DP}(A) \leqslant \alpha_{DP}(B)$,

(iv) $\alpha_{DP}(A \cup \{x\}) = \alpha_{DP}(A)$, for all $A \in P_b(X)$ and $x \in X$.

For example α_K and α_H are α_{DP}. We have

THEOREM 8.1. (see A m a n n [3], B a e [4], F u c h s s t e i n e r [1? s e k i [15] and [16], R u s [28]). *Let (X, d) be a bounded complete metric s $f:X\to X$ a continuous α_{DP}—condensing and δ — condensing mapping. $F_f = \{x^*\}$.*

Proof. Let (X, S, M) be as in Example 2.2. We take $\theta = \alpha_{DP}$, $\eta(A)$ Then (θ, η) is a compatible pair with (X, S, M). The theorem follows from Theorem 6.1. From the Theorem 6.2. we have

THEOREM 8.2. *Let (X, d) be a complete metric space, $f:X\to X$ a conti α_{DP}—condensing and δ—condensing mapping. If $f(X)$ is a bounded subset then $F_f = \{x^*\}$.*

From the Theorem 8.2. and Lemma 1.3.3. in [28] it follows

THEOREM 8.3. *Let (X, d) be a complete metric space and $f:X\to X$ a map We suppose that there exists $n_0 \in \mathbf{N}^*$ such that*

(i) *$f^{n_0}(X)$ is a bounded subset of X,*

(ii) *$f^{n_0}:X\to X$ is continuous,*

(iii) *$f^{n_0}:X\to X$ is α_{DP}—condensing*

(iv) *$f^{n_0}:X\to X$ is δ—condensing.*

Then $F_f = \{x^*\}$.

We have

THEOREM 8.4. (see de B l a s i [6] and J o n e s [17]). *Let (X, d) be a b ded complete metric space, $\alpha_{DP}:P_b(X)\to\mathbf{R}_+$ a Daneš—Pasicki measure of non pactness, $\eta:P(X)\to P(X)$ a closure operator and $f:X\to X$ continuous α_{DP}- densing mapping. We suppose that $\alpha_{DP}(\eta(A)) = \alpha_{DP}(A)$, $A \in P_b(X)$. Then at least one fixed point if any one of the following conditions is satisfied:*

(i) *f is reducible on $Z_{\alpha_{DP}} \cap F_\eta$,*

(ii) *every $Y \in Z_{\alpha_{DP}} \cap F_\eta$ has the fixed point property with respect to C(Y*

Proof. The case (i) follows from the Theorem 5.1.

The case (ii). We remark that $(X, Z_{\alpha DP} \cap F_\eta, M)$ where $M(Y) = C(Y$ is a fixed point structure and (α_{DP}, η), is a compatible pair with this fixed structure.

Remark 8.1. Let (X, d) be a complete metric space. Let $\delta_2(x_1, x_2, x_3)$ the area of the triangle $\Delta(x_1, x_2, x_3)$. For $Y \in P_b(X)$ let $\delta_2(Y) := \sup \{\delta_2(x_1, x_2, x_3) | x_1, x_2, x_3 \in Y\}$. If we take $\theta = \delta_2$ and $\eta(A) = \bar{A}$, from the Theorem 6.1. we have the Theorem 3.1. in T i n e o [39].

9. **Convex metric spaces.** We begin the following consideration with

DEFINITION 9.1. (T a k a h a s h i [37]). Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \to X$ is said to be a *convexity structure* on X if W is continuous and for every $x, y \in X$, $\lambda \in [0, 1]$, we have

$d(u, W(x, y, ; \lambda) \leqslant \lambda d(u, x) + (1 - \lambda)\, d(u, y)$, for all $u \in X$ In this case (X, d, W) is said to be a *convex metric space.*

DEFINITION 9.2. (R u s [34]). Let (X, d, W) be a convex metric space and $Z \subset P(X)$. A mapping $\beta : Z \to \mathbf{R}_+$ is a *weak measure of nonconvexity* if $\beta(A)=0$ implies $\bar{A} \in P_{cv}(X)$.

DEFINITION 9.3. (R u s [34]). Let (X, d, W) be a convex metric space and $Z \subset P(X)$. A mapping $\gamma : Z \to \mathbf{R}_+$ is a *measure of non compact-convexity* if

(i) $\gamma(A) = 0$ implies $\bar{A} \in P_{cp,cv}(X)$

(ii) $\gamma(\bar{A}) = \gamma(A)$, for all $A \in Z$,

(iii) $\gamma|_{Z \cap P_{b,cl}(X)}$ is a mapping with the intersection property (see [30]).

We have

THEOREM 9.1. (see [29], [18] and [34]). *Let (X, d, W) be a bounded convex complete metric space. Let α_{DP} be a Daneš—Pasicki measure of noncompactness on X and β a weak measure of nonconvexity on X. Let $f : X \to X$ be a continuous mapping. We suppose that*

(i) f is α_{DP}*—condensing,*

(ii) *f is β—condensing*

Then $F = \{x^*\}$.

Proof. Let $M(Y) := \{g : Y \to Y \mid g$ is continuous and β—condensing$\}$. We remark that $(X, P_{cp,cv}(X), M)$ is a fixed point structure (see Theorem 24 in [18]). Let $\eta(Y) = \bar{Y}$. Then the pair (α_{DP}, η) is a compatible pair with $(X, P_{cp,cv}(X), M)$. The theorem follows from the Theorem 6.1.

THEOREM 9.2. (see [34]). *Let (X, d, W) be a bounded convex complete metric space, $\gamma : P(X) \to \mathbf{R}_+$ a measure of non compactconvexity on X and $f : X \to X$ a continuous (γ, φ)—contraction. Then $F_f = \{x^*\}$.*

Proof. Let $M(Y) = C(Y, Y)$ and $\eta(Y) = \bar{Y}$. Then $(X, P_{cp,cv}(X), M)$ is a fixed point structure and (γ, η) is a compatible pair with (X, S, M). The theorem follows from the Theorem A in [35].

10. **Locally convex spaces.** Let X be a locally convex topological vector space. Let (X, S, M) be as in Example 2.4. and θ as in Example 3.4. Then the pair $(\theta, \overline{\text{co}})$ is a compatible pair with (X, S, M). From the Theorem 6.1. we have.

THEOREM 10.1. (see [25]). *Let X be a Hausdorff locally convex linear topological space, Y be a nonempty bounded complete convex subset of X, and $f : Y \to Y$*

a continuous θ—condensing mapping. If $f(Y)$ is a bounded subset of Y, then F_f is a nonempty, compact subset.

11. **Nonself mappings.** We have

THEOREM 11.1. (see [33]) *Let (X, S, M) be a fixed point structure and $(\theta, \eta)(\theta : Z \to (O, \leqslant))$ a compatible pair with (X, S, M). Let $Y \in \eta(Z)$, $f : Y \to X$ and $\rho : X \to Y$ a retraction. We suppose that*

(i) *$A \in Z$, $x \in X$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$,*

(ii) *f is a strong θ—condensing* (as in Definition 4.4.),

(iii) *f is retractible onto Y by ρ and $\rho \circ f \in M(Y)$,*

(iv) *ρ is θ—necexpansive.*

Then $F_f \neq \emptyset$ and if $F_f \in Z$, then $\theta(F_f) = 0$.

Proof. We remark that ρ $o f : Y \to Y$ is strong θ—condensing. By the Theorem 6.1., $F_{\rho o f} \neq \emptyset$. From the condition (iii) it follows that $F_{\rho o f} = F_f \neq \emptyset$. Let $F_f \in Z$. From $f(F_f) = F_f$ and the condition (ii) we have $\theta(F_f) = 0$.

12. **Asymptotic fixed point theorems.** At the end of this paper we formulate the following.

Conjecture 12.1 Let (X, S, M) be a fixed point structure and (θ, η) $(\theta : Z \to \to (O, \leqslant))$ a compatible pair with (X, S, M). Let $f \in M(X)$. We suppose that

(i) $A \in Z$, $x \in X$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$,

(ii) f is θ—condensing,

(iii) there exists $m \in \mathbb{N}^*$ such that $f^m(X) \in Z$.

Then $F_f \neq \emptyset$, and if $F_f \in Z$, then $\theta(F_f) = 0$.

Remark 12.1 For $(X, S. M)$ as in Example 2.1, see Rus [28].

Remark 12.2. For (X, S, M) as in Example 2.3, see Browder [7], Eells-Fournier [11], Nussbaum [22], Rus [35], ...

Conjecture 12.2 Let (X, S, M) be a fixed point structure and (θ, η) a compatible pair with (X, S, M). Let $f \in M(X)$ be such that

(i) $A \in Z$, $x \in X$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$,

(ii) there exists $m \in \mathbb{N}^*$ such that $f^m(X) \in Z$ and f^m is θ—condensing.

Then $F_f \neq \emptyset$ and if $F_f \in Z$, then $\theta(F_f) = 0$.

Remark 12.3. For (X, S, M) as in Example 23 see Browder [7], Eells-Fournier [11], Nussbaum [22], Rus [35], ...

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A GENERALIZATION OF PEETRE—RUS THEOREM

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REZUMAT. — O generalizare a teoremei lui Peetre-Rus. Scopul acestei lucrări este demonstrarea unor teoreme de coincidenţă pentru aplicaţii multivoce, din care desprindem ca şi consecinţe, generalizări ale unor rezultate date în [2], [3] şi [4].

1. **Introduction.** The purpose of this paper is to prove a coincidence theorem, similar to Peetre—Rus theorem, for pM-proximate multivalued mappings. Then we obtain some results, that generalize theorems from [2], [3] and [4], by relaxing the continuitiy.

Let (X, d) and (Y, ρ) two metric spaces. Let A, B be two nonempty subsets of X. Let $D(A, B) := \{\inf d(x, y) \mid x \in A \text{ and } y \in B\}$. A multivalued mapping $F: X —\circ Y$ is called upper semicontinuous if for each $a \in X$ and for each $\varepsilon > 0$ there is a neighborhood V of a such that $x \in V$ and $y \in F(x)$ imply $D(F(a), y) < \varepsilon$.

DEFINITION 1. ([2]). Let φ, ψ be two functions of $\mathbf{R}_+$ into itself. We say that ψ is *φ-summable* if for each $t \in \mathbf{R}_+$, the sequence $\{\varphi^n(t)\}_{n \in N}$ converges to 0 and the sequence $\left\{\sum_{i=1}^{n} \psi(\varphi^i(t))\right\}_{n \in \mathbf{N}}$ is convergent.

DEFINITION 2. Two multivalued mappings F, G of X into Y are said to be *pM-proximate* if there exist increasing functions φ, ψ of $\mathbf{R}_+$ into itself and $M > 0$ satisfying the following conditions:

1°. ψ is φ-summable;

2°. there exists $x \in X$ such that $D(F(x), G(x)) \leqslant M$;

3°. there exists a mapping $p: X \to X$ such that $d(x, p(x)) \leqslant \psi(M)$ and $D(F(p(x)), G(p(x)))x \leqslant \varphi(M)$, for every $x \in X$.

2. **Basic results.** We begin with the following lemma.

LEMMA 1. *If F and G are pM-proximate multivalued mappings of a complete metric space (X, d) into a metric space (Y, ρ), then there is a convergent sequence $\{x_n\}_{n \in \mathbf{N}}$ in X such that*

$D(F(x_n), G(x_n)) \to 0$, *as* $n \to \infty$.

Proof. There are increasing functions φ, ψ of $\mathbf{R}_+$ into itself satisfying the conditions 1° and 3°. From 2° it results that there exists $x_0 \in X$ with the property

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$D(F(x_0), G(x_0)) \leqslant M$. From 3° we have that there exists a mapping $p: X \to X$ such that:

$$d(x_0, p(x_0)) \leqslant \psi(M) \quad \text{and}$$

$$D(F(p(x_0)), G(p(x_0))) \leqslant \varphi(M)$$

We denote $p(x_0) = x_1 \in X$. For x_1, using 2° and 3° it follows that:

$$d(x_1, p(x_1)) \leqslant \psi(\varphi(M)) \quad \text{and}$$

$$D(F(p(x_1)), G(p(x_1))) \leqslant \varphi^2(M)$$

We denote $p(x_1) = x_2 \in X$. Thus, we obtain the sequence $(x_n)_{n \in \mathbf{N}} \subset X$ with the following property:

$$x_n = p^n(x_0), \; n \geqslant 1 \qquad (\alpha)$$

$$D(F(x_n), G(x_n)) \leqslant \varphi^n(M) \qquad (\beta)$$

$$d(x_n, x_{n+1}) \leqslant \psi(\varphi^n(M)) \qquad (\gamma)$$

From (γ), $(x_n)_{n \in \mathbf{N}}$ is a Cauchy sequence, hence $(x_n)_{n \in \mathbf{N}}$ convreges to $a \in X$. From (β) we have $D(F(x_n), (G(x_n)) \to 0$, as $n \to \infty$. The proof is complete.

THEOREM 1. *Let F, G be pM-proximate multivalued mappings of a complete metric space (X, d) into a metric space (Y, ρ). If G is upper semicontinuous then there exists an $a \in X$ and a sequence $(x_n)_{n \in \mathbf{N}}$ in X such that $x_n \to a$ and $D(F(x_n), G(a)) \to 0$, as $n \to \infty$.*

Proof. By the lemma there is a convergent sequence $(x_n)_{n \in \mathbf{N}}$ in X such that $D(F(x_n), G(x_n)) \to 0$, as $n \to \infty$.

Let a be the element of X to which $\{x_n\}_{n \in \mathbf{N}}$ converges and $\varepsilon > 0$. Then the upper semicontinuous of G guarantees the existence of a neighborhood V of a such that $x \in V$ and $y \in G(x)$ imply $D(G(a), y) < \frac{\varepsilon}{2}$. Since $x_n \to a$ and $D(F(x_n), G(x_n)) \to 0$, as $n \to \infty$, we can find an $m \in \mathbf{N}$ such that $x_n \in V$ and $D(F(x_n), G(x_n)) < \frac{\varepsilon}{2}$ for every $n \geqslant m$. Hence, if $n \geqslant m$ we have $\rho(y, y') < \frac{\varepsilon}{2}$ for some $y \in F(x_n)$ and $y' \in G(x_n)$. On the other hand, since x_n is in V, we have $\rho(y', y'') < \frac{\varepsilon}{2}$ for some $y'' \in G(a)$ and consequently we obtain $\rho(y, y'') \leqslant \rho(y, y') + \rho(y', y'') < \varepsilon$, which implies $D(F(x_n), G(a)) < \varepsilon$.

THEOREM 2. *Let F, G be pM-proximate multivalued mappings of a complete metric space (X, d) into a metric space (Y, ρ). If G is upper semicontinuous and $G(x)$ is compact for each $x \in X$, then there exist $a \in X$, $b \in G(a)$, a sequence $(x_n)_{n \in \mathbf{N}}$ in X and a sequence $(y_n)_{n \in \mathbf{N}}$ in Y such that $x_n \to a$, $y_n \to b$ and $y_n \in F(x))$ for every $n \in \mathbf{N}$.*
If in addition, the graph of F is closed then $F(a) \cap G(a) \neq \emptyset$ for some $a \in X$.

Proof. It follows from Theorem 1 that there are $a \in X$ and a sequence $(x_n)_{n \in \mathbf{N}}$ in X such that $x_n \to a$ and $D(F(x_n), G(a)) \to 0$ as $n \to \infty$. Hence we can find a mapping k of $\mathbf{N}$ into itself such that $k(n) \geqslant n$ and $D(F(x\quad), G(a)) < \frac{1}{n}$ Consequently the set $T(n) = \left\{(y, y') \in F(x_{k(n)}) \times G(a) \mid \rho(y, y') < \frac{1}{n}\right\}$ is nonempty for each $n \in \mathbf{N}$.

Let s be a choise function for the family $\{T(n) \mid n \in \mathbf{N}\}$ and consider the projections p, q defined by $p(y, y') = y$ and $q(y, y') = y'$, for every $(y, y') \in Y \times Y$. Then since $G(a)$ is compact, the sequence $\{q(s(n))\}_{n \in \mathbf{N}}$ in $G(a)$ has a subsequence $\{q(s(n_i))\}_{n \in \mathbf{N}}$ which converges to some $b \in G(a)$. Hence for every $\varepsilon > 0$, there exists an $m \in \mathbf{N}$ such that $\rho(q(s(n_i)), b) < \frac{\varepsilon}{2}$ and $\frac{1}{n_i} < \frac{\varepsilon}{2}$ for all $i \geqslant m$.

This shows that $p(s(n_i)) \to b$ as $i \to \infty$, since $\rho(p(s(n_i)), b) \leqslant \rho(p(s(n_i)), q(s(n_i))) + \rho(q(s(n_i)), b)$. Therefore the sequence $(x_{k(ni)})_{i \in \mathbf{N}}$ and $(p(s(n_i)))_{i \in \mathbf{N}}$ satisfy the required conditions.

Now we shall turn to prove the second part of our theorem. By the first part, we see that there are $a \in X$, $b \in G(a)$, a sequence $(x_n)_{n \in \mathbf{N}}$ in X and a sequence $(y_n)_{n \in \mathbf{N}}$ in Y such that $x_n \to a$, $y_n \to b$ and $y_n \in F(x_n)$ for every $n \in \mathbf{N}$.

Since the graph of F is closed and the sequence $\{(x_n, y_n)\}_{n \in \mathbf{N}}$ in $X \times Y$ converges to (a, b) under the product topology on $X \times Y$, (a, b) belongs to the graph of F, and so we have $b \in F(a) \cap G(a)$, proving the teorem.

THEOREM 3. *If F and G are pM-proximate upper semicontinuous multivalued mappings of a complete metric space (X, d) into a metric space (Y, ρ), then $D(F(a), G(a)) = 0$ for some $a \in X$.*

Proof. By Theorem 1, there are $a \in X$ and a sequence $(x_n)_{n \in \mathbf{N}}$ in X such that $x_n \to a$ and $D(F(x_n), G(a)) \to 0$ as $n \to \infty$. Let $\varepsilon > 0$. Then because of the upper semicontinuity of F we can find a neighborhood V of a such that $x \in V$ and $y \in F(x)$ imply $D(F(a), y) < \frac{\varepsilon}{2}$.

Hence there is a $n \in \mathbf{N}$ such that $x_n \in V$ and $D(F(x_n, G(a)) < \frac{\varepsilon}{2}$. Consequently, we have $\rho(y', y'') < \frac{\varepsilon}{2}$, for some $y' \in F(x_n)$ and $y'' \in G(a)$. On the other hand, since x_n is in V, we have $\rho(y, y') < \frac{\varepsilon}{2}$ for some $y \in F(a)$. It follows that $\rho(y, y'') < \varepsilon$ or $D(F(a), G(a)) < \varepsilon$. The proof is complete.

3. **Consequences.** The following theorem plays an important role in what follows.

THEOREM 4. *Let $f_1, f_2, \ldots, f_m, g_1, g_2, \ldots, g_m$ be mappings of a complete metric space (X, d) into a metric space (Y, ρ). Suppose that there exist increasing functions φ, ψ of $\mathbf{R}_+$ into itself and $M > 0$ satisfying the following conditions:*

(i) *ψ is φ-summable;*

(ii) *there exists $x \in X$ such that*

$$\sum_1^m \rho(f_i(x), g_i(x)) \leqslant M;$$

(iii) *there exists a mapping* $p : X \to X$ *such that*

$d(x, p(x)) \leqslant \psi(M)$ *and*

$$\sum_{i=1}^{m} \rho(f_i(p(x)), g_i(p(x))) \leqslant \varphi(M), \text{ for every } x \in X.$$

Then the following statements hold.

1°. *If* $g_1, \ldots, g_m$ *are continuous, then there exist an* $a \in X$ *and a sequen* $(x_n)_{n\in\mathbb{N}}$ *in* X *such that* $x_n \to a$ *and* $f_i(x_n) \to g_i(a)$ *for every* $i \in \{1, \ldots, m\}$.

2°. *If the graph of each,* f_i, $i \in \{1, \ldots, m\}$ *is closed and* g_i, $i \in \{1, \ldots, m$ *are continuous, then there exists an* $a \in X$ *such that* $f_i(a) = g_i(a)$ *for every* $i \in$ $\in \{1, \ldots, m\}$.

Proof. Consider the metric space (Y^m, ρ') with the metric ρ' defined b $\rho'((y_1, \ldots, y_m), (y'_1, \ldots, y'_m)) = \sum_{i=1}^{m} \rho(y_i, y'_i)$ for every $(y_1, \ldots, y_m)$, $(y'_1, \ldots$ $y'_m) \in Y^m$ and define two multivalued mapings F, G of X into Y^m by $F(x) = \{(f_1(x), \ldots, f_m(x))\}$ and $G(x) = \{(g_1(x), \ldots, g_m(x))\}$ for every $x \in X$

Then since $D'(F(x), G(x)) = \sum_{i=1}^{m} \rho(f_i(x), g_i(x))$ for every $x \in X$, the hypothes of the theorem shows that F and G are pM-proximate. Now, if $a \in X$ an $\varepsilon > 0$, then there is a neighborhood V of a such that: $\rho(g_i(a), g_i(x)) < \frac{\varepsilon}{m}$ fo every $x \in V$ and for every $i \in \{1, \ldots, m\}$ and so

$D'(G(a), G(x)) = \sum_{i=1}^{m} \rho(g_i(a), g_i(x)) < \varepsilon$ for all $x \in X$. Thus G us upper semi- continuous.

On the other hand, if the graph of each f, $i \in \{1, \ldots, m\}$ is closed, the graph of F is also closed. Therefore the first part of our theorem follows from Theorem 1 and the second part from Theorem 2. The proof is complete.

Theorem 4 is a generalization of Peetre—Rus theorem as it follows from the following result:

THEOREM 5. *Let* f, g *be two mappings of a complete metric space* (X, d) *into a metric space* (Y, ρ). *Suppose that there exist two increasing functions* φ, ψ *of* $\mathbb{R}^+$ *into itself and* $M > 0$, *satisfying the following conditions*

(i) ψ *is* φ-*summable*

(ii) *there exists* $x \in X$ *such that* $\rho(f(x), g(x)) \leqslant M$

(iii) *there exists a mapping* $p : X \to X$ *such that for every* $x \in X$ *we hav*

$d(x, p(x)) \leqslant \psi(M)$ *and* $\rho(f(p(x)), g(p(x))) \leqslant \varphi(M)$

(iv) g *is continuous and the graph of* f *is closed.*

Then there exists $a \in X$ *such that* $f(a) = g(a)$.

Proof. In Theorem 4 we put $m = 1$, $f_1 = f$, $g_1 = g$.

From Theorem 5, we have the following surjectivity theorem for a mapping not necessarely continuous, which extends Theorem 6 of [2] and Theorem 6 of [3].

THEOREM 6. *Let f be a mapping of a complete metric space (X, d) into a metric space (Y, ρ). Suppose that for any $y \in Y$ there exist two increasing functions φ, ψ of $\mathbf{R}_+$ into itself and $M > 0$, satisfying the following conditions:*

(i) *ψ is φ-summable;*

(ii) *there exists $x \in X$ such that $\rho(f(x), y) \leqslant M$;*

(iii) *there exists a mapping $p : X \to X$ such that for any $x \in X$ we have*

$$d(x, p(x)) \leqslant \psi(M) \quad and \quad \rho(f(p(x)), y) \leqslant \varphi(M);$$

(iv) *the graph of f is closed.*

Then f is surjective.

Proof. It suffices to let in Theorem 5 $g(x) = y$ for every $x \in X$.

From Theorem 5, it follows also a fixed point theorem.

THEOREM 7. *Let f be a mapping of a complete metric space (X, d) into itself and φ, ψ two increasing functions of $\mathbf{R}_+$ into itself. Suppose that there exists $M > 0$ such that the following conditions are satisfied:*

(i) *ψ is φ-summable;*

(ii) *there exists $x \in X$ such that $d(f(x), x) \leqslant M$;*

(iii) *there exists a mapping $p : X \to X$ such that:*

$$d(x, p(x)) \leqslant \psi(M) \quad and \quad d(f(p(x)), x) \leqslant \varphi(M);$$

(iv) *the graph of f is closed.*

Then f has a fixed point.

Proof. In Theorem 5 we put $g(x) = x$, for each $x \in X$.

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ERROR ESTIMATES IN THE APPROXIMATION OF THE FIXED POINTS FOR A CLASS OF φ-CONTRACTIONS

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REZUMAT. — Estimări ale erorii în aproximarea punctelor fixe pentru o clasă de φ-contracţii. Lucrarea prezintă o clasă de φ-contracţii, cu φ funcţie de comparaţie care verifică condiţia de convergenţă (c), pentru care estimarea erorii de aproximare a punctului fix prin metoda aproximaţiilor succesive este dată de aceeaşi formulă ca şi cea din teorema de contracţie a lui Banach ([3]).

1. **Introduction.** The paper shows that, for the class of φ-contractions with φ a comparison function which satisfies a convenient convergence condition (c): There exist the numbers k_0 and α, $0 < \alpha < 1$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} a_k$ such that

$$\varphi^{k+1}(r) \leqslant \alpha[\varphi^k(r) + a_k], \text{ for each } k \geqslant k_0 \text{ and } r \in \mathbf{R}_+,$$

the estimation of aproximation error of the fixed point by means of the succesive approximations method is given by the same formulas as in the Banach's contractions theorem ([3]).

To this end, we make use of a generalization of the ratio (or D'Alembert's) test for the series of positive terms, established in [1].

Let (X, d) be a complete metric space, $f: X \to X$ a mapping and $\varphi: \mathbf{R}^+ \to \mathbf{R}^+$ a monotone increasing comparison function, such as:

$$d(f(x), f(y)) \leqslant \varphi(d(x, y)), \text{ for all } x, y \in X. \tag{1}$$

We construct the sequence of succesive approximations, $(x_n)_{n \in N}, x_n = f(x_{n-1})$ $n \geqslant 1$ and $x_0 \in X$, and we obtain from (1), using the montonicity of φ, that, (see [3], p. 80).

$$d(x_n, x_{n+p}) \leqslant \sum_{k=n}^{n+p-1} \varphi^k(d(x_0, x_1)) \text{ for each } p \geqslant 1, \quad n \in \mathbf{N}^*. \tag{2}$$

If the series of positive terms

$$\sum_{k=1}^{\infty} \varphi^k(r), \tag{3}$$

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converges, for every $r \in \mathbf{R}_+$, then the sequence $(x_n)_{n \in \mathrm{N}}$ is a Cauchy sequence, hence $(x_n)_{n \in \mathrm{N}}$ is convergent for all $x_0 \in X$.

The main purpose of this paper is to prove that, if φ fulfils the condition (c), then the series (3) converges for all $r \in \mathbf{R}^+$ and, consequently, we have the estimation (5).

The sequence $(x_n)_{n \in \mathrm{N}}$ is a Cauchy sequence even when φ satisfies more weaker conditions (see [4], Theorem 3.3.1) which are, generally, insufficent to assure the convergence of the series (3).

However, in this case, in order to evaluate the approximation error, we need some additional hypotheses (see [4], Remark 3.3.1).

2. **A necessary and sufficient test for the convergence of the series of decreasing positive terms.** In [1] has been given the following generalization of the ratio test.

THEOREM 1. *Let* $\sum_{n=1}^{\infty} u_n$ *be an infinite series of positive terms. If there exists convergent series of nonnegative terms* $\sum_{n=1}^{\infty} v_n$ *and two numbers* k, n_0, *such as*

$$\frac{u_{n+1}}{u_n + v_n} \leqslant k < 1, \ for \ n \geqslant n_0 \tag{4}$$

then the series $\sum_{n=1}^{\infty} u_n$ *is convergent.*

Remarks. 2. Recall that a series is of positive (nonnegative) terms if all its terms are strictly positive (respectively positive, and an infinity of them may be equal to zero).

3. The Theorem 1 applies in some typical situations when the ratio test fails (see [1]). Obviously, the ratio test is obtained from Theorem 1, for $v_n = 0$, $n \in \mathbf{N}$.

In what follows we give a short new proof of Theorem 1:

From (4) we obtain, by an elementary calculation,

$$u_n \leqslant u_1 k^{n-1} + (v_1 k^{n-1} + v_2 k^{n-2} + \ldots + v_{n-1} k).$$

In view of Martens's theorem ([2]), to prove Theorem 1, it suffices to observe that Σk^n and Σv_n are both absolutely convergent.

For the series of decreasing positive terms, we can prove the converse of theorem 1. Thus, we obtain:

THEOREM 2. *A series* $\sum_{n=1}^{\infty} u_n$ *of decreasing positive terms converges if and only if there exists a convergent series of nonnegative terms* $\sum_{n=1}^{\infty} v_n$, *and two numbers* k, n_0, *such as the condition* (4) *is satisfied.*

Proof. The sufficiency follows from Theorem 1. To prove the necessity is enough to take $v_n = a u_n$, with $a > 0$.

Remark 4. Throughout this paper we shall consider series of decreas positive terms, because, for any comparison function, $\varphi(t) < t$ implies $\varphi^{k+1}(t) \leqslant \varphi^k(t)$.

3. **The error evaluation.** For the definitions and basic properties conc ning comparison functions we refer to [4].

DEFINITION 1. ([4]) A monotone increasing function $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ wh satisfies the condition

(i) $(\varphi^n(r))_{n\in\mathbf{N}}$ converges to 0, as $n \to \infty$, for any $r \in \mathbf{R}_+$, is called *comparis function.*

DEFINITION 2. ([4]) Let (X, d) be a metric space. A mapping $f : X \to$ is called φ-*contraction* if and only if there exists a comparison function φ so th (1) is fulfilled.

DEFINITION 3. A monotone increasing function $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ which s tisfies the condition (c) is called *(c)-comparison function.*

Every (c)-comparison function is a comparison function.

Proof. Applying Theorem 1, we deduce that series (3) converges for a $r \in \mathbf{R}^+$, hence $\varphi^k(t)$ tends to zero as k tends to infinity, which proves lemm

Remark 3. It is not quite obvious that every φ-contraction is a continuo mapping. The following property of a comparison function.

$$\varphi(t) < t, \text{ for each } t < 0$$

(see [4], lemma 3.1.3) suggests a way to remove any doubts. The main resu of this paper is given by

THEOREM 3. *Let (X, d) be a complete metric space and $f : X \to X$ a φ-contra tion with φ (c)-comparison function.*

Then

1) $F_f = \{x^*\}$;

2) *The sequence $(x_n)_{n\in\mathbf{N}}$, $x_n = f(x_{n-1})$, $n \geqslant 1$, $x_0 \in X$ converges to x^*, for an* $x_0 \in X$;

3) *We have*

$$d(x_n, x^*) \leqslant s(d(x_0, x_1)) - S_{n-1}(d(x_0, x_1)), \tag{5}$$

where $s(r)$, $S_{n-1}(r)$ denote the sum, respectively the partial sum of rank $n - 1$ of th series (3);

4) *If, in addition, φ is subadditive and there exists a mapping $g : X \to X$ and $\eta > 0$ such as*

$$d(f(x), g(x)) \leqslant \eta, \text{ for any } x \in X,$$

then

$$d(y_n, x^*) \leqslant \eta + s(\eta) + s(d(x , x_1)) - S_{n-1}(d(x_0, x_1)), \tag{6}$$

where $y_n = g(x)$.

Proof. Since φ is (c)-comparison function, the convergence of $(x_n)_{n\in\mathbf{N}}$ follow immediately from (2). Let x^* be its limit.

Then, the continuity of f, leads to $x^* = f(x^*)$ and x^* is the unique fixed point of f. Thus 1) and 2) are proved.

Also, (5) follows from (2), latting $p \to \infty$.

Finally, we have

$$d(y_n, x^*) \leqslant d(y_n, x_n) + d(x_n, x^*)$$

and

$$d(y_n, x_n) \leqslant \eta + \varphi(d(y_{n-1}, x_{n-1})) \leqslant \ldots \leqslant \eta + \varphi(\eta) + \ldots + \varphi^n(\eta).$$

Since $S_n(\eta) \leqslant s(\eta)$, the preceding inequalities together with (5), give the required estimation (6).

Remarks. 5. If $\varphi(t) = at$, $0 < a < 1$, then, from Theorem 3 we obtain Theorem 3.2.1 [2];

6. Theorem 2 shows that the estimation (5) holds if and only if φ is a (c)-comparison function.

7. Finally, let us observe that $s(r) - S_{n-1}(r)$ is the remainder of rank n of the series (3). By (5) we deduce that $(x_n)_{n \in \mathbf{N}}$ converges to x^* no more quickly than the series (3) to s.

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ON SOME INTERPOLATION PROCEDURE OF SCATTERED D

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REZUMAT. — Asupra unui procedeu de interpolare a unor date neregulat În lucrare se consideră problema interpolării unei funcţii de două variabile definită pe un domeniu plan oarecare avînd informaţii despre funcţie pe o reţea neregulată de puncte.

0. Let D be a domain in $\mathbf{R}^2$ and f a real-valued function on D. that there are given the values f_i, $f_i = f(x_i, y_i)$ of f at a set of po $y_i)$, $i = 0, 1, \ldots, N$ located in D.

One considers the following fitting data problem: find a function g on D, possible from a set of functions (say A), which interpolates f at i.e. $g(x_i, y_i) = f(x_i, y_i)$, $i = 0, 1, \ldots, N$.

This problem is largely trated when D is a rectangle and the dat lie on a rectangular grid. If the given points are (x_i, y_i), $i = 0, 1,$ $j = 0, 1, \ldots, n$ then the usual solution in this rectangular case is t product of the univariate Lagrange operators L_m^x and L_n^y correspo the nodes x_i, $i = 0, 1, \ldots, m$ respectively y_j, $j = 0, 1, \ldots, m$, i.e

$$(L_m^x \otimes L_n^y f)(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{u(x)}{(x - x_i)u'(x_i)} \frac{v(y)}{(y - y_j)v'(y_j)} \cdot f(x_i, y_j)$$

where $u(x) = (x - x_0) \cdot \ldots \cdot (x - x_m)$ and $v(y) = (y - y_0) \cdot \ldots \cdot (y - y_n$ have the tensor product interpolation formula:

$$f = L_m^x \otimes L_n^y f + R_m^x \oplus R_n^y f$$

with „$\oplus$" the boolean sum $(R_m^x \oplus R_n^y = R_m^x + R_n^y - R_m^x \otimes R_n^y)$.

When D is of unusual shape and when the data points are i scattered throghout D the problem becomes more dificult.

1. The natural way to look for a solution, in this general case, neralize the Lagrange's formula (1).

A first such generalization is given by J. F. Steffensen

$$f = P_1 f + R_1 f$$

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$$(P_1f)(x, y) = \sum_{i=0}^{m}\sum_{j=0}^{n_i} \frac{u(x)}{(x - x_i)u'(x_i)} \cdot \frac{v_i(y)}{(y - y_i)v_i'(y_j)} f(x_i, y_j)$$

:re $v(y) = (y - y_0)\cdot \ldots \cdot(y - y_{n_i})$ and:

$$(R_1f)(x, y) = u(x)[x, x_0, \ldots, x_m ; f(x, y)] +$$

$$+ \sum_{i=0}^{m} \frac{u(x)}{(x - x_i)u'(x_i)} [y, y_0, \ldots, y_{n_i} ; f(x_i,)].$$

If is obviously that for $n_0 = \ldots = n_m = n$ the Steffensen's formula (2) omes the tensor product formula (1).

Remark 1. The Steffensen's polynomial P_1f does not solve the given pro-m in the general case.

In 1957 D. D. Stancu [15] gave a new generalization of the tensor iduct procedure, which generalizes the Steffensen's method in the same ie, nomenly:

$$f = P_2f + R_2f \tag{3}$$

en:

$$(P_2f)(x, y) = \sum_{i=0}^{m}\sum_{j=0}^{ni} \frac{u(x)}{(x - x_i)u'(x_i)} \frac{v_i(y)}{(y - y_{ij})v_i'(y_{ij})} f(x_i,y_j)$$

h $v_i(y) = (y - y_{i0})\cdot \ldots \cdot(y - y_{i,n_i})$ and:

$$(R_2f)(x, y) = u(x)[x, x_0, \ldots, x_m ; f(\cdot, y)] +$$

$$+ \sum_{i=0}^{ni} \frac{u(x)}{(x - x_i)u'(x_i)} [y, y_{i0}, \ldots, y_{i,n_i} ; f(x_i, \cdot)].$$

Remark 2. The polynomial P_2f is a solution for the considered problem Indeed, let X, $X = \{(x_i; y_i) \mid i = 0, 1, \ldots, N\}$ be the set of the given ints and X_i, $X_i \subseteq X$, the set of all points $(x_k, y_k) \in X$ with $x_k = x_i$, $=0, 1, \ldots, n_i$. If the distinct abscisses are $x_0, x_1, \ldots, x_m$ $(x_0 < x_1 < \ldots < x_m)$ and $|X_i| = n_i + 1$ then we can write $X_i = \{(x_i, y_{ij}) \mid j = 0, 1, \ldots, n_i\}$ for $i = 0, 1, \ldots, m$. First, using the operator L_m^x, we have:

$$f = L_m^x f + R_m^x f$$

:h:

$$(L_m^x f)(x, y) = \sum_{i=0}^{m} \frac{u(x)}{(x - x_i)u_i'(x)} f(x_i, y)$$

w, if to each function $f(x_i, \cdot)$, $i = 0, 1, \ldots, m$ is applied the operator corresponding to the nodes y_{ij}, $j = 0, 1, \ldots, n_i$, one obtains the Stancu's erpolation formula (3).

Remark 3. If $L^y_{n_0} = \ldots = L^y_{n_m} = L^y_n$(i.e. $n_0 = \ldots n_m = n$ and y_{0j} $= \ldots = y_{mj} = y_j$ for $j = 0, 1, \ldots, n$) then $P_2 = L^x_n \otimes L^y_n$, hence P_2 is lization of the tensor product operator.

Next, formula (3) was generalized [16], taking instead of the L operators L^x_m and $L^y_{n_i}$, $i = 0, 1, \ldots, m$ some arbitrary linear oper and B^y_{ij}, $j = 0, 1, \ldots, n_i$.

2. An interesting procedure, which is largely used, for the consid blem was introduced by H. Shepard in 1964 [12]. The Shepard tor S is defined by the formula:

$$(S_0 f)(x, y) = \sum_{i=1}^{N} A_i(x, y) \cdot f(x_i, y_i)$$

where:

$$A_i(x, y) = \prod_{\substack{j=1 \\ j \neq i}}^{N} r_j^{\mu}(x, y) / (\sum_{k=1}^{N} \prod_{\substack{j=1 \\ j \neq k}}^{N} r_j^{\mu}(x, y)$$

or:

$$A_i(x, y) = 1/(\sum_{k=1}^{N} (r_i(x, y)/r_k(x, y))^{\mu})$$

with:

$$r_i(x, y) = [(x, x_i)^2 + (y - y_i)^2]^{1/2}$$

and $\mu \in \mathbf{R}_+$. The cardinality property of the functions A_i $(A_i(x_j, y_j)$ $i, j = 0, 1, \ldots, N)$ implies that $S_0 f$ interpolates the function f at t (x_i, y_i), $i = 0, 1, \ldots, N$.

The Shepard's formula was generalized in order to interpolate the values f_i of the given function f but also the values of certain of vates, $D^{p,q}f$, $p, q \in \mathbf{N}$, at (x_i, y_i), $i \in \{0, 1, \ldots, N\}$.. As an example a function is:

$$(S_m f)(x, y) = \sum_{i=0}^{m} A_i(x, y)(T_m f)(x, y; x_i, y_i)$$

where $(T_m f)(\cdot,\cdot, ; x_i, y_i)$ is the Taylor's polinomial of the total deg sociated to the function f and to the node (x_i, y_i). Thus, for $\mu >$

$$(D^{p,q} S_m f)(x_i, y_i) = (D^{p,q} f)(x_i, y_i), \; i = 0, 1, \ldots, N$$

for each (p, q) with $p + q \leqslant m$.

3. An extension of scattered data interpolation problem. Let $I_k f$ given information on the function f at (x_k, y_k), $k = 0, 1, \ldots, m$. Th information are the values of the function f and of certain of its at the points (x_k, y_k). The given information at (x_k, y_i) will be de $I_{ki} f$, while $I^x_{kj} f$ respectively $I^y_j f$ will be used to mark the partial in regard to x and y.

One considers the following interpolation problem : find a function g, $g \in A$, defined on D such that :

$$I_k g = I_k f,\ k = 0, 1, \ldots, M,\ M \geqslant N.$$

Remark 4. If $I_k f = f(x_k, y_k)$ one obtains the initial problem.

Now, let us consider once more, the partition X_i of X:

$$X_i = \{(x_i, y_{ij}) \mid j = 0, 1, \ldots, n_i\},\ i = 0, 1, \ldots, m.$$

One denotes by P_m^x the interpolation operator defined by:

$$(P_m^x f)(x, y) = \sum_{k=0}^{m} \varphi_k(x) I_k^x f \tag{5}$$

where φ_k are the cardinal interpolation functions ($I_j \varphi_k = \delta_{jk}$; $j, k = 0, 1, \ldots, m$) and by $P_{n_k}^y$ the operator defined by:

$$(P_{n_k}^y f)(x, y) = \sum_{j=0}^{n_k} \varphi_{kj}(y) I_{kj}^y f \tag{6}$$

with $\varphi_{kj}(y_{kq}) = \delta_{jq}$. Using the operators P_m^x and $P_{n_k}^y$, $k = 0, 1, \ldots, m$ we define the scattered data interpolation operator S_M:

$$(S_M f)(x, y) = \sum_{k=0}^{m} \sum_{j=0}^{nk} \varphi_k(x) \varphi_{kj}(y) I_{kj} f. \tag{7}$$

We also have the corresponding scattered data interpolation formula:

$$f = S_M f + R_M f \tag{8}$$

where $R_M f$ is the remainder term.

Remark 5. If $I_{kj} f = f(x_k, y_{kj})$, $k = 0, 1, \ldots, m$; $j = 0, 1, \ldots, n_k$, then from (8) one obtains the formula (1).

An important characteristic of an approximation operator is its degree of exactness.

THEOREM 1. *If the degree of exactness of the operators* P_m^x *and* $P_{n_k}^y$, $k = 0, 1, \ldots, m$ *are* r *respectively* $s_0, s_1, \ldots, s_m$ *then the degree of exactness of the operator* S_M *is* (r, s) *where*: $s = \min\{s_0, s_1, \ldots, s_m\}$.

Proof. Taking into account the linearity of the operator S_M it is sufficiently to test that $S_M e_{pq} = e_{pq}$ for all $p = 0, 1, \ldots r$ $q = 0, 1, \ldots, s$ and $S_M e_{r+1,v} \neq e_{r+1,v}$ or $S_M e_{\mu,s+1} \neq e_{\mu,s+1}$ for some $\mu \in \{0, 1, \ldots, r\}$ or $v \in \{0, 1, \ldots, s\}$ with $e_{pq}(x, y) = x^p y^q$. We have:

$$(S_M e_{pq})(x, y) = \sum_{k=0}^{m} \varphi_k(x) x_k^p \sum_{j=0}^{nk} \varphi_{kj}(y) y_{kj}^q.$$

But:

$$\sum_{k=0}^{nk} \varphi_k(y) y_{kj}^q = y^q,\ q = 0, 1, \ldots, s_k \text{ and} \sum_{j=0}^{nk} \varphi_{kj}(y) y_{kj}^{s+1} \neq y^{s+1}$$

(the exactness degree of $P^y_{n_k}$ is s_k) and:

$$\sum_{k=0}^{m} \varphi_k(x) x_k^p = x_k^p,\ p = 0, 1, \ldots, r \text{ and} \sum_{k=0}^{m} \varphi_k(x) x^{r+1} \neq x^{r+1}.$$

(the exactness degree of P^x_m is r) and the proof follows.

DEFINITION 1. If all the operators $P_{n_0}, \ldots, P_{n_m}$ have the same of exactness ($s_0 = s_1 = \ldots = s_m$) then the operator S_M is calles a *homog* with regard to y operator.

Remark 6. A tensor product operator is homogeneous with regard t variables.

It is obviously that, from the error of approximation point of v homogeneous operator is preferable. But generally it is not the case. It d on the distribution of the interpolation points (x_i, y_i), $i = 0, 1, \ldots$ D. So, having in mind the given informations on the function f, we are l for the situation in which the scattered data interpolation operator to closed as posible to the partial homogeneous case. To this end, let us d the operator S_M by $S^{x,y}_M$ which means that first it is applied to f the op P^x_m (with regard to x) and then $P^y_{n_k}$, $k = 0, 1, \ldots, m$ (with regard t So, on the same way, we can construct a symmetrical operator $S^{y,x}_M$ d by:

$$(S^{x,y}_M f)(x, y) = \sum_{j=0}^{n} \sum_{k=0}^{m_j} \varphi_j(y) \varphi_{jk}(x) I_{jk} f$$

where $(n + 1)$ is the number of all distinct ordinates y_j of the given p (x_i, y_i), $i = 0, 1, \ldots, N$, $(m_i + 1)$ is the number of the points with the nate y_j and $(x_{j0}, y_j), \ldots, (x_{j,m_j}, y_j)$ are the corresponding points. Sur $m_0 = m_1 = \ldots = m_n$ then $S^{y,x}_M$ defined by (9) is a homogeneous (with r to x) operator.

So, the operator $S^{x,y}_M$ or $S^{y,s}_M$ is selected taking into account the po of the points (x_i, y_i), $i = 0, 1, \ldots, N$ in the domain D. But for this tion we can also have into attention the dependence of the function f on each va le. Indeed, a function of several varables depends, in general, on each variable in a different way. For example, if f is a polynomial of a reaso degree in one of its variable, say x, then it is recomandable to use the op with a better approximation in the variable y.

Remark 7. An interesting scattered data interpolation operator is = $(S^{x,y}_M + S^{y,x}_M)/2$.

Remark 8. If the information $I_k f$ are the values of the function (x_k, y_k); $I_k f = f(x_k, y_k)$, $k = 0, 1, \ldots, N$, then the usual operators P^x_m $P^y_{n_i}$, $i = 0, 1, \ldots, m$ are the Lagrange's interpolation operators, polin spline operators or some rational interpolation operators.

Remark 9. If we are looking for a scattered data approximation operator based on given information, then the univariate operators can be taken as some linear positive operator (Bernsten's type operators, deminishing spline operator, etc.).

4. **Numerical results.** The propose of this section is to illustrate the above scattered data interpolation procedures on some concret examples. To this end we consider as a test function the following:

$$f(x, y) = -(x^4 + y^4 + x^2y^2 + 2x^2 + 2y^2)$$

with its graph is in Fig. 1, the information points as in the Fig. 2 and the information $I_k f$ be $(I_k f) = f(P_k)$, $k = 1, 2, \ldots 17$.

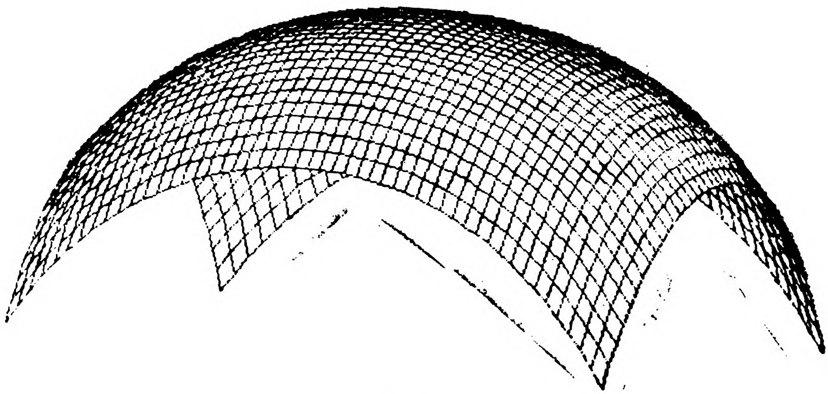


Fig. 1.

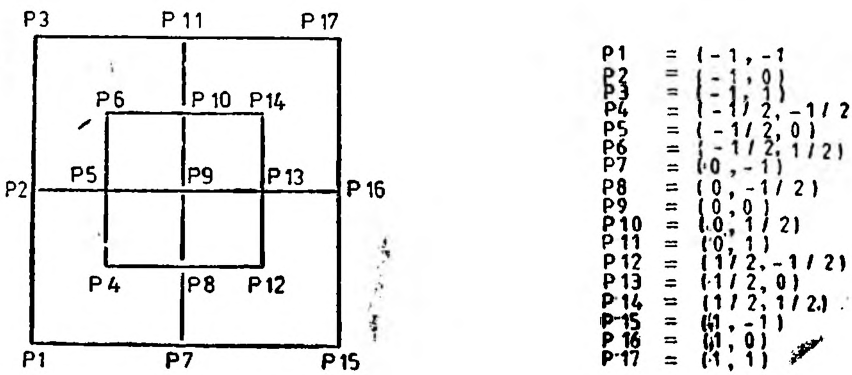


Fig. 2.

A. Let P_m^x be the cubic spline S_3^x corresponding to the nodes -1, -
0, 0. 5, 1 that will be denoted by S_3^x $(-1,\ -0.\ 5,\ 0,\ 0.5,\ 1)$, i.e.

$$(S_3^x f)(x) = \sum_{i=1}^{5} s_i(x) f(x_i,\ y)$$

with s_i the fundamental cubic spline and let $P_{n_i}^y$, $i = 1, \ldots, 5$ the
lowing operators: $P_{n_1}^y = P_{n_5}^y = L_2^y(-1,\ 0,\ 1)$, $P_{n_2}^y = P_{n_4}^y = L_2^y(-0.5,\ 0$
and $P_{n_3}^y = S_3^y(-1,\ -0.5,\ 0,\ 0.5,\ 1)$. One obtains the so called Spline-La
$(S_{SL}^{x,y} f)$ interpolation function with the graph in Fig. 3.

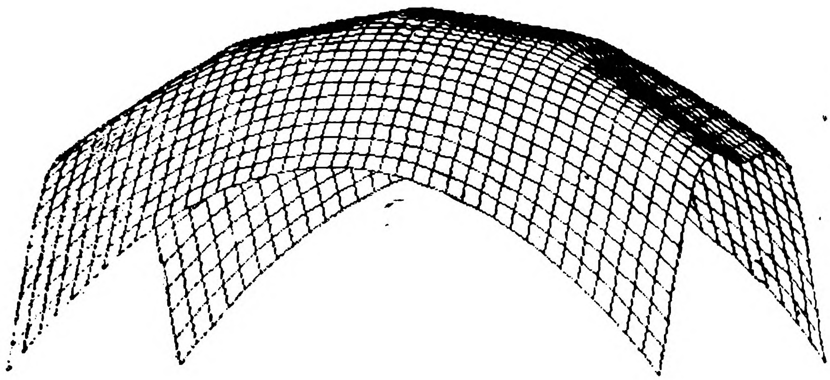
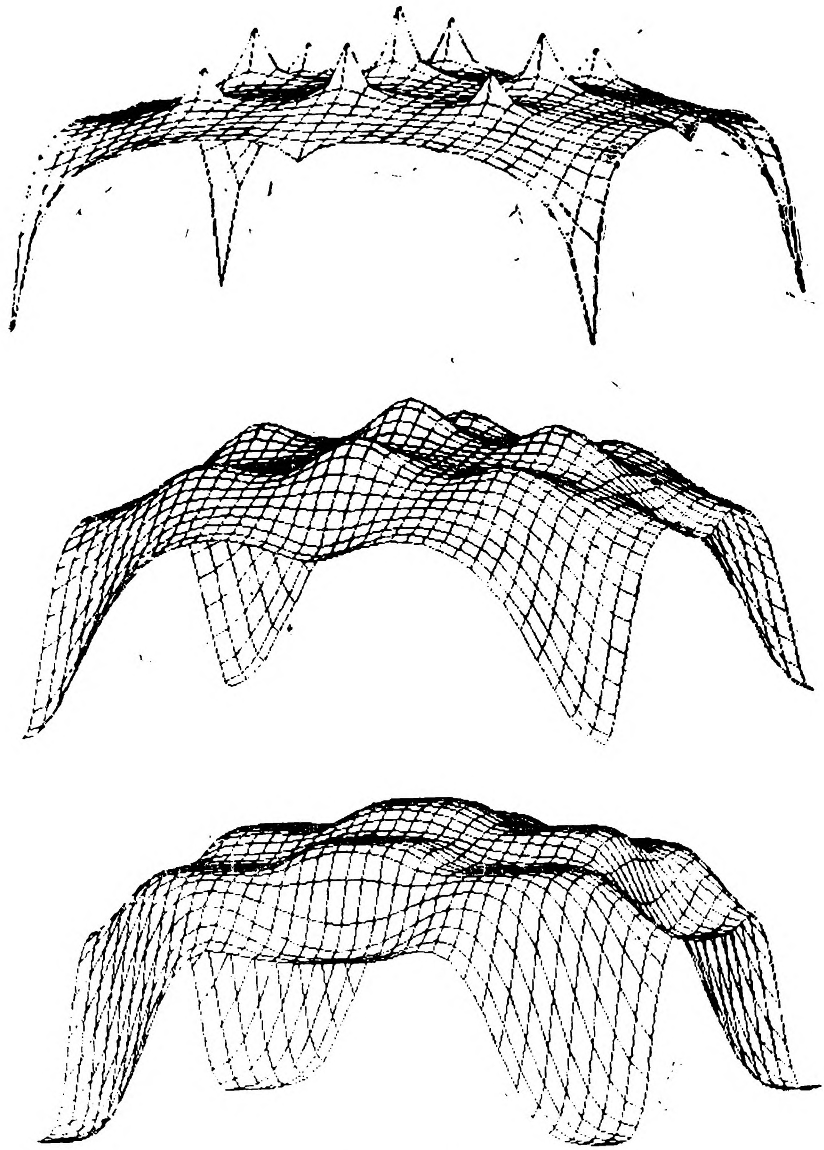


Fig. 3.

In the figs. 4—6, it is given the graph of Shepard's function S_μ
$\mu = 1,\ 2$ respectively 4.

As we can see in the given figurs, for the considered example, the sc
red data interpolation function $S_{SL}^{x,y} f$ (fig. 3) is a better approximatio
the function f (fig. 1) than the Shepard's function $S_0 f$ for each $\mu = 1$,
(fig. 4—6). Also, the computational complexity of the function $S_{SL}^{x,y} f$ is
sentially lower than the computational complexity of Shepard's functio



F i g. 4—6

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NOTE SUR L'ENTROPIE AUX POIDS ET LE PRINCIPE DU MAXIMUM DE L'INFORMATION

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REZUMAT. — Notă asupra entropiei şi a principiului de maxim al teoriei informaţiei. Lucrarea se ocupă cu determinarea distribuţiei continue ce maximizează entropia cu o anumită pondere. Considerîndu-se apoi diferite ponderi se obţin mai multe distribuţii probabilistice cunoscute.

1. Des possibilités pour déterminer ou retrouver des distributions probabilistiques, en utilisant le principe du maximum de l'information pour l'entropie aux poids, sont présentées dans cette note.

2. Soit X une variable aléatoire continue unidimensionnelle a la densité de distribution $f(x)$ sur $I \subseteq \mathbf{R}$, $f(x) > 0, \int_I f(x)\, dx = 1$. On suppose sconnu le principe du maximum de l'information et ses applications (voir [1]).

PROPOSITION. *La distribution continue qui maximise l'entropie aux poids* $u(x)$, $u: I \to \mathbf{R}^*$,

$$G(X, u) = -\int_I u(x)\, f(x)\, \ln f(x)\, dx, \tag{1}$$

à la condition

$$\int f(x)\, dx = 1,\ f(x) > 0, \tag{2}$$

est donnée par

$$f(x) = \exp\left(-\frac{\alpha}{u(x)}\right) \tag{3}$$

où α *est la solution unique de l'équation* (2) *pour la fonction* (3).

Preuve. En considérant la fonction de Lagrange

$$F(f(x), \alpha) = G(X, u) - \alpha \int_I f(x)\, dx = \int_I f(x) \ln \frac{e^{-\alpha}}{(f(x))^{u(x)}}\, dx$$

puisque $\ln x \leqslant x - 1$ avec égalité si et seulement si $x = 1$ on observe que

$$F(f(x), \alpha) \leqslant \int_I f(x)\left(\frac{e^{-\alpha}}{(f(x))^{u(x)}} - 1\right) dx$$

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avec égalité si et seulement si $(f(x))^{u(x)} = e^{-\alpha}$ et par conséquent on déd la relation (3) et l'affirmation de la proposition est vérifiée.

Par suite de cette proposition on peut déduire sans difficultés les résultats suivants :

CONSÉQUENCE 1. *Si* $I = (c, \infty)$ et $u(x) = (a+b\ \ln(x-c) + d(x-c)^p)^{-1}$, $x \in I$, $a, b, c \in \mathbf{R}$, $d, p > 0$, $\alpha b < 1$, *alors*

$$f(x) = p\left(\frac{x-c}{q}\right)^{-\alpha b} e^{-\left(\frac{x-c}{q}\right)^p} / q\Gamma\left(\frac{1-\alpha b}{p}\right) \qquad (4)$$

où $\alpha\ dq^p = 1$ *et* α *est la solution de l'équation* $q^{1-\alpha b}\ \Gamma\left(\frac{1-\alpha b}{p}\right) = pe^{\alpha a}$.

À la suite de la conséquence 1 on déduit :

1) Si a, b, c, d et p sont donnés de sorte que $\alpha = \frac{1-p}{b}$, alors on trouve la distribution de Weibull avec

$$f(x) = \frac{p}{q}\left(\frac{x-c}{q}\right)^{p-1} \exp\left(-\left(\frac{x-c}{q}\right)^p\right)$$

2) Si $c = 0$ et $p = 1$ tandis que a, b et d sont donnés de sorte que $\alpha = \frac{1-m}{b} = \frac{1}{dn}$, $m > 0$, $n > 0$, alors on trouve la distribution gamma généralisée avec

$$f(x) = x^{m-1}\ e^{-\frac{x}{n}} / n^m\Gamma(m)$$

d'où, pour $n = 1$, on a la distribution gamma simple.

3) Si $c = 0$ et $p = 1$ tandis que a, b, et d sont donnés de sorte que $\alpha = \frac{2-m}{2b} = \frac{1}{2\ dn}$, $m \in N$, $n > 0$, alors on trouve la distribution $\chi^2_m(n)$ avec

$$f(x) = x^{\frac{m}{2}-1}\ e^{-\frac{x}{2n}} / (2n)^{\frac{m}{2}}\ \Gamma\left(\frac{m}{2}\right)$$

4) Si $b = c = 0$ et $p = 1$ tandis que a et d sont donnés de sorte que $\alpha d = 1$ alors on trouve la distribution exponentielle avec

$$f(x) = ue^{-ux},\quad u = a/d$$

CONSÉQUENCE 2. *Si* $I = [a, b]$ *et* $u(x) = k$, *on a la distribution uniforme.*

CONSÉQUENCE 3. *Si* $I = \mathbf{R}$ *et* $u(x) = (a + d|x|)^{-1}$, $a > 0$, $d > 0$ *on déduit la distribution de Laplace.*

CONSÉQUENCE 4. *Si* $I = \mathbf{R}$ *et* $u(x) = (a + d(x-c)^2)^{-1}$ *tandis que a et* $d > 0$ *sont donnés de sorte que* $2\alpha d^3 = 1$, *alors on trouve la distribution normale avec*

$$f(x) = \frac{1}{d\sqrt{2\pi}} \exp\left\{-\frac{(x-c)^2}{2d^2}\right\}.$$

CONSÉQUENCE 5. *Si* $I = (0, \infty)$ *et* $u(x) = (a + b \ \ln x + c \ln(1 + dx^n))^{-1}$, $x \in I$, $a, b, c, d \in \mathbf{R}^*$, $n > 0$, $\alpha b < 1$, $\alpha b + \alpha nc > 1$, *alors*

$$f(x) = nx^{-\alpha b}(1+dx^n)^{-\alpha c}/d^{\frac{\alpha b-1}{n}} B\left(\frac{1-\alpha b}{n}, \frac{\alpha b + \alpha nc - 1}{n}\right) \qquad (5)$$

où α *est la solution de l'équation* $d^{\frac{\alpha b-1}{n}} c^{-\alpha a} B\left(\frac{1-\alpha b}{n}, \frac{\alpha b+\alpha nc-1}{n}\right) = 1.$

À la suite de la conséquence 5 on déduit :

1) Si $c \in \mathbf{N}$, $n \in \mathbf{N}$ tandis que a et b sont donnés de sorte que $\alpha = \frac{1-n}{b}$, alors on trouve la distribution de Burr de parametre d avec

$$f(x) = ndx^{n-1}(1+dx^n)^{-s}/B(1, \ s-1), \ s = \frac{c(1-n)}{b}$$

d'où, pour $d = 1$, on trouve la distribution de Burr simple

2) Si $n = 1$ et $b = 0$ tandis que a et c sont donnés de sorte que $\alpha c = 2$, alors on trouve la distribution rationnelle avec

$$f(x) = d(1 + dx)^{-2}$$

3) Si $n = 1$ et $d = \frac{n_1}{n_2}$, $n_1, n_2 \in \mathbf{N}^*$, tandis que a, b et c sont donnés de sorte que $\alpha = \frac{n_1+n_2}{2c}$, alors on trouve la distribution de Snedecor avec

$$f(x) = \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} x^{\frac{n_1}{2}-1}\left(1+\frac{n_1}{n_2}x\right)^{-\frac{n_1+n_2}{2}} / B\left(\frac{n_1}{2}, \frac{n_2}{2}\right).$$

CONSÉQUENCE 6. Si $I = \mathbf{R}$ *et* $u(x) = (a + b \ \ln(c + dx^2))^{-1}$, $x \in I$, $a, b \in \mathbf{R}$, $c > 0$, $d > 0$ *et* $\alpha b > 1/2$, *alors*

$$f(x) = \sqrt{d}\,\Gamma(\alpha b)\left(1 + x^2\frac{d}{c}\right)^{-\alpha b} / \sqrt{\pi c}\,\Gamma\left(\alpha b - \frac{1}{2}\right) \qquad (6)$$

où α *est solution de l'équation* $c^{\frac{1}{2}-\alpha b} e^{-\alpha a} B\left(\frac{1}{2}, \alpha b - \frac{1}{2}\right) = d.$

À la suite de la conséquence 6 on déduit :

1) Si $c = n \in \mathbf{N}$, $d = 1$ tandis que a et b sont donnés de sorte que $\alpha = \frac{n+1}{2b}$, alors on trouve la distribution de Student avec

$$f(x) = \Gamma\left(\frac{n+1}{2}\right)\left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}} / \sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)$$

2) Si $c = n = d = 1$ tandis que a et b sont donnée de sorte que $\alpha = \frac{1}{b}$, alors on trouve la distribution de Cauchy avec $f(x) = \frac{1}{\pi(1+x^2)}$.

CONSÉQUENCE 7. *Si* $I = (0, 1)$ *et* $u(x) = (a + b \ln x + c \ln(1$
$a, b, c \in \mathbf{R}$, $\alpha b < 1$, $\alpha c < 1$, *alors*

$$f(x) = x^{-\alpha b}(1-x)^{-\alpha c}/B(1-\alpha b,\ 1-\alpha c)$$

où α *est la solution de l'équation* $B(1-\alpha b,\ 1-\alpha c) = e^{\alpha a}$.

À la suite de la conséquence 7 on déduit que si a, b et c sont
de sorte que $\alpha = \frac{1-p}{b} = \frac{1-q}{c}$, $p > 0$, $q > 0$, alors on trouve la dist
beta avec

$$f(x) = x^{p-1}(1-x)^{q-1}/B(p,\ q)$$

De la même façon on peut trouver des autres distributions probab
a partir des relations (2) et (3), utilisables dans l'étude des systèmes c
ques, particulièrement des systémes de communications ou systèmes de
gies aléatoires.

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G. Schaar, M. Sonntag, H-M. Teihert, **Hamiltonian Properties of Products of iraphs and Digraphs,** Teubner-Texte zur Matheatik, Band 108, BSB Teubner, Leipzig, 1988, 48 pp.

In this book the authors present a survey the Hamiltonian properties of products of aphs and directed graphs. This properties of e two kind of graphs (undirected and directed) treated in the different parts. A general ict, called B-product, of graphs in defined, which the all well-known products (Car-D, lexicographic, normal product, disjunction, sian sum) can be derived. Beside the known s in the literature up to 1986, the book tains the research results of the authors o. At the end of the two parts 66 + 26 bibliographical references are listed.

Indices of definitions and notations make he book easy to use.

Z. KÁSA

Hervé, Michel, **Analyticity in Infinite Dimensional Spaces,** de Gruyter Studies in Mathematics 10, Walter de Gruyter, Berlin — New ork 1989, 206 pp.

In the last twenty years the theory of analyticity in infinite dimensions marked remarable achievements. The theory contains results imilar with those in finite dimensions, but are so striking differences. The analyticity in infinite imensions developed in two principal direcions: analytic functions with values in Banach paces and analytic functions with values in ocally convex spaces (LCS). The first direction s reflected in some books published mainly y North-Holland Editors (see e.g. J. Mujica, omplex analysis in Banach spaces). The present ook is devoted to the second directions, the ain unifying themes being the notions of anatic map with values in a sequentially complete .S and that of plurisubharmonicity, a notion nsidered first by P. Lelong (Séminaire Lelong 69, LNM vol. 116, Springer V.).

The book is divided into six chapters: Some topological preliminaries; 2 Gâteaux analyticity; 3. Analyticity, or Fréchet analyicity; 4. Plurisubharmonic functions; 5. Problems involving plurisubharmonic functions; 6. Analytic maps from a given domain to another one.

The book is very well and clearly written and brings together many results speaded in various journals or seminar lecture notes. It is an excellent monograph which can be used by a graduate student desiring to enter this very active and attractive field of research, as well by the specialists as a very good reference source.

S. COBZAŞ

Iosip E. Pečarić, **Convex Functions. Inequalities.** (Serbian), Naucna Knjiga, Beograd 1989, 243 pp.

The book is divided into three chapters. Chapter I. Convex Functions — is dealing with: convex functions of one variable, convex functions on normed spaces, Jensen-convex functions, Wright—convex functions, convex functions of higher order and convex functions with respect to a Chebyshev system.

In Chapter II. Convex Functions. General Inequalities, the method of convex functions is systematically applied to prove some classical inequalities and their recent refinements as inequalities of Jensen, Jensen-Steffensen, Hermite-Hadamard, Jensen — type inequalities for n-convex functions, inequalities of Popoviciu, Burkill, Vasic, Favard, Gauss-Wincler, Chebyshev, Grüssov, Young et al.

In the last chapter, Chapter III. Particular Inequalities, the general inequalities proved in the second chapter are applied to some concrete problems in analysis and probability theory.

The book is based on a large bibliography, including eight monographs and an almost complete list of research papers tracing the evoluation of the subject from its very beginning (the pioneering work of J.L.W.V. Jensen 1905) and up to 1987 (the list of references includes many papers only submitted for publication in 1987).

The book will be a very useful guide for all interested in this domain of investigation — convexity and inequalities — involded in almost all branches of analysis and other areas of mathematics.

V. MIHEŞAN

K. Schmüdgen, **Unbounded Operator Algebras and Representation Theory.** Akademie Verlag, Mathematische Monographien Bd. 77, Berlin 1990, 380 pp.

The book is devoted to the theory of *—algebras of unbounded operators in

Hilbert space (called O* — algebras) and to their * — representations. These algebras occur in a natural way in unitary representation theory of Lie groups and in the Wightman formulation of quantum field theory but they are also relevant for other disciplines as the theory of von Neumann algebras, distribution theory, non-commutative probability and non-commutative moment problem. Although some notions as that of weak bounded commutant appeared in quantum field theory early in the sixties, a sistematic study of O* — algebras began only in the seventies, by the efforts of H. J. Bochers, G. Lassner R. T. Powers, A. Uhlmann, A. N. Vasiliev and the author himself.

An O* — algebra is a * — algebra $\mathfrak{A}$ with unit, of linear operators defined on a common dense linear subspace $\mathfrak{D}$ of a Hilbert space and leaving $\mathfrak{D}$ invariant. The multiplication in $\mathfrak{A}$ is the composition of the operators and the involution a^+ is the restriction of the usual Hilbert space adjoint a^* to $\mathfrak{D}$.

The aim of the book is to provide an account of the present day situation of the theory of O* — algebras and their * — representations, with a special emphasis on the topological theory, which is more developed than other parts of the theory. Applications to physics are not included.

The book is divided into two parts. Part I. O* — Algebras and Topologies, develops the basics of the theory of O* — algebras and of the topologies on the domains and on the algebras. Here are included also some topics from the theory of * — representations, involving primarily the study of topologies or the structure of O* — algebras, such as the continuity of * — representations, the realization of the generalized Calkin algebra and the abstract characterization of O* — algebras. This par contains seven chapters headed as follows: 1. Preliminaries, 2. O* — Algebras and Topologies, 3. Spaces of Linear Mappings, 4. Topologies for O — families with metrizable Graph Topologies, 5. Ultraweakly Continuous Linear Functionals and Duality Theory, 6. The Generalized Calkin Algebra and * — Algebra $L^*(\mathfrak{D})$, 7. Commutants.

The representation theory is treated in Part II. * — Representations, which contains five chapters: 8. Basics of * — Representations, 9. Self-Adjoint Representations of Commutative * — Algebras, 10. Integrable Representations of Enveloping Algebras, 11. n-Positivity and Complete Positivity of * — Representations, 12. Integral Decompositions of * — Representations and States.

Each chapter ends with a section entitled Notes and containing references to the sources of main results and examples contained in th text and also to similar problems.

Beside the basic material the book contair also many examples and counter examples helj ing to delimit the general theory. Often tl original proofs have been improved some erro have been corrected and some results generaliz Also, sometimes the terminology and notati have been changed and several new concep were introduced.

Written by one of the founders of th theory, this excellent book can be used by new-comers for the introduction to the sub as well as a reference book by the specialis

S. COB7

Algorithms for aproximation. II. Edi by J. C. Mason and M. G. Cox. Chapma and Hall, London—New York, 1990, 514 p. ISBN 0—412—34580—3.

The papers included in this volume a based on the proceedings of the Second Inter tional Conference on Algorithms for Appr mation, held at Royal Military College of Scier Shrivenham, July 1988. The 41 papers inser in this volume have been arranged into th primary parts. Part One: Development of Alg rithms; Part Two: Aplications and Part Thr Catalogue of Algorithms. The first two pa have been subdivided into eight section (1) Spline approximation; (2) Polynomial a piecewise polynomial approximation; (3) Inte polation; (4) Smoothing and constrai thods; (5) Complex approximation; (6) puter-aiedd design and geometric mo (7) Applications in other disciplin papers presented by the invited speak vere a broad spectrum of the general approximation theory and numerical These were written by several famous m ticians from UK, USA and many other c We mention the following names: John John Mason, Michael Powell, Alastair Wolfgang Dahmen, Tom Lyche, Eric Larry, Schumaker, Lloyd Trefethen Barrodale.

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D. D. S

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