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THE THEORY AND APPLICATIONS OF SECOND-ORDER DIFFERENTIAL SUBORDINATIONS

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REZUMAT. — Teoria şi aplicaţiile subordonărilor diferenţiale de ordinul doi. În această lucrare se prezintă o sinteză a unor rezultate recente ale autorilor privind teoria generală a subordonărilor diferenţiale de ordinul doi, precum şi unele aplicaţii ale acesteia în teoria geometrică a funcţiilor analitice.

Fie Ω şi Δ două submulţimi ale planului complex **C**, fie p o funcţie analitică în discul unitate U şi fie $\Psi : \mathbf{C}^3 \times U \to \mathbf{C}$. Partea centrală a lucrării are ca scop rezolvarea următoarelor trei probleme privind implicaţia (4).

Problema 1. Dîndu-se Ω şi Δ, să se găsească condiţii asupra funcţiei Ψ, astfel ca (4) să aibă loc. O astfel de funcţie se numeşte funcţie admisibilă.

Problema 2. Dîndu-se Ψ şi Ω să se găsească Δ astfel ca (4) să aibă loc.

Problema 3. Dîndu-se Ψ şi Δ, să se găsească Ω astfel ca (4) să aibă loc.

Se dau soluţii ale acestor probleme în cazul cînd Δ este un domeniu simplu-conex, $\Delta \neq \mathbf{C}$. Dacă şi Ω este un domeniu simplu-conex, $\Omega \neq \mathbf{C}$, atunci implicaţia (4) se scrie sub forma (5), unde h şi q sînt reprezentări conforme ale discului unitate pe Ω şi respectiv pe Δ. Se consideră, în mod special, cazurile cînd Δ este un disc sau un semiplan.

Se prezintă unele aplicaţii ale teoriei generale la construirea unor operatori care conservă funcţiile cu partea reală pozitivă, operatori de mediere, operatori care conservă stelaritatea şi operatori care conservă subordonarea. De asemenea se aplică rezultatele generale la studierea subordonărilor diferenţiale de tip Briot-Bouquet, care prezintă un interes deosebit în teoria geometrică a funcţiilor analitice.

I. GENERAL THEORY

1. Introduction. In the field of differential equations of real-valued functions there are many examples of differential inequalities that have important applications in the general theory. In these cases, bounds on a function f are often determined from an inequality involving several of the derivatives of f. As

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a very simple, example, consider a function f which is twice continuously differentiable on $I = (-1, 1)$ and suppose that the differential operator $D[f](t) = [t^2f(t) + t^3]'' = t^2f''(t) + 4tf'(t) + 2f(t) + 6t$ satisfies

$$0 < D[f](t) < 2, \text{ for } t \in I. \tag{1}$$

It is easy to show that $-1 < f(t) < 2$, for $t \in I$. This result can be rewritten as

$$D[f](I) \subset (0, 2) \Rightarrow f(I) \subset (-1, 2). \tag{2}$$

In two articles in 1978 [13] and 1981 [14] the authors extended these ideas involving differential inequalities for real-valued functions to complex-valued functions. Since those two articles, the authors and many others have enriched this new field. In this survey article we will describe some of these new results and their applications. Although many of the concepts to be discussed also pertain to higher order differential expressions, we restrict our attention in this article to first and second-order differential expressions.

A differential inequality of the form (1) does not have a direct analog for complex-valued functions, i.e. we cannot merely replace the real-valued function $f(t)$ in (1) with a complex-valued function $f(z)$. However, the first inclusion relation of (2) does have a natural complex analog such as

$$D[f](U) \subset \Omega,$$

with $D[f](z) = z^2f''(z) + 4zf'(z) + 2f(z) + 6z$, where U, $\Omega \subset \mathbf{C}$ and U is the unit disk. If $f: U \to \mathbf{C}$ satisfies this inclusions, then analogously to (2) we can ask if there is a "smallest" set $\Delta \subset \mathbf{C}$ such that ?

$$D[f](U) \subset \Omega \Rightarrow f(U) \subset \Delta. \tag{3}$$

There are two other problems that are associated with (3). Given Ω and Δ, does there exist a class of functions satisfying (3). And secondly, given f and Δ, does there exist a "largest" set Ω satisfying (3). In these three problems, for this very elementary example, we see some of the ideas that have been used in developing the theory of differential subordinations. These problems will be generalized below and their solutions will be described in this article.

Let Ω and Δ be any sets in $\mathbf{C}$, let p be analytic in the unit disk U, with $p(0) = a$, and let $\psi(r, s, t; z): \mathbf{C}^3 \times U \to \mathbf{C}$. The heart of this article deals with the following implication

$$\{\psi(p(z), zp'(z), z^2p''(z); z) \mid z \in U\} \subset \Omega \Rightarrow p(U) \subset \Delta. \tag{4}$$

Note that (3) is of this form with $\psi(r, s, t; z) = t + 4s + 2r + 6z$. We can now state the three problems that characterize the theory of differential subordinations in the complex plane.

PROBLEM 1. Given Ω and Δ, find conditions on ψ so that (4) holds. We call such a ψ an *admissible function.*

PROBLEM 2. Given ψ and Ω, find Δ such that (4) holds. Furthermore, find the "smallest" such Δ.

PROBLEM 3. Given ψ and Δ, find Ω such that (4) holds. Furthermore, find the "largest" such Ω.

If either Ω or Δ in (4) is a simply connected domain then (4) can be rewritten in terms of subordination. Recall that if f and F are analytic in U and F is univalent in U then f is subordinate to F, written $f(z) \prec F(z)$ or $f \prec F$, if $f(0) = F(0)$ and $f(U) \subset F(U)$.

If Δ is a simply connected domain containing the point a and $\Delta \neq \mathbf{C}$, then there is a conformal mapping q of U onto Δ such that $q(0) = a$. In this case (4) can be rewritten as

$$\{\psi(p(z), zp'(z), z^2p''(z); z) \mid z \in U\} \subset \Omega \Rightarrow p(z) \prec q(z).$$

If Ω is also a simply connected domain and $\Omega \neq \mathbf{C}$, then there is a conformal mapping h of U onto Ω such that $h(0) = \psi(a, 0, 0; 0)$. If in addition $\Psi(p(z), zp'(z), z^2p''(z); z)$ is analytic in U then (4) can be rewritten as

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \Rightarrow p(z) \prec q(z). \tag{5}$$

This last result leads us to some of the important definitions that will be used throughout this article.

DEFINITION 1.1. Let $\Psi: \mathbf{C}^3 \times U \to \mathbf{C}$ and let h be univalent in U. If p is analytic in U and satisfies the *(second-order) differential subordination*

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{6}$$

then p is called a *solution of the differential subordination*. The univalent function q is called a *dominant of the solutions of the differential subordination*, or more simply a *dominant of the differential subordination* if $p \prec q$ for all p satisfying (6). A dominant $\tilde{q}$ which satisfies $\tilde{q} \prec q$ for all dominants q of (6) is said to be the *best dominant* of (6). (Note that the best dominant is unique up to a rotation of U).

Let Ω be a set in $\mathbf{C}$ and suppose (6) is replaced by

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega, \text{ for } z \in U. \tag{6'}$$

Although this is a "differential inclusion" and $\psi(p, zp', z^2p''; z)$ may not be analytic in U, we shall also refer to (6′) as a *(second-order) differential subordination*, and use the same definitions of solution, dominant and best dominant as given in Definition 1.

In the case when Ω and Δ in (4) are simply connected domains, we have seen that (4) can be rewritten in terms of subordinations such as given in (5). Using this and Definition 1 we can restate Problems 1—3 as follows:

PROBLEM 1′. Given univalent functions h and q, find a class of admissible functions $\Psi[h, q]$ such that (5) holds.

PROBLEM 2′. Given the differential subordination (6), find a dominant of (6). Moreover, find the best dominant.

PROBLEM 3′. Given ψ and dominant q, find the largest class of univalent functions h so that (5) holds.

Solutions to Problems 1, 1′, 2 and 2′ will be given in Section 3, while a solution for special cases of Problem 3 and 3′ will be given in Section 8.

We close this introduction by mentioning some special cases of the differential subordination (6) that appeared in the literature prior to our first two articles [13, 14]. These were formulated using different terminologies and their methods of proof were quite different from those that will be presented in this article.

In 1935 G. M. Goluzin [7] considered the simple first order differential subordination $zp'(z) \prec h(z)$. He showed that if h is convex then $p(z) \prec q(z) = \int_0^z h(t)t^{-1}\,dt$, and this q is the best dominant. In 1970 T. Suffridge [32, p. 777] showed that Goluzin's result is true if h is starlike.

In 1947 R. Robinson [30, p. 22] considered the differential subordination $p(z) + zp'(z) \prec h(z)$. He showed that if h and $q(z) = z^{-1}\int_0^z h(t)\,dt$ are univalent then q is the best dominant, at least for $|z| < 1/5$.

In 1975 D. Hallenbeck and S. Ruscheweyh [9, p. 192] considered the differential subordination $p(z) + zp'(z)/\gamma \prec h(z)$, when $\gamma \neq 0$ and $\operatorname{Re}\gamma \geq 0$. They showed that if h is convex, then $q(z) = \gamma z^{-\gamma}\int_0^z h(t)t^{\gamma-1}\,dt$ is the best dominant.

2. **Preliminary Lemmas.** In this section we list the main lemmas that will be needed to prove the theorems of the next section. Proofs will be omitted, but references will be indicated. For $z_0 = r_0 e^{i\theta_0}$ with $0 < r_0 < 1$, we let $U_{r_0} = \{z : |z| < r_0\}$.

LEMMA 2.1. [13, p. 290] *Let $f(z) = a_n z^n + a_{n+1}z^{n+1} + \ldots$ be continuous on $\bar{U}_{r_0}$ and analytic on $U_{r_0} \cup \{z_0\}$ with $f(z) \not\equiv 0$ and $n \geq 1$. If $|f(z_0)| = \max\{|f(z)| : z \in \bar{U}_{r_0}\}$ then there exists an $m \geqslant n$ such that*

(i) $z_0 f'(z_0)/f(z_0) = m$, *and*

(ii) $\operatorname{Re}[z_0 f''(z_0)/f'(z_0)] + 1 \geq m$.

This lemma is based on the fact that $f(U_{r_0})$ lies inside the disk $|w| \leq R = |f(z_0)|$ and that the boundary of $f(U_{r_0})$ is tangent to the circle $|w| = R$. A special version of part (i), with $z_0 = f(z_0) = 1$, appeared in 1925 as a problem of K. Löwner in [29, Problem 291, p. 141].

We need to extend the ideas in this lemma, by replacing the disk $|w| \leq R$ with a more general region Δ. We can achieve this by first introducing the following class of functions.

DEFINITION 2.2. We denote by Q the set of functions q that are analytic and injective on $\bar{U} \setminus E(q)$, where

$$E(q) = \{\zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$.

If $q \in Q$ the domain $\Delta = q(U)$ is simply connected and its boundary consists of either a simple closed regular curve or the union (possibly infinite) of pairwise disjoint simple regular curves, each of which converges to ∞ in both directions. The functions $q_1(z) = z$ and $q_2(z) = (1 + z)/(1 - z)$ are examples of these two cases.

LEMMA 2.3. [14, p. 158] *Let* $q \in Q$ *with* $q(0) = a$, *and let* $p(z) = a + p_n z^n + \ldots$ *be analytic in* U *with* $p(z) \not\equiv a$ *and* $n \geq 1$. *If there exist points* $z_0 \in U$ *and* $\zeta_0 \in \partial U \setminus E(q)$ *such that* $p(z_0) = q(\zeta_0)$ *and* $p(U_{r_0}) \subset q(U)$, *where* $r_0 = |z_0|$, *then there exists an* $m \geqslant n$ *such that*

(i) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$, *and*

(ii) $\mathrm{Re}[z_0 p''(z_0)/p'(z_0) + 1] \geq m\,\mathrm{Re}[\zeta_0 q''(\zeta_0)/q'(\zeta_0) + 1]$.

Note that (i) is a relation between the outer normals to the curves $p(|z| = r_0)$ and $q(\partial U)$ at their point of tangency, while (ii) is a relation between the curvatures of these two curves at their point of tangency.

We next discuss two important cases of Lemma 2.3 corresponding to $q(U)$ being a disk, and $q(U)$ being a half-plane.

Case 1. The Disk $= \{w : |w| < M\}$. If we let

$$q(z) = M(Mz + a)/(M + \bar{a}z),$$

with $M > 0$, and $|a| < M$ then $q(U) = \Delta = U_M$, $q(0) = a$, $E(q) = \varphi$ and $q \in Q$. If there are points $z_0 \in U$, $\zeta_0 \in \partial U$ such that $p(z_0) = q(\zeta_0)$ and $|p(z)| < M$ for $|z| < |z_0|$, then $|p(z_0)| = |q(\zeta_0)| = M$,

$$\zeta_0 = q^{-1}(p(z_0)) = M[p(z_0) - a]/[M^2 - \bar{a}p(z_0)],$$

$$\zeta_0 q'(\zeta_0) = [M^2 - \bar{a}p(z_0)][p(z_0) - a]/[M^2 - |a|^2], \tag{7}$$

and

$$\mathrm{Re}[\zeta_0 q''(\zeta_0)/q'(\zeta_0) + 1] = |p(z_0) - a|^2/[M^2 - |a|^2]. \tag{8}$$

Using these results in Lemma 2.3 we obtain:

LEMMA 2.3′. *Let* $p(z) = a + p_n z^n + \ldots$ *be analytic in* U *with* $p(z) \not\equiv a$ *and* $n \geq 1$. *If there exists* $z_0 \in U$ *such that* $|p(z_0)| = \mathrm{Max}\{|p(z)| : |z| \lneqq |z_0|\}$ *then*

(i) $z_0 p'(z_0)/p(z_0) \geq n|p(z_0) - a|^2/[|p(z_0)|^2 - |a|^2]$ *and*

(ii) $\mathrm{Re}[z_0 p''(z_0)/p'(z_0) + 1] \geq n|p(z_0) - a|^2/[|p(z_0)|^2 - |a|^2]$.

For $a = 0$ this lemma reduces to Lemma 2.1.

Case 2. The Half-Plane $= \{w : \mathrm{Re}\, w > \alpha,\ \alpha\ real\}$. If we let

$$q(z) = [a - (2\alpha - \bar{a})z]/[1 - z]$$

with $\mathrm{Re}\, a > \alpha$ then $q(U) = \Delta$, $q(0) = a$, $E(q) = \{1\}$ and $q \in Q$. If there are points $z_0 \in U$ and $\zeta_0 \in \partial U \setminus \{1\}$ such that $p(z_0) = q(\zeta_0)$ and $\mathrm{Re}\, p(z) > \alpha$ for $|z| < |z_0|$, then $\mathrm{Re}\, p(z_0) = \alpha$,

$$\zeta_0 = q^{-1}(p(z_0)) = [p(z_0) - a]/[p(z_0) - (2\alpha - \bar{a})],$$
$$\zeta_0 q'(\zeta_0) = -|a - p(z_0)|^2/2\mathrm{Re}[a - p(z_0)] \tag{9}$$

and

$$\mathrm{Re}[\zeta_0 q''(\zeta_0)/q'(\zeta_0) + 1] = 0 \tag{10}$$

Using these results in Lemma 2.3 we obtain;

LEMMA 2.3″. *Let* $p(z) = a + p_n z^n + \dots$ *be analytic in* U *with* $p(z) \not\equiv a$ *and* $n \geq 1$. *If there exists* $z_0 \in U$ *such that* $\operatorname{Re} p(z_0) = \operatorname{Min}\{\operatorname{Re} p(z) : |z| \leq |z_0|\}$ *then*

(i) $z_0 p'(z_0) \leqslant -n|a - p(z_0)|^2/2\operatorname{Re}[a - p(z_0)]$ *and*

(ii) $\operatorname{Re} z_0 p''(z_0)/p'(z_0) + 1 \geq 0$.

Remarks. 1. Since $z_0 p'(z_0)$ is real and negative, the inequality (ii) can be replaced by

(ii′) $\operatorname{Re}[z_0^2 p''(z_0)] + z_0 p'(z_0) \leqslant 0$.

2. If $\alpha = 0$ and $a = 1$ the inequality (i) becomes

(i′) $z_0 p'(z_0) \leqslant -n(1 - p^2(z_0))/2 \leqslant -n/2$

LEMMA 2.4. *Let* $q \in Q$, *with* $q(0) = a$, *and let* $p(z) = a + p_n z^n + \dots$ *be analytic in* U *with* $p(z) \not\equiv a$ *and* $n \geq 1$. *If* $p \not\prec q$, *then there exist points* $z_0 = r_0 e^{i\theta_0} \in U$ *and* $\zeta_0 \in \partial U \setminus E(q)$ *and an* $m \geq n$ *for which*

(i) $p(U_{r_0}) \subset q(U)$

(ii) $p(z_0) = q(\zeta_0)$,

(iii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$, *and*

(iv) $\operatorname{Re}[z_0 p''(z_0)/p'(z_0) + 1] \geq m \operatorname{Re}[\zeta_0 q''(\zeta_0)/q'(\zeta_0) + 1]$.

Proof. Since $p(0) = q(0)$, and p and q are analytic on U, we can define

$$r_0 = \sup\{r : p(U_r) \subset q(U)\}.$$

Since $p \not\prec q$ we have $p(U) \not\subset q(U)$. Thus for $0 < r_0 < 1$ we get $p(U_{r_0}) \subset q(U)$ and $p(\bar{U}_{r_0}) \not\subset q(U)$. Since $p(\bar{U}_{r_0}) \subset \overline{q(\bar{U})}$ there exists $z_0 \in \partial U_{r_0}$ such that $p(z_0) \in \partial q(U)$. This implies there exists $\zeta_0 \in \partial U \setminus E(q)$ such that $p(z_0) = q(\zeta_0)$. The conclusions of this lemma now follow by applying Lemma 2.3.

3. Admissible Functions and Fundamental Theorems. In this section we define the class of functions ψ for which we intend to prove (4).

DEFINITION 3.1. Let Ω be a set in $\mathbf{C}$, $q \in Q$ and n be a positive integer. We define the *class of admissible functions* $\Psi_n[\Omega, q]$ to be those functions $\psi : \mathbf{C}^3 \times U \to \mathbf{C}$ that satisfy the following *admissibility condition*:

$$\begin{gathered}\psi(r, s, t; z) \notin \Omega \text{ when } r = q(\zeta),\ s = m\zeta q'(\zeta), \\ \operatorname{Re}[t/s + 1] \geq m\operatorname{Re}[\zeta q''(\zeta)/q'(\zeta) + 1] \text{ and } z \in U, \\ \text{for } \zeta \in \partial U \setminus E(q) \text{ and } m \geq n.\end{gathered} \tag{11}$$

We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

In the special case when $\Omega \neq \mathbf{C}$ is a simply connected domain and h is a conformal mapping of U onto Ω we denote the class by $\Psi_n[h, q]$.

Remarks. 1. If $\psi : \mathbf{C}^2 \times U \to \mathbf{C}$ the condition (11) becomes $\psi(r, s; z) \notin \Omega$, when $r = q(\zeta)$, $s = m\zeta q'(\zeta)$ and $z \in U$, for $\zeta \in \partial U \setminus E(q)$ and $m \geq n$.

2. If $\Omega \subset \tilde{\Omega}$ then $\Psi_n[\tilde{\Omega}, q] \subset \Psi_n[\Omega, q]$, that is, enlarging Ω decreases the class. Also note that $\Psi_n[\Omega, q] \subset \Psi_{n+1}[\Omega, q]$.

THEOREM 3.2. *Let* $\psi \in \Psi_n[\Omega, q]$ *with* $q(0) = a$.

If $p(z) = a + p_n z^n + \dots$ *is analytic in* U *and satisfies*

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega,\ z \in U \tag{12}$$

then $p \prec q$.

Proof. Assume that $p \not\prec q$. By Lemma 2.4 there exist points $z_0 \in U$ and $\zeta_0 \in \partial U \setminus E(q)$ and an $m \geq n$ that satisfy (i) — (iv) of Lemma 2.4. Using these conditions with $r = p(z_0)$, $s = z_0 p'(z_0)$, $t = z_0^2 p''(z_0)$ and $z = z_0$ in Definition 3.1 we obtain

$$\psi(p(z_0), z_0 p'(z_0), z_0^2 p''(z_0) ; z_0) \notin \Omega.$$

Since this contradicts (8) we must have $p \prec q$.

Remarks. 1. The conclusion of Theorem 3.2 also holds if (12) is replaced by

$$\psi(p(z), zp'(z), z^2 p''(z) ; w(z)) \in \Omega, \ z \in U, \tag{12'}$$

for any function $w(z)$ mapping U into U.

2. The conditions on $\Psi_n[\Omega, q]$ are fairly general, and as a result, for a given ψ in the class there may not exist an analytic function p satisfying (12). As an example, let $q(z) = (1 + z)/(1 - z)$, $\Omega = q(U)$ and $\psi(r, s, t ; z) = -r^2 s$. A simple computation shows that $\psi \in \Psi[q(U), q]$. In this case (12) becomes

$$\mathrm{Re}[-(p(z))^2 \cdot zp'(z)] > 0,$$

but there is no analytic function p that satisfies this inequality at $z = 0$. However, in the applications of the theorem that we will present, the existence of a p satisfying (12) will be clear. In the particular case when Ω is a domain in $\mathbf{C}$, $\psi(a, 0, 0 ; 0) \in \Omega$ and ψ is continuous in a neighborhood of $(a, 0, 0 ; 0)$, then $p(z) = a + p_n z^n$ will satisfy (12) for sufficiently small $|p_n|$.

On checking the definitions of Q and $\Psi_n[\Omega, q]$ we see that the hypothesis of Theorem 3.2 requires that q behave very nicely on its boundary. If this is not the case or if the behavior of q on its boundary is not known, it may still be possible to prove that $p \prec q$ by the following limiting procedure.

COROLLARY 3.3. *Let $\Omega \subset \mathbf{C}$ and let q be univalent in U. Let $\psi \in \Psi_n[\Omega, q_\rho]$, for some $\rho \in (0, 1)$, where $q_\rho(z) = q(\rho z)$. If $p(z) = a + p_n z^n + \ldots$ is analytic in U and $\psi(p(z), zp'(z), z^2 p''(z) ; z) \in \Omega$ for $z \in U$, then $p \prec q$.*

Proof. The function q_ρ is univalent on $\bar{U}$, and hence $E(q_\rho)$ is empty and $q_\rho \in Q$. The class $\Psi_n[\Omega, q_\rho]$ is an admissible class and from Theorem 1 we obtain $p \prec q_\rho$. Since $q_\rho \prec q$ we deduce $p \prec q$.

We next list the special case when $\Omega \neq \mathbf{C}$ is a simply connected domain. The proof follows immediately from Theorem 3.2.

THEOREM 3.4. *Let $\psi \in \Psi_n[h, q]$, with $q(0) = a$ and $\psi(a ; 0, 0 ; 0) = h(0)$. If $p(z) = a + p_n z^n + \ldots$ and $\psi(p(z), zp'(z), z^2 p''(z) ; z)$ are analytic in U, and*

$$\psi(p(z), zp'(z) \ z^2 p''(z) ; z) \prec h(z) \tag{13}$$

then $p \prec q$.

An analogue of Corollary 3.3 can be given for $\Psi_n[h, q] = \Psi_n[h(U), q]$.

COROLLARY 3.5. *Let h and q be univalent in U with $q(0) = a$. Let $\psi: \mathbf{C}^3 \times U \to \mathbf{C}$, with $\psi(a, 0, 0 ; 0) = h(0)$, satisfy one of the following conditions:*

(i) $\psi \in \Psi_n[h, q_\rho]$, *for some $\rho \in (0, 1)$ or*

(ii) *there exists $\rho_0 \in (0, 1)$ such that $\psi \in \Psi_n[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$,* *where* $q_\rho(z) = q(\rho z)$ *and* $h_\rho(z) = h(\rho z)$. *If* $p(z) = a + p_n z^n + \dots$ *and* $\psi(p(z),$ $zp'(z), z^2p''(z); z)$ *are analytic in* U *and*

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z),$$

then $p \prec q$.

Proof. Case (i). By applying Theorem 3.4 we obtain $p \prec q_\rho$. Since $q_\rho \prec q$ we deduce $p \prec q$.

Case (ii). If we let $p_\rho(z) = p(\rho z)$ we have

$$\psi(p_\rho(z), zp'_\rho(z), z^2p''_\rho(z); \rho z) = \psi(p(\rho z), \rho zp'(\rho z), \rho^2z^2p''(\rho z); \rho z) \in h_\rho(U)$$

for $z \in U$. By using Theorem 3.2 and Remark 1 following it, with $w(z) = \rho z$, we obtain $p_\rho(z) \prec q_\rho(z)$ for $\rho \in (\rho_0, 1)$. By letting $\rho \to 1^-$ we obtain $p \prec q$.

If $n = 1$ and q is a dominant and solution of (13) then q will be the best dominant. Using this result together with Theorem 3.3 and Corollary 3.5 yields the following theorem.

THEOREM 3.6. *Let* h *be univalent in* U, *and let* $\psi : \mathbf{C}^3 \times U \to \mathbf{C}$. *Suppose that the differential equation*

$$\psi(q(z), zq'(z), z^2q''(z); z) = h(z)$$

has a solution q *and one of the following conditions is satisfied*

(i) $q \in Q$ *and* $\psi \in \Psi[h, q]$,

(ii) q *is univalent in* U *and* $\psi \in \Psi[h, q_\rho]$ *for some* $\rho \in (0, 1)$, *or*

(iii) q *is univalent in* U *and there exists* $\rho_0 \in (0, 1)$ *such that*

$$\psi \in \Psi[h_\rho, q_\rho] \text{ for all } \rho \in (\rho_0, 1).$$

If $p(z) = q(0) + p_1 z + \dots$ *and* $\psi(p, zp', z^2p''; z)$ *are analytic in* U *and if* p *is a solution of* (13), *then* $p \prec q$ *and* q *is the best dominant.*

From the above theorem we see that the problem of finding best dominants corresponds to finding univalent solutions of differential equations. We will take advantage of this in several of the applications in the next chapter.

Theorem 3.2 can be used to show that the solutions of certain second order differential equations are contained in a certain set.

THEOREM 3.7. *Let* $\psi \in \Psi_n[\Omega, q]$ *and let* f *be an analytic function satisfying* $f(U) \subset \Omega$. *If the differential equation*

$$\psi(p(z), zp'(z), z^2p''(z); z) = f(z)$$

has a solution $p(z)$ *analytic in* U *with* $p(0) = q(0)$ *then* $p \prec q$.

4. **Special Cases: the Disc and Half-Plane.** In this section we will apply the theorems of the last section to the particular cases corresponding to $q(U)$ being a disk and $q(U)$ being a half-plane. Some preliminary results for these two cases have been presented in Section 2.

Case 1. The Disk $\Delta = \{w : |w| < M\}$. The function

$$q(z) = M(Mz + a)/(M + \bar{a}z),$$

with $M > 0$ and $|a| < M$ satisfies $q(U) = \Delta$, $q(0) = a$ and $q \in Q$. We first determine the class of admissible functions, as defined in Definition 3.1, for this particular q. We set $\Psi_n[\Omega, M, a] = \Psi_n[\Omega, q]$ and in the special case when $\Omega = \Delta$ we denote the class by $\Psi_n[M, a]$. Since $q(\zeta) = Me^{i\theta}$ with $\theta \in \mathbb{R}$ when $|\zeta| = 1$, by using (7) and (8) the condition of admissibility (11) becomes

$$\begin{gathered}\psi(r, s, t; z) \notin \Omega \text{ when } r = Me^{i\theta},\\ s = mMe^{i\theta}|M - \bar{a}e^{i\theta}|^2/(M^2 - |a|^2),\\ \operatorname{Re}(t/s + 1) \geq m|M - \bar{a}e^{i\theta}|^2/(M^2 - |a|^2),\\ \text{for } z \in U,\ \theta \in \mathbf{R} \text{ and } m \geq n.\end{gathered} \tag{14}$$

If $a = 0$ then (14) simplifies to

$$\begin{cases}\psi(Me^{i\theta}, Ke^{i\theta}, L; z) \notin \Omega \text{ when } K \geq nM,\\ \operatorname{Re}(Le^{-i\theta}) \geq (n-1)K,\ z \in U \text{ and } \theta \in \mathbf{R},\end{cases} \tag{14'}$$

a condition much easier to check.

THEOREM 4.1. *Let $p(z) = a + p_n z^n + \ldots$ be analytic in U. If $\psi \in \Psi_n[\Omega, M, a]$ then*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega \Rightarrow |p(z)| < M.$$

If $\psi \in \Psi_n[M, a]$ then

$$|\psi(p(z), zp'(z), z^2p''(z); z)| < M \Rightarrow |p(z)| < M.$$

The proof of this theorem follows immediately by applying Theorem 3.2.

EXAMPLE 4.2. Let $a = 0$, $n = 1$, $\Omega = h(U)$ where $h(z) = 2Mz$, and $\psi(r, s, t; z) = r + s + t$. We first show that $\psi \in \Psi[h(U), M, 0]$, that is, that admissibility condition (14′) is satisfied. This follows since

$$\begin{aligned}|\psi(Me^{i\theta}, Ke^{i\theta}, L; z)| &= |M + K + Le^{-i\theta}| \geq M + K + \operatorname{Re}(Le^{-i\theta})\\ &\geq M + Mn + (n-1)nM = M(1 + n^2) \geq 2M\end{aligned}$$

when $K \geq nM$ and $\operatorname{Re}(Le^{-i\theta}) \geq (n-1)K$. By Theorem 4.1 we deduce the following result. If $p(z)$ is analytic in U with $p(0) = 0$ then

$$|p(z) + zp'(z) + z^2p''(z)| < 2M \Rightarrow |p(z)| < M.$$

We can use Theorem 3.6 to present a different proof of this result, and to also show that this result is sharp. The differential equation

$$q(z) + zq'(z) + z^2q''(z) = 2Mz,$$

has the univalent solution $q(z) = Mz$. In order to use Theorem 3.6 we need to show that $\psi \in \Psi[2Mz, Mz]$. For $r = M\zeta$, $s = mM\zeta$ and $\operatorname{Re}[t/s + 1] \geq m$, for $|\zeta| = 1$ and $m \geq 1$ we have

$$\begin{aligned}|\psi(r, s, t)| &= |M\zeta + Mm\zeta + t| = M|1 + m + mt/s|\\ &\geq M(1 + m + m^2 - m) = M(1 + m^2) \geq 2M.\end{aligned}$$

Hence $\psi \in \Psi[2Mz, Mz]$, and by Theorem 3.6

$$p(z) + zp'(z) + z^2p''(z) \prec 2Mz \Rightarrow p(z) \prec Mz,$$

and $q(z) = Mz$ is the best dominant.

Case 2. The Half-Plane $\Delta = \{w : \operatorname{Re} w > 0\}$. The function

$$q(z) = (a + \bar{a}z)/(1 - z)$$

with $\operatorname{Re} a > 0$ satisfies $q(U) = \Delta$, $q(0) = a$, $E(q) = \{1\}$, and $q \in Q$. We first determine the class of admissible functions, as defined in Definition 3.1, for this particular q. We set $\Psi_n\{\Omega, a\} = \Psi_n[\Omega, q]$ and in the special case when $\Omega = \Delta$ we denote the class by $\Psi_n\{a\}$. Since $\operatorname{Re} q(\zeta) = 0$ when $\zeta \in \partial U \setminus \{1\}$, by using (9) and (10) the condition of admissibility (11) becomes

$$\psi(i\sigma, \tau, \mu + i\eta; z) \notin \Omega, \text{ for } z \in U \text{ and for } \sigma, \tau, \mu, \eta \text{ satisfying } \tau \leq -n|a - i\sigma|^2/2\operatorname{Re} a \text{ and } \tau + \mu \leq 0. \tag{15}$$

If $a = 1$ then (15) simplifies to

$$\psi(i\sigma, \tau, \mu + i\eta; z) \notin \Omega; \text{ for } z \in U, \text{ and for real } \sigma, \tau, \mu, \eta \text{ satisfying } \tau \leq -n(1 + \sigma^2)/2 \text{ and } \tau + \mu \leq 0. \tag{15'}$$

The proof of the following theorem follows immediately by applying Theorem 3.2.

THEOREM 4.3. *Let* $p(z) = a + p_nz^n + \ldots$ *be analytic in* U. *If* $\psi \in \Psi_n\{\Omega, a\}$ *then*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega \Rightarrow \operatorname{Re} p(z) > 0.$$

If $\psi \in \Psi_n\{a\}$ *then*

$$\operatorname{Re} \psi(p(z), zp'(z), z^2p''(z); z) > 0 \Rightarrow \operatorname{Re} p(z) > 0.$$

EXAMPLES 4.4. a) A simple check of (15') shows that $\psi(r, s, t; z) = r + s + t \in \Psi\{1\}$. Thus if $p(z) = 1 + p_1z + \ldots$ is analytic in U then

$$\operatorname{Re}[p(z) + zp'(z) + z^2p''(z)] > 0 \Rightarrow \operatorname{Re} p(z) > 0.$$

b) Let $\psi(r, s, t; z) = r + B(z)s$, where $B : U \to \mathbb{C}$ and $\operatorname{Re} B(z) > 0$. A simple check of (15) shows that $\psi \in \Psi\{a\}$ with $\operatorname{Re} a > 0$. Thus if $p(z) = a + p_1z + \cdots$ is analytic in U then

$$\operatorname{Re}[p(z) + B(z)zp'(z)] > 0 \Rightarrow \operatorname{Re} p(z) > 0.$$

c) Let $\psi(r, s, t; z) = t + 3s - r^2 + 1$. A simple check of (15') shows that $\psi \notin \Psi_1\{1\}$, but $\psi \in \Psi_2\{1\}$. Thus if $p(z) = 1 + p_2z^2 + \cdots$ is analytic in U then

$$\operatorname{Re}[z^2p''(z) + 3zp'(z) - p^2(z) + 1] > 0 \Rightarrow \operatorname{Re} p(z) > 0.$$

Other examples of similar differential inequalities may be found in [10], [12], [13] and [15].

II. APPLICATIONS

5. Differential and Integral Operators Preserving Functions with Positive Real Part. In the next two sections we will be interested in determining dominants of the *second-order linear differential subordination*

$$A(z)z^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \in \Omega \qquad (16)$$

for $z \in U$, where $\Omega \subset \mathbf{C}$, and A, B, C and D are complex-valued functions defined on U. In this section we let Ω be a set in $\{w \mid \operatorname{Re} w > 0\}$ and let $q(z) = (1+z)/(1-z)$ be a dominant of (16). We will determine conditions on A, B, C and D corresponding to this particular Ω and q. This situation corresponds to Problem 1 of Section 1.

THEOREM 5.1. *Let $A(z) = A \geq 0$ and suppose that $B, C, D: U \to \mathbf{C}$ and satisfy*

$$\operatorname{Re} B(z) \geq A \text{ and } [\operatorname{Im} C(z)]^2 \leq [\operatorname{Re} B(z) - A] \cdot \operatorname{Re}[B(z) - A - 2D(z)]. \qquad (17)$$

If p is analytic in U with $p(0) = 1$, and if

$$\operatorname{Re}[Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z)] > 0, \qquad (18)$$

then $\operatorname{Re} p(z) > 0$.

Proof. If we let $\psi(r, s, t; z) = At + B(z)s + C(z)r + D(z)$, then the conclusion will follow from Theorem 3.1 if we show that $\psi \in \Psi[\Omega, q]$, where $\Omega = \{w \mid \operatorname{Re} w > 0\}$ and $q = (1+z)/(1-z)$. This follows from (17), and Definition 3.1 or (15′) since

$$\begin{aligned}\operatorname{Re} \psi(i\sigma, \tau, \mu + i\eta; z) &= A\mu + \tau \operatorname{Re} B(z) - \sigma \operatorname{Im} C(z) + \operatorname{Re} D(z) \leq \\ &\leq \tau[\operatorname{Re} B(z) - A] - \sigma \operatorname{Im} C(z) + \operatorname{Re} D(z) \leq \\ &\leq [-(1+\sigma^2)/2][\operatorname{Re} B(z) - A] - \sigma \operatorname{Im} C(z) + \operatorname{Re} D(z) = \\ &= -\{[\operatorname{Re} B(z) - A]\sigma^2 + 2[\operatorname{Im} C(z)]\sigma + \operatorname{Re}[B(z) - A - 2D(z)]\}/2 \leqslant 0,\end{aligned}$$

for $z \in U$, and for real σ, τ, μ, η satisfying $\tau \leq -(1+\sigma^2)/2$ and $\tau + \mu \leq 0$. Hence $\psi \in \Psi[\Omega, q]$, $p \prec q$ and $\operatorname{Re} p(z) > 0$.

If $A = 0$ and $D(z) \equiv 0$ then Theorem 5.1 reduces to the following first order result [20, Theorem 8].

COROLLARY 5.2. *Let $B(z)$ and $C(z)$ be functions defined on U, with*

$$|\operatorname{Im} C(z)| \leq \operatorname{Re} B(z). \qquad (19)$$

If p is analytic in U with $p(0) = 1$, and if

$$\operatorname{Re}[B(z) \cdot zp'(z) + C(z) \cdot p(z)] > 0, \qquad (20)$$

then $\operatorname{Re} p(z) > 0$.

We can apply Corollary 5.2 to obtain a corresponding result for integrals [20, Theorem 9].

THEOREM 5.3. *Let $\gamma \neq 0$ be a complex number with* $\operatorname{Re}\gamma \geq 0$, *and let* φ *and* Φ *be analytic in* U, *with* $\varphi(z)\cdot\Phi(z) \neq 0$, $\varphi(0) = \Phi(0)$, *and*

$$|\operatorname{Im}[(\gamma\Phi(z) + z\Phi'(z))/\gamma\varphi(z)]| \leq \operatorname{Re}[\Phi(z)/\gamma\varphi(z)]. \tag{21}$$

Let f be analytic in U with $f(0) = 1$ and $\operatorname{Re} f(z) > 0$, *for* $z \in U$. *If F is defined by*

$$F(z) = \gamma z^{-\gamma}\,\Phi(z)^{-1}\int_0^z f(t)t^{\gamma-1}\varphi(t)\,dt, \tag{22}$$

then F is analytic in U, $F(0) = 1$ *and* $\operatorname{Re} F(z) > 0$ *for* $z \in U$.

Proof. If we let $B(z) = \Phi(z)/\gamma\varphi(z)$ and $C(z) = [\gamma\Phi(z) + z\Phi'(z)]/\gamma\varphi(z)$, then condition (21) implies condition (19). By differentiating (22) we obtain

$$\operatorname{Re}[B(z)\cdot zF'(z) + C(z)\cdot F(z)] = \operatorname{Re} f(z) > 0.$$

Hence (20) of Corollary 5.2 is satisfied with $p = F$, and we conclude that $\operatorname{Re} F(z) > 0$.

If we let $\varphi = \Phi$ and $\gamma > 0$ then (21) reduces to

$$|\operatorname{Im} z\varphi'(z)/\varphi(z)| \leq 1. \tag{23}$$

In this case we deduce

$$\operatorname{Re} f(z) > 0 \Rightarrow \operatorname{Re}\left[z^{-\gamma}\,\varphi(z)^{-1}\int_0^z f(t)t^{\gamma-1}\varphi(t)\,dt\right] > 0.$$

EXAMPLE 5.4. The function $\varphi(z) = e^{\lambda z}$ satisfies (23) for $|\lambda| \leq 1$. In this case we obtain

$$\operatorname{Re} f(z) > 0 \Rightarrow \operatorname{Re}\left[z^{-\gamma}\,e^{-\lambda z}\int_0^z f(t)t^{\gamma-1}\,e^{\lambda t}\,dt\right] > 0. \tag{24}$$

Corollary 5.2 involves a first order linear differential subordination and its integral analog is the linear operator given in Theorem 5.3. The second order linear differential subordination given in Theorem 5.1 also has an integral analog. However, in this case the second order differential subordination gives rise to a double integral.

THEOREM 5.5. *Let β and γ be complex numbers with* $\beta\gamma > 0$, $\operatorname{Re}\beta \geq 0$, $\operatorname{Re}\gamma \geqslant 0$, *let φ and Φ be analytic in U with* $\varphi(z)\cdot\Phi(z) \neq 0$, $\varphi(0) = \Phi(0)$ *and let w be analytic in U with* $w(0) = 0$. *Suppose that (17) holds with*

$$\begin{cases} A \equiv 1/\beta\gamma,\ D(z) \equiv -w(z), \\ B(z) \equiv [\beta + \gamma + 1 + z\varphi'(z)/\varphi(z) + z\Phi'(z)/\Phi(z)]/\beta\gamma, \text{ and} \\ C(z) \equiv [(\beta + z\Phi'(z)/\Phi(z))(\gamma + z\varphi'(z)/\varphi(z)) + z(z\varphi'(z)/\varphi(z))']/\beta\gamma. \end{cases} \tag{25}$$

Let f be analytic in U with $f(0) = 1$ *and* $\operatorname{Re} f(z) > 0$ *for* $z \in U$. *If F is defined by*

$$F(z) = \frac{\beta\gamma}{z^{\gamma}\varphi(z)} \int_0^z \frac{\varphi(t)}{\Phi(t)} t^{\gamma-\beta-1} \int_0^t [f(s) + w(s)]\, \Phi(s) s^{\beta-1}\, ds\, dt, \tag{26}$$

then F is analytic in U, $F(0) = 1$ *and* $\operatorname{Re} F(z) > 0$ *in U.*

Proof. By differentiating (26) and using (25) we obtain $\operatorname{Re}[Az^2F''(z) + B(z)zF'(z) + C(z)F(z) + D(z)] = \operatorname{Re} f(z) > 0$. Since the conditions of (17) hold, we apply Theorem 5.1 with $p = F$ to conclude that $\operatorname{Re} F(z) > 0$ in U.

Note that if $D(z) = -w(z) \equiv 0$ the second-order linear differential subordination (18) gives rise to a linear (double) integral operator $F = I(f)$ given by (26).

If we let $\varphi(z) \equiv 1$, $w(z) \equiv 0$, $\beta > 0$ and $\gamma > 0$ then from (25) we have $D(z) \equiv 0$,

$$B(z) = [\beta + \gamma + 1 + z\Phi'(z)/\Phi(z)]/\beta\gamma \text{ and}$$
$$C(z) = \gamma[\beta + z\Phi'(z)/\Phi(z)]/\beta\gamma.$$

In this case we obtain the following corollary.

COROLLARY 5.6. *Let* $\beta > 0$, $\gamma > 0$, *and let* Φ *be analytic in U with* $\Phi(z) \neq 0$. *Suppose that*

$$\gamma \left| \operatorname{Im} \frac{z\Phi'(z)}{\Phi(z)} \right| \le \operatorname{Re}\left[\beta + \gamma + \frac{z\Phi'(z)}{\Phi(z)}\right]. \tag{27}$$

Let f be analytic in U with $f(0) = 1$ *and* $\operatorname{Re} f(z) > 0$ *in U.*
If F is defined by

$$F(z) = \beta\gamma z^{-\gamma} \int_0^z t^{\gamma-\beta-1} \Phi(t)^{-1} \int_0^t f(s)\, \Phi(s)\, s^{\beta-1}\, ds\, dt,$$

then F is analytic in U, $F(0) = 1$ *and* $\operatorname{Re} F(z) > 0$ *in U.*

The complicated condition (27) has a simple geometric interpretation. If we let $w = z\Phi'(z)/\Phi(z) = u + iv$ then (27) becomes

$$\gamma |v| \le \beta + \gamma + u.$$

Hence (27) requires that $z\Phi'/\Phi$ lies in the closed sector $S(\beta, \gamma)$ containing the origin and bounded by the lines

$$\gamma |v| = \beta + \gamma + u.$$

If we take $\Phi(z) = e^{\lambda z}$ then $z\Phi'(z)/\Phi(z) = \lambda z$. Since the distance δ from the origin to the boundary of the sector $S(\beta, \gamma)$ is given by

$$\delta = (\beta + \gamma)/(1 + \gamma^2)^{1/2}, \tag{28}$$

we obtain the following example.

EXAMPLE 5.7. If $|\lambda| \leq \delta$ where δ is given by (28) then

$$\operatorname{Re} f(z) > 0 \Rightarrow \operatorname{Re}\left[z^{-\gamma}\int_0^z e^{-\lambda t}\, t^{\gamma-\beta-1}\int_0^t f(s)\, e^{\lambda s}\, s^{\beta-1}\, ds\, dt\right] > 0.$$

In the particular case $\beta = \gamma = 1$, we deduce that for $|\lambda| \leq \sqrt{2}$ we have

$$\operatorname{Re} f(z) > 0 \Rightarrow \operatorname{Re}\left[z^{-1}\int_0^z e^{-\lambda t}\, t^{-1}\int_0^t f(s)\, e^{\lambda s}\, ds\, dt\right] > 0. \tag{29}$$

Note that in Example 5.4 we can apply (24) twice with $\gamma = 1$ to obtain (29). However, by using this method (29) will be valid only for $|\lambda| \leq 1$.

6. **Averaging Integral Operators.** In this section we will use a second-order linear differential subordination of the form

$$A(z)z^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z)$$

to obtain a generalized complex-valued version of the First Mean-Value Theorem for Riemann Integrals. Recall that in the real case if f and h are continuous on $J = [0, 1)$, with $h(x) \geqslant 0$, then there exists $c \in (0, x)$ such that

$$\int_0^x f(t)\, h(t)\, dt = f(c)\int_0^x h(t)\, dt.$$

If we let $h(x) = g'(x)$ with $g(0) = 0$, then we obtain

$$g(x)^{-1}\int_0^x f(t)\, g'(t)\, dt = f(c) \in f(J) \tag{30}$$

for all $x \in J$.

We will obtain a complex-valued analog of (30) of the form

$$g(z)^{-1}\int_0^z f(w)\, g'(w)\, dw \in f(U) \tag{31}$$

for all $z \in U$, where f and g are analytic on U and satisfy some simple conditions.

In the remainder of this section we let H be the set of functions that are analytic in U, we let $H_0 = \{h \in H : h(0) = 0\}$ and when we refer to a convex function we assume that it is univalent. In addition, if G is a set in $\mathbf{C}$ then we denote the convex hull of G by $\operatorname{co} G$.

If we denote the integral in (31) by $I[f]$ then (31) can be rewritten as

$$I[f](U) \subset f(U). \tag{31'}$$

This in turn implies that

$$I[f](U) \subset \operatorname{co} f(U). \tag{32}$$

This generalized complex-valued version of Riemann's mean-value theorem will be analyzed in this section. Note that (32) reduces to (31') if f is a convex function. In light of this discussion we make the following definition.

DEFINITION 6.1. If $K \subset H$ and if an operator $I: K \to H$ satisfies

$$I[f](U) \subset \operatorname{co} f(U), \text{ with } I[f](0) = f(0) \tag{33}$$

for all $f \in K$, then I is said to be an *averaging (or mean-value) operator on* K.

A simple characterization of such operators is given in the next theorem.

THEOREM 6.2. *Let $I: K \to H$ satisfy $I[f](0) = f(0)$. A necessary and sufficient condition for I to be an averaging operator on K is that for all convex h*

$$f \prec h \Rightarrow I[f] \prec h. \tag{34}$$

Proof. If I is an averaging operator then condition (34) follows immediately from condition (33). Next suppose that condition (34) holds. If $\operatorname{co} f(U) = \mathbf{C}$ then condition (33) holds. If $\operatorname{co} f(U) \neq \mathbf{C}$ then there exists a convex function $h: U \to \operatorname{co} f(U)$ with $h(0) = f(0)$ and $h(U) = \operatorname{co} f(U)$. Using this h in (34) we obtain (33).

We will obtain a large group of averaging operators as an application of the following theorem on second-order differential subordinations.

THEOREM 6.3 [19]. *Let $h \in H_0$ be convex and let $A \geqslant 0$. Suppose that $k > 4/|h'(0)|$ and that $B(z)$, $C(z)$ and $D(z)$ are analytic in U and satisfy*

$$\operatorname{Re} B(z) \geqslant A + |C(z) - 1| - \operatorname{Re}[C(z) - 1] + k|D(z)|, \tag{35}$$

for $z \in U$. If $p \in H_0$ satisfies the differential subordination

$$Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z), \tag{36}$$

then $p(z) \prec h(z)$.

Proof. We will use Corollary 3.5 part (ii) to prove this theorem. Since $k|h'(0)| > 4$, there is a ρ_0, $0 < \rho_0 < 1$, such that $(1 + \rho_0)^2/\rho_0 = k|h'(0)|$ and

$$4 < (1 + \rho)^2/\rho < k|h'(0)|, \text{ for } \rho_0 < \rho < 1. \tag{37}$$

If we set $\psi(r, s, t; z) = At + B(z)s + C(z)r + D(z)$ and $h_\rho(z) = h(\rho z)$ then the conclusion will follow from Corollary 3.5 if we show that $\psi \in \Psi[h_\rho, h_\rho]$ for $\rho \in (\rho_0, 1)$. Since h_ρ is convex on $\bar{U}$, according to Definition 3.1 it is sufficient to show

$$\psi(r, s, t; z) \notin h_\rho(U), \text{ when } r = h_\rho(\zeta),\ s = m\zeta h_\rho'(\zeta),$$
$$\operatorname{Re}[t/\zeta h_\rho'(\zeta)] \geqslant -m \text{ and } z \in U, \text{ for } |\zeta| = 1 \text{ and } m \geqslant 1. \tag{38}$$

If we set $V \equiv [\psi(r, s, t; z) - h_\rho(\zeta)]/\zeta h_\rho'(\zeta)$ then

$$V = \frac{At}{\zeta h_\rho'(\zeta)} + \frac{B(z)s}{\zeta h_\rho'(\zeta)} + \frac{C(z)r - h_\rho(\zeta)}{\zeta h_\rho'(\zeta)} + \frac{D(z)}{\zeta h_\rho(\zeta)} \tag{39}$$

and

$$\psi(r, s, t; z) = h_\rho(\zeta) + V\zeta h_\rho'(\zeta). \tag{40}$$

We first show that $\operatorname{Re} V > 0$ which will lead to the proof of (38).

Since h_ρ is convex and $h_\rho(0) = 0$ we have $\text{Re}[\zeta h'_\rho(\zeta)/h_\rho(\zeta)] \geq 1/2$ for $|\zeta| = 1$ [8, p. 176], or equivalently

$$|h_\rho(\zeta)/\zeta h'_\rho(\zeta) - 1| \leq 1. \tag{41}$$

If W and Z are complex numbers and $|Z - 1| \leq 1$ then

$$\text{Re}\, WZ = \text{Re}[W(Z - 1)] \geq \text{Re}\, W - |W|.$$

Using this inequality with $W = C(z) - 1$ and $Z = h_\rho(\zeta)/\zeta h'_\rho(\zeta)$ we obtain from (41) that

$$\text{Re}\{[C(z) - 1]h_\rho(\zeta)/\zeta h'_\rho(\zeta)\} \geq \text{Re}[C(z) - 1] - |C(z) - 1|. \tag{42}$$

Using the distortion theorem for convex functions [8, p. 118] we have $|h'(w)| \geq |h'(0)|/(1 + \rho)^2$ for $|w| = \rho < 1$. If we set $w = \rho\zeta$ we obtain

$$|\zeta h'_\rho(\zeta)| \geq \rho |h'(0)|/(1 + \rho)^2, \text{ for } |\zeta| = 1. \tag{43}$$

If we use the conditions on r, s, t and z in (38) together with (35), (39), (42) and (43) we obtain

$$\text{Re}\, V \geq A(-m) + \text{Re}[B(z)]m + \text{Re}\{[C(z)-1]h_\rho(\zeta)/\zeta h'_\rho(\zeta)]\} - |D(z)/\zeta h'_\rho(\zeta)| \geq (m - 1)\{|C(z) - 1| - \text{Re}[C(z) - 1]\} + m[k - (1 + \rho)^2/\rho \cdot |h'(0)|]\,|D(z)|.$$

From (37) and the fact that $m \geq 1$ we obtain $\text{Re}\, V > 0$, or equivalently that $|\arg V| < \pi/2$.

Using this in (40) together with the fact that $\zeta h'_\rho(\zeta)$ is the outward normal to the boundary of the convex domain $h_\rho(U)$ we obtain $\psi(r, s, t; z) \notin h_\rho(U)$, which completes the proof of the theorem.

This theorem is an example of a solution of Problem 1' referred to in Section 1.

Note that condition (36) implies that $D(0) = 0$. The special case $D(0) \equiv 0$ was proved in [20, Theorem 2].

If $C(z) \equiv 1$, the condition $h(0) = p(0) = 0$ can be replaced by $h(0) = p(0)$ and Theorem 6.3 simplifies to:

COROLLARY 6.4. *Let h be convex in U and let $A \geq 0$. Suppose $k > 4/|h'(0)|$ and that $B \in H$, $D \in H_0$* and

$$\text{Re}\, B(z) \geq A + k|D(z)|,$$

for $z \in U$. If $p \in H$ with $p(0) = h(0)$ and if p satisfies

$$Az^2p''(z) + B(z)zp'(z) + p(z) + D(z) \prec h(z)$$

then $p(z) \prec h(z)$.

We are now prepared to obtain some classes of integral averaging operators. Our method involves integrating the differential equation

$$Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) = f(z) \tag{44}$$

and applying Theorems 6.2 and 6.3. We first handle the simpler first-order case corresponding to $A = 0$ and $D(z) \equiv 0$.

THEOREM 6.5. *Let* $\gamma \in \mathbf{C}$ *with* $\gamma \neq -1, -2, \ldots$ *and let* $\varphi, \Phi \in H$ *with* $\varphi(z)\Phi(z) \neq 0$ *for* $z \in U$. *If*

$$\operatorname{Re} B(z) \geqslant |C(z) - 1| - \operatorname{Re}[C(z) - 1], \; z \in U, \tag{45}$$

where $B(z) = \Phi(z)/\varphi(z)$ *and* $C(z) = [\gamma\Phi(z) + z\Phi'(z)]/\varphi(z)$, *then the integral operator* I *defined by*

$$I[f](z) = z^{-\gamma}\,\Phi(z)^{-1}\int_0^z f(t)t^{\gamma-1}\varphi(t)\,dt \tag{46}$$

is an averaging operator on H_0.

Proof. The restriction on γ, φ and Φ imply that $I[f]$ is analytic on U and $I[f](0) = 0$. Let $h \in H_0$ be convex and suppose $f \prec h$. According to (34), the operator will be an averaging operator on H_0 if $I[f] \prec h$.

If we let $p(z) = I[f](z)$ and differentiate (46) we obtain

$$B(z)zp'(z) + C(z)p(z) = f(z) \prec h(z).$$

We now apply Theorem 6.3 with $A = 0$ and $D(z) \equiv 0$. Since (45) implies (35) we obtain $p = I[f] \prec h$. Hence according to Theorem 6.2, I is an averaging operator.

If we set $\varphi(z) = g(z)^{\gamma-1}g'(z)z^{1-\gamma}$ and $\Phi(z) = g(z)^{\gamma}z^{-\gamma}\gamma^{-1}$ in Theorem 6.5 then condition (45) simplifies and we obtain the following corollary.

COROLLARY 6.6. *Let* $\gamma \in \mathbf{C}$ *with* $\operatorname{Re}\gamma > 0$, *and let* $g \in H_0$ *with* $\operatorname{Re}[\gamma z g'(z)/g(z)] > 0$ *in* U. *If* I *is defined by*

$$I[f](z) = \gamma g(z)^{-\gamma}\int_0^z f(t)\,g(t)^{\gamma-1}g'(t)\,dt, \tag{47}$$

then I *is an averaging operator on* H.

If we set $\gamma = 1$ in (47) we obtain

$$I[f](z) = g(z)^{-1}\int_0^z f(t)g'(t)\,dt \in \operatorname{co} f(U).$$

This is a generalized complex analog of the mean-value theorem of real analysis as referred to in (30) and (31). In fact, we obtain exactly (31) when $\operatorname{Re}[zg'(z)/g(z)] > 0$ and f is convex.

EXAMPLE 6.7. If we set $\gamma = 1$ and $g(z) = z^{\alpha}$ in (47) we obtain the following averaging operator on H

$$I[f](z) = \alpha z^{-\alpha}\int_0^z f(t)\,t^{\alpha-1}\,dt, \qquad \operatorname{Re}\alpha > 0.$$

In particular, for $\alpha = 1$ this becomes

$$\frac{1}{z}\int_0^z f(t)\,dt \in \operatorname{co} f(U),$$

that is, the "average value" of f lies in the convex hull of $f(U)$. This is one of the reasons we call such operators as averaging operators.

EXAMPLE 6.8. If we set $\gamma = 0$, $\varphi(z) \equiv 1/2$ and $\Phi(z) \equiv 1$ in Theorem 6.5, then $B(z) \equiv 2$, $C(z) \equiv 0$ and thus (44) is satisfied. Hence the operator

$$I[f](z) = \frac{1}{2}\int_0^z \frac{f(t)}{t}\,dt$$

is an averaging operator on H_0, i.e. $I[f](U) \subset \operatorname{co} f(U)$.

This last operator is very similar to the classical Alexander operator [1]

$$I[f](z) = \int_0^z \frac{f(t)}{t}\,dt. \tag{48}$$

However, this operator is not an averaging operator on H_0. If we set $F(z) = I[f](z)$ and $f(z) = z + z^2/4$, where I is given by (48), then $F(z) = z + z^2/8$. Since $F(1) < f(1)$, $F(-1) < f(-1)$ and the functions f and F are convex, we have $F(U) \not\subset f(U) = \operatorname{co} f(U)$.

Theorem 6.5 and Corollary 6.6 were obtained by integrating (44) when $A = 0$ and $D(z) \equiv 0$. We can obtain a more general (first-order) averaging operator by allowing a more general $D(z)$. The following theorem and corollary are analogs of Theorem 6.5 and Corollary 6.6. Their proofs are similar and will be omitted.

THEOREM 6.9. *Let* $\gamma \in \mathbf{C}$ *with* $\gamma \neq -1, -2, \ldots$, *let* $\varphi, \Phi \in H$ *with* $\varphi(z) \cdot \Phi(z) \neq 0$ *for* $z \in U$, *and let* $\theta \in H_0$. *Let* $I : H_0 \to H_0$ *be defined by*

$$I[f](z) = z^{-\gamma}\Phi(z)^{-1}\int_0^z f(t)\,t^{\gamma-1}\varphi(t)\,dt, \tag{49}$$

and let $B(z) = \Phi(z)/\varphi(z)$ *and* $C(z) = [\gamma\Phi(z) + z\Phi'(z)]/\varphi(z)$. *If*

$$\operatorname{Re} B(z) \geq |C(z) - 1| - \operatorname{Re}[C(z) - 1] + 4|\theta(z)| \tag{50}$$

for $z \in U$, *then the operator* $J : H_0 \to H_0$ *defined by*

$$J_\theta[f] = I[f] + f'(0)I[\theta] \tag{51}$$

is an averaging operator on H_0.

COROLLARY 6.10. *Let* $\gamma \in \mathbf{C}$ *with* $\operatorname{Re}\gamma > 0$, let $\theta \in H_0$ *and let* $g \in H_0$ *with*

$$\operatorname{Re}\left[\frac{1}{\gamma}\cdot\frac{g(z)}{zg'(z)}\right] \geq 4|\theta(z)|$$

for $z \in U$. *If* $I[f]$ *is defined by* (49) *and* $J_0[f]$ *is defined by* (51), *then* $J_\theta[f]$ *is an averaging operator on* H.

EXAMPLE 6.11. If we let $\gamma = 0$ and $\Phi(z) \equiv 1$ in Theorem 6.9 then $B(z) = 1/\varphi(z)$, $C(z) \equiv 0$ and (50) simplifies to

$$\operatorname{Re}[1/\varphi(z)] \geqslant 2[1 + 2|\theta(z)|]$$

for $z \in U$. Hence the operator

$$J_0[f](z) = \int_0^z [f(t) + f'(0)\,\theta(t)]\,t^{-1}\varphi(t)\,dt$$

is an averaging operator on H_0. As a particular case take $\varphi(z) \equiv \frac{1}{6}$ to obtain: if $\theta \in H_0$ with $|\theta(z)| \leqslant |z|$ then

$$J_0[f](z) = \frac{1}{6}\int_0^z \frac{[f(t) + f'(0)\theta(t)]}{t}\,dt$$

is an averaging operator on H_0.

We next consider integrating (44) for the case $A > 0$. This involves two integrations which leads to "second-order" integral averaging operators. The proofs of the following theorem and corollary follow from Theorems 6.2 and 6.3 and will be omitted. See [26] for full details.

THEOREM 6.12. *Let* $\alpha > 0$, *let* $\beta, \gamma \in \mathbf{C}$ *with* $\operatorname{Re}\beta > -1$ *and* $\operatorname{Re}\gamma > -1$ *and let* $\varphi, \Phi \in H$ *with* $\varphi(z)\Phi(z) \neq 0$ *in* U. *Let* $I: H_0 \to H_0$ *be defined by*

$$I[f](z) = \frac{1}{\alpha z^\gamma \varphi(z)} \int_0^z \frac{\varphi(t)}{\Phi(t)} t^{\gamma-\beta-1} \int_0^t f(s)\,\Phi(s)\,s^{\beta-1}\,ds\,dt,$$

let

$$B(z) = \alpha[\beta + \gamma + 1 + z\Phi'(z)/\Phi(z) + z\varphi'(z)/\varphi(z)]$$

and

$$C(z) = \alpha[(\beta + z\Phi'(z)/\Phi(z))(\gamma + z\varphi'(z)/\varphi(z)) + z(z\varphi'(z)/\varphi(z))'].$$

If $\theta \in H_0$ *and*

$$\operatorname{Re} B(z) \geqslant \alpha + |C(z) - 1| - \operatorname{Re}[C(z) - 1] + 4|\theta(z)|$$

for $z \in U$, *then the operator*

$$J_0[f] = I[f] + f'(0)I[\theta]$$

is an averaging operator on H_0.

We remark that $C(z) \equiv 1$ if and only if φ and Φ are constant and $\alpha\beta\gamma = 1$, i.e. $1/\alpha = \beta\gamma > 0$. In this case we have $B(z) = \alpha(\beta + \gamma + 1)$ and Theorem 6.12 simplifies to.

COROLLARY 6.13. *Let* $\beta, \gamma \in \mathbf{C}$ *with* $\operatorname{Re}\beta > 0$, $\operatorname{Re}\gamma > 0$ *and* $\beta\gamma > 0$. *If* $\theta \in H_0$ *satisfies*

$$|\theta(z)| \leqslant \operatorname{Re}(\beta + \gamma)/4\beta\gamma,\ z \in U,$$

then the operator

$$J_\theta[f](z) = \beta\gamma z^{-\gamma}\int_0^z t^{\gamma-\beta-1}\int_0^t [f(s) + f'(0)\,\theta(s)]\,s^{\beta-1}\,ds\,dt$$

is an averaging operator on H.

7. **Analytic and Starlike Integral Operators.** In this section we will discuss the application of differential subordinations to the field of integral operators that map certain classes of analytic functions into univalent functions.

We first define some of the standard subclasses that will be used in this section and in the remainder of this article. Let $A = \{f \in H : f(0) = 0, f'(0) = 1\}$, and let S denote the subset of A consisting of univalent functions.

Let $f \in A$. If there exists a real number σ, $|\sigma| < \pi/2$, such that $\operatorname{Re}[e^{i\sigma} \cdot zf'(z)/f(z)] > 0$ for $z \in U$, then f is said to be *spirallike*. We represent the class if such functions by $\hat{S}$. If $0 \leq \beta < 1$ and $\operatorname{Re}[zf'(z)/f(z)] > \beta$, then f is said to be *starlike of order* β. We represent the class of such functions by $S^*[\beta]$; $S^* \equiv S^*[0]$ is the class of *starlike* functions. The class of *convex* functions, denoted by K, are those $f \in A$ for which $\operatorname{Re}[zf''(z)/f'(z) + 1] > 0$, for $z \in U$. It is well known that $K \subset S^*[1/2] \subset S^* \subset \hat{S} \subset S$.

There is a long history to the study of integral operators mapping a subclass of A into another subclass of A. The first such operator,

$$I[f](z) = \int_0^z [f(t)/t]\,dt,$$

was introduced by J. W. Alexander in 1915 [1]. In that article he showed that $I[S^*] = K$. Since then, extensions of Alexander operator and many other types of integral operators have been investigated by many authors. (See [21] for a short historical review). Most of these operators are special cases of a more general operator of the form

$$I[f](z) = \left[\frac{\beta+\gamma}{z^\gamma\,\Phi(z)}\int_0^z f^\alpha(t)\,\varphi(t)\,t^{\delta-1}\,dt\right]^{1/\beta} \tag{52}$$

This operator was investigated by the authors together with M. Reade [22] and [23]. A key element in the analysis was the use of a differential inequality given in [10]. In that article α, β, γ and δ were restricted to be real, while f was restricted to be in S^* or K.

In this section we consider the operator given by (52), but now allow α, β, γ and δ to be complex and allow f to be in more general subsets of A. A new method of proof, employing a differential subordination, provides an extension and sharpening of all previous results. The key differential subordination is given in Lemma 7.2. We need to first introduce a special mapping from U onto a slit domain.

DEFINITION 7.1. Let c be a complex number such that $\operatorname{Re} c > 0$ and

$$N = N(c) = [\,|c|(1 + 2\operatorname{Re} c)^{1/2} + \operatorname{Im} c]/\operatorname{Re} c.$$

If h is the univalent function $h(z) = 2Nz/(1 - z^2)$ and $b = h^{-1}(c)$, then we define the "open door" function Q_c as

$$Q_c(z) = h[(z + b)/(1 + \bar{b}z)], \quad z \in U.$$

From its definition we see that Q_c is univalent, $Q_c(0) = c$, and $Q_c(U) = h(U)$ is the complex plane slit along the half-lines $\operatorname{Re} w = 0$, $\operatorname{Im} w \geqslant N$ and $\operatorname{Re} w = 0$, $\operatorname{Im} w \leqslant -N$. The reason for the title "open door" will become apparent in the remarks following Lemma 7.2.

LEMMA 7.2. [25]. *Let Q_c be the function given by Definition 7.1 and let $B(z)$ be an analytic function in U satisfying*

$$B(z) \prec Q_c(z). \tag{53}$$

If p is analytic in U, $p(0) = 1/c$ and p satisfies the differential equation

$$zp'(z) + B(z)p(z) = 1 \tag{54}$$

then $\operatorname{Re} p(z) > 0$ *in* U.

The condition $\operatorname{Re} B(z) > 0$ for $z \in U$ implies condition (53) and so in this case we also have $\operatorname{Re} p(z) > 0$ in U. This "right-half plane" condition on B played a crucial role in several of the authors' previous articles on differential inequalities and integral operators. The new condition (53) involving the slit mapping Q_c opens the left-half plane through the opening ("open door") between the two slits of Q_c. This essentially doubles the region of variability of B and is a significant extention of earlier results. In the remainder of this article we shall extend some of our previous results by using this lemma.

The following theorem provides conditions for which the function $F = I[f]$ defined by (52) will be an analytic function. This result is a significant extension of a similar result in [23, Theorem 1].

THEOREM 7.3. [21]. *Let φ and Φ be analytic in U, with $\varphi(z)\Phi(z) \neq 0$ in U, and $\varphi(0) = \Phi(0) = 1$. Let α, β, γ and δ be complex numbers with $\beta \neq 0$, $\alpha + \delta = \beta + \gamma$ and $\operatorname{Re}(\alpha + \delta) > 0$. If $f \in A$ satisfies*

$$\alpha \frac{zf'(z)}{f(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec Q_{\alpha+\delta}(z), \tag{55}$$

where Q_c is defined by (53) *and F is given by* (52), *then* $F \in A$, $F(z)/z \neq 0$ *and*

$$\operatorname{Re}\left[\beta \frac{zF'(z)}{F(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0, \quad \text{for } z \in U. \tag{56}$$

Proof. From (55) we see that $f(z)/z \neq 0$ in U. Since $\operatorname{Re}(\alpha + \delta) > 0$, the function p defined by

$$p(z) = \frac{1}{z^{\alpha+\delta}\Phi(z)}\left[\frac{f(z)}{z}\right]^{-\alpha} \int_0^z \left[\frac{f(t)}{t}\right]^{\alpha} t^{\alpha+\delta-1}\varphi(t)\,dt = \frac{1}{z^{\delta}\Phi(z)f^{\alpha}(z)} \int_0^z f^{\alpha}(t)\, t^{\delta-1}\varphi(t)\,dt \tag{57}$$

is analytic in U with $p(0) = 1/(\alpha + \delta)$. By differentiating (57) we deduce that p satisfies the differential equation (54) with $c = \alpha + \delta$ and $B(z) = \alpha z f'(z)/f(z) + z\varphi'(z)/\varphi(z) + \delta$. By using (55) together with $p(0) = 1/(\alpha + \delta)$ we see that p satisfies Lemma 7.2 and so we have $\operatorname{Re} p(z) > 0$. Since $p(z) \neq 0$ in U, if we let

$$g(z) = \frac{1}{\beta}\left[\frac{1}{p(z)} - \frac{z\Phi'(z)}{\Phi(z)} - \gamma\right] \tag{58}$$

then g is analytic in U and $g(0) = 1$. From (52) and (57) we easily obtain

$$F(z) = z\left[\frac{(\beta + \gamma)p(z)\,\varphi(z)}{\Phi(z)}\right]^{1/\beta}\left[\frac{f(z)}{z}\right]^{\alpha/\beta} = z + \dots$$

Hence $F \in A$ and $F(z)/z \neq 0$. A simple computation shows that $zF'(z)/F(z) = g(z)$, and using this together with (58) we obtain

$$\operatorname{Re}[\beta zF'(z)/F(z) + z\Phi'(z)/\Phi(z) + \gamma] = \operatorname{Re}[1/p(z)] > 0$$

for $z \in U$, which completes the proof of the theorem.

The condition

$$\operatorname{Re}[\alpha z f'(z)/f(z) + z\varphi'(z)/\varphi(z) + \delta] > 0, \tag{59}$$

for $z \in U$ implies condition (55). This "right-half plane" condition given in (59) played a key role in the authors' previous article in defining integral operators of the form (52) [23, Theorem 1]. Many new results taking advantage of the "open door" condition (55) are given in [21].

If $\alpha = \beta$, $\gamma = \delta$ and $\varphi \equiv \Phi \equiv 1$ then Theorem 7.3 reduces to Theorem 2 in [25].

By employing condition (56) we can use Theorem 7.3 to otbain operators $I: A \to S$. A typical example is given in the following corollary, which follows from Theorem 7.3 by taking $\gamma = 0$ and $\Phi(z) \equiv 1$.

COROLLARY 7.4. *Let α and δ be complex numbers with* $\operatorname{Re}(\alpha + \delta) > 0$ *and let φ be analytic in U with $\varphi(0) = 1$ and $\varphi(z) \neq 0$ in U. If $f \in A$ satisfies* (55) *and if F is defined by*

$$F(z) = I[f](z) = \left[(\alpha + \delta)\int_0^z f^{\alpha}(t)\, t^{\delta - 1}\varphi(t)\, dt\right]^{1/(\alpha+\delta)}$$

then $F \in A$, $\operatorname{Re}[(\alpha + \delta)zF'(z)/F(z)] > 0$ *in U and $F \in S$. Moreover, if $\alpha + \delta > 0$ then $F \in S^*$.*

If we let $\delta = 0$ and $\varphi \equiv 1$ in this corollary we obtain the following example.

EXAMPLE 7.5. If $\operatorname{Re}\alpha > 0$, $f \in A$ and

$$\alpha z f'(z)/f(z) \prec Q_{\alpha}(z)$$

then

$$F(z) = I[f](z) = \left[\alpha\int_0^z f^{\alpha}(t)\, t^{-1}\, dt\right]^{1/\alpha}$$

is a spirallike function. If, in addition, α is real then F is a starlike function [25, Corollary 2.1].

By placing some additional restrictions on the function f in (52) it is possible to determine integral operators I such that $I(S^*) \subset S^*$ and $I(K) \subset S^*$. By placing restrictions on the functions f and φ in (52) it is possible to obtain integral operators I such that $I(S^* \times K) \subset S^*$ and $I(K \times K) \subset S^*$. See [21] for a complete discussion.

Special cases of (52) for which $I[S^*] \subset S^*$ and $I[\hat{S}] \subset \hat{S}$ were considered in [27] and [18] respectively. In these articles the order of starlikeness of $I[S^*]$ and order of spirallikeness of $I[\hat{S}]$ were obtained.

8. **Briot — Bouquet Differential Subordinations.** Let β and γ be complex numbers, let $h(z)$ be univalent in U, and let $p(z) = h(0) + p_1 z + \dots$ be analytic in U and satisfy

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z).$$

This first-order differential subordination is said to be of Briot—Bouquet type. This particular differential subordination has some special properties and has a surprising number of applications in the theory of univalent functions. This differential subordination and generalizations of it have been studied by the authors in [6], [16] and [17]. With suitable conditions on β, γ and h, dominants and best dominants were obtained by using the general method of differential subordinations. In each of the aforementioned papers an earlier version of Lemma 7.2 was employed with the more restrictive condition $\operatorname{Re} B(z) > 0$ replacing condition (53). We will state the results in this section using the more general form of the lemma; proofs will be omitted since they are very similar to the original proofs.

We first show an example of a problem in univalent function theory that reduces to finding a best dominant for the Briot—Bouquet differential subordination given above. Let β and γ be complex numbers with $\beta \neq 0$, let Q_c be the univalent function given in Definition 7.1, and define the following subclasses of analytic functions:

$$K_{\beta,\gamma} = \{f \in A : \operatorname{Re}(\beta z f'(z)/f(z) + \gamma) > 0\},$$
$$H_{\beta,\gamma} = \{f \in A : \beta z f'(z)/f(z) + \gamma \prec Q_{\beta+\gamma}\}. \tag{60}$$

As we've indicated in our discussion following Lemma 7.2, because of the open door function $Q_{\beta+\gamma}$ we have $K_{\beta,\gamma} \subset H_{\beta,\gamma}$.

If we set $\varphi \equiv \Phi \equiv 1$, $\alpha = \beta$ and $\delta = \gamma$ in (52) we obtain the operator

$$F(z) = I[f](z) = \left[\frac{\beta+\gamma}{z^{\gamma}} \int_0^z f^{\beta}(t)\, t^{\gamma-1}\, dt\right]^{1/\beta}. \tag{61}$$

If $f \in H_{\beta,\gamma}$ then from (60) and Theorem 7.3 we deduce that $F \in K_{\beta,\gamma}$, that is $I : H_{\beta,\gamma} \to K_{\beta,\gamma}$. A natural question to consider is the following: what would be the effect on F if an additional condition were imposed on f? For example, suppose h is univalent with $h(0) = 1$ and f satisfies $zf'(z)/f(z) \prec h(z)$. Does (61) imply the existence of another univalent function $q(z)$ with $q(0)=1$ such that

$$\frac{zf'(z)}{f(z)} \prec h(z) \Rightarrow \frac{zF'(z)}{F(z)} \prec q(z)? \tag{62}$$

If we set $p(z) = zF'(z)/F(z)$ then from (61) we obtain the Briot—Bouquet differential equation

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} = \frac{zf'(z)}{f(z)}.$$

Hence finding a function q satisfying (62) is equivalent to finding dominants of the Briot—Bouquet differential subordination

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z). \tag{63}$$

Finding the best dominant of this subordination will yield the sharp q solving (62). In [31] S. Ruscheweyh and V. Singh gave a partial answer to this problem in the special case when $h(z) = (1 + z)/(1 - z)$, $\beta > 0$ and $\mathrm{Re}\,\gamma \geq 0$.

We turn now to some results concerning dominants and best dominants of the Briot—Bouquet differential subordination.

THEOREM 8.1. [6]. *Let h be convex in U, with* $\mathrm{Re}[\beta h(z) + \gamma] > 0$ in U. *If p is analytic in U with $p(0) = h(0)$ and if p satisfies* (63) *then $p(z) \prec h(z)$.*

As a corollary of this theorem we obtain the following affirmative answer to the problem presented in (62).

COROLLARY 8.2. [14]. *Let h be convex in U with* $\mathrm{Re}[\beta h(z) + \gamma] > 0$ *in U and $h(0) = 1$. If $f \in A$ and $zf'(z)/f(z) \prec h(z)$, then F as given by* (61) *satisfies $zF'(z)/F(z) \prec h(z)$.*

Special cases of this theorem and its corollary with $h(z) = (1 + z)/(1 - z)$ and particular values of β and γ had been proved by many authors ([11], [2], [33] and [6]). In all of the cases the formulation and the technique of proof was very different from that of the Briot—Bouquet differential subordination.

We can obtain the sharp version of Theorem 8.1 and Corollary 8.2 if we impose a condition on the Briot—Bouquet differential equation.

THEOREM 8.3. [6]. *Let h be convex and suppose that the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z), \quad q(0) = h(0), \tag{64}$$

has a univalent solution q, which satisfies $q \prec h$. If p is analytic in U and satisfies (63) *then $p \prec q$, and q is the best dominant.*

Note that because of Theorem 8.1 we can replace the condition $q \prec h$ in Theorem 8.3 by the condition $\mathrm{Re}[\beta h(z) + \gamma] > 0$ in U [6].

COROLLARY 8.4. [14]. *Let $f \in A$ and let $g \in K_{\beta,\gamma}$ be such that $zg'(z)/g(z)$ is convex. Let $F = I[f]$ and $G = I[g]$, where I is given by* (61), *and suppose $zG'(z)/G(z)$ is univelant in U. If $zf'(z)/f(z) \prec zg'(z)/g(z)$ then $zF'(z)/F(z) \prec zG'(z)/G(z)$, and this result is sharp.*

In [6] a direct proof is given that the Briot—Bouquet differential equation (64) with $h(z) = [1 + (1 - 2\alpha)z]/(1 - z)$, $\beta > 0$ and $-\mathrm{Re}(\gamma/\beta) \leqslant \alpha < 1$ has a univalent solution. In fact, in this case and the general case, if $q(0) = a$, then (64) has a formal solution given by

$$q(z) = \begin{cases} H^{\gamma}(z)\left[\beta \int\limits_0^z H^{\gamma}(t)\, t^{-1}\, dt\right]^{-1} - \gamma/\beta, & \text{if } a = 0, \\ z^{\gamma}[H(z)]^{\beta a}\left[\beta \int\limits_0^z H^{\beta a}(t)\, t^{\gamma-1}\, dt\right]^{-1} - \gamma/\beta, & \text{if } a \neq 0, \end{cases} \tag{65}$$

where

$$H(z) = \begin{cases} z \exp \dfrac{\beta}{\gamma} \int\limits_0^z \dfrac{h(t)}{t}\, dt, & \text{if } a = 0, \\ z \exp \int\limits_0^z \dfrac{h(t) - a}{at}\, dt, & \text{if } a \neq 0. \end{cases}$$

Conditions for the analyticity and univalence of this q are given in the following two theorems.

THEOREM 8.5. *Let h be analytic in U with $h(0) = a$. If*

$$\mathrm{Re}(\beta a + \gamma) > 0 \text{ and } \beta h(z) + \gamma \prec Q_{\beta a + \gamma}(z) \tag{66}$$

where Q_c is given by Definition 7.1, then the solution q of (64) given by (65) is analytic in U and satisfies $\mathrm{Re}(\beta q(z) + \gamma) > 0$ in U.

Although some generality is lost, condition (66) can be replaced by $\mathrm{Re}[\beta h(z) + \gamma] > 0$. in U [16].

THEOREM 8.6. *Let h be analytic in U with $h(0) = a$, $h'(0) \neq 0$ and $\mathrm{Re}(\beta a + \gamma) > 0$. If $P(z) \equiv \beta h(z) + \gamma$ satisfies*

(i) $P \prec Q_{\beta a + \gamma}$

(ii) $P + 1 + zP''/P' - 2zP'/P \prec Q_{\beta a + \gamma + 1}$, *and*

(iii) zP'/P *is starlike* (log P *is convex*)

then the solution q of (64) given by (65) is univalent and satisfies $\mathrm{Re}[\beta q(z) + \gamma] > 0$ in U.

COROLLARY 8.7 [16]. *Let h be analytic in U with $h'(0) \neq 0$. If $P(z) \equiv \beta h(z) + \gamma$ satisfies*

(i) $\mathrm{Re}\, P(z) > 0$ *for $z \in U$, and*

(ii) $1/P$ *and* log P *are convex in U,*

then the solution q of (64) given by (65) is univalent and satisfies $\mathrm{Re}[\beta q(z) + \gamma] > 0$ in U.

Theorem 8.3 requires that $h(z)$ be convex. If this is the case, then $P(z) \equiv \beta h(z) + \gamma$ will also be convex. A simple computation shows that P and

$1/P$ convex imply $\log P$ is also convex. Because of this we can combine Theorem 8.3 and Corollary 8.7 and obtain the following more explicit result.

THEOREM 8.8. *Let h be analytic in U with $h'(0) \neq 0$. If $P(z) \equiv \beta h(z) + \gamma$ satisfies*

(i) $\operatorname{Re} P(z) > 0$ *for* $z \in U$, *and*

(ii) P *and* $1/P$ *are convex in* U,

then the solution q of (64) *given by* (65) *is univalent and is the best dominant of* (63).

Using this theorem we can now give a complete answer to the problem posed in (62).

COROLLARY 8.9. *Let $f \in A$, let $g \in K_{\beta,\gamma}$ and let $F \equiv I[f]$ and $G \equiv I[g]$, where I is given by* (61). *If $P(z) \equiv \beta z g'(z)/g(z) + \gamma$ satisfies conditions* (i) *and* (ii) *of Theorem* 8.8, *then*

$$\frac{zf'(z)}{f(z)} \prec \frac{zg'(z)}{g(z)} \Rightarrow \frac{zF'(z)}{F(z)} \prec \frac{zG'(z)}{G(z)},$$

and this latter subordination is sharp.

Several generalizations of the basic Briot—Bouquet differential subordination (63) have recently appeared ([17], [3], [4], [5] and [28]).

In [17] the authors considered generalizations of the form

$$\theta[p(z)] + zp'(z)\eta[p(z)] \prec h(z), \tag{67}$$

where $\theta(w)$ and $\eta(w)$ are analytic in a domain $D \supset p(U)$, and $h(z)$ is univalent. (If $\theta(w) = w$, $\eta(w) = (\beta w + \gamma)^{-1}$ and h is convex then (67) reduces to the Briot—Bouquet differential subordination discussed previously). They determined conditions on θ, η and h so that dominants and best dominants of (67) could be obtained, that is for $\psi(r, s) \equiv \theta(r) + s\eta(r)$ and h they found a q such that

$$\psi[p(z), zp'(z)] \prec h(z) \Rightarrow p(z) \prec q(z). \tag{68}$$

The authors in [17] also considered the converse problem: given a univalent function q and a function $\psi: \mathbf{C}^2 \to \mathbf{C}$, then determine the largest class of functions h such that (68) will hold. This is an example of Problem 3′ referred to in Section 1. Typical of their results is the following theorem, which has many applications.

THEOREM 8.10 [17, Theorem 3]. *Let q be univalent in U and let θ and η be analytic in a domain D containing $q(U)$, with $\eta(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\eta[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that*

(i) Q *is starlike in* U, *and*

(ii) $\operatorname{Re} \dfrac{zh'(z)}{Q(z)} = \operatorname{Re}\left[\dfrac{\theta'[q(z)]}{\eta[q(z)]} + \dfrac{zQ'(z)}{Q(z)}\right] > 0, \quad z \in U.$

If p is analytic in U, with $p(0) = q(0)$, $p(U) \subset D$ and

$$\theta[p(z)] + zp'(z)\eta[p(z)] \prec h(z) = \theta[q(z)] + zq'(z)\eta[q(z)] \tag{69}$$

then $p \prec q$, and q is the best dominant of (69).

EXAMPLE 8.11. If we chose $q(z) = (1+z)/(1-z)$, and if we take $\theta(w) = w$ and $\eta(w) = 1/w$, then Theorem 8.10 will be satisfied and we have: if $p(z)$ is analytic in U, with $p(0) = 1$, then

$$p(z) + \frac{zp'(z)}{p(z)} \prec \frac{1+z}{1-z} + \frac{2z}{1-z^2} \Rightarrow p(z) \prec \frac{1+z}{1-z},$$

and $q(z) = (1+z)/(1-z)$ is the best dominant.

An interesting application of this differential subordination shows that if f is analytic in U, with $f(0) = 0$ and $|f''(z)/f'(z)| \leqslant 2$ for $z \in U$, then f is a starlike function [17, Theorem 4].

9. **Integral Operators Preserving Subordination.** Let $E \subset H$ and let I be an integral operator $I : E \to H$. In this section we apply the theory of differential subordinations to determine conditions under which

$$f \prec g \Rightarrow I[f] \prec I[g]. \tag{70}$$

We will call such operators *subordination-preserving*.

A few subordination-preserving integral operators have already appeared in the literature. In 1935 G. M. Goluzin [7] considered the operator I: $H_0 \to H$ defined by

$$I[f](z) = \int_0^z f(t)\, t^{-1}\, dt.$$

He showed that if g is convex then (70) is satisfied. In 1970 T. Suffridge [32] extended this result to the case when g is starlike. In 1981 the authors [14], using the theory of differential subordinations, extended these results by considering the operator $I : H_0 \to H$ defined by

$$I[f](z) = \left[\int_0^z f^{\beta}(t)\, t^{-1}\, dt\right]^{1/\beta}.$$

They showed that if $\beta \geqslant 1$ and if g is starlike then (70) is satisfied.

In 1947 R. Robinson [30] considered the differential subordination

$$[zF(z)]' \prec [zG(z)]',$$

with $F(0) = G(0)$ and showed that $F(rz) \prec G(rz)$, for $r \leqslant 1/5$. If we let $f(z) = [zF(z)]'$ and $g(z) = [zG(z)]'$ then this result can be rewritten as

$$f(z) \prec g(z) \Rightarrow I[f](rz) \prec I[g](rz) \tag{71}$$

for $r \leqslant 1/5$, where $I : H \to H$ is defined by

$$I[f](z) = \frac{1}{z}\int_0^z f(t)\, dt \tag{72}$$

and g is univalent in U. We will extend this result in this section.

In 1975 D. Hallenbeck and S. Ruscheweyh [9] showed that if $\gamma \neq 0$, $\operatorname{Re}\gamma \geqslant 0$, and g is convex then the operator $I : H \to H$, defined by

$$I[f](z) = z^{-\gamma}\int_0^z f(t)\, t^{\gamma-1}\, dt,$$

satisfies (70). Another proof of this result using differential subordinations is given in [14].

We can generalize all of the above operators by considering the operator $I : E \to H$ defined by

$$I[f](z) = \left[z^{-\gamma}\int_0^z f^{\beta}(t)\, t^{\gamma-1}\, dt\right]^{1/\beta}. \tag{73}$$

The two main theorems of this section show that the operator (73) is subordination-preserving on appropriate subsets of H and for suitable complex constants β and γ. We first define the appropriate subsets. Let β and γ be complex numbers with $\operatorname{Re}\beta > 0$ and $\operatorname{Re}\gamma \geqslant 0$, and let $E_{\beta,\gamma}$ be defined as follows:

$$E_{\beta,\gamma} = \begin{cases} H, & \text{if } \beta = 1,\ \gamma \neq 0 \\ \{f \in H;\ f(0) = 0\}, & \text{if } \beta = 1,\ \gamma = 0 \\ \{f \in H;\ f(z) = z^j h(z),\ h(z) \neq 0,\ j \geqslant 1\}, & \text{if } 1/\beta \in N\setminus\{1\} \\ \{f \in H;\ f(0) = 0,\ f'(0) \neq 0,\ \operatorname{Re}(\beta z f'/f + \gamma) > 0\}, & \text{otherwise.} \end{cases}$$

A straightforward examination of the first three cases shows that the operator I as given by (73) is well defined. That I is well defined in the last case follows from Theorem 7.3. Note that $E_{\beta,0}$ with $\beta > 0$ and $1/\beta \notin N$ is the class of starlike functions.

THEOREM 9.1 [24]. *Let $f \in E_{\beta,0}$ with $\beta > 0$, and let $g(z) = b_1 z + b_2 z^2 + \cdots$ be starlike in U. If the operator $I : E_{\beta,0} \to H$ is defined by*

$$I[f](z) = \left[\int_0^z f^{\beta}(t)\, t^{-1}\, dt\right]^{1/\beta},$$

then $I[g]$ is univalent and $f \prec g \Rightarrow I[f] \prec I[g]$.

THEOREM 9.2 [24]. *Let $\operatorname{Re}\beta > 0$, $\operatorname{Re}\gamma \geqslant 0$ and $f, g \in E_{\beta,\gamma}$ with $g'(0) \neq 0$. Let*

$$\operatorname{Re}\left[(\beta - 1)\frac{zg'(z)}{g(z)} + \left(\frac{zg''(z)}{g'(z)} + 1\right)\right] > -\delta, \qquad z \in U,$$

where $\delta = \operatorname{Min}\{\operatorname{Re}\gamma,\ 2\operatorname{Re}\beta\operatorname{Re}\gamma/(|\beta+\gamma| + |\beta-\gamma|)^2\}$. If the operator $I : E_{\beta,\gamma} \to H$ is defined by (73) then $I[g]$ is univalent and $f \prec g \Rightarrow I[f] \prec I[g]$.

Each of these theorems was proved by using the theory of differential subordinations. Special choices of g in each of these theorems can be used to obtain new distortion theorems for some classes of analytic functions. As examples of each of them we present respectively the following examples:

EXAMPLE 9.3. If f is analytic in U then

$$f(z) \prec \frac{z}{(1+z)^2} \Rightarrow \left[\int_0^z \frac{f^{1/2}(t)}{t}\,dt\right]^2 \prec [2 \arctan z^{1/2}]^2.$$

In addition,

$$-\pi/2 < -2 \arctan r^{1/2} \leqslant \operatorname{Re}\int_0^z \frac{f^{1/2}(t)}{t}\,dt \leqslant 2 \arctan r^{1/2} < \pi/2.$$

for $|z| \leqslant r$.

EXAMPLE 9.4. If f is analytic in U then

$$f(z) \prec \frac{z}{1+z} \Rightarrow \left[\frac{1}{z}\int_0^z \left[\frac{f(t)}{t}\right]^{1/2} dt\right]^2 \prec \frac{4z}{(1+\sqrt{1+z})^2}.$$

If we set $\beta = \gamma = 1$ in Theorem 9.2 then $\delta = 1/2$ and we obtain the following particular result.

COROLLARY 9.5. *If* $f, g \in H$, $g'(0) \neq 0$ *and*

$$\operatorname{Re}\left[\frac{zg''(z)}{g'(z)} + 1\right] > -\frac{1}{2}, \tag{74}$$

then $I[g]$ *is univalent and*

$$f(z) \prec g(z) \Rightarrow \frac{1}{z}\int_0^z f(t)\,dt \prec \frac{1}{z}\int_0^z g(t)\,dt.$$

This corollary improves the result of D. Hallenbeck and S. Ruscheweyh [9, p. 192], who proved the conclusion with (74) replaced by $\operatorname{Re}[zg''(z)/g'(z) + 1] > 0$ in U.

Corollary 9.5 can be used to improve the Robinson result given by (71) and (72). If g is univalent in U, then it is easy to show that (74) holds for $|z| \leqslant r_0 = 4 - \sqrt{13} = .3944\ldots$ Using $f(r_0 z) \prec g(r_0 z)$ in Corollary 9.5 we deduce the following result.

THEOREM 9.6. *If* g *is univalent in* U *and* $I: H \to H$ *is defined by* (72) *then*

$$f(z) \prec g(z) \Rightarrow I[f](rz) < I[g](rz)$$

for $r \leqslant 4 - \sqrt{13} = .3944\ldots$

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ON THE OLOVYANIŠNIKOV–SCHOENBERG THEOREM AND SOME SIMILAR RESULTS

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REZUMAT. — Asupra teoremei Olovyanišnikov-Schoenberg şi cîteva rezultate similare. În lucrare se dau generalizări ale inegalităţii Olovyanišnikov-Schoenberg pentru funcţii k-convexe precum şi cîteva rezultate similare.

0. The following result is given in [3] and [4].

THEOREM A. *Let g be k-convex for $k = 0, \ldots, n+1$ on $(-\infty, 0]$, and assume that g satisfies*

$$g(0) \leqslant M \text{ and } g^{(n)}(0)/n! \leqslant M_n.$$

Then

$$g^{(j)}(x)/j! \leqslant \binom{n}{j} M_0^{(n-j)/n} M_n^{j/n} \; (j = 1, \ldots, n-1). \tag{1}$$

Equality holds in (1) for some $1 \leqslant j \leqslant n-1$ iff

$$g(x) = M_n(l + x)^n \text{ on } [-l, 0] \text{ and } g \equiv 0 \text{ in } (-\infty\ -l) \tag{2}$$

where $l^n = M_0/M_n$.

We shall use the notation:

$[x_0, \ldots, x_n]f$ is divided defference of f;
$f \in W^n_\infty[a, b] := \{f : f^{(n-1)}$ is absolutely continuous and $f^{(n)} \in L_\infty[a, b]\}$;

where the norm in L_∞ is given by $||f||_\infty = \text{esssup}\{|f(x)| : x \in [a, b]\}$,

$P \in \Pi_n$ implies P is a polynomial of degree at most n.

A simple proof of Theorem A is given in [1], where the following result is used:

THEOREM B. *Let $h \in W^n_\infty[a, b]$ and assume that*
a) *h is n-convex on $[a, b]$ and*
b) *there are points $a \leqslant \xi_0 \leqslant \ldots \leqslant \xi_{n-1} \leqslant b$ such that*

$$[\xi_k, \ldots, \xi_{n-1}]h \geqslant 0 \text{ for every } k = 0, \ldots, n-1.$$

Then $h^{(k)}(b) \geqslant 0$ for every $k = 0, \ldots, n-1$. If for some $0 \leqslant m \leqslant n-1$ $h^{(m)}(b) = 0$ and $\xi_{n-m-1} < b$ then $h \in \Pi_{m-1}$ on $[\xi_0, b]$.

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1. First we shall give generalization of Theorem 1 from [1].

THEOREM 1. *Let* $f, g \in W^n_\infty[a, b]$ *and suppose that:*

$$g(b) \leqslant f(b),\ g^{(k)}(a) \geqslant f^{(k)}(a)\ \ (k = 0, \ldots, n-2) \tag{3}$$

and

$$[x_0, \ldots, x_{n-1}, b]f \geqslant [x_0, \ldots, x_{n-1}, b]g \text{ for all } a \leqslant x_0 \leqslant \ldots \leqslant x_{n-1} \leqslant b \tag{4}$$

with $x_0 \neq x_{n-1}$.

Then $g^{(k)}(b) \leqslant f^{(k)}(b)$ $(k = 1, \ldots, n-1)$. *If* $g^{(m)}(b) = f^{(m)}(b)$ for *some* $1 \leqslant m \leqslant n-1$, *then* $g \equiv f$.

Proof. Let in Theorem B be $n \to n-1$, $h \to [x, b]\,H$, $H = f - g$, $\xi_i = a$ $(i = 0, \ldots, n-2)$. Then $h^{(k)}(b) \geqslant 0$ for $k = 0, \ldots, n-2$ become $f^{(k)}(b) \geqslant g^{(k)}(b)$ for $k = 1, \ldots, n-1$ because

$$(b-a)^k \underbrace{[a, \ldots, a]}_{k \text{ times}}([x, b]H) = \underbrace{[a, \ldots, a,}_{k \text{ times}} b]H(b-a)^k =$$

$$= H(b) - \sum_{j=0}^{k-1} H^{(j)}(a)(b-a)^j/j! \geqslant 0, \text{ for } k = 1, \ldots, n-1.$$

Now, assume that $H^{(m)}(b) = 0$ for some $1 \leqslant m \leqslant n-1$. This is equivalent to $h^{(m-1)}(b) = 0$, then by Theorem B, $h \in \Pi_{m-2}$, i.e., $H \in \Pi_{m-1}$, and thus

$$0 = (b-a)^m \underbrace{[a, \ldots, a,}_{m \text{ times}} b]H = H(b) - \sum_{j=0}^{m-1} H^{(j)}(a)(b-a)^j/j!.$$

Since by (3) $H(b) \geqslant 0$ and $H^{(j)}(a) \leqslant 0$ for each j in the sum, we get $H^{(j)}(a) = 0$ $(j = 0, \ldots, m-1)$ and hence $H \equiv 0$.

COROLLARY 1.1. *Let* $f, g \in C^n[a, b]$ *and suppose that* (3) *and*

$$f^{(n)}(x) \geqslant g^{(n)}(x) \text{ for all } x \in [a, b]$$

hold. Then the conclusion of Theorem 1 is also valid.

Proof. We shall use the following theorem of representation due to D. V. Ionescu (see [2] p. 54):

$$\underbrace{[x_0, \ldots, x_0,}_{n_0 \text{ times}} \underbrace{x_1, \ldots, x_1,}_{n_1 \text{ times}} \ldots, \underbrace{x_k, \ldots, x_k}_{n_k \text{ times}}]f = \int_{x_0}^{x_k} \varphi(s) f^{(n)}(s)\, ds$$

where $x_0 < x_1 < \ldots < x_k$, $n_0 + n_1 + \ldots + n_k = n+1$ and $f \in C^n[x_0, x_k]$.

By this fact, the condition (4) is equivalent to

$$\int_{x_0}^{b} \varphi(s)\,(f^{(n)}(s) - g^{(n)}(s))\, ds \geqslant 0$$

and since φ is positive on (x_0, x_k) (see [2] p. 74), then (4′) implies (4) and Corollary is proven.

COROLLARY 1.2. *If in addition to the conditions of Theorem 1, $g^{(k)}$ is an increasing function for any $1 \leqslant k \leqslant n-1$, then $g^{(k)}(x) \leqslant f^{(k)}(b)$ for all $x \in [a, b]$.*

COROLLARY 1.3. *If g is k-convex for $k = 0, \ldots, n$ on $(-\infty, 0]$ and satisfies:*

$$g(0) \leqslant M_0 \text{ and } 1!/n[x_0, \ldots, x_{n-1}, 0]\, g \leqslant M_n \tag{5}$$

for all $-l \leqslant x_0 \leqslant \ldots \leqslant x_{n-1} \leqslant 0$ where $l^n = M_0/M_n$, then (1) is valid with equality if g is given by (2).

COROLLARY 1.4. *If g is k-convex for $k = 0, \ldots, n$ on $(-\infty, 0]$ and satisfies*

$$g(0) \leqslant M_0 \text{ and } 1/n!\, g^{(n)}(x) \leqslant M_n \ (\text{for all } x \in [-l, 0]) \tag{6}$$

then (1) is valid with the same conditions for equality as in Corollary 1.3.

Proof. This is a simple consequence of Corollary 1.3.

Remark 1. If the function g is also $(n+1)$-convex, then $g^{(n)}$ is nondecreasing and the second condition in (6) is equivalent to $g^{(n)}(0)/n! \leqslant M_n$ what is equivalent with Theorem A.

2. The following theorem is valid.

THEOREM 2. *Let $f, g \in W^n_\infty[a, b]$ and suppose that*

$$|g(b)| \leqslant M_n(b-a)^n = M_0,\ g^{(k)}(a) = 0 \ (k = 0, \ldots, n-2) \tag{7}$$

and

$$|1/n!\,[x_0, \ldots, x_{n-1}, b]g| \leqslant M_n \text{ for all } a \leqslant x_0 \leqslant \ldots \leqslant x_{n-1} \leqslant b. \tag{8}$$

Then the following inequality holds:

$$|1/j!\, g^{(j)}(b)| \leqslant \binom{n}{j} M_n(b-a)^{n-j} = \binom{n}{j} M_0^{(n-j)/n} M_n^{j/n}. \tag{9}$$

Proof. Let in Theorem 1 be $f(x) = M_n(x-a)^n$ then (3) and (4) become

$$g(b) \leqslant M_n(b-a)^n,\ g^{(k)}(a) \geqslant 0 \tag{10}$$

$$[x_0, \ldots, x_{n-1}, b]g \leqslant M_n \tag{11}$$

and $g^{(k)}(b) \leqslant f^{(k)}(b)$ $(k = 1, 2, \ldots, n-1)$ become

$$1/k!\, g^{(k)}(b) \leqslant \binom{n}{k} M_n(b-a)^{n-k}. \tag{12}$$

Put in Theorem 1; $g(x) = -M_n(x-a)^n$, $f \to g$, then (3) and (4) become

$$g(b) \geqslant -M_n(b-a)^n,\ g^{(k)}(a) \leqslant 0 \tag{13}$$

$$[x_0, \ldots, x_{n-1}, b]\, g \geqslant -M_n \tag{14}$$

and in this case we have:

$$1/k!\, g^{(k)}(b) \geqslant -\binom{n}{k} M_n(b-a)^{n-k}. \tag{15}$$

It is obvious that (10) and (13) are equivalent to (7), (11) and (14) with (8), and (12) and (15) with (9).

COROLLARY 2.1. *Let* $f, g \in W^n_\infty[a, b]$ *and suppose that* (7) *and*

$$|1/n![x_0, \ldots, x_n]g| \leqslant M_n \text{ for all } a \leqslant x_0 \leqslant \ldots \leqslant x_n \leqslant b \tag{16}$$

are valid. Then (9) *is valid too.*

COROLLARY 2.2. *Let* $g \in C^n[a, b]$ *and suppose that*

$$|g(b)| \leqslant \|g^{(n)}\|_\infty/(n!)^2(b-a)^n, \; g^{(k)}(a) = 0 \tag{17}$$

$$(k = 0, \ldots, n-2).$$

Then the following inequality holds

$$|g^{(j)}(b)| \leqslant \|g^{(n)}\|_\infty/(n!(n-j)!)(b-a)^{n-j} \text{ for all } 0 \leqslant j \leqslant n. \tag{18}$$

Proof. It is well known that:

$$|[x_0, x_1, \ldots, x_{n-1}, b]g| \leqslant \|g^{(n)}\|_\infty/n! \text{ (see for example [2], p. 76).}$$

Putting $M_n := 1/(n!)^2 \|g^{(n)}\|_\infty$ in (7) and (8), we get by Theorem 2:

$$|1/j!\, g^{(j)}(b)| \leqslant n!/((n-j)!j!)\; 1/(n!)^2 \; \|g^{(n)}\|_\infty (b-a)^{n-j}$$

what implies (18).

3. We start to the following definition:

DEFINITION 1. A mapping $f: [a, b] \to \mathbb{R}$ is called *n-FZ-convex on* $[a, b]$ if there are points: $a \leqslant \xi_0 \leqslant \ldots \leqslant \xi_n \leqslant b$, such that

$$[\xi_k, \ldots, \xi_n]f \geqslant 0 \text{ for all } k = 0, 1, \ldots, n. \tag{19}$$

It is clear the fact that if f is k-convex ($k = 0, 1, \ldots, n$) on the interval $[a, b]$ then f is n-FZ-convex on the every subinterval $[a, x]$ of $[a, b]$.

The following characterization theorem is valid.

THEOREM 3. *Let* $f \in C^n[a, b]$. *Then the following statements are equivalent*:

(i) f *is* k*-convex* ($k = 0, 1, \ldots, n$) *on* $[a, b]$;

(ii) f *is* n*-convex on* $[a, b]$ *and* $(n-1)$*-FZ-convex on every subinterval* $[a, x]$ *of* $[a, b]$.

Proof. "(i) ⇒ (ii)". It's obvious.

"(ii) ⇒ (i)". If f is n-convex on $[a, b]$ and $(n-1)$-FZ-convex on every subinterval $[a, x]$, then by Theorem of Farwing—Zwick (see Theorem B) we obtain $f^{(k)}(x) \geqslant 0$ for all $k = 0, \ldots, (n-1)$, i.e., f is k-convex ($k = 0, \ldots, n$) on $[a, b]$ and the theorem is proven.

Now, we give another definition.

DEFINITION 2. A mapping $f \in C^n[a, b]$ is called *n-Taylor convex on* $[a, b]$ if $f(x) \geqslant 0$ on $[a, b]$ and:

$$f(x) \geqslant f(a) + (x-a)/1!f^{(1)}(a) + \ldots + (x-a)^{n-1}/(n-1)!f^{(n-1)}(a) \tag{20}$$

$$\text{for all } x \in [a, b] \; (n \geqslant 1)$$

Remark 2. It is easy to see that if $f \in C^n[a, b]$ is n-convex on $[a, b]$ then f is n-Taylor convex on $[a, b]$.

Indeed, since:

$$f(x) - [f(a) + (x-a)/1!\,f^{(1)}(a) + \ldots + (x-a)^{n-1}/(n-1)!\,f^{(n-1)}(a)] = f^{(n)}(\xi_x)/n!(x-a)^n, \ \xi_x \in [a, x],$$

and $f^{(n)}(\xi_x) \geqslant 0$ for every $x \in [a, b]$, the Remark is proven.

The following lemma is also valid.

LEMMA. *Let $f \in C^n[a, b]$. If f is k-Taylor convex ($k = 0, \ldots, n-1$) on $[a, b]$, then f is $(n-1)$-FZ-convex on every subinterval $[a, x]$ of $[a, b]$.*

Proof. We start to the following equality

$$[a, a, \ldots, a, x] f (x-a)^m = f(x) - \sum_{j=0}^{m-1} f^{(j)}(a)(x-a)^j/j!$$

for all $1 \leqslant m \leqslant n-1$ and $x \in [a, b]$.

Since f is k-Taylor convex ($k = 0, \ldots, n-1$) we have for $1 \leqslant m \leqslant n-1$:

$$f(x) \geqslant \sum_{j=0}^{m-1} f^{(j)}(a)(x-a)^j/j! \text{ for all } x \in [a, b]$$

and putting $\xi_0 = a, \ldots, \xi_{n-2} = a, \ \xi_{n-1} = x$ it results

$$[\xi_k, \ldots, \xi_{n-1}] f \geqslant 0 \text{ for all } k = 0, 1, \ldots, n-1,$$

i.e., f is $(n-1)$-FZ-convex on every subinterval $[a, x]$ of $[a, b]$.

THEOREM 4. *Let $f \in C^n[a, b]$. Then the following assertions are equivalent:*

(i) *f is k-convex ($k = 0, 1, \ldots, n$) on $[a, b]$;*

(ii) *f is n-convex and k-Taylor convex ($k = 0, 1, \ldots, n-1$) on $[a, b]$.*

The proof follows by Theorem 3 and by the above lemma. We omit the details.

COROLLARY. *Let $f, g \in C^n[a, b]$. If*

1. *$f^{(n)} \geqslant g^{(n)}$ on $[a, b]$;*

2. *$f - g$ is k-Taylor convex ($k = 0, 1, \ldots, n-1$) on $[a, b]$, then $f^{(k)} \geqslant g^{(k)}$ on $[a, b]$ for all $k = 0, 1, \ldots, n$.*

Remark 3. Let $f \in C^2[a, b]$. Then the following sentences are equivalent

(i) f is nonnegative, nondecreasing and convex on $[a, b]$;

(ii) f is convex on $[a, b]$ and for all $x \in [a, b]$ there are points $a \leqslant \xi_0 \leqslant \xi_1 \leqslant \xi_2 \leqslant x$ such that:

$$[\xi_0, \xi_1, \xi_2] f, \ [\xi_1, \xi_2] f, \ f(\xi_2) \geqslant 0;$$

(iii) f is nonnegative convex on $[a, b]$ and for all $x \in [a, b]$

$$f(x) \geqslant f(a).$$

The proof follows by Theorem 3 and 4 for $n = 2$.

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A GENERALIZATION OF BÉZIER SURFACES

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REZUMAT. — **O generalizare a suprafeţelor Bézier.** În lucrare se dă o generalizare a suprafeţelor Bézier, înlocuind funcţiile $b_{p,k}$ cu funcţiile $w_{p,k,r}$ (vezi [5]).

A Bézier surface associated to a set of $(m+1)(n+1)$ points $P_{ij} \in \mathbb{R}^3$ is represented by the vectorial equation (see [1])

$$B(u, v) = \sum_{i=0}^{m}\sum_{j=0}^{n} b_{m,i}(u) b_{n,j}(v) P_{ij}, \tag{1}$$

where

$$b_{p,k}(t) = \binom{p}{k} t^k (1-t)^{p-k}, \quad t \in [0, 1]. \tag{2}$$

The Bézier surface (1) lies only on the points P_{00}, P_{m0}, P_{0n} and P_{mn}; the others P_{ij} $i = \overline{1, m-1}$, $j = \overline{1, n-1}$ determine the derivatives of vectorial function B along the curves $u = c_1$, $c_1 \in \{0, 1\}$ and $v = c_2$, $c_2 \in \{0, 1\}$ respectively; also, these points are used to define twist vectors on the points P_{00}, P_{m0}, P_{0n} and P_{mn}.

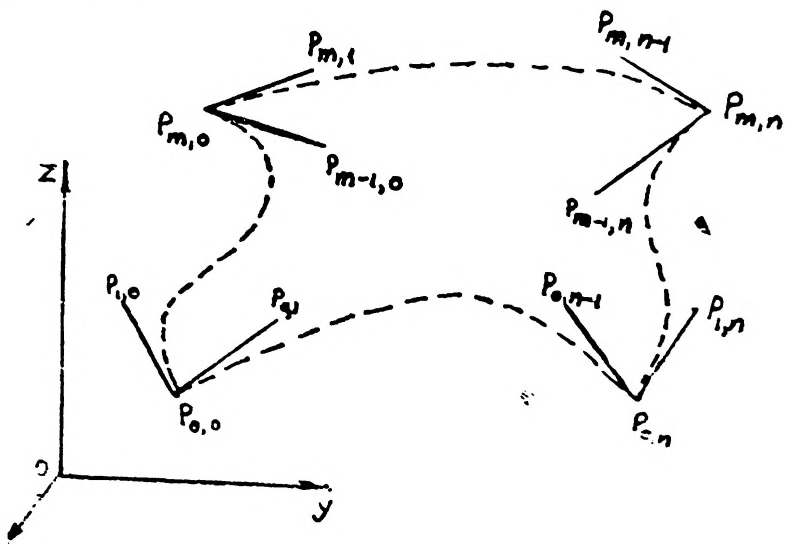

Fig. 1

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Indeed, from (1) and (2) one deduces

$$B(0,0) = P_{00}, \quad B(0,1) = P_{0n}, \quad B(1,0) = P_{m0}, \quad B(1,1) = P_{mn} \tag{3}$$

and taking account by [1] and [3] we obtain

$$\begin{aligned}
B^{(p,0)}(0,v) &= A_m^p \sum_{i=0}^{p} \sum_{j=0}^{n} (-1)^{p-i} \binom{p}{i} b_{n,j}(v) P_{ij} \\
B^{(p,0)}(1,v) &= A_m^p \sum_{i=0}^{p} \sum_{j=0}^{n} (-1)^{i} \binom{p}{i} b_{n,j}(v) P_{m-i,j} \\
B^{(0,q)}(u,0) &= A_n^q \sum_{i=0}^{m} \sum_{j=0}^{q} (-1)^{q-j} \binom{q}{j} b_{m,i}(u) P_{ij} \\
B^{(0,q)}(u,1) &= A_n^q \sum_{i=0}^{m} \sum_{j=0}^{q} (-1)^{j} \binom{q}{j} b_{m,i}(u) P_{i,n-j} \\
B^{(p,q)}(0,0) &= A_m^p A_n^q \sum_{i=0}^{p} \sum_{j=0}^{q} (-1)^{p+q-i-j} \binom{p}{i} \binom{q}{j} P_{ij} \\
B^{(p,q)}(0,1) &= A_m^p A_n^q \sum_{i=0}^{p} \sum_{j=0}^{q} (-1)^{p-i+j} \binom{p}{i} \binom{q}{j} P_{i,n-j} \\
B^{(p,q)}(1,0) &= A_m^p A_n^q \sum_{i=0}^{p} \sum_{j=0}^{q} (-1)^{q+i-j} \binom{p}{i} \binom{q}{j} P_{m-i,j} \\
B^{(p,q)}(1,1) &= A_m^p A_n^q \sum_{i=0}^{p} \sum_{j=0}^{q} (-1)^{i+j} \binom{p}{i} \binom{q}{j} P_{m-i,n-j},
\end{aligned} \tag{4}$$

where

$$A_n^k = \frac{n!}{(n-k)!}.$$

In this paper we give a generalization of Bézier surface (1), changing the functions $b_{p,k}$ with functions $w_{p,k,r}$ (see [5]), where

$$w_{p,k,r}(t) = \begin{cases} \binom{p-r}{k} t^k (1-t)^{p-r-k+1}, & \text{if } 0 \leqslant k < r, \\ \binom{p-r}{k} t^k (1-t)^{p-r-k+1} + \binom{p-r}{k-r} t^{k-r+1} (1-t)^{p-r}, & \text{if } r \leqslant k \leqslant p-r, \\ \binom{p-r}{k-r} t^{k-r+1} (1-t)^{p-k}, & \text{if } p-r < k \leqslant p, \end{cases} \tag{5}$$

r being a non-negative integer such that $2r < p$.

Taking in view [3], we have succesively

$$S(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} w_{m,r,i}(u) w_{n,p,j}(v) P_{ij} =$$

$$= \sum_{i=0}^{m} w_{m,r,i}(u) \sum_{j=0}^{n-s} b_{n-s,j}(v) [\underbrace{P_{ij} + v(P_{i,s+j} - P_{ij})}_{Q_{ij}}] =$$

$$= \sum_{j=0}^{n-s} b_{n-s,j}(v) \sum_{i=0}^{m-r} b_{m-r,i}(u) [Q_{ij} + u(Q_{r+i} - Q_{ij})] =$$

$$= \sum_{i=0}^{m-r} \sum_{j=0}^{n-s} b_{m-r,i}(u) b_{n-s,j}(v) \{P_{ij} + v(P_{i,s+j} - P_{ij}) +$$

$$+ u[P_{r+i,j} + v(P_{r+i,s+j} - P_{r+i,j}) - P_{ij} - v(P_{i,s+j} - P_{ij})]\} =$$

$$= \sum_{i=0}^{m-r} \sum_{j=0}^{n-s} b_{m-r,i}(u) b_{n-s,j}(v) [P_{ij} + u(P_{r+i,j} - P_{ij}) +$$

$$+ v(P_{i,s+j} - P_{ij}) + uv(P_{r+i,s+j} - P_{i,s+j} - P_{r+i,j} + P_{ij})].$$

Therefore, we have obtaind the following vectorial equation

$$S(u, v) = \sum_{i=0}^{m-r} \sum_{j=0}^{n-s} b_{m-r,i}(u) b_{n-s,j}(v) [P_{ij} + u(P_{r+i,j} - P_{ij}) +$$

$$+ v(P_{i,s+j} - P_{ij}) + uv(P_{r+i,s+j} - P_{i,s+j} - P_{r+i,j} + P_{ij})], \tag{6}$$

which turns to equation (1) for $(r, s) \in \{0, 1\} \times \{0, 1\}$.

Using [3], formulas (7) and (8), we deduce

$$S^{(p,0)}(0, v) = A^{p}_{m-r} \sum_{j=0}^{n} w_{n,s,j}(v) A_j$$

$$S^{(p,0)}(1, v) = A^{p}_{m-r} \sum_{j=0}^{n} w_{n,s,j}(v) B_j$$

$$S^{(0,q)}(u, 0) = A^{q}_{n-s} \sum_{i=0}^{m} w_{m,r,i}(u) C_i$$

$$S^{(0,q)}(u, 1) = A^{q}_{n-s} \sum_{i=0}^{m} w_{m,r,i}(u) D_i \tag{7}$$

$$S^{(p,q)}(0, 0) = A^{p}_{m-r} A^{q}_{n-s} \sum_{j=0}^{q} (-1)^{q-j} \binom{q}{j} A_j = A^{p}_{m-r} A^{q}_{n-s} \sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} C_i$$

$$S^{(p,q)}(0, 1) = A^{p}_{m-r} A^{q}_{n-s} \sum_{j=0}^{q} (-1)^{j} \binom{q}{j} A_{n-j} = A^{p}_{m-r} A^{q}_{n-s} \sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} D_i$$

$$S^{(p,q)}(1,0) = A^p_{m-r}A^q_{n-s}\sum_{j=0}^{q}(-1)^{q-j}\binom{q}{j}B_j = A^p_{m-r}A^q_{n-s}\sum_{i=0}^{p}(-1)^i\binom{p}{i}C_{m-i}$$

$$S^{(p,q)}(1,1) = A^p_{m-r}A^q_{n-s}\sum_{j=0}^{q}(-1)^j\binom{q}{j}B_{n-j} = A^p_{m-r}A^q_{n-s}\sum_{i=0}^{p}(-1)^i\binom{p}{i}D_{m-i}$$

where,

$$A_j = P_{p,j} + p\sum_{i=0}^{p-1}(-1)^{p-i}\binom{p-1}{i}\left(\frac{P_{ij}}{p-i} + \frac{P_{ij} - P_{r+i,j}}{m-r-p+1}\right)$$

$$B_j = (-1)^p P_{m-p,j} + p\sum_{i=0}^{p-1}(-1)^i\binom{p-1}{i}\left(\frac{P_{m-i,j}}{p-i} + \frac{P_{m-i,j} - P_{m-r-i,j}}{m-r-p+1}\right)$$

$$C_i = P_{i,q} + q\sum_{j=0}^{q-1}(-1)^{q-j}\binom{q-1}{j}\left(\frac{P_{ij}}{q-j} + \frac{P_{ij} - P_{i,s+j}}{n-s-q-1}\right)$$

$$D_i = (-1)^q P_{i,n-q} + q\sum_{j=0}^{q-1}(-1)^j\binom{q-1}{j}\left(\frac{P_{i,n-j}}{q-j} + \frac{P_{i,n-j} - P_{i,n-s-j}}{n-s-q+1}\right)$$

Easily one verifies that for $(r, s) \in \{0, 1\} \times \{0, 1\}$, formulas (7) become (4).

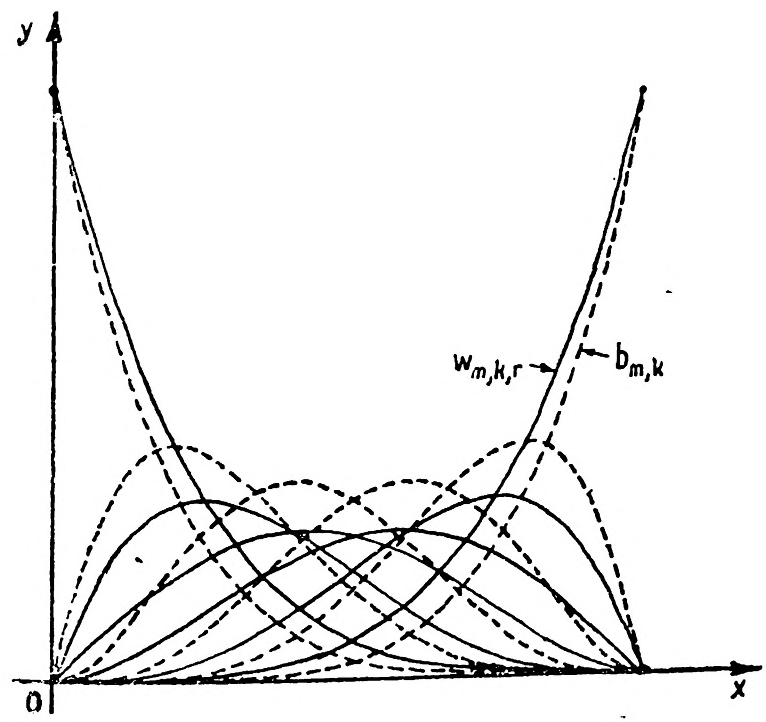

Fig. 2

Therefore, the Bézier surface represented by vectorial equation (6) generalizes the Bézier surface represented by vectorial equation (1).

Formulas (7) give us the possibility to control the derivatives of vectorial function S, either by points P_{ij} or by the natural numbers r and s.

In figure 2 are represented functions $b_{5,k}$, $k=\overline{0,5}$, defined at (2) (dotted curves) and functions $w_{5,k,2}$, $k=\overline{0,5}$ defined at (5) (continuous curves).

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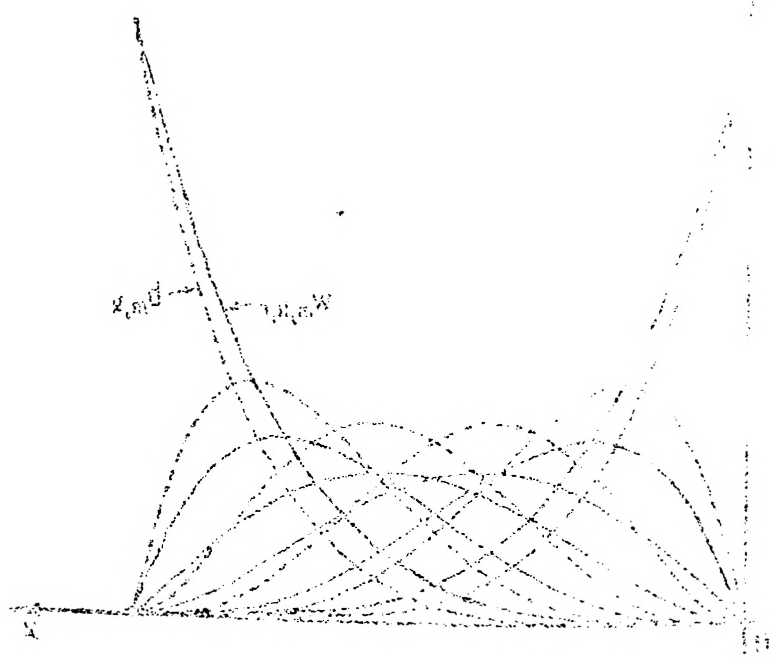

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THÉORÈMES DE POINT FIXE DANS LES ENSEMBLES CONVEXES

FLORICA VOICU*

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REZUMAT. — Teoreme de punct fix pe mulţimi convexe. Fie $\mathfrak{X}$ un spaţiu liniar complet reticulat şi $K \subset \mathfrak{X}$ o submulţime nevidă şi convexă. În această lucrare se stabilesc unele rezultate pentru aplicaţii Kannan definite pe K cu valori în K. Terminologia şi notaţiile sînt cele din [2].

1. Introduction. Soient $\mathfrak{X}$ un espace linéaire complètement réticulé et $K \subset \mathfrak{X}$, K non-vide et convexe. Dans cette note on donnent certains résultats pour les applications Kannan défines sur K à valeurs dans K. La terminologie et les notations sont celles de [2]. En particulier, on désigne par $\mathbf{N}$ l'ensemble des nombres naturels.

2. Soit $\mathfrak{X}$ un espace linéaire complètement réticulé satisfaisant la suivant condition :

(C*) Pour toute suite généralisée (o)—borné, décroissante $\{A_\delta\}_{\delta\in\Delta}$ de sous — ensembles non-vides de $\mathfrak{X}$, convexes et (o^*)—fermés on a $\bigcap_{\delta\in\Delta} A_\delta \neq \Phi$.

Exemple. Tout espace $\mathfrak{X}$ de type (KB) possède la propriété (C*) (par exemple : $V((0, 1))$; (l^p), (L^p) pour $1 \leqslant p < \infty$.)

DÉFINITION Soient $\mathfrak{X}$ un espace linéaire complètement réticulé et $K \subset \mathfrak{X}$, K non-vide et convexe. Une application $T : K \to K$ s'appelle *application Kannan sur* K s'il existe $\alpha \in \left]0, \frac{1}{2}\right[$ tel que :

$$|T(x) - T(y)| \leqslant \alpha(|x - T(x)| + |y - T(y)|) \ (\forall) \ x, y \in K \tag{1}$$

THÉORÈME 1. *Soit $\mathfrak{X}$ un espace linéaire complétement réticulé qui possède la propriété (C*) et soit K un sous-ensemble non-vide de $\mathfrak{X}$ (o)—borné, (o^*)—fermé et convexe. Soit T une applications Kannan sur K. Si :*

$$\sup_{y\in F} |T(y) - y| \neq \sup_{x,y\in F} |x - y| \tag{2}$$

pour tout sous-ensemble $F \subset K$, (o^*)—fermé et convexe, qui a au moins deux éléments et $T(F) \subset F$ alors T a un point fixe unique dans K.

Démonstration. Soit

$$\mathfrak{F}_K = \{F / F \subset K, \ K \text{ convexe}, \ (o^*)\text{—fermé}, \ T(F) \subset F\}$$

La famille $\mathfrak{F}_K$ est non-vide et possède un element minimal G. Si $G = \{x^*\}$ alors $T(x^*) = x^*$.

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Supposons que $G \neq \{x^*\}$. Alors quelque soient $x, y \in G$ on a :

$$|T(x) - T(y)| \leqslant \alpha(|x - T(x)| + |y - T(y)|) \leqslant 2\alpha \sup_{y \in G} |T(y) - y| \leqslant$$
$$\leqslant \sup_{y \in G} |T(y) - y|$$

d'où on a :

$$T(G) \subset G_0 = \{z \in K / |z - T(x)| \leqslant \sup_{y \in G} |y - T(y)|, \ (\forall)\ x \in G\}$$

On observe que l'ensemble G_0 est (o^*)—fermé et que l'ensemble $G \cap G_0$ est T—invariant, c'est-à-dire, $T(G \cap G_0) \subset G \cap G_0$. D'autre part, G est minimal et par conséquent il résulte que :

$$G \subset G_0.$$

Alors pour tout $x \in G$ on a :

$$\sup_{y \in G} |T(x) - y| \leqslant \sup_{y \in G} |T(y) - y| \tag{3}$$

Considérons l'ensemble

$$G_1 = \{z \in G / \sup_{y \in G} |z - y| \leqslant \sup_{y \in G} |T(y) - y|\}$$

On observe que

$$G_1 \neq \Phi$$

Soient $u, v \in G_1$ et $\lambda \in [0, 1]$. Alors :

$$|\lambda u + (1 - \lambda)v - y| = |\lambda u - \lambda y + (1 - \lambda)v - (1 - \lambda)y| \leqslant$$
$$\leqslant \lambda |u - y| + (1 - \lambda)|v - y| \quad (\forall)\ y \in G$$

d'où il résulte que

$$\sup_{y \in G} |\lambda u + (1 - \lambda)v - y| \leqslant \lambda \sup_{y \in G} |u - y| + (1 - \lambda) \sup_{y \in G} |v - y| \leqslant$$
$$\leqslant \lambda \sup_{y \in G} |y - T(y)| + (1 - \lambda) \sup_{y \in G} |y - T(y)| =$$
$$= \sup_{y \in G} |y - T(y)|$$

$$\sup_{y \in G} |\lambda u + (1 - \lambda)v - y| \leqslant \sup_{y \in G} |y - T(y)| \tag{4}$$

De (4) on a

$$\lambda u + (1 - \lambda)v \in G$$

c'est-à-dire G_1 est un ensemble convexe.

Nous supposons maintenant que $z \in \bar{G}_1$, ou $\bar{G}_1$ est (o^*)—fermeture de G_1. Soit $\{z_n\}_{n \in \mathbb{N}}$ une suite d'elements de G_1 telle que $z_n \xrightarrow{o^*} z$.

Soit $\{z_{j_n}\}_{n \in \mathbb{N}}$ une sous-suite quelconque de la suite $\{z_n\}_{n \in \mathbb{N}}$. Alors il existe une sous-suite $\{z_{j_{k_n}}\}$ telle que

$$z_{j_{k_n}} \xrightarrow{o} z$$

On a
$$|z_{j_{k_n}} - y| \leqslant \sup_{y\in G} |z_{j_{k_n}} - y| \leqslant \sup_{y\in G} |y - T(y)| \ (\forall)\, y \in G \tag{5}$$

Considérant la limite, pour $n \to \infty$, dans la relation (5) on obtient
$$\sup_{y\in G} |z - y| \leqslant \sup_{y\in G} |y - T(y)|$$

Donc $z \in G_1$, c'est-à-dire G_1 est un ensemble (o^*)—fermé. Pour tout $z \in G_1$ de l'inegalité (3) on obtient
$$\sup_{y\in G} |T(x) - y| \leqslant \sup_{y\in G} |T(y) - y| \tag{3'}$$

d'ou il résulte que
$$T(z) \in G_1 \text{ pour tout } z \in G_1, \text{ c'est-à-dire } T(G_1) \subset G_1.$$

Des relations (3), (3') et la définition de G_1 il résulte
$$\sup_{u,v\in G_1} |u - v| \leqslant \sup_{y\in G} |y - T(y)| \leqslant \sup_{u,v\in G} |u - v| \tag{6}$$

De (6) il résulte que G est un sous-ensemble propre de G_1 convexe et (o^*)— fermé, en contradiction avec la condition que G est minimal. Montrons l'unicité du point fixe pour T. Soient $x^* = T(x^*)$ et $y^* = T(y^*)$. Alors:
$$|x^* - y^*| = |T(x^*) - T(y^*)| \leqslant \alpha(|x^* - T(x^*)| + |y^* - T(y^*)|) = 0$$

Il résulte que $x^* = y^*$.
Le théorème est démontré.

THÉORÈME 2. *Soient $\mathfrak{X}$ un espace linéaire complètement réticulé, $K \subset \mathfrak{X}$ un sous-ensemble non-vide, convexe, (o^*)—fermé, (o)—borné et $T : K \to K$ une application Kannan. Soit* $\alpha \in \left]0, \frac{1}{2}\right[$ *tel que:*
$$\frac{1}{\alpha} \sup_{y\in F} |y - T(y)| < \sup_{x,y\in F} |x - y| \tag{7}$$

pour toute sous-ensemble $F \subset K$, convexe, (o^*)—fermé, $T(F) \subset F$ et qui possède au moins deux elements.
S'il esiste un sous-ensemble K^* de K, convexe, $T(K^*) \subset K^*$, (o^*)—fermé, minimal, alors T a un point fixe unique dans K.

Démonstration. Si l'ensemble $K^* = \{x^*\}$ alors le point x^* est un point fixe pour T.
Supposons que l'ensemble K^* possède au moins deux èlèments. Pour tous x, $x_0 \in K^*$ et $\lambda \in [0, 1]$ on a
$$\lambda x + (1 - \lambda)x_0 \in K^*$$

et pour tout $y \in K^*$ on a
$$|T(y) - T(x_0)| \leqslant \alpha(|T(y) - y| + |T(x_0) - x_0|) \leqslant$$
$$\leqslant 2\alpha \sup_{y\in K^*} |y - T(y)| \leqslant \sup_{y\in K^*} |y - T(y)|$$

Parce que l'ensemble

$$C = \{z \in K / |z - T(x_0)| \leqslant \sup_{y \in K^*} |y - T(y)|, \ x_0 \in K^*\}$$

est (o^*)—fermé et $T(K^*) \subseteq C$ il résulte que

$$T(K^* \cap C) \text{ est convexe et } T(K^* \cap C) \subset K^* \cap C$$

De la minimalitée de K^* il résulte $K^* \subset C$. Donc

$$\sup_{y \in K^*} |y - T(x_0)| \leqslant \sup_{y \in K^*} |y - T(y)| \tag{8}$$

Soit

$$K' = \{z \in K^* / \sup_{y \in K^*} |z - y| \leqslant 2 \sup_{y \in K^*} |y - T(y)|\}$$

Pour $y \in K^*$ on a

$$|x_0 - y| \leqslant |x_0 - T(x_0)| + |T(x_0) - y| \leqslant 2 \sup_{y \in K^*} |y - T(y)| \tag{9}$$

De la relation (9) il resulte $x_0 \in K'$.
Pour $z \in K'$ on a

$$|T(z) - y| \leqslant |T(z) - T(x_0)| + |T(x_0) - y| \leqslant$$
$$\leqslant \sup_{y \in K^*} |y - T(y)| + \sup_{y \in K^*} |y - T(y)| = 2 \sup_{y \in K^*} |y - T(y)| \tag{10}$$

De la relation (10) il résulte que l'ensemble K' est T—invariant. Pour x, $x^0 \in K'$, $y \in K^*$ et $\lambda \in [0, 1]$ on a

$$|\lambda x + (1 - \lambda) x_0 - y| \leqslant \lambda |x - y| + (1 - \lambda) |x_0 - y| \leqslant 2 \sup_{y \in K^*} |y - T(y)|$$

Donc

$$\sup_{y \in K^*} |\lambda x + (1 - \lambda) x_0 - y| \leqslant 2 \sup_{y \in K^*} |y - T(y)| \tag{11}$$

Il résulte que

$$\lambda x + (1 - \lambda) x_0 \in K'$$

et donc l'ensemble K' est convexe.
Supposons que $z \in \overline{K}'$ ou $\overline{K}'$ est (o^*)—fermeture de K'. Alors il existe une suite $\{z_n\}_{n \in N} \subset K'$ telle que

$$z_n \xrightarrow{0^*} z$$

et

$$\sup_{y \in K^*} |z_{j_{k_n}} - y| \leqslant 2 \sup_{y \in K^*} |y - T(y)| \tag{12}$$

Considérant la limite pour $n \to \infty$, de (12) on obtient

$$\sup_{y \in K^*} |z - y| \leqslant 2 \sup_{y \in K^*} |y - T(y)| \tag{13}$$

De (13) il résulte que $z \in K'$.

Mais, par hypothèse, on a

$$\sup_{a,b\in K'} |a-b| \leqslant 2\sup_{y\in K^*} |y-T(y)| \leqslant 2\alpha \sup_{a,b\in K^*} |a-b| < \sup_{a,b\in K^*} |a-b|$$

Donc K' est un sous-ensemble propre de K^*, convexe, T-invariant, (o^*)— fermé, en contradiction avec la minimalitée de K^*. L'unicité du point fixe résulte immediatment de (1).

Le théorème est démontré.

3. THÉORÈME 3. *Soient $\mathfrak{X}$ un espace linéaire σ — réticulé et $K \subset \mathfrak{X}$ convexe et (o) — compacte. Soit $T: K \to K$ une application non-expansive, c'est-à-dire :*

$$|T(x) - T(y)| \leqslant |x-y| (\forall)\ x, y \in K \tag{14}$$

Dans cettes conditions il résulte que T a un point fixe unique.

Démonstration. Soient $x, y \in K$ et $\lambda_n = 1 - \dfrac{1}{n}$, $n \in \mathbf{N}$

Definissons les applications $T_n : K \to K$ par la formule

$$T_n(x) = \lambda_n T(x) + (1-\lambda_n)y \tag{15}$$

Les applications T_n sont des contractions, parce que

$$|T_n(u_1) - T_n(u_2)| = |\lambda_n T(u_1) + (1-\lambda_n)y - \lambda_n T(u_2) - (1-\lambda_n)y| =$$
$$= \lambda_n |T(u_1) - T(u_2)| \leqslant \left(1 - \frac{1}{n}\right)|u_1 - u_2|,$$
$$|T_n(u_1) - T_n(u_2)| \leqslant \left(1 - \frac{1}{n}\right)|u_1 - u_2| \quad (\forall)\ u_1, u_2 \in K \tag{16}$$

Il résulte, donc, qu'il existe un point fixe unique x_n^*,

$$x_n^* = T_n(x_n^*) \text{ pour tout } n \in \mathbf{N}.$$

Mais le sous-ensemble K étant (o)—compacte, il résulte que la suite $\{x_n^*\}_{n\in\mathbf{N}}$ possède une sous-suite $\{x_{j_n}^*\}_{n\in\mathbf{N}}$ telle que

$$x_{j_n}^* \xrightarrow{o} x^* \in K \tag{17}$$

D'autre part avec (14) on a

$$x_{j_n}^* = T_{j_n}(x_{j_n}^*) = \lambda_{j_n} T(x_{j_n}^*) + (1-\lambda_{j_n})y \tag{18}$$

Mais l'application T étant non-expansive, est (o)—continue et donc de (17) on a

$$T(x_{j_n}^*) \xrightarrow{o} T(x^*) \tag{19}$$

Considérant la limite, pour $n \to \infty$, de (18) on obtient

$$x^* = T(x^*) \tag{20}$$

L'unicité du point fixe résulte immediatment de (14). Le théorème est démonstré.

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THE FLOW WITHOUT LOSSES ON CASCADE BLADES OF INCOMPRESSIBLE FLUID

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REZUMAT. — Studiul curgerii fluidelor incompresibile în reţele spaţiale de profile. Studiul curgerii (repartiţiei de viteze şi presiuni) pe palelele unor reţele spaţiale de profile (cazul rotoarelor de turbină Francis şi al pompelor diagonale) este destul de dificil de efectuat. El se poate face prin metoda diferenţelor finite, prin metoda elementului finit sau prin transformări conforme. Profilele realizate în aceste reţele spaţiale au bordul de fugă ascuţit, ceea ce face ca metoda elementului finit şi a diferenţelor finite să fie destul de dificil de aplicat. În continuare, în lucrare se va prezenta a treia metodă. Se va utiliza întîi o transformare conformă a reţelei spaţiale într-o reţea liniară de profile. Folosind o a doua transformare conformă, reţeaua de profile se transformă într-un contur rectangular, extradosul profilelor fiind cuprins în latura superioară a conturului iar intradosul lor pe cea inferioară. Acest procedeu prezintă avantajul posibilităţii de a studia în oricîte puncte dorim repartiţia de viteze şi presiuni în vecinătatea bordului de fugă.

1. Introduction. The flow without losses of incompressible fluid into axial cascade blades may be study with:

a) Finites differences-method used for small Reynold's number (laminar flow) and for rounded trailing edge.

b) Using the finite element aproximation, when the surface between blades is discretized by conforming linear triangular elements.

c) The conformable representation method.

Further in this report will be presented the third method.

First we use a conformable representation of the three-dimensional cascade blades (surface S_1) with variable breadth of layer into a system fields, containing any number of two-dimensional cascade blades. Then we use a second conformable transformation of the linear, two-dimensional cascade blades into a rectangular fields (ξ, η).

No restrictions are placed on the shape of the boundaries, which may even be time-dependent. Such representation is best accomplished when the boundary is such that it is coincident with some coordinate line (ξ and η). On this case interpolation may be done.

The avoiding of interpolation is particularly important for boundaries with strong curvature or slope discontinuities.

In many differential systems the boundary conditions are the dominant influence on the character of the solution.

To simplify the ecuations and the stream lines, the limit conditions must be constant in time.

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The differential method is a good one on the rectangular areas. For a detached blade the method was studied in [1].

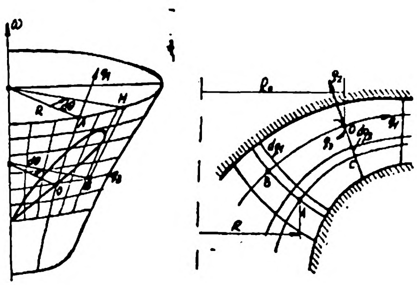
2. The conformable transformation of the three-dimensional cascade blades into a two-dimensional one. The three-dimensional cascade blades are used on the rotor blade of Francis hydraulic turbine and on diagonal pumps.

The meridian lines also depend, in the same time, of the (z, r) coordinates of the points which are on them.

The surface S_1 is an axial-symmetrical-one. The intersection between S_1 and the turbine shapes is a three-dimensional cascade blade. The three-dimensional cascade blades may be transformed into a two-dimensional (x, y) plane with the relations:

$$x = r_0 \cdot \theta_m \tag{1}$$

$$y = r_0 \cdot \int_0^m \frac{dm}{r}$$



where

$$r_0 = m_L \Big/ \int_0^{m_L} \frac{dm}{r} \tag{2}$$

Fig. 1

(for m, r — see Fig. 1) — m_L — the highest value of m on the outlet line; — θ — the angle made by the stream line between the entrance and the outlet line.

The l-spring of shape and the β_0 angle of blade on the (x, y) fields surface are

$$l = \sqrt{m_L^2 + r_0^2\,\theta_0^2}$$

$$\beta_0 = \text{arctg}\,[m_L/(r_0 \cdot \theta_0)] \tag{3}$$

If $N =$ the number of blades,

$$t = \frac{2\pi\, r_0}{N} \tag{4}$$

and t/l — the thickness of the cascade blade

$$t/l = \frac{2\pi\, r_0}{l \cdot N} \tag{5}$$

The components C_u and C_r of the absolute velocity of the cascade blades become V_x and V_y on (x, y) fields.

$$V_x = C_u \cdot \frac{r}{r_0}, \qquad V_y = -\,C_m \cdot \frac{r}{r_0}. \tag{6}$$

With $h =$ normal on fields, $h = \frac{\Delta n(r)}{\Delta n(r_0)}$, the velocity becomes (function of stream lines)

$$V_x = 1/h \cdot \frac{\partial \psi}{\partial y}, \qquad V_y = -1/h \cdot \frac{\partial \psi}{\partial x}. \tag{7}$$

Then we may say: on S_1 surface the absolute motion is a potential one, so that the flow equations on (x, y) fields is potential too.

$$\frac{\partial V_x}{\partial y} - \frac{\partial V_y}{\partial x} = 0 \tag{8}$$

Substituting (7) into (8), there results a partial differential equation of the stream function

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - \frac{1}{h} \cdot \frac{\partial h}{\partial y} \cdot \frac{\partial \psi}{\partial y} = 0 \tag{9}$$

The components are:

— on the $1-1'$ line, the outside value of the stream line becomes:

$$\left(\frac{\partial \psi}{\partial x}\right)_{11'} = -h_1 V_{1y} = h_1 \cdot C_{m_1} \cdot \frac{r_1}{r_0} \tag{10}$$

$$\left(\frac{\partial \psi}{\partial y}\right)_{11'} = h_1 \cdot V_{1x} = h_1 \cdot C_{u_1} \cdot \frac{r_1}{r_0} \tag{10'}$$

— on the $2-2'$ line, the outside value of the stream line becomes:

$$\left(\frac{\partial \psi}{\partial x}\right)_{22'} = -h_2 \cdot V_{2y} = h_2 \cdot C_{m_2} \cdot r_2/r_0 \tag{11}$$

$$\left(\frac{\partial \psi}{\partial y}\right)_{22'} = h_2 \cdot V_{2x} = h_2(V_{1x} - \Gamma/t) \tag{11'}$$

— on the boundary $C^+ \cup C^-$

$$\left.\frac{\partial \psi}{\partial s}\right|_{C^+ \cup C^-} = \omega \cdot r_0 \left(\frac{r}{r_0}\right)^2 \cdot h \cdot \frac{dy}{ds} \tag{12}$$

— on the congruents lines $24 \cup 13$ and $2'4' \cup 1'3'$ in the points with the same y-ordinates, since the difference is the same, the stream line value is constant, as it follows:

$$\psi|_{2'4' \cup 1'3'} = \psi|_{24 \cup 13} + t \cdot h_1 \cdot C_{m_1} \cdot r_1/r_0 \tag{13}$$

3. The conformable transformation of the plan area into a rectangular one. The boundary $11'3'C^-4'2'24C^+311'$ interior on the closed boundary D on the (x, y) fields is transformed into a rectangular region D^- on the plane (ξ, η), as shown in Fig. 2.

The general transformation from the physical plane (x, y) to the transformed plane (ξ, η) is given by $\xi = \xi(x, y)$, $\eta = \eta(x, y)$.

Similarly, the inverse transformation is given by $x = x(\xi, \eta)$, $y = y(\xi, \eta)$.

The intrados profiles C^- — on the physical plane transforms into lower side of the rectangular transformed plane $(\eta = \eta_{min})$ and the extrados C^+ is transformed to the upper side of the rectangular transformed plane $(\eta = \eta_{max})$.

The order is shown in Fig. 3.

The $\xi(x, y)$ and $\eta(x, y)$ functions of the conformable transformation are harmonics on D fields,

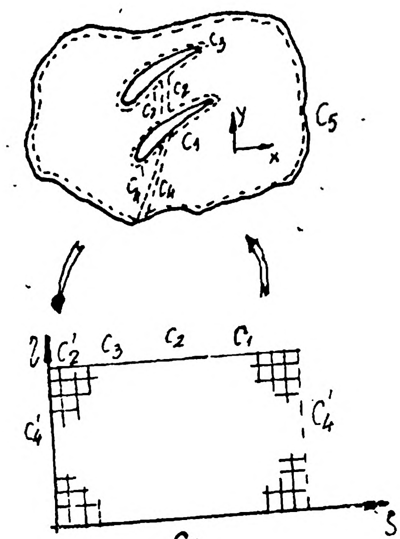


Fig. 2

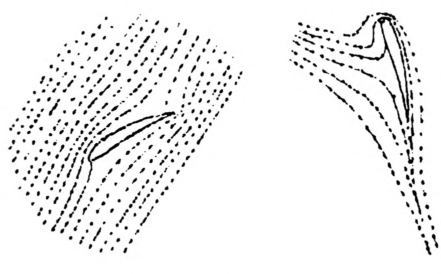


Fig. 3

The basic idea of the transformation is to generate transformation functions such that all boundaries are coincident with coordinate lines, the natural coordinates (ξ, η) are taken as solutions of some suitable elliptic boundary value problem with one of these coordinates constant on the boundaries.

Using Laplace's equation as the generating elliptic system, we have:

$$\left.\begin{aligned} \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} = 0 \\ \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} = 0 \end{aligned}\right\} \text{ on } D \text{ fields} \qquad (14)$$

The functions $x(\xi, \eta)$ and $y(\xi, \eta)$ are solutions of a differential system:

$$\alpha \frac{\partial^2 x}{\partial \xi^2} - 2\beta \frac{\partial^2 x}{\partial \xi \partial \eta} + \gamma \cdot \frac{\partial^2 x}{\partial \eta^2} = 0 \qquad (15)$$

$$\alpha \frac{\partial^2 y}{\partial \xi^2} - 2\beta \frac{\partial^2 y}{\partial \xi \partial \eta} + \gamma \cdot \frac{\partial^2 y}{\partial \eta^2} = 0 \qquad (15')$$

where

$$\begin{aligned} \alpha &= \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 \\ \beta &= \frac{\partial x}{\partial \xi} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi} \cdot \frac{\partial y}{\partial \eta} \\ \gamma &= \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 \end{aligned} \qquad (16)$$

The Jacobian of the transformation is

$$J = \frac{\partial x}{\partial \xi} \cdot \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \cdot \frac{\partial y}{\partial \xi} \qquad (16')$$

If we replace in (9) the relation:

$$\alpha \frac{\partial^2 \psi}{\partial \xi^2} - 2\beta \cdot \frac{\partial^2 \psi}{\partial \xi \partial \eta} + \gamma \cdot \frac{\partial^2 \psi}{\partial \eta^2} + H(y) \cdot J \left(\frac{\partial x}{\partial \xi} \cdot \frac{\partial \psi}{\partial \eta} - \frac{\partial x}{\partial \eta} \cdot \frac{\partial \psi}{\partial \xi}\right) = 0 \qquad (17)$$

where:

$$H(y) = \frac{1}{h} \cdot \frac{dh}{dy} \qquad (17')$$

4. The utilisation of differentiales variables on rectangular fields. If we notice the values:

$$\xi_{min} = \eta_{min} = 1 \; ; \; \xi_{max} = M \; ; \; \eta_{max} = N : \Delta\xi = \Delta\eta = 1$$

we obtain a set of partial differential equations of U-function in points (i, j) on the natural coordinate system which we may solve.

The points (i, j) are inner the rectangular fields. All calculations are made on the centre of their differences.

$$
\begin{aligned}
\left(\frac{\partial u}{\partial \xi}\right)_{i,j} &= \frac{1}{2}\left(u_{i+1,j} - u_{i-1,j}\right) \\
\left(\frac{\partial u}{\partial \eta}\right)_{i,j} &= \frac{1}{2}\left(u_{i,j+1} - u_{i,j-1}\right) \\
\left(\frac{\partial^2 u}{\partial \xi^2}\right)_{i,j} &= u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \qquad \begin{matrix} i = 2, M-1 \\ j = 2, N-1 \end{matrix} \\
\left(\frac{\partial^2 u}{\partial \eta^2}\right)_{i,j} &= u_{i,j+1} - 2u_{i,j} + u_{i,j-1} \\
\left(\frac{\partial^2 u}{\partial \xi \partial \eta}\right)_{i,j} &= \frac{1}{4}\left(u_{i+1,j+1} - u_{u+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}\right)
\end{aligned}
\tag{18}
$$

The (18) equation for $\frac{\partial u}{\partial \xi}$ is available and for $j = 1$ when $i \neq 1, M$. For $j = 1$ and $i = 1$ or $i = M$, $\frac{\partial u}{\partial \xi}$ is approximated using second-order, central differences for derivatives

$$
\begin{aligned}
\left(\frac{\partial u}{\partial \xi}\right)_{1,i} &= \frac{1}{2}\left(-u_{3,1} + 4u_{2,1} - 3u_{1,1}\right) \\
\left(\frac{\partial u}{\partial \xi}\right)_{M,1} &= \frac{1}{2}\left(u_{M-2,1} - 4u_{M-1,1} + 3u_{M,1}\right)
\end{aligned}
\tag{19}
$$

The variation for $\frac{\partial u}{\partial \eta}$ for $j = 1$ is:

$$
\left(\frac{\partial u}{\partial \eta}\right)_{i,1} = \frac{1}{2}\left(-u_{i,3} + 4u_{i,2} - 3u_{i,1}\right) \tag{20}
$$

The (18) equation for $\frac{\partial u}{\partial \xi}$ is available and for $j = N$ and $i \neq 1, M$. For $j = N$ and $i = 1, M$ we obtain:

$$
\begin{aligned}
\left(\frac{\partial u}{\partial \xi}\right)_{1,N} &= \frac{1}{2}\left(-u_{3,N} + 4u_{2,N} - 3u_{1,N}\right) \\
\left(\frac{\partial u}{\partial \xi}\right)_{M,N} &= \frac{1}{2}\left(u_{M-2,N} - 4u_{M-1,N} + 3u_{M,N}\right) \\
\left(\frac{\partial u}{\partial \eta}\right)_{i,N} &= \frac{1}{2}\left(u_{i,N-2} - 4u_{i,N-1} + 3u_{i,N}\right)
\end{aligned}
\tag{21}
$$

Then, using the (19) equations into (15) it results:

$$
x_0 = a_1(x_1 + x_8 - x_3 - x_6) + a_2(x_2 + x_7) + a_4(x_4 + x_5) \tag{22}
$$

where the coefficients a_1, a_2, a_4 are

$$a_1 = -\frac{\beta_{ij}}{4(\alpha_{ij}+\gamma_{ij})}$$

$$a_2 = \frac{\gamma_{ij}}{2(\alpha_{ij}+\gamma_{ij})}; \qquad a_4 = \frac{\alpha_{ij}}{2(\alpha_{ij}+\gamma_{ij})} \tag{22'}$$

Into Fig. 4 the flow potential is indicated by points. Then the (17) equation becomes:

$$\psi_0 = a_1(\psi_1 + \psi_8 - \psi_3 - \psi_6) + a_2(\psi_2 + \psi_7) + b_2(-\psi_2 + \psi_7) + \\ + a_4(\psi_4 + \psi_5) + b_4(\psi_4 - \psi_5) \tag{23}$$

where

$$b_2 = \frac{H_j \cdot J_{ij} \cdot \left|\frac{\partial x}{\partial \xi}\right|_{ij}}{4(\alpha_{ij}+\gamma_{ij})} \qquad b_4 = \frac{H_j \cdot J_{ij} \cdot \left|\frac{\partial x}{\partial \eta}\right|_{ij}}{4(\alpha_{ij}+\gamma_{ij})} \tag{23'}$$

The points of intersection of the zero streamline with airfoils agree with the analytic solutions.

We require that the η-coordinate be equal to the same constant on all the interior boundaries, i.e. on all "bodies" in the fields.

All bodies except one are split into two segments. Each cut appears twice on the transformed field boundary, the two segments corresponding to the two "sides" of the cut, in the physical plane.

There are no discontinuities in the streamlines across the cut between the bodies.

The slip boundary for equation (22) will be:

— on the line 1—1'

$$\left.\begin{aligned} x(M,j) &= l\cdot\cos\beta_0 + t(N_2 - j)/(N_2 - N_1) \\ y(M,j) &= l\sin\beta_0 + d \end{aligned}\right\} j = N_1, N_2 \tag{24}$$

where $d \geqslant \max(1, t)$

— on the line 2—2'

$$\left.\begin{aligned} x(1,j) &= t(N_2 - j)/(N_2 - N_1) \\ y(1,j) &= -d \end{aligned}\right\} j = N_1, N_2 \tag{25}$$

— on the line 1'—3'

$$\left.\begin{aligned} x(M,j) &= l\cdot\cos\beta_0 + t \\ y(M,j) &= l\cdot\sin\beta_0 + d(j-1)/(N_1 - 1) \end{aligned}\right\} j = 1, N_1 - 1 \tag{26}$$

— on the line 1—3

$$\left.\begin{aligned} x(M,j) &= l\cdot\cos\beta_0 \\ y(M,J) &= l\cdot\sin\beta_0 + d(N-j)/(N-N_2) \end{aligned}\right\} j = N_2+1, N_1-1 \tag{27}$$

— on the line 4—2

$$\left.\begin{aligned} x(1, j) &= 0 \\ y(L, j) &= -d(N - j)/(N - N_2) \end{aligned}\right\} j = N_2+1, N_1-1 \tag{28}$$

— on the line 4′—2′

$$\left.\begin{aligned} x(1, j) &= t \\ y(1, j) &= d(1 - j)/(N_1 - 1) \end{aligned}\right\} j = 1, N_1-1 \tag{29}$$

— on the extrados $C^+(3-4)$

$$\left.\begin{aligned} x(i, N) &= x^+(M - i + 1) \\ y(i, N) &= y^+(M - i + 1) \end{aligned}\right\} i = 1, M \tag{30}$$

— on the intrados $C^-(3'-4')$

$$\left.\begin{aligned} x(1, 1) &= x^-(M - i + 1) \\ y(1, 1) &= y^-(M - i + L) \end{aligned}\right\} i = 1, M \tag{31}$$

The solving of the (23) equations system finish with the slip boundary.

5. Conclusions.

The method affords to study the mouvement into a Francis hydraulic turbine. The treatment of fields with complex boundaries and any number of bodies is not inherently more difficult than problems with simply geometry. The method affords a natural means of treating problems with moving boundaries, since the computational fields remains stredy in any case and no interpolation is required.

Finaly, the complete coupling of the partial differential equations for the coordinate system with those of the physical problem of interest, so that the coordinate system as such, is effectively eliminated.

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NEW RESULTS CONCERNING THE SATELLITE MOTION INTO THE GEOMAGNETIC FIELD

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REZUMAT. — **Noi rezultate privind mişcarea unui satelit în cîmpul geomagnetic.** Pornind de la expresia potenţialului geomagnetic, se studiază influenţa separată a celui de-al doilea termen al părţii nedipolare a cîmpului magnetic terestru asupra mişcării sateliţilor artificiali. Se stabilesc formule analitice aproximative atît pentru variaţiile elementelor orbitale, cît şi pentru diferenţa dintre perioada nodală şi perioada kepleriană corespunzătoare. Este avut în vedere numai cazul orbitelor de excentricitate mică.

1. **Introduction.** The perturbations caused in the motion of an artificial satellite by the interaction between the main geomagnetic field and the electrical charge of the satellite have been studied by different authors with some simplifying assumption like: the coincidence between the geomagnetic axis and the terrestrial rotation axis, the symmetry of the geomagnetic field with respect to this axis, the coincidence between the geomagnetic field and a dipole one, etc. We have studied previously [2]—[4] such perturbations by estimating analytically the difference between the nodal period of a satellite and the corresponding keplerian period, difference due to the geomagnetic field influence. Keeping the first two above mentioned assumptions, we have taken into account either the dipolic part of the geomagnetic field, or this one together with the first term of the nondipolic part.

Knowing that the second term of the nondipolic part of the geomagnetic field is of the same order of magnitude as the first term of the nondipolic part, it seemed us interesting to investigate the separate influence of this term upon the satellite motion. We shall consider only the satellites moving in quasi-circular orbits.

Starting from the known expression of the geomagnteic potential (see e.g. [1]), and using S c h n a l 's [5] formulae for determining the components of the disturbing acceleration, we shall estimate analytically the difference ΔT_{Ω} between the nodal period and the corresponding keplerian one, caused only by the second term of the nondipolic par of the main geomagnetic field.

2. **Motion equations.** We start from the Newton-Euler system written in the matrix form:

$$\begin{bmatrix} dp/du \\ d\Omega/du \\ di/du \\ dq/du \\ dk/du \\ dt/du \end{bmatrix} = Zr^3 \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & B/(Dp) \\ 0 & 0 & 0 & A/p \\ 0 & B/r & A/r + (A+q)/p & CBk/(Dp) \\ 0 & -A/r & B/r + (B+k)/p & -CBq/(Dp) \\ \sqrt{\mu/p}/r & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ S \\ T \\ W \end{bmatrix} \qquad (1)$$

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where $Y = \{p, \Omega, i, q = e\cos\omega, k = e\sin\omega\}$ are the orbital elements in the usual notations, μ is the gravitational constant multiplied by the Earth's mass, r is the geocentric radius vector of the satellite, S, T, W are the classical (radial, transversal and binormal, respectively) components of the disturbing acceleration while Z has the expression:

$$Z = (\mu - \sqrt{\mu/p}\, r^2 C(d\Omega/dt))^{-1}; \tag{2}$$

we have also used the abbreviating notations:

$$A = \cos u,\ B = \sin u,\ C = \cos i,\ D = \sin i, \tag{3}$$

in which u is the argument of latitude.

Using the formulae given by Sehnal [5], we have determined the components S, T, W of the disturbing acceleration due only to the considered term in the geomagnetic potential expansion. These ones, also written in a matrix form, have the expressions:

$$\begin{bmatrix} S \\ T \\ W \end{bmatrix} = K_1F_3 \begin{bmatrix} 3CD_1/r & 0 & 0 \\ 0 & -3CBD_1/p & 3CAD_1/p \\ 4DBD_2/r & 3DABD_1/p & -3DA^2D_1/p \end{bmatrix} \begin{bmatrix} 1 \\ q \\ k \end{bmatrix} \tag{4}$$

where we have denoted:

$$\begin{aligned} K_1 &= (R/r)^5\sqrt{\mu p}/2, \quad F_3 = (Q/m)g_{30}, \\ D_1 &= 1 - 5D^2B^2, \quad D_2 = D_1 + 2, \end{aligned} \tag{5}$$

formulae in which R is the terrestrial radius, Q and m are respectively the electrical charge and the mass of the satellite, while g_{30} is a constant featuring the second term of the nondipolic part in the geomagnetic potential expansion.

Now we consider the orbit equation written in polar coordinates: $r = p/(1 + e\cos v)$, where v is the true anomaly. This equation can also be written: $r = p/(1 + Aq + Bk)$, and, since only quasi-circular orbits are considered, we have the approximate relation:

$$r^n = p^n(1 - nAq - nBk), \tag{6}$$

where q^n, k^n (for $n \geqslant 2$) and the product qk have been neglected.

With (4)—(6), the Newton-Euler system (1) becomes:

$$\begin{bmatrix} dp/du \\ d\Omega/du \\ di/du \\ dq/du \\ dk/du \\ dt/du \end{bmatrix} = ZK_2F_3 \begin{bmatrix} 0 & -6CBD_1p & 6CAD_1p \\ 4B^2D_2 & 3AB^2D_3 & 3B^3D_3 - 3BD_1 \\ 4DABD_2 & 3DA^2BD_3 & 3DAB^2D_3 - 3DAD_1 \\ 3CBD_1 & 6CABD_1 & 2CB^2D_4 + 6CD_1 \\ 3CAD_1 & -2CB^2D_4 - 12CA^2D_1 & -6CABD_1 \\ 2(p/R)^5/F_3 & -4A(p/R)^5/F_3 & -4B(p/R)^5/F_3 \end{bmatrix} \begin{bmatrix} 1 \\ q \\ k \end{bmatrix} \tag{7}$$

where we have introduced the simplyfying notations:

$$K_2 = \sqrt{\mu/p}(R^5/2)/p^3, \qquad D_3 = 5D_1 + 8, \qquad D_4 = 5D_1 + 4. \tag{8}$$

3. Variations of the orbital elements. The variation of an orbital element $y \in Y$ (due to an arbitrary disturbing factor) between the initial (u_0) and current (u) position is given by the integral:

$$\Delta y = \int_{u_0}^{u} (dy/du) du, \qquad y \in Y, \tag{9}$$

where the integrands are provided by the Newton-Euler equations.

Considering the second term of the nondipolic part of the geomagnetic potential as being the disturbing factor, the integrands in (9) are given by (7). Performing the integrals obtained in this manner by the successive approximations method (limiting us to the first order approximation), with $Z \cong 1/\mu$ we have determined the variations of the orbital elements. These variations written in a matrix form, are:

$$\begin{bmatrix} \Delta p \\ \Delta\Omega \\ \Delta i \\ \Delta q \\ \Delta k \end{bmatrix} = K_3 F_3 [a_{jn}] \begin{bmatrix} 1 \\ q_0 \\ k_0 \end{bmatrix}, \quad j = \overline{1,5}, \quad n = \overline{1,3}, \tag{10}$$

where $K_3 = p^{-7/2}\mu^{-1/2}R^5/2$, while the elements of the matrix $[a_{jn}]$ are given by:

$$\begin{aligned}
a_{11} &= 0, \\
a_{12} &= 2C_0(5D_0^2(A^3 - A_0^5) + 3(1 - 5D_0^2)(A - A_0))p_0, \\
a_{13} &= -2C_0(5D_0^2(B^3 - B_0^3) - 3(B - B_0))p_0, \\
a_{21} &= 5D_0^2(AB^3 - A_0B_0^3) + 3(5D_0^2/2 - 2)(AB - A_0B_0 - (u - u_0)), \\
a_{22} &= -15D_0^2(B^5 - B_0^5) + 13(B^3 - B_0^3), \\
a_{23} &= 15D_0^2(A^5 - A_0^5) + (13 - 45D_0^2)(A^3 - A_3^0) + 12(5D_0^2 - 3)(A - A_0), \\
a_{31} &= -5D_0^3(B^4 - B_0^4) + 6D_0(B^2 - B_0^2), \\
a_{32} &= -15D_0^3(A^6 - A_0^5) + D_0(25D_0^2 - 13)(A^3 - A_0^3), \\
a_{33} &= -15D_0^3(B^5 - B_0^5) + D_0(5D_0^2 + 13)(B^3 - B_0^3) - 3D_0(B - B_0), \\
a_{41} &= 5C_0D_0^2(A^3 - A_0^3) + 3C_0(5D_0^2\,1 -)\ (A - A_0), \\
a_{42} &= -15C_0D_0^2(B^4 - B_0^4)/2 + 3C_0(B^2 - B_0^2), \\
a_{43} &= 25C_0D_0^2(AB^3 - A_0B_0^3)/2 + 9C_0(15D_0^2/4 - 1)(AB - A_0B_0) - \\
&\quad -15C_0(9D\ /4 - 1)(u - u_0), \\
a_{51} &= 5C_0D_0^2(B^3 - B_0^3) - 3C_0(B - B_0), \\
a_{52} &= 5C_0D_0^2(AB^3 - A_0B_0^3)/2 - 3C_0(35D_0^2/4 - 1)(AB - A_0B_0) + \\
&\quad + 15C_0(7D_0^2/4 - 1)(u - u_0), \\
a_{53} &= 15C_0D_0^2(B^4 - B_0^4)/2 - 3C_0(B^2 - B_0^2).
\end{aligned} \tag{11}$$

In these expressions, the supplementary index „o" fixes the values of the respective quantities at the initial epoch (t_0) or position (u_0).

4. Difference between the two periods. As a consequence of the action of an arbitrary distribution factor, the nodal period of a satellite will differ from the corresponding keplerian period by a quantity ΔT_Ω which can be written as the sum (see e.g. [3]):

$$\Delta T_\Omega = \sum_{j=1}^{4} I_j, \tag{12}$$

where:

$$\begin{aligned} I_j &= \alpha p_0^\beta \mu^{-1/2} \int_0^{2\pi} (1 + Aq_0 + Bk_0)^{-(\beta+3/2)} \gamma \Delta y du, \quad j = \overline{1,3} \\ I_4 &\cdot \int_0^{2\pi} (\partial(r^4 C(d\Omega/dt)/(\mu p))/\partial\sigma)\sigma\, du, \end{aligned} \tag{13}$$

in which σ is a small parameter featuring the disturbing factor (in our case we take $\sigma = F_3$), while α, β, γ are, according to $y \in Y$:

$$\begin{aligned} y = p &\Rightarrow \alpha = 3/2, & \beta &= 1/2, & \gamma &= 1; \\ y = q &\Rightarrow \alpha = -2, & \beta &= 3/2, & \gamma &= A; \\ y = k &\Rightarrow \alpha = -2, & \beta &= 3/2, & \gamma &= B. \end{aligned} \tag{14}$$

Starting from (10)—(11) and taking permanently into account the approximation (6), we have performed the integrals (13). The results, written in a matrix form, are:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = K_4 \begin{bmatrix} 0 & -5D_0^2 A_0^3 + 3(5D_0^2 - 1)A_0 & 5D_0^2 B_0^3 - 3B_0 \\ 1 - 15D_0^2/4 & 5D_0^2 A_0^3 + 3(5D_0^2 - 1)A_0 & 0 \\ 1 - 5D_0^2/4 & 5(7D_0^2/2 - 2) & -5D_0^2 B_0^3 + 3B_0 \\ 2 - 5D_0^2/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ q_0 \\ k_0 \end{bmatrix} \tag{15}$$

where we have used the notation:

$$K_4 = 3\pi F_3 R^6 \mu^{-1} p_0^{-2} C_0. \tag{16}$$

Finally, performing the sum (12) by using (15), we obtain the following equivalent expressions for the difference between the perturbed nodal period and the keplerian one:

$$\Delta T_\Omega = K_4(4 - 15D_0^2/2 + (5(6A_0 + 7/2)D_0^2 - 2(3A_0 + 5))q_0), \tag{17}$$

$$\Delta T_\Omega = K_4((15C_0^2 - 7)/2 + (3(8A_0 + 5/2) - 5(6A_0 + 7/2)C_0^2)q_0). \tag{18}$$

Observe the fact that for $C_0 = 0$ (namely $i_0 = 90°$) the factor K_4 vanishes. In other words, the nodal period of a satellite moving in a polar orbit is not affected by the second term of the nondipolic part of the geomagnetic potential.

Also observe that for circular orbits ($e_0 = 0 \Rightarrow q_0 = 0$), the expression of the difference ΔT_Ω becomes:

$$\Delta T_\Omega = K_4(4 - 15D_0^2/2) = K_4(15C_0^2 - 7)/2, \tag{19}$$

where p_0 in the expression of K_4 represents the radius of the circular orbit.

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A SYSTOLIC NETWORK FOR NUMERICAL INTEGRATION

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REZUMAT. — O reţea sistolică pentru integrarea numerică. În lucrare se propune o reţea sistolică capabilă să implementeze o clasă largă de metode de integrare. Este studiată influenţa ordinei de calcul al integralelor asupra timpului de răspuns, atunci cînd evaluarea funcţiilor de integrat se face în interiorul reţelei.

1. Introduction. This paper presents systolic ([6]—[7]) implementations of a large classs of numerical integration methods having the form

$$I = c\sum_{k=1}^{n} w_i f(x_i), \tag{1}$$

where c is a factor, $w_1, \ldots, w_n$ are the weights and $x_1, \ldots, x_n$ are the nodes, all these elements depending on the type of the method. The class $\mathcal{M}$ of these methods includes Newton-Cotes integration (trapezoidal rule, Simpson's rule, etc.), Gaussian integration (Gauss-Lobatto, Gauss-Legendre, Gauss-Laguerre, Gauss-Hermite, Gauss-Chebyshew, etc.) ([8]), the methods of undetermined coefficients, etc. We suppose that f is a real function given by an arithmetic expression or by tabulated data. More precisely, we present a systolic network able to compute with a constant period

$$I(k) \approx \int_{a_k}^{b_k} f_k(x)\, dx,$$

so that $I(k)$ is obtained by applying a prescribed rule $r_k \in \mathcal{M}$, $k = 1, \ldots, K$.

The case of the trapezoidal and Simpson formulas is studied in [1], while some variants of the Romberg method are sumarized in [2].

In Section 2, we describe the systolic network performing the above task and give correctness proof of its working. Section 3 shows how the response time of the network depends on the order of computing the integrals when the computation of functions is done inside of the network. The possibilities to treat some specific situations are discussed in the last section.

2. The systolic network. Further, the clock tick (CT) is the time to perform a division or both a multiplication and an addition. We shall designate by t the time which is a count of the number of CTs. If L is the label of an input (output) then $L(t)$ is the value circulating through this during the t — th

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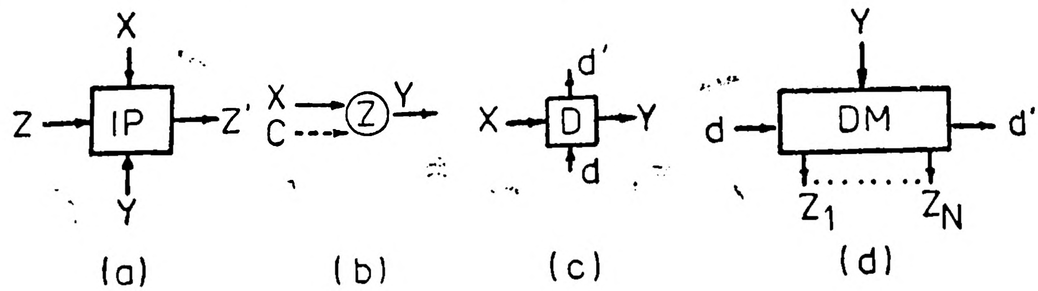
pulse number. The processing elements (PE) used in this work are depicted in Figure 1. Their work is stated by the following equations:

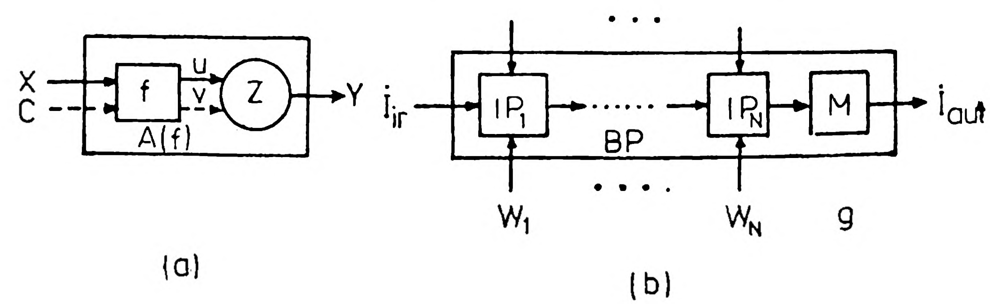
(a) $IP: z'(t+1) = z(t) + x(t)y(t)$;
(b) Z: if $c(t) = 1$ then $y(t+1) = x(t)$, and $y(t) = 0$ for $c(t) = 0$;
(c) $D: d'(t+1) = d(t)$, if $d(t) = 1$ then $y(t+1) = x(t)$, while for $d(t) = 0$ y does not emit;
(d) $d'(t+1) = d(t)$ and if $d(t) = 1$ and $d(t+h) = 0$, $h = 1, \ldots, N-1$, then $z_k(t+k) = y(t+k-1)$, $k = 1, \ldots, N$

(DM is a demultiplexor and contains only D PEs).



The subarray denoted by "f" in Figure 2(a) has a response time equal to $T(f)$ CTs and 1 CT as period. It performs $u(t + T(f) + h) = f(x(t+h))$, $h \geqslant 0$, while c-input receives "1" as a control signal, and $v(t + T(f) + h) = u(t+h)$, $h \geqslant 0$. The manner to design such an array is described in [3] and [4]. The working of $A(f)$-cell is stated by

Proposition 1. If $c(t+h-1) = 1$, $h = 1, \ldots, N_1$, $c(t + N_1 + h - 1) = 0$, $h = 1, \ldots, N_2$, $X(t+h-1) = x_h$, $h = 1, \ldots, N_1$, *then* $Y(t + T(f) + j) = f(x_j)$, $j = 1, \ldots, N_1$ *and* $u(t + T(f) + N_1 + h) = 0$, $h = 1, \ldots, N_2$.

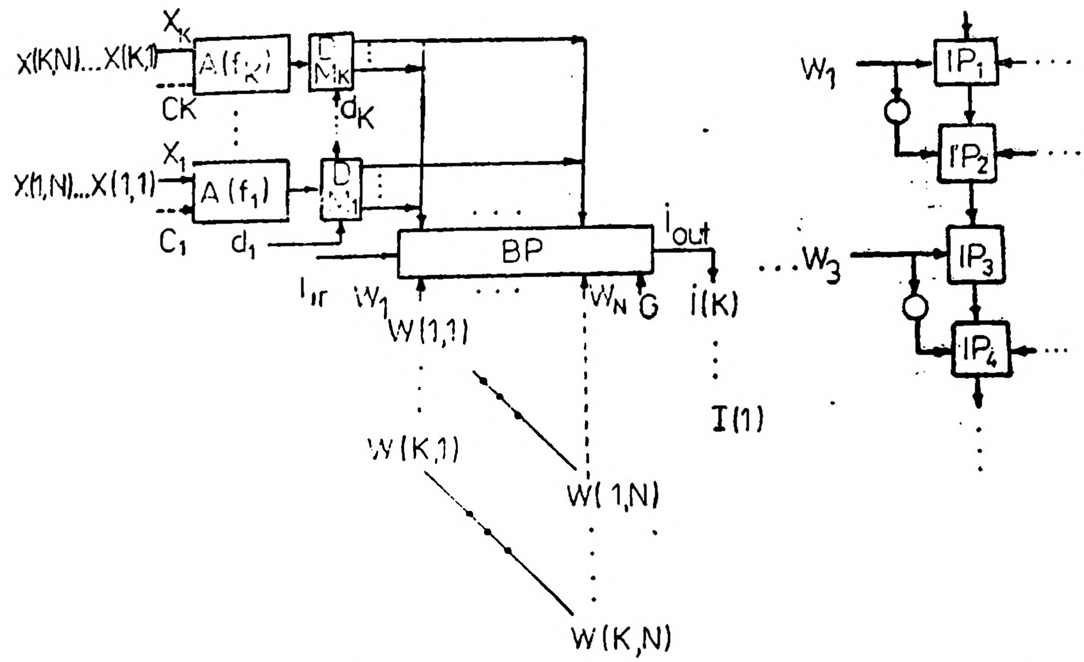


The basic pipe (BP) to pipeline (1) is depicted in Figure 2(b). It contains the IP processors $IP_1, \ldots, IP_N$, while M is multiplier. Our aim is to give a systolic implementation of

$$I(k) = g(k) \sum_{i=1}^{N(k)} w(k, i)\, f_k(x(k, i)), \quad k = 1, \ldots, K, \tag{2}$$

where $g(k)$, $N(k)$, $w(k, i)$ and $x(k, i)$, $i = 1, \ldots, N(k)$, are the factor, the number of terms, the weights and the nodes in $[a_k, b_k]$, respectively, required by the applying of the formula $r_k \in \mathfrak{M}$, $k = 1, \ldots, K$. The systolic network able to do this, SN, is presented in Figure 3(a). We suppose that $N(k) \leqslant N$, $k = 1, \ldots, K$.

Let us analyse the manner in which the own activity of SN is synchronized with the I/O operations. Let t_k be the time when I_{out} sends its k-th output, $k = 1, \ldots, K$. By assuming that each PE in SN is active during every CT, it results that $t_k = t_0 + k$, $k = 1, \ldots, K$, where t_0 will be determined by a certain initial condition. If $J(k) = I_{out}(t_k)$, we remark that $J(k)$ enters IP_1 as zero, accumulates the terms obtained from the inputs y and w, is multiplied by the value entering M, and finally emerges from I_{out}. The value $J(k)$ reaches M



(a) (b)

during $t_k - 1$ CT, thus $G(t_k - 1) = g(k)$, and it enters IP_i during $t_k - N + i - 2$ CT, $i = 1, \ldots, N$. In order to compute correctly (2), we need $Y_i(t_k - N + i - 2) = f_k(x(k, i))$, for $i = 1, \ldots, N(k)$, $Y_i(t_k - N + i - 2) = 0$, for $i = N(k) + 1, \ldots, N$ if $N(k) < N$, while $I_{in}(t_k - N - 1) = 0$. Also, it is required that $W_i(t_k - N + i - 2) = w(k, i)$, $i = 1, \ldots, N(k)$ and $W_i(t_k - N + i - 2) = 0$, for $i = N(k) + 1, \ldots, N$ if $N(k) < N$. Now, because $Y_1(t_1 - N - 1) = f_1(x(1, 1))$ we need $d_1(t_1 - N - 2) = 1$ and $d_1(s) = 0$ for $s > t_1 - N - 2$. From the definition of DM, we obtain $d_{k-1}(t_{k-1} - N - 2) = d'_{k-1}(t_{k-1} - N - 1) = d_k(t_k - N - 2)$, thus $d_k(t_k - N - 2) = 1$, $k = 2, \ldots, K$. On the other hand, $f_k(x(k,1))$

emerges from DM_k during $t_k - N - 1$ CT, thus it enters DM_k at $t_k - N - 2$ CT, i.e. exactly when d_k – input receives "1". Because the response time of $A(f_k)$ is $T(f_k) + 1$, it results that $X_k(t_k - T(f_k) - N - 3) = x(k, 1)$ and therefore $X_k(t_k - T(f_k) - N - 4 + i) = x(k, i)$ and $c_k(t_k - T(f_k) - N - 4 + i) = 1$, $i = 1, \ldots, N(k)$ and $c_k(t_k - T(f_k) - N - 4 + i) = 0$, $i = 1 + N(k), \ldots, N$. If all inputs receive the above values at the right time then it is clear that $J(k) = I(k)$, $k = 1, \ldots, K$.

Let us denote by $S(p)$ the moment in which the first value is pumped through input p. We obtain $S(G) = t_0$, $S(W_i) = t_0 - N + i - 1$, $i = 1, \ldots, N$, $S(I_{in}) = t_0 - N$, $S(d_1) = t_0 - N - 1$, $S(X_k) = S(C_k) = t_0 + k - T(f_k) - N - 3$, $k = 1, \ldots, K$. Clearly, $S(d_1) < S(W_i)$, $i = 1, \ldots, N$, $S(d_1) < S(I_{in})$, $S(G)$. Let us consider $P = \{d_1, X_1, \ldots, X_K\}$. The value of t_0 is given by the condition

$$\min_{p \in P} S(p) = 0. \tag{3}$$

Consequently, we can state

THEOREM 2. *If* $G(t_k - 1) = g(k)$, $I_{in}(t_k - N - 1) = 0$, $W_i(t_k - N + i - 2) = w(k, i)$, $X_k(t_k - T(f_k) - N - 4 + i) = x(k, i)$, $C_k(t_k - T(f_k) - N - 4 + i) = 1$, *for* $i = 1, \ldots, N(k)$, *and* $W_i(t_k - N + i - 2) = 0$, $C_k(t_k - T(f_k) - N - 4 + i) = 0$ *for* $i = N(k) + 1, \ldots, N$, $k = 1, \ldots, K$ *then* $I_{out}(t_k) = I(k)$, $k = 1, \ldots, K$, *where* $t_k = t_0 + k$, $k = 1, \ldots, K$ *and* t_0 *is given by* (3).

The period of SN is 1 CT, while the response time of SN is $RT(SN) = t_0 + 1$. The entire processing takes $t_0 + K$ CTs.

3. The influence of the processing order. Further, we analyse two cases when the control of the left side of SN is more simple allowing to the control signal to circulate along a line in order to activate the cells $A(f_k)$, $k = 1, \ldots, K$. These situations are given by the order to compute the integrals and we show how the response time of SN can be better precised and eventually reduced.

THEOREM 3. *If* $T(f_k) < T(f_{k+1})$, $k = 1, \ldots, K - 1$ *then*

(i) $S(X_k) \geq S(X_{k+1})$, $k = 1, \ldots, K - 1$;

(ii) *if* $K \geq T_{max} + 2$ *then* $RT(SN) = N + 2$;

(iii) *if* $K < T_{max} + 2$ *then* $RT(SN) = T_{max} - K + N + 4$, *where* $T_{max} = \max_k T(f_k)$.

Proof. (i) From Theorem 2, it results $S(X_{k+1}) - S(X_k) = 1 + T(f_k) - T(f_{k+1}) \leq 0$, because $T(f_k) - T(f_{k+1}) \leq -1$, $k = 1, \ldots, K - 1$.

(ii) From (3) and (i), it results $\min_{p \in P} S(p) = \min(S(d_1), S(X_K)) = \min(t_0 - N - 1, t_0 + K - T_{max} - N - 3)$ because $T(f_K) = T_{max}$. Now, $K \geq T_{max} + 2$ implies $S(d_1) \leq S(X_K)$, thus $t_0 = N + 1$ and $RT(SN) = N + 2$.

(iii) Now, $K < T_{max} + 2$ implies $S(d_1) > S(X_K)$ and therefore $t_0 + K - T_{max} - N - 3 = 0$ and $RT(SN) = T_{max} - K + N + 4$.

Let us remark that (i) from Theorem 3 implies that the control signal can move along a line from $A(f_{k+1})$ to $A(f_k)$, $k = K - 1, \ldots, 1$.

Now, let us suppose that $1 \leq k_1 < k_2 < \ldots < k_s \leq K$, $n_r = k_{r+1} - k_r$, $r = 1, \ldots, s - 1$, $n_s = K - k_s + 1$, so that $T(f_{k_r}) = T(f_{k_r+j})$, $j = 1, \ldots, n_r - 1$

and $$T(f_{k_r}) < T(f_{k_{r+1}}),\ r = 1, \ldots, s-1.$$

THEOREM 4. *If* $T(f_{k_r}) = T(f_{k_r+j}),\ j = 1, \ldots, n_r - 1$ *and* $T(f_{k_{r+1}}) \geq T(f_{k_r})$, $r = 1, \ldots, s-1$, *then*

(i) $S(X_k) \geq S(X_{k_{r+1}}),\ r = 1, \ldots, s-1$ *and* $S(X_{k_r+j}) = S(X_{k_r}) + j,\ j = 1, \ldots, n_r - 1,\ r = 1, \ldots, s$;

(ii) *if* $k_s \geq T_{\max} + 2$ *then* $RT(SN) = N + 2$;

(iii) *if* $k_s < T_{\max} + 2$ *then* $RT(SN) = T_{\max} - k_s + N + 1$, *with* $T_{\max} = \max_k T(f_k)$.

Proof. (i) We have $S(X_{k_{r+1}}) - S(X_{k_r}) = k_{r+1} - k_r + T(f_{k_r}) - T(f_{k_{r+1}}) = n_r + T(f_{k_r}) - T(f_{k_{r+1}}) \leq 0,\ r = 1, \ldots, s-1$. On the other hand, $S(X_{k_r+j}) = S(X_{k_r+j-1}) + 1$, because $T(f_{k_r+j}) = T(f_{k_r+j-1})$, and therefore

$$S(X_{k_r+j}) = S(X_{k_r}) + j,\ j = 1, \ldots, n_r - 1, r = 1, \ldots, s.$$

(ii) Because $S(X_{k_r+j}) > S(X_{k_r}),\ j = 1, \ldots, n_r - 1$, it results that

$\min_{p \in P} S(p) = \min\,(S(d_1), S(X_{k_1}), \ldots, S(X_{k_s}))$ and from (i) we obtain $\min_{p \in P} S(p) = \min\,(S(d_1), S(X_{k_s}))$. The rest of the proof is similar to that of Theorem 3.

A consequence of Theorem 4 is that the control signal moves from $X_{k_{r+1}}$ to X_{k_r}, but circulates reversely from X_{k_r+j-1} to X_{k_r+j}, $j = 1, \ldots, n_r - 1$, $r = 1, \ldots, s-1$.

THEOREM 5. *If* $T(f_k) \geq T(f_{k+1}),\ k = 1, \ldots, K-1$ *then*

(i) $S(X_k) < S(X_{k+1}),\ k = 1, \ldots, K-1$;

(ii) $RT(SN) = T_{\max} + N + 3$.

Proof. (i) $S(X_{k+1}) - S(X_k) = 1 + T(f_k) - T(f_{k+1}) \geq 1,\ k = 1, \ldots, K-1.$

(ii) Now, we obtain $\min_{p \in P} S(p) = \min\,(S(d_1), S(X_1)) = \min\,(t_0 - N - 1,\ t_0 - T_{\max} - N - 2)$, because $T(f_1) = T_{\max}$, thus $t_0 = T_{\max} + N + 2$.

The control signal moves now from $A(f_k)$ to $A(f_{k+1}),\ k = 1, \ldots, K-1$.

4. **Some remarks.** If $I(k),\ k = 1, \ldots, K$, are to be computed using a single method, then the weights and the factor could be charged by a reset command in $N + 1$ registers keeping these values during the entire computation. Thus N pins could be saved.

If $f_k = f$ but $a_k \neq a_{k'}$ or $b_k \neq b_{k'},\ k \neq k',\ k, k' = 1, \ldots, K$, then from Theorem 4 we obtain $RT(SN) = T(f) + N + 3$.

The entire computation takes $TT(K) = RT(SN) + K - 1$ CTs. If there exists a constant q so that $T(f_k) \leq q,\ k = 1, \ldots$, it results that $\lim TT(K)/K = 1$, as K tends to infinity, i.e. for larger values of K the computation needs 1 CT per integral.

Observe that the response time given by Theorem 5 dominates the response times given by Theorems 3–4, thus we obtain the appropriate processing order.

The case of tabulated data can be treated by considering that the response time of $A(f_k)$ is zero, $k = 1, \ldots, K$. By applying Theorem 4 we obtain $RT(SN)$

$= N + 2$. Another way is to send the f_k — values to Y-inputs. Consequently, the DMs can be saved and $RT(SN) = N + 1$. The case of tabulated data is an appropriate one for implementing quadrature formulas containing the values of the derivatives of the processed functions ([5]); it suffices to regard the derivative values as function values.

The network SN can implement some cubature formulas, too.

The case when $w(k, i) = w(k, N(k) - i + 1)$, $i = 1, \ldots, N(k)$, and $N(k) = 2m$ (e.g. Newton-Côtes formulas) leads to a variant of SN obtained by replacing the even numbered w-ways by a single 1-bit wide control way and by adding some $1CT$-delay processors (denoted by a circle) as it is shown in Figure 3 (b). The f_k-values must be given in the order $f_k(x(k, i))$, $f_k(x(k, N(k) - i + 1))$, $i = 1, \ldots, m$. The $w(k, 2i - 1)$ — value is introduced through W_{2i-1}, $i = 1, \ldots, m$, one value every $2CTs$. The case when $N(k) = 2m + 1$ can be treated in a similar manner.

Let us remark that SN can process formulas for which the number of terms exceed N by splitting the summ into parts and by adding these parts outside or inside of SN (in this last case an adder is to be placed at the right end of the basic pipe). Also, a copy of the array is to be included for each part. The case when a function is given by different arithmetic expressions on prescribed intervals, could be treated in a similar manner.

The network can support any value of K by adding new A and DM cells at the top.

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REDUCING OF VARIANCE BY SPLINE FUNCTIONS IN MONTE CARLO INTEGRATION

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REZUMAT. — Reducerea dispersiei prin funcţii spline în integrarea Monte Carlo. Reducerea estimatorului folosit în evaluarea Monte Carlo a unei integrale definite este una din problemele importante în integrarea numerică a funcţiilor pe cale probabilistică. În această lucrare se studiază acest aspect din integrarea Monte Carlo. Se consideră două metode de reducere a dispersiei, metoda separării părţii esenţiale şi metoda alegerii esenţiale. În cele două metode se folosesc funcţii părţii esenţiale şi metoda alegerii esenţiale. În cele două metode se folosesc funcţii spline care conservă unele proprietăţi ale funcţiei integrate. De asemenea, se prezintă două utilizări combinate ale celor două metode mai sus numite. În final sînt date unele rezultate numerice în comparaţie cu integrarea numerică Monte Carlo clasică.

0. Introduction. A definite integral can be estimated by probabilistic considerations, so-called Monte Carlo integration methods, and these methods are preferably when multiple integral is considered. Moreover, as usual the mathematical software of computers possesses fast generators of random numbers, which are necessary in te Monte Carlo methods.

In Monte Carlo integration the definite integral is looked as expectation of a certain random variable, and this is an unknown parameter. To estimate this parameter, i.e. the definite integral, a sampling from the random variable which was considered is performed, and then one takes an unbiased estimation for this parameter. Generally, this method is not fast-converging ratio to volume sampling, and efficiency depends on the variance of estimator. To increase the efficiency must to reduce as much as possible this variance.

Two important methods for reducing of the variance are presented in [8]: method of control variates, and method of importance sampling. The Bernstein polynomials for reducing of the variance were considered in [10], using the above mentioned methods, when the definite integral on the unit hypercube is considered. We have considered in [2] the same thing in the case when the domain of integration is the n-dimensional simplex.

In this paper we cosider the reducing of variance by the two methods and combinations of these using the spline functions namely the Schoenberg's spline operator. This operator was introduced in [11] and then it was generalized in bidimensional case in [6]. At the end some numerical results are presented in unidimensional case, when the two methods are applied to reduce of the variance. Two combined schemes are considered too. All these techniques are reported to crude Monte Carlo method.

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1. Schoenberg's spline operator. Let $\mathbf{D}_s$ be the s-dimensional unit hypercube, i.e. $\mathbf{D}_s = [0, 1]^s$, and the integer vectors $\mathbf{m} = (m_1, m_2, \ldots, m_s)$, $\mathbf{k} = (k_1, k_2, \ldots, k_s)$, where $m_p \geqslant 1$, $k_p \geqslant 2$, $p = \overline{1, s}$. We define a rectangular partition of $\mathbf{D}_s$ denoted by Δ which is obtained by one-dimensional partitions:

$$\Delta_p : t_1^{(p)} = \ldots = t_{k_p}^{(p)} = 0 < t_{k_p+1}^{(p)} < \ldots < t_{k_p+m_p-1}^{(p)} < 1 = $$
$$= t_{k_p+m_p}^{(p)} = \ldots = t_{2k_p+m_p-1}^{(p)}, \ p = \overline{1, s}.$$

If one considers the multi-index set

$$\mathbf{J} = \{\mathbf{j} = (j_1, \ldots, j_s) \mid j_p = \overline{1, m_p + 2k_p - 1}, \ p = \overline{1, s}\},$$

then

$$\Delta = \mathop{\times}_{p=1}^{s} \Delta_p = \{\mathbf{t}_\mathbf{j} = (t_{j_1}^{(1)}, \ldots, t_{j_s}^{(s)}) \mid \mathbf{j} \in \mathbf{J}\}.$$

The points of Δ are named *the knots* of partition Δ. One considers too the multi-index set

$$\mathbf{I} = \{\mathbf{i} = (i_1, \ldots, i_s) \mid i_p = \overline{1, m_p + k_p - 1}, \ p = \overline{1, s}\}.$$

Using the knots of partition Δ one defines *the nodes*

$$\xi_{\mathbf{i},\mathbf{k}} = (\xi_{i_1,k_1}^{(1)}, \ldots, \xi_{i_s,k_s}^{(s)}), \ \mathbf{i} = (i_1, \ldots, i_s) \in \mathbf{I},$$

with

$$\xi_{i_p,k_p}^{(p)} = (t_{i_p+1}^{(p)} + \ldots + t_{i_p+k_p-1}^{(p)})/(k_p - 1), \ i_p = \overline{1, m_p + k_p - 1}, \ p = \overline{1, s}.$$

Also, by the knots of partition Δ one defines *the (s-dimensional) B-spline functions*

$$\mathbf{M}_{\mathbf{i},\mathbf{k}}(\mathbf{x}) = M_{i_1,k_1}^{(1)}(x_1) \ldots M_{i_s,k_s}^{(s)}(x_s),$$

with $\mathbf{i} = (i_1, \ldots, i_s) \in \mathbf{I}$, $\mathbf{x} = (x_1, \ldots, x_s) \in \mathbf{D}_s$, and

$$M_{i_p,k_p}^{(p)}(x_p) = [t_{i_p}^{(p)}, \ldots, t_{i_p+k_p}^{(p)}; \ k_p(t - x_p)_+^{k_p-1}], \ i_p = \overline{1, m_p + k_p - 1}, \ p = \overline{1, s},$$

the (one-dimensional) B-spline functions. We denote by $[z_1, \ldots, z_{r+1}; h(t)]$ the r-th divided difference relative to the knots $z_1, \ldots, z_{r+1}$ and the function $h(t)$.

We consider too *the normalized B-spline functions* defined by:

$$\mathbf{N}_{\mathbf{i},\mathbf{k}}(\mathbf{x}) = N_{i_1,k_1}^{(1)}(x_1) \ldots N_{i_s,k_s}^{(s)}(x_s),$$

with $\mathbf{i} = (i_1, \ldots, i_s) \in \mathbf{I}$, $\mathbf{x} = (x_1, \ldots, x_s) \in \mathbf{D}_s$, and

$$N_{i_p,k_p}^{(p)}(x_p) = \frac{t_{i_p+k_p}^{(p)} - t_{i_p}^{(p)}}{k_p} M_{i_p,k_p}^{(p)}(x_p),$$

the (one-dimensional) normalized B-spline function. Taking into account these considerations we have that

$$\mathbf{N}_{\mathbf{i},\mathbf{k}}(\mathbf{x}) = \prod_{p=1}^{s} \frac{t^{(p)}_{i_p+k_p} - t^{(p)}_{i_p}}{k_p} \mathbf{M}_{\mathbf{i},\mathbf{k}}(\mathbf{x}), \ \mathbf{i} = (i_1, \ldots, i_s) \in \mathbf{I}, \ \mathbf{x} \in \mathbf{D}_s.$$

The (s-dimensional) Schoenberg's operator relative to a real function f defined on $\mathbf{D}_s$ is given by

$$S_\Delta(f)(\mathbf{x}) = \sum_{\mathbf{i} \in \mathbf{I}} \mathbf{N}_{\mathbf{i},\mathbf{k}}(\mathbf{x}) f(\xi_{\mathbf{i},\mathbf{k}}), \ \mathbf{x} \in \mathbf{D}_s.$$

We recall some important properties of this operator:

(i) $S_\Delta(f)$ is a polynomial spline of degree $k_p - 1$ in the p-th nedeterminate.

(ii) $S_\Delta(f)$ is a positive linear operator.

(iii) $S_\Delta(e_0) = e_0$ and $S_\Delta(e_p) = e_p$, $p = \overline{1, s}$, where $e_0(\mathbf{x}) = 1$ and $e_p(\mathbf{x}) = x_p$, with $\mathbf{x} = (x_1, \ldots, x_s) \in \mathbf{D}_s$.

(iv) If $m_p = 1$ then $S_\Delta(f)$ is a polynomial of degree $k_p - 1$ in the p-th nedeterminate, and so if $m_p = 1$ for the all $p = \overline{1, s}$, then $S_\Delta(f)$ is a polynomial in the each nedeterminate, moreover it is known that in this late case $S_\Delta(f)$ is s-dimensional Bernstein polynomial corresponding to the function f.

(v) $S_\Delta(f)$ converges uniformly to the function f as $\frac{|\Delta_1|}{k_1} + \ldots + \frac{|\Delta_s|}{k_s} \to 0$, where $|\Delta_p|$ denotes the norm of the partition Δ_p.

(vi) Taking into account that $\int_{\mathbf{D}_s} \mathbf{M}_{\mathbf{i},\mathbf{k}}(\mathbf{x}) d\mathbf{x} = 1$, and using the relation between the s-dimensional B-spline functions and one-dimensional B-spline functions we can to consider that $\mathbf{M}_{\mathbf{i},\mathbf{k}}(\mathbf{x})$ is a probability density function of some s-dimensional random vector with independent random components (the values of the random vector are taken in $\mathbf{D}_s$). Also we have that

$$T = \int_{\mathbf{D}_s} S_\Delta(f)(\mathbf{x}) d\mathbf{x} = \sum_{\mathbf{i} \in \mathbf{I}} a_{\mathbf{i}} f(\xi_{\mathbf{i},\mathbf{k}}),$$

where

$$a_{\mathbf{i}} = \prod_{p=1} \frac{t^{(p)}_{i_p+k_p} - t^{(p)}_{i_p}}{k_p}, \quad \mathbf{i} = (i_1, \ldots, i_s) \in \mathbf{I},$$

and $T > 0$ when $f > 0$.

2. Crude Monte Carlo method. Let f be an absolute integrable function defined on the domain $\mathbf{D}_s$. The approximating value of the integral

$$I = \int_{\mathbf{D}_s} f(\mathbf{x})\, d\mathbf{x}$$

can be obtained by probabilistic interpretation of this integral as the expectation of the random variable $f(\mathbf{X})$, where $\mathbf{X} = (X_1, \ldots, X_s)$ is an uniform distributed random vector over the unit hypercube $\mathbf{D}_s$.

One performs a sample with volume N which is uniform distributed over $\mathbf{D}_s$. Let $\mathbf{X}^{(k)} = (X_1^{(k)}, \ldots, X_s^{(k)})$, $k = \overline{1, N}$, be this sampling of random vectors. Then the sampling function

$$\alpha_N = \frac{1}{N} \sum_{k=1}^{N} f(\mathbf{X}^{(k)})$$

is an unbiased estimation function for the parameter I, i.e. for the considered definite integral. The variance of α_N is σ_C^2/N, where σ_C^2 is the variance of $f(\mathbf{X})$. Thus, α_N converges with probability one to I as $N \to \infty$.

Taking into account these probabilistic considerations one has that

$$\int_{\mathbf{D}_s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \approx \frac{1}{N} \sum_{k=1}^{N} f(x_1^{(k)}, \ldots, x_s^{(k)}),$$

where $(x_1^{(k)}, \ldots, x_s^{(k)})$, $k = 1, N$, are independent uniformly random number vectors over $\mathbf{D}_s$.

3. Method of control variates. This method consists in rewrite the integral I in the form

$$I = \int_{\mathbf{D}_s} [f(\mathbf{x}) - g(\mathbf{x})]\,\mathrm{d}\mathbf{x} + \int_{\mathbf{D}_s} g(\mathbf{x})\,\mathrm{d}\mathbf{x},$$

where $g(\mathbf{x})$ is chosen such as to be theoretically integrated over $\mathbf{D}_s$ and to mimic the behaviour of the function f. Such that, the estimation of I is reduced to estimate of the integral

$$I_1 = \int_{\mathbf{D}_s} [f(\mathbf{x}) - g(\mathbf{x})]\,\mathrm{d}\mathbf{x}.$$

Because $S_\Delta(f)$ has these required properties, we take $g = S_\Delta(f)$, and so $I = I_1 + T$.

To evaluate the integral I_1 one considers the estimation function

$$\beta_N = \frac{1}{N} \sum_{k=1}^{N} e(\mathbf{X}^{(k)}),$$

where $e = f - S_\Delta(f)$, and $\mathbf{X}^{(k)} = (X_1^{(k)}, \ldots, X_s^{(k)})$, $k = \overline{1, N}$, are independent and identically uniform distributed random vectors over $\mathbf{D}_s$. This estimating function is an unbiased estimating function for the parameter I_1. The variance of β_N is σ_V^2/N, with σ_V^2 the variance of $e(\mathbf{X})$, where $\mathbf{X} = (X_1, \ldots, X_s)$ is a uniform distributed random vector over $\mathbf{D}_s$.

Therefore we have that

$$\int_{\mathbf{D}_s} [f(\mathbf{x}) - S_\Delta(f)(\mathbf{x})]\,\mathbf{dx} \approx \widehat{\beta}_N,$$

with

$$\widehat{\beta}_N = \frac{1}{N}\sum_{k=1}^{N} e(\mathbf{x}^{(k)}) = \frac{1}{N}\sum_{k=1}^{N} [f(\mathbf{x}^{(k)}) - S_\Delta(f)(\mathbf{x}^{(k)})],$$

where $\mathbf{x}^{(k)} = (x_1^{(k)}, \ldots, x_s^{(k)})$, $k = \overline{1, N}$, are uniform random number vectors over $\mathbf{D}_s$. It results that

$$\int_{\mathbf{D}_s} f(\mathbf{x})\mathbf{dx} \approx \widehat{\beta}_N + T,$$

where T is calculated by formula which is presented in the section one.

To generate the uniform number vectors over the unit hypercube $\mathbf{D}_s$ a lot of methods are known [12].

4. Method of importance sampling. This method consists to consider a new density function g, that mimics the properties of the function f and to rewrite the integral I in the form

$$I = \int_{\mathbf{D}_s} h(\mathbf{x})\, g(\mathbf{x})\, \mathbf{dx},$$

where $h = f/g$.

We consider $g = \widetilde{S}_\Delta(f)$ with $\widetilde{S}_\Delta(f) = \widetilde{S}_\Delta(f)/T$. Taking into account that $S_\Delta(f)$ is a positive linear operator we have that $\widetilde{S}_\Delta(f) > 0$ when $f > 0$. Alternatively, an appropriate constant is added to f. Further on the positivity of the function f is supposed.

To evaluate the integral I one considers the estimation function

$$\gamma_N = \frac{1}{N}\sum_{k=1}^{N} h(\mathbf{X}^{(k)}) = \frac{1}{N}\sum_{k=1}^{N} f(\mathbf{X}^{(k)})/\widetilde{S}_\Delta(f)(\mathbf{X}^{(k)}),$$

where $\mathbf{X}^{(k)} = (X_1^{(k)}, \ldots, X_s^{(k)})$, $k = \overline{1, N}$, are independent and identically distributed random vectors, with the probability density function $\widetilde{S}_\Delta(f)$ over $\mathbf{D}_s$. The estimating function γ_N is an unbiased estimating function for the parameter I and the variance of γ_N is σ_S^2/N, with σ_S^2 the variance of $h(\mathbf{X})$, where the random vector $\mathbf{X} = (X_1, \ldots, X_s)$ is $\widetilde{S}_\Delta(f)$ distributed over $\mathbf{D}_s$. Therefore $I \approx \widehat{\gamma}_N$, where

$$\widehat{\gamma}_N = \frac{1}{N}\sum_{k=1}^{N} h(\mathbf{x}^{(k)}) = \frac{T}{N}\sum_{k=1}^{N} \frac{f(x^{(k)})}{S_\Delta(f)(x^{(k)})},$$

with $\mathbf{x}^{(k)} = (x_1^{(k)}, \ldots, x_s^{(k)})$, $k = \overline{1, N}$, random number vectors $\widetilde{S}_\Delta(f)$ distributed over $\mathbf{D}_s$.

Let $A(z_1, \ldots, z_{k+1})$ be an algorithm which generates the uni-dimensional B-spline distribution given by probability density

$$M_k(t; z_1, \ldots, z_{k+1}) = [z_1, \ldots, z_{k+1}; k(z-t)_+^{k-1}], \quad t \in [0, 1].$$

An open problem is to construct a simple algorithm $A(z_1, \ldots, z_{k+1})$ which generates the B-spline distribution. If a such algorithm is given, then the generation of the $\tilde{S}_\Delta(f)$ distribution can be obtained. Namely, one considers an urn which contains balls of $M = (k_1 + m_1 - 1) \ldots (k_s + m_s - 1)$ colours denoted by $\mathbf{i} = (i_1, \ldots, i_s) \in \mathbf{I}$. Let $B_\mathbf{i}$ be the event of drawing out a ball labeled by $\mathbf{i}$ and one considers that the probability of this event is

$$P(B_\mathbf{i}) = a_\mathbf{i} f(\xi_{\mathbf{i},\mathbf{k}}) / \sum_{\mathbf{i} \in \mathbf{I}} a_\mathbf{i} f(\xi_{\mathbf{i},\mathbf{k}}).$$

If a ball of the colour $\mathbf{i}$ is drawing then one generates the independent random numbers x_p, $p = \overline{1, s}$, by the respectively algorithm $A(t_{i_p}^{(p)}, \ldots, t_{i_p+k_p}^{(p)})$, $p = \overline{1, s}$, $(i_1, \ldots, i_s) = \mathbf{i}$. These random numbers are the values of the components of the random vector $\mathbf{X} = (X_1, \ldots, X_s)$ which is $\tilde{S}_\Delta(f)$ distributed over $\mathbf{D}_s$.

Indeed, let F be the distribution function of the random vector $\mathbf{X} = (X_1, \ldots, X_s)$ and ρ corresponding density function. Using the total probability formula we have that

$$F(\mathbf{x}) = F(x_1, \ldots, x_s) = P(X_1 < x_1, \ldots, X_s < x_s) =$$

$$= \sum_{\mathbf{i} \in \mathbf{I}} P(B_\mathbf{i}) P(X_1 < x_1, \ldots, X_s < x_s \mid B_\mathbf{i}) = \sum_{\mathbf{i} \in \mathbf{I}} P(B_\mathbf{i}) \prod_{p=1}^{s} P(X_p < x_p \mid B_\mathbf{i}).$$

But $P(X_p < x_p \mid B_\mathbf{i})$ is the distribution function which has accordingly probability density $M_{i_p,k_p}^{(p)}$, for corresponding $p = \overline{1, s}$. Such that, we have

$$\rho(\mathbf{x}) = \rho(x_1, \ldots, x_s) = \sum_{\mathbf{i} \in \mathbf{I}} \frac{a_\mathbf{i} f(\xi_{\mathbf{i},\mathbf{k}})}{\sum_{\mathbf{j} \in \mathbf{I}} a_\mathbf{j} f(\xi_{\mathbf{j},\mathbf{k}})} \prod_{p=1}^{s} M_{i_p,k_p}^{(p)}(x_p) =$$

$$= \frac{1}{T} \sum_{\mathbf{i} \in \mathbf{I}} f(\xi_{\mathbf{i},\mathbf{k}}) \prod_{p=1}^{s} N_{i_p,k_p}^{(p)}(x_p) = \tilde{S}_\Delta(f)(\mathbf{x}),$$

therefore $\rho = \tilde{S}_\Delta(f)$.

On the basis of these results we give the following algorithm to generate a random vector which is $\tilde{S}_\Delta(f)$ distributed over the unit hypercube $\mathbf{D}_s$:

Step 1. A correspondence one-to-one one defines $r: \{1, 2, \ldots, M\} \to \mathbf{I} = \{(i_1, \ldots, i_s) \mid i_p = \overline{1, m_p + k_p - 1},\ p = \overline{1, s}\}$, where $M = \prod_{p=1}^{s} (m_p + k_p - 1)$.

Step 2. Calculate $q_k = P(B_{r(k)})$, $k = \overline{1, M}$.

Step 3. Generate uniform x over $(0,1)$.

Step 4. If $x \in [q_1 + \ldots + q_{i-1}, q_1 + \ldots + q_i)$, then for $p = \overline{1, s}$, generate x_p by algorithm $A(t^{(p)}_{i_p}, \ldots, t^{(p)}_{i_p+k_p})$, with $r(i) = (i_1, \ldots, i_s)$.

Step 5. $(x_1, \ldots, x_s)$ is $\widetilde{S}_\Delta(f)$ distributed.

5. **Importance sampling-control variates method.** The two methods to reduce of variance are applied successively, in the first one applies the method of importance sampling and then the method of control variates is applied. In this case one writes the integral I in the form $I = I_2 + I_3$, with

$$I_2 = \int_{\mathbf{D}_s} [h(\mathbf{x}) - S_\Delta(h)(\mathbf{x})]\, \widetilde{S}_\Delta(f)(\mathbf{x})\,\mathbf{dx}$$

and

$$I_3 = \int_{\mathbf{D}_s} S_\Delta(h)(\mathbf{x})\, \widetilde{S}_\Delta(f)(f)\,\mathbf{dx},$$

where $h = f/\widetilde{S}_\Delta(f)$.

The integral I_3 can be calculated using the formula

$$\int_{\mathbf{D}_s} S_\Delta(u)(\mathbf{x})\, S_\Delta(v)(\mathbf{x})\,\mathbf{dx} = \sum_{\mathbf{i}\in\mathbf{I}} \sum_{\mathbf{j}\in\mathbf{I}} u(\xi_{\mathbf{i},\mathbf{k}})\, v(\xi_{\mathbf{j},\mathbf{k}}) A^{(\mathbf{k})}_{\mathbf{i},\mathbf{j}},$$

where

$$A^{(\mathbf{k})}_{\mathbf{i},\mathbf{j}} = \int_{\mathbf{D}_s} \mathbf{N}_{\mathbf{i},\mathbf{k}}(\mathbf{x})\,\mathbf{N}_{\mathbf{j},\mathbf{k}}(\mathbf{x})\,\mathbf{dx} = \prod_{p=1}^{s}\left(\int_0^1 N^{(p)}_{i_p,k_p}(x) N^{(p)}_{j_p,k_p}(x)\,dx\right).$$

In [3], [5] is presented a method to calculate the integral of the product of two B-spline functions, and so the coefficients $A^{(\mathbf{k})}_{\mathbf{i},\mathbf{j}}$ are obtained.

To estimate the integral I_2 one considers the estimating function

$$\delta_N = \frac{1}{N}\sum_{k=1}^{N} u(\mathbf{X}^{(k)}),$$

where $u = h - S_\Delta(h)$, and $\mathbf{X}^{(k)} = (X^{(k)}_1, \ldots, X^{(k)}_s)$, $k = \overline{1, N}$, are independent and identically distributed random vectors with probability density function $\widetilde{S}_\Delta(f)$. Of course, the function f must to be positively, otherwise it is incremented by a suitable constant. The estimating function is unbiased for the parameter I_2 and it has the variance σ^2_{SV}/N, where σ^2_{SV} is the variance of $u(\mathbf{X})$ with $\mathbf{X}$ a random vector $\widetilde{S}_\Delta(f)$ distributed over $\mathbf{D}_s$.

Therefore, we have that

$$\int_{\mathbf{D}_s} f(\mathbf{x})\,\mathbf{dx} \approx \widehat{\delta}_N + I_3$$

with

$$\hat{\delta}_N = \frac{1}{N}\sum_{k=1}^{N} u(\mathbf{x}^{(k)}) = \frac{T}{N}\sum_{k=1}^{N}\left[\frac{f(\mathbf{x}^{(k)})}{S_\Delta(f)(\mathbf{x}^{(k)})} - S_\Delta\left(\frac{f}{S_\Delta(f)}\right)(\mathbf{x}^{(k)})\right]$$

where $\mathbf{x}^{(k)} = (x_1^{(k)}, \ldots, x_s^{(k)})$, $k = \overline{1, N}$, are random number vectors $\tilde{S}_\Delta(f)$ distributed over $\mathbf{D}_s$. To generate these random number vectors one follows the algorithm presented in the previous section.

6. Control variates-importance sampling method. This second combining scheme applies in the first the method of control variates and then the importance sampling method is applied. It is supposed that $e = f - S_\Delta(f)$ is nonnegative. The integral I is writed in the form $I = I_4 + T$, where

$$I_4 = \int_{\mathbf{D}_s} z(\mathbf{x})\, \tilde{S}_\Delta(e)(\mathbf{x})\, \mathrm{d}\mathbf{x},$$

with $z = e/\tilde{S}_\Delta(e)$.

The evaluating of the integral I is reduced to evaluate the integral I_4. If the random vectors $\mathbf{X}^{(k)} = (X_1^{(k)}, \ldots, X_s^{(k)})$, $k = \overline{1, N}$, are independent and identically $\tilde{S}_\Delta(e)$ distributed over $\mathbf{D}_s$, then the estimating function

$$\varepsilon_N = \frac{1}{N}\sum_{k=1}^{N} z(\mathbf{X}^{(k)})$$

is an unbiased estimating function for the parameter I_4, and it has the variance σ_{CS}^2/N, with σ_{CS}^2 the variance of $z(\mathbf{X})$, where $\mathbf{X}$ is a random vector $\tilde{S}_\Delta(e)$ distributed over $\mathbf{D}_s$.

Using these results we have that

$$\int_{\mathbf{D}_s} f(\mathbf{x})\, \mathrm{d}\mathbf{x} \approx \hat{\varepsilon}_N + T,$$

where

$$\hat{\varepsilon}_N = \frac{1}{N}\sum_{k=1}^{N} z(\mathbf{x}^{(k)}) = \frac{Q}{N}\sum_{k=1}^{N}\frac{f(\mathbf{x}^{(k)}) - S_\Delta(f)(\mathbf{x}^{(k)})}{S_\Delta(f)(\mathbf{x}^{(k)}) - S_\Delta(S_\Delta f)(\mathbf{x}^{(k)})},$$

with $\mathbf{x}^{(k)} = (x_1^{(k)}, \ldots, x_s^{(k)})$, $k = \overline{1, N}$, random number vectors which have the probability density function $\tilde{S}_\Delta(e)$ over $\mathbf{D}_s$ and

$$Q = \sum_{\mathbf{i}\in\mathbf{I}} a_{\mathbf{i}}\, [f(\xi_{\mathbf{i},\mathbf{k}}) - S_\Delta(f)(\xi_{\mathbf{i},\mathbf{k}})] = T - \sum_{\mathbf{i}\in\mathbf{I}} a_{\mathbf{i}}\, S_\Delta(f)(\xi_{\mathbf{i},\mathbf{k}}).$$

To generate these random number vectors one proposes the algorithm presented in the section four, by replacing the function f with the function e.

7. Numerical experiments. The numerical results are considered in uni-dimensional case. We have considered the functions

$$f_1(x) = 1/(x + 1) \text{ and } f_2(x) = 1/(x^2 + 3).$$

It was used reducing of variance by the two methods and by the two combining schemes. The results are presented in the enclosed table. The variances of the estimators were computed by numerical methods using in double

Type of scheme	k	$m = 1$ (1)	(2)	$m = 2$ (1)	(2)	$m = 3$ (1)	(2)
				$f_1(x) = 1/(x + 1)$			
Crude Monte Carlo		0.6931D+00	100.	0.6931D+00	100.	0.6931D+00	100.
Control variates	2	0.6595D−03	3.37	0.1066D−03	0.55	0.2563D−04	0.13
	3	0.2076D−03	1.06	0.4509D−04	0.23	0.1393D−04	0.07
	4	0.9779D−04	0.50	0.3703D−04	0.19	0.1587D−04	0.08
Importance sampling	2	0.5960D−03	3.05	0.6794D−04	0.35	0.1552D−04	0.08
	3	0.1685D−03	0.86	0.3578D−04	0.18	0.9229D−05	0.05
	4	0.7652D−04	0.39	0.3115D−04	0.16	0.1294D−04	0.07
Importance sampling-control variates	2	0.5960D−03	3.05	0.6793D−04	0.35	0.1552D−04	0.08
	3	0.5096D−04	0.26	0.2567D−05	0.01	0.7617D−06	0.004
	4	0.1333D−04	0.07	0.2231D−05	0.01	0.4050D−06	0.002
Control variates-importance sampling	3	0.7782D−04	0.40	0.6509D−05	0.03	0.4997D−06	0.003
	4	0.2215D−04	0.11	0.1531D−05	0.008	0.1530D−06	0.0008
				$f_2(x) = 1/(x^2 + 3)$			
Crude Monte Carlo		0.3023D+00	100.	0.3023D+00	100.	0.3023D+00	100.
Control variates	2	0.2531D−04	3.81	0.3616D−05	0.54	0.7549D−06	0.11
	3	0.8105D−05	1.22	0.2113D−05	0.32	0.6229D−06	0.09
	4	0.4004D−05	0.60	0.1740D−05	0.26	0.7669D−06	0.12
Importance sampling	2	0.2380D−04	3.58	0.3045D−05	0.46	0.6311D−06	0.09
	3	0.7104D−05	1.07	0.1833D−05	0.28	0.5126D−06	0.08
	4	0.3464D−05	0.52	0.1531D−05	0.23	0.6676D−06	0.10
Importance sampling-control variates	2	0.2380D−04	3.58	0.3044D−05	0.46	0.4688D−07	0.007
	3	0.2902D−05	0.44	0.2308D−06	0.03	0.3948D−07	0.006
	4	0.8781D−06	0.13	0.1818D−06	0.03	0.4319D−07	0.006
Control variates-importance sampling	3	0.2660D−05	0.40	0.1577D−06	0.02	0.4102D−07	0.006
	4	0.7214D−06	0.11	0.9750D−07	0.01	0.2464D−07	0.004

(1) — variance, (2) — per cent of crude Monte Carlo method.

precision a romanian computer CORAL−4030, i.e. σ_C^2, σ_V^2, σ_S^2, σ_{SV}^2, and σ_{CS}^2 respectively. In all the cases the variances were reported to variance of the crude Monte Carlo method.

Remarks;

1. The case $m = 1$ corresponds to the Bernstein polynomial of the degree $k - 1$.

2. The combining scheme control variates-importance sampling is not applicably in the case $k = 2$. In this case $S_\Delta(e) = S_\Delta(f) - S_\Delta(S_\Delta(f)) = 0$, and it is wanting in the table.

3. To compute the values of the B-spline functions we have used the methods presented in [4], and which are based on the recurrence formula of divided differences.

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RECENZII

J. Wloka, **Partial Differential Equations**, Cambridge University Press, Cambridge 1987, 517 pp.

The book is concerned with boundary value problems for partial differential equations (PDE). In order to make it self-contained the author included with complete definitions and proofs, less familiar material from functional analysis. So, the first chapter entitled Sobolev Spaces, is a brief but fairly complete introduction to distribution theory including partition of unity, convolution, Fourier transformation. The Sobolev spaces are considered only in the L^2-case, which is sufficient for the linear case considered in the book, the L^p-theory being characteristic for the nonlinear case. The author adopts the Slobodeckii approach to W_2^s-spaces, avoiding the interpolation theory and working in an elementary way. Using, as an essential tool, the Fourier transformation he proves the basic facts from the theory of Sobolev spaces-embedding theorems, Sobolev spaces on manifolds, Trace operators. In the last section of this chapter, using the weak sequential compactness of bounded sets in the separable Hilbert space W_2^1, the author proves an approximation theorem for derivatives by difference quotients which is used later to prove reqularity theorems for the solutions of PDE.

The study of PDE begins in the second chapter, Elliptic Differential Operators. All the boundary value problems are supposed to satisfy Lopatinskii-Shapiro covering condition which is presented in several equivalent forms. It turns out that for all classifical boundary value problems this condition is fulfilled, the examples being worked through individually. The theory of Fredholm operators developed in § 12 is used to prove an abstract form of Weyl's lemma and some index theorems for elliptic boundary value problems (in § 13). The main theorem of this section states the equivalence between the ellipticity of a boundary value problem, the a priori estimate, the Fredholm property and smoothability. The chapter ends with two Green's formulae which are applied to study normal and Dirichlet boundary value problems and adjoint boundary value problems.

In the third chapter, Strongly Elliptic Differential Operators and the Method of Variations, after presenting some functional analytic tools as Gelfand triples, Lax-Milgram theorem, V-elliptic and V-coercive forms, Green operators, the author proves the main facts about strongly elliptic defferential operators: the theorems of Garding and Agmon, regulatrity of the solutions. Although the book is concerned mainly with linear problems, the nonlinear ones are touched up in § 22 via Brouwer and Schauder fixed point theorems, which are proved in details.

Parabolic differential operators are treated in the fourth chapter. In order to define distributions with values in a Hilbert Space and corresponding Sobolev spaces, the author presents (with complete definitions and proofs) Pettis' measurability theorem and Bochner integral for functions with values in a Hilbert Space. These results are used to prove an abstract existence theorem for parabolic equations, the regularity of the solutions and their differentiability with respect to t and x. Hyperbolic differential equations are treated in Chapter V.

The last chapter, Chapter VI, Difference Processes for the Calculation of the Solutions of the Partial Differential Equations, is based on the lectures talked by the author at the University of Kiel in 1979 and prepared by R. Janssen. The aim of this chapter is to show that replacing the derivateves by difference quotients it is actually possible to solve numerically partial differential equations and this is the simplest way to do this reducing it to a system of linear equations which can be solved by usual methods.

Published first in German, University of Kiel 1975, the book is a valuable contribution to the theory of partial differential equations. Written by an eminent specialist in functional analysis it reflects the taste and the preferences of the author, i.e. replacing, whenever possible the hard analysis by soft analysis and reducing the huge estimation machinery to an unavoidable minimum. The author achieves masterly this task and the result is an excellent book PDE which we recommend warmly to all interested in this very active field of investigation.

Ş. COBZAŞ

G. Beauquet and M. Pogu, **Programmation des elements finis (P_1, 2D)**, Cepadues-Editions, Toulouse—France, 1987, 114 pp.

The work is an excellent textbook for those willing to initiate in FEM programming. It is elementary but at the same time very accurate.

It needs only basic knowledge of programming in a scientific language. From a mathematical point of view it needs notions about double integrals and some simple notions concerning variational calculus.

However, it is self contained, progressive, with an extremely clear exposition.

The authors gradually consider the Dirichlet and Dirichlet-Neumann problems for Laplace and Poisson equations (this last case also for equations having the right hand side of divergential form). They also consider some simple nonlinear problems.

All these are presented for plane domains ($2D$) with finite elements of first degree (P_1).

Some specific operations regarding FEM are treated as follows: the domain triangulation, local and global network labelling ,the construction of matrix (stiff) of linear system corresponding in a first approximation to a programmable relation, the loading of this matrix, the solving of linear systems using iterative methods and finally, only in a second approximation, the writing of the linear system corresponding to variational formulation.

Each operation is a program module. A flow diagram is given for each module and some of them are detailed to program lines.

The work is full of varied and intere[...] examples beeing thus very attractive.

C. I. GHEORGH[...]

M a r i a H a s s e, **Grundbegriffe der Me[...]genlehre und Logik**, B.G. Teubner Verlag 198[9] 84 pp.

The presented book represents the 10-[...] edition (begining with 1965). Compared with the four preceding editions, none completion [...] modification. The fact shows that this work is foundamental in mathematics, useful for students, teachers and all interested in the fou[...]tions of mathematics.

The book contains: Basical notion[...] propositional logic, sets, subsets, operations[...] tions (equivalences, orderings), function[...] nary structures ("operative Mengen"), o[...] sets, closure operators and systems, topol[...] space.

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The material is systematical expos[...] that in a little volume the author gives [...] quantity of information.

N. [...]

CRONIC[...]

I. Publicaţii ale seminariilor de cercetare ale catedrelor de matematică (seria de preprinturi):

Preprint 1—1988, *Seminar on Mechanics* (edited by I. Pop and T. Petrilă);

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Preprint 10—1988, *Seminar on Cele[...] Mechanics and Space Research* (edited by A. [...]

II. Manifestări ştiinţifice organizate de cated[...] de matematică în 1988:

1. *Şedinţele de comunicări lunare ale c[...]drelor de matematică*;
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