STUDIA
UNIVERSITATIS BABEȘ-BOLYAI

MATHEMATICA

2

1989

CLUJ-NAPOCA

Anul XXXIV 1989

STUDIA
UNIVERSITATIS BABEŞ-BOLYAI
MATHEMATICA
2

Redacţia: 3400 CLUJ-NAPOCA, str. M. Kogălniceanu, 1 ● Telefon 1 61 01

SUMAR — CONTENTS

Recenzii — Book Reviews

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXIV, 2, 1989

AN ASYMPTOTIC FORMULA CONCERNING THE UNITARY DIVISOR SUM FUNCTION

LÁSZLÓ TÓTH*

Received: January 17, 1989

REZUMAT. — **O formulă asimptotică referitoare la funcţia sumă a divizorilor unitari.** În lucrare se stabileşte formula asimptotică (21) care reprezintă analogul unitar al unei formule a lui Ramanujan demonstrată în [8].

1. **Introduction.** It is well-known that a divisor $d > 0$ of a positive integer n is called unitary if $n = de$ and $(d, e) = 1$. Let $\sigma_s^*(n)$ denote, as usual, the sum of the s-th powers of all unitary divisors of n and let $\sigma_1^*(n) = \sigma^*(n)$ be the sum of all unitary divisors of n, $\sigma_0^*(n) = \tau^*(n)$ be the number of the unitary divisors of n.

In this paper we establish an asymptotic formula for the sum $\sum_{n \leq x} \sigma_s^{*2}(n)$, where $s > 0$, using an elementary method based on the convolutional identity of lemma 1. For $s = 1$ we obtain, as a corollary, the unitary analogue of Ramanujan's formula

$$\sum_{n \leq x} \sigma^2(n) = \frac{5}{6} \zeta(3) x^3 + 0(x^2 \log^2 x), \tag{1}$$

([5], eq. 19), where $\sigma(n)$ denotes the sum of the divisors of n and $\zeta(z)$ is the Riemann Zeta function.

2. **Preliminaries.** The unitary convolution of the arithmetical functions f and g is defined by

$$(f \cdot g)(n) = \sum_{\substack{de = n \\ (d,e)=1}} f(d)\, g(e) \tag{2}$$

The unitary convolution of two multiplicative functions is also multiplicative ([2], lemma 6.1). Let $U(n) = 1$ and $E_s(n) = n^s$ for all n, hence we have $\sigma_s^* = U \cdot E_s$ and $\tau^* = U \cdot U$.

LEMMA 1.

$$\sigma_s^{*2}(n) = \sum_{\substack{de = n \\ (d,e)=1}} d^s\, \tau^*(d)\, \sigma_{2s}^*(e) \tag{3}$$

Proof. It is easy to verify that for any multiplicative functions f, g, h and k we have $(f \cdot g)(h \cdot k) = fh \cdot fk \cdot gh \cdot gk$ ([7], theorem 2). Now, if $f = h = U$

* Str. N. Golescu 5, 3900 Satu Mare, Romania

and $g = k = E_s$ we obtain

$$\sigma_s^{*2} = (U \cdot E_s)^2 = U \cdot E_s \cdot E_s \cdot E_{2s} = E_s(U \cdot U) \cdot U \cdot E_{2s} = E_s\,\tau^* \cdot \sigma_{2s}^*,$$

which proves the lemma.

Remark 1. A direct proof of (3) is the following. Both sides of this identity are multiplicative, hence it is enough to verify it for $n = p^a$, a prime power. We have

$$\sum_{\substack{de = p^a \\ (d,e)=1}} d^s \tau^*(d)\, \sigma_{2s}^*(e) = \sigma_{2s}^*(p^a) + p^{as}\tau^*(p^a) = p^{2as} + 1 + 2p^{as} = (p^{as}+1)^2 = \sigma_s^{*2}(p^a).$$

We need the following familiar formulas:

LEMMA 2.

$$\sum_{n \leqslant x} n^s = \frac{x^{s+1}}{s+1} + 0(x^s), \quad s \geqslant 0 \tag{4}$$

$$\sum_{n \leqslant x} \frac{1}{n^s} = 0(x^{1-s}), \quad 0 < s < 1 \tag{5}$$

$$\sum_{n \leqslant x} \frac{1}{n} = 0(\log x) \tag{6}$$

$$\sum_{n \leqslant x} \frac{1}{n^s} = \zeta(s) + 0\left(\frac{1}{x^{s-1}}\right), \quad s > 1 \tag{7}$$

$$\sum_{n > x} \frac{1}{n^s} = 0\left(\frac{1}{x^{s-1}}\right), \quad s > 1. \tag{8}$$

Let φ denote the Euler totient function and μ the Möbius function, for which

$$\varphi(n) = n \sum_{d \mid n} \frac{\mu(d)}{d} \tag{9}$$

$$\sum_{d \mid n} \mu(d) = I(n) \equiv \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases} \tag{10}$$

LEMMA 3. (cf. [1], lemma 2.3). *For* $s \geqslant 0$.

$$\sum_{\substack{n \leqslant x \\ (n,k)=1}} n^s = \frac{x^{s+1}\varphi(k)}{(s+1)k} + 0(x^s\tau(k)), \tag{11}$$

where $\tau(k)$ *denotes the number of the divisors of* k.

Proof. By (10) and (4) we have

$$\sum_{\substack{n \leqslant x \\ (n,k)=1}} n^s = \sum_{n \leqslant x} n^s I((n,k)) = \sum_{n \leqslant x} n^s \sum_{\substack{d \mid (n,k) \\ de = n}} \mu(d) = \sum_{d \mid k} d^s \mu(d) \sum_{e \leqslant \frac{x}{d}} e^s =$$

$$= \sum_{d \mid k} d^s \mu(d) \left\{\frac{1}{s+1}\left(\frac{x}{d}\right)^{s+1} + 0\left(\left(\frac{x}{d}\right)^s\right)\right\} = \frac{x^{s+1}}{s+1} \sum_{d \mid k} \frac{\mu(d)}{d} + 0(x^s \sum_{d \mid k} 1) =$$

$$= \frac{x^{s+1}\varphi(k)}{(s+1)k} + 0(x^s\tau(k)),$$

using (9) too.

Let $J_s(n) = n^s \prod_{p \mid n} \left(1 - \frac{1}{p^s}\right)$, $s > 0$ denote the Jordan-type functions ([3], p. 147), where $J_1(n) \equiv \varphi(n)$.

LEMMA 4. *For* $s > 0$

$$\sum_{\substack{n=1 \\ (n,k)=1}}^{\infty} \frac{\varphi(n)}{n^{s+2}} \equiv \sum_{n=1}^{\infty} \frac{\varphi(n)\, I((n,k))}{n^{s+2}} = \frac{\zeta(s+1) k\, J_{s+1}(k)}{\zeta(s+2)\, J_{s+2}(k)} \tag{12}$$

Proof. For $s > 0$ the series is absolutely convergent because $\frac{\varphi(n)}{n^{s+2}} \leqslant \leqslant \frac{n}{n^{s+2}} = \frac{1}{n^{s+1}}$, and the general term is a multiplicative function of n. Hence the series can be expanded into an infinite product of Euler type ([3], §17.4):

$$\sum_{n=1}^{\infty} \frac{\varphi(n)\, I((n,k))}{n^{s+2}} = \prod_{p \nmid k} \left(\sum_{i=0}^{\infty} \frac{\varphi(p^i)}{p^{i(s+2)}} \right) = \prod_{p \nmid k} \left(1 + \frac{p-1}{p^{s+2}} + \frac{p(p-1)}{p^{2(s+2)}} + \frac{p^2(p-1)}{p^{3(s+2)}} + \dots \right) =$$

$$= \prod_{p \nmid k} \left(1 + \frac{p-1}{p^{s+2}} \left(1 + \frac{1}{p^{s+1}} + \frac{1}{p^{2(s+1)}} + \dots \right)\right) = \prod_{p \nmid k} \left(1 + \frac{p-1}{p^{s+2}} \left(1 - \frac{1}{p^{s+1}}\right)^{-1}\right) =$$

$$= \prod_{p \nmid k} \left(1 - \frac{1}{p^{s+2}}\right) \Big/ \prod_{p \nmid k} \left(1 - \frac{1}{p^{s+1}}\right) = \frac{\zeta(s+1)}{\zeta(s+2)} \prod_{p \mid k} \left(1 - \frac{1}{p^{s+1}}\right) \Big/ \prod_{p \mid k} \left(1 - \frac{1}{p^{s+2}}\right) =$$

$$= \frac{\zeta(s+1) k\, J_{s+1}(k)}{\zeta(s+2)\, J_{s+2}(k)}.$$

LEMMA 5.

$$\sum_{n \leqslant x} \frac{\tau(n)}{n^s} = 0\,(x^{1-s} \log x), \quad 0 < s < 1 \tag{13}$$

$$\sum_{n \leqslant x} \frac{\tau(n)}{n} = 0\,(\log^2 x) \tag{14}$$

$$\sum_{n \leqslant x} \frac{\tau(n)}{n^s} = \zeta^2(s) + 0\left(\frac{\log}{x^{s-1}}\right), \quad s > 1 \tag{15}$$

Proof.

$$F_s(x) \equiv \sum_{n \leqslant x} \frac{\tau(n)}{n^s} = \sum_{de = n \leqslant x} \frac{1}{(de)^s} = \sum_{d \leqslant x} \frac{1}{d^s} \sum_{e \leqslant \frac{x}{d}} \frac{1}{e^s}.$$

Hence for

$$0 < s < 1 : F_s(x) = \sum_{d \leqslant x} \frac{1}{d^s}\, 0\left(\left(\frac{x}{d}\right)^{1-s}\right) = 0\left(x^{1-s} \sum_{d \leqslant x} \frac{1}{d}\right) =$$

$= 0\,(x^{1-s} \log x)$ by (5) and (6); $F_1(x) = \sum_{d \leqslant x} \frac{1}{d}\, 0\left(\log \frac{x}{d}\right) =$

$= 0\left(\log x \sum_{d \leqslant x} \frac{1}{d}\right) = 0\,(\log^2 x)$ by (6); and for $s > 1 : F_s(x) =$

$$= \sum_{d \leq x} \frac{1}{d^s} \left\{ \zeta(s) + 0\left(\left(\frac{d}{x} \right)^{s-1} \right) \right\} = \zeta(s) \sum_{d \leq x} \frac{1}{d^s} + 0 \left(\frac{1}{x^{s-1}} \sum_{d \leq x} \frac{1}{d} \right) =$$

$$= \zeta^2(s) + 0\left(\frac{1}{x^{s-1}} \right) + 0 \left(\frac{\log x}{x^{s-1}} \right) = \zeta^2(s) + 0\left(\frac{\log x}{x^{s-1}} \right) \text{ by (7) and (6).}$$

LEMMA 6. *For* $s > 0$

$$\sum_{\substack{n \leq x \\ (n,k)=1}} \sigma_s^* (n) = \frac{\zeta(s+1)\, \varphi(k)\, J_{s+1}(k)}{(s+1)\, \zeta(s+2)\, J_{s+2}(k)} x^{s+1} + 0\, (A_s(x)\, \tau(k)), \tag{16}$$

where $A_s(x) = x^s$, $x \log^2 x$ *or* $x \log x$ *according as* $s > 1$, $s = 1$ *or* $s < 1$.

Proof. Using lemma 3:

$$\sum_{\substack{n \leq x \\ (n,k)=1}} \sigma_s^* (n) = \sum_{\substack{n \leq x \\ (n,k)=1}} \sum_{\substack{de=n \\ (d,e)=1}} e^s = \sum_{\substack{d \leq x \\ (d,k)=1}} \sum_{\substack{e \leq \frac{x}{d} \\ (e,kd)=1}} e^s =$$

$$= \sum_{\substack{d \leq x \\ (d,k)=1}} \left\{ \frac{\left(\frac{x}{d} \right)^{s+1} \varphi(kd)}{(s+1)\, kd} + 0\left(\left(\frac{x}{d} \right)^s \tau(kd) \right) \right\} = \frac{x^{s+1} \varphi(k)}{(s+1)k} \sum_{\substack{d \leq x \\ (d,k)=1}} \frac{\varphi(d)}{d^{s+2}} + 0 \left(x^s \tau(k) \sum_{d \leq x} \frac{\tau(d)}{d^s} \right) =$$

$$= \frac{x^{s+1} \varphi(k)}{(s+1)k} \sum_{\substack{d=1 \\ (d,k)=1}}^{\infty} \frac{\varphi(d)}{d^{s+2}} + 0 \left(x^{s+1} \sum_{d > x} \frac{\varphi(d)}{d^{s+2}} \right) + 0(x^s\, \tau(k) \sum_{d \leq x} \frac{\tau(d)}{d^s}).$$

And now by lemma 4 the main term is

$$\frac{\zeta(s+1)\, \varphi(k)\, J_{s+1}(k)}{(s+1)\, \zeta(s+2)\, J_{s+2}(k)} x^{s+1},$$

the first 0-term is

$$0 \left(x^{s+1} \sum_{d > x} \frac{1}{d^{s+1}} \right) = 0\, (x)$$

by (8) and the second 0-term is

for $s > 1 : 0\, (x^s\, \tau(k))$; for $s = 1 : 0 \left(x\, \tau(k) \sum_{d \leq x} \frac{\tau(d)}{d} \right) = 0\, (x \log^2 x\, \tau(k))$

and for $s < 1 : 0\, (x^s\, \tau(k)\, x^{1-s} \log x) = 0\, (x \log x\, \tau(k))$ by lemma 5.

LEMMA 7. *For* $s > 0$ *the series*

$$\sum_{n=1}^{\infty} \frac{\tau^*(n)\, \varphi(n)\, J_{2s+1}(n)}{n^{s+1}\, J_{2s+2}(n)}$$

is absolutely convergent and its sum is given by $\zeta(s+1)\zeta(2s+2)\, \alpha_s$, where

$$\alpha_s \equiv \prod_p \left(1 + \frac{1}{p^{s+1}} - \frac{2}{p^{s+2}} - \frac{1}{p^{2s+2}} - \frac{2}{p^{3s+2}} + \frac{3}{p^{3s+3}} \right), \tag{17}$$

the product being extended over the primes p.

Proof. $\varphi(n)\, J_{2s+1}(n) / J_{2s+2}(n) \leqslant 1$ for all n, hence the general term is $\leqslant \frac{\tau^*(n)}{n^{s+1}} \leqslant \frac{\tau(n)}{n^{s+1}}$ and the series is absolutely convergent (cf. (15)). The general term is multiplicative in n, so the series can be expanded into an infinite product of Euler type:

$$\sum_{n=1}^{\infty} \frac{\tau^*(n)\, \varphi(n)\, J_{2s+1}(n)}{n^{s+1}\, J_{2s+2}(n)} = \prod_p \left(\sum_{i=0}^{\infty} \frac{\tau^*(p^i)\, \varphi(p^i)\, J_{2s+1}(p^i)}{p^{i(s+1)}\, J_{2s+2}(p^i)} \right) =$$

$$= \prod_p \left(1 + \frac{2(p-1)(p^{2s+1}-1)}{p^{s+1}(p^{2s+2}-1)} + \frac{2p(p-1)p^{2s+1}(p^{2s+1}-1)}{p^{2(s+1)}\, p^{2s+2}(p^{2s+2}-1)} + \ldots \right) =$$

$$= \prod_p \left(1 + \frac{2(p-1)(p^{2s+1}-1)}{p^{s+1}(p^{2s+2}-1)} \left(1 + \frac{1}{p^{s+1}} + \frac{1}{p^{2(s+1)}} + \ldots \right) \right) =$$

$$= \prod_p \left(1 + 2(p-1)(p^{2s+1}-1)\, p^{-3(s+1)} \left(1 - \frac{1}{p^{s+1}}\right)^{-1} \left(1 - \frac{1}{p^{2s+2}}\right)^{-1} \right)$$

and the lemma follows on factoring out

$$\prod_p \left(1 - \frac{1}{p^{s+1}}\right)^{-1} \prod_p \left(1 - \frac{1}{p^{2s+2}}\right)^{-1} = \zeta(s+1)\, \zeta(2s+2).$$

LEMMA 8.

$$\sum_{n \leqslant x} \frac{\tau^2(n)}{n^s} = 0\,(x^{1-s} \log^3 x), \quad 0 < s < 1 \tag{18}$$

$$\sum_{n \leqslant x} \frac{\tau^2(n)}{n} = 0\,(\log^4 x) \tag{19}$$

$$\sum_{n \leqslant x} \frac{\tau^2(n)}{n^s} = 0\,(1), \quad s > 1. \tag{20}$$

Proof. The Dirichlet convolution of the arithmetical functions f and g is defined by

$$(f * g)(n) = \sum_{de=n} f(d)\, g(e).$$

We have $\tau^2 = U * U * U * \mu^2 = \tau^* * \tau$ by [4], hence

$$G_s(x) \equiv \sum_{n \leqslant x} \frac{\tau^2(n)}{n^s} = \sum_{n \leqslant x} \frac{1}{n^s} \sum_{de=n} \tau^*(d)\, \tau(e) = \sum_{d \leqslant x} \frac{\tau^*(d)}{d^s} \sum_{e \leqslant \frac{x}{d}} \frac{\tau(e)}{e^s}$$

and for

$$0 < s < 1,\ G_s(x) = \sum_{d \leqslant x} \frac{\tau^*(d)}{d^s}\, 0\left(\left(\frac{x}{d}\right)^{1-s} \log \frac{x}{d} \right) = 0\left(x^{1-s} \log x \sum_{d \leqslant x} \frac{\tau(d)}{d} \right) = 0(x^{1-s} \log^3 x)$$

by (13) and (14); for $s = 1$

$$G_1(x) = \sum_{d \leqslant x} \frac{\tau^*(d)}{d}\, 0\left(\log^2 \frac{x}{d}\right) = 0\left(\log^2 x \sum_{d \leqslant x} \frac{\tau(d)}{d}\right) = 0(\log^4 x) \text{ by (14);}$$

and for

$$s > 1,\ G_s(x) = \sum_{d \leqslant x} \frac{\tau^*(d)}{d^s} 0(1) = 0\left(\sum_{d \leqslant x} \frac{\tau(d)}{d^s}\right) = 0\ (1) \text{ by } (15).$$

Remark 2. It is well-known Ramanujan's formula ([5], eq. 1),

$$\sum_{n=1}^{\infty} \frac{\tau^2(n)}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)} \text{ for } s > 1.$$

3. The theorem. Now we are ready to prove the following

THEOREM. *For* $s > 0$

$$\sum_{n \leqslant x} \sigma_s^{*2}(n) = \frac{\zeta(s+1)\,\zeta(2s+1)\,\alpha_s}{2s+1}\, x^{2s+1} + 0\,(B_s(x)), \tag{21}$$

where α_s *is defined by* (17) *and*

$$B_s(x) = x^{2s},\ x^2 \log^4 x,\ x^{s+1} \log^3 x,\ x^{\frac{3}{2}} \log^5 x \text{ or } x^{s+1} \log^4 x$$

according as

$$s > 1,\ s = 1,\ \frac{1}{2} < s < 1,\ s = \frac{1}{2} \text{ or } s < \frac{1}{2}.$$

Proof. By lemmas 1 and 6,

$$\sum_{n \leqslant x} \sigma_s^{*2}(n) = \sum_{d \leqslant x} d^s\, \tau^*(d) \sum_{\substack{c \leqslant \frac{x}{d} \\ (c,d)=1}} \sigma_{2s}^*(c) =$$

$$= \sum_{d \leqslant x} d^s\, \tau^*(d) \left\{ \frac{\zeta(2s+1)\,\varphi(d)\,J_{2s+1}(d)}{(2s+1)\,\zeta(2s+2)\,J_{2s+2}(d)} \left(\frac{x}{d}\right)^{2s+1} + 0\left(A_{2s}\left(\frac{x}{d}\right)\tau(d)\right)\right\} =$$

$$= \frac{\zeta(2s+1)\,x^{2s+1}}{(2s+1)\,\zeta(2s+2)} \sum_{d \leqslant x} \frac{\tau^*(d)\,\varphi(d)\,J_{2s+1}(d)}{d^{s+1}\,J_{2s+2}(d)} + 0\left(\sum_{d \leqslant x} A_{2s}\left(\frac{x}{d}\right) d^s\, \tau^*(d)\,\tau(d)\right) =$$

$$= \frac{\zeta(2s+1)\,x^{2s+1}}{(2s+1)\,\zeta(2s+2)} \sum_{d=1}^{\infty} \frac{\tau^*(d)\,\varphi(d)\,J_{2s+1}(d)}{d^{s+1}\,J_{2s+2}(d)} + 0\left(x^{2s+1} \sum_{d > x} \frac{\tau^*(d)\,\varphi(d)\,J_{2s+1}(d)}{d^{s+1}\,J_{2s+2}(d)}\right) +$$

$$+ 0\left(\sum_{d \leqslant x} A_{2s}\left(\frac{x}{d}\right) d^s\, \tau^2(d)\right).$$

Now by lemma 7 the main term is $\frac{1}{2s+1}\,\zeta(s+1)\,\zeta(2s+1)\,\alpha_s\, x^{2s+1}$, the first remain term is

$$0\left(x^{2s+1} \sum_{d > x} \frac{\tau(d)}{d^{s+1}}\right) = 0\left(x^{2s+1}\, \frac{\log x}{x^s}\right) = 0\,(x^{s+1} \log x)$$

by (15) and the second remain term is (using lemma 8): for $s > 1$

$$0\left(\sum_{d\leqslant x}\left(\frac{x}{d}\right)^{2s} d^s\, \tau^2(d)\right) = 0\left(x^{2s}\sum_{d\leqslant x}\frac{\tau^2(d)}{d^s}\right) = 0(x^{2s}) \text{ by (20); for } s = 1$$

$$0\left(\sum_{d\leqslant x}\left(\frac{x}{d}\right)^{2} d\tau^2(d)\right) = 0\left(x^2\sum_{d\leqslant x}\frac{\tau^2(d)}{d}\right) = 0(x^2\log^4 x) \text{ by (19); for } \frac{1}{2} < s < 1$$

$$0\left(\sum_{d\leqslant x}\left(\frac{x}{d}\right)^{2s} d^s\, \tau^2(d)\right) = 0\left(x^{2s}\sum_{d\leqslant x}\frac{\tau^2(d)}{d^s}\right) = 0(x^{s+1}\log^3 x) \text{ by (18); for } s = \frac{1}{2}$$

$$0\left(\sum_{d\leqslant x}\frac{x}{d}\log^2\left(\frac{x}{d}\right) d^{\frac{1}{2}}\, \tau^2(d)\right) = 0\left(x\log^2 x\sum_{d\leqslant x}\frac{\tau^2(d)}{d^{\frac{1}{2}}}\right) = 0\left(x^{\frac{3}{2}}\log^5 x\right)$$

by (18) and for $s < \frac{1}{2}$ it is

$$0\left(\sum_{d\leqslant x}\frac{x}{d}\log\left(\frac{x}{d}\right) d^s\, \tau^2(d)\right) = 0\left(x\log x\sum_{d\leqslant x}\frac{\tau^2(d)}{d^{1-s}}\right) = 0\left(x^{s+1}\log^4 x\right)$$

also by (18) and the proof is complete.

Remark 3. This result is the unitary analogue of the following asymptotic formula referring for $\sigma_s(n)$, the sum of the s-th powers of the divisors of n:

$$\sum_{n\leqslant x}\sigma_s^2(n) = \frac{\zeta(2s+1)\,\zeta^2(s+1)}{(2s+1)\ \zeta(2s+2)}\,x^{2s+1} + 0\,(C_s(x)),$$

where

$$C_s(x) = x^{2s},\ x^2\log^2 x,\ x^{\frac{3}{2}}\log^2 x \quad\text{or}\quad x^{s+1}\log x$$

according as

$$s > 1,\ \ s = 1,\ s = \frac{1}{2} \text{ or } s < 1 \text{ and } s \neq \frac{1}{2} \qquad ([8],\ \text{eq. } (3.5)).$$

COROLLARY ($s = 1$).

$$\sum_{n\leqslant x}\sigma^{*2}(n) = \frac{\zeta(3)\,\pi^2\alpha}{18}\,x^3 + 0(x^2\log^4 x) \tag{22}$$

where

$$\alpha \equiv \alpha_1 = \prod_p\left(1 + \frac{1}{p^2} - \frac{2}{p^3} - \frac{1}{p^4} - \frac{2}{p^5} + \frac{3}{p^6}\right).$$

REFERENCES

1. J. Chidambaraswamy, *Sum functions of unitary and semiunitary divisors*, J. Indian Math. Soc., **31** (1957), 117—126.
2. E. Cohen, *Arithmetical functions associated with the unitary divisors of an integer*, Math. Z., **74** (1960), 66—80.
3. L. E. Dickson, *History of the theory of numbers*, vol. I, Chelsea, New York, 1952.

4. G. Nettler, *Problem E*: 2235, Amer. Math. Monthly, **77** (1970), 522.
5. S. Ramanujan, *Some formulae in the analytic theory of numbers*, Messenger Math., **45** (1916), 81–84.
6. R. Sita Rama Chandra Rao, D. Suryanarayana, *On* $\sum_{n \leq x} \sigma^*(n)$ *and* $\sum_{n \leq x} \sigma^*(n)$, Proc. Amer. Math. Soc., **41** (1973), 61–66.
7. M. V. Subbarao, *Arithmetic functions and distributivity*, Amer. Math. Monthly, **75** (1968), 984–988.
8. L. Tóth, *Generalizations of an asymptotic formula of Ramanujan*, Studia Univ. Babeş-Bolyai, Mathematica **31** (1986), 9–15.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXIV, 2, 1989

ON NONLINEAR INTEGRAL INEQUALITIES OF TWO INDEPENDENT VARIABLES

SEVER S. DRAGOMIR* and NICOLETA M. IONESCU**

Received: March 9, 1989

REZUMAT. — Asupra unor inegalităţi integrale neliniare în două variabile. În lucrare sînt stabilite mai multe inegalităţi de tip Gronwall pentru funcţii de două variabile.

There exists an extensive literature concerning various generalizations of the Gronwall inequality in the case of two or more independent variables (see for example the recent works [1]—[5]).

In paper [2] we proved that if Φ, A, B are nonnegative continuous functions defined on $\mathbf{R}_+^2$ and $L:\mathbf{R}_+^3\to\mathbf{R}_+$ is continuous and satisfies the condition:
(L) $0 \leqslant L(x,y,t) - L(x,y,s) \leqslant M(x,y,s)(t-s)$ for all $x, y \geqslant 0$ and $t \geqslant s \geqslant 0$, where M is nonnegative continuous on $\mathbf{R}_+^3$, and Φ verifies the inequality:

$$\Phi(x,y) \leqslant A(x,y) + B(x,y)\int_0^x\int_0^y L(s,t,\Phi(s,t))\,ds dt;\ \ x,y \geqslant 0 \tag{1}$$

then

$$\Phi(x,y) \leqslant A(x,y) + B(x,y)\left(\exp\int_0^x\int_0^y \Delta(s,t)\,ds dt - 1\right) \tag{2}$$

where the mapping Δ is given by

$$\Delta(x,y) := [L^2(x,y,A(x,y)) + M^2(x,y,A(x,y))\,B^2(x,y)]^{1/2}$$

for all $x, y \geqslant 0$.

If Φ, A, B are as above and $D:\mathbf{R}_+^3\to\mathbf{R}_+$ is a continuous function satisfying the relation
(D) D is differentiable on $[0,\infty)^3$, $\partial D(x,y,t)/\partial t$ is nonnegative on $[0,\infty)^3$ and there exists a continuous function $P:\mathbf{R}_+^3\to\mathbf{R}_+$ with the property $\partial D(x,y,t)/\partial t \leqslant P(x,y,\tau)$ for all $x,y \geqslant 0$ and $t \geqslant \tau \geqslant 0$,
then for every Φ satisfying the integral inequality

$$\Phi(x,y) \leqslant A(x,y) + B(x,y)\int_0^x\int_0^y D(s,t,\Phi(s,t))\,ds\,dt,\quad x,y \geqslant 0 \tag{1'}$$

* *Secondary School, 1600 Băile Herculane, Caraş-Severin County, Romania*
** *Secondary School, 1612 Mehadia, Caraş-Severin County, Romania*

the following estimation holds

$$\Phi(x, y) \leqslant A(x, y) + B(x, y) \left(\exp \int_0^x \int_0^y U(s, t)\, ds\, dt - 1 \right), \quad x, y \geqslant 0 \tag{2}$$

where the function U is defined in what follows:

$$U(x, y) := [D^2(x, y, A(x, y)) + P^2(x, y, A(x, y))\, B^2(x, y)]^{1/2}, \quad x, y \geqslant 0.$$

For the consequences of these integral inequalities see [2] and [3].

Further, we shall point out another bound for $\Phi(x, y)$ by the use of the following result due to Adrian Corduneanu (see for example [1]):

LEMMA 1. *If the continuous function u satisfies the inequality*

$$u(x, y) \leqslant f(x, y) + \int_0^x \int_0^y b(s, t)\, u(s, t)\, ds\, dt, \quad x, y \geqslant 0 \tag{3}$$

where f is continuous and monotone nondecreasing with respect to each variable, b is continuous and nonnegative, then it follows for $x, y \geqslant 0$:

$$u(x, y) \leqslant f(x, y) \left[1 + \int_0^x \int_0^y b(s, t) \exp \left(\int_s^x \int_t^y b(\tau, \eta)\, d\tau\, d\eta \right) ds\, dt \right]. \tag{4}$$

This result was obtained using the notion of resolvent kernel of the theory of Volterra linear integral equations. For generalizations of this fact we send to [1].

THEOREM 1. *Let Φ, A, B be nonnegative continuous on $\mathbf{R}_+^2$ and $L : \mathbf{R}_+^3 \to \mathbf{R}_+$ a continuous function satisfying (L). If Φ verifies the integral inequality* (1), *then we have the bound:*

$$\Phi(x, y) \leqslant A(x, y) + B(x, y)\, \overline{U}(x, y) \int_0^x \int_0^y L(s, t, A(s, t))\, ds\, dt, \quad x, y \geqslant 0 \tag{5}$$

where $\overline{U}$ is given by

$$\overline{U}(x, y) := 1 + \int_0^x \int_0^y M(s, t, A(s, t))\, B(s, t) \exp \left(\int_s^x \int_t^y M(\tau, \eta, A(\tau, \eta))\, B(\tau, \eta)\, d\tau\, d\eta \right) ds\, dt \tag{6}$$

Proof. Let $\Psi(x, y) := \int_0^x \int_0^y L(s, t, \Phi(s, t))\, ds\, dt$ for $x, y \geqslant 0$. Then

$$\begin{aligned} \Psi''_{xy}(x, y) &= L(x, y, \Phi(x, y)) \leqslant L(x, y, A(x, y) + B(x, y)\, \Psi(x, y)) \leqslant \\ &\leqslant L(x, y, A(x, y)) + M(x, y, A(x, y))\, B(x, y) \Psi(x, y) \end{aligned} \tag{7}$$

for all $x, y \geqslant 0$.

Integrating (7) on the rectangle $0 \leqslant s \leqslant x,\ 0 \leqslant t \leqslant y$, we get

$$\Psi(x, y) \leqslant \int_0^x\int_0^y L(s, t, A(s, t))\, ds dt + \int_0^x\int_0^y M(s, t, A(s, t))\, B(s, t)\, \Psi(s, t)\, ds\, dt$$

for all $x, y \geqslant 0$.

Applying Lemma 1, for $u(x, y) := \Psi(x, y)$, $f(x, y) := \int_0^x\int_0^y L(s, t, A(s, t)) ds dt$, $b(x, y) := M(x, y, A(x, y))\, B(x, y)$, $x, y \geqslant 0$ we deduce that

$$\Psi(x, y) \leqslant \bar{U}(x, y) \int_0^x\int_0^y L(s, t, A(s, t)\, ds dt, \quad x, y \geqslant 0,$$

where $\bar{U}(x, y)$ is given in (6), what implies the bound (5).

The theorem is proven.

Now, we shall give some particular cases which are important in applications.

COROLLARY 1.1. *Let Φ, A, B be as above and $G : \mathbf{R}_+^3 \to \mathbf{R}_+$ a continuous mapping satisfying the assumption*

(G) $\quad 0 \leqslant G(x, y, t) - G(x, y, s) \leqslant N(x, y)\, (t - s)$ *for all* $x, y \geqslant 0$ *and* $t \geqslant s \geqslant 0$

where N is nonnegative continuous on $\mathbf{R}_+^2$.

If Φ verifies the integral inequality

$$\Phi(x, y) \leqslant A(x, y) + B(x, y) \int_0^x\int_0^y G(s, t, \Phi(s, t))\, ds\, dt, \quad x, y \geqslant 0 \tag{8}$$

then we have the bound

$$\Phi(x, y) \leqslant A(x, y) + B(x, y) V(x, y) \int_0^x\int_0^y G(s, t, A(s, t))\, ds\, dt, \quad x, y \geqslant 0 \tag{9}$$

where V is given by

$$V(x, y) := 1 + \int_0^x\int_0^y N(s, t)\, B(s, t) \exp\left(\int_s^x\int_t^y N(\tau, \eta)\, B(\tau, \eta)\, d\tau d\eta\right) dsdt$$

for all $x, y \geqslant 0$.

The second result is embodied in:

COROLLARY 1.2. *Let Φ, A, B, C be nonnegative continuous on $\mathbf{R}_+^2$ and $H : \mathbf{R}_+ \to \mathbf{R}_+$ a function satisfying the following Lipschitz's type condition:*

(H) $\quad 0 \leqslant H(t) - H(s) \leqslant M(t - s)$ *for all $t \geqslant s \geqslant 0$, where $M > 0$.*

If Φ *verifies the inequality*

$$\Phi(x, y) \leqslant A(x, y) + B(x, y) \int_0^x \int_0^y C(s, t)\, H(\Phi(s, t))\, ds dt, \quad x, y \geqslant 0 \tag{10}$$

then it follows

$$\Phi(x, y) \leqslant A(x, y) + B(x, y)\, W(x, y) \int_0^x \int_0^y C(s, t)\, H(A(s, t))\, ds\, dt, \quad x, y \geqslant 0 \tag{11}$$

where

$$W(x, y) := 1 + M \int_0^x \int_0^y C(s, t)\, B(s, t) \exp\left(M \int_s^x \int_t^y C(\tau, \eta) B(\tau, \eta)\, d\tau d\eta\right) ds\, dt$$

for all $x, y \geqslant 0$.

Remark 1. By the inequality (16) of [1] we deduce that

$$\bar{U}(x, y) \leqslant \exp\left(\int_0^x \int_0^y M(\tau, \eta, A(\tau, \eta))\, B(\tau, \eta)\, d\tau\, d\eta\right) \quad \text{for } x, y \geqslant 0,$$

and then the inequality (6) implies

$$\Phi(x, y) \leqslant A(x, y) + B(x, y) \int_0^x \int_0^y L(s, t, A(s, t))\, ds\, dt \times$$

$$\times \exp\left(\int_0^x \int_0^y M(s, t, A(s, t))\, B(s, t)\, ds dt\right) \tag{6'}$$

for all $x, y \geqslant 0$.

The same observations are valid refering to the inequalities (9) and (11). We omit the details.

Now, we state and prove the second main result of our paper.

THEOREM 2. *Let* Φ, A, B *be as above and* $D : \mathbf{R}_+^3 \to \mathbf{R}_+$ *a continuous function satisfying the condition*

(D) *the partial derivative* $\partial D(x, y, t)/\partial t$ *exists and is nonnegative on* $\mathbf{R}_+^3$ *and there exists a mapping* $P : \mathbf{R}_+^3 \to \mathbf{R}_+$ *with the property*

$$\partial D(x, y, p)/\partial t \leqslant P(x, y, s) \quad \text{for } x, y \geqslant 0 \text{ and } p \geqslant s \geqslant 0. \tag{12}$$

If Φ *verifies the integral inequality*

$$\Phi(x, y) \leqslant A(x, y) + B(x, y) \int_0^x \int_0^y D(s, t, \Phi(s, t))\, ds dt, \quad x, y \geqslant 0 \tag{13}$$

then the following estimation is valid

$$\Phi(x, y) \leqslant A(x, y) + B(x, y)\, \overline{\overline{U}}(x, y) \int_0^x \int_0^y D(s, t, A(s, t))\, ds\, dt \tag{14}$$

where $\overline{\overline{U}}(x, y)$ *is given by*

$$\overline{\overline{U}}(x,y) := 1 + \int_0^x \int_0^y P(s, t, A(s, t))\, B(s, t) \exp\left(\int_s^x \int_t^y P(\tau, \eta, A(\tau, \eta)) B(\tau, \eta)\, d\tau\, d\eta\right) ds\, dt$$

for all $x, y \geqslant 0$.

Proof. By Lagrange's theorem, for every $p \geqslant s \geqslant 0$ and $x, y \geqslant 0$, there exists a $\mu \in (s, p)$ such that

$$D(x, y, p) - D(x, y, s) = \partial D(x, y, \mu)/\partial t(p - s).$$

Since

$$0 \leqslant \partial D(x, y, \mu)/\partial t \leqslant P(x, y, s)$$

we obtain

$$0 \leqslant D(x, y, p) - D(x, y, s) \leqslant P(x, y, s)(p - s)$$

for all $x, y \geqslant 0$ and $p \geqslant s \geqslant 0$.

Applying Theorem 1 for $L(x, y, p) := D(x, y, p)$ and $M(x, y, s) = P(x, y, s)$, we obtain the bound (14)

The theorem is proven.

The following corollaries are important in applications.

COROLLARY 2.1. *Let* Φ, A, B *be as above and* $I : \mathbb{R}_+^3 \to \mathbb{R}_+$ *a continuous function satisfying the condition:*
(I) *the partial derivative* $\partial I(x, y, t)/\partial t$ *exists on* $\mathbb{R}_+^3$, *is nonnegative continuous on* $\mathbb{R}_+^3$ *and*

$$\partial I(x, y, p)/\partial t \leqslant \partial I(x, y, s)/\partial t \text{ for all } x, y \geqslant 0 \text{ and } p \geqslant s \geqslant 0. \tag{15}$$

If Φ *is a solution of the following integral inequality*

$$\Phi(x, y) \leqslant A(x, y) + B(x, y) \int_0^x \int_0^y I(s, p, \Phi(s, p))\, ds\, dp, \quad x, y \geqslant 0, \tag{16}$$

then we have the bound:

$$\Phi(x, y) \leqslant A(x, y) + B(x, y)\, \overline{V}(x, y) \int_0^x \int_0^y I(s, p, A(s, p))\, ds\, dp \tag{17}$$

where $\overline{V}$ *is given by*

$$\overline{V}(x,y) := 1 + \int_0^x\int_0^y \partial I(s,p,A(s,p))/\partial t\, B(s,p) \times$$

$$\times \exp\left(\int_s^x\int_t^y \partial I(\tau,\mu,A(\tau,\mu))/\partial t\, B(\tau,\mu)\,d\tau\, d\mu\right) ds\, dt$$

for $x, y \geqslant 0$.

COROLLARY 2.2. *Let* Φ, A, B, C *be nonnegative continuous and* $K : \mathbf{R}_+ \to \mathbf{R}_+$ *with the property*

(K) K *is monotone nondecreasing and derivable on* $\mathbf{R}_+$ *with the derivative* dK/dt *monotone nonincreasing on* $\mathbf{R}_+$.

If Φ *satisfies the inequality:*

$$\Phi(x,y) \leqslant A(x,y) + B(x,y)\int_0^x\int_0^y C(s,t)\,K(\Phi(s,t))\,dsdt,\quad x,y \geqslant 0 \tag{18}$$

then it follows

$$\Phi(x,y) \leqslant A(x,y) + B(x,y)\,\overline{W}(x,y)\int_0^x\int_0^y C(s,t)\,K(A(s,t))\,ds\,dt \tag{19}$$

where $\overline{W}$ *is given by*

$$\overline{W}(x,y) := 1 + \int_0^x\int_0^y dK/dt A(s,p))\,C(s,p)\,B(s,p) \times$$

$$\times \exp\left(\int_s^x\int_p^y dK/dt\,(A(\tau,\eta))\,C(\tau,\eta)\,B(\tau,\eta)\,d\tau\,d\eta\right) ds\,dp,\qquad x,y \geqslant 0.$$

Remark 2. By the inequality (16) of [1] we deduce that

$$\overline{\overline{U}}(x,y) \leqslant \exp\left(\int_0^x\int_0^y P(\tau,\eta,A(\tau,\eta))\,B(\tau,\eta)\,d\tau\,d\eta\right) \text{ for } x,y \geqslant 0$$

and then the inequality (14) implies the bound

$$\Phi(x,y \leqslant A(x,y) + B(x,y)\int_0^x\int_0^y D(s,t,A(s,t))\,dsdt \times$$

$$\times \exp\left(\int_0^x\int_0^y P(s,t,A(s,t))\,B(s,t)\,ds\,dt\right),\quad x,y \geqslant 0.$$

The same observations are valid refering to the inequalities (17) and (19). We omit the details.

Finally, we shall point out some natural consequences of the above corollaries.

CONSEQUENCES 1. *Let* Φ, A, B, C be nonnegative continuous on $\mathbf{R}^2_+$, $A(x,y) > 0$ for all $x, y \geq 0$ and $r \in [0,1]$. If Φ verifies the inequality:

$$\Phi(x,y) \leq A(x,y) + B(x,y)\int_0^x\int_0^y C(s,t)\Phi^r(x,y)\,ds\,dt, \quad x,y \geq 0 \tag{20}$$

then we have the estimation

$$\Phi(x,y) \leq A(x,y) + B(x,y)\,R(x,y)\int_0^x\int_0^y C(s,t)\,A^r(s,t)\,ds\,dt, \quad x,y \geq 0 \tag{21}$$

where R is given by

$$R(x,y) := 1 + r\int_0^x\int_0^y \frac{C(s,t)\,B(s,t)}{A(s,t)^{1-r}}\exp\left(r\int_s^x\int_t^y \frac{C(\tau,\eta)\,B(\tau,\eta)}{A(\tau,\eta)^{1-r}}\,d\tau\,d\eta\right)ds\,dt$$

for all $x, y \geq 0$.

2. Let Φ, A, B, C be as above. If Φ verifies the inequality

$$\Phi(x,y) \leq A(x,y) + B(x,y)\int_0^x\int_0^y C(s,t)\ln(\Phi(s,t)+1)\,ds\,dt \tag{22}$$

then we have the bound

$$\Phi(x,y) \leq A(x,y) + B(x,y)\,Q(x,y)\int_0^x\int_0^y C(s,t)\ln(A(s,t)+1)\,ds\,dt \tag{23}$$

where Q is given by

$$Q(x,y) := 1 + \int_0^x\int_0^y \frac{C(s,t)\,B(s,t)}{A(s,t)+1}\exp\left(\int_s^x\int_t^y \frac{C(\tau,\eta)\cdot B(\tau,\eta)}{A(\tau,\eta)+1}\,d\tau\,d\eta\right)ds\,dt$$

for all $x, y \geq 0$.

REFERENCES

1. A. Corduneanu, *An integral inequality in two independent variables*, Anal. Şt. Univ. „Al. I. Cuza", Iaşi, **29** (3) (1983), 69–74.
2. S. S. Dragomir, N. M. Ionescu, *The Wendroff type lemmas and applications* (I), Itin. Sem. on Func. Eq., Approx. and Convex., Cluj-Napoca (1987), 135–142.
3. S. S. Dragomir, N. M. Ionescu, *The Wendroff type lemmas and applications* (III); *Ibidem* (1988), 169–174.
4. A. M. Fink, *Wendroff's inequalities*, Nonlinear Analysis, Theory, Methods and Applications, **8** (1981), 873–874.
5. V. M. Singare, *Some integral inequalities in two independent variables*, Mathematics Seminar Notes, **10** (1982), 775–783.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXIV, 2, 1989

ALPHA-CONVEX INTEGRAL OPERATOR AND STRONGLY-STARLIKE FUNCTIONS

PETRU T. MOCANU*

Received: February 12, 1989

REZUMAT. — Operatorul integral alfa-convex şi funcţii tare stelate. Rezultatul principial al lucrării este conţinut în următoarea teoremă.

TEOREMA 1. Fie $\alpha > 0$, $0 < \beta \leqslant 2$ şi fie g o funcţie olomorfă în discul unitate U, $g(0) = g'(0) - 1 = 0$, care satisface condiţia

$$\frac{zg'(z)}{g(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta} + \frac{2\alpha\beta z}{1-z^2}$$

Dacă $f = I_\alpha(g)$ este definită de (1) atunci

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \beta\frac{\pi}{2}, \qquad z \in U.$$

1. Introduction. Let A denote the set of functions $f(z) = z + a_2z^2 + \ldots$ that are analytic in the unit disc $U = \{z ; |z| < 1\}$. A function $f \in A$ is called strongly-starlike of order β, $0 < \beta \leqslant 1$, if $|\arg [zf'(z)/f(z)]| < \beta\pi/2$, for $z \in U$. Let denote by $S^*[\beta]$ the class of all such functions. Since our main result holds for all $\beta \in (0, 2]$, we continue to use the notation $S^*[\beta]$ for $\beta \in (1, 2]$, although a function in this class is not necesarely starlike, not even univalent.

For $g \in A$ and $\alpha > 0$, let $f = I_\alpha(g)$ be defined by

$$f(z) = \left(\frac{1}{\alpha}\int_0^z g^{1/\alpha}(w)\, w^{-1}\, dw\right)^{\alpha}, \quad z \in U. \tag{1}$$

This integral operator was introduced in [4] and in [3] and [4] it was proved that $I_\alpha(S^*[1]) \subset S^*[1]$, where $S^*[1] = S^*$ is the usual class of starlike functions. The class $I_\alpha(S^*)$ is the class of α-convex functions and $I_1(S^*)$ is the usual class of convex functions.

From a more general result obtained in [1, Theorem 1] it is easy to show that $I_\alpha(S^*[\beta]) \subset S^*[\beta]$, for $\alpha > 0$ and $0 < \beta \leqslant 1$.

If we denote

$$J(\alpha, f; z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right), \tag{2}$$

* University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania

then from (1) we deduce

$$J(\alpha, f; z) = \frac{zg'(z)}{g(z)} \tag{3}$$

and the above mentioned result can be restated as

$$f \in A \text{ and } |\arg J(\alpha, f; z)| < \beta \frac{\pi}{2} \Rightarrow \left|\arg \frac{zf'(z)}{f(z)}\right| < \beta \frac{\pi}{2}$$

or, in terms of subordination, as

$$f \in A \text{ and } J(\alpha, f; z) \prec \left(\frac{1+z}{1-z}\right)^{\beta} \Rightarrow \frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta} \tag{4}$$

In the particular case $\alpha = 1$, in [6] we improved the implication (4) by the following result, which holds for all $\beta \in (0, 2]$.

THEOREM A. *If* $0 < \beta \leqslant 2$ *and* $f \in A$ *satisfies*

$$1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta} + \frac{2\beta z}{1-z^2},$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta}.$$

An equivalent form of Theorem A is given by

THEOREM B. *Let* $0 < \beta \leqslant 2$ *and let* $g \in A$ *satisfy*

$$\frac{zg'(z)}{g(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta} + \frac{2\beta z}{1-z^2}.$$

If $f = I_1(g)$ *is defined by*

$$f(z) = \int_0^z \frac{g(w)}{w}\, dw,$$

then $f \in S^*[\beta]$.

For $\beta = 1$, this result reduces to the „open door" theorem proved in [5, Corollary 2.1]

In this paper we extend Theorem B to the α-convex integral operator I_α defined by (1). Actually we obtain the following result.

THEOREM 1. *Let* $\alpha > 0$, $0 < \beta \leqslant 2$ *and let* $g \in A$ *satisfy*

$$\frac{zg'(z)}{g(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta} + \frac{2\alpha\beta z}{1-z^2}.$$

If $f = I_\alpha(g)$ *is defined by* (1), *then* $f \in S^*[\beta]$.

Some consequences of this main result are obtained.

2. **Preliminaries.** In some of our results we use the following restricted definition of subordination. If F and G are analytic functions in U, then F is subordinate to G, written $F \prec G$, or $F(z) \prec G(z)$, if G is univalent, $F(0) = G(0)$ and $F(U) \subset G(U)$.

The proof of Theorem 1 is based on the following lemma.

LEMMA. *Let $\alpha > 0$, $0 < \beta \leqslant 2$ and let P be an analytic function such that*

$$P(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta} + \frac{2\alpha\beta z}{1-z^2} \equiv h(z) \tag{6}$$

If p is analytic in U, $p(0) = 1$ and satisfies the differential equation

$$\alpha z p'(z) + P(z)p(z) = 1 \tag{7}$$

then

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}.$$

The proof of this lemma is similar to the proof of Lemma 2 in [6] and will be omitted.

The domain $h(U)$ is symmetric with respect to the real axis. Therefore, if $z = e^{i\theta}$, then in order to obtain the boundary of $h(U)$ it is sufficient to suppose $0 \leqslant \theta \leqslant \pi$.

Letting $\operatorname{ctg}(\theta/2) = t$ and $h(e^{i\theta}) = u + iv$, we find

$$\begin{cases} u = at^{\beta} \\ v = bt^{\beta} + \dfrac{\alpha\beta(1+t^2)}{2t}, \qquad t \geqslant 0, \end{cases} \tag{8}$$

where $a = \cos(\beta\pi/2)$ and $b = \sin(\beta\pi/2)$.

If $\beta = 1$ then $u = 0$ and $v = t + \alpha(1+t^2)/2t \geqslant \sqrt{\alpha(\alpha+2)}$ and we deduce that $h(U)$ is the complex plane slit along the half-lines $u = 0$ and $|v| \geqslant \sqrt{\alpha(\alpha+2)}$.

We note that $u > 0$, if $0 < \beta < 1$ and $u < 0$, if $1 < \beta \leqslant 2$. In the last two cases it is possible to eliminate the parameter t in (8) and to obtain v as the following function of u.

$$v = v(u) \equiv \frac{b}{a}u + \frac{\alpha\beta}{2}\left[\left(\frac{u}{a}\right)^{1/\beta} + \left(\frac{u}{a}\right)^{-1/\beta}\right], \quad \begin{cases} u > 0, \text{ if } 0 < \beta < 1 \\ u < 0, \text{ if } 1 < \beta \leqslant 2 \end{cases} \tag{9}$$

We also have

$$\lim_{u\to 0} v(u) = \lim_{u\to\pm\infty} v(u) = +\infty.$$

In all the above cases we deduce that h is univalent in U.

3. Proof of Theorem 1. Let $g \in A$ satisfy (5), i.e.

$$\frac{zg'(z)}{g(z)} \prec h(z) \equiv \left(\frac{1+z}{1-z}\right)^{\beta} + \frac{2\alpha\beta z}{1-z^2}.$$

Since (5) implies $g(z)/z \neq 0$, we can define

$$p(z) = \frac{1}{\alpha g^{1/\alpha}(z)} \int_0^z g^{1/\alpha}(w)\, w^{-1}\, dw = \frac{1}{\alpha}\int_0^1 \left[\frac{g(tz)}{g(z)}\right]^{1/\alpha} t^{-1}\, dt, \tag{10}$$

where all powers are principal ones. The function p is analytic in U and $p(0)=1$. From (10) we easily deduce that p satisfies the diferential equation (7) with

$$P(z) = \frac{zg'(z)}{g(z)}. \tag{11}$$

Since P satisfies (6), by Lemma we deduce

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}. \tag{12}$$

The subordination (12) implies $p(z) \neq 0$ in U, hence we can define the analytic function $[p(z)]^{\alpha} = 1 + \ldots,\ z \in U$. Therefore the function f defined by

$$f(z) = g(z)[p(z)]^{\alpha} \tag{13}$$

is analytic in U and from (10) we deduce that f is given by (1), i.e. $f = I_{\alpha}(g)$. On the other hand, from (13), by using (11) and (7), we obtain

$$\frac{zf'(z)}{f(z)} = P(z) + \alpha\frac{zp'(z)}{p(z)} = \frac{1}{p(z)}.$$

Hence from (12) we obtain

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1-z}{1+z}\right)^{\beta},$$

which shows that $f \in S^*[\beta]$.

4. **Equivalent forms of Theorem 1.** By using (2) and (3), Theorem 1 can be restated in the following equivalent form.

THEOREM 2. *Let* $\alpha > 0,\ 0 < \beta \leqslant 2$ *and let* $f \in A$ *satisfy*

$$J(\alpha, f; z) \prec h(z) \equiv \left(\frac{1+z}{1-z}\right)^{\beta} + \frac{2\alpha\beta z}{1-z^2}. \tag{14}$$

Then

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \beta\frac{\pi}{2},\ \text{i.e. } f \in S^*[\beta].$$

If we let

$$k(z) = z\exp\int_0^z\left[\left(\frac{1+w}{1-w}\right)^{\beta} - 1\right]\frac{dw}{w}, \tag{15}$$

then

$$J(\alpha, k; z) = h(z)$$

and we obtain the following symmetric equivalent form of Theorem 1.

THEOREM 3. *Let* $\alpha > 0,\ 0 < \beta \leqslant 2$ *and let* $f \in A$ *satisfy*

$$J(\alpha, f; z) \prec J(\alpha, k; z),$$

where k *is given by* (15). *Then*

$$J(0, f; z) \prec J(0, k; z).$$

Remarks. (i) For $\alpha > 0$ and $\beta = 1$, Theorem 3 reduces to Corollary 2.2 in [5].

(ii) For $\alpha = 1$ and $0 < \beta \leqslant 1$, Theorem 3 reduces to Example (c) in [2].

(iii) For $\alpha = 1$ and $0 < \beta \leqslant 2$, Theorem 3 was proved in [6, Theorem 1].

5. **Corollaries.** By choosing certain subdomains of $h(U)$, from the main result we can deduce some interesting consequences.

COROLLARY 1. *Let $\alpha > 0$, $0 < \beta < 1$ and let γ be defined by*

$$\operatorname{tg}\frac{\gamma\pi}{2} = \operatorname{tg}\frac{\beta\pi}{2} + \frac{\alpha\beta}{(1-\beta)\cos\frac{\beta\pi}{2}}\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1+\beta}{2}} \tag{16}$$

If $f \in A$ satisfies

$$|\arg J(\alpha, f; z)| < \gamma\frac{\pi}{2}, \quad z \in U, \tag{17}$$

then

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \beta\frac{\pi}{2}, \quad z \in U.$$

Proof. From (9) we deduce

$$E(u) = \frac{v(u)}{u} = \frac{b}{a} + \frac{\alpha\beta}{2}\left[a^{-1/\beta}u^{1/\beta-1} + a^{1/\beta}u^{-1/\beta-1}\right]$$

with $a = \cos(\beta\pi/2)$ and $b = \sin(\beta\pi/2)$.

We have

$$E'(u) = \frac{\alpha}{2}\left[(1-\beta)a^{-1/\beta}u^{1/\beta-2} - (1+\beta)a^{1/\beta}u^{-1/\beta-2}\right].$$

The equation $E'(u) = 0$ has the unique root

$$u_0 = \left(\frac{1+\beta}{1-\beta}\right)^{\beta/2}\cos\frac{\beta\pi}{2}$$

and

$$E(u_0) = \frac{b}{a} + \frac{\alpha\beta}{2a}\left[\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1-\beta}{2}} + \left(\frac{1-\beta}{1+\beta}\right)^{\frac{1+\beta}{2}}\right] = \operatorname{tg}\frac{\beta\pi}{2} + \frac{\alpha\beta}{(1-\beta)\cos\frac{\beta\pi}{2}}\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1+\beta}{2}}.$$

We deduce that the sector $\{w;\ |\arg w| < \gamma\pi/2\}$, where γ is given by (16) is the largest sector in the right half-plane which lies in $h(U)$. Hence (17) implies (14) and the conclusion of Corollary 1 follows from Theorem 2.

If we let $p(z) = zf'(z)/f(z)$, then Corollary 1 can be restated as

COROLLARY 2. *Let $\alpha > 0$, $0 < \beta < 1$ and let γ be defined by* (16). *If p is analytic in U, with $p(0) = 1$, then*

$$p(z) + \alpha\frac{zp'(z)}{p(z)} \prec \left(\frac{1+z}{1-z}\right)^{\gamma} \Rightarrow p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}.$$

Example 1. If we take $\alpha = 1$ and $\beta = 1/2$ in (16), then we deduce tg $(\gamma\pi/2) = 1 + (4/27)^{1/4} = 1.6204 \ldots$, hence $\gamma\pi/2 = 1.0178 \ldots$ (58° 32 ...) and $\gamma = 0.6479\ldots$ Therefore by Corollary 1 we obtain

$$f \in A \text{ and } \left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < 1.0178 \Rightarrow \left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4}$$

If we take $\beta = 1/2$ in Theorem 2, then we obtain

COROLLARY 3. *If $\alpha > 0$ and if $f \in A$ satisfies*

$$|\operatorname{Im} J(\alpha, f; z)| < \delta, \ z \in U, \tag{18}$$

where $\delta = \delta(\alpha)$ is the smallest positive root of the equation

$$64\alpha x^4 + 32x^3 - 32\alpha^3x^2 - 72\alpha^2x - \alpha(27 - 4\alpha^4) = 0. \tag{19}$$

Then

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4}, \quad z \in U.$$

Proof. From (9) we deduce

$$v = v(u) = u + \frac{\alpha}{4}\left(2u^2 + \frac{1}{2u^2}\right), \quad u > 0.$$

It is clear that $\delta = \min\{v(u); u > 0\}$ is the bigest positive number x such that

$$v(u) \geqslant x, \text{ for all } u > 0. \tag{20}$$

If we let

$$H(u) = 4\alpha u^4 + 8u^3 - 8xu^2 + \alpha,$$

then (20) is equivalent to $H(u) \geqslant 0$, for all $u > 0$. The equation $H'(u) = 0$ has the unique positive root

$$u_0 = \frac{1}{4\alpha}[\sqrt{9 + 16\alpha x} - 3]$$

and

$$H(u_0) = \min\{H(u); u > 0\} =$$

$$= \frac{1}{8\alpha^3}[(9 + 16\alpha x)^{3/2} - 32\alpha^2x^2 - 72\alpha x - 27 + 8\alpha^4]$$

The equation $H(u_0) = 0$ yields (19) and we deduce that the strip $\{w; |\operatorname{Im} w| < \delta\}$ is the largest strip parallel to the real axis which lies in $h(U)$. Hence (18) implies (14), with $\beta = 1/2$, and the conclusion of Corollary 3 follows from Theorem 2.

Example 1. If we let $\alpha = 1$ in Corollary 2, then we obtain

$$f \in A \text{ and } \left|\operatorname{Im} \frac{zf''(z)}{f'(z)}\right| < \delta = 1.114 \ldots \Rightarrow \left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4},$$

where $\delta = 1.114 \ldots$ is the positive root of the equation

$$8x^4 + 4x^3 - 4x^2 - 9x = 23/8.$$

This result is a slight improvement of Corollary 1.1 in [6].

If we let $\beta = 2$ in (9), we easily deduce $v \geqslant 2\alpha$ and by Theorem 2 we obtain

COROLLARY 4. *If* $\alpha > 0$ *and if* $f \in A$ *satisfies*

$$|\operatorname{Im} J(\alpha, f; z)| < 2\alpha,$$

then

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \pi.$$

For $\alpha = 1$ this last result reduces to Corollary 1.2 in [6].

REFERENCES

1. P. Eenigenburg, S. S. Miller, P. T. Mocanu, M. O. Reade, *On a Briot-Bouquet differential subordination*, ISNM 64, General Inequalities 3 (1983), 339–348.
2. S. S. Miller, P. T. Mocanu, *Marx-Strohhäcker differential subordination systems*, Proc. Amer. Math. Soc. 99, 3 (1987), 527–534.
3. S. S. Miller, P. T. Mocanu, M. O. Reade, *Bazilevič functuons and generalized convexity*, Rev. Roumaine Math Pures Appl., 19 (1974), 213–224.
4. P. T. Mocanu, *Une propriété de convexité généralisée dans la théorie de la représentation conforme*, Mathematica (Cluj), 11 (34), 1 (1969), 127–133.
5. P. T. Mocanu, *Some integral operators and starlike functions*, Rev. Roumaine Math.. Pures Appl., 31, 3 (1986), 231–235.
6. P. T. Mocanu, *On strongly-starlike and strongly-convex functions*, Studia Univ. Babeş-Bolyai, Math., 31, 4 (1986), 16–21.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXIV, 2, 1989

MEASURE OF NONCOMPACTNESS AND SECOND ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

RADU PRECUP*

Received: January 28, 1989

REZUMAT. — Măsura de necompactitate şi ecuaţiile diferenţiale de ordinul al doilea cu argument modificat. În această lucrare este studiată existenţa şi unicitatea soluţiei problemei lui Dirichlet (1.1), (1.2) într-un spaţiu Banach. Această problemă este privită ca un caz particular al problemei lui Dirichlet (3.2) pentru ecuaţia funcţional-diferenţială (3.1). Principalul rezultat referitor la existenţa soluţiei problemei (3.1), (3.2) este conţinut în Teorema 1, în care aplicaţiei din membrul drept al ecuaţiei (3.1) i se cere să satisfacă o condiţie mai slabă decît aceea de a fi compactă. Această condiţie se exprimă cu ajutorul măsurii de necompactitate a lui Kuratowski.

1. **Introduction.** This paper deals with the boundary value problem

$$u''(t) = f(t, u(t), u'(t), u(g_1(t)), \ldots, u(g_m(t))), \quad t \in I, \tag{1.1}$$

$$u(t) = \varphi(t), \; t \in I' \setminus \text{int } I, \tag{1.2}$$

in a real Banach space X, where $I = [a, b]$, $I' = [a', b']$, $a' \leqslant a < b \leqslant b'$, f is a continuous mapping from $I \times X^{m+2}$ into X, $g_i (i = 1, \ldots, m)$ are continuous functions from I into I' and φ is a continuous function from $I' \setminus$ int I into X.

By a solution to problem (1.1), (1.2) we mean a function $u \in C^2(I; X) \cap C(I'; X)$ satisfying conditions (1.1) and (1.2).

For $u \in C(I; X)$ let us denote by u^i $(i = 1, \ldots, m)$ the function from I into X, $u^i(t) = u(g_i(t))$ if $g_i(t) \in I$ and $u^i(t) = \varphi(g_i(t))$ otherwise.

Let us consider the mapping $h: I \times X \times X \times C(I; X) \to X$,

$$h(t, x, y; u) = f(t, x, y, u^1(t), \ldots, u^m(t)), \tag{1.3}$$

for $t \in I$, $x, y \in X$ and $u \in C(I; X)$.

A function u is a solution to (1.1), (1.2) if and only if $u \in C^2(I; X)$ and satisfies

$$u''(t) = h(t, u(t), u'(t); u), \quad t \in I, \tag{1.4}$$

$$u(a) = \varphi(a), \; u(b) = \varphi(b). \tag{1.5}$$

In the particular case when $g_i(I) \subset I$, $i = 1, \ldots, m$, by the continuity of f it follows that h is also continuous. In this case, the existence of solutions to problem (1.4), (1.5) was established by K. Schmitt and R. Thompson [17] assuming in addition the compactness of h and also by us [14] under more general additional conditions on h.

* University of Cluj-Napoca, Faculty of Mathematics and Physics, 3100 Cluj-Napoca, Romania

Generally, for $g_i(I) \subset I'$, $i = 1, \ldots, m$, the mapping (1.3) may be not continuous on $I \times X \times X \times C(I; X)$. Nevertheless, its restriction to the subset $I \times X \times X \times C_b$, where

$$C_b = \{u \in C(I; X) : u(a) = \varphi(a),\ u(b) = \varphi(b)\},$$

is continuous. The main result on the existence of solutions to (1.4), (1.5), Theorem 1, requires that h be $\alpha-$ Lipschitz (α being the Kuratowski measure of noncompactness). The proof of Theorem 1 uses the topological transversality theorem (Lenay-Schauder's alternative) for condensing mappings, which has been proved in [14] without using the topological degree. In addition, we make use of a priori bounds technique. Similar methods have been used by K. Schmitt and R. Thompson [17], R. Thompson [18], A. Granas, R. Guenther and J. Lee [8]. Theorem 1 may be compared with the results obtained by V. Lakshmikantham [11] and J. Chandra, V. Lakshmikantham, A. Mitchell [5].

In particular, sufficient conditions for that the problem (1.1), (1.2) have solutions are given. These conditions are relaxed in case $X = \mathbf{R}^n$.

The existence theorems are stated in Section 3 and the main result, Theorem 1, is proved in Section 4. In Section 5 a uniqueness theorem is given.

2. **Preliminaires.** Let X be a real Banach space, X^* its dual. We shall denote both the norm in X and its dual norm in X^* by $|\cdot|$. The value of $x^* \in X^*$ at $x \in X$ will be denoted by (x^*, x). In case $X = \mathbf{R}^n$ the bilinear functional $(.,.)$ stands for the scalar product.

Denote $\|u\| = \max(|u(t)| : t \in I)$ for $u \in C = C(I; X)$, $\|u\|_1 = \max(\|u\|, \|u'\|)$ for $u \in C^1 = C^1(I; X)$ and $\|u\|_2 = \max(\|u\|, \|u'\|, \|u''\|)$ for $u \in C^2 = C^2(I; X)$.

Let J be the duality mapping of X, i.e. $J : X \to 2^{X^*}$,

$$Jx = \{x^* \in X^* : (x^*, x) = |x|^2 = |x^*|^2\},\ x \in X.$$

Recall that

$$(x^*, y - x) \leqslant \frac{1}{2}|y|^2 - \frac{1}{2}|x|^2, \tag{2.1}$$

for all $x, y \in X$ and $x^* \in Jx$.

Let us denote by α the Kuratowski measure of noncompactness; for each bounded subset A of a Banach space one has

$$\alpha(A) = \inf\{\delta > 0 : A \text{ can be convered by finitely many sets of diameter } \leqslant \delta\}.$$

With a view to avoid any confusion we will denote by α_n the Kuratowski measure of noncompactness on the Banach space $C^n(I; X)$ endowed with the norm $\|u\|_n = \max(\|u\|, \ldots, \|u^{(n)}\|)$. Only for the space $C(I; X)$ the Kuratowski measure of noncompactness will be simply denoted by α instead of α_0.

Clearly, if A is a bounded subset of $C(I; X)$ then $\alpha(A(t)) \leqslant \alpha(A)$ for all $t \in I$, where $A(t) = \{u(t) : u \in A\} \subset X$. Moreover, we have

LEMMA 1. *If A is a bounded equicontinuous subset of $C(I; X)$ then*

$$\alpha(A) = \alpha(A(I)) = \sup(\alpha(A(t)) : t \in I), \tag{2.2}$$

where $A(I) = \{u(t) : u \in A, t \in I\} \subset X$.

This result has been proved by A. Ambrosetti [2] and generalizes the classical Ascoli—Arzelà Theorem.

If A is a bounded subset of $C^n(I; X)$ $(n \geq 1)$ then

$$\alpha_n(A) = \max (\alpha(A), \alpha(A'), \ldots, \alpha(A^{(n)})), \tag{2.3}$$

where

$$A^{(i)} = \{u^{(i)} : u \in A\} \subset C(I; X), \ i = 0, 1, \ldots, n.$$

Also

$$\alpha_{i-1}(A) \leq \alpha_i(A), \tag{2.4}$$

for $i = 1, \ldots, n$.

Let Y be a closed convex subset of X and Z an arbitrary subset of X. A continuous mapping $F : Z \to Y$ is said to be (α, ρ) — *Lipschitz*, $\rho \geq 0$, if for every bounded subset A of Z, $F(A)$ is bounded and

$$\alpha(F(A)) \leq \rho\alpha(A).$$

F is called α — *Lipschitz* if there exists $\rho \geq 0$ such that F be (α, ρ) — Lipschitz. F is said to be *condensing* if for every bounded subset A of Z, $F(A)$ is bounded and if $\alpha(A) > 0$ then

$$\alpha(F(A)) < \alpha(A).$$

Let $U \subset Y$ be bounded and open in Y and let $\mathcal{A}(\bar{U}; Y)$ be the set of all condensing mappings $F : \bar{U} \to Y$ which are fixed point free on the boundary ∂U of U. A mapping $F \in \mathcal{A}(\bar{U}; Y)$ is said to be *essential* if each mapping of $\mathcal{A}(\bar{U}; Y)$ which coincides with F on ∂U has at least one fixed point in U.

In this connection, the following lemma will be used latter (for the proof see Lemma 2.1 in [14]).

LEMMA 2. *For each fixed $x_0 \in U$ the mapping $F : \bar{U} \to Y$, $Fx = x_0$ for all $x \in \bar{U}$, is essential.*

Two mappings $F_0, F_1 \in \mathcal{A}(\bar{U}; Y)$ are said to be *homotopic* if there exists $H: [0, 1] \times \bar{U} \to Y$ such that $H_\lambda = H(\lambda, .) \in \mathcal{A}(\bar{U}; Y)$ for all $\lambda \in [0, 1]$, $H_0 = F_0$, $H_1 = F_1$ and $H(., x) : [0, 1] \to Y$ is continuous uniformly with respect to $x \in \bar{U}$.

We also note the following topological transversality theorem (Leray—Schauder's alternative) for condensing mappings.

LEMMA 3. *Let $F_0, F_1 \in \mathcal{A}(\bar{U}; Y)$ be two homotopic mappings. Then F_0 is essential if and only if F_1 is essential.*

The proof of this lemma can be found in [14]. It reproduces with some specifical changes that of the topological transversality theorem for completely continuous mappings (see [6] and [7]).

For other properties of measures of noncompactness and other results on condensing mappings we refer to the book of R. R. Ahmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina and B. N. Sadovskii [1].

3. Existence theorems. Let us consider the problem

$$u''(t) = h(t, u(t), u'(t); u), \quad t \in I, \tag{3.1}$$

$$u(a) = r, \ u(b) = s, \tag{3.2}$$

where r and s are two fixed elements of X.

Let $C_b = \{u \in C(I;X) : u(a) = r, u(b) = s\}$, $C_b^1 = C_b \cap C^1$ and $C_b^2 = C_b \cap C^2$. We shall consider on C_b, C_b^1 and C_b^2 the topologies induced by those of C, C^1 and C^2, respectively.

The main existence result is

THEOREM 1. *Assume that*

(i) $h \in C(I \times X \times X \times C_b^1; X)$ *and* h *is uniformly continuous on* $I \times A_1 \times A_2 \times A_3$ *whenever* A_1, A_2 *are bounded subsets of* X *and* $A_3 \subset C_b^2$ *is bounded in* C^1.

(ii) *There exists* ρ *such that*

$$0 \leqslant \rho < \min (8/(b-a)^2, 2/(b-a), 1) \tag{3.3}$$

and

$$\alpha(h(t, A_1, A_2, A_3)) \leqslant \rho \max (\alpha(A_1), \alpha(A_2), \alpha_1(A_3)), \tag{3.4}$$

whenever $t \in I$, A_1 *and* A_2 *are bounded subsets of* X *and* $A_3 \subset C_b^2$ *is bounded in* C^1.

(iii) *For each* $x \in X$ *satisfying* $|x| > M \geqslant \max(|r|, |s|)$ *there exists* $x^* \in Jx$ *such that*

$$(x^*, h(t, x, y; z)) > 0, \tag{3.5}$$

for all $t \in]a, b[$, $y \in X$ *satisfying* $(x^*, y) = 0$ *and* $z \in C_b^2$ *with* $\|z\| = |x|$.

(iv) *There exists a nondecreasing function*

$$\Psi : [0, +\infty[\to]0, +\infty[$$

such that

$$\liminf_{t \to +\infty} t^2/\Psi(t) > 4M \tag{3.6}$$

and

$$|h(t, x, y; z)| \leqslant \Psi(|y|), \tag{3.7}$$

for all $t \in I$, $x, y \in X$ *and* $z \in C_b^2$ *satisfying* $|x| \leqslant \|z\| \leqslant M$.

Then equation (3.1) *has at least one solution* $u \in C_b^2$.

Remark 1. If $h : I \times X \times X \times C_b^1 \to X$ is completely continuous then condition (i) in Theorem 1 holds, condition (3.4) holds with $\rho = 0$ and (3.5) may be relaxed as follows:

$$(x^*, h(t, x, y; z)) \geqslant 0. \tag{3.8}$$

Indeed, in this case, for $1/n < \min(8/(b-a)^2, 2/(b-a), 1)$ the mapping $h_n(t, x, y; z) = h(t, x, y; z) + (1/n)x$ satisfies the hypothesis of Theorem 1 with $\rho = 1/n$ and $\Psi + M$ instead of Ψ. In consequence, by Theorem 1, the equation

$$u''(t) = h_n(t, u(t), u'(t); u), \ t \in I \tag{3.9}$$

has at least one solution $u_n \in C_b^2$. Let $\{u_n\}_{n\geq n_0}$ be a sequence of solutions to (3.9), where $1/n_0 < \min\,(8/(b-a)^2, 2/(b-a), 1)$. As follows from the proof of Theorem 1, the set $\{u_n\}_{n\geq n_0}$ is bounded in C^2. Hence $\{u_n\}_{n\geq n_0}$ is equicontinuous in C^1. On the other hand, if $G_n : I \times I \to \mathbf{R}$ is the Green's function associated with the scalar problem $y'' - (1/n)y = f(t)$, $y(a) = y(b) = 0$ and u_b^n is the unique solution from C_b^2 to the equation $u'' - (1/n)\,u = 0$, then we must have

$$u_n(t) = -\int_a^b G_n(t, \xi)\, h(\xi, u_n(\xi), u_n'(\xi)\,;\, u_n)d\xi + u_b^n(t). \tag{3.10}$$

Whence, using the compactness of h we obtain that the sets $\{u_n(t)\}$ and $\{u_n'(t)\}$ are precompact in X for each $t \in I$. Thus, by the Ascoli—Arzelà Theorem the sequence $\{u_n\}$ has a subsequence which converges in C^1; its limit is a solution to (3.1) as follows also by (3.10).

As regards the existence of solutions to (1.1), (1.2) we have the following result

COROLLARY 1. *Suppose that*

(i) *f is uniformly continuous on each bounded subset of* $I \times X^{m+2}$, $g_i \in C(I \cdot I')$, $i = 1, \ldots, m$ *and* $\varphi \in C(I' \setminus \operatorname{int} I\,;\, X)$.

(ii) *There exists* ρ *satisfying condition* (3.3) *and*

$$\alpha(f(t, A_1, A_2, \ldots, A_{m+2})) \leq \rho \max\,(\alpha(A_i) : i = 1, 2, \ldots, m+2), \tag{3.11}$$

whenever $t \in I$ *and* A_i, $i = 1, 2, \ldots, m+2$ *are bounded subsets of* X.

(iii) *For each* $x \in X$ *satisfying* $\|x\| > M \geq \max\,(\|\varphi(t)\| : t \in I' \setminus \operatorname{int} I)$ *there exists* $x^* \in \mathfrak{J}x$ *such that*

$$(x^*, f(t, x, y, x^1, \ldots, x^m)) > 0, \tag{3.12}$$

for all $t \in \,]a, b[$ *and all* $x, y, x^1, \ldots, x^m \in X$ *satisfying* $(x^*, y) = 0$ *and* $\|x^i\| \leq \leq \|x\|$, $i = 1, \ldots, m$.

(iv) *There exists a nondecreasing function*
$\Psi : [0, +\infty[\, \to\,]0, +\infty[$ *satisfying condition* (3.6) *and*

$$\|f(t, x, y, x^1, \ldots, x^m)\| \leq \Psi(\|y\|), \tag{3.13}$$

for all $t \in I$ *and all* $x, y, x^1, \ldots, x^m \in X$ *with* $\|x\| \leq M$ *and* $\|x^i\| \leq M$, $i = = 1, \ldots, m$.

Then problem (1.1), (1.2) *has at least one solution.*

Remark 2. Theorem 1 and Corollary 1 remain true if we consider certain other measures of noncompactness instead of the Kuratowski measure of noncompactness.

On finite-dimensional spaces some requirements of the hypothesis of Theorem 1 may be lessened as follows.

THEOREM 2. *Assume that*

(i) *The mapping* $h : I \times \mathbf{R}^n \times \mathbf{R}^n \times C_b \to \mathbf{R}^n$ *is continuous.*

(ii) *There exists* $M \geq \max\,(\|r\|, \|s\|)$ *such that*

$$(x, h(t, x, y\,;\, z)) \geq 0, \tag{3.14}$$

for all $t \in I$, $x \in \mathbf{R}^n$ *with* $\|x\| > M$, $y \in \mathbf{R}^n$ *with* $(x, y) = 0$ *and* $z \in C_b^2$ *satisfying* $\|z\| = \|x\|$.

(iii) *For each* $j \in \{1, \ldots, n\}$ *and each* $M' > 0$ *there is a function* $\Psi_{j,M'}$: $[0, +\infty[\to]0, +\infty[$ *such that* $t/\Psi_{j,M'}(t)$ *is locally integrable on* $[K, +\infty[$, *where* $K = \|r - s\|/(b - a)$,

$$\int_K^{\infty} t / \Psi_{j,M'}(t)\, dt > 2M \tag{3.15}$$

and

$$|h_j(t, x, y; z)| \leqslant \Psi_{j,M'}(|y_j|), \tag{3.16}$$

whenever $t \in I$, $x \in \mathbf{R}^n$ *and* $z \in C_b^2$ *satisfy* $\|x\| \leqslant \|z\| \leqslant M$ *and* $y \in \mathbf{R}^n$, $y = (y_1, \ldots, y_n)$ *satisfies* $|y_i| \leqslant M'$ *for all* $i \leqslant j - 1$.

Then the system (3.1) *has at least one solution* $u \in C_b^2$.

This result may be compared with Theorem 2.4 in [8], chap. V. Its proof follows easily by that of Theorem 1 if we take into account Lemma 5.6 in [8], chap. II and Remark 1.

As a consequence of Theorem 2 we have

COROLLARY 2. *Let the following conditions hold*

(i) $f \in C(I \times (\mathbf{R}^n)^{m+2}; \mathbf{R}^n)$, $g_i \in C(I; I')$, $i = 1, \ldots, m$ *and*

$$\varphi \in C(I' \setminus \operatorname{int} I; \mathbf{R}^n).$$

(ii) *There exists* $M \geqslant \max(\|\varphi(t)\| : t \in I' \setminus \operatorname{int} I)$ *such that*

$$(x, f(t, x, y, x^1, \ldots, x^m)) \geqslant 0, \tag{3.17}$$

for all $t \in I$, $x \in \mathbf{R}^n$ *with* $\|x\| > M$, $y \in \mathbf{R}^n$ *with* $(x, y) = 0$ *and all* $x^i \in \mathbf{R}^n$ *satisfying* $\|x^i\| \leqslant \|x\|$, $i = 1, \ldots, m$.

(iii) *For each* $j \in \{1, \ldots, n\}$ *and each* $M' > 0$ *there is a function* $\Psi_{j,M'} : [0, +\infty[\to]0, +\infty[$ *such that* $t/\Psi_{j,M'}(t)$ *is locally integrable on* $[K, +\infty[$, *satisfies condition* (3.15) *and*

$$|f_j(t, x, y, x^1, \ldots, x^m)| \leqslant \Psi_{j,M'}(|y_j|), \tag{3.18}$$

for every $t \in I$, $x, x^1, \ldots, x^m \in \mathbf{R}^n$ *satisfying* $\|x\| \leqslant M$, $\|x^i\| \leqslant M$, $i = 1, \ldots, m$ *and any* $y \in \mathbf{R}^n$ *with* $|y_i| \leqslant M'$ *for all* $i \leqslant j - 1$.

Then the problem (1.1), (1.2) *has at least one solution.*

4. **Proofs.** For the proof of Theorem 1 we need some lemmas referring to the a priori bounds on solutions of equation (3.1).

LEMMA 4. *Assume that conditions* (i) *and* (iii) *from Theorem* 1 *hold. Then any solution* $u \in C_b^2$ *of equation* (3.1) *satisfies the inequality*

$$\|u\| \leqslant M. \tag{4.1}$$

Proof. Let $u \in C_b^2$ be a solution of (3.1) and let $t_0 \in I$ be such that $\|u\| = \|u(t_0)\|$. If $t_0 = a$ or $t_0 = b$ then (4.1) follows by $M \geqslant \max(\|r\|, \|s\|)$.

Let $t_0 \in]a, b[$. Then we have $(x_0^*, u'(t_0)) = 0$ for any $x_0^* \in \mathfrak{J}u(t_0)$.

Assume, a contrario, that $\|u(t_0)\| > M$. Then, by (3.5), there exists $x_0^* \in \mathfrak{J}u(t_0)$ such that

$$(x_0^*, h(t_0, u(t_0), u'(t_0) ; u)) > 0.$$

Since h is continuous there is $\delta > 0$ such that

$$(x_0^*, h(t_0 + \lambda, u(t_0 + \lambda), u'(t_0 + \lambda) ; u)) > 0,$$

whenever $|\lambda| < \delta$ and $t_0 + \lambda \in I$. This implies that

$$(x_0^*, u''(t_0 + \lambda)) > 0 \text{ for } |\lambda| < \delta \text{ with } t_0 + \lambda \in I.$$

Hence, by using the Taylor's formula

$$u(t_0 + \lambda) - u(t_0) = \lambda u'(t_0) + (\lambda^2/2)u''(t_0 + \mu),$$

where $\mu = \mu(\lambda)$ lies between t_0 and $t_0 + \lambda$, we deduce that

$$(x_0^*, u(t_0 + \lambda) - u(t_0)) > 0 \text{ for } |\lambda| < \delta,\ \lambda \neq 0 \text{ with } t_0 + \lambda \in I.$$

On the other hand, since $x_0^* \in \mathfrak{J}u(t_0)$, by (2.1), we must have

$$(x_0^*, u(t_0 + \lambda) - u(t_0)) \leqslant \frac{1}{2}\|u(t_0 + \lambda)\|^2 - \frac{1}{2}\|u(t_0)\|^2 \leqslant 0,$$

which contradicts the previous inequality. Therefore $\|u(t_0)\| \leqslant M$ and the proof is complete.

The next lemma is due to K. Schmitt and R. Thompson [17] and it will be used to derive a priori bounds on derivatives of solutions of equation (3.1).

LEMMA 5. *Let* $\Psi : [0, +\infty[\to]0, +\infty[$ *be a nondecreasing function satisfying condition* (3.6) *and let* M *be a positive number. Then there exists a positive constant* M_1 *(depending only on* Ψ *and* M*) such that, if* $u \in C^2(I ; X)$ *is such that* $\|u\| \leqslant M$ *and* $\|u''\| \leqslant \Psi(\|u'\|)$*, then*

$$\|u'\| \leqslant M_1. \tag{4.2}$$

Let $G : I \times I \to \mathbf{R}$ be the Green's function associated with the scalar boundary value problem $y'' = f(t)$, $y(a) = y(b) = 0$. We have

$$G(t, \xi) = \frac{(\xi - a)(b - t)}{b - a} \text{ for } \xi \leqslant t$$

$$= \frac{(t - a)(b - \xi)}{b - a} \text{ for } \xi > t.$$

Define the linear integral operator $N : C \to C^2$,

$$(Nu)(t) = -\int_a^b G(t, \xi)\, u(\xi)\, d\xi, \qquad t \in I.$$

We have

$$\|Nu\|_2 \leqslant \max((b - a)^2/8,\ (b - a)/2,\ 1), \tag{4.3}$$

for all $u \in C$ having $\|u\| \leqslant 1$.

If we assume that $h: I \times X \times X \times C_b \to X$ is continuous we may also define the operator $F: C_b^2 \to C$,

$$(Fu)(t) = h(t, u(t), u'(t); u), \quad t \in I.$$

Let u_b be the unique solution from C_b^2 to the equation $u'' = 0$ and define the operator $T: C_b^2 \to C_b^2$,

$$Tu = NFu + u_b, \; u \in C_b^2. \tag{4.4}$$

LEMMA 6. *If the mapping $NF: C_b^2 \to C^2$ is condensing and if there exists $\bar{M} > 0$ such that $\|u\|_2 < \bar{M}$ for any solution $u \in C_b^2$ to the equation*

$$u''(t) = \lambda h(t, u(t), u'(t); u), \quad t \in I \tag{4.5}$$

and for all $\lambda \in [0, 1]$, then the problem (3.1), (3.2) *has at least one solution.*

Proof. A function u is a solution to problem (3.1), (3.2) if and only if

$$u(t) = -\int_a^b G(t, \xi)\, h(\xi, u(\xi), u'(\xi); u)d\xi + u_b(t), \quad t \in I$$

or, equivalently, if and only if it is a fixed point of T, i.e.

$$u = NFu + u_b. \tag{4.6}$$

Similarly, $u \in C_b^2$ is a solution to (4.5) if and only if

$$u = \lambda NFu + u_b. \tag{4.7}$$

Let $U = \{u \in C_b^2 : \|u\|_2 < \bar{M}\}$. Clearly C_b^2 is a convex closed subset of the Banach space C^2 and U is open in C_b^2. By Lemma 2 the mapping $H_0: \bar{U} \to C_b^2$, $H_0 u = u_b$ for all $u \in \bar{U}$, is essential. Also, if we define $H_\lambda: \bar{U} \to C_b^2$, $H_\lambda u = \lambda NFu + u_b$ we see that $H_\lambda \in \mathcal{A}(\bar{U}; C_b^2)$ for all $\lambda \in [0, 1]$. Moreover, since NF is condensing and $\bar{U}$ is bounded we have that $NF(\bar{U})$ is a bounded subset of C^2 and in consequence the mapping $H(.; u): [0, 1] \to C_b^2$ is continuous uniformly with respect to $u \in \bar{U}$. Thus H_0 and H_1 are homotopic and by Lemma 3 it follows that H_1 is also essential. Therefore $T(= H_1)$ has at least one fixed point, as desired.

Proof of Theorem 1. We will prove first that the mapping $F: C_b^2 \to C$ is (α, ρ) — Lipschitz.

First of all let us show that by (i) and (ii) we have

$$\alpha(h(I, A_1, A_2; A_3)) \leqslant \rho \max(\alpha(A_1), \alpha(A_2), \alpha_1(A_3)), \tag{4.8}$$

whenever A_1, A_2 are bounded in X and $A_3 \subset C_b^2$ is bounded in C^1. To this end, let $\varepsilon > 0$ be arbitrary fixed. Then, by the uniform continuity of h assumed in (i), it follows that for each $\bar{t} \in I$ there is a neighbourhood $V(\bar{t}; \varepsilon)$ of $\bar{t}$ such that

$$|h(t, x, y; z) - h(\bar{t}, x, y; z)| < \varepsilon$$

for all $t \in V(\bar{t}; \varepsilon)$, $x \in A_1, y \in A_2$ and $z \in A_3$. Consequently

$$\alpha(h(V(\bar{t}; \varepsilon), A_1, A_2; A_3)) \leqslant \alpha(h(\bar{t}, A_1, A_2; A_3)) + 2\varepsilon.$$

This, by (3.4) and the compactness of I, yields

$$\alpha(h(I, A_1, A_2 ; A_3)) \leqslant \rho \max (\alpha(A_1), \alpha(A_2), \alpha_1(A_3)) + 2\varepsilon.$$

Now letting $\varepsilon \to 0$ we get (4.8) as desired.

The continuity of F follows easily by that of h.

Let D be an arbitrary bounded subset of C_b^2. If we apply (4.8) to $A_1 = D(I)$, $A_2 = D'(I)$ and $A_3 = D$ we see that $F(D)$ is bounded. Further we will show that

$$\alpha(F(D)) \leqslant \rho\alpha_2(D). \tag{4.9}$$

Since D is bounded in C^2, the sets D and D' are equicontinuous families of functions. Hence, by Lemma 1, we have

$$\alpha(D) = \sup (\alpha(D(t)) : t \in I),\ \alpha(D') = \sup (\alpha(D'(t)) : t \in I). \tag{4.10}$$

Moreover, the equicontinuity of D and D' together with the uniform continuity of h assumed in (i), implie that $F(D)$ is also an equicontinuous family of functions. Thus

$$\alpha(F(D)) = \sup (\alpha(F(D)(t)) : t \in I). \tag{4.11}$$

But, by (3.4), (2.3) and (2.4), we have

$$\alpha(F(D)(t)) = \alpha(\{h(t, u(t), u'(t) ; u) : u \in D\}) \leqslant \rho \max (\alpha(D(t)),$$
$$\alpha(D'(t)), \alpha_1(D)) = \rho\alpha_1(D) \leqslant \rho\alpha_2(D),$$

for all $t \in I$. Whence, (4.9) follows by (4.11).

Therefore the mapping F is (α, ρ) — Lipschitz as claimed.

Further, by (4.3) and (4.9), we get

$$\alpha_2(NF(D)) \leqslant \rho \max ((b - a)^2/8,\ (b - a)/2,\ 1)\alpha_2(D),$$

whence we may claim that NF is condensing.

Now, according to Lemma 6, we have only to prove the boundedness in C^2 of the set of solutions to equation (4.5).

For each $\lambda \in]0, 1]$ the function λh satisfies the hypothesis of Lemma 4. Thus $\|u\| \leqslant M$ for any solution $u \in C_b^2$ to equation (4.5) and for all $\lambda \in]0,1]$. In addition, since $M \geqslant \max (|r|, |s|)$ we see that u_b, the unique solution in C_b^2 to equation (4.5) for $\lambda = 0$, also satisfies $\|u_b\| \leqslant M$.

Further, according to assumption (iv) and Lemma 5, there exists a constant M_1 such that $\|u'\| \leqslant M_1$ for any solution $u \in C_b^2$ to (4.5) and all $\lambda \in [0, 1]$.

Finally, if we put

$$D = \{u \in C_b^2 : u \text{ is a solution to (4.5) for a certain } \lambda \in [0, 1]\}$$

and we apply (4.8) to $A_1 = D(I)$, $A_2 = D'(I)$, $A_3 = D$, then we obtain that the set $D''(I)$ is bounded in X. Hence, there exists a constant M_2 such that $\|u''\| \leqslant M_2$ for any solution $u \in C_b^2$ to equation (4.5) and all $\lambda \in [0, 1]$. The proof of Theorem 1 is now complete.

5. **Uniqueness.** We will establish the uniqueness of solution to equation

$$u''(t) = h(t, u(t), u'(t)) + A(t, u), \quad t \in I, \tag{5.1}$$

together with the boundary conditions (3.2), where h maps $I \times X \times X$ into X and A maps $I \times C_b^2$ into X.

The uniqueness is established under some monotonicity conditions.

THEOREM 3. *Suppose that the following conditions are satisfied:*

(i) $$(x^*, h(t, x + x^1, y + y^1) - h(t, x^1, y^1)) > 0 \qquad (\geqslant 0), \tag{5.2}$$

for all $t \in]a, b[$, $x, x^1 \in X$ *with* $x \neq 0$, $x^* \in \daleth x$ *and all* $y, y^1 \in X$ *satisfying* $(x^*, y) = 0$.

(ii) $$(x^*, A(t_0, u_1) - A(t_0, u_2)) \geqslant 0 \quad (> 0), \tag{5.3}$$

for all $u_1, u_2 \in C_b^2$, $u_1 \neq u_2$, $t_0 \in]a, b[$ *such that* $|u_1(t_0) - u_2(t_0)| = \|u_1 - u_2\|$ *and all* $x^* \in \daleth(u_1(t_0) - u_2(t_0))$.

Then problem (5.1), (3.2) *has at most one solution* $u \in C_b^2$.

Proof. Let u_1 and u_2 be two solutions to (5.1), (3.2) and let $u = u_1 - u_2$. If $u_1 \neq u_2$ then it whould exist $t_0 \in]a, b[$ such that $|u(t_0)| = \|u\| > 0$. This would imply that $(x^*, u'(t_0)) = 0$ and

$$(x^*, u''(t_0)) \leqslant 0, \tag{5.4}$$

for all $x^* \in \daleth u(t_0)$. On the other hand, by (5.2) and (5.3) we should have

$$(x^*, u''(t_0)) = (x^*, h(t_0, u_1(t_0), u_1'(t_0)) - h(t_0, u_2(t_0), u_2'(t_0))) +$$
$$+ (x^*, A(t_0, u_1) - A(t_0, u_2)) > 0,$$

which would contradict (5.4). Thus $u_1 = u_2$ and the theorem is proved.

As an application we will establish the uniqueness of solution to problem (1.1), (1.2) in the particular case when equation (1.1) has the form

$$u''(t) = h(t, u(t), u'(t)) + q(t)\, u(t) + \sum_{i=1}^{m} q_i(t)\, u(g_i(t)), \tag{5.5}$$

where q and q_i, $i = 1, \ldots, m$ are real functions defined on I.

COROLLARY 3. *Suppose that h satisfies condition* (i) *from Theorem 2 and*

$$q_i(t) \leqslant 0, \quad q(t) + \sum_{i=1}^{m} q_i(t) \geqslant 0 \quad (> 0), \tag{5.6}$$

for all $t \in]a, b[$.

Then problem (5.5), (1.2) *has at most one solution.*

Proof. Apply Theorem 3 to

$$A(t, u) = q(t)\, u(t) + \sum_{i=1}^{m} q_i(t)\, u^i(t).$$

Note that Theorem 3 and Corollary 3 generalize Theorem 6 and Theorem 7 in [15].

REFERENCES

1. Ahmerov, R. R., Kamenskii, M. I., Potapov, A. S., Rodkina, A. E., Sadovskii, B.N., *Measure of noncompactness and condensing operators* (Russian), Novosibirsk, Nauka, 1986.
2. Ambrosetti, A., *Un theorema di esistenza per le equazioni differenziali negli spazi di Banach*, Rend. Sem. Mat. Univ. Padova, 39 (1967), 349–361.
3. Bernfeld, S. R., Lakshmikantham, V., *An introduction to nonlinear boundary value problems*, Academic Press, New York, London, 1974.
4. Bernstein, S. N., *Sur les équations du calcul des variations*, Ann. Sci. Ecole Norm. Sup. 29 (1912), 431–485.
5. Chandra, J., Lakshmikantham, V., Mitchell, A., *Existence of solutions of boundary value problems for nonlinear second order systems in a Banach space*, J. Nonlinear Analysis 2 (1978), 157–168.
6. Dugundji, J., Granas, A., *Fixed point theory*, vol. 1., Warszawa, 1982.
7. Granas, A., *Sur la méthode de continuité de Poincaré*, C. R. Acad. Sci., Paris 282 (1976), 983–985.
8. Granas, A., Guenther, R., Lee, J., *Nonlinear boundary value problems for ordinary differential equations*, Dissertationes Mathematicae, CCXIV, Warszawa, 1985.
9. Hartman, P., *On boundary value problem for systems of ordinary nonlinear second order differential equations*, Trans. Amer. Math. Soc. 96 (1960), 493–509.
10. Hartman, P., *Ordinary differential equations*, John Wiley & Sons, Inc., New York, 1964.
11. Lakshmikantham, V., *Abstract boundary value problems*, Nonlinear Equations in Abstract Spaces (Ed. V. Lakshmikantham), Academic Press, New York, San Francisco, London, 1978, 117–123.
12. Lasota, A., Yorke, J. A., *Existence of solutions of two-point boundary value problems for nonlinear systems*, J. Differential Equations 11 (1972), 509–518.
13. Mawhin, J., *Nonlinear boundary value problems for ordinary differential equations from Schauder theorem to stable homotopy*, Université Catholique de Louvain, Rapport No. 86 (1976).
14. Precup, R., *Nonlinear boundary value problems for infinite systems of second-order functional differential equations*, Univ. Babeş-Bolyai, Cluj-Napoca, Preprint Nr. 8, 1988, 17–30.
15. Rus, A. I., *Maximum principles for some nonlinear differential equations with deviating arguments*, Studia Univ. „Babeş-Bolyai" 32, No. 2 (1987), 53–57.
16. Scorza-Dragoni, G., *Sul problema dei valori ai limiti per i sistemi di equazioni diferenziali del secondo ordine*, Bull. U.M.I. 14 (1935), 225–230.
17. Schmitt, K., Thompson, R., *Boundary value problems for infinite systems of second-order differential equations*, J. Differential Equations 18 (1975), 277–295.
18. Thompson, R. C., *Differential inequalities for infinite second order systems and an application to the method of lines*, J. Differential Equations 17 (1975), 421–434.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXIV, 2, 1989

AN APPLICATION OF FRYSZKOWSKI SELECTION THEOREM TO THE DARBOUX PROBLEM FOR A MULTIVALUED EQUATION

GEORGETA TEODORU*

Received: September 20, 1988

REZUMAT. — O aplicaţie a teoremei de selecţie a lui Fryszkowski la problema lui Darboux pentru o ecuaţie multivocă. În lucrare se consideră problema lui Darboux pentru ecuaţia hiperbolică multivocă $\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z)$, $F: D \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ $D = [0, a] \times [0, b]$, unde F este o aplicaţie inferior semicontinuă ale cărei valori sînt submulţimi închise, nu neapărat convexe, ale lui $\mathbf{R}^n$. Teorema de selecţie a lui Fryszkowski şi teorema de punct fix a lui Schauder sînt folosite pentru a demonstra existenţa unei soluţii locale a problemei considerate.

1. **Introduction.** The Darboux problem for the hyperbolic multivalued equation

$$\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z), F: D \times \mathbf{R}^n \to 2^{\mathbf{R}^n}, D = [0, a] \times [0, b]$$

and the notions of the local and global solution were defined in [15], [16].

Under the hypotheses of Carathéodory type, F haing convex and compact values, using Kakutani-Ky Fan fixed point theorem in [15], [16] we prove the existence of a local solution for the posed problem. In [17] it is proved, under the supplementary restrictions, that the local solution may be prolonged in D, obtaining a global solution. In view of the characterization theorem of the global solution [7], [15], [18] and the existence of a sequence of functions $f_n: D \to \mathbf{R}^n$ uniformly convergent on compacts to F [6], [7], [15], [19] we consider in [15], [19] a sequence of approximatting equations and we obtain a global solution of the Darboux proglem as the uniform limit of the sequence of the solutions for the approximating equations.

In [20], [21] we prove selection theorems, namely the existence of some continuous selections for every function $(x, y) \to F(x, y, z(x, y)), (x, y) \in D$, relatively to a given family of continuous functions $\{z(x, y)\}$, $(x, y) \to z(x, y)$ under the hypothesis that F takes compact not necessarily convex values, F continuous [20], F satisfies conditions of Carathéodory type [21]. As Corollaries, we obtain existence theorems of some global solutions, via Schauder fixed point theorem and the existence of the continuous selections.

In [22] we give three existence theorems of the global solutions for the Darboux problem, also in the case of not necessarily convex values for F, F being Lipschitzian mapping with respect to z. The results are obtained by the method of the successive approximations, using two selection theorems [10],

* Polytechnic Institute of Iaşi, Department of Mathematics, 6600 Iaşi, Romania

[11]. In [23] under the assumption that F is Lipschitzian mapping with respect to z, with possibly nonconvex values, we obtain a global solution of the Darboux problem, using the contraction principle of Covitz and Nadler, [8].

In this paper in a manner used in [1], [13], [20], [21] we obtain the existence of a local solution, using the Fryszkowski selection theorem [9], [13] and Schauder fixed point theorem, under the assumption that F is lower semicontinuous, taking not necessarily convex values.

2. **Preliminary results.** Let U be a topological space, Y a metric space and let $F: U \to 2^Y$ be a nonempty and closed valued mapping. We say that F is *lower semicontinuous* (*l.s.c*) on U if $F^{-1}(B) = \{u \in U | F(u) \cap B \neq \emptyset\}$ is open whenever B is open, [2], [11], [13].

A nonempty and closed valued mapping $H: [0, T] \to 2^Y$ is called *weakly measurable* if $H^{-1}(B) = \{t \in [0, T] | H(t) \cap B \neq \emptyset\}$ is Lebesgue measurable whenever B is open, [11], [13].

The next result is an instance of a general selection theorem due to Kuratowski and Ryll-Nardzewski, [12], [13].

THEOREM 2.1. *Let X a separable real Banach space and let $H: [0, T] \to 2^X$ be a nonempty and closed valued mapping. If H is weakly measurable then it has at least one measurable selection, i.e. there exists at least one measurable function $h: [0, T] \to X$ such that $h(t) \in H(t)$ a.e. for $t \in (0, T)$*, [13], [5; Theorem III 6, p. 65].

COROLLARY 2.1. *Let X be a separable real Banach space, let C be a nonempty and closed subset in X and let $F: [0, T] \times C \to 2^X$ be a nonempty and closed valued mapping which is l.s.c. on $[0, T] \times C$. Then, for each continuous function $u: [0, T] \to C$, the mapping $F \circ u: [0, T] \to 2^X$, $(F\circ u)(t) = F(t, u(t))$ for each $t \in [0, T]$, has at least one measurable selection*, [13].

Proof. Obviously $F\circ u$ is l.s.c. on $[0, T]$ and hence it is weakly measurable Thus Theorem 2.1 applies and this completes the proof, [13].

Let K be a nonempty subset in $C([0, T]; X)$ and let $\mathfrak{F}: K \to 2^{L^1([0,T];X)}$ be a nonempty and closed valued mapping. We say that $\mathfrak{F}$ is *decomposable* if for each $u \in K$, each $f, g \in \mathfrak{F}(u)$ and each measurable subset E in $[0, T]$ we have

$$f \cdot \chi_E + g \cdot \chi_{[0,T]-E} \in \mathfrak{F}(u),$$

where χ_E is the characteristic function of E, [3], [13].

In the proof of our existence result it is used the following specific form of a selection theorem due to Fryszkowski, [9], [13].

THEOREM 2.2. *Let X be a separable real Banach space, let K be a compact subset in $C([0, T]; X)$ and let $\mathfrak{F}: K \to 2^{L^1([0,T];X)}$ be a nonempty and closed valued mapping which is l.s.c. and decomposable. Then there exists at least one continuous function $f: K \to L^1([0, T]; X)$ such that $f(u) \in \mathfrak{F}(u)$ for each $u \in K$*, [13].

3. **The main result.** In this section we shall use the following hypotheses:

(H_1) $F: D \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ is a nonempty and closed valued mapping which is l.s.c. on $D \times \mathbf{R}^n$.

(H_2) There exists $M > 0$ such that

$$\sup \{\|\zeta\| \mid \zeta \in F(x, y, z)\} \leq M$$

for every $(x, y, z) \in D \times \mathbf{R}^n$, $(x, y) \in D$, $\|z\| \leqslant C$, $C > 0$.

(H_3) The functions $\sigma \in AC([0, a]; \mathbf{R}^n)$, $\tau \in AC([0, b]; \mathbf{R}^n)$ satisfy the condition $\sigma(0) = \tau(0)$.

Remark. From (H_3) it results that the function $\alpha : D \to \mathbf{R}^n$ defined by

$$\alpha(x, y) = \sigma(x) + \tau(y) - \sigma(0), \ (x, y) \in D, \tag{1}$$

is absolutely continuous in the Carathéodory sense on D; $\alpha \in C^*(D; \mathbf{R}^n)$, [4].

DEFINITION. [15], [16]. The Darboux problem for the equation

$$\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z), \ (x, y, z) \in D \times \mathbf{R}^n, \tag{2}$$

consists in the determination of a solution for (2), i.e. of absolutely continuous in the Carathéodory sense function on D, $z \in C^*(D; \mathbf{R}^n)$, [4]; which satisfies the equation (2) a.e. for $(x, y) \in D$ and the conditions

$$\begin{cases} z(x, 0) = \sigma(x), \ 0 \leqslant x \leqslant a, \\ z(0, y) = \tau(y), \ 0 \leqslant y \leqslant b. \end{cases} \qquad \sigma(0) = \tau(0). \tag{3}$$

Let K be the set of absolutely continuous functions $z : D \to \mathbf{R}^n$, [4] which satisfy the inequality

$$\|z(x, y) - \alpha(x, y)\| \leqslant r, \ \text{where } r > 0, \tag{4}$$

and the conditions (3).

The relation $z \in K$ implies $z \in C(D; \mathbf{R}^n)$. We observe that $\frac{\partial^2 z}{\partial x \partial y}$ exists a.e. $(x, y) \in D$, as $z \in C^*(D; \mathbf{R}^n)$, [4].

Integrating $\frac{\partial^2 z}{\partial x \partial y}$ and using (1) and (3) it follows

$$z(x, y) = \alpha(x, y) + \int_0^x \int_0^y \frac{\partial^2 z(\xi, \eta)}{\partial \xi \partial \eta} d\xi \, d\eta, \ (x, y) \in D. \tag{5}$$

THEOREM 3.1. *Let be the hypotheses* $(H_1)-(H_3)$ *satisfied.*
Then, the Darboux problem (3) + (4) has a local solution on $[0, x_0] \times [0, y_0] \subseteq D$.

Proof. Let us define $\mathcal{F} : K \to 2^{L^1(D; \mathbf{R}^n)}$ by

$$\mathcal{F}(z) = \{f \in L^1(D; \mathbf{R}^n) \mid f(x, y) \in F(x, y, z(x, y)), \text{ a.e. for } (x, y) \in D\} \tag{6}$$

for each $z \in K$. In view of the hypotheses (H_1), (H_2), of (4) and of Corollary 2.1, $\mathcal{F}$ has nonempty and closed values. Since F is l.s.c. the mapping $\mathcal{F}$ is l.s.c. too. In addition, $\mathcal{F}$ is decomposable. Then, by Theorem 2.2, it follows that there exists a continuous function $f : K \to L^1(D; \mathbf{R}^n)$ such that

$$f(z)(x, y) \in F(x, y, z(x, y)) \text{ for each } z \in K \text{ and a.e. for } (x, y) \in D. \tag{7}$$

Let be $h(z) : D \to \mathbf{R}^n$, $z \in K$, the continuous function defined by

$$h(z)(x, y) = \alpha(x, y) + \int_0^x \int_0^y f(z)(\xi, \eta) \, d\xi \, d\eta, \quad (x, y) \in D. \tag{8}$$

By the hypothesis (H_2) and the properties of absolutely continuous functions [14, p. 328], we have

$$\begin{cases} \sigma(x) = \sigma(0) + \int_0^x \sigma'(\xi)\, d\xi, & 0 \leqslant x \leqslant a, \\ \tau(y) = \tau(0) + \int_0^y \tau'(\eta)\, d\eta, & 0 \leqslant y \leqslant b, \end{cases} \tag{9}$$

hence

$$h(z)(x, y) = \int_0^x \int_0^y f(z)(\xi, \eta)\, d\xi d\eta + \int_0^x \sigma'(\xi)\, d\xi + \int_0^y \tau'(\eta) d\eta + C_0, \quad (x, y) \in D, \tag{8'}$$

where $C_0 = \sigma(0) = \tau(0)$.

Then $h(z) \in C^*(D; \mathbf{R}^n)$ for each $z \in K$, i.e. $h(z)$ is absolutely continuous in the Carathéodory sense on D, [4]. One obtains $h(z) \in K$ hence $h(K) \subset K$. Indeed, we have

$$\| h(z)(x, y) - \alpha(x, y) \| = \left\| \int_0^x \int_0^y f(z)(\xi, \eta) d\xi\, d\eta \right\| \leqslant \int_0^x \int_0^y \| f(z)(\xi, \eta) \|\, d\xi\, d\eta \leqslant$$

$$\leqslant Mxy \leqslant Mx_0 y_0 \leqslant r, \text{ for } x \leqslant x_0,\ y \leqslant y_0. \tag{10}$$

Choose $(x_0, y_0) \in D$ such that the condition

$$Mx_0 y_0 \leqslant r \tag{11}$$

holds.

Since h is continuous and K is compact, from Mazur's theorem [13], [24, Theorem 1.2.9 p. 8] it follows that the set $K_0 = \overline{\text{conv}}\ h(K)$ is compact and convex. Using the Schauder fixed point theorem, it follows that there exists $\bar{z} \in K_0$ such that $\bar{z} = h(\bar{z})$, i.e. $h(\bar{z})(x, y) = \bar{z}(x, y)$, $(x, y) \in [0, x_0] \times [0, y_0]$. This implies from (7), (8) that $\frac{\partial^2 \bar{z}}{\partial x \partial y} = f(\bar{z})(x, y) \in F(x, y, \bar{z}(x, y))$, a. e. for $(x, y) \in D$, $0 \leqslant x \leqslant x_0$, $0 \leqslant y \leqslant y_0$ and $\bar{z}(x, 0) = \sigma(x)$, $0 \leqslant x \leqslant x_0$, $\bar{z}(0, y) = \tau(y)$, $0 \leqslant y \leqslant y_0$, therefore (2) and (3) hold for $\bar{z}$, consequently $\bar{z}$ is a solution of the Darboux problem (2) + (3). The proof is complete.

REFERENCES

1. H. A. Antosiewicz, A. Cellina, *Continuous Selections and Differential Relations*, J. Diff. Eqs., 19 (1975), 386—398.
2. C. Berge, *Espaces topologiques, fonctions multivoques*, Dunod, Paris, 1966.
3. A. V. Bogatirev, *Continuous branches of multivalued mappings with nonconvex right-hand side* (Russian), Math. Sb., 120 (162), 1983, 344—353.
4. C. Carathéodory, *Vorlesungen über Reelle Funktionen*, Chelsea Publishing Company, 1948.

5. C. Castaing, M. Valadier, *Convex analysis and measurable multifunctions*, Lecture Notes in Mathematics, 580, Springer Verlag, 1977.
6. A. Cellina, *Approximation of set valued functions and fixed point theorems*, Ann. Mat. Pura Apl., 82 (1969), 17–24.
7. A. Cellina, *Multivalued differential equations and ordinary differential equations*, SIAM J. Appl. Math. 18 (1970), 533–538.
8. H. Covitz, S. B. Nadler Jr., *Multivalued contraction mappings in generalized metric space*, Israel J. Math., 8 (1970), 5–11.
9. A. Fryszkowski, *Continuous selections for a class of nonconvex multivalued maps*, Studia Mathematica, 76 (1983), 163–174.
10. H. Hermes, *The generalized differential equation $\dot{x} \in R(t, x)$*, Advanc. Math., 4 (1970), 149–169.
11. C. J. Himmelberg, F. S. Van Vleck, *Lipschitzian generalized differential equations*, Rend. Sem. Mat. Univ. Padova, 48 (1973), 159–169.
12. K. Kuratowski, C. Ryll-Nardzewski, *A general theorem on selectors*, Bull. Acad. Pol. Sci, 13 (1965), 397–403.
13. E. Mitidieri, I. Vrabie – *Differential Equations governed by nonconvex perturbations of m-accretive operators*, „Al. I. Cuza" University of Iaşi, Preprint Series in Mathematics of „A. Myller" Mathematical Seminary, No 2, 1988.
14. I. P. Natanson, *Teoria funcţiilor de variabilă reală*, Ed. Tehn. Bucureşti, 1957 (traducere din limba rusă).
15. G. Teodoru, *Studiul soluţiilor ecuaţiilor de forma $\partial^2 z/\partial x \partial y \in F(x, y, z)$*, Teză de doctorat, Univ. „Al. I. Cuza" Iaşi, 1984.
16. G. Teodoru, *Le problème de Darboux pour une équation aux dérivées partielles multivoque*, An. Şt. Univ. Iaşi, s. Ia, Mat., 31 (1985), 173–176.
17. G. Teodoru, *Sur le problème de Darboux pour l'équation $\partial^2 z/\partial x \partial y \in F(x, y, z)$*, An. şt. Univ. Iaşi, s. Ia Mat., 32 (1986), 41–49.
18. G. Teodoru, *A characterization of the solutions of the Darboux problem for the equation $\partial^2 z/\partial x \partial y \in F(x, y, z)$*, An. şt. Univ. Iaşi, s. Ia, Mat., 33 (1987), 34–38.
19. G. Teodoru, *Approximation of the solutions of the Darboux problem for the equation $\partial^2 z/\partial x \partial y \in F(x, y, z)$*, An. şt. Univ. Iaşi, s. Ia, Mat., 34 (1988), 31–36.
20. G. Teodoru, *Continuous selections for multivalued functions*, Proc. Itinerant Seminar on Functional Equations, Approximation and Convexity, Cluj-Napoca „Babeş-Bolyai" University, Faculty of Mathematics and Physics, Research Seminaries, Preprint no. 7, 1986, 273–278.
21. G. Teodoru, *Continuous selections for multifunctions satisfying the Caracthéodory type conditions. The Darboux problem associated to a multivalued equation*, Proc. Itinerant Seminar on Functional Equations, Approximation and Convexity, Cluj-Napoca, „Babeş-Bolyai" University, Faculty of Mathematics and Physics, Research Seminaries, Preprint no 6, 1987, 281–286.
22. G. Teodoru, *The Darboux problem associated with a Lipschitzian hyperbolic multivalued equation*, Bul. Inst. Polit. Iaşi, s. I Mec. Mat. (to appear).
23. G. Teodoru, *An application of the contraction principle of Covitz and Nadler to the Darboux problem for a multivalued equation*, An. şt. Univ. Iaşi, s. Ia, Mat. (to appear).
24. I. I. Vrabie, *Compactness method for nonlinear evolutions*, Pitman Monographs and Surveys in Pure and Applied Mathematics Longman Scientific and Technical, 1987.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXIV, 2, 1989

APPROXIMATION OF CONTINUOUSLY DIFFERENTIABLE FUNCTIONS BY MONOTONOUS SEQUENCES OF POLYNOMIALS IN TWO VARIABLES

SORIN GH. GAL*

Received: January 6, 1989

REZUMAT. — Aproximarea funcţiilor continuu diferenţiabile de două variabile prin şiruri monotone de polinoame. Scopul acestei lucrări este acela de a extinde rezultatele din [1] şi [3] la cazul a două variabile, obţinindu-se astfel o aproximare prin şiruri monotoane de polinoame de două variabile.

1. Introduction. Let $\Delta = [a, b] \times [a, b] \subset R^2$ be a bidimensional compact interval $(a, b \in R)$ and for $p \in N \cup \{0\} = N_0$ let's denote by $C^p(\Delta)$ the linear space of real-valued functions $f(x, y)$ defined on Δ, with all partial derivatives $(\partial^{i+j}f/(\partial x^i \partial y^j))(x, y)$, $i, j \geqslant 0$, $i + j \leqslant p$, continuous for $(x, y) \in \Delta$. Also, for $r = (i, j) \in N_0 \times N_0$ let's denote by $|r| = i + j$ and $(D^r f)(x, y) = (\partial^{|r|} f/(\partial x^i \partial y^j))(x, y)$, $(x, y) \in \Delta$.

Regarding the approximation by polynomials in $C^p(\Delta)$, the following Bernstein's theorem is well-known (for a more general variant (see e.g. [4], p. 104):

THEOREM 1.1. (Bernstein) *If $f \in C^p(\Delta)$, then there exists a sequence of polynomials in two variables $(S_m(x, y))_m$ such that $(D^r S_m)(x, y) \overset{m}{\to} (D^r f)(x, y)$ uniformly for $(x, y) \in \Delta$, for each $r \in N_0 \times N_0$ with $|r| \leqslant p$.*

The aim of the present note is to extend the results of [1], [3] to the case of two variables, obtaining thus an approximation result by monotonous sequences of polynomials in two variables.

2. Preliminaries. Let's consider the numbers $s_i \in \{-1, 1\}$, $i = \overline{0, p}$ fixed and the polynomial

$$R(x) = \sum_{i=0}^{p} 2^{p-i} \cdot s^i \cdot x^i / i!, \quad x \in [0, 1]. \tag{1}$$

LEMMA 2.1. *There takes place: $s_i \cdot R^{(i)}(x) \geqslant 1$ for any $x \in [0, 1]$ and any $i = 0, 1, \ldots, p$.*

Proof. It is easy to show that $R^{(i)}(x) = \sum_{j=0}^{p-i} 2^{p-(i+j)} \cdot s_{i+j} \cdot x^j/j! = 2^{p-i} \cdot s_i + 2^{p-i-1} \cdot s_{i+1} \cdot x + \ldots + s_p \cdot x^{p-i}/(p-i)!$. Then, because $2^{p-i} - 1 = 2^{p-i-1} + 2^{p-i-2} + \ldots + 2 + 1 = \sum_{j=1}^{p-i} 2^{p-(i+j)}$ and $|s_i \cdot 2^{p-(i+j)} \cdot s_{i+j} \cdot x^j/(j!)| \leqslant 2^{p-(i+j)}$, $\forall i = \overline{0, p}$, $\forall j = \overline{0, p-i}$, $\forall x \in [0, 1]$, we obtain:

* „Înfrăţirea" Calculation Office, Tuberozelor Str., no. 26, 3700 Oradea, Romania.

$s_i \cdot R^{(i)}(x) = 2^{p-i} + s_i \cdot 2^{p-i-1} \cdot s_{i+1} \cdot x + \ldots + s_i \cdot s_p \cdot x^{p-i}/(p-i)! = (2^{p-i-1} + 2^{p-i-2} + \ldots + 2 + 1) + 1(s_i \cdot 2^{p-i-1} \cdot s_{i+1} \cdot x + \ldots + s_i \cdot s_p \cdot x^{p-i}/(p-i)!) = 2^{p-i-1} + s_i \cdot 2^{p-i-1} + s_{i+1} \cdot x) + \ldots + (1 + s_i s_q \cdot x^{p-i}/(p-i)!) + 1 \geqslant 1$, $\forall x \in [0, 1]$, $\forall i = \overline{0, p}$.

3. **Main Result.** For simplicity, we will consider $\Delta = [0, 1] \times [0, 1]$.

There takes place:

THEOREM 3.1. *If $f \in C^p(\Delta)$ and $s_i^{(1)}, s_j^{(2)} \in \{-1, +1\}$ $i = \overline{0, p}$, $j = \overline{0, p}$ are fixed, then there exists a polynomial sequence in two variables $P_n(x, y)$, $n \in N$ such that for each $k \in \{0, 1, \ldots, p\}$, $k = i + j$, we have:*

$\frac{\partial^k P_n}{\partial x^i \partial y^j} \xrightarrow{n} \frac{\partial^k f}{\partial x^i \partial y^j}$ *uniformly on* Δ *and* $s_{i,j} \cdot \frac{\partial^k P_{n+1}(x, y)}{\partial x^i \partial y^j} < s_{i,j} \cdot \frac{\partial^k P_n(x, y)}{\partial x^i \partial y^j}$, *for any* $n = 1, 2, \ldots$, *and any* $(x, y) \in \Delta$, *where* $s_{i,j} = s_i^{(1)}, s_j^{(2)}$, $i = \overline{0, p}$, $j = \overline{0, p}$.

Proof. Let $f \in C_p(\Delta)$ be. If we denote $|||f||| = \max_{|r| \leqslant p} \{\sup \{|D^r f(x, y)|; (x, y) \in \Delta\}\}$ it is known that $|||\cdot|||$ is a norm on $C^p(\Delta)$.

In this case, the previous Bernstein's result (theorem 1.1.) may be formulated also: "if $f \in C_p(\Delta)$ then there exists a sequence of polynomials in two variables $(S_m(x, y))_m$ such that

$$|||S_m - f||| \to 0, \text{ when } m \to \infty\text{''}. \tag{2}$$

From (2), follows immediately that for each $n \in N$, there exists an index m_n such that

$$|||S_{m_n} - f||| < 1/n^2.$$

If we denote by $T_n(x, y) = S_{m_n}(x, y)$, $n = 1, 2, \ldots, (x, y) \in \Delta$, taking into account the definition of norm $|||\cdot|||$, from the above inequality we obtain immediately

$$\left| \frac{\partial^k T_n(x, y)}{\partial x^i \partial y^j} - \frac{\partial^k f(x, y)}{\partial x^i \partial y^j} \right| < 1/n^2, \tag{3}$$

for any $n = 1, 2, \ldots$, any $(x, y) \in \Delta$ and each $k \in \{0, 1, \ldots, p\}$, with $k = i + j$.

Because in general we have $|D^r T_n - D^r T_{n+1}| \leqslant |D^r T_n - D^r f| + |D^r f - D^r T_{n+1}|$, from (3) we obtain

$$\left| \frac{\partial^k T_n(x, y)}{\partial x^i \cdot \partial y^j} - \frac{\partial^k T_{n+1}(x, y)}{\partial x^i \partial y^j} \right| < 1/n^2 + 1/(n+1)^2 < 2/n^2.$$

Taking into account that $|s_{i,j}| = 1$, $i = \overline{0, p}$, $j = \overline{0, p}$, from the above inequality, we obtain

$$\left| s_{i,j} \cdot \frac{\partial^k T_n(x, y)}{\partial x^i \partial y^j} - s_{i,j} \cdot \frac{\partial^k T_{n+1}(x, y)}{\partial x^i \partial y^j} \right| < 2/n^2, \tag{4}$$

$(x, y) \in \Delta$, $n = 1, 2, \ldots, k \in \{0, 1, \ldots, p\}$, $k = i + j$.

Let's introduce the notations: $R_1(x) = \sum_{i=0}^{p} 2^{p-i} \cdot s_i^{(1)} x^i/i!$, $R_2(y) = \sum_{j=0}^{p} 2^{p-j} \cdot s_j^{(2)} \cdot y^j/(j!)$, $a_n = 2 \cdot \sum_{k=n}^{\infty} 1/k^2$ and $P_n(x, y) = T_n(x, y) + a_n \cdot R_1(x) \cdot R_2(y)$.

We have: $\frac{\partial^k P_n(x, y)}{\partial x^i \partial y^j} = \frac{\partial^k T_n(x, y)}{\partial x^i \partial y^j} + a_n \cdot R_1^{(i)}(x) \cdot R_2^{(j)}(y)$, from where, taking nto account (3) and $a_n \searrow 0$, there results immediately

$\frac{\partial^k P_n(x, y)}{\partial x^i \partial y^j} \xrightarrow{n} \frac{\partial^k f}{\partial x^i \partial y^j}$ uniformly on Δ, for each $k \in \{0, 1, \ldots, p\}$, $k = i + j$.

Then,
$$s_{i,j}\left(\frac{\partial^k P_n}{\partial x^i \partial y^j} - \frac{\partial^k P_{n+1}}{\partial x^i \partial y^j}\right) = s_{i,j} \cdot \left(\frac{\partial^k T_n}{\partial x^i \partial y^j} - \frac{\partial^k T_{n+1}}{\partial x^i \partial y^j}\right) + s_{i,j} \cdot R_1^{(i)}(x) \cdot R_2^{(j)}(y) \cdot (a_n - a_{n+1}) = s_{i,j} \cdot \left(\frac{\partial^k T_n}{\partial x^i \partial y^j} - \frac{\partial^k T_{n+1}}{\partial x^i \partial y^j}\right) + s_i^{(1)} \cdot R_1^{(i)}(x) \cdot s_j^{(2)} \cdot R_2^{(j)}(y) \cdot (2/n^2).$$

But, from the lemma 2.1. we have $s_i^{(1)} \cdot R_1^{(i)}(x) \geqslant 1$, for any $x \in [0, 1]$ and $s_j^{(2)} \cdot R_2^{(j)}(y) \geqslant 1$ fot any $y \in [0, 1]$. In conclusion, according to (4), we obtain
$$s_{i,j} \cdot \left(\frac{\partial^k P_n}{\partial x^i \partial y^j} - \frac{\partial^k P_{n+1}}{\partial x^i \partial y^j}\right) \geqslant s_{i,j}\left(\frac{\partial^k T_n}{\partial x^i \partial y^j} - \frac{\partial^k T_{n+1}}{\partial x^i \partial y^j}\right) + 2/n^2 > 0, \text{ for any } (x, y) \in \Delta,$$
any $n = 1, 2, \ldots,$ and each $k \in \{0, 1, \ldots, p\}$, with $k = i + j$.

Remarks. 1). The theorem 3.1. may be extended without any problems in the case of functions with more than two variables.

2). The case when f is a function of one variable was proved in [3] (see also [1]).

3). In spite of the fact that in general, the construction of the polynomials $P_n(x, y)$, $n \in N$, is an open question, in certain particular cases, they may be made up, exactly as in the case of a one variable (see [2]).

4). In the particular case $p = 0$, for constructively obtaining monotonous sequences of polynomials in two variables, we can use for instance the bivariate Bernstein—Stancu operators (see e.g. [5] p. 270 — 271 and [6]).

REFERENCES

1. Gal, S. G., *Sur l'approximation par des polynômes dans C^p[0,1]*, "Studia Univ. Babeş—Bolyai, Ser. Mathematica" (Cluj, XXVII (1982), 1, 57—60.
2. Gal, S. G., *On the Approximation by Polynomials in C^q[0,1]*, "J. Approx. Theory", 42 (1984), 1, 27—29.
3. Gal, S. G., *Extensions d'un résultat sur l'approximation par des polynômes dans C^q[0,1]* (submitted).
4. Păltineanu, G., *Elements of Approximation Theory of continuous functions* (Romanian), Edit. Acad. R. S. Romania, Bucureşti, 1982.
5. Stancu, D. D., *Numerical Analysis* (Romanian), "Babeş—Bolyai" University Press, vol. I, Cluj-Napoca, 1977.
6. Stancu, D. D., *Approximation of Bivariate Functions by Means of Some Bernstein — type Operators*, Multivariate Approximation, Proc. Sympos. Durham, 1977, p. 189—208, New York — San Francisco — London, Academic Press, 1978.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXIV, 2, 1989

AN ABSTRACT KOROVKIN TYPE THEOREM AND APPLICATIONS

D. ANDRICA* and C. MUSTĂŢA*

Received: January 26, 1989

REZUMAT. — O teoremă abstractă de tip Korovkin şi aplicaţii. În lucrare se obţin teoreme de tip Korovkin pentru spaţiul $C(X)$, unde X este un spaţiu metric compact (Teoremele 2 şi 3). Se aplică rezultatele obţinute pentru cazul cînd X este o submulţime compactă a unui spaţiu prehilbertian şi se dau delimitări ale diferenţei $\|B_n(f) - f\|$, unde B_n este operatorul lui Bernstein-Lototsky-Schnabl.

The well known Korovkin's theorem (see e.g. [1]) asserts that if $(L_n)_{n\geq 1}$ is a sequence of positive linear operators, acting from $C[a, b]$ to $C[a, b]$ and such that $(L_n(e_k))_{n\geq 1}$ converges uniformly to e_k, for $k = 0, 1, 2$, where $e_k(t) = t^k$, $t \in [a, b]$, then the sequence $(L_n(f))_{n\geq 1}$ converges uniformly to f, for every $f \in C[a, b]$.

This theorem was extended and generalized in many directions. One direction is to replace the above mentioned system of test functions by other systems of functions, which led to the theory of so called Korovkin subspaces. Another direction is to consider functions defined on more general compact spaces than the interval $[a, b]$, first of all on compact subsets of $\mathbf{R}^m$.

The aim of this paper is to give Korovkin type theorems for the space $C(X)$, where X is a compact metric space. As application, supposing that X is a compact convex subset of a Hilbert space, one obtains evaluations of the order of approximation by the Bernstein — Lototsky — Schnabl operator, similar to those given in [4].

If (X, d) is a compact metric space, denote by $C(X) = C(X, \mathbf{R})$ the space of all real-valued continuous functions defined on X and by Lip (X) the subspace of $C(X)$ formed of all real-valued Lipschitz functions defined on X. Equiped, as usually, with the uniform norm $\|f\| = \sup\{|f(x)| : x \in X\}$, $f \in C(X)$, the space $C(X)$ is a Banach space.

Our first result is a density theorem:

THEOREM 1. *The subspace* Lip (X) *is dense in* $C(X)$, *with respect to the uniform norm.*

Proof. The assertion of the theorem will follow from the Stone—Weierstrass theorem if we shall show that Lip (X) is a subalgebra of $C(X)$ containing the constant functions and separating the points of X.

If $f, g \in$ Lip (X) then

$$|(f\cdot g)(x) - (f\cdot g)(y)| \leq |f(x)|\cdot|g(x) - g(y)| + |g(y)|\cdot|f(x) - f(y)| \leq$$
$$\leq (\|f\|\cdot K_g + \|g\|\cdot K_f)\cdot d(x, y),$$

* University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania

for all $x, y \in X$, where K_f and K_g are Lipschitz constants for f and g, respectively. Therefore $f \cdot g \in \mathrm{Lip}(X)$ and since $\mathrm{Lip}(X)$ is a subspace of $C(X)$ it follows that $\mathrm{Lip}(X)$ is a subalgebra of the algebra $C(X)$.

As the constant functions are obviously in $\mathrm{Lip}(X)$ to finish the proof we have only to show that the algebra $\mathrm{Lip}(X)$ separates the points of X. For $x, y \in X$, $x \neq y$ let $f: X \to \mathbf{R}$ be defined by $f(z) = d(z, y)$, $z \in X$. Then

$$|f(z_1) - f(z_2)| = |d(z_1, y) - d(z_2, y)| \leqslant d(z_1, z_2), \quad z_1, z_2 \in X,$$

which shows that f is in $\mathrm{Lip}(X)$, $f(y) = d(y, y) = 0$ and $f(x) = d(x, y) > 0$. Theorem is proved.

A *Markov operator* L on $C(X)$ is a positive linear operator $L: C(X) \to C(X)$ such that $L(e_0) = e_0$, where $e_0(x) = 1$, $x \in X$, i.e. L preserves the constant functions.

In the following we shall need the following simple lemma:

LEMMA 1. *If L is a Markov operator acting on $C(X)$ then* $\|L\| = 1$.

Proof. Taking into account the positivity of L and applying L to the inequalities $-\|f\| \cdot e_0 \leqslant f \leqslant \|f\| \cdot e_0$, we obtain $-\|f\| \cdot e_0 \leqslant L(f) \leqslant \|f\| \cdot e_0$, so that $\|L(f)\| \leqslant \|f\|$, for all $f \in C(X)$. As $\|L(e_0)\| = \|e_0\| = 1$ it follows $\|L\| = 1$. Lemma is proved.

If $(L_n)_{n \geqslant 1}$ is a sequence of Markov operators acting on $C(X)$, let

$$\begin{aligned} \alpha_n(x) &= L_n(d(.,x);\ x), \\ \beta_n(x) &= L_n(d^2(.,x);\ x), \end{aligned} \tag{1}$$

for all $x \in X$ and $n = 1, 2, \ldots$.

Our first Korovkin type theorem is the following:

THEOREM 2. *Let $(L_n)_{n \geqslant 1}$ be a sequence of Markov operators acting on $C(X)$. If $(\alpha_n(x))_{n \geqslant 1}$ converges to zero, uniformly with respect to $x \in X$, then $(L_n(f))_{n \geqslant 1}$ converges uniformly to f, for all $f \in C(X)$.*

Proof. Let $f \in \mathrm{Lip}(X)$ and let $K_f \geqslant 0$ be a Lipschitz constant for f, i.e.

$$|f(x) - f(y)| \leqslant K_f \cdot d(x, y),$$

for all $x, y \in X$. This inequality can be rewritten in the form:

$$-K_f \cdot d(.,x) \leqslant f(\cdot) - f(x).e_0 \leqslant K_f \cdot d(.,x),$$

for all $x \in X$. Applying to these inequalities the operator L_n and taking into account the positivity of L_n and the notations (1), one obtains:

$$-K_f \cdot \alpha_n(x) \leqslant L_n(f; x) - f(x) \leqslant K_f \cdot \alpha_n(x)$$

for all $x \in X$, or equivalently,

$$|L_n(f; x) - f(x)| \leqslant K_f |\alpha_n(x)|, \tag{2}$$

for all $x \in X$. Since, by the hypothesis of the theorem the sequence $(\alpha_n(x))_{n \geqslant 1}$ tends to zero, uniformly for $x \in X$, the inequality (2) implies that $(L_n(f))_{n \geqslant 1}$ tends uniformly to f.

By Theorem 1 the space $\mathrm{Lip}(X)$ is dense in $C(X)$ with respect to the uniform norm on $C(X)$ and by Lemma 1, $\|L_n\| = 1$, $n = 1, 2, \ldots$ so that by the

Banach–Steinhaus theorem, the sequence $(L_n(f))_{n\geq 1}$ tends uniformly to f, for all $f \in C(X)$. The theorem is proved.

THEOREM 3. *Let $(L_n)_{n\geq 1}$ be a sequence of Markov operators acting on $C(X)$. If $\beta_n(x)$ is defined by (1) and the sequence $(\beta_n(x))_{n\geq 1}$ tends to zero, uniformly with respect to $x \in X$, then the sequence $(L_n(f))_{n\geq 1}$ tends uniformly to f, for all $f \in C(X)$.*

If $f \in$ Lip (X) then, furthermore

$$\| L_n(f) - f \| \leqslant K_f \cdot \sqrt{\| \beta_n \|}, \tag{3}$$

for all $n = 1, 2, \ldots$.

Proof. We have $L_n(e_0) = e_0$ and

$$0 \leqslant L_n((t \cdot f - e_0)^2) = t^2 L_n(f^2) - 2t \cdot L_n(f) + e_0 \quad .$$

for all $t \in R$, implying

$$[L_n(f)]^2 \leqslant L_n(f^2),$$

for all $f \in C(X)$. Applying this inequality to the function $f = d(.,x)$, one obtains:

$$(L_n(d(.,x);x))^2 \leqslant L_n(d^2(.,x);x), \tag{4}$$

for all $x \in X$. Taking into account the notations (1), it follows that the sequence $(\alpha_n(x))_{n\geq 1}$ converges to zero, uniformly for $x \in X$, provided that the sequence $(\beta_n(x))_{n\geq 1}$ converges to zero uniformly for $x \in X$. The first assertion of the theorem follows now from Theorem 2.

The inequality (2), obtained in the proof of Theorem 2, implies

$$\| L_n(f) - f \| \leqslant K_f \cdot \| \alpha_n \|,$$

for all $f \in$ Lip (X). By the inequality (4), $\| \alpha_n \| \leqslant \sqrt{\| \beta_n \|}$, so that

$$\| L_n(f) - f \| \leqslant K_f \cdot \sqrt{\| \beta_n \|},$$

which ends the proof of the theorem.

Now, let H be a real pre-Hilbert space with inner product $\langle .,. \rangle$. For $t \in H$ fixed let the function $e_t : H \to \mathbf{R}$ be defined by $e_t(x) = \langle x, t \rangle$, $x \in H$, and let $e : H \to \mathbf{R}$ be defined by $e(x) = \langle x, x \rangle = \| x \|^2$, $x \in H$.

THEOREM 4. *Let X be a compact subset of the pre-Hilbert space H and let $(L_n)_{n\geq 1}$ be a sequence of Markov operators acting on $C(X)$. If $(L_n(e))_{n\geq 1}$ converges uniformly to e and the sequence $(L_n(e_x;x))_{n\geq 1}$ converges to $e(x)$, uniformly for $x \in X$, then the sequence $(L_n(f))_{n\geq 1}$ converges uniformly to f, for all $f \in C(X)$.*

Proof. We have

$$\| t - x \|^2 = e(t) - 2e_x(t) + e(x).$$

Considering x fixed and t variable, applying the operator L_n to this equality and evaluating at the point $t = x$, one obtains:

$$\begin{aligned}\beta_n(x) = L_n(\| \cdot - x \|^2; x) &= L_n(e;x) - 2L_n(e_x;x) + e(x) = \\ &= L_n(e;x) - e(x) - 2[L_n(e_x;x) - e(x)].\end{aligned} \tag{5}$$

Taking into account the hypotheses of the theorem it follows that the sequence $(\beta_n(x))_{n\geq 1}$ converges to zero uniformly for $x \in X$, and Theorem 4 follows from Theorem 3.

Remark. If $f \in \mathrm{Lip}\,(X)$ then

$$\| L_n(f) - f \| \leqslant K_f \sqrt{\| a_n - 2b_n \|}, \tag{6}$$

where $a_n(x) = L_n(e\,;\,x) - e(x)$ and $b_n(x) = L_n(e_x\,;\,x) - e(x)$, for $x \in X$ and $n = 1, 2, \ldots$.

COROLLARY 1. (Korovkin's theorem). *If $(L_n)_{n\geq 1}$ is a sequence of Markov operators acting on $C[a, b]$ such that $L_n(e_1) \xrightarrow{u} e_1$, $L_n(e_2) \xrightarrow{u} e_2$, where $e_1(x) = x$ and $e_2(x) = x^2$, $x \in [a, b]$, then $(L_n(f))_{n\geq 1}$ converges uniformly to f, for all $f \in C[a, b]$.*

Proof. In Theorem 4 take $H = \mathbf{R}$, $X = [a, b]$ and the inner product be the usual multiplication in $\mathbf{R}$, $\langle x, y\rangle = x \cdot y$. Then $e(x) = x^2 = e_2(x)$, $e_t(x) = t \cdot x = t \cdot e_1(x)$ and $L_n(e_t\,;\,x) = t \cdot L_n(e_1\,;\,x)$. By hypothesis $L_n(e) = L_n(e_2) \xrightarrow{u} e_2 = e$. The corrollary will follow from Theorem 4 if we show that $L_n(e_x\,;\,x) \to x^2$ uniformly for $x \in [a, b]$. By hypothesis $L_n(e_1) \xrightarrow{u} e_1$, so that if $\varepsilon > 0$ is given, there exists $n_\varepsilon \in N$ such that $|L_n(e_1\,;\,x) - x| < \varepsilon/M$ for all $n \geqslant n_\varepsilon$ and all $x \in [a, b]$, where $M = \max\,(|a|, |b|)$. Consequently $|L_n(e_2\,;\,x) - tx| = |t| \cdot |L_n(e_1\,;\,x) - x| < \varepsilon$, for all $n \geqslant n_\varepsilon$ and all x and t in $[a, b]$. In particular for $t = x$, one obtains $|L_n(e_x;\,x) - x^2| < \varepsilon$, for all $n \geqslant n_\varepsilon$ and all $x \in [a, b]$, which shows that the sequence $(L_n(e_x\,;\,x))_{n\geq 1}$ converges to $e(x)$, uniformly for $x \in [a, b]$. The corollary is proved.

If $L_n = B_n$, where B_n denotes the Bernstein polynomial operator defined by

$$B_n(f\,;\,x) = \sum_{k=0}^{n} \binom{n}{k} (1 - x)^{n-k} x^k f\left(\frac{k}{n}\right),\ x \in [0, 1],\ f \in C[0, 1],$$

then

$$B_n(e_1\,;\,x) = e_1(x) \text{ and } B_n(e_2\,;\,x) = e_2(x) + \frac{e_1(x) - e_2(x)}{2}.$$

The delimitation (6) gives

$$\| B_n(f) - f \| \leqslant K_f \cdot \frac{1}{2\cdot\sqrt{n}},$$

for all $f \in \mathrm{Lip}\,[0, 1]$.

Applications. 1°. In the Hilbert space $\mathbf{R}^m$ consider a compact convex set X with nonvoid interior. For $f \in C^1(X)$ (the space of all real-valued continuously differentiable functions on X) and $u \in \mathbf{R}^m$, denote by $\nabla f(u)$ the gradient vector of f at the point u, i.e.

$$\nabla f(u) = \left(\frac{\partial f}{\partial x_1}(u), \ldots, \frac{\partial f}{\partial x_m}(u)\right).$$

LEMMA 2. *If $f \in C^1(X)$ then $f \in \mathrm{Lip}\,(X)$ and* $K_f = \max\limits_{u \in X} \| \nabla f(u) \|$.

Proof. Let $x, y \in X$, $x \neq y$. The mean value theorem implies the existence of a point $u \in X$ (which is an internal point of the segment joining x and y) such that

$$f(x) - f(y) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(u) \cdot (x_i - y_i) = \langle \nabla f(u),\ x - y \rangle.$$

Applying now the Schwarz inequality, one obtains

$$|f(x) - f(y)| = \|\nabla f(u)\| \cdot \|x - y\| \leqslant (\max_{u \in X} \|\nabla f(u)\|) \cdot \|x - y\|.$$

COROLLARY 2. *If $(L_n)_{n \geqslant 1}$ is a sequence of Markov operators acting on $C(X)$, where X is a compact convex subset of $\mathbf{R}^m$ with non-void interior, then*

$$|L_n(f; x) - f(x)| \leqslant \max_{u \in X} \|\nabla f(u)\| \cdot \sqrt{L_n(\|\cdot - x\|^2; x)}, \tag{7}$$

for all $f \in C^1(X)$.

Proof. By Lemma 2, the inequality (7) is a consequence of the inequality (3) (see also (1) for the definition of β_n).

2°. *The Bernstein—Lototsky—Schnabl operator.* If X is a compact space, S is a subspace of $C(X)$ such that $e_0 \in S$ (remind that $e_0(x) = 1,\ x \in X$), L is a Markov operator on $C(X)$ and x is a point in X then a Radon probability measure ν_x on X is called an $L(S)$ — *representing measure* for x if

$$L(f; x) = \int_X f \, d\nu_x,$$

for all $f \in S$.

Suppose from now on that X is a compact convex subset of a pre-Hilbert space H and let $A(X)$ be the space of all real-valued continuous affine functions defined on X. Let $V = (V_n)_{n \geqslant 1}$ be a sequence of Markov operators on $C(X)$ and let $M(V) = \{\nu_{x,n} : n \geqslant 1,\ x \in X\}$ be a set of Radon probability measures on X such that $\nu_{x,n}$ is an $V_n(A(X))$ — representing measure for x, for all $x \in X$ and $n = 1, 2, \ldots$. Suppose further that the family $M(V)$ is such that the functions $E_n : X \to \mathbf{R}$ defined by $E_n(x) = \nu_{x,n}(e)$, $x \in X$, are continuous for all $n = 1, 2, \ldots$. Let $P = (p_{n,j})_{n,j \geqslant 1}$ be a lower triangular stochastic matrix i.e. an infinite matrix such that $p_{n,j} \geqslant 0$ for all $n, j \geqslant 1$,

$$\sum_{j=1}^{n} p_{n,j} = 1 \quad \text{and} \quad p_{n,j} = 0$$

for all $j > n$. If $\rho = (\rho_n)_{n \geqslant 1}$ is a sequence of continuous functions $\rho_n : X \to [0, 1]$, $n = 1, 2, \ldots$, define

$$\nu_{x,n,\rho}^{(V)} = \rho_n(x)\, \nu_{x,n} + (1 - \rho_n(x))\, \varepsilon_x \circ V_n,$$

where ε_t denotes the Dirac measure on X centered at $t \in X$. Let also $\pi_{n,P} : X^n \to X$ be defined by

$$\pi_{n,P}(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} p_{n,j} \cdot x_j,$$

for $(x_1, x_2, \ldots, x_n) \in X^n$.

The Bernstein—Lototski—Schnabl operator with respect to $M(V)$, P and ρ is defined by

$$B_n(f;x)=\int_{X^n} f\circ\pi_{n,P}\,d\bigotimes_{1\le j\le n}\nu^{(V)}_{x,j},$$

for all $x\in X$ and all $f\in C(X)$. It follows that B_n is a Markov operator on $C(X)$ and straightforward calculations (see [5]) show that

$$B_n(e_y;x)=\sum_{j=1}^{n}p_{n,j}\cdot\rho_j(x)\langle x,y\rangle+\sum_{j=1}^{n}p_{n,j}\cdot (1-\rho_j(x))\langle x,y\rangle=\langle x,y\rangle=e_y(x), \tag{8}$$

for all $x,y\in X$ and

$$B_n(e;x)=\sum_{j=1}^{n}p^2_{n,j}\,[\rho_j(x)\,\nu_{x,j}(e)+(1-\rho_j(x))\cdot V_j(e;x)]+\left(1-\sum_{j=1}^{n}p^2_{n,j}\right)e(x), \tag{9}$$

for all $x\in X$. Here, the functions $e_y, e: X\to\mathbf{R}$ are defined as above by $e_y(x)=\langle x,y\rangle$ and $e(x)=\langle x,x\rangle$, $x\in X$.

As a consequence of the general convergence theorems one obtains the following result:

THEOREM 5. *If*

$$\lim_{n\to\infty}\sum_{j=1}^{n}p^2_{n,j}\,\rho_j(x)\cdot(\nu_{x,j}(e)-V_j(e;x))+\sum_{j=1}^{n}p^2_{n,j}(V_j(e;x)-e(x))=0, \tag{10}$$

uniformly for $x\in X$, *then* $(B_n(f))_{n\ge1}$ *converges uniformly to* f, *for all* $f\in C(X)$. *If* $f\in \mathrm{Lip}\,(X)$ *then furthermore:*

$$\|B_n(f)-f\|\le K_f\left\|\sum_{j=1}^{n}p^2_{n,j}\,\rho_j\,E_j-V_j(e)+\sum_{j=1}^{n}p^2_{n,j}(V_j(e)-e)\right\|^{1/2}, \tag{11}$$

where $E_j(x)=\nu_{x,j}(e)$, $x\in X$.

Proof. The convergence result follows from Theorem 4. Indeed, by (9),

$$B_n(e;x)-e(x)=\sum_{j=1}^{n}p^2_{n,j}[\rho_j(x)\,\nu_{x,j}(e)-V_j(e;x)]+\sum_{j=1}^{n}p^2_{n,j}(V_j(e;x)-e(x)),$$

and the condition (10) of the theorem implies that $(B_n(e;x))_{n\ge1}$ converges to $e(x)$, uniformly for $x\in X$. The equality (8) gives for $y=x$, $B_n(e_x;x)=\langle x,x\rangle=e(x)$, for all $x\in X$ and $n=1,2,\ldots$. The hypotheses of Theorem 4 are all fulfilled and, consequently, the sequence $(B_n(f))_{n\ge1}$ converges uniformly to f, for all $f\in C(X)$.

The equalities (5) (for $L_n = B_n$), (8) and (9) give:

$$\beta_n(x) = B_n(e;x) - 2B_n(e_x;x) + e(x) =$$
$$= \sum_{j=1}^{n} p_{n,j}^2[\rho_j(x)\,\nu_{x,j}(e) + (1-\rho_j(x))\cdot V_j(e;x)] +$$
$$+ \left(1 - \sum_{j=1}^{n} p_{n,j}^2\right)e(x) - 2e(x) + e(x) =$$
$$= \sum_{j=1}^{n} p_{n,j}^2[\rho_j(x)\,\nu_{x,j}(e) + (1-\rho_j(x))\,V_j(e;x)] - \sum_{j=1}^{n} p_{n,j}^2\, e(x) =$$
$$= \sum_{j=1}^{n} p_{n,j}^2\,\rho_j(x)\,[E_j(x) - V_j(e;x)] + \sum_{j=1}^{n} p_{n,j}^2[V_j(e;x) - e(x)].$$

It follows that the delimitation (11) is a consequence of the delimitation (3) from Theorem 3.

COROLLARY 3. *If* $\nu_{x,j} = \nu_x$ *for* $j = 1, 2, \ldots$ *and* $\rho_j(x) = 1$, $x \in X$, $j = 1, 2, \ldots$ *then the condition* (10) *from Theorem* 5 *reduces to*

$$\lim_{n\to\infty} \sum_{j=1}^{n} p_{n,j}^2 = 0, \tag{12}$$

and the delimitation (11) *takes the form*

$$\| B_n(f) - f \| \leqslant K_f \left(\sum_{j=1}^{n} p_{n,j}^2 \| E - e \| \right)^{1/2}, \tag{13}$$

where $E : X \to \mathbf{R}$ *is defined by* $E(x) = \nu_x(e)$, $x \in X$.

Proof. The first assertion of the Corollary follows from the following delimitation for the expression involved in the condition (10), for $\rho_j(x) = 1$ and $\nu_{x,j}(e) = \nu_x(e) = E(x)$:

$$\left| \sum_{j=1}^{n} p_{n,j}^2[E(x) - V_j(e;x)] + \sum_{j=1}^{n} p_{n,j}^2[V_j(e;x) - e(x)] \right| \leqslant$$
$$\leqslant \sum_{j=1}^{n} p_{n,j}^2 \| E - V_j(e) \| + \sum_{j=1}^{n} p_{n,j}^2 \| V_j(e) - e \| \leqslant$$
$$\leqslant (\| E \| + 2 \| V_j(e) \| + \| e \|) \sum_{j=1}^{n} p_{n,j}^2 \leqslant (\| E \| + 3) \sum_{j=1}^{n} p_{n,j}^2.$$

The delimitation (13) follows immediately from (11), taking $\rho_j = 1$ and $E_j = E$.

REFERENCES

1. De Vore, Ronald A., *The Approximation of Continuous Functions by Positive Linear Operators*, Springer-Verlag, Berlin—Heidelberg—New York, 1972.
2. Mamedov, R. G., *On the order of the approximation of differentiable functions by linear operators* (Russian), Doklady SSSR 128, (1959), 674—676.
3. Shisha, O., Mond, B., *The degree of convergence of sequences of linear positive operators*, Proc. Nat. Acad. Sci. USA 60, (1968), 1196—1200.
4. Nishishiraho, T., *The degree of convergence of positive linear operators*, Tôhoku Math. Journal 29 (1977), 81—89.
5. Nishishiraho, T., *Convergence of positive linear approximation processes*, Tôhoku Math. Journal 35 (1983), 441—458.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXIV, 2, 1989

ON APPROXIMATIVELY SOLVING CERTAIN OPERATOR EQUATIONS

M. BALÁZS* and G. GOLDNER*

Received: March 2, 1989

REZUMAT. — Asupra rezolvării aproximative a unor ecuaţii operatoriale. În lucrare este prezentată o metodă convergentă pentru rezolvarea aproximativă a ecuaţiilor operatoriale în spaţii Banach. În algoritm nu intervine derivata, iar domeniul şi codomeniul operatorului care intervine pot fi din spaţii distincte.

0. In order to solve the operator equations in Banach spaces, A. S. Sergheev [4] generalized the well-known method of chords. This method presents the disadvantage of a small rate of convergence. For avoiding this short-coming some authors had studied the generalization of classical Steffensen's method ([1], [3], [5]), where without the using the derivative, the same rapidity as in the Newton method is obtained. But this method is applicable only to the self-mappings of Banach spaces.

The aim of the present paper is to give a quadratically method convergent, where the derivative is not used in algorithm and the domain and the range of the mapping can be in distinct sets, evidently with the studied operators having peculiar properties.

1. Let X be a Banach space, Y a normed space, the mapping $P: X \to Y$, and the equation

$$P(x) = 0. \tag{1.1}$$

We suppose that there exists $a \in \mathbf{R}$, $0 < a < 1$, such that for every x in a neighbourhood V of a point $x_0 \in X$ the inequality

$$\| P(ax) \| \leqslant a \| P(x) \| \tag{1.2}$$

holds. We remark that if $0 \in V$, then this inequality implies $P(0) = 0$. We show the existence of mappings satisfying this property.

Exemple 1. Let X be the set $\mathfrak{M}_n(\mathbf{R})$ of the square matrixes of n-th order defined on $\mathbf{R}$, $Y = \mathbf{R}$, and the mapping $P: X \to Y$ given for every $M \in \mathfrak{M}_n(\mathbf{R})$ by $P(M) = \det(M)$. We have $\| P(aM) \| = a^n |\det(M)| < a |\det(M)|$ for all M in X and a in $]0, 1[$.

Exemple 2. Let be $A \in \mathfrak{M}_n(\mathbf{R})$ the matrix of a given quadratic form. We consider the mapping $P: \mathbf{R}^n \to \mathbf{R}$ given for all x in X by $P(x) = \langle Ax, x \rangle$. Obviously $P(ax) = \langle A(ax), ax \rangle = a^2 \langle Ax, x \rangle$, and for every x in $\mathbf{R}^n$ and a in $]0, 1[$ we have $\| P(ax) \| = |a^2 \langle Ax, x \rangle| = a^2 |\langle Ax, x \rangle| < a \| P(x) \|$.

* University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania

Exemple 3. Let X be a normed space, Y a Hilbert space and the mapping $f: X \to Y$ with $f(0) = 0$. We define the mapping $P: X \to \mathbf{R}$ by the equality $P(x) = \sqrt{\langle f(x), f(x)\rangle} = \|f(x)\|$ for every x in X. If the inequality $\langle D^2 f(x)h^2, f(x)\rangle > 0$, then P is convex, hence $P(ax) = P(ax + (1-a)0) \leqslant aP(x) + (1 - a)P(0) = aP(x)$, and this implies for every $x \in X$ and a in $]0, 1[$ that $\| P(ax) \| \leqslant a \| P(x) \|$.

Exemple 4. Let be $P: \mathbf{R}\setminus\{0\} \to \mathbf{R}$,

$$P(x) = \prod_{n=1}^{\infty}\left(1 - \frac{\sqrt{2}}{2^{n-1}x}\right).$$

The infinite product is convergent, which is implied by the absolute convergence of the serie $\sum_{n\geqslant 1} \frac{\sqrt{2}}{2^{n-1}x}$ for every $x \in R\setminus\{0\}$:

$$\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{\frac{\sqrt{2}}{2^n x}}{\frac{\sqrt{2}}{2^{n-1}x}}\right| = \frac{1}{2} < 1, \quad x \in \mathbf{R},\ x \neq 0.$$

We have:

$$P\left(\frac{1}{2}x\right) = \left[\prod_{n=1}^{\infty}\left(1 - \frac{\sqrt{2}}{2^{n-1}x}\right)\right]\left(1 - \frac{2\sqrt{2}}{x}\right)$$

$$\left|P\left(\frac{1}{2}x\right)\right| = \left|1 - \frac{2\sqrt{2}}{x}\right| \cdot |P(x)|$$

$$\left|1 - \frac{2\sqrt{2}}{x}\right| \leqslant \frac{1}{2} \Leftrightarrow -\frac{1}{2} \leqslant 1 - \frac{2\sqrt{2}}{x} \leqslant \frac{1}{2} \Leftrightarrow \begin{cases} -\frac{3}{2} \leqslant -\frac{2\sqrt{2}}{x} \\ -\frac{2\sqrt{2}}{x} \leqslant -\frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} \frac{3x - 4\sqrt{2}}{x} \geqslant 0 \\ \frac{x - 4\sqrt{2}}{x} \leqslant 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x \in \left]-\infty, 0\right[\cup \left[\frac{4\sqrt{2}}{3}, +\infty\right[\\ x \in \left]0, 4\sqrt{2}\right] \end{cases} \Leftrightarrow x \in \left[\frac{4\sqrt{2}}{3}, 4\sqrt{2}\right].$$

Hence for $x_0 \in \left]\frac{4\sqrt{2}}{3}, 4\sqrt{2}\right[$, $x_0 \in V \subseteq \left]\frac{4\sqrt{2}}{3}, 4\sqrt{2}\right[$, and $a = \frac{1}{2}$ this mapping satisfies condition (1.2).

Remark. More generally, puting

$$P(x) = \prod_{n=1}^{\infty}\left(1 - \frac{x^* \lambda^{n-1}}{x}\right) \quad \text{with } x^* > 0$$

we have for all $\lambda \in]0, 1[$

$$P(x) = \left(1 - \frac{x^*}{x}\right)\left(1 - \frac{x^*}{x}\lambda\right)\left(1 - \frac{x^*}{x}\lambda^2\right)\cdots$$

$$P(\lambda x) = \left(1 - \frac{x^*}{\lambda x}\right)\left(1 - \frac{x^*}{x}\right)\left(1 - \frac{x^*}{x}\lambda\right)\cdots$$

$$\frac{|P(\lambda x)|}{|P(x)|} \leqslant \left|1 - \frac{x^*}{\lambda x}\right| \leqslant \lambda$$

$$\left|1 - \frac{x^*}{\lambda x}\right| \leqslant \lambda \Leftrightarrow -\lambda \leqslant 1 - \frac{x^*}{x\lambda} \leqslant \lambda \Leftrightarrow \begin{cases} 1 + \lambda \geqslant \dfrac{x^*}{x\lambda} \\ 1 - \lambda \leqslant \dfrac{x^*}{x\lambda} \end{cases} \Leftrightarrow \begin{cases} \dfrac{(1+\lambda)\lambda x - x^*}{x} \geqslant 0 \\ \dfrac{(1-\lambda)\lambda x - x^*}{x} \leqslant 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x \in]-\infty, 0[\cup \left[\dfrac{x^*}{\lambda(1+\lambda)}, +\infty\right[\\ x \in]0, \dfrac{x^*}{(1-\lambda)\lambda}\right] \end{cases} \Leftrightarrow x \in \left[\frac{x^*}{\lambda(1+\lambda)}, \frac{x^*}{(1-\lambda)\lambda}\right].$$

So $x_0 \in \left]\dfrac{x^*}{\lambda(1+\lambda)}, \dfrac{x^*}{(1-\lambda)\lambda}\right[$, and $x_0 \in V \subseteq \left]\dfrac{x^*}{\lambda(1+\lambda)}, \dfrac{x^*}{(1-\lambda)\lambda}\right[$, the inequality (1.2) holds for every $\lambda \in]0, 1[$.

2. Let X and Y be normed spaces and the mapping $P: X \to Y$, and let denote $\mathfrak{L}(X, Y)$ the set of the linear and continuous mappings of X in Y. The mapping $[u, v; P] \in \mathfrak{L}(X, Y)$ is a divided difference of first order of P in the distinct points $u, v \in X$ if $[u, v; P](u - v) = P(u) - P(v)$. We consider the symmetrical divided differences, i.e. $[u, v; P] = [v, u; P]$. For divided differences of the second and higher order the symmetry is supposed too. We use for divided difference of second order the equality $[u, v, w; P](u - v) = [u, w; P] - [v, w; P]$ (see e.g. [1], [2], [4]).

Supposing the continuity of the mapping P and the completeness of X, we prove the quadratical convergence of the sequence given by

$$x_{n+1} = x_n - [x_n, ax_n; P]^{-1}P(x_n) := x_n - \Gamma_n P(x_n), \quad n \in \mathbb{N} \tag{2.1}$$

(with x_0 in X given, $P(x_0) \neq 0$) to a solution of the equation (1.1).

THEOREM. *We suppose that there exist $x_0 \in X$, $a \in]0, 1[$ and the real, strictly positive constants B_0, η_0 and M such that the following conditions are satisfied:*

(i) *The mapping $[x_0, ax_0; P]^{-1} := \Gamma_0$ exists and $\|\Gamma_0\| \leqslant B_0$;*

(ii) *$\|P(x_0)\| \leqslant \eta_0$ and $\|P(ax)\| \leqslant a\|P(x)\|$ for every x in the closed sphere $S[x_0, r] = \{x \in X : \|x - x_0\| \leqslant r\}$, where $r = 2B_0\eta_0(1 + a)$;*

(iii) *for every u, v, w of $S[x_0, r]$ we have $\|[u, v, w; P]\| \leqslant M$;*

(iv) *$h_0 = B_0^2 M(1 + a)\eta_0 \leqslant \dfrac{1}{3}$.*

Then the equation (1.1) *has a solution* x^* *in* $S[x_0, r]$ *which is the limit of the sequence given by* (2.1), *the rapidity of convergence being given by the inequality*

$$\| x^* - x_n \| \leqslant 3B_0 \eta_0 \left(\frac{2}{3}\right)^{n+2} \cdot \left(\frac{3}{4}\right)^{2^n} \cdot (3h_0)^{2^n} \cdot s$$

where

$$s = \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k-1} \cdot \left(\frac{3}{4}\right)^{2^n \cdot (2^{k-1}-1)}.$$

Proof. By hypothesis (i) we can construct the approximation x_1, and by the conditions (i), (ii) and (iii), considering the formulas (2.1) we obtain for x_1:

$$\begin{aligned}
&||x_1 - x_0|| = ||\Gamma_0 P(x_0)|| \leqslant B_0\eta_0;\\
&||x_1 - ax_0|| = ||x_0 - ax_0 - \Gamma_0 P(x_0)|| = ||[x_0, ax_0; P](x_0 - ax_0) -\\
&- \Gamma_0 P(x_0)|| = ||\Gamma_0\{[x_0, ax_0; P](x_0 - ax_0) - P(x_0)\}|| =\\
&= ||\Gamma_0[P(x_0) - P(ax_0) - P(x_0)]|| \leqslant B_0||P(ax_0)|| \leqslant B_0 a\eta_0 < B_0\eta_0;\\
&||ax_1 - ax_0|| \leqslant aB_0\eta_0 < B_0\eta_0;\\
&||ax_1 - x_0|| = ||ax_0 - x_0 - a\Gamma_0 P(x_0)|| = ||\Gamma_0[x_0, ax_0; P](ax_0 - x_0) -\\
&- a\Gamma_0 P(x_0)|| = ||\Gamma_0\{P(ax_0) - P(x_0) - aP(x_0)\}|| \leqslant B_0\eta_0(1 + 2a);\\
&||x_0 - ax_0|| = ||\Gamma_0[x_0, ax_0; P](x_0 - ax_0)|| \leqslant B_0\eta_0(1 + a).
\end{aligned}$$

These inequalities show that ax_0, x_1, ax_1 belong to $S[x_0, r]$. The hypothesis (i)—(iv) are satisfied for x_1, when the constants B_0, η_0 are replaced by the B_1, η_1 which will be established.

a) Indeed we have

$$\begin{aligned}
&I - \Gamma_0[x_1, ax_1; P] = I - [x_0, ax_0; P]^{-1}[x_1, ax_1; P] =\\
&= [x_0, ax_0; P]^{-1}[x_0, ax_0; P] - [x_0, ax_0; P]^{-1}[x_1, ax_1; P] =\\
&= \Gamma_0\{[x_0, ax_0; P] - [x_1, ax_1; P]\} = \Gamma_0\{[x_0, ax_1; P] - [ax_0, x_1; P] +\\
&+ [ax_0, x_1; P] - [x_1, ax_1; P]\} = \Gamma_0\{[x_0, ax_0, x_1; P](x_0 - x_1) +\\
&+ [ax_0, x_1, ax_1; P](ax_0 - ax_1)\}
\end{aligned}$$

It results:

$$||I - \Gamma_0[x_1, ax_1; P]|| \leqslant B_0^2 M(1 + a)\eta_0 = h_0 \leqslant \frac{1}{3}$$

and so we obtain

$$||\Gamma_0[x_1, ax_1; P]^{-1}|| \leqslant \frac{1}{1 - h_0} \leqslant \frac{3}{2}$$

Using the equality

$$\{\Gamma_0[x_1, ax_1; P]\}^{-1}\Gamma_0 = [x_1, ax_1; P]^{-1} = \Gamma_1$$

we obtain that the mapping Γ_1 exists and

$$||\Gamma_1|| \leqslant \frac{B_0}{1 - h_0} = B_1 \leqslant \frac{3}{2} B_0 \quad (B_0 < B_1)$$

b) The definition of the diveded differences and the reccurent formula (2.1) give for $n = 1$

$$[x_0, ax_0, x_1; P](x_1 - ax_0)(x_1 - x_0) = \{[x_0, x_1; P] - [x_0, ax_0; P]\}(x_1 - x_0) = $$
$$= [x_0, x_1; P](x_1 - x_0) - [x_0, ax_0; P](x_1 - x_0) = P(x_1) - P(x_0) + P(x_0) = P(x_1)$$

and it results that

$$||P(x_1)|| = ||[x_0, ax_0, x_1; P](x_1 - ax_0)(x_1 - x_0)|| \leqslant$$
$$\leqslant M||x_1 - ax_0|| \cdot ||x_1 - x_0|| \leqslant MB_0^2 a\eta_0^2 = B_0^2 M\eta_0 a\eta_0 < h_0\eta_0 = \eta_1 \leqslant \frac{1}{3}\eta_0$$

c) We have

$$h_1 = B_1^2 M(1 + a)\eta_1 = \frac{B_0^2 M}{(1 - h_0)^2}(1 + a)\, h_0\eta_0 = \frac{h_0^2}{(1 - h_0)^2} \leqslant \left(\frac{3}{2}\right)^2 h_0^2 = \frac{1}{4} < \frac{1}{3}$$

Using the result a) and formula (2.1) we can define the approximation x_2, which by the results a)—c) has the following properties:

$$||x_2 - x_1|| = ||\Gamma_1 P(x_1)|| \leqslant ||\Gamma_1|| \cdot ||P(x_1)|| \leqslant B_1\eta_1 = \frac{B_0}{1 - h_0} h_0\eta_0 \leqslant \frac{1}{2} B_0\eta_0;$$

$$||x_2 - ax_1|| = ||x_1 - ax_1 - \Gamma_1 P(x_1)|| = ||\Gamma_1\{[x_1, ax_1; P](x_1 - ax_1) -$$
$$- P(x_1)\}|| = ||\Gamma_1[P(x_1) - P(ax_1) - P(x_1)]|| \leqslant B_1||P(ax_1)|| \leqslant B_1 a\eta_1 \leqslant$$
$$\leqslant \frac{1}{2} B_0\eta_0;$$

$$||ax_2 - ax_1|| = a||x_2 - x_1|| \leqslant aB_1\eta_1 \leqslant \frac{1}{2} B_0\eta_0 a < \frac{1}{2} B_0\eta_0;$$

$$||ax_2 - x_1|| = ||ax_1 - x_1 - a\Gamma_1 P(x_1)|| = ||\Gamma_1[x_1, ax_1; P](ax_1 - x_1) -$$
$$- a\Gamma_1 P(x_1)|| \leqslant B_1||P(ax_1) - P(x_1) - aP(x_1)|| \leqslant B_1\eta_1(1 + 2a) \leqslant$$
$$\leqslant \frac{1}{2} B_0\eta_0(1 + 2a) < \frac{1}{2} B_0\eta_0 + B_0\eta_0 a;$$

$$||x_1 - ax_1|| = ||\Gamma_1[x_1, ax_1; P](x_1 - ax_1)|| \leqslant B_1\eta_1(1 + a) \leqslant \frac{1}{2} B_0\eta_0(1 + a)$$

These inequalities show that the points ax_1, x_2, ax_2 belong to the sphere $S[x_0, r]$. Indeed we have

$$||x_0 - ax_1|| \leqslant B_0\eta_0(1 + 2a);$$

$$||x_0 - x_2|| \leqslant ||x_0 - x_1|| + ||x_1 - x_2|| \leqslant B_0\eta_0\left(1 + \frac{1}{2}\right) < 2B_0\eta_0;$$

$$||x_0 - ax_2|| \leqslant ||x_0 - ax_1|| + ||ax_1 - ax_2|| \leqslant B_0\eta_0 + \frac{1}{2} B_0\eta_0 + 2B_0\eta_0 a =$$
$$= \frac{3}{2} B_0\eta_0 + 2B_0\eta_0 a < 2B_0\eta_0(1 + a),$$

Replacing the constants B_1, η_1 with $B_2 = \frac{B_1}{1-h_1} \leqslant \frac{3}{2}$, and $\eta_2 = h_1\eta_1$ the conditions (i)—(iv) of the Theorem are verified for x_2. We have

$$I - \Gamma_1[x_2, ax_2; P] = I - [x_1, ax_1; P]^{-1}[x_2, ax_2; P] =$$
$$= \Gamma_1\{[x_1, ax_1; P] - [ax_1, x_2; P] + [ax_1, x_2; P] + [x_2, ax_2; P] =$$
$$= \Gamma_1\{[x_1, ax_1, x_2; P](x_1 - x_2) + [ax_1, x_2, ax_2; P](ax_1 - ax_2]\}$$

which implies

$$||I - \Gamma_1[x_2, ax_2; P]|| \leqslant B_1^2 M(1 + a)\eta_1 = h_1 \leqslant \frac{1}{3}$$

and so

$$||\Gamma_1[x_2, ax_2; P]^{-1}|| \leqslant \frac{1}{1-h_1} \leqslant \frac{3}{2}$$

By the evident equality

$$(\Gamma_1[x_2, ax_2; P])^{-1}\Gamma_1 = [x_2, ax_2; P]^{-1} = \Gamma_2$$

it results the existence of the mapping Γ_2 and

$$||\Gamma_2|| \leqslant \frac{B_1}{1-h_1} \qquad B_2 \leqslant \frac{3}{2} B_1 \leqslant \left(\frac{3}{2}\right)^2 B_0 \qquad (B_1 < B_2)$$

As above we obtain

$$P(x_2) = [x_1, ax_1, x_2; P](x_2 - ax_1)(x_2 - x_1)$$

and it results

$$||P(x_2)|| = ||[x_1, ax_1, x_2; P](x_2 - ax_1)(x_2 - x_1)|| \leqslant$$
$$\leqslant M||x_2 - ax_1|| \cdot ||x_2 - x_1|| \leqslant MB_1^2 a\eta_1^2 = B_1^2 M\eta_1 a\eta_1 < h_1\eta_1 =$$
$$= \eta_2 < \frac{1}{3}\eta_1 \leqslant \left(\frac{1}{3}\right)^2 \eta_0$$

and

$$B_2\eta_2 \leqslant \frac{1}{2^2} B_0 \eta_0$$

We have

$$h_2 = B_2^2 M(1 + a)\eta_2 = \frac{B_1^2}{(1-h_1)^2} M(1 + a) h_1 \eta_1 =$$
$$= \frac{h_1^2}{(1-h_1)^2} = \left(\frac{3}{2}\right)^2 \cdot h_1^2 \leqslant \left(\frac{3}{2}\right)^2\left(\frac{1}{3}\right)^2 = \frac{1}{4} < \frac{1}{3}.$$

By induction we can prove that the point x_n given by (2.1) exists for every $n \in \mathbf{N}$ and it satisfies the inequalities

$$||x_n - x_{n-1}|| \leqslant B_{n-1}\eta_{n-1} \leqslant \frac{1}{2^{n-1}} B_0 \eta_0;$$
$$||x_n - ax_{n-1}|| \leqslant aB_{n-1}\eta_{n-1} < \frac{1}{2^{n-1}} B_0\eta_0; \quad ||ax_n - ax_{n-1}|| < \frac{1}{2^{n-1}} B_0\eta_0;$$
$$||ax_n - x_{n-1}|| \leqslant B_{n-1}\eta_{n-1}(1 + 2a) < \frac{1}{2^{n-1}} B_0\eta_0(1 + 2a)$$

Which show that ax_{n-1}, x_n and ax_n belong to $S[x_0, r]$. The hypothesis of the Theorem for x_n and ax_n are satisfied and the following relations hold:

$$
\begin{aligned}
&\|\Gamma_n\| \leqslant \frac{B_{n-1}}{1-h_{n-1}} = B_n \leqslant \left(\frac{3}{2}\right)^n \cdot B_0; \\
&\|P(x_n)\| \leqslant h_{n-1}\,\eta_{n-1} = \eta_n \leqslant \frac{\eta_0}{3^n}; \qquad (2.2)\\
&h_n = \frac{h_{n-1}^2}{(1-h_{n-1})^2} \leqslant \left(\frac{3}{2}\right)^2 \cdot h_{n-1}^2.
\end{aligned}
$$

By the relations (2.2) it results that

$$
\begin{aligned}
&h_n \leqslant \left(\frac{3}{2}\right)^{2(2^n-1)} \cdot h_0^{2^n}; \\
&\eta_n \leqslant 3\eta_0 \left(\frac{2}{3}\right)^{2(n+1)} \cdot \left(\frac{3}{4}\right)^{2^n} \cdot (3h_0)^{2^n-1}; \qquad (2.3)\\
&\|x_{n+1} - x_n\| \leqslant B_n\,\eta_n \leqslant \frac{1}{2^n} B_0\eta_0
\end{aligned}
$$

Using the properties of the norm by the relations (2.3) we obtain succesively

$$
\begin{aligned}
&\|x_{n+p} - x_n\| \leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + \ldots + \|x_{n+p-1} - x_{n+p}\| \leqslant \\
&\leqslant B_n\,\eta_n + B_{n+1}\,\eta_{n+1} + \ldots + B_{n+p-1}\,\eta_{n+p-1} \leqslant \\
&\leqslant 3B_0\eta_0 \left(\frac{2}{3}\right)^{n+2} \cdot \left(\frac{3}{4}\right)^{2^n} \cdot (3h_0)^{2^n-1}\left[1 + \frac{2}{3}\left(\frac{3}{4}\right)^{2^n} \cdot (3h_0)^{2^n} + \ldots \right.\\
&\left.\ldots + \left(\frac{2}{3}\right)^{p-1} \cdot \left(\frac{3}{4}\right)^{2^n(2^{p-1}-1)} \cdot (3h_0)^{2^n(2^{p-1}-1)}\right] < \\
&< 3B_0\eta_0 \left(\frac{2}{3}\right)^{n+2} \cdot \left(\frac{3}{4}\right)^{2^n} \cdot (3h_0)^{2^n} \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k-1} \cdot \left(\frac{3}{4}\right)^{2^n(2^{k-1}-1)}
\end{aligned}
$$

where the series

$$\sum_{k\geqslant 1} \left(\frac{2}{3}\right)^{k-1} \cdot \left(\frac{3}{4}\right)^{2^n(2^{k-1}-1)}$$

evidently converges, and its sum s is majorized by the sum of the series

$$\sum_{k\geqslant 1} \left(\frac{2}{3}\right)^{k-1} \cdot \left(\frac{3}{4}\right)^{2^{k+1}-4}.$$

For all $p \in \mathbf{N}$ we have hence

$$\|x_{n+p} - x_n\| \leqslant 3B_0\,\eta_0\, s \left(\frac{2}{3}\right)^{n+2} \cdot \left(\frac{3}{4}\right)^{2^n} \cdot (3h_0)^{2^n} \qquad (2.4)$$

and for $n \to \infty$ we have $||x_{n+p} - x_n|| \to 0$ independently of p, i.e. the sequence (x_n) is a Cauchy sequence. X being Banach space, it exists the point $x^* \in X$ with $\lim_{n\to\infty} x_n = x^*$. By (2.4) for $p \to \infty$ we obtain the rapidity of convergence. We can prove that x_n and x^* are in $S[x_0, r]$:

$$||x_0 - x_n|| \leqslant ||x_0 - x_1|| + ||x_1 - x_2|| + \ldots + ||x_{n-1} - x_n|| \leqslant$$
$$\leqslant B_0\eta_0 + \frac{1}{2} B_0\eta_0 + \frac{1}{2^2} B_0\eta_0 + \ldots + \frac{1}{2^{n-1}} B_0\eta_0 < 2B_0\eta_0$$
$$||x_0 - ax_n|| \leqslant ||x_0 - ax_1|| + ||ax_1 - ax_2|| + \ldots + ||ax_{n-1} - ax_n|| \leqslant$$
$$\leqslant B_0\eta_0(1 + 2a) + \frac{1}{2} B_0\eta_0 + \frac{1}{2^2} B_0\eta_0 + \ldots + \frac{1}{2^{n-1}} B_0\eta_0 \leqslant$$
$$= 2B_0\eta_0 + 2aB_0\eta_0 = 2B_0\eta_0(1 + a)$$

The divided differences $[x_n, ax_n; P]$ are bounded in $S[x_0, r]$:

$$[x_n, ax_n; P] = [x_n, ax_n; P] - [ax_n, x_{n-1}; P] + [ax_n, x_{n-1}; P] -$$
$$- [x_{n-1}, x_{n-2}; P] + [x_{n-1}, x_{n-2}; P] - [x_{n-2}, x_{n-3}; P] + \ldots$$
$$\ldots + [x_1, x_0; P] + [x_1, x_0; P] = [x_{n-1}, x_n, ax_n; P](x_n - x_{n-1}) +$$
$$+ [x_{n-2}, x_{n-1}, ax_n; P](ax_n - x_{n-2}) + [x_{n-3}, x_{n-2}, x_{n-1}; P](x_{n-1} - x_{n-3}) + \ldots$$
$$\ldots + [x_0, x_1, x_2; P](x_2 - x_0) + [x_1, x_0; P]$$
$$||[x_n, ax_n; P]|| \leqslant M(||x_n - x_{n-1}|| + a||x_n - x_{n-1}|| + ||ax_{n-1} - x_{n-2}|| +$$
$$+ ||x_{n-1} - x_{n-2}|| + ||x_{n-2} - x_{n-3}|| + \ldots + ||x_2 - x_1|| + ||x_1 - x_0||) +$$
$$+ ||[x_0, x_1; P]|| \leqslant M[2B_0\eta_0 + \frac{a}{2^{n-1}} B_0\eta_0 + \frac{1}{2^{n-2}} B_0\eta_0(1 + 2a)] +$$
$$+ ||[x_0, x_1; P]|| \leqslant K$$

Taking in account this fact, in (2.1) for $n \to \infty$ we obtain that x^* is a solution of equation (1.1).

Remarks. **1**. Condition (iii) of the Theorem can be replaced by (iii′) For every $u, v, w \in S[x_0, r]$ we have

$$||[u, v; P] - [u, w; P]|| \leqslant M||v - w||$$

without the change of the conclusions or of the proof.

2. The sequence $(\tilde{x}_n)$ given by the formula

$$\tilde{x}_{n+1} = ax_n - \Gamma_n P(ax_n)$$

is the same with the sequence given by (2.1). Indeed we have

$$x_{n+1} - \tilde{x}_{n+1} = x_n - ax_n - [\Gamma_n P(x_n) - \Gamma_n P(ax_n)] =$$
$$= \Gamma_n [x_n, ax_n; P](x_n - ax_n) - \Gamma_n [P(x_n) - P(ax_n)] =$$
$$= \Gamma_n [P(x_n) - P(ax_n) - P(x_n) + P(ax_n)] = 0$$

3. If $x^* \neq 0$ is a solution of equation (1.1), then $a^n x^*$ $(n \in \mathbf{N})$ are the solutions of this equation and $\lim_{n\to\infty} a^n x^* = 0$.

REFERENCES

1. B a l á z s, M., *Contributions to the Study of the Solving of the Equations in Banach Spaces* (Romanian), Doctor Thesis, Studia Univ. Babeş-Bolyai, Fasc. 2, pp. 89–90 (1970).
2. B a l á z s, M., G o l d n e r, G., *On Existence of Divided Differences in Linear Spaces*, Rev. d'Anal. Numér. Théor. Approx., **2**, pp. 59 (1973).
3. P ă v ă l o i u, I., *Sur la méthode de Steffensen pour la résolution des equations opérationelles non-linéaires*, Rev. Roum. des Math. Pure Appl., **1**, XIII, pp. 149–158 (1968).
4. S e r g h e e v, A. S., *About the method of chords* (Russian), Sibirsk Mat. J., **XI**. (2), pp. 282–289 (1961).
5. U l'm. S., *Algorithm of the Generalized Steffensen's Method* (Russian), Izv. Akad. Nauk Estonsk. S.S.R., **3**, pp. 435–443 (1965).

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXIV, 2, 1989

BASIC PROBLEMS OF THE METRIC FIXED POINT THEORY REVISITED (I)

IOAN A. RUS*

Received: February 12, 1989

REZUMAT. — Problemele de bază ale teoriei metrice a punctului fix revizitate (I). În 1979 în lucrarea [45] am formulat anumite probleme deschise, în legătură cu metoda aproximaţiilor succesive, probleme ce în opinia autorului sînt fundamentale în teoria metrică a punctului fix. În prezenta lucrare se reiau aceste probleme din perspectiva rezultatelor obţinute în perioada 1980—1988. Prima parte a lucrării se referă la aplicaţii univoce.

1. **Introduction.** In 1979 in the paper [45] we formulated some basic problems in the metric fixed point theory. The aim of this paper is to analyse these problems from the perspective of the results given in 1980—1988. In the first part of the paper we refer to single valued mappings.

Throughout the paper we follow terminologies and notations in [47].

We begin with some old results

2. **Results.** The following are the principal results of the metric fixed point theory in 1922—1979:

THEOREM 2.1. (Picard, Banach, Caccioppoli). *Let (X, d) be a complete metric space and $f: X \to X$ an a-contraction. Then*

(i) $$F_f = \{x^*\},$$

(ii) $$f^n(x_0) \to x^* \text{ as } n \to \infty.$$

THEOREM 2.2. (Bessaga). *Let X be a nonempty set and $f: X \to X$ a mapping such that*

$$F_f = F_{f^n} = \{x^*\},$$

for all $n \in \mathbb{N}$. Let $a \in]0,1[$. Then there exists a metric d on X such that

(i) *(X, d) is a complete metric space,*

(ii) *$f: (X, d) \to (X, d)$ is an a-contraction.*

THEOREM 2.3. (Janos). *Let (X, d) be a compact metric space and $f: X \to X$ a mapping. We suppose*

(1) $$f \text{ is continuous},$$

(2) $$\bigcap_{n\in\mathbb{N}} f^n(X) = \{x^*\}$$

* University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania

Then for each $a \in]0, 1[$ *there exists a metric* ρ *on* X *such that.*

(i) d *and* ρ *are topologic equivalent,*

(ii) $f: (X, \rho) \to (X, \rho)$ *is an a-contraction.*

THEOREM 2.4. (Meyers). *Let* (X, τ) *be a metrizable topological space and* $f: X \to X$ *a continuous mapping. We suppose*

(1) $F_f = \{x^*\}$,

(2) $f^n(x_0) \to x^*$ *as* $n \to \infty$, *for all* $x_0 \in X$,

(3) *there exists an open neighborhood* U *of* x^* *such that for any open neighborhood* V *of* x^* *there exists an* $n(V) > 0$ *such that* $f^n(U) \subset V$.

Then for each $a \in]0, 1[$ *there exists a metric* d *on* X *compatible with* τ *such that* $f: (X, d) \to (X, d)$ *is an a-contraction.*

THEOREM 2.5. (Banach). *Let* X *be a Banach space and* $f: X \to X$ *an a-contraction. Then* $1_X - f$ *is a homeomorphism.*

THEOREM 2.6. (Caristi). *Let* (X, d) *be a complete metric space and* $f: X \to X$ *such that for some lower semicontinuous* $\varphi: X \to \mathbf{R}^+$

$$d(x, f(x)) \leqslant \varphi(x) - \varphi(f(x)),$$

for all $x \in X$.

Then f *has a fixed point. If* f *is continuous then* $(f^n(x_0))_{n \in \mathbf{N}}$ *converges to a fixed point of* f.

THEOREM 2.7. (Dugundji). *Let* (X, d) *be a complete metric space and* $f: X \to X$ *such that* $d(f(x)), f(y)) \leqslant d(x, y)$, *for all* $x, y \in X$. *If the mapping* $g: X \times X \to \mathbf{R}_+$ $(x, y) \mapsto d(x, y) - d(f(x), f(y))$ *is positive definite mod* $\Delta(X)$, *then* f *has a unique fixed point* x^* *and* $f^n(x_0) \to x^*$ *as* $n \to \infty$, *for all* $x_0 \in X$.

From these results the following definitions arise.

3. **Definitions.** DEFINITION 3.1. (see [47], [49], [50]). Let (X, d) be a metric space. A mapping $f: X \to X$ is a (*strict*) *Picard mapping* if there exists $x^* \in X$ such that $F_f = \{x^*\}$ and $(f^n(x_0))_{n \in \mathbf{N}}$ converges to x^* (uniformly) for all $x_0 \in X$.

DEFINITION 3.2. (see [47], [49], [50]). Let (X, d) be a metric space. A mapping $f: X \to X$ is (*strict*) *weakly Picard mapping* if $(f^n(x_0))_{n \in \mathbf{N}}$ converges (uniformly) for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of f.

DEFINITION 3.3. (see [47], [49], [50]). Let X be a nonempty set. A mapping $f: X \to X$ is a *Bessaga mapping* if there exists $x^* \in X$ such that $F_{f^n} = \{x^*\}$, for all $n \in \mathbf{N}$.

DEFINITION 3.4. (see [47], [49], [50]). Let X be a nonempty set. A mapping $f: X \to X$ is a *Janos mapping* if $\bigcap_{n \in \mathbf{N}} f^n(X) = \{x^*\}$.

DEFINITION 3.5. (see [47], [49], [50].)Let X be a Banach space. A mapping $f: X \to X$ is a *Banach mapping* if

(i) f is a Picard mapping,

(ii) $1_X - f$ is a homeomorphism.

The following problems are the basic problems of the metric fixed point theory.

4. **Problem 1.** Problem 1 a. Let (X, d) be a complete metric space. Which generalized contractions $f: X \to X$ are

(a) (strict) Picard mappings?

(b) (strict) weakly Picard mappings?

(c) Bessaga mappings?

(d) Janos mappings?

For some results for the Problem 1 a see: [9], [12], [18], [19], [23], [27], [28], [32], [36], [44], [45], [47], [54].

Problem 1 b. Let (X, d) be a bounded complete metric space. Which generalized contraction $f: X \to X$ are

(a) (strict) Picard mappings?

(b) (strict) weakly Picard mappings?

(c) Bessaga mappings?

(d) Janos mappings?

The main results for this problem are:

THEOREM 4.1. (see [46]). *Let (X, d) be a bounded complete metric space, $f: X \to X$ and $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$. We suppose that*

(i) *φ is a comparison function (i.e., φ is increasing and $\varphi^n(t) \to 0$ as $n \to \infty$ for all $t > 0$),*

(ii) *$\delta(f(A)) \leqslant \varphi(\delta(A))$, for all $A \subset X$ such that $f(A) \subset A$ (i.e., f is a (δ, φ)-contraction).*

Then

(a) *f is a Picard mapping,*

(b) *f is a Janos mapping.*

Proof. (a) Let $A_1 := f(X), \ldots, A_{n+1} = f(A_n), \ldots$ We have $A_{n+1} \subset A_n$, $A_n = \bar{A}_n$, and $f(A_n) \subset A_n$, $n \in \mathbb{N}$.

On the other hand

$\delta(A_{n+1}) = \delta(f(A_n)) = \delta(f(A_n)) \leqslant \varphi(\delta(A_n)) \leqslant \ldots \leqslant \varphi^n(\delta(X)) \to 0$ as $n \to \infty$. Thus we have $A_\infty := \bigcap A_n = \{x^*\}$ and $f(A_\infty) \subset A_\infty$, i.e., $F_f = \{x^*\}$.

Let $x_0 \in X$ and $B_n := \{f^n(x_0), f^{n+1}(x_0), \ldots, x^*\}$.

From $f(B_n) = B_{n+1} \subset B_n$, and $\delta(B_{n+1}) \leqslant \varphi(\delta(B_n))$, it follows that $\delta(B_n) \to 0$ as $n \to \infty$, i.e., $f^n(x_0 \to x^*$ as $n \to \infty$.

(b) $x^* \in \bigcap_{n \in \mathbb{N}} f^n(X) \subset A_\infty = \{x^*\}$.

THEOREM 4.2. *Let (X, d) be a bounded complete metric space and $f: X \to X$ a mapping with the property that there exists $n \in \mathbb{N}^*$ such that f^n is a (δ, φ) — contraction. Then*

(a) *f is a Picard mapping,*

(b) *f is a Janos mapping.*

Proof. (a) + (b). From the Theorem 4.1. we have

$$F_{f^n} = \{x^*\} \text{ and } \delta(f^{nk}(X)) \to 0 \text{ as } k \to \infty.$$

On the other hand

$$X \supset f(X) \supset \cdots \supset f^n(X) \supset \cdots \supset f^{n_k}(X) \supset \cdots$$

Thus we have (a) + (b).

For other results for the Problem 1 b see [11], [18], [36], [44], [45], [46], [47], [58].

Problem 1 c. Let (X, d) be a compact metric space. Which generalized contractions $f: X \to X$ are

(a) (strict) Picard mappings?
(b) (strict) weakly Picard mappings?
(c) Bessaga mappings?
(d) Janos mappings?

The main result for the Problem 1 c is

THEOREM 4.3. *Let (X, d) be a compact metric space and $f: X \to X$ a continuous mapping such that $\delta(f(A)) < \delta(A)$, for all $A \subset X$, for which $f(A) \subset A$ and $\delta(A) \neq 0$.*

Then:

(i) *f is a strict Picard mapping,*

(ii) *f is a Janos mapping.*

Proof. (i) + (ii). We remark that card $F_f \leqslant 1$. On the other hand by a lemma of Martelli [30] there exists a nonempty closed set $Y \subset X$ such that $f(Y) = Y$. This implies $\delta(Y) = 0$. Thus $Y = \{x^*\}$, i.e., $F_f = \{x^*\}$. Let $A_\infty : \bigcap_{n \in N} f^n(X)$. From the continuity of f we have (see [29]):

(a) $f(A_\infty) = A_\infty$. This implies $\{\delta(A_\infty) = 0$, i. e., $A_\infty = \{x^*\}$.

(b) $\delta(f^n(X)) \to 0$ as $n \to \infty$.

From (a) and (b) we have (i) and (ii).

For other results for the Problem 1 c. see: [18], [29], [36], [44], [45], [47], [48].

5. **Problem 2.** Let (X, d) be a complete metric space, (Y, τ) a topological space and $f: X \times Y \to X$ a continuous mapping. We suppose that $F_{f(.,y)} = \{x_y^*\}$. Which are the generalized contraction $f(.,y): X \to X$, $y \in Y$, such that the mapping $P: Y \to X$, $y \mapsto x_y^*$ is continuous?

Let us consider some results for this problem.

THEOREM 5.1. (see [47]). *Let (X, d) be a complete metric space, (Y, τ) a topological space and let $f: X \times Y \to X$ be a continuous mapping for which there exists a strict comparison function φ such that*

$$d(f(x_1, y), f(x_2, y)) \leqslant \varphi(d(x_1, x_2))$$

for all $x_1, x_2 \in X$ and $y \in Y$.

Then the mapping P is continuous.

THEOREM 5.2. (see [45]). *Let $f: X \times Y \to X$ be a continuous mapping. If there exist $a, b \in \mathbf{R}_+$, $a + 2b < 1$, such that*

$d(f(x_1, y), f(x_2, y)) \leqslant ad(x_1, x_2) + bd(x_1, f(x_1, y)) + d(x_2, f(x_2, y))$, for all $x_1, x_2 \in X$ and $y \in Y$, then the mapping P is continuous.

For other results for the Problem 2 see [17], [18], [27], [36], [45], [47] [48].

6. **Problem 3.** Problem 3a. Let (X, d) be a complete metric space and $f, f_n : X \to X$, $n \in \mathbf{N}$, be such that:

(i) f_n converges uniformly to f,

(ii) $F_f = \{x^*\}$,

(iii) $F_{f^n} \neq \emptyset$.

Let $x_n^* \in F_{f^n}$. Which are the generalized contractions f such that $(x_n^*)_{n \in \mathbf{N}}$ converges to x^*?

We have

THEOREM 6.1. (see [47]). *If f is a strict φ-contraction then $x_n^* \to x^*$ as $n \to \infty$*

DEFINITION 6.1. Let (X, d) be a complete metric space and $f, f_n : X \to X$, $n \in \mathbf{N}$. The sequence $(f_n)_{n \in \mathbf{N}}$ is *asymptotical uniform convergent* to f if for each $\varepsilon > 0$ there exists $n_0(\varepsilon)$, $m_0(\varepsilon) \in \mathbf{N}$ such that $d(f_n^m(x), f^m(x)) < \varepsilon$, for all $n \geqslant n_0(\varepsilon)$, $m \geqslant m_0(\varepsilon)$, and all $x \in X$.

If the sequence $(f_n)_{n \in \mathbf{N}}$ converges asimptotical uniform to f then we denote

$$f_n \overset{a}{\rightrightarrows} f$$

Example 6.1. If f is a contraction and $f_n \rightrightarrows f$, the $f_n \overset{a}{\rightrightarrows} f$.

Example 6.2. Let (X, d) be a bounded complete metric space. If f is a Kannan mapping and $f_n \rightrightarrows f$, then $f_n \overset{a}{\rightrightarrows} f$.

Now we can give.

THEOREM 6.2. ([49]). *Let (X, d) be a complete metric space and $f, f_n : X \to X$, $n \in \mathbf{N}$. We suppose that*

(i) *f is a Picard mapping $(F_f = \{x^*\})$*

(ii) $f_n \overset{a}{\rightrightarrows} f$

(iii) *$F_{f_n} \neq \emptyset (x_n^* \in F_{f_n})$ for all $n \in \mathbf{N}$.*

Then $x_n^* \to x^*$ as $n \to \infty$.

Proof. From (ii) and (iii) we have

$$d(x_n^*, x^*) = d(f_n^m(x_n^*), f^m(x^*)) \leqslant d(f_n^m(x_n^*), f^m(x_n^*)) + d(f^m(x_n^*), f^m(x^*)).$$

Let $\varepsilon > 0$, and $n_0(\varepsilon)$, $m_0(\varepsilon)$ be such that

$$d(f_n^m(x_n^*), f^m(x_n^*)) < \frac{\varepsilon}{2}, \text{ for all } n \geqslant n_0(\varepsilon),\ m \geqslant m_0(\varepsilon).$$

From (i) we have that for each $n \geqslant n_0(\varepsilon)$ there exists $m(n;\varepsilon) \in \mathbf{N}$ such that

$$d(f^{m(n;\varepsilon)}(x_n^*), f^{m(n;\varepsilon)}(x^*)) < \frac{\varepsilon}{2}.$$

The proof is complete.

Problem 3 b. Let (X, d) be a complete metric space and $f, f_n : X \to X$, $n \in \mathbb{N}$. We suppose that:

(i) f, f_n are generalized contractions,
(ii) $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f_0,
(iii) $F_{f_n} = \{x_n^*\}$, $F_f = \{x^*\}$, $n \in \mathbb{N}$

Does $x_n^* \to x^*$ as $n \to \infty$?

Problem 3 c. Let (X, d) and (X, d_n), $n \in \mathbb{N}$ be complete metric spaces. We suppose that

(i) $f_n : (X, d_n) \to (X, d_n)$, $f : (x, d) \to (X, d)$ are generalized contractions,
(ii) $d_n \rightrightarrows d$,
(iii) $f_n \xrightarrow{d} f$
(iv) $F_{f_n} = \{x_n^*\}$, $F_f = \{x^*\}$, $n \in \mathbb{N}$

Does $x_n^* \xrightarrow{d} x^*$ as $n \to \infty$?

For some results for these problems see: [9], [16], [18], [32], [36], [46], [47].

7. **Problem 4.** Let $f : (X, d) \to (X, d)$ be a Picard mapping. Let $g : X \to X$ be a mapping which approximates the mapping f. More precisely we assume that

$d(f(x), g(x)) \leqslant \eta$, with given $\eta \in \mathbb{R}_+$, for all $x \in X$. Let $y_n = g^n(x_0)$. The problem is to give an estimate for $d(x^*, y_n)$.

For some results for the Problem 4 see [45] and [47].

8. **Problem 5.** Let (X, d) be a complete metric space and $f : X \to X$ a Lipschitz Beşsaga mapping. A problem of practical interest is „to construct" an equivalent metric ρ on X such that $f : (X, \rho) \to (X, \rho)$ is a Picard mapping.

For some practical results for this probleme see: [8], [6], [18], [24], [36], [45], [47], [49].

9. **Problem 6.** Let (X, d) be a complete metric space, $Y \subset X$ a closed subset of X and $f : Y \to X$ a mapping which satisfies a "boundary condition" (i.e., $f(\partial Y) \cap Y$; Leray-Schauder (if X is a Banach space and $Y = B(0; r)$), etc.). Which are the generalized contractions $f : Y \to X$ such that $F_f \neq \emptyset$.

Now some results for this problem.

THEOREM 9.1. (see [43] and [52]). *Let X be a Banach space and $f : B(0; r) \to X$ a weakly inward contraction mapping. Then $F_f = \{x^*\}$.*

THEOREM 9.2. (see [18]). *Let X be a Banach space and $f : B(0; r) \to X$ be a contraction such that $f(x) = -f(-x)$ for all $x \in \partial B(0; r)$. Then f has a unique fixed point.*

For other results for the Problem 6 see: [1], [2], [18], [43], [52].

10. **Problem 7.** Let $(X, \|\cdot\|)$ be a Banach space. Which generalized contractions $f : X \to X$ are Banach mappings?

For some results given by Danes, Kolomy, Dugundji and Granas, Ha, Rus, Mureşan see: [47], [18], [22], [31].

11. **Problem 8.** Determine all generalized contraction pair f, g such that
(i) $F_f \cap F_g = \{x^*\}$.
(ii) $F_f = F_g = \{x^*\}$.
(iii) f and g are Bessaga mappings.
(iv) f and g are Picard mappings
(v) f and g are Janos mappings.

We have

THEOREM 10.1. (see [45]). *Let (X, d) be a complete metric space and $f, g: X \to X$ two mapping for which there exists $a \in \left]0, \frac{1}{2}\right[$ such that*

$$d(f(x), g(y)) \leqslant a\,[d(x, f(x)) + d(y, g(y))],$$

for all $x, y \in X$. Then
(i) $F_f = F_g = \{x^*\}$.
(ii) *f and g are Picard mappings.*

For other results for this problem see [14], [15], [33], [34], [38], [40], [45], [47], [53].

12. **Remarks.** *Remark 12.1.* Three technique of proofs are now available in the metric fixed point theory:

(i) the method of successive approximations,
(ii) the method of maximal elements. [4], [11], [17], [18], [37], [48], [55], [56].
(iii) the method of Dugundji [18], [47].

In this paper we refer to the method of successive approximations.

Remark 12.2. For some results and problems in the coincidence point theory see: [18], [47], [7].

Remark 12.3. For some results and problems in the fixed point theory of nonexpansive mappings see: [43], [21], [26].

Remark 12.4. For the fixed point theory of expanding (dilatation) mappings see: [20], [14], [25], [41], [45].

Remark 12.5. For the fixed point theory in generalized metric spaces see: [47], [45], [27], [37], [42].

Remark 12.6. For some application of the metric fixed point theory see: [3], [5], [6], [8], [9], [10], [18], [39], [47].

REFERENCES

1. Algethmy A., Sehgal V. M.; *A convergence theorem for nonlinear contraction and fixed points*, Math. Japonica, 27 (1982), 113–116.
2. Assad N. A., *On some nonself mappings in Banach spaces*, Math. Japonica, 33 (1988), 501–515.
3. Bassanini P., Galaverni M., *Contractioni multiple; sistemi iperbolici, e problema del laser*, Atti Sem. Mat. Fis. Univ. Modena, 31 (1982), 32–50.

4. Bollenbacher A., Hics T. L., *A fixed point theorem revisited*, Proceed. A.M.S., 102 (1988), No. 4, 898–900.
5. Brown R. F. (ed.), *Fixed point theory and its applications*, A.M.S. Providence, 1988.
6. Bushell P. J., *The Cayley-Hilbert metric and positive operators*, Linear algebra and its applications, 84 (1986), 271–280).
7. Cho Y. J., Singh S. L., *A coincidence theorem and fixed point theorems in Saks spaces*, Kobe J. Math., 3 (1986), 1–6.
8. Corduneanu C., *Bielecki's method in the theory of integral equations*, Ann Univ. Marie Curie-Skladowska, 38 (1984), Nr. 2, 23–40.
9. Czerwick S., *Fixed point theorems and special solutions of functional equations*, Univ. Slaski, Katowice, 1980.
10. D'Apuzzo L., *φ – contractive mappings and the functional equation $y = f(x, y)$* (Italian), Ricerche Mat., 34 (1985), nr. 1, 5–25.
11. Deimling K., *Nonlinear functional analysis*, Springer-Verlag, Berlin, 1985.
12. Delbosco D., *An unified approach for the contractive mappings*, Tnânabha, 16 (1986), 1–11.
13. Dhage B. C., Dhobale V. V., *A note on expasive mappings*, J. Math. Phys. Sci 20 (1986) no. 4, 349–352.
14. Dien N. H., *Some common fixed point theorems in metric spaces*, Preprint 3–1985 (Univ. of Cluj-Napoca), 15–36.
15. Ding X. P., *On common fixed point of a pair of contractive mappings- some open problems* (Chinese), J. Math. Res Exposition 4 (1984), no. 2, 25–30.
16. Dominguez B. T., *Generic existence of nonempty compact set of fixed points*, J. Math. Anal. and Appl., 90 (1982), 421–430.
17. Downing D. J., *Nonlinear mapping theory and the continuation of solutions of equations under homotopy*, Nonlinear Anal., 4 (1980), 27–30.
18. Dugundji J., Granas A., *Fixed point theory*, Warszawa, 1982.
19. Fam K. A., *On global convergence of iterative processes* (Russian), Mat. Zametki 29 (1981), 923–928.
20. Gillespie A. A., Williams B. B., *Fixed point theorem for expanding maps*, The Univ. of Texas at Arlington, Tech. Report, No. 154, 1981.
21. Goebel K., Kirk. W. A. *Iteration process for nonexpansive mappings*, Contemporary Math., Nr. 21, 1983, 115–123.
22. Ha C.–W., *Obn the Banach's homeomorphism theorem*, Arch. Math., 37 (1981), 450–453.
23. Hegedüs M., Szilágyi T., *Equivalent conditions and a new fixed point theorem in the theory of contractive type mappings*, Math. Japonica, 25 (1980), nr. 1, 147–157.
24. Heikkilä S., Roach G. F., *On equivalent norms and the contractions mapping principle*, Nonlinear Anal, 8 (1984), 1241–1252.
25. Khan M. A., Khan M. S., Sessa S., *Some theorems on expansion mappings and their fixed points*, Demonstratio Math., 19 (1986) Nr. 3, 673–683.
26. Kuczumow T., *Nonexpansive mappings and isometries of Hilbert n-balls with hyperbolic metrics*, Univ. Marie Curie-Skladowskiej, Lublin, 1987.
27. Kwapisz M., *General inequalities and fixed point problems*, General Inequalities 2, 341 – 353, Birkhäuser Verlag Basel, 1980.
28. Leader S., *Two convergence principles with aplpications to fixed points in metric spaces*, Nonlinear Anal, 6 (1982), 531–538.
29. Leader S., *Uniformly contractive fixed points in compact metric space*, Proceed. A.M.S., 86 (1982), 153–158.
30. Martelli M., *A lemma on maps of a compact topological space and an application to fixed point theory*, Atti Accad. Lincei, 49 (1970), 242–243.
31. Mureşan S. A., *On a fixed point problem*, Univ. of Cluj-Napoca, Preprint Nr. 3, 1983, 138–141.
32. Mureşan V., *Basic problem for Maia-Perov fixed point theorem*, Univ. of Cluj-Napoca, Preprint Nr. 3, 1988, 43–48.
33. Nhan D. T., *Pair of nonlinear contraction mappings. Common fixed points*, Studia Univ. Babeş-Bolyai, Mathematica, 26 (1981), No. 1, 34–51.
34. Nhan D. T., *Some metrical fixed point theorems*, Mathematica, 24 (1982), 85–98.
35. Palczewski B., Miczko A., *Converses of generalized Banach contraction principle and remarks on mappings with a contractive iterate at the point*, Zbornik Radova, 17 (1987), 71–91.
36. Papini P. L., *Una bibliografia Italiana sui punti fissi* (1970–1983), Univ. Bologna, 1983.

37. Park S., *Some applications of Ekeland's variational principle to fixed point theory*, Approximation theory and application, 159—172, Res. Notes in Math. 133, Pitman, Boston, 1985.
38. Park S., *Some general theorems on common fixed points*, Kobe J. Math., 4 (1987), 141—145.
39. Petruşel A., *Fredholm-Volterra integral equation and Maia's theorem*, Univ. of Cluj-Napoca, Preprint Nr. 3, 1988, 79—82.
40. Popa V., *Common fixed points of commuting mappings*, Mathematica, 29 (1987), No. 1, 67—71.
41. Popa V., *Fixed point theorem for expansion mappings*, Univ. of Cluj-Napoca. Preprint Nr. 3, 1987, 25—30.
42. Precup. R. *A fixed point theorem of Maia type in syntopogenous spages*, Univ. of. Cluj-Mapoca, Preprint. Nr. 3, 1988, 49—70.
43. Reich S., *Some problems and results in fixed point theory*, Contemporary Math., 21 (1983), 179—187.
44. Rhoades B. E., *Contractive definitions revisited*, Contemporary Math., 21 (1983), 189—205.
45. Rus I. A., *Metrical fixed point theorems*, Univ. of Cluj-Napoca, Cluj-Napoca, 1979.
46. Rus I. A., *Generalized φ-contractions*, Mathématica, 24 (1982), 175—178.
47. Rus I. A., *Generalized contractions*, Univ. of Cluj-Napoca, Preprint nr. 3, 1983, 1—130.
48. Rus. I. A., *Seminar on fixed point theory : fifteen years of activity*, Univ. of Cluj-Napoca, Preprint nr. 3, 1984, 1—19.
49. Rus. I. A., *Bessaga mappings*, Proceed. Colloq. Approx. Opt., Cluj-Napoca, 1984, 165—172.
50. Rus I. A., *Picard mappings : results and problems*, Itin. Sem. Funct. Eq. Approx. Conv., Cluj-Napoca, 1987, 55—64.
51. Rus I. A., *Picard mappings. I.*, Studia Univ. Babeş-Bolyai, Math. 33 (1988), Nr. 2, 70—73.
52. Rus I. A., *Fixed points of retractible mappings*, Univ. of Cluj-Napoca, Preprint Nr. 2, 1988, 163—166.
53. Tan D. H., Nhan D. T., *Common fixed points of two mappings of contractive type*, Acta Math., Vietnamica 5 (1980), Nr. 1, 150—160.
54. Tasković M. R., *A characterization of the class of contraction type mappings*, Kobe J. Math., 2 (1985), 45—55.
55. Turinici M., *Maximal elements in a class of order complete metric spaces*, Math. Japonia 25 (1980), 511—517.
56. Turinici M., *Maximal elements in ordered metric spaces*, Anal. Univ. Bucureşti, Matematica, 31 (1982), 89—94.
57. Ver Eecke P., *Applications du calcul differentiel*, Press Univ. de France, 1985.
58. Walter W., *Remarks on a paper by F. Browder about contraction*, Nonlinear Analysis, 5 (1981), Nr. 1, 21—25.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXIV, 2, 1989

ON THE SEMI-SYMMETRIC AND QUARTER-SYMMETRIC CONNECTIONS

P. ENGHIŞ*

Received: March 20, 1989

REZUMAT. — Asupra conexiunilor semi-simetrice şi sfert-simetrice. În lucrare se stabilesc mai multe proprietăţi în spaţii cu conexiune semi-simetrică sau sfert-simetrică.

Let A_n be a space with affine connection. Noting by Γ^i_{jk} the components of the affine connection, in a coordinate system and by $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$ the components of the torsion tensor, one say [7] that the connection Γ is semi-symmetric if there exists a covariant vectorial field S_i so that:

$$T^i_{jk} = S_j\delta^i_k - S_k\delta^i_j \tag{1}$$

and quarter-symmetric if there exists a covariant vectorial field S_i and a tensorial field t^i_j so that:

$$T^i_{jk} = S_j t^i_k - S_k t^i_j \tag{2}$$

if we apply in (1) a contraction in i and j it follows

$$T_k = T^i_{ik} = S_k(1 - n) \tag{3}$$

where T_k is the torsion vector or the Vrînceanu vector.

Taking count of (3) in (1) it follows

$$T^i_{jk} = \frac{1}{n-1}\,[T_k\,\delta^i_j - T_j\,\delta^i_k] \tag{4}$$

PROPOSITION 1. *In a space $A_n (n > 1)$ with semi-symmetric connection, the torsion is determined by the torsion vector.*

PROPOSITION 2. *In a space A_n with semi-symmetric connection, the torsion beeing determined by the torsion vector as in the case of the spaces A_2, the rezults from the volume I of "Lessons of Differential Geometry" one extend oner these spaces too.*

If the space A_n, is of recurrent torsion

$$T^i_{jk,r} = \Psi_r\, T^i_{jk} \tag{5}$$

where comma denotes the covariant derivation with respect to Γ, [2], by contractingin i and j it foollows

$$T_{k,r} = \Psi_r\, T_k \tag{6}$$

that is the Vranceanu's vector is also recurrent with the same vector.

* University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania

Let us notice now [5] that it the spaces with semi-symmetric connection the converse is also true, because from (4), through covariant derivation and taking count of (6) and (4), it follows (5).

Let us suppose now that we have $T_{i,j} = T_{j,i}$, so Γ connection is a E-connection [4] or an Enghiş-Stavre connection [8]. In this case S_i is gradient [7] and from (3) it follows:

Remark. In a space A_n with semi-symmetric connection, the Vrânceanu's vector is gradient.

It is known [7] that in a space A_n with semi-symmetric E-connection take place the relations:

$$T^h_{ij,k} + T^h_{jk,i} + T^h_{ki,j} = 0 \tag{7}$$

In the case of T-recurrency the relations (7) become:

$$\Psi_k T^h_{ij} + \Psi_i T^h_{jk} + \Psi_j T^h_{ki} = 0 \tag{8}$$

which are Walker-like relations [10]. From (8) contracting in h and k and taking (4) into account, it follows:

$$(2 - n)\, \Psi_h T^h_{ij} = 0 \tag{9}$$

PROPOSITION 3. *In a T-recurrent A_n space, $(n > 2)$, with a semi-symmetric E-connection, the vector of T-recurrency is solution of the system (9).*

It is known [3] that the T-recurrents A_n spaces, with T-recurrency vector the gradient of a function Ψ are metrizable, the metric tensor beeing $g_{ij} = e^{-2\Psi} T_{ij}$, where $T_{ij} = T^s_{ik} T^k_{sj}$.

PROPOSITION 4. *In the spaces A_n, T-recurrents with semi-symmetric E-connection, one attach to Γ a metric connection between them, taking place the relation:*

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + S_j \delta^i_k - g_{jk} S^i, \quad S^i = g^{ih} S_h \tag{10}$$

and for these spaces are valid the results obtained together with P. Stavre [6].

From the quarter-symmetric spaces (2) let us consider those for which the tensorial field t^i_j is covariantly constant and S_k is gradient. For these spaces, it follows immediately the relations (7). So:

PROPOSITION 5. *In a space A_n with quarter-symmetric connection in which the tensorial field t^i_j is covariantly constant and the vectorial field S_k, gradient relations (7) take place.*

For these spaces, if in addition the field S_k is recurrent of vector Ψ_r, it is known [5] that they are T-recurrents of vector Ψ_r. Teking count of this in (7) it follows (8) and therefore.

PROPOSITION 6. *In a space A_n, T-recurrent with quarter-symmetric connection in which the tensorial field t^i_j is covariantly constant and S_k gradient, relations (8) take place.*

In (8) contracting in i and j we have:

$$(T^i_{kr} + T_k \delta^i_r - T_r \delta^i_k)\, \Psi_i = 0 \tag{11}$$

PROPOSITION 7. *In a space A_n, T-recurrent with quarter-symmetric connection in which the tensorial field t_j^i is covariantly constant and S_k gradient, the T-recurrency vector is solution of the system (11).*

PROPOSITION 8. *A necessary condition for a space A_n with quarter-symmetric connection with the tensorial field t_j^i covariantly constant and S_k gradient to be T-recurrent is that*

$$\text{rank} \| T^i_{jk} - T_k \delta^i_j - T_j \delta^i_k \| < n.$$

Remark. The spaces A_n T-recurrents with T-recurrency gradient beeing metrizable, it follows that for those spaces with quarter-symmetric connection with t_j^i covariantly constant and S_k gradient, proposition 4 is true.

REFERENCES

1. Eisenhart, L. P., *Non Riemannian Geometry*, Am. Math. Ser. Coll. Publ. VIII (1927).
2. Enghiş, P., *Sur les espaces à connexion affine avec torsion récurrente*, Studia Univ. Babeş-Bolyai ser. Math. Mec. fasc. 1 (1973), 13–16.
3. Enghiş, P., *On the metrizability of the spaces A_n T-recurrents*, Studia Univ. Babeş-Bolyai ser. Math. (1976), 11–14
4. Enghiş, P., *Semi-symmetric E-connections*, Studia Univ. Cluj-Napoca Math. XXIX–1984, 66–69.
5. Enghiş, P., Lup, I., *On the T-recurrency of some semi-symmetric and quarter-symmetric connections*, Studia Univ. Babeş-Bolyai Math. XXX 1985, 24–26.
6. Enghiş, P., Stavre, P., *Problèmes de récurrence pour des connexions semi-symmetriques metriques*, Studia Univ. Cluj-Napoca XXXI, (1) (1986) p. 6–14.
7. Golab, S., *On semi-symmetric and quarter-symmetric linear connections*, Tensor M.S. 29, 1975, 249–254.
8. Stavre, P., *On the canonical form of a connection*, Studii şi cercetări mat. 1, 1970, 73–80.
9. Vrânceanu, G., *Lessons of differential Geometry*, vol. 1, Ed. Acad. R.P.R. 1952.
10. Walker, A. G., *On Ruse's spaces of recurrent curvature*, Proc. London Math. Soc. 2, 1950, 36–64.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXIV, 2, 1989

ON A METHOD TO OBTAIN THE GENERAL CONNECTION TRANSFORMATION

P. STAVRE*

Received: December 28, 1987

REZUMAT. — Asupra unei metode pentru obţinerea transformării conexe generale. În această lucrare se dă o clasificare a transformărilor conexe aplicînd teoria punctelor fixe ale aplicaţiei f_Ω.

1. The map f_Ω. Let M be a C^∞ — differentiable, real n — dimensional manifold, $(\mathfrak{U}, \varphi) = (\mathfrak{U}, \varphi(x) = x^i)$ a local chart on M and $g(g_{ij})$ a tensor field of type (0,2) symmetric and nondegenerate. We consider the Obata's operators

$$\Omega_{sj}^{ir} = \frac{1}{2}(\delta_s^r \delta_j^r - g_{sj} g^{ir}); \quad \overset{*}{\Omega}{}_{sj}^{ir} = \frac{1}{2}(\delta_s^i \delta_j^r + g_{sj} g^{ir}). \tag{1}$$

DEFINITION (1.1). The map $f_\Omega : \mathcal{T}_2^1(M) \to \mathcal{T}_2^1(M)$ given by

$$f_\Omega : X_{jk}^i \to Y_{jk}^i = f_\Omega(X_{jk}^i) \overset{def.}{=\!=} \Omega_{sj}^{ir} X_{rk}^s; \quad \forall X \in \mathcal{T}_2^1(M), \tag{2}$$

is called a Ω-*map*.

If

$$\bar{\mathcal{T}}_2^1(M) \overset{def.}{=\!=} \{A_{jk}^i | A_{jk}^i \in \mathcal{T}_2^1(M);\ \overset{*}{\Omega}{}_{sj}^{ir} A_{rk}^s = 0\} \tag{3}$$

we have:

PROPOSITION (1.1). *The set of all fixed points of f_Ω is given by $\bar{\mathcal{T}}_2^1(M)$.*

Example (1.1). If $\tau_2^1(M)$ is the set of the tensor fields $\tau \in \mathcal{T}_2^1(M)$, with properties $\tau_{jk}^i = -\tau_{kj}^i$ and

$$\underset{1}{Y}{}_{jk}^i = \tau_{jk}^i + g^{ir} g_{sj} \tau_{kr}^s + g^{ir} g_{sk} \tau_{jr}^s \tag{3}$$

then $f_\Omega(Y) = Y \in \bar{\mathcal{T}}_2^1(M)$.

Example (1.2). If D is a linear connection on M and

$$\underset{2}{Y}{}_{jk}^i = \frac{1}{2} g^{ia}(g_{ak/j} - g_{jk/a}) \tag{4}$$

where (/) denotes the covariant derivative with respect to D, then $\underset{2}{Y}$ is a fixed point of f_Ω.

* *University of Craiova, 1100 Craiova, Romania*

Example (1.3). If $\omega = \tau_k dx^k$ is a $1-$form, on M, and

$$\underset{3}{Y}{}^i_{jk} = \delta^i_k \tau_j - g_{jk}\tau^i ; \quad \tau^i = g^{ia}\tau_a \tag{5}$$

then $f_\Omega(\underset{3}{Y}) = \underset{3}{Y}(\underset{3}{Y}$ is a fixed point of $f_\Omega)$

Evidently, the map $\varphi : \tau^1_2(M) \to \bar{\mathcal{T}}^1_2(M)$ given by

$$\varphi : \tau^i_{jk} \to \varphi\,(\tau^i_{jk}) = \underset{1}{Y}{}^i_{jk} \tag{6}$$

is a bijective map.

DEFINITION (1.2). If $X, Y \in \mathcal{T}^1_2(M)$, X is Ω *— equivalent with* Y (or $X \overset{\Omega}{\sim} Y$) iff $f_\Omega(X) = f_\Omega(Y)$.

This relation is a equivalence relation. Let $\mathcal{T}^1_2(M)/\overset{\Omega}{\sim}$ be set of all equivalence class, and $\theta_A = \{X^i_{jk} \mid X \in \mathcal{T}^1_2(M) ;\ X^i_{jk} = A^i_{jk} + \mathring{\Omega}^{ir}_{sj} Y^s_{rk}\}$ where $Y \in \mathcal{T}^1_2(M)$, A is fixed $(A \in \bar{\mathcal{T}}^1_2(M))$. If $A = 0$ we have

$$Q_0 = \{X^i_{jk} \mid X \in \mathcal{T}^1_2(M) ;\ X^i_{jk} = \mathring{\Omega}^{ir}_{sj} Y^s_{rk}\}\ \forall Y \in \mathcal{T}^1_2(M).$$

If $A_1 \neq A_2$ then $Q_{A_1} \neq Q_{A_2}$ and $Q_{A_1} \cap Q_{A_2} = \Phi$. We have;

PROPOSITION (1.2). *The sets* $\mathcal{T}^1_2(M)/\overset{\Omega}{\sim}$ *and* $\bar{\mathcal{T}}^1_2(M)$ *are equivalent* $(\tau^1_2(M) \leftrightarrow \leftrightarrow \bar{\mathcal{T}}^1_2(M))$ *by map*

$$\bar{h}_\Omega : A \to Q_A ;\ \forall A \in \bar{\mathcal{T}}^1_2(M). \tag{7}$$

We obtain: $\tau^1_2(M) \overset{\varphi}{\leftrightarrow} \bar{\mathcal{T}}^1_2(M) \overset{\bar{h}_\Omega}{\leftrightarrow} \mathcal{T}^1_2(M)/\overset{\Omega}{\sim}$, and $g_\Omega \overset{\text{def.}}{=} \bar{h}_\Omega \cdot \varphi : \tau^1_2(M) \leftrightarrow \mathcal{T}^1_2(M)/\overset{\Omega}{\sim}$.
From (3) we have

$$\tau^i_{jk} = \frac{1}{2}\,(\underset{1}{Y}{}^i_{jk} - \underset{1}{Y}{}^i_{kj}). \tag{8}$$

We have:

THEOREM (1.1). *If* $\mathcal{U}^i_{jk} \in \tau^1_2(M)$ *is given, then the tensorial system*

$$Y^i_{jk} - Y^i_{kj} = \mathcal{U}^i_{jk} \tag{9}$$

has a unique solution in $\bar{\mathcal{T}}^1_2(M)$; *it is* $\underset{0}{Y} = \frac{1}{2}\varphi(\mathcal{U})$ *or.*

$$\underset{0}{Y}{}^i_{jk} = \frac{1}{2}\,(\mathcal{U}^i_{jk} + g^{ia} g_{bj} \mathcal{U}^b_{ka} + g^{ia} g_{bk} \mathcal{U}^b_{ja}) \tag{10}$$

THEOREM (1.2). *If* $\tau^i_{jk} \in \tau^1_2(M)$ *is given then the tensorial system:*

$$\Omega^{ir}_{sj} X^s_{rk} - \Omega^{ir}_{sk} X^s_{rj} = 2\tau^i_{jk}, \tag{11}$$

has a unique solution in $\bar{\mathcal{T}}^1_2(M)$; *it is* $X = A = \varphi(\tau)$. *The general solution is a class* Q_A, *where* $A = \varphi(\tau)$ *is given by* (3)

We have:

$$f_\Omega(\tau) = \tau \Leftrightarrow g_{sj}\tau^s_{bk} + g_{sk}\tau^s_{bj} = 0; \quad \forall \tau \in \tau^1_2(M) \tag{12}$$

THEOREM (1.3). *The general solution of system (9) is given by* $Y^i_{jk} = \tau^i_{jk} + V^i_{jk}$ *where* $\tau^i_{jk} = \frac{1}{2}\mathfrak{A}^i_{jk}$ *and* V^i_{jk} *is an arbitrary symmetric tensor* $(V^i_{jk} = V^i_{kj})$.

If $V \neq 0$ then $f_\Omega(V) \neq V$. The solution Y^i_{jk} is solution in $\mathscr{T}^1_2(M)$ iff:

$$V^i_{jk} = g^{ia}g_{bj}\tau^b_{ka} + g^{ai}g_{bk}\tau^b_{ja}. \tag{13}$$

We obtain:

THEOREM (1.4). *The general solution of system (9) is,*

$$Y^i_{jk} = \underset{0}{Y^i_{jk}} + S^i_{jk} \tag{14}$$

where S^i_{jk} *is an arbitrary symmetric tensor* $(S^i_{kj} = S^i_{jk})$.

2. **The set** $\mathscr{T}$. Let $D = (\Gamma^i_{jk})$ be a linear connection with the torsion tensor $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$ and a general transformation $T: D \to \overline{D}$, given by.

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + Y^i_{jk}; \quad Y^i_{jk} \in \mathscr{T}^1_2(M). \tag{1}$$

DEFINITION (2.1). The transformation, $T: D \to \overline{D}$ is called a Ω-*transformation* if Y^i_{jk} is a fixed point of f_Ω.

DEINIFTION (2.1). The transformation T is called a *non* Ω — *transformation* if Y cannot a fixed point of f_Ω.

Let $\mathscr{T}_\Omega$ the set of Ω — transformations and $\mathscr{T}_{\text{non}\,\Omega}$ the set of non — Ω — transformations.

LEMMA S *(of Separation). We have* $\mathscr{T}_\Omega \cap \mathscr{T}_{\text{non}\,\Omega} = \Phi$, $\mathscr{T}_\Omega \cup \mathscr{T}_{\text{non}\,\Omega} = \mathscr{T}$, *where* $\mathscr{T}$ *is the general group of connection transformations.*

From (2.1) we have

$$Y^i_{jk} - Y^i_{kj} = 2\tau^i_{jk}, \tag{2}$$

where $\tau^i_{jk} = \frac{1}{2}(\overline{T}^i_{jk} - T^i_{jk}) \in \tau^1_2(M)$.

From theorem (1.1) and (2.2) we obtain:

THEOREM (2.1). A *necesary and sufficient condition that a transformation* $T \in \mathscr{T}$ *be a* Ω — *transformation* $(T \in \mathscr{T}_\Omega)$ *is given by*

$$Y = \varphi(\tau) \tag{3}$$

or

$$Y = f_\Omega(X) \tag{4}$$

where X *is arbitrary in* $Q_{A=\varphi(\tau)}$.

From theorem (1.4), and (2.2) ve have:

THEOREM (2.2). *The transformation* $T \in \mathscr{T}$ *is a non* Ω — *transformation* $(T \in \mathscr{T}_{\text{non}\,\Omega})$, *iff:*

$$Y = \varphi(\tau) + S, \ \forall S \neq 0, \ S^i_{jk} = S^i_{kj} \tag{5}$$

or,

$$Y = f_\Omega(X) + S, \quad \forall X \in Q_{A=\varphi(\tau)} \tag{6}$$

THEOREM (2.3) (of Separation). *Every $T \in \mathfrak{T}$ is given by*(2.1) *and* (2.3) *(respectively* (2.4)*) or is given by* (2.1) *and* (2.5) *(respectively* (2.6)*).*

THEOREM (2.4). *A necessary and sufficient condition that a transformation $T \in \mathfrak{T}$, be a T_Ω — transformation is that $g_{ij/k}$ be invariant.*

Proof. Since (where (//) denotes the covariant derivative with respect to $\bar{D}$)

$$g_{ij//k} = g_{ij/k} - (Y^s_{ik}\, g_{sj} + Y^s_{jk}\, g_{is}) \tag{7}$$

from $T \in \mathfrak{T}_\Omega$, we have; $Y = \varphi(\tau)$ and, $g_{ij//k} = g_{ij/k}$. (8)

Conversely, from (2.7) and (2.8) we have $Y = \varphi(\tau)$ and $Y \in \bar{\mathfrak{T}}^1_2(M)$. Hence $T \in \mathfrak{T}_\Omega$.

THEOREM 2.5. *A necessary and sufficient condition that a transformation T, be a Ω — transformation is* .

$$\bar{G}^i_{jk} \overset{\text{def}}{=} \frac{1}{2}\, g^{ia}(g_{aj//k} + g_{ak//j} - g_{jk//a}) = \frac{1}{2}\, g^{ia}(g_{aj/k} + g_{ak/j} - g_{jk/a}) \overset{def}{=} G^i_{jk} \tag{9}$$

(G^i_{jk} is an invariant of T)

Proof. From (2.8) we obtain (2.9). Conversely, from (1.1) and (2.5) we have;

$$g_{ij//k} = g_{ij/k} - S^i_{jk}\, g_{is} - S^s_{ik}\, g_{sj} \tag{10}$$

since $\varphi(\tau) \in \bar{\mathfrak{T}}^1_2(M)$. One obtains

$$\bar{G}^i_{jk} = G^i_{jk} - S^i_{jk}. \tag{11}$$

From (2.9), (2.11) we have $S^i_{jk} = 0$ and $Y = \varphi(\tau)$. We obtain: $T = T_\Omega$.

From (2.5) and (2.11) we have:

THEOREM (2.6). *A necessary and sufficient condition that a transformation T, be a non Ω — transformation is given by $g_{ij//k} \neq g_{ij/k}$.*

The general theorem. *Every $T \in \mathfrak{T}$ is*

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \varphi(\tau^i_{jk}) \tag{12}$$

iff $g_{ij//k} = g_{ij/k}$, or

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \varphi(\tau^i_{jk}) + G^i_{jk} - \bar{G}^i_{jk} \tag{13}$$

iff $g_{ij//k} \neq g_{ij/k}$ where $\tau = \frac{1}{2}(\bar{T} - T)$, T is the torsion of fixed connection D, G is given by

$$G^i_{jk} = \frac{1}{2}\, g^{ia}(g_{aj/k} + g_{ak/j} - g_{jk/a}), \tag{14}$$

$\bar{T} \in \tau^1_2(M)$ is arbitrary, and $\bar{G}^i_{jk}$ is an arbitrary simmetric tensor ($\bar{G}^i_{jk} = G^i_{kj}$).

For a fixed $(D, \bar{D})$, $\bar{T}$ is the torsion of $\bar{D}$ and $\bar{G}^i_{jk}$ is given by,

$$\bar{G}^i_{jk} = \frac{1}{2}\, g^{ia}(g_{aj//k} + g_{ak//j} - g_{jk//a}) \tag{15}$$

The classical results. a). The set of all metrical connection transformations ($g_{ij/|k} = 0$; $g_{ij//k} = 0$) is given by (2.12)

b) If $\bar{G} = 0$, from (2.13), we have the set of all metrical connections generated by D.

c) If $I^i_{jk} = T^i_{jk} - \frac{1}{n-1}(\delta^i_j T_k - \delta^i_k T_j)$ then the set of all transformations, $T \in \mathcal{T}$, characterized by Schouten's invariant $I = \bar{I}$ is given by

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_k \alpha_j - g_{jk} \alpha^i + G^i_{jk} - \bar{G}^i_{jk} \tag{16}$$

since

$$\varphi(\tau^i_k) = \delta^i_k \alpha_j - g_{jk} \alpha^i; \qquad \alpha^i = g^{ir} \alpha_r \tag{17}$$

d) The set of all conformal transformations is given by (2.13) where

$$G^i_{jk} - \bar{G}^i_{jk} = \delta^i_k \lambda_j + \delta^i_j \lambda_k - g_{jk} \lambda^i; \qquad \lambda^i = g^{ia} \lambda_a \tag{18}$$

since $g_{ij/k} = 2\tau_k g_{ij}$; $g_{ij//k} = 2\omega_k g_{ij}$ and $G - \bar{G}$ is given by (2.18) where $\lambda_k = = \tau_k - \omega : k$

e) If D is the Levi — Civita connection then the Weyl transformation is given by $\tau = 0$, $G = 0$ and $g_{ij//k} \neq g_{ij/k} = 0$, $-\bar{G}^i_{jk} = \delta^i_j \mu_k + \delta^i_k \mu_j$. We have $T \in \mathcal{T}_{\text{non}\,\Omega}$, given by,

$$\bar{\Gamma}^i_{jk} = \{^{\,i}_{jk}\} + \delta^i_j \mu_k + \delta^i_k \mu_j. \tag{19}$$

f) If $g_{ij//k} = g_{ij/k}$, $\bar{\Gamma}$ is a semi-symmetric connection $\varphi(\bar{\Gamma}^i_{jk}) = \delta^i_k \alpha_j - g_{jk} \alpha^i$ and Γ is given by the Weyl projective tranaformation,

$$\Gamma^i_{jk} = \{^{\,i}_{jk}\} + \delta^i_j \rho_k + \delta^i_k \rho_j. \tag{20}$$

from (12), we have

$$\bar{\Gamma}^i_{jk} = \{^{\,i}_{jk}\} + \delta^i_j \rho_k + \delta^i_k \rho_j + \delta^i_k \alpha_j - g_{jk} \alpha^i. \tag{21}$$

It is the transformation obtained by S m a r a n d a [4].

From (2.4) we have, if $g_{ij//k} = g_{ij/k}$,

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \Omega^{ir}_{sj} X^s_{rk} \tag{22}$$

where X is arbitrary. For a fixed $(D, \bar{D})$ X is arbitrary in $Q_{A=\varphi(\tau)}$.

From (6) we have, the general transformation;

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \Omega^{ir}_{sj} X^s_{rk} + G^i_{jk} - \bar{G}^i_{jk} \tag{23}$$

where X is arbitrary and $\bar{G}^i_{jk} = \bar{G}^i_{kj}$ is arbitrary. For a fixed $(D, \bar{D})$, X is arbitrary in $Q_{A=\varphi(\tau)}$ and $\bar{G}$ is (2.15).

From (2.18) and (2.23) we obtain the transformation of R. M i r o n [3].

REFERENCES

1. Eisenhart, L. P., *Riemannian Geometry*, Princeton Univ. Press (1949).
2. Kobayashi, S., Nomizu, K. *Fundations of Differential Geometry, I.* Interscience Publishers, New-York (1953).
3. Miron, R., *Asupra conexiunilor compatibile cu structuri conform aproape simplectice sau conform metrice*, An. Univ. din Timişoara vol. V (1970) 127–133.
4. Andonie, C., Smaranda, D., *Certains connections semi-symmetric*. Tensor (N.S) vol. 31, 8–12.
5. Stavre, P., Klepp, F., f_Ω *mappings with fixed points*, Simp. de Mat. Timişoara, 1–2 XI (1985), p. 145–148.

RECENZII

Nonlinear Analysis and Applications, Edited by V. Lakshmikantham, Lecture notes in pure and applied mathematics vol. 109, 1987 Marcel Dekker, Inc.

The volume consists of the proceedings of the seventh International Conference on „Nonlinear Analysis and Applications" held at the University of Texas at Arlington in the summer of 1986. It contains 82 papers which show the main trends in theory and applications of Nonlinear Analysis; in the list of contributors one finds the names of many well-known scientists in the field.

The book opens with a dedication signed by K. Deimling to Professor V. Lakshmikantham on the occasion of his 60 th birthday and of this seventh conference organized by him and his team. The preface contains the aim of the editor „to put together the works of a wide range of mathematical scientists".

Indeed, both pure and applied mathematicians will enjoy the proceedings, some of which reflect the joint efforts of mathematicians, biologists and engineers. A set of papers deal with modeling of combustion problems, neutral networks, ecological and biological systems, population dynamics and other applications. Controlability and optimization problems, Hamiltonian systems, ill-posed problems, scattering theory and Navier-Stokes equations are considered in several papers. Various kinds of nonlinear ordinary equations (with and without delay), as well as partial differential and integro-differential ones are the subject of a great number of papers. One finds also recent results on fixed point theory, dynamical systems and numerical methods using finite element and iterations.

The reader may use the final index to find directely the works related to his field of activity.

This outstanding volume will be of great help to pure and applied mathematicians working in nonlinear analysis, biomathematicians and engineers.

MIRA-CRISTIANA ANISIU

T. Morgan, **Geometric Measure Theory. A beginner's guide**, Academic Press, New York 1987, 145 pp.

Geometric measure theory can be roughly described as measure theory dealing with finite dimensional not necessarily smooth maps and surfaces. Combining methods from multilinear algebra, differential geometry, topology and measure theory, geometric measure theory is an important and relatively new domain of research, with subtantial applications to elliptic variational problems, including area minimizing problem (Plateau's problem). As a spectacular application, one can mention the original proof of the positive mass conjecture in cosmology by R. Schoen and S—T. Yau, Comm. Math. Phys. 65 (1979), 45—76. As a starting point for the modern geometric measure theory the author quotes the fundational paper of H. Federer and W. Fleming, Annal of Math. 72 (1960), 458—520, culminating with the monumental treatise of H. Federer, Geometric Measure Theory (GMT), Springer Verlag, 1969 (Russian translation, Moscow 1987). But it is not easy to read Federer's treatise and it is the aim of this marvellous book to make. Federer's text more accesible and to provide the newcomer with the basic material needed to confront the literature, to understand and contribute to the subject. The author achieves masterly this task.

Not all the proofs are given in detail (some are omitted at all and are referred to GMT, as a basic souce) but the author marks the main steps of the proofs stressing on the ideas which lie behind. the technical and formal aspects. For a better understanding of the subject the book is provided with numerous excellent illustrations (drawn by J. Bredt).

The basic tools used in the book are rectifiable currents and Hausdorff measure H_m. A rectifiable current is an m-dimensional surface S in R^n, where the relevant function $f: R^m \to R^n$ need not be smooth but merely Lipschitz. By integration on S smooth differential forms φ, the set S may be viewed as a current, i.e. a linear functional on differential forms. Via Stoke's theorem — $(\partial S)(\varphi) = S(d\varphi)$ — one defines a boundary operator ∂ acting from m-dimensional rectifiable currents to $(m-1)$-dimensional currents. The use of rectifiable currents has some advantage concerning restrictions of the types of singularities and on topological complexity and the lack of compactness, which occur in the smooth case. It is interesting to note that the solution of the area minimizing problem in the class of rectifiable m-dimensional currents in R^{m+1} is a smooth embedded manifold, for $2 \leqslant m \leqslant 6$, but, in higher dimensions, singularities can occur.

Now, more exactly on the contents. The book is divided into a Preface and 12 chapters headed as follows: 1. Geometric measure theory (outlining the purpose and the general concepts of geometric measure theory); 2. Measures; 3. Lipschitz functions and rectifiable sets; 4. Normal and rectifiable currents; 5. The compactness theorem and the existence of area-minimizing surfaces; 6. Examples of area-minimizing surfaces; 7. The approximation theorem; 8. Survey of regularity results; 9. Monotonicity and oriented tangent cones; 10. The regularity of area-minimizing hypersurfaces; 11. Flat chains modulo ν, varifolds and (M, ε, δ)-minimal sets; 12. Miscellaneous useful results. Each chapter is endowed with a set of exercises, completing the main text, whose solutions are given at the end of the book. A basic bibliography, an index of symbols, a subject index and a name index are also included.

In conclusion, this is a remarkable book which can be warmly recommended to all interested to learn and to apply the powerful methods of geometric measure theory. In reviewer opinion, the book is also useful to the specialists in the field for a better understanding of the subjects and for numerous illustrated examples.

S. COBZAŞ

C. Bennett, R. Sharpley, **Interpolation of Operators**, Academic Press, New York 1988, 465 pp.

The modern theory of interpolation of operators is an important branch of functional analysis with many applications to approximation theory, Fourier analysis, partial differential equations etc. At its foundation lie three classical theorems: M. Riesz convexity theorem (1926), its complex and operatorial version proved by G.O. Thorin (1939) and J. Marcinkiewicz interpolation theorem (1939) (J. Marcinkiewicz announced this theorem in Comptes Rendus — Paris, and a proof was published only in 1956 by A. Zygmund). Thorin's technique has given rise to the complex method of interpolation and Marcinkiewicz's to the real method.

The present book is about the real method, tarting from its origins, that is, through the heory of spaces of measurable functions, which s developed in the first two chapters of the book: 1. Banach Function Spaces, and 2. Rearrangement Invariant Function Spaces. The Banach function spaces are defined by a function norm on the space of all measurable and a.e. finite real valued functions on a σ-finite measure space and they are natural extensions of classical Lebesgue space L^p, $1 \leqslant p \leqslant \infty$. Among the topics treated in these chapters we mention: absolute continuity of the norm, duality and reflexivity, separability, the spaces $L^1 + L^\infty$ and $L^1 \cap L^\infty$, measure preserving transformations.

The study of the interpolation of operators begins in Chapter 3. Interpolation of Operators on Rearrangement — Invariant Spaces, with the general definition of interpolation spaces and continuity with the interpolation between L^1 and L^∞ and the Hardy-Littlewood maximal operator. The Hilbert transform, which plays a decisive role in questions concerning norm-convergence of Fourier series and Fourier integrals, and also motivates the weak-type interpolation theory, is treated by a direct real variable approach (including existence and L^p-boundedness). At the end of chapter, the general results and methods are brought together to prove some norm-convergence theorems for Fourier series in rearrangement-invariant function spaces.

Chapter 4. The classical Interpolation Theorems, deals with Riesz and Riesz-Thorin convexity theorems, analytic families of operators, Zygmund spaces L Log L and $L_{r,p}$, weak-type theory and Orlicz spaces.

The last chapter of the book, Chapter 5. The K-Method, is devoted to J. Peetre's method of J — and K-functionals, with applications to Besov and Sobolev spaces, Re H^1 and BMO, BMO and weak-L^∞ and to interpolation between L^1 and BMO and between H^1 and H^∞.

Each chapter is endowed with a set of exercises, completing the base text and ends with a section of notes containing bibliographical comments and references for further investigations.

The book is clearly written and very well organized (a list of notations and an index are included). The included topics are carefully choosen and well motivated, helping the reader to understand the subject bringing him to the frontiers of current research.

We recommend it warmly to all interested in interpolation theory of operators and its applications.

S. COBZAŞ

ÎNTREPRINDEREA POLIGRAFICĂ CLUJ,
Municipiul Cluj-Napoca, Cd. nr. 396/1989

În cel de al XXXIV-lea an (1989), *Studia Universitatis Babeş-Bolyai* apare în specialităţile:

matematică

fizică

chimie

geologie-geografie

biologie

filosofie

ştiinţe economice

ştiinţe juridice

istorie

filologie

In the XXXIV-th year of its publication (1989), *Studia Universitatis Babeş-Bolyai* is issued as follows:

mathematics

physics

chemistry

geology-geography

biology

philosophy

economic sciences

juridical sciences

history

philology

Dans sa XXXIV-e année (1988), *Studia Universitatis Babeş-Bolyai* paraît dans les spécialités:

mathématiques

physique

chimie

géologie-géographie

biologie

philosophie

sciences économiques

sciences juridiques

histoire

philologie

43 875

Abonamentele se fac la oficiile poştale, prin factorii poştali şi prin difuzorii de presă, iar pentru străinătate prin „ROMPRESFILATELIA", sectorul export-import presă, P. O. Box 12—201, telex. 10376 prsfir, Bucureşti, Calea Griviţei nr. 64—66

Lei 35