STUDIA
UNIVERSITATIS BABEȘ-BOLYAI

MATHEMATICA

4

1988

CLUJ-NAPOCA

ANUL XXXIII 1988

STUDIA
UNIVERSITATIS BABEȘ-BOLYAI
MATHEMATICA
4

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UDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXIII, 4, 1988

SYSTÉMES D'AXIOMES POUR LES TREILLIS NON-COMMUTATIFS DE TYPE (S)

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Manuscrit reçu le 25 Mai 1988

REZUMAT — Sisteme de axiome pentru laticile necomutative de tip (S). Într-o lucrare anterioară [2] au fost introduse şi studiate laticile. necomutative de tip (S). În această lucrare se prezintă 16 sisteme de axiome logic echivalente pentru această clasă specială de latici necomutative.

Le triplet $(L, \wedge, \vee)$, où L est un ensemble et $\wedge$ et $\vee$ sont deux opérations binaires définies en L, s'appelera de treillis non-commutatif de type (S) si pour tout $a, b. c \in L$ vérifie les axiomes:

$$(A).\quad \begin{cases} (a \wedge b) \wedge c = a \wedge (b \wedge c) \\ (a \vee b) \vee c = a \vee (b \vee c) \end{cases}$$

$$(B).\quad \begin{cases} a \wedge (a \vee b) = a \\ a \vee (a \wedge b) = a \end{cases}$$

$$(S_1).\quad \begin{cases} a \wedge (b \vee c) = a \wedge (c \vee b) \\ a \vee (b \wedge c) = a \vee (c \wedge b). \end{cases}$$

Observons que, si $(L, \wedge, \vee)$ est treillis non-commutatif de type (S), alors pour tout $a, b, c \in L$ sont vraies les égalités:

$$(C).\quad \begin{cases} a \wedge (b \vee a) = a \\ a \vee (b \wedge a) = a. \end{cases}$$

Les treillis non-commutatifs de type (S) on été introduits pour la première fois en [2].

Dans ce travail nous monterons que cette classe spéciale de treillis non-commutatifs peut être caractérisée aussi par d'autres systèmes d'axiomes.

(1.1). *Si $(L, \wedge, \wedge)$ satisfait les axiomes (A), (B) et (C), alors pour tout $a, b, c \in L$ sont vraies les égalités:*

$$(i).\quad \begin{cases} a \wedge a = a \\ a \vee a = a \end{cases}$$

$$(ii).\quad \begin{cases} a \wedge b \wedge a = a \wedge b \\ a \vee b \vee a = a \vee b \end{cases}$$

$$(iii).\quad \begin{cases} a \wedge b = (a \wedge b) \vee (b \wedge a) \\ a \vee b = (a \vee b) \wedge (b \vee a). \end{cases}$$

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Démonstration. (*i*). En utilisant les axiomes (B) on obtient que, pour $a \in L$ nous avons $a \wedge a = a \wedge (a \vee (a \wedge b)) = a$ et $a \vee a = a \vee (a \vee (a \vee b)) = a$ donc en $(L, \wedge, \vee)$ sont vraies les égalités (*i*), c'est-à-dire les lois de l'idempotence.

(*ii*). En utilisant les axiomes (B) et (C) on appercoit que, pour tout $a, b \in L$ nous avons $a \wedge b \wedge a = (a \wedge b) \wedge (a \vee (a \wedge b)) = a \wedge b$ et $a \vee b \vee a = (a \vee b) \vee (a \wedge (a \vee b)) = a \vee b$.

(*iii*). En utilisant les axiomes (A), (C) et les propriétés (*i*) et (*ii*) on obtient que, pour tout $a, b \in L$ nous avons $a \wedge b = (a \wedge b) \vee ((b \wedge a) \wedge (a \wedge b)) = (a \wedge b) \vee (b \wedge (a \wedge a) \wedge b) = (a \wedge b) \vee (b \wedge a \wedge b) = (a \wedge b) \vee (b \wedge a)$ et $a \vee b = (a \vee b) \wedge ((b \vee a) \vee (a \vee b)) = (a \vee b) \wedge (b \vee (a \vee a) \vee b) = (a \vee b) \wedge (b \vee a \vee b) = (a \vee b) \wedge (b \vee a)$.

(1.2). *Si $(L, \wedge, \vee)$ satisfait les axiomes (A), (B) et (C), alors pour tout $a, b, c \in L$ les suivantes 16 affirmations sont logiquement équivalentes :*

$$(S_1). \begin{cases} a \wedge (b \vee c) = a \wedge (c \vee b) \\ a \vee (b \wedge c) = a \vee (c \wedge b) \end{cases}$$

$$(S_2). \begin{cases} a = a \wedge b \Rightarrow c \vee b = c \vee a \vee b \\ a = a \vee b \Rightarrow c \wedge b = c \wedge a \wedge b \end{cases}$$

$$(S_3). \begin{cases} a = a \wedge b \text{ et } c = c \wedge d \Rightarrow b \vee d = b \vee a \vee c \vee d \\ a = a \vee b \text{ et } c = c \vee d \Rightarrow b \wedge d = b \wedge a \wedge c \wedge d \end{cases}$$

$$(S_4). \begin{cases} a = a \wedge b \text{ et } d = d \vee c \Rightarrow b \vee d = b \vee a \vee c \vee d \\ a = a \vee b \text{ et } d = d \wedge c \Rightarrow b \wedge d = b \wedge a \wedge c \wedge d \end{cases}$$

$$(S_5). \begin{cases} a = a \wedge b \text{ et } c = c \wedge d \Rightarrow b \vee d = b \vee c \vee a \vee d \\ a = a \vee b \text{ et } c = c \vee d \Rightarrow b \wedge d = b \wedge c \wedge a \wedge d \end{cases}$$

$$(S_6). \begin{cases} b = b \vee a \text{ et } c = c \wedge d \Rightarrow b \vee d = b \vee c \vee a \vee d \\ b = b \wedge a \text{ et } c = c \vee d \Rightarrow b \wedge d = b \wedge c \wedge a \wedge d \end{cases}$$

$$(S_7). \begin{cases} a = a \wedge b \text{ et } d = d \vee c \Rightarrow b \vee d = b \vee c \vee a \vee d \\ a = a \vee b \text{ et } d = d \wedge c \Rightarrow b \wedge d = b \wedge c \wedge a \wedge d \end{cases}$$

$$(S_8). \begin{cases} a = a \wedge b \text{ et } c = c \wedge d \Rightarrow b \vee d = b \vee c \vee d \vee a \\ a = a \vee b \text{ et } c = c \vee d \Rightarrow b \wedge d = b \wedge c \wedge d \wedge a \end{cases}$$

$$(S_9). \begin{cases} b = b \vee a \text{ et } c = c \wedge d \Rightarrow b \vee d = b \vee c \vee d \vee a \\ b = b \wedge a \text{ et } c = c \vee d \Rightarrow b \wedge d = b \wedge c \wedge d \wedge a \end{cases}$$

$$(S_{10}). \begin{cases} a = a \wedge b \text{ et } d = d \vee c \Rightarrow b \vee d = b \vee c \vee d \vee a \\ a = a \vee b \text{ et } d = d \wedge c \Rightarrow b \wedge d = b \wedge c \wedge d \wedge a \end{cases}$$

$$(S_{11}). \begin{cases} a = a \wedge b \text{ et } c = d \wedge c \Rightarrow d \vee b = d \vee a \vee b \vee c \\ a = a \vee b \text{ et } c = d \vee c \Rightarrow d \wedge b = d \wedge a \wedge b \wedge c \end{cases}$$

(S_{12}). $\begin{cases} b = b \vee a \text{ et } c = d \wedge c \Rightarrow d \vee b = d \vee a \vee b \vee c \\ b = b \wedge a \text{ et } c = d \vee c \Rightarrow d \wedge b = d \wedge a \wedge b \wedge c \end{cases}$

(S_{13}). $\begin{cases} a = a \wedge b \text{ et } c = d \wedge c \Rightarrow d \vee b = d \vee a \vee c \vee b \\ a = a \vee b \text{ et } c = d \vee c \Rightarrow d \wedge b = d \wedge a \wedge c \wedge b \end{cases}$

(S_{14}). $\begin{cases} b = b \vee a \text{ et } c = d \wedge c \Rightarrow d \vee b = d \vee a \vee c \vee b \\ b = b \wedge a \text{ et } c = d \vee c \Rightarrow d \wedge b = d \wedge a \wedge c \wedge b \end{cases}$

(S_{15}). $\begin{cases} a = a \wedge b \text{ et } c = d \wedge c \Rightarrow d \vee b = d \vee c \vee a \vee b \\ a = a \vee b \text{ et } c = d \vee c \Rightarrow d \wedge b = d \wedge c \wedge a \wedge b \end{cases}$

(S_{16}). $\begin{cases} b = b \vee a \text{ et } c = d \wedge c \Rightarrow d \vee b = d \vee c \vee a \vee b \\ b = b \wedge a \text{ et } c = d \vee c \Rightarrow d \wedge b = d \wedge c \wedge a \wedge b. \end{cases}$

Démonstration. Dans cette démonstration on utilisera les propriétés (i), (ii) et (iii) de (1.1). Ensuite, perceque le système d'axiomes (A), (B), (C) est autoduale, cela suffira à démontrer seulement l'une de ces deux affirmations qui apparaissent dans l'écriture de (S_i), $i = 1, 2, \ldots, 16$.

$(S_1) \Rightarrow (S_2)$. Si les égalités (S_1) sont vraies et pour $a, b \in L$ nous avons $a = a \wedge b$, alors :

$$\begin{aligned} c \vee b &= c \vee (b \vee (a \wedge b)) \\ &= c \vee ((b \vee (a \wedge b)) \wedge ((a \wedge b) \vee b)) \\ &= c \vee ((a \wedge b) \vee b) \wedge (b \vee (a \wedge b))) \\ &= c \vee ((a \wedge b) \vee b) \\ &= c \vee (a \wedge b) \wedge b \\ &= c \vee a \vee b, \end{aligned}$$

donc (S_2) est vraie aussi.

$(S_2) \Rightarrow (S_3)$. Si (S_2) est vraie, alors dans l'égalité évidente :

$(d \vee (c \wedge d)) \wedge ((c \wedge d) \vee d) = (d \vee (c \wedge d)) \wedge ((c \wedge d) \vee d)) \wedge$
$\wedge (((c \wedge d) \vee d \wedge (d \vee (c \wedge d)))$ on obtient que pour tout $a, b, c, d \in L$ sont vraie l'égalité :

$$b \vee (((c \wedge d) \vee d) \wedge (d \vee (c \wedge d))) = b \vee ((d \vee (c \wedge d)) \wedge ((c \wedge d) \vee d))$$

En conséquence, si pour $a, b, c, d \in L$ nous avons $a = a \wedge b$ et $c = c \wedge d$, alors :

$$\begin{aligned} b \vee d &= b \vee d \vee (c \wedge d) \\ &= b \vee ((d \vee (c \wedge d)) \wedge ((c \wedge d) \vee d)) \\ &= b \vee (((c \wedge d) \vee d) \wedge (d \vee (c \wedge d))) \\ &= b \vee c \wedge d) \vee d \\ &= b \vee (a \wedge b) \vee (c \wedge d) \vee d \\ &= b \vee a \vee c \vee d, \end{aligned}$$

donc (S_3) est vraie aussi.

$(S_3) \Rightarrow (S_4)$. Si (S_3) este vraie, alor des égalités évidentes $b = b \wedge b$, $(c \wedge d) \vee d = ((c \wedge d) \vee d) \wedge (d \vee (c \wedge d))$ on obtient $b \vee d \vee (c \wedge d) = b \vee (c \wedge d) \vee d$. En conséquence, si pour $a, b, c, d \in L$ nous avons $a = a \wedge b$, $d = d \vee c$, alors $b \vee d = b \vee d \vee (c \wedge d) = b \vee (c \wedge d) \vee d = b \vee (c \wedge (d \vee c)) \vee d = b \vee c \vee d = b \vee (a \wedge b) \vee c \vee d = b \vee a \vee c \vee d$, donc (S_4) est vraie aussi.

$(S_4) \Rightarrow (S_5)$. Si (S_4) este vraie, alors des égalités évidentes $b = b \wedge b$, $d \vee (c \wedge d) = (d \vee (c \wedge d)) \vee ((c \wedge d) \vee d)$ on obtient $b \vee d \vee (c \wedge d) = b \vee (c \wedge d) \vee d$. En conséquence, si pour $a, b, c, d \in L$ nous avons $a = a \wedge b$ et $c = c \wedge d$, alors $b \vee d = b \vee d \vee (c \wedge d) = b \vee (c \wedge d) \vee d = b \vee (c \wedge d) \vee b \vee d = b \vee (c \wedge d) \vee b \vee (a \wedge b) \vee d = b \vee (c \wedge d) \vee (a \wedge b) \vee d = b \vee c \vee a \vee d$, donc (S_5) est vraie aussi.

$(S_5) \Rightarrow (S_6)$. Si (S_5) est vraie, alors des égalités évidentes $a = a \wedge a$ et $(c \wedge d) \vee d = ((c \wedge d) \vee d) \wedge (d \vee (c \wedge d))$ on obtient $a \vee d \vee (c \wedge d) = a \vee (c \wedge d) \vee d$. En conséquence, si pour $a, b, c, d \in L$ nous avond $b = b \wedge a$ et $c = c \wedge d$, alors $b \vee d = b \vee a \vee d = b \vee a \vee d \vee (c \wedge d) = b \vee a \vee (c \wedge d) \vee d = b \vee a \vee (c \wedge d) \vee a \vee d = b \vee c \vee a \vee d$, donc (S_6) est vraie aussi.

$(S_6) \Rightarrow (S_7)$. Si (S_6) est vraie, alors des égalités évidentes $b = b \vee b$ et $d \vee (c \wedge d) = (d \vee (c \wedge d)) \wedge ((c \wedge d) \vee d)$ on obtient $b \vee d \vee (c \wedge d) = b \vee (c \wedge d) \vee d$. En conséquence, si pour $a, b, c, d \in L$ nous avons $a = a \wedge b$ et $d = d \vee c$, alors $b \vee d = b \vee d \vee (c \wedge d) = b \vee (c \wedge d) \vee d = b \vee (c \wedge (d \vee c)) \vee d = b \vee c \vee d = b \vee c \vee b \vee d = b \vee c \vee b \vee (a \wedge b) \vee d = b \vee c \vee (a \wedge b) \vee d = b \vee c \vee a \vee d$, donc (S_7) est vraie aussi.

$(S_7) \Rightarrow (S_8)$. Si (S_7) est vraie, alors des égalités évidentes $b = b \wedge b$ et $d \vee (c \wedge d) = (d \vee (c \wedge d)) \vee ((c \wedge d) \vee d)$ on obtient $b \vee d \vee (c \wedge d) = b \vee (c \wedge d) \vee d$. En conséquence, si pour $a, b, c, d \in L$ nous avons $a = a \wedge b$ et $c = c \wedge d$, alors $b \vee d = b \vee d \vee (c \wedge d) = b \vee (c \wedge d) \vee d = b \vee (c \wedge d) \vee d \vee b = b \vee (c \wedge d) \vee d \vee b \vee (a \wedge b) = b \vee (c \wedge d) \vee d \vee (a \wedge b) = b \vee c \vee d \vee a$, donc (S_8) este vraie aussi.

$(S_8) \Rightarrow (S_9)$. Si (S_8) est vraie, alors des égalités évidentes $b = b \wedge b$ et $(c \wedge d) \vee d = ((c \wedge d) \vee d) \wedge (d \vee (c \wedge d))$ on obtient $b \vee d \vee (c \wedge d) = b \vee (c \wedge d) \vee d$. En conséquence, si pour $a, b, c, d \in L$ nous avons $b = b \vee a$ et $c = c \wedge d$, alors $b \vee d = b \vee d \vee (c \wedge d) = b \vee (c \wedge d) \vee d = b \vee a \vee (c \wedge d) \vee d = b \vee a \vee (c \wedge d) \vee d \vee a = b \vee c \vee d \vee a$, donc (S_9) est vraie aussi.

$(S_9) \Rightarrow (S_{10})$. Si (S_9) est vraie, alors des égalités évidentes $b = b \vee b$ et $(c \wedge d) \vee d = ((c \wedge d) \vee d) \wedge (d \vee (c \wedge d))$ on obtient $b \vee d \vee (c \wedge d) = b \vee (c \wedge d) \vee d$. En conséquence, si pour $a, b, c, d \in L$ nous avons $a = a \wedge b$ et $d = d \vee c$, alors $b \vee d = b \vee d \vee (c \wedge d) = b \vee (c \wedge d) \vee d = b \vee (c \wedge (d \vee c)) \vee d = b \vee c \vee d = b \vee c \vee d \vee b = b \vee c \vee d \vee b \vee (a \wedge b) = b \vee c \vee d \vee (a \wedge d) = b \vee c \vee d \vee a$, donc (S_{10}) est vraie aussi.

$(S_{10}) \Rightarrow (S_{11})$. Si (S_{10}) est vraie, alors des égalités évidentes $d = d \wedge d$ et $b \vee (a \wedge b) = (b \vee (a \wedge b)) \vee ((a \wedge b) \vee b)$ on obtient $d \vee b \vee (a \wedge b) = d \vee (a \wedge b) \vee b$. En conséquence, si pour $a, b, c, d \in L$ nous avons $a = a \wedge b$ et $c = d \wedge c$, alors $d \vee b = d \vee b \vee (a \wedge b) = d \vee (a \wedge b) \vee b = d \vee (a \wedge b) \vee b \vee d = d \vee (a \wedge b) \vee b \vee d \vee (d \wedge c) = d \vee (a \wedge b) \vee (d \wedge c) = d \vee a \vee b \vee c$, donc (S_{11}) est vrie aussi.

$(S_{11}) \Rightarrow (S_{12})$. Si (S_{11}) est vraie, alors des égalités évidemtes $(a \wedge b) \vee b = ((a \wedge b) \vee b) \wedge (b \vee (a \wedge b))$ et $d = d \wedge d$ on obtient $d \vee b \vee (a \wedge b) = d \vee (a \wedge b) \vee b$. En conséquence, si pour $a, b, c, d \in L$ nous avons $b = b \vee a$ et $c = d \wedge c$, alors $d \vee b = d \vee b \vee (a \wedge b) = d \vee (a \wedge b) \vee b = d \vee (a \wedge (b \vee a)) \vee b = d \vee a \vee b = d \vee a \vee b \vee d = d \vee a \vee b \vee d \vee (d \wedge c) = d \vee a \vee b \vee (d \wedge c) = d \vee a \vee b \vee c$, donc (S_{12}) est vraie aussi.

$(S_{12}) \Rightarrow (S_{13})$. Si (S_{12}) est vraie, alors des égalités évidentes $b \vee (a \wedge b) = (b \vee (a \wedge b)) \vee ((a \wedge b) \vee b)$ et $d = d \wedge d$ on obtient $d \vee b \vee (a \wedge b) = d \vee (a \wedge b) \vee b$. En conséquence, si pour $a, b, c, d \in L$ nous avons $a = a \wedge b$ et $c = d \wedge c$, alors $d \vee b = d \vee b \vee (a \wedge b) = d \vee (a \wedge b) \vee b = d \vee (a \wedge b) \vee d \vee b = d \vee (a \wedge b) \vee d \vee (d \wedge c) \vee b = d \vee (a \wedge b) \vee (d \wedge c) \vee b = d \vee a \vee c \vee b$, donc (S_{13}) est vriae aussi.

$(S_{13}) \Rightarrow (S_{14})$. Si (S_{13}) est vraie, alors des égalités évidentes $(a \wedge b) \vee b = ((a \wedge b) \vee b) \wedge (b \vee (a \wedge b))$ et $d = d \wedge d$ on obtient $d \vee b \vee (a \wedge b) = d \vee (a \wedge b) \vee b$. En conséquence, si pour $a, b, c, d \in L$ nous avons $b = b \vee a$ et $c = d \wedge c$, alors $d \vee b = d \vee b \vee (a \wedge b) = d \vee (a \wedge b) \vee b = d \vee (a \wedge (b \vee a)) \vee b = d \vee a \vee b = d \vee a \vee d \vee b = d \vee a \vee d \vee (d \wedge c) \vee b = d \vee a \vee (d \wedge c) \vee b = d \vee a \vee c \vee b$, donc (S_{14}) est vraie aussi.

$(S_{14}) \Rightarrow (S_{1}\ )$. Si (S_{14}) est vraie, alors des égalités évidentes $b \vee (a \wedge b) = (b \vee (a \wedge b)) \vee ((a \wedge b) \vee b)$ et $d = d \wedge d$ on obtient $d \vee b \vee (a \wedge b) = d \vee (a \wedge b) \vee b$. En conséquence, si pour $a, b, c, d \in L$ nous avons $a = a \wedge b$ et $c = d \wedge c$, alors $d \vee b = d \vee b \vee (a \wedge b) = d \vee (a \wedge b) \vee b = d \vee (d \wedge c) \vee (a \wedge b) \vee b = d \vee c \vee a \vee b$, donc (S_{15}) est vraie eussi.

$(S_{15}) \Rightarrow (S_{16})$. Si (S_{14}) est vraie, alors égalités évidentes $(a \wedge b) \vee b = ((a \wedge b) \vee b) \wedge (b \vee (a \wedge b))$ et $d = d \wedge d$ on obtient $d \vee b \vee (a \wedge b) = d \vee (a \wedge b) \vee b$. En conséquence, si pour $a, b, c, d \in L$ nous avons $b = b \vee a$ et $c = d \wedge c$, alors $d \vee b = d \vee b \vee (a \wedge b) = d \vee (a \wedge b) \vee b = d \vee (a \wedge (b \vee a)) \vee b = d \vee a \vee b = d \vee (d \wedge c) \vee a \vee b = d \vee c \vee a \vee b$, donc (S_{16}) este vriaie aussi.

$(S_{16}) \Rightarrow (S_{1})$. Si (S_{16}) est vraie, alors des égalités évidentes $b \vee c = (b \vee c) \wedge (c \vee b)$ et $a = a \vee a$ on obtient $a \wedge (b \vee c) = a \wedge (c \vee b)$, donc (S_1) est vraie aussi.

En conséquence, les treillis non-commutatifs de type (S) peuvent être définis par les suivants 16 systèmes d'axiomes logiquement équivalentes:

$\mathfrak{S}_1 = \{(A), (B), (S_1)\}$

$\mathfrak{S}_i = \{(A), (B), (C), (S_i)\}$, où $i = 2, 3, \ldots, 16$.

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STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXIII, 4, 1988

ON THE IRRATIONALITY OF SOME ALTERNATING SERIES

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Received: February 1, 1988

REZUMAT. — Despre iraţionalitatea unor serii alternate. Obiectul acestei lucrări este de a demonstra cîteva teoreme pentru iraţionalitatea unor serii alternate. În cazuri speciale rezultate de acest gen au mai fost găsite în [1], [3], [4], [11].

1. The aim of this paper is to find some irrationality criteria for alternating series. H. F. Sandham [5] has proved the formula $C = \sum_{n=1}^{\infty} (-1)^n \frac{[\ln n/\ln 2]}{n}$ where C is Euler's constant and $[x]$ denotes the integer part of the real number x. It is well-known that the old problem of the irrationality of C is still open and there are only a few results related to this problem. In the light of the above mentioned formula it would be interesting to prove theorems on the irrationality of alternating series. Some result of this type on special series have been obtained by A. Oppenheim [4], M.R. Spiegel [11], R. Breusch [1], P. Bundschuh [2] and N. J. Lord [3].

2. THEOREM 1. *Let (a_n) be a sequence of positive integers such that $a_n(a_1a_2 \dots a_{n-1})^2 \to \infty$ for $n \to \infty$. Then the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{a_n(a_1 \dots a_{n-1})^2}$ is irrational.*

Proof. This is based on the following result:

Lemma 1. Let $(r_n) = (h_n / k_n)$ be a sequence of rational numbers satisfying: (i) $1 \leq k_1 < k_2 < \dots < k_n \dots$ (ii) $r_2 < r_4 < r_6 < \dots < r_7 < r_5 < r_3 < r_1$ (iii) $|h_{n+1} k_n - k_{n+1}h_n| = 1$ for $n = 1, 2, 3, \dots$ Then the sequence (r_n) is convergent and its limit is an irrational number.

Proof. In view of (i) we have $k_n \to \infty$ as $n \to \infty$, hence by (iii), written in the form: $|r_{n+1} - r_n| = 1/k_nk_{n+1}$, we obtain: $|r_{n+1} - r_n| \to 0$ $(n \to \infty)$. Then using (ii), it is well-known (see e.g. [7]) there exists $\lim_{n\to\infty} r_n = \theta$, where θ is situated between r_n and r_{n+1} for all $n = 1, 2, 3, \dots$. This means that $0 < |\theta - r_n| < |r_{n+1} - r_n| = 1/k_nk_{n+1}$. Supposing now θ rational, i.e. $\theta = a/b$ with integers a, b $(b > 0)$, the above written inequality yields: $0 < |ak_n - bh_n| < b/k_{n+1} < 1$ for sufficiently large n (because of $k_n \to \infty$ for $n \to \infty$). Here $ak_n - bh_n$ is an integer number, a contradiction which finishes the proof.

For the proof of the theorem consider the series $\sum_{n+1}^{\infty} (-1)^{n-1} \frac{u_n}{v_n}$ where $u_n = (a_1a_2 \dots a_{n-1})^{-2}$, $v_n = a_n$. Then the sequence (u_n/v_n) is strictly decreasing to zero, thus Leibnitz's theorem assures the convergence of the series. Take $r_n =$

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$= h_n/k_n = \sum_{i=1}^{n} (-1)^{i-1} u_i/v_i$, where $k_n = v_1 v_2 \dots v_n = a_1 a_2 \dots a_n$. The equality $|r_{n+1} - r_n| = u_{n+1} / v_{n+1} = (a_{n+1})^{-1} (a_1 \dots a_n)^{-2} = 1/k_n k_{n+1}$ and simple observations assure the conditions of the applicability of lemma 1. Thus $\theta = \lim_{n\to\infty} r_n = \sum_{i=1}^{\infty} (-1)^{i-1} u_i/v_i$ is irrational.

THEOREM 2. *Let (b_n), (a_n) be two sequences of positive integers having the following properties: (1) $n \mid a_1 a_2 \dots a_n$ for every $n = 1, 2, 3, \dots$ (2) $b_{n+1}/a_{n+1} < b_n < a_n$ for $n = 1, 2, 3, \dots$*

Then the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{b_k}{a_1 a_2 \dots a_n}$ is irrational.

Proof. Condition (1) implies $a_1 a_2 \dots a_n \to \infty$ $(n \to \infty)$ and by (2) we obtain $x_n = b_n / a_1 a_2 \dots a_n < 1/a_1 a_2 \dots a_{n-1} \to 0$. The left-side inequality of (2) implies the monotonicity of (x_n), so by the Leibnitz criterion the series is convergent. Suppose that its value is a rational number m/k. Then by multiplication with $a_1 a_2 \dots a_n$ we get:

$$\frac{m a_1 a_2 \dots a_k}{k} = \sum_{n=1}^{k} (-1)^{n-1} \frac{a_1 a_2 \dots a_k}{a_1 a_2 \dots a_n} b_n + (-1)^k \left(\frac{b_{k+1}}{a_{k+1}} - \frac{b_{k+1}}{a_{k+2} a_{k+2}} + \dots \right)$$

This relation and (1) imply that $\alpha =$ the alternating series on the right hand side, is an integer number. This series has its first term a positive number and the general term $u_n = b_{k+n} / a_{k+1} \dots a_{k+n}$. Using (2) we infer that $u_{n+1} < u_n$ and similarly, $u_n < 1/a_{k+1} \dots a_{k+n-1} \to 0$ $(n \to \infty)$ so the Leibnitz criterion implies that the above series is convergent, having a sum between $b_{k+1}/a_{k+1} - b_{k+2} / a_{k+1} a_{k+2}$ and b_{k+1}/a_{k+2} (see e.g. [7]). The inequality $b_{k+1} > b_{k+2} / a_{k+2}$ yields $0 < b_{k+1} / a_{k+1} - b_{k+2} / a_{k+1} a_{k+2} = (b_{k+1} a_{k+2} - b_{k+2}) / a_{k+1} a_{k+2}$. On the other hand one has $b_{k+1} / a_{k+1} < 1$. Thus we have obtainde that for the integer number α we have $\in (0, 1)$, a trivial contradiction.

THEOREM 3. *Let $f: N \to R$ be an arithmetical function with the properly $\lim_{n\to\infty} n f(n) = 0$. Further let $(v_n) = (b_n/a_n)$ a decreasing sequence of rational numbers with $\lim_{n\to\infty} b_n/a_n = 0$. Suppose also that the inequality $v_{k+1} \leqslant f(a_1 a_2 \dots a_k)$ holds true for all $k = 1, 2, 3, \dots$*

Then the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{b_n}{a_n}$ is irrational.

Proof. For the proof we need the following known lemma [8]:

Lemma 2. If $n f(n) \to 0$ and the inequality $|\alpha - p/q| \leqslant f(q)$ is satisfied for an infinity of mutually different rational numbers p/q, then α is an irrational number.

Take $p_k/q_k = b_1/a_1 - b_2/a_2 + \dots + (-1)^{k-1} b_k/a_k$ and $\theta = \lim_{n\to\infty} p_k/q_k$. Then we have $|\theta - p_k/q_k| = b_{k+1}/a_{k+1} - b_{k+2}/a_{k+2} + \dots < b_{k+1}/a_{k+1} = v_{k+1} \leqslant f(a_1 a_2 \dots a_k)$. Now lemma 2 implies at once the theorem.

THEOREM 4. *Let (b_n) and (a_n) be two sequences of positive integers satisfying $a_{n+1} \geqslant a_n^2 \cdot \frac{b_{n+1}}{b_n} + a_n \cdot \frac{b_{n+1}}{b_n}$ for all large n. Suppose that here is no equality for all n, sufficiently large. Then the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{b_n}{a_n}$ is irrational.*

Proof. We use the following theorem of H. Tietze [12]:
If the elements of a continued fraction

$$b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \frac{a_3|}{|b_3} + \dots \ (a_n \neq 0)$$

are integer numbers such that for all sufficiently large n one has

$$b_n \geqslant |a_n|$$

$$b_n \geqslant |a_n| + 1 \text{ for } a_{n+1} < 0 \qquad (*)$$

then the continued fraction is convergent having an irrational value, with exception when for all large n we have

$$a_n < 0, \ b_n = |a_n| + 1 \qquad (**)$$

In this case the continued fraction or is divergent or has a rational value.

In order to prove the theorem, we shall use the fact that the series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n/a_n$ and the continued fraction $\frac{b_1|}{|a_1} + \frac{a_1^2 b_2|}{|a_2 b_1 - a_1 b_2} + \frac{a_2^2 b_3|}{|a_3 b_2 - a_2 b_1} + \dots$ are equivalent. This follows e.g. from the known theorem ([6], p. 206) which asserts that the series $\sum_{n=0}^{\infty} c_n$ and the continued fraction

$$c_0 + \frac{c_1|}{|1} - \frac{\frac{c_2}{c_1}\Big|}{\Big|1 + \frac{c_2}{c_1}} - \frac{\frac{c_3}{c_2}\Big|}{\Big|1 + \frac{c_3}{c_2}} - \dots$$

are equivalent, by putting

$$c_n = (-1)^{n-1} b_n/a_n, \ n \geqslant 1 \text{ and } c_0 = 0.$$

On other argument which implies also this remark is based on the identity $\frac{1}{x} - \frac{1}{y} = \frac{1}{x + \frac{x^2}{y - x}}$. Apply this relation for $x = a_1/b_1$, $y = 1/\sum_{i \geqslant 2} (-1)^{i-1} b_i/a_i$ and then for $x = -a_2/b_2$, $y = 1/\sum_{i \geqslant 3} (-1)^{i-1} \cdot b_i/a_i$; etc.

In view of $(*)$ one can write: $a_{n+1} b_n - a_n b_{n+1} \geqslant a_n^2 b_{n+1}$ which is exactly the supposed inequality. Tietze's theorem implies (also using $(**)$) the convergence of the continued fraction which has an irrational value.

Finally, we prove a theorem for absolute convergent seies, using an idea by N. J. Lord [3]

THEOREM 5. *Let* (a_n), (b_n) *be two sequences of integers,* $a_n \neq 0$, $b_n \neq 0$ $(n = 1, 2, 3, \ldots)$ *satisfying:* (1) $2 |a_n| < \left|\frac{a_{n+1}}{b_{n+1}}\right|$ *for all* $n = 1, 2, 3, \ldots$ (2) $a_n | a_{n+1}$ *for all* n *(i.e.* (a_n) *is a factorial sequence, see* [9]), (3) *every integer is a divisor of some* a_n.

Then $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ *is convergent and its sum is irrational.*

Proof. First remark that (1) implies $|a_{n+1}| > |a_n| \cdot \left|\frac{b_{n+1}}{b_n}\right|$ which, by the ratio test, guarantees the absolute convergence of the above sum to A, say. Next note, that (1) implies also that $|a_{n+k}| > 2^k |a_n| \cdot |b_{n+k}|$ for all $n, k = 1, 2, 3, \ldots$. Hence

$$\left| a_n \cdot \frac{b_{n+1}}{a_{n+1}} + a_n \cdot \frac{b_{n+2}}{a_{n+2}} + \ldots \right| < \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1 \qquad (*)$$

Now assume that A is rational. Then, by (3) $a_n \cdot A$ is an integer for some n. Also $a_n\left(\frac{b_1}{a_1} + \ldots + \frac{b_n}{a_n}\right)$ is an integer, by (2). Therefore $S = \left| a_n A - a_n\left(\frac{b_1}{a_1} + \ldots + \frac{b_n}{a_n}\right)\right|$ also is an integer. But $S = \left| a_n \left(\frac{b_{n+1}}{a_{n+1}} + \frac{b_{n+2}}{a_{n+2}} + \ldots\right)\right| = \left| a_n \cdot \frac{b_{n+1}}{a_{n+1}} \right| \cdot \left| \left(1 + \frac{b_{n+2}}{a_{n+2}} \cdot \frac{a_{n+1}}{b_{n+1}} + \frac{b_{n+3}}{a_{n+3}} \cdot \frac{a_{n+1}}{b_{n+1}} + \ldots\right)\right|$

Here the secons term $< 1 + \frac{1}{|b_{n+2}|} \cdot |(b_{n+2}/a_{n+2})\, a_{n+1} + (b_{n+3}/a_{n+3}) \cdot a_{n+1} + \ldots| = 1 + \frac{1}{|b_{n+1}|} \cdot |\Sigma| \underset{(*)}{<} 1 + \frac{1}{|b_{n+1}|} \leqslant 2$, i.e. $S < 1$ by (1). On the other hand one has $\left|\frac{1}{b_{n+1}} \cdot \Sigma\right| < 1$ by $(*)$ and $|b_{n+1}| \geqslant 1$.

Thus $1 + \frac{1}{b_{n+1}} \Sigma > 0$, and $0 < S < 1$ and so S cannot be an integer. This contradiction shows that A is irrational.

Finally, we mention without proof the following result [10] based on ideas used for proving theorems 2 and 5:

THEOREM 6. *Let* $1 < a_1 \leqslant a_2 \leqslant a_3 \leqslant \ldots$ *be a sequence of integers with the properties* $a_n \to \infty$ $(n \to \infty)$ *and* $n \mid a_1 a_2 \ldots a_n$ *for all* $n = 1, 2, 3, \ldots$ *Let* (b_n) *be a bounded sequence of integers such that* $b_n \neq 0$ *for infinitely many values of* n.

Then the series $\sum_{n=1}^{\infty} \frac{b_n}{a_1 a_2 \ldots a_n}$ *is irrational.*

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STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXIII, 4, 1988

AN EXTENSION OF BECKER'S UNIVALENCE CRITERION

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Received: June 1, 1988

REZUMAT. — Generalizare a criteriului de univalenţă al lui Becker. În acest articol se obţin condiţii suficiente de univalenţă pentru funcţii olomorfe în discul unitate, utilizînd metoda lanţurilor de subordonare.

1. DEFINITION 1. Let $f(z)$ and $g(z)$ be two analytic functions in $U=\{z:|z|<1\}$. We say that $f(z)$ is *subordinate* to $g(z)$, written $f(z)\prec g(z)$, or $f\prec g$, if there exists a function $\varphi(z)$ analytic in U which satisfies $\varphi(0)=0$, $|\varphi(z)|<1$ and:

$$f(z)=g(\varphi(z)),\quad |z|<1. \tag{1}$$

DEFINITION 2. A function $L(z, t)$, $z\in U$, $t\in I=[0,+\infty)$ is a *Loewner chain*, or a *subordination chain* if $L(z,t)$ is analytic and univalent in U for all $t\in I$, is continuously differentiable on I for all $z\in U$, and for all $t_1, t_2\in I$, $0\leqslant t_1<t_2$, $L(z,t_1)\prec L(z,t_2)$.

It is known the next univalence criterion, due to Becker [1]:

THEOREM *If $f(z)$ is analytic in* U, $f(0)=f'(0)-1=0$ *and:*

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right|\leqslant 1,\quad \forall z\in U, \tag{2}$$

then the function $f(z)$ is univalent in U.

We will need the following theorem to prove our results.

THEOREM ([5], [6]). *If there exists $r_0\in(0,1]$ such that the function*

$$L(z,t)=a_1(t)z+\ldots,\ a_1(t)\neq 0,\ \lim_{t\to\infty} a_1(t)=\infty \tag{3}$$

be analytic in $U_{r_0}=\{z:|z|<r_0\}$ for all $t\in I$, continuously differentiable on I for all $z\in U_{r_0}$ and if there exists an analytic function $p(z,t)$ in U for any $t\in I$, continuous on I for all $z\in U$ such that:

$$\operatorname{Re} p(z,t)>0, \tag{4}$$

and

$$z\,\frac{\partial L(z,t)}{\partial z}=p(z,t)\,\frac{\partial L(z,t)}{\partial t},\ \textit{for all } z\in U,\ t\in I, \tag{5}$$

then the function $L(z,t)$ is a subordination chain.

2. In this note we obtain a sufficient condition for univalence of an analytic function in the unit disc, which generalizes the well-known condition due to Becker [1], using the notion of subordination chain.

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THEOREM 1. *Let $f(z)$, $g(z)$ be two analytic functions in U* with $f(0)=g(0)=f'(0)-1=g'(0)-1=0$, *$g'(z)\neq 0$ for all $z\in U$, and λ be a complex number*, $\operatorname{Re}\lambda>1$. *If:*

$$\left|\frac{f'(z)}{g'(z)}-\frac{\lambda}{2}\right|<\frac{|\lambda|}{2}, \tag{6}$$

and

$$\left||z|^{\lambda}\left(\frac{f'(z)}{g'(z)}-1\right)+(1-|z|^{\lambda})\frac{zg''(z)}{g'(z)}+1-\frac{\lambda}{2}\right|\leqslant\frac{|\lambda|}{2}, \tag{7}$$

for all $z\in U$, then the function $f(z)$ is univalent in U.

Proof. Let $L(z,t)$ be the function defined by:

$$L(z,t)=f(e^{-t}z)+(e^{\lambda t}-1)\,e^{-t}zg'(e^{-t}z) \tag{8}$$

which is analytic in U for $t\in I$.

We will prove that $L(z,t)$ is a subordination chain.

Because $f(z)$ and $g(z)$ are analytic functions in U, $L(z,t)$ is continuously differentiable on I for any $z\in U$, $(r_0=1)$.

We remark that $L(z,t)=a_1(t)z+\ldots$, where $a_1(t)=e^{(\lambda-1)t}$, $a_1(t)\neq 0$ for any $t\in I$ and $|a_1(t)|=e^{(\operatorname{Re}\lambda-1)t}$. We have $\operatorname{Re}\lambda-1>0$ and then $\lim\limits_{t\to\infty}|a_1(t)|=\infty$, hence $\lim\limits_{t\to\infty}a_1(t)=\infty$.

From (8) we obtain:

$$\frac{\partial L(z,t)}{\partial z}=e^{-t}f'(e^{-t}z)+(e^{\lambda t}-1)\;e^{-t}g'(e^{-t}z)+(e^{\lambda t}-1)\,e^{-2t}z\cdot g''(e^{-t}z), \tag{9}$$

and

$$\frac{\partial L(z,t)}{\partial t}=-e^{-t}z\cdot f'(e^{-t}z)+\lambda e^{\lambda t}\cdot e^{-t}z\cdot g'(e^{-t}z)-$$
$$-(e^{\lambda t}-1)\;[e^{-t}zg'(e^{-t}z)+e^{-2t}z^2\;g''(e^{-t}z)]. \tag{10}$$

Let $p(z,t)$ be the function defined for $z\in U$, $t\in I$, by:

$$p(z,t)=z\frac{\partial L(z,t)}{\partial z}\Big/\frac{\partial L(z,t)}{\partial t}. \tag{11}$$

Replacing (9) and (10) in (11) we obtain:

$$p(z,t)=\frac{e^{-t}z[f'(e^{-t}z)+(e^{\lambda t}-1)(g'(e^{-t}z)+e^{-t}zg''(e^{-t}z))]}{e^{-t}z[\lambda e^{\lambda t}g'(e^{-t}z)-f'(e^{-t}z)-(e^{\lambda t}-1)(g'(e^{-t}z)+e^{-t}zg''(e^{-t}z))]} \tag{12}$$

In order to prove that the function $p(z,t)$ is analytic, with positive real part in U for any $t\in I$ and continuous on I for all $z\in U$, it is sufficient to prove that the function

$$w(z,t)=\frac{p(z,t)-1}{p(z,t)+1}. \tag{13}$$

is analytic in U, continuous on I, and

$$|w(z,t)|<1 \text{ for all } z\in U,\ t\in I. \tag{14}$$

From (11) and (13) we have:

$$w(z, t) = \frac{z\dfrac{\partial L(z,t)}{\partial z} - \dfrac{\partial L(z,t)}{\partial t}}{z\dfrac{\partial L(z,t)}{\partial z} + \dfrac{\partial L(z,t)}{\partial t}}.$$

Replacing (9) and (10) we obtain:

$$w(z, t) = \frac{2}{\lambda} - 1 + \frac{2}{\lambda e^{\lambda t}}\left[\frac{f'(e^{-t}z)}{g'(e^{-t}z)} - 1\right] + \frac{2}{\lambda e^{\lambda t}}(e^{\lambda t} - 1)\frac{e^{-t}z \cdot g''(e^{-t}z)}{g'(e^{-t}z)}. \quad (15)$$

For $z = 0$ the inequality (14) becomes:

$$\left|\frac{2}{\lambda} - 1\right| < 1,$$

which is equivalent to $\operatorname{Re} \lambda > 1$.

For $z \neq 0$ and $t = 0$ the inequality (14) becomes:

$$\left|\frac{2}{\lambda}\,\frac{f'(z)}{g'(z)} - 1\right| < 1,$$

which is equivalent to (6).

For $z \neq 0$ and $t > 0$ the function $w(z, t)$ is analytic in $\overline{U}$ because $|e^{-t}z| \leqslant |e^{-t}| < 1$ for any $z \in \overline{U}$. Then:

$$|w(z, t)| < \max_{|z|=1} |w(z, t)| = |w(e^{i\theta}, t)|, \quad (16)$$

where θ is a real number.

It is sufficient that

$$|w(e^{i\theta}, t)| \leqslant 1, \quad \forall t \in I. \quad (17)$$

If $u = e^{-t} e^{i\theta}$, $u \in U$, then $|u| = e^{-t}$, $t = -\ln|u|$ and from (15) we have:

$$w(e^{i\theta}, t) = \frac{2}{\lambda} - 1 + \frac{2|u|^{\lambda}}{\lambda}\left[\frac{f'(|u|e^{i\theta})}{g'(|u|e^{i\theta})} - 1\right] + \frac{2|u|^{\lambda}}{\lambda}\left(\frac{1}{|u|^{\lambda}} - 1\right)\frac{|u|e^{i\theta}g''(|u|e^{i\theta})}{g'(|u|e^{i\theta})}.$$

The inequality (17) becomes:

$$\left| |u|^{\lambda}\left[\frac{f'(u)}{g'(u)} - 1\right] + (1 - |u|^{\lambda})\frac{u g''(u)}{g'(u)} + 1 - \frac{\lambda}{2}\right| \leqslant \frac{|\lambda|}{2}. \quad (18)$$

Because $u \in U$, from (16) and (7) it results that $|w(z, t)| < 1$ for any $z \in U$ and $t \in I$.

Then, from Theorem B the function $L(z, t)$ is a subordination chain and hence the analytic function $L(z, 0) = f(z)$ is univalent in U.

Remarks. 1) For $\lambda = 2$ and $f(z) = g(z)$, Theorem 1 becomes Theorem A, that is the Becker's univalence criterion [1].

2) If λ is a real number, $\lambda > 1$, then Theorem 1 follows from the generalization of the Becker's univalence criterion obtained by Lewandowski [2].

3) Theorem 1 from [4], for $\alpha = 1$ and $m + 1 = \lambda \in C$ becomes Theorem 1 from this note.

COROLARY 1. *Let* $f(z)$, $g(z)$ *be two analytic functions in* U with $f(0) = g(0) = f'(0) - 1 = g'(0) - 1 = 0$, $g'(z) \neq 0$ *for all* $z \in U$, *and* λ *be a real number* $\lambda > 1$. *If:*

$$\left|\frac{f'(z)}{g'(z)} - \frac{\lambda}{2}\right| < \frac{\lambda}{2}, \tag{19}$$

and

$$\left|\frac{zg''(z)}{g'(z)} + 1 - \frac{\lambda}{2}\right| \leqslant \frac{\lambda}{2}, \quad \forall z \in U, \tag{20}$$

then $f(z)$ *is an univalent function in* U.

Proof. From (19) and (20) it follows that the conditions (6) and (7) are satisfied. Hence, by Theorem 1 it results that $f(z)$ is a univalent function in U.

Geometrically the conditions (19) and (20) show that $\frac{f'(z)}{g'(z)}$ and $\frac{zg''(z)}{g'(z)} + 1$ belong to the disc with the centre in the point $\frac{\lambda}{2}$ and with the radius $\frac{\lambda}{2}$. If $\lambda \to +\infty$ this disc becomes the right semiplane. In this case it results:

COROLARY 2. *Let* $f(z)$, $g(z)$ *be two analytic functions in* U *with* $f(0) = g(0) = f'(0) - 1 = g'(0) - 1$, $g'(z) \neq 0$ *for all* $z \in U$. *If:*

$$\operatorname{Re}\frac{f'(z)}{g'(z)} > 0 \tag{21}$$

and

$$\operatorname{Re}\frac{zg''(z)}{g'(z)} + 1 > 0 \ \textit{for all} \ z \in U, \tag{22}$$

then $f(z)$ *is a univalent function in* U.

Remark. 4) This result was obtained by Kaplan for the close-to-convex functions [2].

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BOUNDARY VALUE PROBLEMS WITH DERIVATIVES IN THE THEORY OF FUNCTIONS OF TWO COMPLEX VARIABLES

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Received: March 25, 1988

REZUMAT. — Probleme la limită cu derivate în teoria funcţiilor de două variabile complexe. În această lucrare problemele la limită de tip Riemann—Hilbert pentru două variabile complexe studiate în lucrarea [1] se extind la cazul cînd în aceste probleme apar limitele derivatelor de un ordin anumit al unei funcţii olomorfe de două variabile complexe. Se enunţă şi se rezolvă problema la limită omogenă de tip Riemann—Hilbert şi se enunţă o problemă neomogenă de acelaşi tip, ambele pentru cazul a două variabile complexe.

1. **Introduction.** In a previous paper [1] we have formulated and solved a homogeneous Riemann—Hilbert boundary value problem for analytic functions of two complex variables. A non-homogeneous Riemann—Hilbert problem has been also formulated, but yet not solved.

Our purpose is to develop in what follows analogous boundary value problems containing this time the limiting values of the derivatives of a certain order of a function of two complex variables toward the considered boundary. In order to carry out these problems we take into account as in [1] a space of two complex variables z_1 and z_2, and denote it by $D = D_1 \times D_2$ which we assume to be bounded by the contour $L = L_1 \times L_2$. Here D_1 is a simple connected domain in the plane of z_1 and L_1 a closed smooth contour surrounding D_1, while D_2 is an analogous domain in the omplex plane of z_2, encircled by a closed smooth contour L_2 in this plane. The contour $L = L_1 \times L_2$ is supposed as built up by a finite number of closed smooth non-intersecting contours $L_1^{(j)}$, $L_2^{(j)}$, $(j = 0, 1, 2, \ldots)$, where $L_1^{(j)}$ or $L_2^{(j)}$, $(j = 0, 1, 2, \ldots)$, are contours in the z_1-or z_2-planes respectively. It is assumed that the contour $L_1^{(0)}$ encircles all the contours $L_1^{(j)}$, $(j = 1, 2, \ldots)$, while $L_2^{(0)}$ in its turn encircles all the contours $L_2^{(j)}$, $(j = 1, 2, \ldots)$. In this manner the contours $L_1^{(j)}$ $(j = 0, 1, 2, \ldots)$, divide the domain D_1 in an inner domanin D_1^+ and a series of external domains $D_1^{(j)-}$, $(j = 0, 1, 2, \ldots)$, while the contours $L_2^{(j)}$, $(j = 0, 1, 2, \ldots)$, divide the domain D_2 in an inner domains D_2^+ and a series of external domains $D_2^{(j)-}$, $(j = 0, 1, 2, \ldots)$. One usually chooses a positive sense on the contour $L = L_1 \times L_2$, such that the inner domain D_1^+ lies at the left of the contour L_1 and the inner domain D_2^+ lies at the left of the contour L_2. In these circumstances the external domains $D_1^{(j)-}$ lie at the right of the contour L_1, while the external domains $D_2^{(j)-}$ lie at the

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right of L_2. We always have here $j = 0, 1, 2, \ldots$. We also denote the totality of the external domains by D_1^- and D_2^-.

In this complex space, using the same notations as in [1], [3], an analytic function of two complex variables may be represented by a double Cauchy integral

$$\Phi(z_1, z_2) = -\frac{1}{4\pi^2}\int_{L_1}\int_{L_2} \frac{\varphi(\tau_1, \tau_2)d\tau_1 d\tau_2}{(\tau_1 - z_1)(\tau_2 - z_2)}, \tag{1.1}$$

where $\varphi(\tau_1, \tau_2)$ is a function of the two complex variables τ_1, τ_2, which satisfies a Hölder condition on the closed contour $L = L_1 \times L_2$ and z_1, z_2 represent some points of the two complex variables space, but not on the contour.

Let be $\Phi^{++}(t_1, t_2)$, $\Phi^{-+}(t_1, t_2)$, $\Phi^{+-}(t_1, t_2)$, and $\Phi^{--}(t_1, t_2)$ the limiting values of the integral $\Phi(z_1, z_2)$ when z_1 and z_2 aproache some points τ_1, τ_2 on the contours L_1 or L_2 along any paths entirely in one of the domains $D_1^+ \times D_2^+$, $D_1^- \times D_2^+$, $D_1^+ \times D_1^-$ or $D_1^- \times D_2^-$ respectively. These limiting values are given by the generalized Plemelj—Sohotskiĭ formulas or Kakicsev formulas, as we have called them in [1], and which are of the following form [1], [2], [4]:

$$\left.\begin{matrix}\Phi^{++}(t_1, t_2)\\ \Phi^{--}(t_1, t_2)\end{matrix}\right\} = \frac{1}{4}(\varphi \pm S_1\varphi \pm S_2\varphi + S_{12}\varphi), \tag{1.2a}$$

$$\left.\begin{matrix}\Phi^{-+}(t_1, t_2)\\ \Phi^{+-}(t_1, t_2)\end{matrix}\right\} = \frac{1}{4}(-\varphi \pm S_1\varphi \mp S_2\varphi + S_{12}\varphi), \tag{1.2b}$$

where

$$S_1\varphi = \frac{1}{\pi i}\int_{L_1} \frac{\varphi(\tau_1, t_2)}{\tau_1 - t_1}\, d\tau_1, \tag{1.3}$$

$$S_2\varphi = \frac{1}{\pi i}\int_{L_2} \frac{\varphi(t_1, \tau_2)}{\tau_2 - t_2}\, d\tau_2, \tag{1.4}$$

$$S_{12}\varphi = -\frac{1}{\pi^2}\int_{L_1}\int_{L_2} \frac{\varphi(\tau_1, \tau_2)d\tau_1\, d\tau_2}{(\tau_1 - t_1)(\tau_2 - t_2)}. \tag{1.5}$$

We must point out that in all the above formulas (1.2a) — (1.5) the integrals are understooud to be Cauchy principal value integrals.

It is immediately seen that by suitable combinations of the relations (1.2a) — (1.2b), the following new ones may be deduced

$$\Phi^{++}(t_1, t_2) \pm \Phi^{+-}(t_1, t_2) \pm \Phi^{-+}(t_1, t_2) + \Phi^{--}(t_1, t_2) = \begin{cases} S_{12}\varphi \\ \varphi \end{cases} \tag{1.6a}$$

$$\Phi^{++}(t_1, t_2) \mp \Phi^{+-}(t_1, t_2) \pm \Phi^{-+}(t_1, t_2) - \Phi^{--}(t_1, t_2) = \begin{cases} S_1\varphi \\ S_2\varphi \end{cases} \tag{1.6b}$$

Thus, in conclusion $\Phi(z_1, z_2)$ is a sectionally holomorphic function and vanishes at infinity, while the limiting values $\Phi^{++}(t_1, t_2)$, $\Phi^{+-}(t_1, t_2)$, $\Phi^{-+}(t_1, t_2)$, and $\Phi^{--}(t_1, t_2)$ are functions fulfilling a Hölder condition on $L = L_1 \times L_2$.

2. Derivatives of Kakicsev formulas for two complex variables. Let now take the partial derivatives of the expression (1.1). We obtain

$$[\Phi(z_1, z_2)]^{(m,n)} = \frac{\partial^{m+n}\Phi(z_1, z_2)}{\partial^m z_1 \partial^n z_2} = -\frac{m!\,n!}{4\pi^2}\iint_{L_1 L_2} \frac{\varphi(\tau_1, \tau_2)\, d\tau_1\, d\tau_2}{(\tau_1 - z_1)^{m+1}(\tau_2 - z_2)^{n+1}}. \tag{2.1}$$

We apply then the same operations to the formulas (1.2*a*) and (1.2*b*). It follows

$$\left.\begin{matrix}[\Phi^{++}(t_1, t_2)]^{(m,n)}\\ [\Phi^{--}(t_1, t_2)]^{(m,n)}\end{matrix}\right\} = \frac{1}{4}\left(\varphi^{(m,n)} \pm [S_1\varphi]^{(m,n)} \mp [S_2\varphi]^{(m,n)} + [S_{12}\varphi]^{(m,n)}\right), \tag{2.2a}$$

$$\left.\begin{matrix}[\Phi^{-+}(t_1, t_2)]^{(m,n)}\\ [\Phi^{+-}(t_1, t_2)]^{(m,n)}\end{matrix}\right\} = \frac{1}{4}\left(-\varphi^{(m,n)} \pm [S_1\varphi]^{(m,n)} \pm [S_2\varphi]^{(m,n)} + [S_{12}\varphi]^{(m,n)}\right), \tag{2.2b}$$

where

$$[S_1\varphi]^{(m,n)} = \frac{m!}{\pi i}\int_{L_1} \frac{\varphi^{(0,n)}(\tau_1, t_2)}{(\tau_1 - t_1)^{m+1}}\, d\tau_1, \tag{2.3}$$

$$[S_2\varphi]^{(m,n)} = \frac{n!}{\pi i}\int_{L_2} \frac{\varphi^{(m,0)}(t_1, \tau_2)}{(\tau_2 - t_2)^{n+1}}\, d\tau_2, \tag{2.4}$$

$$[S_{12}\varphi]^{(m,n)} = -\frac{m!\,n!}{\pi^2}\iint_{L_1 L_2} \frac{\varphi(\tau_1, \tau_2)\, d\tau_1\, d\tau_2}{(\tau_1 - t_1)^{m+1}(\tau_2 - t_2)^{n+1}}. \tag{2.5}$$

Or, we have assumed from the first that L_1 and L_2 are closed smooth contours, so that by performing in (2.3) — (2.5) the integrals along the contours L_1 and L_2 one obtains respectively

$$[S_1\varphi]^{(m,n)} = \frac{1}{\pi i}\int_{L_1} \frac{\varphi^{(m,n)}(\tau_1, t_2)}{\tau_1 - t_1}\, d\tau_1 = S_1\varphi^{(m,n)}, \tag{2.6}$$

$$[S_2\varphi]^{(m,n)} = \frac{1}{\pi i}\int_{L_2} \frac{\varphi^{(m,n)}(t_1, \tau_2)}{\tau_2 - t_2}\, d\tau_2 = S_2\varphi^{(m,n)}, \tag{2.7}$$

$$[S_{12}\varphi]^{(m,n)} = -\frac{1}{\pi^2}\iint_{L_1 L_2} \frac{\varphi^{(m,n)}(\tau_1, \tau_2)}{(\tau_1 - t_1)(\tau_2 - t_2)}\, d\tau_1\, d\tau_2 = S_{12}\varphi^{(m,n)}. \tag{2.8}$$

Then again by taking partial derivatives of the relations (1.6*a*), we get

$$[\Phi^{++}(t_1, t_2)]^{(m,n)} \pm [\Phi^{+-}(t_1, t_2)]^{(m,n)} \pm [\Phi^{-+}(t_1, t_2)]^{(m,n)} + [\Phi^{--}(t_1, t_2)]^{(m,n)} = \\ = \begin{cases}[S_{12}\varphi]^{(m,n)},\\ \varphi^{(m,n)},\end{cases} \tag{2.9a}$$

$$[\Phi^{++}(t_1, t_2)]^{(m,n)} \mp [\Phi^{+-}(t_1, t_2)]^{(m,n)} \pm [\Phi^{-+}(t_1, t_2)]^{(m,n)} - [\Phi^{--}(t_1, t_2)]^{(m,n)} = \\ = \begin{cases}[S_1\varphi]^{(m,n)},\\ [S_2\varphi]^{(m,n)}.\end{cases} \tag{2.9b}$$

In this formulas the right members are given either by (2.3)—(2.5) or by (2.6)—(2.8).

3. Extension of the formula of S.N.Al'per and Ju. S. Cserskom. S.N.Al'per and Ju. I. Cserskom have proved that for the limiting values of an analytic function $\Phi(z)$ of a single complex variable on the contour L surrounding a domain D in the complex plane of z, the following relation holds [3].

$$[\Phi^{(m)}(t)]^{\pm} = [\Phi^{\pm}(t)]^{(m)}, \tag{3.1}$$

where t is an arbitrary point on L.

We will show in the following that a similar relation also exists for the case of an analytic function $\Phi(z_1, z_2)$ of two complex variables z_1, z_2.

In order to deduce such an extended relation, we begin by establishing a particular one, i.e.

$$[\Phi^{(1,0)}(t_1, t_2)]^{++} = [\Phi^{++}(t_1, t_2)]^{(1,0)}. \tag{3.2}$$

First of all, we introduce the notation

$$H(z_1, z_2) = \Phi^{(1,0)}(z_1, z_2), \tag{3.3}$$

where

$$\Phi^{(1,0)}(z_1, z_2) = -\frac{1}{4\pi^2} \iint\limits_{L_1 L_2} \frac{\varphi(\tau_1, \tau_2)\, d\tau_1\, d\tau_2}{(\tau_1 - t_1)^2(\tau_2 - t_2)} =$$

$$= -\frac{1}{4\pi^2} \iint\limits_{L_1 L_2} \frac{\varphi^{(1,0)}(\tau_1, \tau_2)\, d\tau_1\, d\tau_2}{(\tau_1 - t_1)(\tau_2 - t_2)}, \tag{3.4}$$

so that we may write

$$[\Phi^{(1,0)}(t_1, t_2)]^{++} = H^{++}(t_1, t_2). \tag{3.5}$$

Now we will prove the equality

$$[\Phi^{++}(t_1, t_2)]^{(1,0)} = H^{++}(t_1, t_2). \tag{3.6}$$

But, according to the basic definitions, $[\Phi^{++}(t_1, t_2)]^{(1,0)}$ is continuous inside the domain $D = D_1 \times D_2$ and fulfils a Hölder condition on the contour $L = L_1 \times L_2$. Thus, we have

$$[\Phi^{++}(t_1, t_2)]^{(1,0)} = \lim_{\Delta t_1 \to 0} \frac{\Phi^{++}(t_1 + \Delta t_1, t_2) - \Phi^{++}(t_1, t_2)}{\Delta t_1} \tag{3.7}$$

On the other hand, by virtue of Cauchy's theorem, the integral

$$\Phi^{++}(z_1, t_2) = \int_{z_{10}}^{z_1} H^{++}(\tau_1, t_2)\, d\tau_1 \tag{3.8}$$

is independent of the choice of the path of integration in the domain D_1 and is simgle-valued. So we have

$$\Phi^{++}(t_1 + \Delta t_1, t_2) = \int_{t_{10}}^{t_1+\Delta t_1} H^{++}(\tau_1, t_2)\, d\tau_1, \tag{3.9}$$

and

$$\Phi^{++}(t_1, t_2) = \int_{t_{10}}^{t_1} H^{++}(\tau_1, t_2)\, d\tau_1. \tag{3.10}$$

Inserting into (3.7) from (3.9) and (3.10), we may write succesively

$$[\Phi^{++}(t_1, t_2)]^{(1,0)} = \lim_{\Delta t_1 \to 0} \frac{1}{\Delta t_1} \left\{ \int_{t_{10}}^{t_1+\Delta t_1} H^{++}(\tau_1, t_2) d\tau_1 - \int_{t_{10}}^{t_1} H^{++}(\tau_1, t_2)\, d\tau_1 \right\} =$$

$$= \lim_{\Delta t_1 \to 0} \frac{1}{\Delta t_1} \int_{t_1}^{t_1+\Delta t_1} H^{++}(\tau_1, t_2)\, d\tau_1. \tag{3.11}$$

Let us consider the always positive expression

$$\mathfrak{M}(t_1 + \Delta t_1, t_2) = \left| \frac{\Phi^{++}(t_1 + \Delta t_1, t_2) - \Phi^{++}(t_1, t_2)}{\Delta t_1} - H^{++}(t_1, t_2) \right|. \tag{3.12}$$

Introducing here the expressions (3.9) and (3.10), then taking into account that

$$\Delta t_1 = \int_{t_1}^{t_1+\Delta t_1} d\tau_1, \tag{3.13}$$

we get, if $\Delta t_1 > 0$,

$$\mathfrak{M}(t_1 + \Delta t_1, t_2) = \left| \frac{1}{|\Delta t_1|} \int_{t_{10}}^{t_1+\Delta t_1} H^{++}(\tau_1, t_2)\, d\tau_1 - \right.$$

$$\left. - \frac{1}{|\Delta t_1|} \int_{t_{10}}^{t_1} H^{++}(\tau_1, t_2) d\tau_1 - \frac{1}{|\Delta t_1|} \int_{t_1}^{t_1+\Delta t_1} H^{++}(t_1, t_2)\, d\tau_1 \right| \tag{3.14}$$

$$= \frac{1}{|\Delta t_1|} \left| \int_{t_1}^{t_1+\Delta t_1} \{H^{++}(\tau_1, t_2) - H^{++}(t_1, t_2)\}\, d\tau_1 \right|$$

$$\leqslant \frac{1}{|\Delta t_1|} \max_{\tau_1 \in (t_1, t_1+\Delta t_1)} |H^{++}(\tau_1, t_1) - H^{++}(t_1, t_2)|\, |\Delta t_1|.$$

Hence

$$\mathfrak{M}(t_1 + \Delta t_1, t_2) = \max_{\tau_1 \in t_1+\Delta t_1} |H^{++}(\tau_1, t_2) - H^{++}(t_1, t_2)|. \tag{3.15}$$

However, as $H^{++}(t_1, t_2)$ is a continuous function on the contour, $L = L_1 \times L_2$, we have

$$\lim \mathfrak{M}(t_1+\Delta t_1, t_2) = \lim_{\substack{\Delta t_1 \to 0 \\ \tau_1 \in (t_1, t_1+\Delta t_1)}} \frac{1}{|\Delta t_1|} \max |H^{++}(\tau_1, t_2) - H^{++}(t_1, t_2)| < \varepsilon, \tag{3.16}$$

where ε is any small positive real number approaching zero, if $0 < \Delta t_1 < 1$
Now, coming back to the formula (3.12), we may write

$$\lim_{\Delta t_1 \to 0} M(t_1 + \Delta t_1, t_2) = \lim_{\substack{\Delta t_1 \to 0 \\ \tau_1 \in (t_1, t_1 + \Delta t_1)}} \frac{1}{|\Delta t_1|} \left| \frac{\Phi^{++}(t_1 + \Delta t_1, t_2) - \Phi^{++}(t_1, t_2)}{\Delta t_1} - H^{++}(t_1, t_2) \right| < \varepsilon \qquad (3.17)$$

We conclude therefore that (3.6) holds and by comparing (3.5) with (3.6) it become (3.2). The foregoing demonstration folows that one given in [3] and [5] for an analogous formula to the formula [3.2], but for the case of only one variable.

In the same manner we can prove the relation

$$[\Phi^{++}(t_1, t_2)]^{(0,1)} = [\Phi^{(1,0)}(t_1, t_2)]^{++} \qquad (3.18)$$

By repeating the same operation, we can also state the more general relation

$$[\Phi^{++}(t_1, t_2)]^{(m,n)} = [\Phi^{(m,n)}(t_1, t_2)]^{++} \qquad (3.19)$$

It is obvious that we may demonstrate by the same procedure the relations

$$[\Phi^{--}(t_1, t_2)]^{(m,n)} = [\Phi^{(m,n)}(t_1, t_2)]^{--}, \qquad (3.20)$$

$$[\Phi^{+-}(t_1, t_2)]^{(m,n)} = [\Phi^{(m,n)}(t_1, t_2)]^{+-}, \qquad (3.21)$$

$$[\Phi^{-+}(t_1, t_2)]^{(m,n)} = [\Phi^{(m,n)}(t_1, t_2)]^{-+}. \qquad (3.22)$$

4. **An alternative form of the derived Kakicsev formula.** The results expounded formerly permit us to write down the Kakicsev formulas (2.2*a*) and (2.2*b*) under the following form

$$\left.\begin{matrix}[\Phi^{(m,n)}(t_1, t_2)]^{++} \\ [\Phi^{(m,n)}(t_1, t_2)]^{--}\end{matrix}\right\} = \frac{1}{4}\{\varphi^{(m,n)} \pm S_1\varphi^{(m,n)} \pm S_2\varphi^{(m,n)} + S_{12}\varphi^{(m,n)}\}, \qquad (4.1)$$

$$\left.\begin{matrix}[\Phi^{(m,n)}(t_1, t_2)]^{-+} \\ [\Phi^{(m,n)}(t_1, t_2)]^{+-}\end{matrix}\right\} = \frac{1}{4}\{-\varphi^{(m,n)} \pm S_1\varphi^{(m,n)} \mp S_2\varphi^{(m,n)} + S_{12}\varphi^{(m,n)}\} \qquad (4.2)$$

Alike, we may write down the formulas (2.9*a*), (2.9*b*) in the following manner

$$[\Phi^{(m,n)}(t_1, t_2)]^{++} \pm [\Phi^{(m,n)}(t_1, t_2)]^{+-} \pm [\Phi^{(m,n)}(t_1, t_2)]^{-+} + [\Phi^{(m,n)}(t_1, t_2)]^{--} = \begin{cases} S_{12}\varphi^{(m,n)}, \\ \varphi^{(m,n)} \end{cases} \qquad (4.2a)$$

$$[\Phi^{(m,n)}(t_1, t_2)]^{++} \mp [\Phi^{(m,n)}(t_1, t_2)]^{+-} \pm [\Phi^{(m,n)}(t_1, t_2)]^{-+} - [\Phi^{(m,n)}(t_1, t_2)]^{--} = \begin{cases} S_1\varphi^{(m,n)}, \\ S_2\varphi^{(m,n)} \end{cases} \qquad (4.2b)$$

5. **A homogeneous Riemann-Hilbert boundary value problem with derivatives.** Let be in the space of two complex variables z_1, z_2 the closed smooth contour $L = L_1 \times L_2$, such that the contours L_1 and L_2 are built up from a finite

number of closed smooth noninteracting contours $L_1^{(j)}$ and $L_2^{(j)}$ respectively with $j=0, 1, 2, \ldots$. The other definitions and notations are the same as in our previously cited paper [1], as well as that explained above.

All these points being settled, we can formulate the following extension of the Riemann-Hilbert homogeneous boundary value problem:

In a space of two complex variables z_1, z_2 let be a smooth closed contour $L = L_1 \times L_2$ dividing the space into a domain S^+ situated at its left and into other domain S^- situated at the right of this contour; find then a function $\Phi(z_1, z_2)$ possesing m partial derivatives with respect to z_1 and n partial derivatives with respect to z_2 denoted here by $\Phi^{(m,n)}(z_1, z_2)$, these partial derivatives being assumed sectionally holomorphic with a finite order at infinity (that is when $B_0 \to \infty$) and fulfilling on the contour $L = L_1 \times L_2$ the boundary condition

$$[\Phi^{(m,n)}(t_1, t_2)]^{++} [\Phi^{(m,n)}(t_1, t_2)]^{--} = G(t_1, t_2)[\Phi^{(m,n)}(t_1, t_2)]^{+-} [\Phi^{(m,n)}(t_1, t_2)]^{-+}, \quad (5.1)$$

where $G(t_1, t_2)$ is a given function on the contour $L = L_1 \times L_2$, which fulfils on this contour a Hölder condition and do not vanish on this contour. Here t_1 is a variable on L_1, while t_2 is a variable on L_2.

We can also write (5.1) under the form

$$\frac{[\Phi^{(m,n)}(t_1, t_2)]^{++}[\Phi^{(m,n)}(t_1, t_2)]^{--}}{[\Phi^{(m,n)}(t_1, t_2)]^{+-}[\Phi^{(m,n)}(t_1, t_2)]^{-+}} = G(t_1, t_2) \quad (5.2)$$

By virtue of (3.19) the condition (5.1) may be written as

$$[\Phi^{++}(t_1, t_2)]^{(m,n)}[\Phi^{--}(t_1, t_2)]^{(m,n)} = G(t_1, t_2)[\Phi^{+-}(t_1, t_2)]^{(m,n)}[\Phi^{-+}(t_1, t_2)]^{(m,n)} \quad (5.3)$$

and (5.2) under the form

$$\frac{[\Phi^{++}(t_1, t_2)]^{(m,n)}[\Phi^{--}(t_1, t_2)]^{(m,n)}}{[\Phi^{+-}(t_1, t_2)]^{(m,n)}[\Phi^{-+}(t_1, t_2)]^{(m,n)}} = G(t_1, t_2). \quad (5.4)$$

In order to solve the foregoing formulated Riemann-Hilbert homogeneous boundary value problem (5.1) we introduce the notations

$$F(t_1, t_2) = \Phi^{(m,n)}(t_1, t_2), \quad (5.5)$$
$$\chi(t_1, t_2) = \varphi^{(m,n)}(t_1, t_2),$$

such that

$$\begin{aligned} F^{++}(t_1, t_2) &= [\Phi^{(m,n)}(t_1, t_2)]^{++}, \\ F^{--}(t_1, t_2) &= [\Phi^{(m,n)}(t_1, t_2)]^{--}, \\ F^{+-}(t_1, t_2) &= [\Phi^{(m,n)}(t_1, t_2)]^{+-}, \\ F^{-+}(t_1, t_2) &= [\Phi^{(m,n)}(t_1, t_2)]^{-+}. \end{aligned} \quad (5.6)$$

Then, the formulas (4.1*a*) and (4.1*b*) become

$$\left.\begin{aligned} F^{++}(t_1, t_2) \\ F^{--}(t_1, t_2) \end{aligned}\right\} = \frac{1}{4}\{\chi \pm S_1\chi \pm S_2\chi + S_{12}\chi\}, \quad (5.7a)$$

$$\left.\begin{aligned} F^{-+}(t_1, t_2) \\ F^{+-}(t_1, t_2) \end{aligned}\right\} = \frac{1}{4}\{-\chi \pm S_1\chi \mp S_2\chi + S_{12}\chi\}, \quad (5.7b)$$

while the formulas (4.2a), (4.2b) take the following form

$$F^{++}(t_1, t_2) \pm F^{+-}(t_1, t_2) \pm F^{-+}(t_1, t_2) + F^{--}(t_1, t_2) = \begin{cases} S_{12}\chi \\ \chi \end{cases}, \tag{5.8a}$$

$$F^{++}(t_1, t_2) \mp F^{+-}(t_1, t_2) \pm F^{-+}(t_1, t_2) - F^{--}(t_1, t_2) = \begin{cases} S_1\chi \\ S_2\chi \end{cases}; \tag{5.8b}$$

These last relations (5.7a)—(5.8b) are no other than the Kakicsev formulas for the function

$$F(z_1, z_2) = -\frac{1}{\pi^2}\int_{L_1}\int_{L_2} \frac{\chi(\tau_1, \tau_2)d\tau_1\, d\tau_2}{(\tau_1 - z_1)(\tau_2 - z_2)}, \tag{5.9}$$

With the notations (5.5) the boundary value problem (5.1) may be reduced to the boundary value problem previously investigated in [1], that is

$$F^{++}(t_1, t_2)F^{--}(t_1, t_2) = G(t_1, t_2)\, F^{+-}(t_1, t_2)\, F^{-+}(t_1, t_2). \tag{5.10}$$

As we have shown there, by defininig the new functions

$$\Omega(z_1, z_2) = \begin{cases} \Pi_1(z_1)\,\Pi_2(z_2)\,F(z_1, z_2), & \text{in } D_1^+ \times D_2^+, \\ z_1^{\varkappa_1}\Pi_2(z_2)\,F(z_1, z_2), & \text{in } D_1^- \times D_2^+, \\ \Pi_1(z_1)\,z_2^{\varkappa_2}F(z_1, z_2), & \text{in } D_1^+ \times D_2^-, \\ z_1^{\varkappa_1},\, z_2^{\varkappa_2}F(z_1, z_2), & \text{in } D_1^- \times D_2^-, \end{cases} \tag{5.11}$$

and

$$G_0(t_1, t_2) = t_1^{-\varkappa_1}t_2^{-\varkappa_2}\Pi_1(t_1)\,\Pi_2(t_2)\,G(t_1, t_2), \tag{5.12}$$

where $\varkappa_1$ and $\varkappa_2$ are the indices of the function $\ln G(t_1, t_2)$ with respect to the contour L_1 or L_2, the problem (5.10) may be replaced by the new one

$$\Omega^{++}(t_1, t_2)\,\Omega^{--}(t_1, t_2) = G_0(t_1, t_2)\,\Omega^{+-}(t_1, t_2)\,\Omega^{-+}(t_1, t_2) \tag{5.13}$$

Taking here the logarithm in the both parts of this relation, we obtain

$$[\ln\Omega(t_1, t_2)]^{++} + [\ln\Omega(t_1, t_2)]^{--} - [\ln\Omega(t_1, t_2)]^{+-} - [\ln\Omega(t_1, t_2)]^{-+} = \ln G_2(t_1, t_2). \tag{5.14}$$

This formula corespond to the second Kakicsev formula (4.6a); hence we may write

$$\ln\Omega(z_1, z_2) = -\frac{1}{4\pi^2}\int_{L_1}\int_{L_2} \frac{\ln G_0(\tau_1, \tau_2)d\tau_1 d\tau_2}{(\tau_1 - z_1)(\tau_2 - z_2)}, \tag{5.15}$$

or

$$\Omega(z_1, z_2) = e^{\gamma(z_1, z_2)}, \tag{5.16}$$

where we have set

$$\gamma(z_1, z_2) = -\frac{1}{4\pi^2}\int_{L_1}\int_{L_2} \frac{\ln G_0(\tau_1, \tau_2)d\tau_1 d\tau_2}{(\tau_1 - z_1(\tau_2 - z_2)}. \tag{5.17}$$

However, if we wish to return to the original function $\Phi(z_1, z_2)$, we must introduce the derivatives of the density $\varphi(z_1, z_2)$ defined by the partial differential equation

$$\omega^{(m,n)}(z_1, z_2) = \chi(z_1, z_2) = m!\, n! \ln G_0(t_1, t_2). \tag{5.18}$$

Then the second formula (5.8a) receives an analogous form as the second formula (4.2a), that is

$$[\Phi^{(m,n)}(t_1, t_2)]^{++} - [\Phi^{(m,n)}(t_1, t_2)]^{+-} - [\Phi^{(m,n)}(t_1, t_2)]^{-+} + [\Phi^{(m,n)}(t_1, t_2)]^{--} = \\ = \omega^{(m,n)}(t_1, t_2). \tag{5.19}$$

On the other hand, the integral (5.17) becomes

$$\gamma(z_1, z_2) = \Gamma_0^{(m,n)}(z_1, z_2) = -\frac{1}{4\pi^2 m!\, n!} \int_{L_1}\int_{L_2} \frac{\omega^{(m,n)}(\tau_1, \tau_2) d\tau_1\, d\tau_2}{(\tau_1 - z_1)(\tau_2 - z_2)} =$$

$$= -\frac{1}{4\pi^2} \int_{L_1}\int_{L_2} \frac{\omega(\tau_1, \tau_2)\, d\tau_1\, d\tau_2}{(\tau_1 - z_1)^{mH} (\tau_2 - z_2)^{nH}}, \tag{5.20}$$

such that we may write (5.16) under the form

$$\Omega(z_1, z_2) = e^{\Gamma_0^{(m,n)}(z_1, z_2)}. \tag{5.21}$$

Now, by using this result, together with the first formula (5.5) and the expressions (5.11) we get, denoting by $X^{(m,n)}(z_1, z_2)$ a canonical solution

$$X^{(m,n)}(z_1, z_2) = \begin{cases} \dfrac{1}{\Pi_1(z_1)\Pi_2(z_2)} e^{\Gamma_0^{(m,n)}(z_1, z_2)}, \text{ in } D_1^+ \times D_2^+, \\ \dfrac{1}{\Pi(z_1) z_2^{\varkappa_2}} e^{\Gamma_0^{(m,n)}(z_1, z_2)}, \text{ in } D_1^+ \times D_2^-, \\ \dfrac{1}{z_1^{\varkappa_1} \Pi_2(z_2)} e^{\Gamma_0^{n}(z_1, z_2)}, \text{ in } D_1^- \times D_2^+, \\ \dfrac{1}{z_1^{\varkappa_1} z_2^{\varkappa_2}} e^{\Gamma_0^{(m,n)}(z_1, z_2)}, \text{ in } D_1^- \times D_2^-. \end{cases} \tag{5.22}$$

6. Evaluation of the boundary value functions.

$$[X^{(m,n)}(t_1, t_2)]^{++}, \ [X^{(m,n)}(t_1, t_2)]^{+-}$$
$$[X^{(m,n)}(t_1, t_2)]^{-+}, \ [X^{(m,n)}(t_1, t_2)]^{--}.$$

By virtue of (5.22) we may write down immediately

$$[X^{(m,n)}(t_1, t_2)]^{++} = \frac{1}{\Pi_1(t_1)\,\Pi(t_2)}\, e^{[\Gamma_0^{(m,n)}(t_1,t_2)]^{++}}, \tag{6.1}$$

$$[X^{(m,n)}(t_1, t_2)]^{+-} = \frac{1}{\Pi_1(t_1)\, t_2^{\varkappa_2}}\, e^{[\Gamma_0^{(m,n)}(t_1,t_2)]^{+-}}, \tag{6.2}$$

$$[X^{(m,n)}(t_1, t_2)]^{-+} = \frac{1}{t_1^{\varkappa_1}\,\Pi_2(t_2)}\, e^{[\Gamma_0^{(m,n)}(t_1,t_2)]^{-+}}, \tag{6.3}$$

$$[X^{(m,n)}(t_1, t_2)]^{--} = \frac{1}{t_1^{\varkappa_1} t_2^{\varkappa_2}}\, e^{[\Gamma_0^{(m,n)}(t_1,t_2)]^{--}}. \tag{6.4}$$

Or, if we come back to the expression (5.20), and then we take into account the derived Kakicsev formulas (4.1a) and (4.1b) and the formulas (2.3)—(2.8) too, we deduce

$$\left.\begin{matrix}[\Gamma_0^{(m,n)}(t_1,t_2)]^{++}\\ [\Gamma_0^{(m,n)}(t_1,t_2)]^{--}\end{matrix}\right\} = \frac{1}{4}\left\{\frac{\omega^{(m,n)}(t_1,t_2)}{m!\,n!} \pm \frac{1}{\pi i}\,\frac{m!}{n!}\int_{L_1}\frac{\omega^{(0,n)}(\tau_1,t_1)}{(\tau_1-t_1)^{m+1}}\,d\tau_1 \pm\right.$$

$$\left.\pm \frac{1}{\pi i}\,\frac{n!}{m!}\int_{L_2}\frac{\omega^{(m,0)}(t_1,\tau_1)}{(\tau_2-t_2)^{n+1}}\,d\tau_2\right\} + \Gamma_0^{(m,n)}, \tag{6.5}$$

$$\left.\begin{matrix}[\Gamma_0^{(m,n)}(t_1,t_2)]^{-+}\\ [\Gamma_0^{(m,n)}(t_1,t_2)]^{+-}\end{matrix}\right\} = \frac{1}{4}\left\{-\frac{\omega^{(m,n)}(t_1,t_2)}{m!\,n!} \pm \frac{1}{\pi i}\,\frac{m!}{n!}\int_{L_1}\frac{\omega^{(0,n)}(\tau_1,t_1)}{(\tau_1-t_1)^{m+1}}\,d\tau_1 \mp\right.$$

$$\left.\mp \frac{1}{\pi i}\,\frac{n!}{m!}\int_{L_2}\frac{\omega^{(m,0)}(t_1,\tau_2)}{(\tau_2-t_2)^{n+1}}\,d\tau_2\right\} + \Gamma_0^{(m,n)}. \tag{6.6}$$

Inserting these expressions in (6.1)—(6.4) and recalling the definition (5.18), we become

$$[X^{(m,n)}(t_1, t_2)]^{++} = \frac{\sqrt[4]{G_0(t_1,t_2)}}{\Pi_1(t_1)\,\Pi_2(t_2)}\, e^{\Gamma_0^{(m,n)}(t_1,t_2)}\, e^{Q_+(t_1,t_2)}, \tag{6.7}$$

$$[X^{(m,n)}(t_1, t_2)]^{+-} = \frac{1}{\Pi_1(t_1)\,t_2^{\varkappa_2}\sqrt[4]{G_0(t_1,t_2)}}\, e^{\Gamma_0^{(m,n)}(t_1,t_2)}\, e^{Q_-(t_1,t_2)}, \tag{6.8}$$

$$[X^{(m,n)}(t_1, t_2)]^{-+} = \frac{1}{t_1^{\varkappa_1}\,\Pi_2(t_2)\sqrt[4]{G_0(t_1,t_2)}}\, e^{\Gamma_0^{(m,n)}(t_1,t_2)}\, e^{Q_-(t_1,t_2)}, \tag{6.9}$$

$$[X^{(m,n)}(t_1, t_2)]^{--} = \frac{\sqrt[4]{G_0(t_1,t_2)}}{t_1^{\varkappa_1} t_2^{\varkappa_2}}\, e^{\Gamma_0^{(m,n)}(t_1,t_2)}\, e^{Q_+(t_1,t_2)}, \tag{6.10}$$

where

$$Q_{\pm}(t_1, t_2) = \frac{1}{\pi i}\frac{m!}{n!}\int_{L_1} \frac{\omega^{(0,n)}(\tau_1, t_1)}{(\tau_1 - t_1)} d\tau_1 \pm \frac{1}{\pi i}\frac{n!}{m!}\int_{L_2} \frac{\omega^{(m,0)}(t_1, t_2)}{(\tau_2 - t_2)} d\tau_2 \qquad (6.11)$$

If we introduce in (6.7)—(6.10) the expression (5.12), we obtain

$$[X^{(m,n)}(t_1, t_2)]^{++} = \left[\frac{G(t_1, t_2)}{t_1^{\varkappa_1} t_2^{\varkappa_2} \Pi_1^3(t_1) \Pi_2^3(t_2)}\right]^{1/4} e^{\Gamma_0^{(m,n)}(t_1,t_2)+Q_+(t_1,t_2)}, \qquad (6.12)$$

$$[X^{(m,n)}(t_1, t_2)]^{+-} = \left[\frac{t_1^{\varkappa_1}}{t_2^{3\varkappa_2} \Pi_1^5(t_1) \Pi_2(t_2) G(t_1, t_2)}\right]^{1/4} e^{\Gamma_0^{(m,n)}(t_1,t_2)-Q_-(t_1,t_2)}, \qquad (6.13)$$

$$[X^{(m,n)}(t_1, t_2)]^{-+} = \left[\frac{t_2^{\varkappa_2}}{t_1^{3\varkappa_1} \Pi_1(t_1) \Pi_2^5(t_2) G(t_1, t_2)}\right]^{1/4} e^{\Gamma_0^{(m,n)}(t_1,t_2)+Q_-(t_1,t_2)}, \qquad (6.14)$$

$$[X^{(m,n)}(t_1, t_2)]^{--} = \left[\frac{\Pi_1(t_1)\Pi_2(t_2)\; G(t_1, t_2)}{t_1^{5\varkappa_1} t_2^{5\varkappa_2}}\right]^{1/4} e^{\Gamma_0^{(m,n)}(t_1,t_2)-Q_+(t_1,t_2)}. \qquad (6.15)$$

The sign of the above functions must be given by the relation

$$\frac{\sqrt[4]{G_0(t_1, t_2)}}{\Pi_1(t_1)\,\Pi_2(t_2)} = \left[\frac{1}{\Pi_1(t_1)\,\Pi_2(t_2)}\right]^{3/4} \left[\frac{G(t_1, t_2)}{t_1^{\varkappa_1} t_2^{\varkappa_2}}\right]^{1/4} = \frac{1}{\Pi_1(t_1)\,\Pi_2(t_2)} e^{\frac{1}{4}\ln G_0(t_1, t_2)} \qquad (6.16)$$

7. **A nonhomogeneous Rieman-Hilbert boundary value problem with derivatives.** By a consequent extension we may also formulate in the space of two complex variables a nonhomogeneous Riemann-Hilbert problem with derivatives as follows:

Let be in a space of two complex variables a smooth contour $L = L_1 \times L_2$ which divides this space in domains D^+ and D^-, the domain D^+ being situated at the left of this contour and the domain D^- at its right. It is understood that a positive sense was chosen on the considered contour. Find a function $\Phi(z_1, z_2)$ possessing m partial derivatives with respect to z_1 and n partial derivatives with respect to z_2, these partial derivatives of $\Phi(z_1, z_2)$ being denoted by $\Phi^{(m,n)}(z_1, z_2)$ and assumed as sectionally holomorphic with a finite order at infinity and verifying on the contour $L = L_1 \times L_2$ the boundary conditions

$$[\Phi^{(m,n)}(t_1, t_2)]^{++}\,[\Phi^{(m,n)}(t_1, t_2)]^{--} =$$
$$= G(t_1, t_2)\,[\Phi^{(m,n)}(t_1, t_2)]^{+-}\,[\Phi^{(m,n)}(t_1, t_2)]^{-+} + g(t_1, t_2), \qquad (7.1)$$

where $G(t_1,t_2)$ and $g(t_1, t_2)$ are given functions on the contour $L = L_1 \times L_2$ fulfilling on this contour a Hölder condition; it is also assumed that the function $G(t_1, t_2)$ do not cancel everywhere on the contour, while t_1 and t_2 are variables on L_1 or L_2 respectively.

By taking into account the relations (3.19)—(3.22) one immediately see, that the boundary condition may also be written under the form

$$[\Phi^{++}(t_1, t_2)]^{(m,n)}[\Phi^{--}(t_1, t_2)]^{(m,n)} =$$
$$= G(t_1, t_2)\;[\Phi^{+-}(t_1, t_2)]^{(m,n)}[\Phi^{-+}(t_1, t_2)]^{(m,n)} + g(t_1, t_2). \qquad (7.2)$$

As for the solution of this boundary problem the same remarks as that exposed in [1] for the case $m = 0$, $n = 0$ are here also to be taken into consideration.

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STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXIII, 4, 1988

ON SOME GRONWALL TYPE LEMMAS

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Received: October 12, 1987

REZUMAT. — Asupra unor leme de tip Gronwall. În această lucrare se dau două leme de tip Gronwall şi se stabilesc o seamă de cazuri particulare importante în teoria calitativă a ecuaţiilor diferenţiale şi a celor integrale de tip Volterra.

Introduction. In qualitative theory of differential and Volterra integral equations the Gronwall type lemmas of one variable for the real functions play a very important role.

The first use of Gronwall lemma to establish boundedness and stability is due to R. Bellman. For the ideas and the methods of R. Bellman see [3] where further references are given.

In 1919, T. H. Gronwall [9] proved a remarkable inequality which has attracted and continues to attract considerable attention in the literature.

LEMMA I. (Gronwall) *Let x, ψ, $\varkappa$ be real functions defined in $[a, b]$ and continuous, $\varkappa(t) \geqslant 0$ for $t \in [a, b]$. We suppose that on $[a, b]$ we have the inequality:*

$$x(t) \leqslant \psi(t) + \int_a^t \varkappa(s)x(s)ds. \tag{1}$$

Then

$$x(t) \leqslant \psi(t) + \int_a^t \varkappa(s)\,\psi(s)\,\exp\left[\int_s^t \varkappa(u)du\right] ds \tag{2}$$

in $[a, b]$ (see [1] p. 25, [10] p. 9).

COROLLARY I. 1. *If ψ is differentiable, from (1) it follows that*

$$x(t) \leqslant \psi(a)\,\exp\left(\int_a^t \varkappa(u)du\right) + \int_a^t \exp\left(\int_s^t \varkappa(u)du\right)\psi'(s)\,ds \tag{3}$$

for all $t \in [a, b]$.

COROLLARY I. 2. *If ψ is constant, from*

$$x(t) \leqslant \psi + \int_a^t \varkappa(s)x(s)ds \tag{4}$$

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follows

$$x(t) \leqslant \psi \exp\left(\int_a^t x(u)\,du\right). \tag{5}$$

Another well-known generalization of Gronwall lemma is the following due to I. B i h a r i ([4], [1] p. 26).

LEMMA II. (Bihari) *Let* $x : [a, b] \to \mathbf{R}_+$ *be a continuous function which satisfies the inequality:*

$$x(t) \leqslant M + \int_a^t \psi(s)\,\omega(x(s))\,ds,\ t \in [a, b] \tag{6}$$

where $M \geqslant 0$, $\psi : [a, b] \to \mathbf{R}_+$ *is continuous and* $\omega : \mathbf{R}_+ \to \mathbf{R}_+^*$ *is continuous and monotone-increasing.*

Then the following estimation holds:

$$x(t) \leqslant \Phi^{-1}\left(\Phi(M) + \int_a^t \psi(s)\,ds\right),\ t \in [a, b] \tag{7}$$

where $\Phi : \mathbf{R} \to \mathbf{R}$ *is given by:*

$$\Phi(u) := \int_{u_0}^{u} \frac{ds}{\omega(s)},\ u \in \mathbf{R}. \tag{8}$$

Finally, we shall present another lemma of Gronwall type which is very important in qalitative theory of differential equations for monotone operators in Hilbert spaces ([1] p. 27, [5] Appendice).

LEMMA III. *Let* $x : [a, b] \to \mathbf{R}$ *be a continuous function which satisfies the following relation:*

$$\frac{1}{2}x^2(t) \leqslant \frac{1}{2}x_0^2 + \int_a^t \psi(s)\,x(s)\,ds,\ t \in [a, b] \tag{9}$$

where $x_0 \in \mathbf{R}$ *and* ψ *is nonegative continuous in* $[a, b]$.

Then the following estimation holds:

$$|x(t)| \leqslant |x_0| + \int_a^t \psi(s)\,ds,\ t \in [a, b]. \tag{10}$$

For others results in connection with Gronwall's lemma for the real functions of one variable we send to P. R. B e e s a c k [2], G. I. C a n d i r o v [6], M. M. K o n s t a n t i n o v [11], and B. G. P a c h p a t t e [13–17].

Main results. This section is devoted to the study of some integral inequation of Gronwall type (see (1.1.), (1.4.), (1.6.), (1.8.) and (1.12.)) and to give some lemmas of estimations for the nonnegative continuous solutions of these integral inequations which are interesting in themselves.

The author belives that the inequalities established in this section are new to the literature.

LEMMA 1.1. *Let* $A.\ B: [\alpha, \beta) \to \mathbf{R}_+$, $L: [\alpha, \beta) \times \mathbf{R}_+ \to \mathbf{R}_+$ *be continuous and*

$$0 \leqslant L(t, u) - L(t, v) \leqslant M(t, v)\,(u - v),\ t \in [\alpha, \beta),\ u \geqslant v \geqslant 0$$

where M *is nonnegative continuous on* $[\alpha, \beta) \times \mathbf{R}_+$. (L)

Then for every nonnegative continuous solution of the following integral inequation:

$$x(t) \leqslant A(t) + B(t) \int_\alpha^t L(s, x(s))\, ds,\ t \in [\alpha, \beta) \tag{1.1}$$

we have the estimation

$$x(t) \leqslant A(t) + B(t) \int_\alpha^t L(u, A(u)) \exp\left(\int_u^t M(s, A(s))\, B(s)\, ds\right) du \tag{1.2}$$

for all $t \in [\alpha, \beta)$.

Proof. Let consider the mapping $y: [\alpha, \beta) \to \mathbf{R}_+$ given by $y(t) :=$

$$:= \int_\alpha^t L(s, x(s))\, ds. \text{ Then } y \text{ is differentiable on } (\alpha, \beta),\ y'(t) =$$

$$= L(t, x(t)) \text{ if } t \in (\alpha, \beta) \text{ and } y(\alpha) = 0.$$

By the relation (L) it follows that:

$$y'(t) \leqslant L(t, A(t) + B(t)y)t)) \leqslant L(t, A(t)) + M(t, A(t))\, B(t)\, y(t) \tag{1.3}$$

for all $t \in (\alpha, \beta)$.

Putting $z(t) := y(t) \exp\left(-\int_\alpha^t M(s, A(s))\, B(s)\, ds\right)$, $t \in [\alpha, \beta)$, from (1.3) we obtain the following integral inequation:

$$z'(t) \leqslant L(t, A(t)) \exp\left(-\int_\alpha^t M(s, A(s))\, B(s)\, ds\right),\ t \in [\alpha, \beta).$$

By integration on $[\alpha, t]$, we have:

$$z(t) \leqslant \int_\alpha^t L(u, A(u)) \exp\left(-\int_\alpha^u M(s, A(s))\, B(s)\, ds)\right) du.$$

what implies

$$y(t) \leqslant \int_{\alpha}^{t} L(u, A(u)) \exp\left(\int_{u}^{t} M(s, A(s))\, B(s)\, ds\right) du, \quad t \in [\alpha, \beta)$$

from where results the estimation (1.2)

The lemma is thus proved.

Now we can give the following two corollaries which are evident by the above lemma.

COROLLARY 1.1.1. *Let suppose that* $A, B : [\alpha, \beta) \to \mathbf{R}_+$, $G : [\alpha, \beta) \times \mathbf{R}_+ \to \mathbf{R}_+$ *be continuous and*

$$0 \leqslant G(t, u) - G(t, v) \leqslant N(t)\,(u - v)\,;\ t \in [\alpha, \beta),\ u \geqslant v \geqslant 0$$

where N *is nonnegative continuous on* $[\alpha, \beta)$. (G)

Then for every nonnegative continuous solution of the following integral inequation

$$x(t) \leqslant A(t) + B(t) \int_{\alpha}^{t} G(s, x(s))\, ds, \quad t \in [\alpha, \beta) \tag{1.4}$$

we have the estimation

$$x(t) \leqslant A(t) + B(t) \int_{\alpha}^{t} G(u, A(u)) \exp\left(\int_{u}^{t} N(s) B(s)\, ds\right) du \tag{1.5}$$

for all $t \in [\alpha, \beta)$.

COROLLARY 1.1.2. *Let now* $A, B, C : [\alpha, \beta) \to \mathbf{R}_+$, $H : \mathbf{R}_+ \to \mathbf{R}_+$ *be continuous and* H *satisfies the following condition of Lipschitz type:*

$$0 \leqslant H(u) - H(v) \leqslant M(u - v),\ M > 0,\ u \geqslant v \geqslant 0. \tag{H}$$

Then for every nonnegative continuous solution of the integral inequation

$$x(t) \leqslant A(t) + B(t) \int_{\alpha}^{t} C(s)\, H(x(s))\, ds, \quad t \in [\alpha, \beta) \tag{1.6}$$

we have the estimation

$$x(t) \leqslant A(t) + B(t) \int_{\alpha}^{t} C(u) H(A(u)) \exp\left(M \int_{u}^{t} C(s) B(s) ds\right) du \tag{1.7}$$

for all $t \in [\alpha, \beta)$.

Remark. Putting $H : \mathbf{R}_+ \to \mathbf{R}_+$, $H(x) = x$ one gets Lemma 1. of [7] which gives a natural generalisation of Gronwall lemma.

A very important consequence of Lemma 1.1. for differentiable kernels is the following lemma:

LEMMA 1.2. *Let* A, $B : [\alpha, \beta) \to \mathbf{R}_+$, $D : [\alpha, \beta) \times \mathbf{R}_+ \to \mathbf{R}_+$ *be continuous and the following condition holds:*

D is differentiable on domain $(\alpha, \beta) \times (0, \infty)$, $\frac{\partial D(t, x)}{\partial x}$ *is nonnegative on* $(\alpha, \beta) \times (0, \infty)$ *and there exists a function* $P : [\alpha, \beta) \times \mathbf{R}_+ \to \mathbf{R}_+$ *continuous such that* $\frac{\partial D(t, u)}{\partial x} \leqslant P(t, v)$ *for any* $t \in (\alpha, \beta)$ *and* $u \geqslant v \geqslant 0$. **(D)**

Then for every nonnegative continuous solution of the following integral inequation

$$x(t) \leqslant A(t) + B(t) \int_{\alpha}^{t} D(s, x(s))ds, \quad t \in [\alpha, \beta) \tag{1.8}$$

we have the estimation:

$$x(t) \leqslant A(t) + B(t) \int_{\alpha}^{t} D(u, A(u)) \exp\left(\int_{u}^{t} P(s, A(s))B(s)ds\right)du \tag{1.9}$$

for all $t \in [\alpha, \beta)$.

Proof. Applying Lagrange's theorem for the function D in domain $= (\alpha, \beta) \times (0, \infty)$, for every $u > v > 0$ and $t \in (\alpha, \beta)$ there exists $\mu \in (v, u)$ such that

$$D(t, u) - D(t, v) = \frac{\partial D(t, \mu)}{\partial x}(u - v).$$

Since

$$0 \leqslant \frac{\partial D(t, \mu)}{\partial x} \leqslant P(t, v)$$

we obtain

$$0 \leqslant D(t, u) - D(t, v) \leqslant P(t, v)(u - v)$$

for every $u \geqslant v > 0$ and $t \in (\alpha, \beta)$.

The proof of lemma follows by an argument similar to that in the proof of Lemma 1.1. We omit the details.

COROLLARY 1.2.1. *Let* $A : [\alpha, \beta) \to (0, \infty)$, $B : [\alpha, \beta) \to \mathbf{R}_+$, $I : [\alpha, \beta) \times \mathbf{R}_+ \to \mathbf{R}_+$ *be continuous and I satisfies the relation:*

I is differentiable on domain $(\alpha, \beta) \times (0, \infty)$, $\frac{\partial I(t, x)}{\partial x}$ *is nonnegative continuous on* $(\alpha, \beta) \times (0, \infty)$ *and we have* $\frac{\partial I(t, u)}{\partial x} \leqslant \frac{\partial I(t, v)}{\partial x}$ *for any* $u \geqslant v > 0$ *and* $t \in (\alpha, \beta)$.

Then for every nonnegative continuous solution of the following integral inequation:

$$x(t) \leqslant A(t) + B(t) \int_{\alpha}^{t} I(s, x(s))ds, \; t \in [\alpha, \beta) \tag{1.10}$$

we have

$$x(t) \leqslant A(t) + B(t) \int_{\alpha}^{t} I(u, A(u)) \exp\left(\int_{u}^{t} \frac{\partial I(s, A(s))}{\partial x} B(s)ds\right) du \tag{1.11}$$

in the interval $[\alpha, \beta)$.

Finally, we have

COROLLARY 1.2.2. *Let* $A, B, C : [\alpha, \beta) \to \mathbf{R}_+$, $K : \mathbf{R}_+ \to \mathbf{R}_+$ *be continuous,* $A(t) > 0$ *for all* $t \in [\alpha, \beta)$ *and* K *satisfies the relation:*

K is monotone-increasing and differentiable in domain $(0, \infty)$ *with* $\frac{dK}{dx} : (0, \infty) \to \mathbf{R}_+$ *is monotone-decreasing and continuous in* $(0, \infty)$.

Then for every nonnegative continuous solution of integral inequation

$$x(t) \leqslant A(t) + B(t) \int_{\alpha}^{t} C(s)K(x(s))ds, \; t \in [\alpha, \beta) \tag{1.12}$$

we have the estimation

$$x(t) \leqslant A(t) + B(t) \int_{\alpha}^{t} C(u)K(A(u)) \exp\left(\int_{u}^{t} \frac{dK}{dx}(A(s))B(s)C(s)ds\right) du \tag{1.13}$$

in the interval $[\alpha, \beta)$.

Applications. In this section we shall apply the main results established in the first to obtain some Gronwall type lemmas for particular integral inequations which are important in applications.

1. Let $A, B, C : [\alpha, \beta) \to \mathbf{R}_+$ be continuous and $A(t) > 0$, $r(t) \leqslant 1$ for all $t \in [\alpha, \beta)$. Then for every nonnegative continuous solutions of integral inequation

$$x(t) \leqslant A(t) + B(t) \int_{\alpha}^{t} C(s)x(s)^{r(s)}ds, \; t \in [\alpha, \beta) \tag{r}$$

we have the estimation

$$x(t) \leqslant A(t) + B(t) \int_{\alpha}^{t} C(u)A(u)^{r(u)} \exp\left(\int_{u}^{t} \frac{r(s)C(s)B(s)}{A(s)^{1-r(s)}} ds\right) du \tag{2.1}$$

for all $t \in [\alpha, \beta)$.

Particularly, if r is constant, then

$$x(t) \leqslant A(t) + B(t) \int_{\alpha}^{t} C(s)x(s)^r ds, \quad t \in [\alpha, \beta) \tag{r.c.}$$

implies

$$x(t) \leqslant A(t) + B(t) \int_{\alpha}^{t} C(u)A(u)^r \exp\left(r \int_{u}^{t} \frac{C(s)B(s)}{A(s)^{1-r}} ds\right) du \tag{2.2}$$

in the interval $[\alpha, \beta)$.

The proof is evident by Corollary 1.2.1. for $I(t, x) = x^{r(t)}$, $t \in [\alpha, \beta)$, $x \geqslant 0$.

2. Let $A, B, C : [\alpha, \beta) \to \mathbf{R}_+$ be continuous on $[\alpha, \beta)$. If $X : [\alpha, \beta) \to \mathbf{R}_+$ is continuous and satisfies the relation

$$x(t) \leqslant A(t) + B(t) \int_{\alpha}^{t} C(s)\ln(x(s) + 1)ds, \quad t \in [\alpha, \beta) \tag{ln}$$

then we have the estimation

$$x(t) \leqslant A(t) + B(t) \int_{\alpha}^{t} C(s) \ln(A(s) + 1) \exp\left(\int_{s}^{t} \frac{C(u)B(u)}{A(u) + 1} du\right) ds \tag{2.3}$$

for all $t \in [\alpha, \beta)$.

The proof follows by Corollary 1.2.2. for $K(x) = \ln(x + 1)$, $x \geqslant 0$.

3. Assume that A, B, C are nonnegative continuous in $[\alpha, \beta)$. Then for every $x : [\alpha, \beta) \to \mathbf{R}_+$ a solution of the following integral inequation

$$x(t) \leqslant A(t) + B(t) \int_{\alpha}^{t} C(s) \operatorname{arctg}(x(s))ds, \quad t \in [\alpha, \beta) \tag{arctg}$$

we have the estimation

$$x(t) \leqslant A(t) + B(t) \int_{\alpha}^{t} C(s) \operatorname{arctg}(A(u)) \exp\left(\int_{u}^{t} \frac{B(s)C(s)}{A^2(s) + 1} ds\right) du \tag{2.4}$$

in the interval $[\alpha, \beta)$.

The proof results by Corollary 1.2.2. for $K(x) = \operatorname{arctg} x$, $x \geqslant 0$.

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STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXIII, 4, 1988

THE PICARD PROBLEM ASSOCIATED WITH A LIPSCHITZIAN HYPERBOLIC MULTIVALUED EQUATION

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Received: May 26, 1988

REZUMAT. — Problema lui Picard asociată unei ecuaţii multivoce hiperbolice Lipschitziene. În lucrare se consideră problema lui Picard asociată ecuaţiei multivoce hiperbolice $\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z)$, $F: D \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$, $D = [0, a] \times [0, b]$, unde F este Lipschitziană în raport cu z şi $F(x, y, z)$ este mulţime neconvexă. Pentru această problemă sînt demonstrate trei teoreme de existenţă folosind metoda aproximaţiilor succesive şi teoreme de selecţie.

1. **Introduction.** In this paper we consider the Picard problem, [1], [2], associated with the multivalued hyperbolic equation $\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z)$, $F: D \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$, $D = [0, a] \times [0, b]$, F is Lipschitzian with respect to z, $F(x, y, z)$ is nonconvex set. For this problem we prove three existence theorems by the successive approximations method, using two selection theorems, [3], [4]. The method was applied by A. F. Filippov [5], H. Hermes [3], C. J. Himmelberg and F. S. Van Vleck [4] for the equation $\dot{x} \in F(t, x)$. The obtained results are similar to those contained in [3], [4], [5].

2. **Existence results.** Let be satisfied the following hypotheses:

(H_0) The curve $\gamma: x = \psi(y)$, $0 \leqslant y \leqslant b$, is defined by the function $\psi \in C^1([0, b]; \mathbf{R})$ with the properties

$$\psi(0) = 0,\ 0 \leqslant \psi(y) \leqslant a,\ 0 \leqslant y \leqslant b, \tag{1}$$

(H_1) $F: D \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ is the multivalued application with the values contained in the ball of radius r centered in origin of $\mathbf{R}^n$,

(H_2) $F(x, y, z)$ is compact for every $(x, y, z) \in D \times \mathbf{R}^n$,

(H_3) F is continuous on $D \times \mathbf{R}^n$,

(H_4) F is Lipschitzian with respect to z; there exists a function $k: D \to \mathbf{R}_+$, $k \in L^1(D)$, such that

$$h(F(x, y, z), F(x, y, z')) \leqslant k(x, y)\,||z - z'||,\ (x, y) \in D,\ z, z' \in \mathbf{R}^n, \tag{2}$$

where h is the Hausdorff metric,

(H_5) The functions $P \in AC([0, a]; \mathbf{R}^n)$, $Q \in AC([0, b]; \mathbf{R}^n)$ satisfy the condition $P(0) = Q(0)$, $AC([\alpha_1, \alpha_2]; \mathbf{R}^n)$ being the space of absolutely continuous functions $f: [\alpha_1, \alpha_2] \to \mathbf{R}^n$, endowed with the norm

$$||f|| = \sup_{t \in [\alpha_1, \alpha_2]} ||f(t)|| + \int_{\alpha_1}^{\alpha_2} ||f'(t)||\,dt,$$

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(H_6) There exists an absolutely continuous in the Carathéodory sense function [6, §565—§568], $\xi : D \to \mathbf{R}^n$, $\xi \in C^*(D; \mathbf{R}^n)$ such that

$$\sup_{(x,y)\in D} \rho\left(\frac{\partial^2\xi(x,y)}{\partial x\,\partial y}, F(x, y, \xi(x,y))\right) \leqslant M < +\infty, \text{ for some } M > 0 \tag{3}$$

Remark 1. Such functions ξ exist; for example, if ξ is constant, M may be taken r.

(H_7) The functions $\alpha : D \to \mathbf{R}^n$, $\alpha_0 : D \to \mathbf{R}^n$ defined by

$$\alpha(x, y) = P(x) + Q(y) - P(\psi(y)),\ (x, y) \in D, \tag{4}$$

and

$$\alpha_0(x, y) = \xi(x, 0) + \xi(\psi(y), y) - \xi(\psi(y), 0),\ (x, y) \in D, \tag{5}$$

satisfy the condition

$$||\alpha(x, y) - \alpha_0(x, y)|| \leqslant M_1, \text{ for some } M_1 > 0. \tag{6}$$

Remark 2. The functions α and α_0 are absolutely continuous in the Carathéodory sense on D, [6]; α, $\alpha_0 \in C^*(D; \mathbf{R}^n)$.

DEFINITION [1], [2]. The *Picard problem* for the multivalued hyperbolic equation

$$\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z),\ (x, y, z) \in D \times \mathbf{R}^n, \tag{7}$$

consists in determination of a solution for (7), which is an absolutely continuous in the Carathéodory sense function [6], $z : D \to \mathbf{R}^n$, $z \in C^*(D; \mathbf{R}^n)$, which satisfies a.e. $(x, y) \in D$ the equation (7) and the conditions

$$\begin{cases} z(x, 0) = P(x), & 0 \leqslant x \leqslant a, \\ z(\psi(y), y) = Q(y), & 0 \leqslant y \leqslant b. \end{cases} \tag{8}$$

THEOREM 1. *If the hypotheses $(H_0) - (H_7)$ are satisfied, the Picard problem (7) + (8) has a solution in D.*

Proof. We consider the rectangle $D_0(x, y) \subset D$, $(x, y) \in D$, defined by

$$D_0(x, y) = \{(u, v) | \psi(y) \leqslant u \leqslant x,\ 0 \leqslant v \leqslant y\},\ (x, y) \in D.$$

Integrating $\frac{\partial^2 z}{\partial x\, \partial v}(x, y)$ in the rectangle $D_0(x, y)$, we have

$$\iint\limits_{D_0(x,y)} \frac{\partial^2 z(u, v)}{\partial u\, \partial v}\, du\, dv = \int_0^y dv \int_{\psi(v)}^x \frac{\partial^2 z(u, v)}{\partial u\, \partial v}\, du = z(x, y) - z(x, 0) - z(\psi(y), y) +$$
$$+ z(\psi(y), 0) = z(x, y) - P(x) - Q(y) + P(\psi(y)),\ (x, y) \in D,$$

from we obtain for $(x, y) \in D$,

$$z(x, y) = P(x) + Q(y) - P(\psi(y)) + \iint\limits_{D_0(x,y)} \frac{\partial^2 z(u, v)}{\partial u\, \partial v}\, du\, dv = \alpha(x, y) +$$
$$+ \iint\limits_{D_0(x,y)} \frac{\partial^2 z(u, v)}{\partial u\, \partial v}\, du\, dv, \tag{9}$$

We define the sequence of succesive approximations $\{z_i\}_{i\geq 0}$; let

$$z_0(x, y) = \xi(x, y), (x, y) \in D. \tag{10}$$

According to the Lemma 1.2 [3] there exists a measurable function $\lambda_0 : D \to \mathbf{R}^n$ such that

$$\lambda_0(x, y) \in F(x, y, z_0(x, y)), \text{ a.e. } (x, y) \in D, \tag{11$_0$}$$

and

$$\rho\left(\frac{\partial^2 z_0(x, y)}{\partial x\, \partial y}, F(x, y, z_0(x, y))\right) = \left\| \lambda_0(x, y) - \frac{\partial^2 z_0(x, y)}{\partial x \partial y} \right\|, (x, y) \in D. \tag{12$_0$}$$

The second approximation of the solution is given by

$$z_1(x, y) = \alpha(x, y) + \iint\limits_{D_0(x, y)} \lambda_0(u, v)du\, dv, (x, y) \in D, \tag{13$_1$}$$

from which it follows

$$\frac{\partial^2 z_1(x, y)}{\partial x\, \partial y} = \lambda_0(x, y) \in F(x, y, z_0(x, y)), \text{ a.e. } (x, y) \in D. \tag{14$_1$}$$

Again applying the cited Lemma, it follows the existence of a measurable function $\lambda_1 : D \to \mathbf{R}^n$, having the properties

$$\lambda_1(x, y) \in F(x, y, z_1(x, y)), \text{ a.e. } (x, y) \in D, \tag{11$_1$}$$

and

$$\rho\left(\frac{\partial^2 z_1(x, y)}{\partial x\, \partial y}, F(x, y, z_1(x, y))\right) = \left\| \lambda_1(x, y) - \frac{\partial^2 z_1(x, y)}{\partial x\, \partial y} \right\|, (x, y) \in D. \tag{12$_1$}$$

We define the third approximation of the solution by

$$z_2(x, y) = \alpha(x, y) + \iint\limits_{D_0(x, y)} \lambda_1(u, v)du\, dv, (x, y) \in D, \tag{13$_2$}$$

which implies

$$\frac{\partial^2 z_2(x, y)}{\partial x\, \partial y} = \lambda_1(x, y) \in F(x, y, z_1(x, y)), \text{ a.e. } (x, y) \in D. \tag{14$_2$}$$

In this way we obtain the sequences $\{z_i(x, y)\}_{i\geq 0}$, $\{\lambda_i(x, y)\}_{i\geq 0}$, $(x, y) \in D$ with the properties:

$$\lambda_i(x, y) \in F(x, y, z_i(x, y)), \text{ a.e. } (x, y) \in D, \tag{11$_i$}$$

$i = 0, 1, 2, \ldots$

and

$$\rho\left(\frac{\partial^2 z_i(x, y)}{\partial x\, \partial y}, F(x, y, z_i(x, y))\right) = \left\| \lambda_i(x, y) - \frac{\partial^2 z_i(x, y)}{\partial x\, \partial y} \right\|, (x, y) \in D, \tag{12$_i$}$$

$i = 0, 1, 2, \ldots$

where

$$z_i(x, y) = \alpha(x, y) + \iint\limits_{D_0(x, y)} \lambda_{i-1}(u, v)dudv, \quad (x, y) \in D, \tag{13_i}$$

$i = 1, 2, \ldots$

From the preceding relation it follows

$$\frac{\partial^2 z_i(x, y)}{\partial x \partial y} = \lambda_{i-1}(x, y) \in F(x, y, z_{i-1}(x, y)), \text{ a.e. } (x, y) \in D, \tag{14_i}$$

$i = 1, 2, \ldots.$

The functions $\{\lambda_i\}_{i \geq 0}$ are integrable from (14_i) and the hypotheses (H_1) and (H_6). Indeed, from (2) taking into account (14_i) it results

$$\rho\left(\frac{\partial^2 z_i(x, y)}{\partial x \partial y}, F(x, y, z_i(x, y))\right) = \rho(\lambda_{i-1}(x, y), F(x, y, z_i(x, y))) \leqslant$$
$$\leqslant \rho(\lambda_{i-1}(x, y), F(x, y, z_{i-1}(x, y))) + h(F(x, y, z_{i-1}(x, y)), \tag{15_i}$$
$$F(x, y, z_i(x, y))) \leqslant k(x, y) \,||z_{i-1}(x, y) - z(x, y)||, \quad (x, y) \in D,$$

$i = 2, 3, \ldots,$

and for $i = 1$ we have (3) and (10_0).

After some standard calculation, using the inequality

$$\iint\limits_{00}^{xy} k(s, t) \iint\limits_{00}^{st} k(s_1, t_1) \iint\limits_{00}^{s_1 t_1} k(s_2, t_2) \ldots \iint\limits_{00}^{s_{n-1} t_{n-1}} k(s_n, t_n)\, ds_n\, dt_n \ldots ds_1\, dt_1\, ds\, dt \leqslant$$
$$\leqslant \frac{1}{(n+1)!}\left[\iint\limits_{00}^{xy} k(u, v)dudv\right]^{n+1}, \quad (x, y) \in D, \tag{16}$$

we obtain the following basic estimations:

$$||\lambda_i(x, y) - \lambda_{i-1}(x, y)|| = \left\|\frac{\partial^2 z_{i+1}(x, y)}{\partial x \partial y} - \frac{\partial^2 z_i(x, y)}{\partial x \partial y}\right\| \leqslant$$
$$\leqslant k(x, y)\frac{M_1 + Mab}{(i-1)!}\left[\iint\limits_{00}^{xy} k(u, v)dudv\right]^{i-1}, \quad (x, y) \in D, \tag{17_{i+1}}$$

$i = 1, 2, \ldots$

and

$$||z_{i+1}(x, y) - z_i(x, y)|| \leqslant \frac{M_1 + Mab}{i!}\left[\iint\limits_{00}^{xy} k(u, v)dudv\right]^i, \quad (x, y) \in D, \tag{18_{i+1}}$$

$i = 0, 1, 2, \ldots$

From (17_{i+1}) we conclude that the sequence $\{\lambda_i(x, y)\}_{i\geq 0}$ converges in $L^n_\infty(D)$ to a function $\lambda : D \to \mathbf{R}^n$, and from (18_{i+1}) it follows that the sequence $\{z_i(x, y)\}_{i\geq 0}$ is uniformly convergent to a function $z : D \to \mathbf{R}^n$.

Letting $i \to \infty$ in (12_i), (13_i), (14_i) and using the hypotheses (H_2) and (H_3) it follows that z is absolutely continuous function, [6], $z \in C^*(D; \mathbf{R}^n)$ and satisfies the Picard problem (7) + (8).
We obtain

$$\rho\left(\frac{\partial^2 z(x, y)}{\partial x\partial y},\ F(x, y, z(x, y))\right) = \left\|\lambda(x, y) - \frac{\partial^2 z(x, y)}{\partial x\partial y}\right\|,\ (x, y) \in D, \tag{12}$$

and

$$z(x, y) = \alpha(x, y) + \iint\limits_{D_0(x,\ y)} \lambda(u, v)dudv,\ (x, y) \in D, \tag{13}$$

and

$$\frac{\partial^2 z(x, y)}{\partial x\partial y} = \lambda(x, y) \in F(x, y, z(x, y)),\ \text{a.e. } (x, y) \in D. \tag{14}$$

The function z given by (13) satisfies the equation (7) taking into account (14) and z satisfies the conditions (8).

THEOREM 2. *We suppose satisfied the hypotheses* $(H_0) - (H_5)$, (H_7) *and*
(H_6') *There exists an absolutely continuous function* [6] $\xi : D \to \mathbf{R}^n$, $\xi \in C^*(D; \mathbf{R}^n)$, *such that*

$$\rho\left(\frac{\partial^2 \xi(x, y)}{\partial x\partial y},\ F(x, y, \xi(x, y))\right) \leqslant \varepsilon,\ (x, y) \in D, \tag{3'}$$

or some $\varepsilon > 0$.
Then there exists a solution $z : D \to \mathbf{R}^n$ *of the problem* (7) + (8) *satisfying*

$$||z(x, y) - \xi(x, y)|| \leqslant (M_1 + \varepsilon ab)\exp\left[\int_0^x\!\!\int_0^y k(u, v)dudv\right],\ (x, y) \in D. \tag{19}$$

The proof is similar to those of Theorem 1. We obtain

$$||z_{i+1}(x, y) - z_i(x, y)|| \leqslant \frac{M_1 + \varepsilon ab}{i!}\left[\int_0^x\!\!\int_0^y k(u, v)dudv\right]^i,\ (x, y) \in D, \tag{$18'_{i+1}$}$$

$i = 0, 1, 2, \dots$
Using $(18'_{i+1})$ for $i = 0, 1, 2, \dots, j-1$ and the elementary inequality

$$\sum_{n=0}^{j} \frac{t^n}{n!} \leqslant e^t,\ t \geqslant 0,$$

it follows for every $j = 0, 1, 2, \dots$

$$||z_j(x, y) - \xi(x, y)|| \leqslant (M_1 + \varepsilon ab)\ \exp\left[\int_0^x\!\!\int_0^y k(u, v)dudv\right],\ (x, y) \in D. \tag{$18'_j$}$$

The conclusion (19) it results from $(18_j')$ letting $j \to \infty$ using $z_j(x, y) \to z(x, y)$, uniformly in D, and z is a solution of the problem (7) + (8).

THEOREM 3. *Let be satisfied the hypotheses* (H_0'), $F : D \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ *is a multi-valued application with the properties:*

a) $F(x, y, z)$ *is closed for every* $(x, y, z) \in D \times \mathbf{R}^n$,

b) $F(\cdot, \cdot, z)$ *is measurable for each* $z \in \mathbf{R}^n$,

c) $F(x, y, .)$ *is Lipschitzian with respect to* z; *there exists a function* $k : D \to \mathbf{R}_+$, $k \in L^1(D)$, *such that*

$$h_d(F(x, y, z), F(x, y, z')) \leqslant k(x, y) \| z - z' \|, \quad (x, y) \in D, \ z, z' \in \mathbf{R}^n, \tag{2'}$$

where h_d *is the (generalized) Huasdorff pseudometric* [4], *and* (H_5), (H_0), $H_1)$.

Then, the problem (7) + (8) has a solution in D.

The proof is similar to those of Theorem 1 and use the Proposition 1 and the Corollary 1 [4] to ensure the existence of the measurable functions $v_i : D \to \mathbf{R}^n$, $i = 0, 1, 2, \ldots$, with the properties:

$$v_i(x, y) \in F(x, y, z_i(x, y)), \text{ a.e. } (x, y) \in D, \tag{11$_i'$}$$

$i = 0, 1, 2,$

and

$$\rho\left(\frac{\partial^2 z_i(x, y)}{\partial x \partial y}, F(x, y, z_i(x, y))\right) = \left\| v_i(x, y) - \frac{\partial^2 z_i(x, y)}{\partial x \partial y} \right\|, \quad (x, y) \in D, \tag{12$_i'$}$$

and

$$z_i(x, y) = \alpha(x, y) + \iint_{D_0(x, y)} v_{i-1}(u, v) du dv, \quad (x, y) \in D, \tag{13$_i'$}$$

$i = 1, 2, \ldots$

From $(13_i')$ it follows

$$\frac{\partial^2 z_i(x, y)}{\partial x \partial y} = v_{i-1}(x, y) \in F(x, y, z_{i-1}(x, y)) \text{ a.e. } (x, y) \in D, \tag{14$_i'$}$$

$i = 1, 2, \ldots$

The sequence $\{z_i(x, y)\}_{i \geqslant 0}$ is uniformly convergent to $z : D \to \mathbf{R}^n$ and the sequence $\{v_i(x, y)\}_{i \geqslant 0}$ converges in $L^n_{00}(D)$ to $v : D \to \mathbf{R}^n$.

Letting $i \to \infty$ in $(13_i')$ and $(14_i')$ we obtain

$$z(x, y) = \alpha(x, y) + \iint_{D_0(x, y)} v(u, v) du dv, \quad (x, y) \in D, \tag{13'}$$

and

$$\frac{\partial^2 z(x, y)}{\partial x \partial y} = v(x, y) \in F(x, y, z(x, y)), \text{ a.e. } (x, y) \in D. \tag{14'}$$

For the relation (14′) we use the hypotheses *a)* and *c)* from it results that the graph of F is closed. The function z given by (13′) is a solution of the problem (7) + (8), taking into account (14′).

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STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXIII, 4, 1988

ON THE ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF A CERTAIN IMPLICIT VOLTERRA INTEGRAL EQUATION

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Received: June 1, 1988

REZUMAT. — Asupra comportării asimptotice a soluţiilor unei ecuaţii integrale Volterra implicite. Considerînd o ecuaţie Volterra implicită se stabilesc condiţii pentru ca oricare ar fi ξ o soluţie staţionară a ecuaţiei să existe o soluţie x mărginită a ecuaţiei astfel încît $x(t) \to \xi$ cînd $t \to \infty$.

0. The problem which we propose in this paper, concerning the asymptotic behavior of bounded solutions of an implicit Volterra integral equation, has as premise some works in which their authors study in what conditions the bounded solutions of a such equation approach at infinity the set of stationary solutions of the equation (or in other terms the set of the states of asymptotic equilibrium or the set of strong states).

In the year 1974 S. O. London, ([7]), studying the convergence of the bounded solutions of the scalar equation:

$$x(t) + \int_0^t a(t-s)h(x(s))ds = f(t), \ t \geqslant 0, \tag{E}$$

showed that if $f \in AC(R_+, R)$, $f' \in L^1(R_+, R)$, $\lim\limits_{t\to\infty} f(t) = f_0$, $\lim\limits_{t\to\infty} \operatorname{ess\,sup}\limits_{t\leqslant\tau<\infty} |f'(\tau)| = 0$; $a \in L^1(R_+)$ is nonincreasing; $h \in C(R)$, then every bounded solution of the equation (E) approach at infinity the set of the stater of asymptotic equilibrium:

$$\left\{\xi \in R : \xi + h(\xi) \cdot \int_0^\infty a(s)ds = f_0\right\}$$

Such an asymptotic behaviour for the scalar functional-delay equation

$$x(t) = F(t, x(\alpha_1(t)), \ldots, x(\alpha_k(t)), \int_0^t a(t-s)g(s, x(s))ds), \quad t \geqslant 0$$

was discussed by G. Karakostas in 1981 ([4]) and for the equation in R^n:

$$x(t) = F(t, x(\alpha(t)), \int_0^t a(t-s)g(s, x(s))ds), \ t \geqslant 0$$

in 1982 ([5]), using causal operators.

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The problem is taken again and developed in [6].

Also, V. M. Popov in [9] studies the same problem for the system of convolution type:

$$x + Pg * h(x) + f = 0,$$

using a comparison system as model an introducing the concept of mutable system.

1. In the same background established above of asymptotic behavior of the bounded solutions of an implicit Volterra equation, we put forth the following, problem, modified in comparison with the problem of the mentioned authors: in what conditions, for every stationary solution ξ of the equation, there is a bounded solution x of the equation, so that $\lim_{t\to\infty} x(t) = \xi$?

At first we solve this problem for the nonlinear system in implicit general form:

$$x(t) = F\left(t, x(t), \int_0^t a_1(t-s)h_1(x(s))ds, \ldots, \int_0^t a_k(t-s)h_k(x(s))ds\right), \quad t \geq 0, \quad (SN)$$

after that we refer to the particular case of equation (E) studied by Londen ([7]).

The conditions on the system (SN) are the following:

$F: R_+ \times R^n \times (R^n)^k \to R^n$ is continuous with respect to $(x, y_1, \ldots, y_k) \in R^n \times (R^n)^k$ for every $t \geq 0$ and locally integrable on R_+ for every $(x, y_1, \ldots, y_k)$; $F(t, x, y_1, \ldots, y_k) \to F_0(x, y_1, \ldots, y_k)$, as $t \to \infty$; $a_i \in L^1(R_+, \mathcal{M}_n)$; $h_i \in C(R)$, $1 \leq i \leq k$.

Let Σ be the set of the strong states of the system (SN), which is composed from the solutions of following equation, called the equation of stationary solutions:

$$\xi = F_0(\xi, A_1h_1(\xi), \ldots, A_kh_k(\xi)) \quad (St)$$

where $A_i = \int_0^\infty a_i(t)\,dt$, $1 \leq i \leq k$

As to have sense to put forth the problem mentioned above and try to give a solution for it, is necessary to assure at first that it is possible to establish certain conditions in which the set Σ is nonempty and bounded.

Let the mapping

$G: R^n \times R^n \to R^n$

$G(x, \xi) = F_0(x, A_1 h_1(\xi), \ldots, A_kh_k(\xi))$

PROPOSITION. *Suppose that:*

i) There is a bounded, closed and convex subset $Z \subset R^n$, *so that* $G(Z, Z) \subseteq Z$.

ii) There is $\lambda \in (0, 1)$ *so that:*

$$\|G(x, \xi) - G(x', \xi)\| \leq \lambda\|x - x'\|, \quad \forall x, x', \xi \in Z$$

Then Σ is nonempty and bounded set.

Prof. Let $\tilde{G} : Z \times Z \to Z$ the restriction of the mapping G over $Z \times Z$.

Because for every $\xi \in Z$, the mapping $\tilde{G}(.,\xi) : Z \to Z$ is a strict contraction with the constante λ, from the well-known Banach's fixed point theorem it results that exists a unique point $x(\xi) \in Z$, such that:

$$x(\xi) = G(x(\xi), \xi) \tag{1}$$

As the mapping $\tilde{G}$ is continuous on Z (thanks to continuity of the mapping F_0 anf h_i), it results that the fixed point $x(.)$ is continuous with respect to the parameter $\xi \in Z$ (as is demonstrated by the theorem 3.3.1 from [10])

Then, from the Brouwer's fixed point theorem for the mapping $x(.) : Z \to Z$, it results that exists $\bar{\xi} \in Z$ so that:

$$x(\bar{\xi}) = \bar{\xi} \tag{2}$$

We have, taking account of (1) and (2):

$$\bar{\xi} = x(\bar{\xi}) = \tilde{G}(x(\bar{\xi}), \bar{\xi}) = \tilde{G}(\bar{\xi}, \bar{\xi}) = F_0(\bar{\xi}, A_1 h_1(\bar{\xi}), \ldots, A_k h_k(\bar{\xi}))$$

Therefore $\bar{\xi}$ is a solution of the equation (St), namely Σ is nonempty.

As Σ coincides with the set of fixed points $\bar{\xi}$, found in this way, set which is included in Z, it results that Σ is bounded. □

2. Now we give the answer of the problem posed at the beginning.

THEOREM. *Assume there are fulfilled the following conditions:*

i) Σ *is nonempty and bounded set.*

ii) $F(t, 0, 0, \ldots, 0) = 0, \ \forall t \geq 0$

iii) $$||F(t, x, y_1, \ldots, y_k) - F(t, x', y_1', \ldots, y_k')|| \leq \lambda(t)\,||x - x'|| + \\ + \sum_{i=1}^{k} \mu_i(t)\,||y_i - y_i'||, \ \forall t \geq 0, \ \forall x, x', y_i, y_i' \in R^n$$

where $\lambda, \mu_i \in L^1_{loc}(R_+)$ *are bounded,* $0 \leq \lambda(t) < 1, \ \forall t \geq 0$

iv) $||a_i(t - s)\,[h_i(x) - h_i(x')]|| \leq \gamma_i(t) \cdot ||x_i - x_i'||, \ 0 \leq s \leq t;\ x, x' \in R^n$, *where* $\gamma_i \in L^1_{loc}(R_+)$ *and* $\lim_{t\to\infty} t \cdot \gamma_i(t) = 0, \ 1 \leq i \leq k.$

Then, for every $\xi \in \Sigma$ *exists a bounded solution* $\bar{x} \in L^1_{loc}(R_+, R^n)$ *of the system (SN), such that:*

$$\lim_{t\to\infty} \bar{x}(t) = \xi. \tag{3}$$

Moreover, there is a unique function $z_0 \in L^1_{loc}(R_+)$, *such that* $\bar{x}$ *is the unique solution of the system (SN) subject to condition*

$$||\bar{x}(t) - \xi|| \leq z_0(t), \ t \geq 0$$

which satisfies (3).

Proof. We shall give a constructive demonstration of this theorem.

Let $\xi \in \Sigma$ arbitrarily, fixed and we define, for $t \geqslant 0$:

$$u(t) = \left\| F\left(t, \xi, \int_0^t a_1(t-s)h_1(\xi)\,ds, \ldots, \int_0^t a_k(t-s)\,h_k(\xi)\,ds\right) - \xi \right\|$$

Evidently $u \in L^1_{\mathrm{loc}}(R_+)$

Prove at first that the equation:

$$z(t) = \frac{1}{1-\lambda(t)} \sum_{i=1}^{k} \delta_i(t) \cdot \int_0^t z(s)ds + \frac{u(t)}{1-\lambda(t)} \tag{4}$$

has a unique solution in $L^1_{\mathrm{loc}}(R_+)$, where $\delta_i(t) = \mu_i(t) \cdot \gamma_i(t)$, $t \geqslant 0$.

Integrating the equation (4) over $[0, t]$, it becomes equivalent to the following equation:

$$\zeta(t) = \int_0^t \frac{1}{1-\lambda(s)} \left(\sum_{i=1}^{k} \delta_i(s)\right) \zeta(s)\,ds + \int_0^t \frac{u(s)}{1-\lambda(s)}\,ds, \tag{5}$$

of which solutions, if exist, can be considered in the class $C(R_+)$.

Is sufficiently to prove that the equation (5) has a unique solution. In this purpose we consider the operator

$T: C(R_+) \to C(R_+)$

$$(T\zeta)(t) = \int_0^t \frac{1}{1-\lambda(s)} \left(\sum_{i=1}^{k} \delta_i(s)\right) \zeta(s)\,ds + \int_0^t \frac{u(s)}{1-\lambda(s)}\,ds$$

We introduce in $C(R_+)$ the family of seminorms $\{||\cdot||_n\}$ defined by the formula:

$$||\zeta||_n = \sup_{0 \leqslant t \leqslant n} \{\exp[-\rho\alpha(t)] \cdot |\zeta(t)|\}, \ \rho \geqslant 1$$

where:

$$\alpha(t) = \int_0^t \frac{1}{1-\lambda(s)} \left(\sum_{i=1}^{k} \delta_i(s)\right) ds$$

and we shall show that T is a strict contraction with respect to every seminorm of the family.

Indeed, for any $\zeta_1, \zeta_2 \in C(R_+)$ and $0 \leqslant t \leqslant n$, we have:

$$|(T\zeta_1)(t) - (T\zeta_2)(t)| \leqslant \int_0^t \frac{1}{1-\lambda(s)} \left(\sum_{i=1}^{k} \delta_i(s)\right) |\zeta_1(s) - \zeta_2(s)|\,ds =$$

$$= \int_0^t \frac{1}{1-\lambda(s)} \left(\sum_{i=1}^{k} \delta_i(s)\right) \exp[\rho\alpha(s)] \cdot \exp[-\rho\alpha(s)]\,|\zeta_1(s) -$$

$$- \zeta_2(s)|\,ds \leqslant ||\zeta_1 - \zeta_2||_n \cdot \int_0^t \frac{1}{1-\lambda(s)} \left(\sum_{i=1}^{k} \delta_i(s)\right) \exp[\rho\alpha(s)]\,ds$$

Multiplying by $\exp\,[-\rho\alpha(t)]$, we obtain:

$$\exp\,[-\rho\alpha(t)]\cdot|(T\zeta_1)(t)-(T\zeta_2)(t)|\leqslant||\zeta_1-\zeta_2||_n\cdot\exp\,[-\rho\alpha(t)]\,.$$
$$\cdot\int_0^t\frac{1}{1-\lambda(s)}\left(\sum_{i=1}^k\delta_i(s)\right)\exp\,[\rho\alpha(s)]\,ds=||\zeta_1-\zeta_2||_n\cdot\exp\,[-\rho\alpha(t)]\,.$$
$$\cdot\int_0^t\alpha'(s)\exp\,[\rho\alpha(s)]\,ds=\frac{1}{\rho}\,||\zeta_1-\zeta_2||_n\cdot\exp\,[-\rho\alpha(t)]\,\cdot$$
$$\cdot\{\exp\,[\rho\alpha(t)]-1\}\leqslant\frac{1}{\rho}||\zeta_1-\zeta_2||_n$$

Taking the supremum for $t\in[0,n]$, we obtain:

$$||T\zeta_1-T\zeta_2||_n<\frac{1}{\rho}\,||\zeta_1-\zeta_2||_n,$$

hence T is a strict contraction.

By the generalization of Banach's contraction principle, there exists a unique solution $\zeta_0\in C(R_+)$ of equation (5).

As (4) is equivalent to (5), $z_0(.)=\zeta_0'(.)$ is the unique solution of equation (4) in the space $L^1_{\text{loc}}(R_+)$, namely:

$$z_0(t)=\frac{1}{1-\lambda(t)}\sum_{i=0}^k\delta_i(t)\cdot\int_0^t z_0(s)ds+\frac{u(t)}{1-\lambda(t)}\tag{6}$$

On the other hand, we observe that the unique solution of the equation

$$z(t)=\frac{1}{1-\lambda(t)}\sum_{i=0}^k\delta_i(t)\cdot\int_0^t z(s)\,ds\tag{7}$$

in the class of functions satisfying $0\leqslant z(t)\leqslant z_0(t)$, $t\geqslant0$, is the trivial solution $z=0$.

Indeed, we find without any difficulty that trivial solution satisfies the equation (7). After that suppose that there is another solution of (7) satisfying $0\leqslant\bar z(t)\leqslant z_0(t)$, $t\geqslant0$, namely:

$$\bar z(t)=\frac{1}{1-\lambda(t)}\sum_{i=1}^k\delta_i(t)\cdot\int_0^t\bar z(s)ds\tag{8}$$

Substrating (8) from (6), we obtain, for any $t\geqslant0$:

$$z_0(t)-\bar z(t)=\frac{1}{1-\lambda(t)}\sum_{i=1}^k\delta_i(t)\cdot\int_0^t[z_0(s)-\bar z(s)]ds+\frac{u(t)}{1-\lambda(t)},$$

from where it results that $z_0-\bar z$ is a solution of the equation (4).

As z_0 is the unique solution of (4), we have:

$$z_0(t) - \bar{z}(t) = z_0(t), \quad t \geq 0,$$

namely $\bar{z}(t) = 0, \ t \geq 0$.

Hence the unique solution of (7) satisfying $0 \leq z(t) \leq z_0(t), \ t \geq 0$, is $z = 0$.

Now we define in $L^1_{loc}(R_+)$ the sequence $\{z_n\}$ thus:

$$z_{n+1}(t) = \sum_{i=1}^{k} \delta_i(t) \cdot \int_0^t z_n(s)ds + \lambda(t) \ z_n(t), \ t \geq 0, \ n = 0, 1, \ldots \tag{9}$$

where z_0 is the unique solution of equation (4).

By induction we find easily that for each $t \geq 0$:

$$0 \leq z_{n+1}(t) \leq z\ (t), \quad n = 0, 1, \ldots \tag{10}$$

Indeed, from (6) we have for any $t \geq 0$:

$$z_0(t) \geq \frac{1}{1 - \lambda(t)} \sum_{i=1}^{k} \delta_i(t) \cdot \int_0^t z_0(s)ds,$$

hence:

$$z_1(t) = \sum_{i=1}^{k} \delta_i(t) \cdot \int_0^t z_0(s)ds + \lambda(t) z_0(t) \leq z_0(t),$$

namely (10) is verified for $n = 0$.

Suppose after that $z_{n+1}(t) \leq z\ (t), \ t \geq 0$ and we have:

$$z_{n+1}(t) - z_{n+2}(t) = \sum_{i=1}^{k} \delta_i(t) \cdot \int_0^t [z_n(s) - z_{n+1}(s)]\ ds + \lambda(t)\ [z_n(t) - z_{n+1}(t)] \geq 0,$$

hence $z_{n+2}(t) \leq z_{n+1}\ (t), \ t \geq 0$.

The sequence $\{z_n(t)\}$ being monoton decreasing and lower bounded at zero, for each $t \geq 0$, it converges, namely there is for each $t \geq 0$:

$$\lim_{n \to \infty} z_n(t) = \tilde{z}(t)$$

Taking the limit in (9) as $n \to \infty$, we obtain that the function $\tilde{z}$ satisfies the equation (7) and because $0 \leq \tilde{z}(t) \leq z_0(t), \ t \geq 0$, we have $\tilde{z}(t) = 0, \ t \geq 0$.

Hence $\lim_{n \to \infty} z_n(t) = 0$, for each $t \geq 0$.

Now we define the operator:

$$\Lambda : L^1_{loc}(R_+) \to L^1_{loc}(R_+)$$

$$(\Lambda z)(t) = \sum_{i=1}^{k} \delta_i(t) \cdot \int_0^t z(s)ds + \lambda(t)\ z(t)$$

We find easily that if $u, v \in L^1_{loc}(R_+)$, $u \leqslant v$, then $(\Lambda u)(t) \leqslant (\Lambda v)(t)$, for each $t \geqslant 0$

Define

$$\Phi : L^1_{loc}(R_+, R^n) \to L^1_{loc}(R_+, R^n)$$

$$(\Phi x)(t) = F(t, x(t), \int_0^t a_1(t-s)h_1(x(s))ds, \ldots, \int_0^t a_k(t-s)h_k(x(s))ds)$$

Also, we define in $L^1_{loc}(R_+, R^n)$ the sequence $\{x_n\}$ thus:

$$x_{n+1} = \Phi x_n, \; n = 0, 1, \ldots, \text{ where } x_0 = \xi.$$

For $x, y \in L^1_{loc}(R_+, R^n)$ we have for any $t \geqslant 0$:

$$||(\Phi x)(t) - (\Phi y)(t)|| = ||F(t, x(t), \int_0^t a_1(t-s)h_1(x(s))ds, \ldots$$

$$\ldots, \int_0^t a_k(t-s)h_k(x(s))ds) - F(t, y(t)), \int_0^t a_1(t-s)h_1(y(s))ds, \ldots$$

$$\ldots, \int_0^t a_k(t-s)h_k(y(s))ds)|| \leqslant \lambda(t)||x(t) - y(t)|| + \sum_{i=1} \mu_i(t) \cdot$$

$$\cdot \int_0^t ||a_i(t-s)\;[h_i(x(s)) - h_i\,(y(s))||ds \leqslant \lambda(t)||x(t) - y(t)|| +$$

$$+ \sum_{i=1}^k \delta_i(t) \cdot \int_0^t ||x(s) - y(s)||ds = \Lambda(||x(.) - y(.)||)(t).$$

We have for any $t \geqslant 0$, on the one hand:

$$||x_1(t) - \xi|| = ||(\Phi\xi)(t) - \xi|| = u(t),$$

and on the other hand z_0 being the unique solution of equation (4):

$$z_0(t) = \sum_{i=1}^k \delta_i(t) \cdot \int_0^t z_0(s)ds + \lambda(t)z_0(t) + u(t) = (\Lambda z_0)(t) + u(t)$$

Thus, it results:

$$||x_1(t) - \xi|| \leqslant z_0(t), \; t \geqslant 0$$

After that, for any $t \geqslant 0$:

$$||x_2(t) - x_1(t)|| = ||(\Phi x_1)(t) - (\Phi\xi)(t)|| \leqslant \Lambda(||x(.) - \xi||)(t) =$$
$$= (\Lambda u)(t) = \Lambda(z_0 - \Lambda z_0)(t) \leqslant (\Lambda z_0)(t) = z_1(t).$$

By induction we obtain for any $t \geqslant 0$:

$$||x_{n+1}(t) - x_n(t)|| \leqslant z_n(t), \ n = 0, 1, \ldots$$

and more general:

$$||x_{n+p}(t) - x_n(t)|| \leqslant z_n(t); \ n, p = 0, 1, \ldots \tag{11}$$

As $\lim_{n \to \infty} z_n(t) = 0$ for each $t \geqslant 0$, it results from (11) that the sequence $\{x_n(t)\}$ is Cauchy, hence it converges, say, to $\bar{x}(t)$.

Thus we obtained the function $\bar{x} \in L^1_{loc}(R_+, R^n)$ which is the weak limit of sequence $\{x_n\}$.

It results from (11), as $p \to \infty$:

$$||\bar{x}(t) - x_n(t)|| \leqslant z_n(t), \ t \geqslant 0. \tag{12}$$

Now we prove that $\bar{x}$ is a solution for the system (SN).

We have, for any $t \geqslant 0$:

$$||\bar{x}(t) - (\Phi\bar{x})(t)|| \leqslant ||x(t) - x_{n+1}(t)|| + ||(\Phi x_n)(t) - (\Phi\bar{x})(t)|| \leqslant z_{n+1}(t) +$$
$$+ \Lambda(||x_n(.) - \bar{x}(.)||)(t) \leqslant z_{n+1}(t) + (\Lambda z_n)(t) = 2 \ z_{n+1}(t)$$

From here it results, for $n \to \infty$, that $\bar{x}(t) = (\Phi\bar{x})(t)$, $t \geqslant 0$, namely $\bar{x}$ is a solution of (SN).

From (12), taking $n = 0$, it results that is satisfied inequality:

$$||\bar{x}(t) - \xi|| \leqslant z_0(t), \ t \geqslant 0. \tag{13}$$

To prove the uniqueness of $\bar{x}$, let us suppose that exists another solution $\tilde{x} \in L^1_{loc}(R_+, R^n)$ of (SN) satisfying (13), namely:

$$||\tilde{x}(t) - \xi|| \leqslant z_0(t), \ t \geqslant 0$$

We find easily, by induction, that:

$$||\tilde{x}(t) - x_n(t)|| \leqslant z_n(t), \ t \geqslant 0; \ n = 0, 1, \ldots$$

Indeed, for $n = 0$ the inequality above is verified, after that suppose, for each $t \geqslant 0$:

$$||\tilde{x}(t) - x_n(t)|| \leqslant z_n(t)$$

We have:

$$||\tilde{x}(t) - x_{n+1}(t)|| = ||(\Phi\tilde{x})(t) - (\Phi x_n)(t)|| \leqslant \Lambda(||\tilde{x}(.) - x_n(.)||)(t) \leqslant$$
$$\leqslant (\Lambda z_n)(t) = z_{n+1}(t)$$

Taking the limit as $n \to \infty$, we obtain $\lim_{n \to \infty} x_n(t) = \tilde{x}(t)$, for each $t \geqslant 0$, and from the uniqueness of the limit of sequence $\{x_n(t)\}$ it results that $\bar{x} = \tilde{x}$.

Next we shall show that for every $n \in N$:

$$\lim_{t \to \infty} x_n(t) = \xi.$$

We have at first:

$$\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} F(t, \xi, \int_0^t a_1(t-s)h_1(\xi)ds, \ldots, \int_0^t a_k(t-s)\, h_k(\xi)ds) =$$

$$= F_0(\xi, A_1h_1(\xi), \ldots, A_kh_k(\xi)) = \xi$$

Because h_i is continuous it results that h_i is bounded on Σ and therefore for every $i \in \{1, \ldots, k\}$ the mapping

$$t \to \int_0^t a_i(t-s)\, h_i(\xi)ds$$

is bounded on R_+.

After that, from *ii*) and *iii*) we have, for $t \geqslant 0$:

$$||x_1(t)|| = ||F(t, \xi, \int_0^t a_1(t-s)h_1(\xi)ds, \ldots, \int_0^t a_k(t-s)h_k(\xi)ds)|| \leqslant$$

$$\leqslant \lambda(t)\,||\xi|| + \sum_{i=1}^{k} \mu_i(t) \cdot ||\int_0^t a_i(t-s)h_i(\xi)ds||$$

Hence x_1 is bounded on R_+.

From here results that $h_i(x_1(.))$ is bounded and therefore for every i the mapping:

$$t \to \int_0^t a_i(t-s)\;\; h_i(x_1(s))ds$$

is bounded on R_+.

Using again *ii*) and *iii*) it results that x_2 is bounded.
By induction we obtain therefore that x_n is bounded on R_+, for every $n = 0, 1, \ldots$

Then it results also the fact that $\bar{x}$ is bounded on R_+.

Also $x_{n+1} - x_n$ is bounded for $n = 0, 1, \ldots$, hence:

$$||x_{n+1}(t) - x_n(t)|| \leqslant M_n, \;\forall t \geqslant 0$$

On the other hand, using *iii*) and *iv*) we obtain for $t \geqslant 0$:

$$||x_2(t) - x_1(t)|| \leqslant \lambda(t)\,||x_1(t) - \xi|| + \sum_{i=1}^{k} \delta_i(t) \cdot \int_0^t ||x_1(s) - \xi||ds \leqslant$$

$$\leqslant \lambda(t)\,||x_1(t) - \xi|| + M_0 \cdot \sum_{i=1}^{k} t\delta_i(t)$$

Considering the way in which μ_i and γ_i were defined and taking the limit as $t \to \infty$, we find that:

$$\lim_{t\to\infty} ||x_2(t) - x_1(t)|| = 0$$

By induction, we obtain:

$$\lim_{t\to\infty} ||x_{n+1}(t) - x(t)|| = 0, \ n = 0, 1, \ldots \tag{14}$$

Using the relation (14), now we obtain, for any $t \geqslant 0$:

$$||x_2(t) - \xi|| \leqslant ||x_2(t) - x_1(t)|| + ||x_1(t) - \xi||,$$

and $\lim\limits_{t\to\infty} x_2(t) = \xi$.

Using successively the relation (14), we obtain by induction:

$$\lim_{t\to\infty} x_n(t) = \xi, \ n = 0, 1, \ldots \tag{15}$$

From (15) and the that $\lim\limits_{n\to\infty} x_n(t) = \bar{x}(t)$, $t \geqslant 0$, it results:

$$\lim_{t\to\infty} \bar{x}(t) = \xi$$

3. In the end of this paper we consider a particular case of the system (SN), namely the scalar equation studied by S. O. Londen ([7]):

$$x(t) + \int_0^t a(t-s)h(s))ds = f(t), \ t \geqslant 0 \tag{E}$$

in weaker conditions on f and a, namely: $f \in L^1_{loc}(R_+, R)$, $f(t) \to f_0$, $t \to \infty$, $a \in L^1(R_+, R)$ and $h \in C(R)$.

We attach to the equation (E) the equation of stationary solutions:

$$\xi + \alpha h(\xi) = f_0, \tag{St}$$

where $\alpha = \int_0^\infty a(t)\ dt$ ant let Σ be the set of stationary solutions.

We obtain as an immediate consequence of the theorem above:

COROLLARY. *Assume there are fulfilled the following conditions:*

i) Σ *is nonempty and bounded set.*

ii) $|a(t-s)\ [h(x) - h(x')]| \leqslant \gamma(t)|x - x'|$, $0 \leqslant s \leqslant t$; $x, x' \in R$ *where*

$$\gamma \in L^1_{loc}(R_+) \text{ and } \lim_{t\to\infty} t \cdot \gamma(t) = 0$$

Then the conclusion of theorem above holds.

Remark. The condition *i)* of corollary is fulfilled, for example, if there is a closed interval $I \subset R$ so that h is monoton decreasing on I and $f_0 - \alpha h(I) \subseteq I$.

Then, applying the Knaster's fixed point theorem to mapping $f_0 - \alpha h(.)$ on the interval I, it results that Σ us nonempty and bounded set.

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STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXIII, 4, 1988

A NOTE ON MULTIVALUED AFFINE MAPPINGS

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Received: April 3, 1988

REZUMAT. — Notă asupra aplicaţiilor afine multivoce. Se stabilesc citeva proprietăţi ale aplicaţiilor afine multivoce care pun in evidenţă o relaţie strînsă a acestor aplicaţii cu aplicaţiile afine univoce, pe de o parte, şi cu aplicaţiile liniare multivoce, pe de altă parte. În final, se stabileşte o teoremă de punct fix de tip Markoff—Kakutani pentru aplicaţii multivoce.

1. **Definitions and properties.** Probably the notion of multivalued affine mappings was introduced by P. H. Sach in [6] as a special case of the notion of multivalued convex mappings introduced by B. N. Pshenitchny in [4] and investigated by the author in [7]. In this section we shall prove some elementary properties of multivalued affine mappings and establish a relationship between the latters and multivalued linear mappings introduced by C. Berge in [1] and investigated by the author in [8].

Throughout this section X and Y denote two linear spaces with the origin θ, 2^Y stands for the collection of all nonempty subsets of Y, E — an affine subset of X and R — the real line.

DEFINITION 1 [1]. A mapping $L : X \to 2^Y$ is said to be *linear* if

$$L(x_1 + x_2) = Lx_1 + Lx_2,\ L(rx) = rLx,\ \theta \in L\theta$$

for all $x,\ x_1,\ x_2 \in X,\ r \in R,\ r \neq 0$.

DEFINITION 2 [6]. A mapping $A : E \to 2^Y$ is said to be *affine* if

$$A(rx_1 + (1 - r)x_2) = rAx_1 + (1 - r)Ax_2 \tag{1}$$

for all $x_1,\ x_2 \in E$ and $r \in R$.

Note that (1) implies

$$A(rx_1 + sx_2 + tx_3) = rAx_1 + sAx_2 + tAx_3 \tag{2}$$

for all $x_1,\ x_2,\ x_3 \in E$ and $r,\ s,\ t \in R$ with $r + s + t = 1$.

In the sequel A always denotes an affine multivalued mapping but the word „multivalued" is omitted for simplicity of exposition.

PROPERTY 1. *A is affine if and only if its graph G_A is affine, where $G_A = \{(x, y) \in E \times Y : y \in Ax\}$.*

Proof. The „only if" part being trivial we now prove the „if" part. Let G_A be affine and $y \in rAx_1 + (1 - r)Ax_2$. Then there exist $y_i \in Ax_i$, $i = 1, 2$, such that $y = ry_1 + (1 - r)y_2$. Since $(x_i, y_i) \in G_A$ we have

$$r(x_1, y_1) + (1 - r)(x_2, x_2) = (rx_1 + (1 - r)x_2, y) \in G_A,$$

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i.e. $y \in A(rx_1 + (1-r)x_2)$. From this

$$rAx_1 + (1-r)Ax_2 \subset A(rx_1 + (1-r)x_2).$$

To prove the inverse inclusion let $r \in R$ and $y \in A(rx_1 + (1-r)x_2)$ be given. Without loss of generality we may assume that $r \neq 1$. Taking $y_1 \in Ax_1$ we have $(x_1, y_1) \in G_A$ and hence $(rx_1, ry_1) \in rG_A$. Putting $y_2 = (y - ry_1)/(1-r)$ we get $y = ry_1 + (1-r)y_2$. Because $(rx_1 + (1-r)x_2, y) \in G_A$ one has

$$((1-r)x_2, (1-r)y_2) \in G_A - rG_A = (1-r)G_A,$$

where the last equality is true since G_A is affine and $r \neq 1$. From this $(x_2, y_2) \in G_A$, i.e. $y_2 \in Ax_2$. This shows that $y \in rAx_1 + (1-r)Ax_2$ and hence $A(rx_1 + (1-r)x_2) \subset rAx_1 + (1-r)Ax_2$. The proof is complete.

PROPERTY 2. *If A is affine then Ax is affine for each $x \in X$.*

The proof is straightforward.

PROPERTY 3. *If $A : X \to 2^Y$ is affine then the mapping L defined by $Lx = Ax - A\theta$, $x \in X$, is linear.*

Proof. For $x_1, x_2, \in X$ we have by (2)

$$L(x_1 + x_2) = A(x_1 + x_2) - A\theta = A(x_1 + x_2 - \theta) - A\theta =$$
$$Ax_1 + Ax_2 - A\theta - A\theta = Lx_1 + Lx_2.$$

Since A is affine by Property 2, for $r \neq 0$ we have

$$(1-r)A\theta - A\theta = -rA\theta.$$

From this, for $x \in X$ and $r \neq 0$ one gets

$$L(rx) = A(rx) - A\theta = A(rx + (1-r)\theta) - A\theta =$$
$$= rAx + (1-r)A\theta - A\theta = rAx - rA\theta = rLx.$$

It is clear that $\theta \in A\theta - A\theta = L\theta$. Thus, L is linear as claimed.

PROPERTY 4. *If $A : E \to 2^Y$ is affine then $Ax - Ax = M$ for all $x \in E$ and $Ax = y + M$ for each $y \in Ax$.*

Proof. By (2) for every $x, x' \in E$ we have

$$Ax - Ax = Ax - (Ax + Ax' - Ax') = Ax' - (Ax' + Ax - Ax) =$$
$$Ax' - Ax' = M.$$

Now since $Ax + M = Ax + Ax - Ax = Ax$ we have $y + M \subset Ax$ for every $y \in Ax$. From this $M \subset Ax - y \subset Ax - Ax = M$. Therefore $M = Ax - y$ and the proof is complete.

COROLLARY. *An affine mapping A is single-valued if and only if Ax is a singleton for some $x \in E$.*

Indeed, in this case M must be the singleton $\{\theta\}$.

PROPERTY 5. *Each affine mapping A possesses an affine selection a. Moreover, for each $x \in E$ we have $Ax = \{a(x) : a \in \mathcal{A}\}$, where $\mathcal{A}$ denotes the collection of all selections of A.*

Proof. Let $M = Ax - Ax$ be the subspace mentioned in Property 4, m_0 an element of M and N an alebraic complement of M in Y. Denote $N_0 = N + m_0$. We define a mapping $a : E \to N_0$ by putting $a(x) = Ax \cap N_0$. First we show that a is single-valued. Indeed, if $y_1, y_2 \in a(x)$ then $y_1 - y_2 \in M \cap N = \{\theta\}$, hence $y_1 = y_2$. Since $a(x) \in Ax$ for all $x \in E$, it remains to prove that a is affine. Let $x_1, x_2 \in E$ and $r \in R$ be given. We have

$$a(rx_1 + (1 - r)x_2) = A(rx_1 + (1 - r)x_2) \cap N_0.$$

On the other hand, since N_0 is affine, we get

$$ra(x_1) + (1 - r)a(x_2) \in [rAx_1 + (1 - r)Ax_2] \cap N_0.$$

Combining the above results yields

$$a(rx_1 + (1 - r)x_2) = ra(x_1) + (1 - r)a(x_2).$$

To prove the second claim in Property 5 we take an $y_0 \in Ax$ and fix an algebraic complement N of M. Write $y_0 = m_0 + n_0$ with $m_0 \in M$, $n_0 \in N$, then put $N_0 = N + m_0$. It is obvious that the mapping a_0 defined by $a_0(x) = Ax \cap N_0$ is an affine selection of A with $a_0(x) = y_0$. The proof is complete.

DEFINITION 3 [2]. Let X be a topological space, Y a real topological vector space. A multivalued mapping $T : X \to 2^Y$ is said to be *upper demi-continuous* (briefly, u.d.c.) if for each $x \in X$ and each open half-space H in Y containing Tx there exists a neighbourhood U of x such that $T(U) \subset H$.

PROPERTY 6. *Let $A : E \to 2^Y$ be an affine mapping, where Y is a real locally convex space. If A is u.d.c. then the selection a defined in Property 5 is also u.d.c.*

Proof. Let $a : E \to N_0$ be a selection of A defined in the proof of Property 5. Fix an $x \in E$ and an open half-space H of Y containing $a(x)$. We must show such a neighbourhood U of x that $a(U) \subset H$. By P we denote the hyperplane defining H. Then $S = P \cap N_0$ is an affine subset of Y and obviously, $a(x) \notin S$. Choose an open convex neighbourhood V of θ in Y such that $(a(x) + V) \cap S = \emptyset$. It is clear that we still have $(Ax + V) \cap S = \emptyset$. Since $Ax + V$ is open and convex, by Theorem 2 of Chapter 2 in [5], S can be extended to a hyperplane S' such that $(Ax + V) \cap S' = \emptyset$. The hyperplane S' defines an open half-space H' containing Ax. By the upper demi-continuity of A there is a neighbourhood U of x such that $A(U) \subset H'$, i.e. $A(U) \cap S' = \emptyset$, in particular $a(U) \cap S = \emptyset$. This shows that $a(U) \subset H$ and the proof is complete.

PROPERTY 7. *If $A : E \to 2^X$ is affine then its fixed point set F_A is affine, where $F_A = \{x \in E : x \in Ax\}$. If in addition, X is real locally convex, A is u.d.c. with closed values then F_A is closed.*

Proof. The first claim being obvious we prove now the second one. Let $x \notin F_A$, i.e. $x \notin Ax$. Since Ax is convex and closed, there exist an open half-space H containing Ax and a neighbourhood U_1 of x such that, $U_1 \cap H = \emptyset$. By the upper demi-continuity of A there is a neighbourhood U_2 of x such that $A(U_2) \subset H$. With $U = U_1 \cap U_2$ we have $U \cap A(U) = \emptyset$. From this, $U \cap F_A = \emptyset$ and F_A is closed as desired.

Note that under the same condition using the similar arguments we can prove that G_A (the graph of A) is closed in $E \times X$.

We conclude this section with formulations of the following properties whose easy proofs are omitted.

PROPERTY 8. *If A is affine and $A(E) = Y$ then the inverse mapping A^{-1} defined by $A^{-1}y = \{x \in X : y \in Ax\}$, $y \in Y$, is also affine.*

PROPERTY 9. *If A_1 and A_2 are affine then the sum $A = A_1 + A_2$ defined by $Ax = A_1x + A_2x$, $x \in E$, is also affine.*

PROPERTY 10. *If $A_1 : E \to 2^Y$ and $A_2 : Y \to 2^Z$ are affine then the composition $A = A_2 \cdot A_1$ defined by $Ax = A_2(A_1x)$, $x \in E$, is also affine.*

PROPERTY 11. *If A_1 and A_2 are affine then the intersection $A = A_1 \cap A_2$ defined by $Ax = A_1x \cap A_2x$, $x \in E$, is also affine provided $Ax \neq \emptyset$ for all $x \in E$.*

PROPERTY 12. *If $A_1 : E \to 2^Y$ and $A_2 : E \to 2^Z$ are affine then the Cartesian product $A = A_1 \times A_2$ defined by $Ax = A_1x \times A_2x$, $x \in E$, is also affine.*

2. **A fixed point theorem.** This section is devoted to a multivalued version of the following result known as the Markoff—Kakutani fixed point theorem [3].

THEOREM (Markoff—Kakutani). *Let K be a nonempty convex compact subset of a Hausdorff locally convex space X, $\{a_i : i \in I\}$ a commuting family of continuous (single-valued) affine mappings from K into itself. Then there exists a common fixed point of the family $\{a_i\}$.*

Remark 1. If X is a Hausdorff *real* locally convex space then using the Ky Fan theorem in [2] instead of the well-known Tychonoff fixed point theorem one can show that the continuity in the above theorem can be replaced by the upper demicontinuity.

Note that the condition „commuting" is not sufficient for the existence of common fixed points of multi-valued affine mappings as shown in the following

Example 1. Let $X = R^2$, $K = \{(x_1, x_2) : 0 \leqslant x_1, x_2 \leqslant 1\}$, $A(x_1, x_2) = \left\{\left(\frac{x_1+1}{2}, r\right) : r \in R\right\}$, $B(x_1, x_2) = \{(r, r + x_2) : r \in R\}$. It is easy to verify that $ABx = BAx = R^2$ for each $x \in K$ and $F_A = \{(0, r) : r \in R\}$, $F_B = \{(r, 0) : r \in R\}$.

The above example justifies the following

DEFINITION 4. Two mappings $A, B : X \to 2^X$ are said to be *strictly commuting* if for every $x \in X, y \in Ax$ and $z \in Bx$ we have $Az = By$. A family of mappings is strictly commuting if each pair of its elements is strictly commuting.

Obviously for single-valued mappings the two notions "commuting" and "strictly commuting" coincide.

Before stating the main result of this note we remark that for a strictly commuting family of affine mappings the subspace M defined in Property 4 is common for the whole family, so we can choose a common affine set N_0 mentioned in the proof of Property 5.

THEOREM. *Let X be a Hausdorff real locally convex space and $\{A_i : i \in I\}$ a strictly commuting family of u.d.c. multivalued affine mappings from X into*

2^X with closed values, K a nonempty compact convex subset of X. Suppose that we can choose the set N_0 (mentioned above) such that

$$N_0 \cap K \cap A_i x \neq \emptyset \tag{3}$$

for all $x \in K$ and $i \in I$. Then the common fixed point set of the family $\{A_i\}$ contained in K is nonempty convex and compact.

Proof. Define for each $i \in I$ an affine selection a_i of A_i by putting $a_i(x) = A_i x \cap N_0$. Condition (3) ensures that each a_i maps K into itself, and by Property 6, they are u.d.c. We now prove that they are commuting. Since A_i are strictly commuting for $i, j \in I$ and $x \in K$ we have

$$A_i a_j(x) = A_j a_i(x).$$

From this $a_i(a_j(x)) = A_i a_j(x) \cap N_0 = A_j a_i(x) \cap N_0 = a_j(a_i(x))$ as desired. By the modified version of the Markoff—Kakutani theorem mentioned in Remark 1, the family $\{a_i\}$ has a common fixed point and hence the common fixed point set of the family $\{A_i\}$ is nonempty. The fact that this set is convex and compact follows from Property 7 and the compactness of K. The proof is complete.

Remark 2. In the single-valued case we have $M = \{\theta\}$, $N = N_0 = X$ so the condition (3) is equivalent to the requirement that a_i maps K into itself. So we see that the above theorem is equivalent to the mentioned version of the Markoff—Kakutani theorem.

We conclude this section by noting that the notion „strictly commuting" cannot be weakened, namely if we require only that for each $x \in X$ there exist $y \in Ax$ and $z \in Bx$ such that $Az = By$ then the assertion of the theorem is not valid.

Example 2. Let X, K and B be as in Example 1. Further let $C(x_1, x_2) = \{(r + x_1, r) : r \in R\}$ and $D(x_1, x_2) = \{(r, r + 1 - 2x_1) : r \in R\}$. It is not difficult to verify that each of the pairs (B, C), (C, D), (D, B) satisfies the above condition but they are not strictly commuting. It is also easy to see that they satisfy the condition (3) with $N = \{(r, -r) : r \in R\}$ and $m_0 = \left(\frac{1}{2}, \frac{1}{2}\right)$ but we have $F_B \cap F_C = (0, 0)$, $F_C \cap F_D = (1, 0)$ and $F_D \cap F_B = (0, 1)$.

As for the condition (3) a simple example in R^3 shows that a set N_0 satisfying (3) does not always exist.

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STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXIII, 4, 1988

BIRECURRENCY PROBLEMS IN RIEMANNIAN SPACES

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Received: October 15, 1987

REZUMAT. — Probleme de birecurenţă în spaţii riemanniene. În această lucrare se extind rezultatele din [2] pentru cazul spaţiilor riemanniene birecurente (1, 2). Se dau relaţiile (1,6), (1,7), (1,8) pe care le verifică tensorul de birecurenţă φ_{ij}, se stabilesc relaţiile ce există între spaţiile riemanniene cu tensori birecurenţi, propoziţiile $\overline{2.1-2.15}$ şi schema (2,30).

1. Generalities. Let V_n be a Riemann space of metrics

$$ds^2 = g_{ij}\, dx^i dx^j \tag{1.1}$$

DEFINITION 1.1: The space V_n is called *birecurrent* [3], if there exists a covariant tensor of second order, so that

$$R^i_{jkh,rs} = \varphi_{rs} R^i_{jkh} \tag{1.2}$$

where comma denotes the covariant derivation of second order with respect to the metrics of the space (1.1), and R^i_{jkh} are the components of the curvature tensor.

If in (1.2) we apply a contraction in i and k, we get

$$R_{jh,rs} = \varphi_{rs} R_{jh} \tag{1.3}$$

where R_{jh} are the components of the Ricci tensor.

DEFINITION 1.2: A space V_n that verifies (1.3) is called *Ricci birecurrent*.

From (1.2) and (1.3) it follows:

PROPOSITION 1.1. *A birecurrent V_n space of tensor φ_{rs} is also Ricci-birecurrent with the same birecurrency tensor.*

Remark. The converse of this assertion is not generally true. Here too, one can give necessary and sufficient conditions so that the converse take place, conditions similarly to those from [4].

Transvecting (1.3) by g^{jh} we get:

$$R_{,rs} = \varphi_{rs} R \tag{1.4}$$

where R is the scalar curvature of the space.

We have therefore:

DEFINITION 1.3: A space V_n that verifies (1.4) is called *birecurrent scalar curvature space.*

From (1.2), (1.3) and (1.4) we have

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PROPOSITION 1.2. *A birecurrent or Ricci-birecurrent V_n space is of birecurernt scalar curvature.*

Considering the Bianchi identities of second type

$$R^i_{jkh,r} + R^i_{jhr,k} + R^i_{jrk,h} = 0$$

and applying the covariant derivation of index s, with respect to metrics (1.1), we have:

$$R^i_{jkh,rs} + R^i_{jhr,ks} + R^i_{jrk,hs} = 0 \tag{1.5}$$

Taking count of (1.2) in (1.5) we have:

$$\varphi_{rs} R^i_{jkh} + \varphi_{ks} R^i_{jhr} + \varphi_{hs} R^i_{jrk} = 0 \tag{1.6}$$

and therefore:

PROPOSITION 1.3. *In a birecurrent V_n space relations (1.6) hold for the components of the birecurrency tensor with a fixed index.*

Contracting (1.6) with respect i and r, we have

$$\varphi_{is} R^i_{jkh} - \varphi_{ks} R_{jh} + \varphi_{hs} R_{jk} = 0$$

whence

$$[R^i_{jkh} - \delta^i_k R_{jh} + \delta^i_h R_{jk}] \varphi_{is} = 0 \tag{1.7}$$

In (1.7), transvecting by g^{jk}, we have

$$(\delta^i_h R - 2R^i_h) \varphi_{is} = 0 \tag{1.8}$$

From (1.7) and (1.8) it follows:

PROPOSITION 1.4. *In a birecurrent V_n space, the components of the birecurrency tensor φ_{rs} with a fixed index are solutions of the homogeneous system (1.7) and verify relations (1.8).*

Let us consider a recurrent V_n space

$$R^i_{jkh,r} = \varphi_r R^i_{jkh} \tag{1.9}$$

and let us apply the covariant derivation in both members of the relation (1.9), we have:

$$R^i_{jkh,rs} = \varphi_{r,s} R^i_{jkh} + \varphi_r R^i_{jkh,s} = (\varphi_{r,s} + \varphi_r \varphi_s) R^i_{jkh} \tag{1.10}$$

Noting $\varphi_{r,s} + \varphi_r \varphi_s = \varphi_{rs}$ it follows the well-known

PROPOSITION 1.5. *The recurrent V_n spaces are also birecurrent with birecurrency tensor*

$$\varphi_{rs} = \varphi_{r,s} + \varphi_r \varphi_s$$

2. Relations between riemannian spaces with birecurrent tensors. Let us consider now, in the V_n space the projective curvature tensors

$$P^i_{jkh} = R^i_{jkh} - \frac{1}{n-1} (\delta^i_k R_{jh} - \delta^i_h R_{jk}) \tag{2.1}$$

the conform curvature tensor

$$C^i_{jkh} = R^i_{jkh} - \frac{1}{n-2}(\delta^i_k R_{jh} - \delta^i_h R_{jk} + g_{jh} R^i_k - g_{jk} R^i_h) + \\ + \frac{R}{(n-1)(n-2)}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) \tag{2.2}$$

the coharmonic curvature tensor

$$Z^i_{jkh} = R^i_{jkh} - \frac{1}{n-2}(\delta^i_k R_{jh} - \delta^i_h R_{jk} + g_{jh} R^i_k - g_{jk} R^i_h) \tag{2.3}$$

the concircular curvature tensor

$$T^i_{jkh} = R^i_{jkh} - \frac{R}{n(n-1)}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) \tag{2.4}$$

and the tensors:

$$g^{jh} P^i_{jkh} = P^i_k = \frac{n}{n-1}\left(R^i_k - \frac{R}{n}\delta^i_k\right) \tag{2.5}$$

$$g_{ih} P^i_k = P_{hk} = \frac{n}{n-1}\left(R_{hk} - \frac{R}{n} g_{hk}\right) \tag{2.6}$$

$$Z^i_{jih} = Z_{jh} = -\frac{R}{n-2} g_{jh} \tag{2.7}$$

$$T^i_{jih} = T_{jh} = R_{jh} - \frac{R}{n} g_{jh} \tag{2.8}$$

We have:

DEFINITION 2.1. A V_n space is called *birecurrent projective curvature space, birecurrent conform curvature space, birecurrent coharmonic curvature space, birecurrent concircular curvature space,* if there exists a covariant tensor of second order, φ_{rs} so that we respectively have:

$$P^i_{jkh,rs} = \varphi_{rs} P^i_{jkh} \tag{2.9}$$

$$C^i_{jkh,rs} = \varphi_{rs} C^i_{jkh} \tag{2.10}$$

$$Z^i_{jkh,rs} = \varphi_{rs} Z^i_{jkh} \tag{2.11}$$

$$T^i_{jkh,rs} = \varphi_{rs} T^i_{jkh} \tag{2.12}$$

Remark. The spaces V_n, that verify relations (2.9)—(2.12) we will simply call projective birecurrent, conform birecurrent coharmonic birecurrent and concircular birecurrent spaces.

Derivating covariantly twice the relation (2.1), we have:

$$P^i_{jkh,rs} = R^i_{jkh,rs} - \frac{1}{n-1}(\delta^i_k R_{jh,rs} - \delta^i_h R_{jk,rs}) \tag{2.13}$$

if the space V_n is birecurrent, taking count in (2.13) of (1.2) and (1.3) it follows (2.9) and therefore we have:

PROPOSITION 2.1. *The birecurrent V_n spaces are also projective birecurrent with the same birecurrency tensor.*

For the converse of this assertion, from (1.3), (2.9) and (2.13) we have

$$\varphi_{rs} P^i_{jkh} = R^i_{jkh,rs} - \frac{1}{n-1} \varphi_{rs} (\delta^i_k R_{jh} - \delta^i_h R_{jk})$$

or

$$\varphi_{rs} \left[R^i_{jkh} - \frac{1}{n-1} (\delta^i_h R_{jk} - \delta^i_k R_{jh}) \right] = R^i_{jkh,rs} - \frac{1}{n-1} \varphi_{rs} (\delta^i_k R_{jh} - \delta^i_h R_{jk})$$

whence it follows

$$R^i_{jkh,rs} = \varphi_{rs} R^i_{jkh}$$

and the space is birecurrent. Hence we have

PROPOSITION 2.2. *The projective birecurrent and Ricci-birecurrent V_n spaces, are birecurrent.*

Remark. From proposition 2.2 it follows that a sufficient condition so that the projective birecurrent spaces be birecurrent is that they have to be Ricci-birecurrent.

Transvecting (2.9) by g^{jh} and taking count of (2.5) and (2.6) we have

$$P_{hk,rs} = \varphi_{rs} P_{hk} \tag{2.14}$$

and therefore

PROPOSITION 2.3. *In a projective birecurrent V_n space, the tensor (2.6) is also birecurrent with the same birecurrency tensor.*

Derivating covariantly twice the relation (2.2) and taking count of (1.2) and (1.3) we have

$$C^i_{jkh,rs} = \varphi_{rs} C^i_{jkh} \tag{2.15}$$

and therefore:

PROPOSITION 2.4. *The birecurrent V_n spaces are also conform birecurrent with the same birecurrency tensor.*

For the converse of this assertion, we observe that from (1.3), (1.4) and (2.15) it follows (1.2), therefore:

PROPOSITION 2.5. *The conform birecurrent V_n spaces are birecurrent if and only if they are Ricci-birecurrent with the same birecurrency tensor.*

Derivating covariantly twice the relation (2.3) and taking count of (1.2) and (1.3) it follows

$$Z^i_{jkh,rs} = \varphi_{rs} Z^i_{jkh} \tag{2.16}$$

and hence we have

PROPOSITION 2.6. *The birecurrent V_n spaces are also coharmonic birecurrent with the same birecurrency tensor.*

For the converse if this assertion, like above, it follows that the coharmonic birecurrent and Ricci-birecurrent spaces, are birecurrent.

If in (2.11) one apply a contraction in i and k, it follows

$$Z_{jh,rs} = \varphi_{rs} Z_{jh} \tag{2.17}$$

and therefore

PROPOSITION 2.7. *In a coharmonic birecurrent V_n space, the contracted tensor (2.17) is also birecurrent with the same birecurrency tensor.*

Derivating covariantly twice the relation (2.4) and taking count of (1.2) and (1.4) it follows

$$T^i_{jkh,rs} = \varphi_{rs} T^i_{jkh} \tag{2.18}$$

and therefore

PROPOSITION 2.8. *The birecurrent V_n spaces are also concircular birecurrent with the same birecurrency tensor.*

For the converse, we observe that from (2.18) and (1.4) it follows (1.2).

If in (2.12) one apply a contraction in i and k, it follows:

$$T_{jh,rs} = \varphi_{rs} T_{jh} \tag{2.19}$$

and therefore

PROPOSITION 2.9. *In a concircular birecurrent V_n space, the contracted tensor (2.8) is also birecurrent, with the same birecurrency tensor.*

Considering now the space V_n, projective recurrent, therefore

$$P^i_{jkh,r} = \varphi_r P^i_{jkh}$$

or conform recurrent, that is

$$C^i_{jkh,r} = \varphi_r C^i_{jkh}$$

or coharmonic recurrent

$$Z^i_{jkh,r} = \varphi_r Z^i_{jkh}$$

or concircular recurrent

$$T^i_{jkh,r} = \varphi_r T^i_{jkh}$$

and derivating covariantly once more, we get (2.9, (2.10), (2.11) and (2.12) where

$$\varphi_{rs} = \varphi_{r,s} + \varphi_r \varphi_s \tag{2.20}$$

therefore

PROPOSITION 2.10. *The projective recurrent, conform recurrent, coharmonic recurrent or concircular recurrent V_n spaces are also birecurrent with birecurrency tensor given by (2.20).*

Let us now consider [2]. The relations that holds between the projective curvature, conform curvature, coharmonic curvature, concircular tensors and their contracted:

$$C^i_{jkh} = Z^i_{jkh} - \frac{1}{n-1}(\delta^i_k Z_{jh} - \delta^i_h Z_{jk}) \tag{2.21}$$

$$P^i_{jkh} = T^i_{jkh} - \frac{1}{n-1}(\delta^i_k T_{jh} - \delta^i_h T_{jk}) \tag{2.22}$$

$$T^i_{jkh} = P^i_{jkh} + \frac{1}{n}(\delta^i_k P_{jh} - \delta^i_h P_{jk}) \tag{2.23}$$

$$C^i_{jkh} = T^i_{jkh} - \frac{1}{n-2}(\delta^i_k T_{jh} - \delta^i_h T_{jk} + g_{jh} T^i_k - g_{jk} T^i_h) \tag{2.24}$$

$$C^i_{jkh} = P^i_{jkh} - \frac{1}{n(n-2)}(\delta^i_k P_{jh} - \delta^i_h P_{jk}) - \frac{n-1}{n(n-2)}(g_{jh} P^i_k - g_{jk} P^i_h) \tag{2.25}$$

$$P_{ik} = \frac{n}{n-1} T_{ik} \tag{2.26}$$

$$T_{jh} = R_{jh} + \frac{n-2}{n} Z_{jh} \tag{2.27}$$

$$(n-1)\, P_{jh} = n\, R_{jh} + (n-2)\, Z_{jh} \tag{2.28}$$

From (2.16), (2.17) and (2.21) it follows that the coharmonic birecurrent V_n spaces are also conform birecurrent with the same birecurrency tensor.

For the converse of this assertion, from (2.10). (2.21) and (2.7) it follows:

$$Z^i_{jkh,rs} - \varphi_{rs} Z^i_{jkh} = \frac{1}{(n-1)(n-2)}(\delta^i_k g_{jh} - \delta^i_h g_{jk})(\varphi_{rs} R - R_{,rs}) \tag{2.29}$$

whence

PROPOSITION 2.11. *The necessary and sufficient condition for a conform birecurrent V_n space to be coharmonic birecurrent is that it has to be a birecurrent scalar curvature space, with the same birecurrency tensor or that the scalar curvature to be identically null.*

From (2.14), (2.19), (2.22), (2.23) if follows that the projective birecurrent spaces are also concircular birecurrent and those concircular birecurrent are also projective birecurrent, therefore:

PROPOSITION 2.12. *The projective birecurrent V_n spaces and those concircular birecurrent with the same birecurrency tensor, coincide.*

From (2.24), (2.25), (2.14), (2.19) and (2.9) it follows (2.10).

For the converse, from (2.25) and (2.10) it follows (2.9), if and only if the tensor (2.5) is birecurrent. We have therefore:

PROPOSITION 2.13. *The projective birecurrent or concircular birecurrent V_n spaces are conform birecurrent with the same birecurrency tensor, and the conform birecurrent spaces are projective birecurrent if and only if the tensor (2.5) is birecurrent with the same birecurrency tensor.*

From (2.21), (2.24), (2.25) it follows that the projective birecurrent or concircular birecurrent spaces are coharmonic birecurrent if they are birecurrent scalar curvature or null curvature spaces.

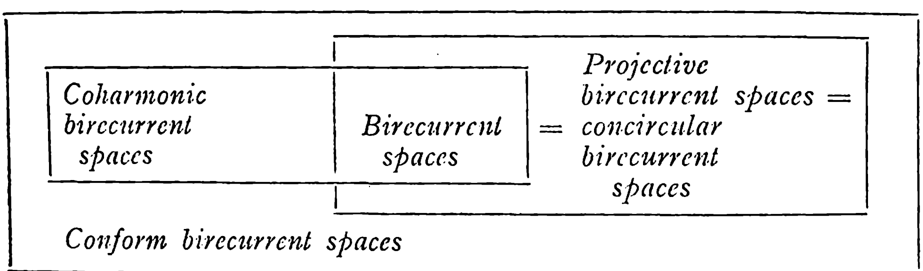
Conversely, the coharmonic birecurrent spaces are projective birecurrent if they are Ricci-birecurrent, therefore birecurrent. Hence we have:

PROPOSITION 2.14. *A projective birecurrent or concircular birecurrent and coharmonic birecurrent V_n space, is birecurrent.*

From the above ir follows

PROPOSITION 2.15. *The coharmonic birecurrent V_n spaces are also conform birecurrent with the same birecurrency tensor. The concircular birecurrent V_n spaces and those projective birecurrent, coincide. The projective birecurrent V_n spaces are also conform birecurrent, and the simultaneous projective birecurrent and coharmonic birecurrent spaces are birecurrent.*

Schematic, this rezult can be outlined as in (2.30).



(2.30)

Remark. From this scheme if follows that between the Riemannian spaces with birecurrent tensor there exists the same relations as those established in [2] for the recurrent V_n spaces.

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STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXIII, 4, 1988

ORBITAL MOTION WITH PERIODICALLY CHANGING GRAVITATIONAL PARAMETER

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Received: February 27, 1988

ABSTRACT. — The initially circular motion perturbed by a periodic variation of the gravitational parameter is studied. The equations of motion in the phase plane, in dimensionless variables, are investigated by means of the stroboscopic method. The stability of the motion is also investigated and the time scale for the unstable motion is estimated. A possible application for the case of a pulsating star is pointed out.

1. The Variable Gravitational Parameter. Consider a body (material point) of mass m, moving in the central gravitational field of another body (material point) of mass M, under the influence of a central perturbing force (or resultant of several such forces) which is described by an inverse square law. Since the resulting force which acts on the body m is central, the orbit of this one lies in a plane determined by the radius vector and the velocity vector at the initial instant. We shall choose this plane as reference plane for the study of the motion. The trajectory of the material point m is described by the differential equation (see [4]):

$$\ddot{r} - C^2/r^3 = -G(M + m)/r^2 + K/r^2, \tag{1}$$

where r is the radius vector of the body m with respect to M, C is the constant angular momentum, G is the gravitational constant, while K/r^2 is the perturbing acceleration due to the above mentioned perturbing force ($K < 0$ if this force is attractive and $K > 0$ if it is repulsive).

Suppose that both the masses M, m and the quantity K (which features the perturbing force) are time-dependent. Moreover, according to some relatively recent cosmological theories, as Brans-Dicke theory (see e.g. [2], [11]), we may suppose a very slow decrease of G with the time. Hence we can write:

$$\begin{aligned} M &= M_0(1 + f_M(t)),\\ m &= m_0\,(1 + f_m(t)),\\ K &= K_0\,(1 + f_K(t)),\\ G &= G_0\,(1 + f_G(t)). \end{aligned} \tag{2}$$

In these conditions, the given problem is equivalent to the Kepler problem is the case of a variable gravitational parameter, namely the equation of motion is:

$$\ddot{r} - C^2/r^3 = -\mu/r^2, \tag{3}$$

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where μ is of the form:

$$\mu = H(1 - f(t)), \tag{4}$$

the constant H being the effective gravity (the constant part of the gravitational parameter, to which one adds the constant part of the quantity K featuring the perturbing force), *i.e.*:

$$H = G_0 (M_0 + m_0) - K_0. \tag{5}$$

Indeed, from (2) we have:

$$G(M + m) - K = H - K_0 f_K + G_0 (1 + f_G) (M_0 f_M + m_0 f_m), \tag{6}$$

or:

$$G(M + m) - K = H(1 - f(t)), \tag{7}$$

where we have denoted:

$$f(t) = \frac{K_0 f_K(t) - G_0(1 + f_G(t))(M_0 f_M(t) + m_0 f_m(t))}{G_0(M_0 + m_0) - K_0}. \tag{8}$$

Replacing (7) in (1) and denoting $\mu = G(M + m) - K$, one obtains directly equation (3).

Obviously, for $f(t) = 0$ we just have the standard Kepler problem.

D u b o s h i n [1] considered the case of the variability of the classical purely gravitational paramter $\mu = G(M + m)$ due to changes in M or m (or both), when this one can be written under the form $\mu = \mu_0 + \tilde{\mu}(t)$, where μ_0 is a constant and $\tilde{\mu}(t)$ is a given function of time. The following cases are discerned: (*a*) only M changes; (*b*) only m changes; (*c*) both M and m change. These three situations can be found again from the general case considered by us, taking $K = 0$, $f_G = 0$ and, respectively: (*a*) $f_M \neq 0$, $f_m = 0$; (*b*) $f_M = 0$, $f_m \neq 0$; (*c*) $f_M \neq 0$, $f_m \neq 0$.

We shall assign to the gravitational parameter μ given by (4) a periodic variation described by the law:

$$\mu = H(1 - \alpha \sin (\nu t)), \tag{9}$$

where α and ν are respectively the amplitude (which is small) and the frequency of the oscillation of μ $(0 < \alpha \ll 1, \nu > 0)$.

2. **Equation of perturbed motion in dimensionless variables.** We shall derive the equation of perturbed motion for the case of an initially circular orbit. Consider $t = 0$ as initial time and $f(0) = 0$; equation (9) fulfils this condition. At this instant $\mu = H =$ constant and we are in the case of the two body problem. Taking into account the relation $p = C^2/\mu$ (where p is the semi-latus rectum of the orbit; $p = r$ for unperturbed circular motion), we find the initial radius of the circular orbit:

$$r_0 = C^2/H. \tag{10}$$

Denoting by x the perturbation of the radius vector $(|x| < 1)$, we write (see also [5], [8]):

$$r = r_0(1 + x). \tag{11}$$

We also define a dimensionless time τ by the relation:

$$t = r_0^{3/2} H^{-1/2} \tau, \tag{12}$$

where the scale factor $r_0^{3/2} H^{-1/2}$ is equal to $T_0/(2\pi)$, T_0 representing the unperturbed period of the circular motion in the two body problem with the constant gravitational parameter H.

Denoting by $T_p = 2\pi/\nu$ the oscillation period of μ and taking into account (12), one obtains:

$$\nu t = k\tau, \tag{13}$$

where $k = T_0/T_p$.

Replacing (10) — (13) into (3) and (9), equation (3) of the trajectory is written in dimensionless variables as follows:

$$d^2x/d\tau^2 = 1/(1+x)^3 - (1 - \alpha \sin(k\tau))/(1+x)^2. \tag{14}$$

This equation can be integrated numerically, for different peculiar cases corresponding to α and k, with pre-established initial conditions for x and $dx/d\tau$.

In order to continue the analytical study, we expand the righthand side of the above equation in power series of x. Equation (14) becomes to third order in x:

$$d^2x/d\tau^2 = -x + 3x^2 - 6x^3 + (1 - 2x + 3x^2 - 4x^3)\alpha \sin(k\tau). \tag{15}$$

3. Equations of motion in the phase plane. In order to obtain the equations of motion in the phase plane $(x, dx/d\tau)$, we introduce two new dimensionless variables [9]:

$$y = x^2 + (dx/d\tau)^2, \tag{16}$$

$$\psi = \tan^{-1}((dx/d\tau)/x). \tag{17}$$

By means of these ones, we can describe the motion in the phase plane in polar coordinates. The relations between the two coordinate systems are:

$$x = y^{1/2} \cos \psi, \qquad dx/d\tau = y^{1/2} \sin \psi. \tag{18}$$

From (16) and (17) we obtain:

$$dy/d\tau = 2x(dx/d\tau) + 2(dx/d\tau)(d^2x/d\tau^2), \tag{19}$$

$$d\psi/d\tau = (1 + (dx/d\tau)^2/x^2)^{-1} d((dx/d\tau)/x)/d\tau, \tag{20}$$

from which, using (18), we readily find:

$$dy/d\tau = 6y^{3/2} \sin \psi \cos^2\psi - 12y^2 \sin \psi \cos^3\psi + (2y^{1/2} \sin \psi - 4y \sin \psi \cos \psi + 6y^{3/2} \sin \psi \cos^2\psi - 8y^2 \sin \psi \cos^3\psi)\alpha \sin(k\tau), \tag{21}$$

$$d\psi/d\tau = -1 + 3y^{1/2} \cos^3\psi - 6y \cos^4\psi + (y^{-1/2} \cos \psi - 2 \cos^2\psi + 3y^{1/2} \cos^3\psi - 4y \cos^4\psi)\alpha \sin(k\tau). \tag{22}$$

This system of coupled non-linear differential equations describes the motion in the phase plane, in polar coordinates.

We have initially considered the amplitude α as being a small parameter. Thus we may write the expansions:

$$y(\tau) = y_0(\tau) + \alpha y_1(\tau) + \dots,$$
$$\psi(\tau) = \psi_0(\tau) + \alpha\psi_1(\tau) + \dots, \tag{23}$$

from which we obtain:

$$dy(\tau)/d\tau = dy_0(\tau)/d\tau + \alpha dy_1(\tau)/d\tau, \tag{24}$$
$$d\psi(\tau)/d\tau = d\psi_0(\tau)/d\tau + \alpha d\psi_1(\tau)/d\tau. \tag{25}$$

We identify the terms of the same order in α between equations (21) and (24) and, respectively, (22) and (25). For the zero order terms, taking also into account the fact that these ones feature the unperturbed motion, we find (see also [9]):

$$y_0(\tau) = y_0 = \text{constant},$$
$$\psi_0(\tau) = \theta_0 - \tau, \tag{26}$$

where y_0 and θ_0 represent the initial conditions. Replacing (26) in the terms of the first order in α, the next approximation is obtained in the form:

$$dy_1/d\tau = 2y_0^{1/2}\sin(k\tau)\sin(\theta_0 - \tau) - 4y_0\sin(k\tau)\sin(\theta_0 - \tau)\cos(\theta_0 - \tau) + $$
$$+ 6y_0^{3/2}\sin(k\tau)\sin(\theta_0 - \tau)\cos^2(\theta_0 - \tau) - 8y_0^2\sin(k\tau)\sin(\theta_0 - \tau)\cos^3(\theta_0 - \tau), \tag{27}$$

$$d\psi_1/d\tau = y_0^{-1/2}\sin(k\tau)\cos(\theta_0 - \tau) - 2\sin(k\tau)\cos^2(\theta_0 - \tau) +$$
$$+ 3y_0^{1/2}\sin(k\tau)\cos^3(\theta_0 - \tau) - 4y_0\sin(k\tau)\cos^4(\theta_0 - \tau). \tag{28}$$

If we introduce the abbreviating notations:

$$S_j = S_j(\tau) = \sin(j\theta_0 + (k - j)\tau),$$
$$S'_j = S'_j(\tau) = \sin(j\theta_0 - (k + j)\tau),$$
$$C_j = C_j(\tau) = \cos(j\theta_0 + (k - j)\tau),$$
$$C'_j = C'_j(\tau) = \cos(j\theta_0 - (k + j)\tau), \tag{29}$$

with $j \in \mathbf{N}$, equations (27) and (28) become respectively:

$$dy_1/d\tau = (y_0^{1/2} + 3y_0^{3/2}/4)(C'_1 - C_1) - (y_0 + y_0^2)(C'_2 - C_2) + (3y_0^{3/2}/4)(C'_3 - C_3) -$$
$$- (y_0^2/2)(C'_4 - C_4), \tag{30}$$

$$d\psi_1/d\tau = (1/2 + 3y_0/4)(S'_0 - S_0) - (y_0^{-1/2}/2 + 9y_0^{1/2}/8)(S'_1 - S_1) +$$
$$+ (1/2 + y_0)(S'_2 - S_2) - (3y_0^{1/2}/8)(S'_3 - S_3) + (y_0/4)(S'_4 - S_4). \tag{31}$$

4. Stroboscopic equations of motion. In order to continue the study of the motion in the phase plane, we shall adopt the stroboscopic method, proposed and developed by Minorsky [3]. The principle of this method is the examination of the orbit not continuously, but at time intervals of 2π. Using a suggestive comparison [9], it is just like we plot the orbit in the phase plane, but keep the picture in darkness, except short flashes of light at the moments $\tau = 2n\pi$, $n \in N$. What we see is the evolution of the phase point, during these flashes, along its trajectory in the phase plane; this is sufficient to investigate the variation of the orbit.

If we integrate equations (30) and (31) for indefinitely large τ, the accuracy would eventually diminish considerably, due to the higher order terms which were neglected. Instead, we shall derive the stroboscopic equations by letting τ vary only between 0 and 2π. The results obtained at $\tau = 2\pi$ constitute initial conditions for the integration between 2π and 4π, and so on [9]. In this way, the differential equation is replaced by a series of difference equations for consecutive intervals of 2π.

As Saslaw [9] shows, one deduce in this manner from (15) an operation which transforms the phase point at $\tau = 2n\pi$ into the phase point at $\tau = 2(n + 1)\pi$, $n \in N$. The accuracy is kept since the righthand members of equations (30) and (31) are periodic. It is true that we see the phase point only at intervals of 2π (and this fact entails the lack of detailed informations about the phase plane motion), but over very large time intervals as against the period 2π ($t = T_0 \Rightarrow \tau = 2\pi$) the difference equations may be considered nearly continuous. At limit, one obtains differential equations (see also [6], [10]) for the orbit evolution in the phase plane every $2n\pi\tau$. The smaller α is, the better this approximation will be.

With these considerations, we define a stroboscopic time by:

$$\hat{\tau} = \alpha\tau. \tag{32}$$

Integrating over a period (between 0 and 2π), we obtain:

$$\Delta\hat{\tau} = 2\pi\alpha. \tag{33}$$

From (24) — (26) we find:

$$\Delta y = \alpha\Delta y_1, \tag{34}$$

$$\Delta\psi = -2\pi + \alpha\Delta\psi_1. \tag{35}$$

We replace the angular variable ψ by its variation modulo 2π, namely θ, and, taking into account (33), we obtain:

$$\Delta y/\Delta\hat{\tau} = (2\pi)^{-1}\Delta y_1, \tag{36}$$

$$\Delta\theta/\Delta\hat{\tau} = (2\pi)^{-1}\Delta\psi_1. \tag{37}$$

As we see, it is in fact an averaging [7]; since Δy_1 and $\Delta\psi_1$ are obtained by integrating (30) and (31), respectively between 0 and 2π.

From the above exposed reasons, we may consider the following limits: $\Delta y/\Delta\hat{\tau} \rightarrow dy/d\hat{\tau}$ and $\Delta\theta/\Delta\hat{\tau} \rightarrow d\theta/d\hat{\tau}$. In this way, we obtain respectively from equations (30) and (31):

$$dy/d\hat{\tau} = (a_1 s_1 - b_1 c_1)y^{1/2} - (a_2 s_2 - b_2 c_2)y + \\ + (3/4)(a_1 s_1 - b_1 c_1 + a_3 s_3 - b_3 c_3)y^{3/2} - \\ - (a_2 s_2 - b_2 c_2 + a_4 s_4/2 - b_4 c_4/2)y^2, \tag{38}$$

$$d\theta/d\hat{\tau} = (1/2)(a_1 c_1 + b_1 s_1)y^{-1/2} - (1/2)(a_0 + a_2 c_2 + b_2 s_2) + \\ + (3/8)(3a_1 c_1 + 3b_1 s_1 + a_3 c_3 + b_3 s_3)y^{1/2} - \\ - (1/4)(3a_0 + 4a_2 c_2 + 4b_2 s_2 + a_4 c_4 + b_4 s_4)y, \tag{39}$$

in which we denoted:

$$\left.\begin{array}{l} s_j = s_j(\theta) = \sin(j\theta), \\ c_j = c_j(\theta) = \cos(j\theta), \end{array}\right. \quad j = \overline{1,4} \tag{40}$$

and:

$$\left.\begin{array}{l} a_j = k(1 - \cos(2\pi k))/(\pi(k^2 - j^2)), \\ b_j = j\sin(2\pi k)/(\pi(k^2 - j^2)). \end{array}\right. \quad j = \overline{0,4} \tag{41}$$

Equations (38) and (39) constitute the stroboscopic differential equations of the motion featured by equation (15). They provide the continuous trajectory which would be obtained if every individual phase point at $2n\pi\alpha$ were elongated towards the two neighbouring such points, and if the elongations were joined smoothly [9]. Of course, this virtual trajectory obtained by passing from discrete points to continuous trait lies in the phase plane.

5. Stability of the motion. As Minorsky [3] shows, the existence of a singular point of the stroboscopic equations constitutes a criterion for the existence of stable periodic motion described by the original equation, in our case equation (15).

The stability of the motion in such a problem was studied by Saslaw [9] for a restricted case of equation (15), namely:

$$d^2x/d\tau^2 + (1 + 2\alpha\sin(k\tau))x = \alpha\sin(k\tau), \tag{42}$$

hence stopping the expansion of equation (14) to the terms of first order in x. In these conditions, the stroboscopic equations (38) and (39) become respectively:

$$dy/d\hat{\tau} = (a_1 s_1 - b_1 c_1)y^{1/2} - (a_2 s_2 - b_2 c_2)y, \tag{43}$$

$$d\theta/d\hat{\tau} = (a_1 c_1 + b_1 s_1)y^{-1/2}/2 - (a_0 + a_2 c_2 + b_2 c_2)/2. \tag{44}$$

Examining equations (38) and (39), and taking into account the notations (41), one notices that for $k \in \mathbf{N}^*$ all coefficients a_j, b_j vanish, except those for which $k = j$. Generalizing for higher order terms, there will always exist nonzero coefficients a_j, b_j. Therefore, for natural values of k the motion is unstable due to resonance effects.

The case $k \notin \mathbf{N}^*$ was studied by Saslaw [9] on the basis of equations (43) and (44). He showed that for $\theta = \theta_e$ and $y = y_e$, given respectively by the relations:

$$\tan \theta_e = (2/k) \sin(2\pi k)/(3 - \cos(2\pi k)), \tag{45}$$

$$y_e = \frac{(a_1 \tan \theta_e - b_1)^2 (1 + \tan^2 \theta_e)}{(2a_2 \tan \theta_e + b_2(\tan^2 \theta_e - 1))^2}, \tag{46}$$

one obtains $dy/d\hat{\tau} = d\theta/d\hat{\tau} = 0$. He also showed that for $k < 0.845$ the motion is stable, while for $k > 0.845$ is unstable to perturbations of x and $dx/d\tau$.

In the more general case of the stroboscopic equations (38) and (39) originated by equation (15), or in the still more general case of equation (14), the study of the stability becomes more complicated. For such cases we intend to perform a numerical investigation, based of course on the critical value $k = 0.845$ found in [9] for the mentioned restricted case.

6. **Direction for the instability of motion.** Consider the unstable orbit; we ask in what direction will it move with respect to the initially circular orbit? In other words, we are interested in the evolution of x; if x decreases, the orbit moves inward from the initial orbit, while if x increases the orbit moves outward.

The direction of this displacement depends on the initial value θ_i of the phase. We shall consider the case when y has very small values. The derivative $dy/d\hat{\tau}$ must be positive since x and $dx/d\tau$ must be real. In this case, the sign of the derivative is given by the term in $y^{1/2}$ in the right-hand side of (38). Restricting this part of the equation to:

$$dy/d\hat{\tau} = (a_1 \sin \theta - b_1 \cos \theta) y^{1/2}, \tag{47}$$

one notices the existence of a critical value:

$$\theta_{cr} = \tan^{-1}(b_1/a_1), \tag{48}$$

for which the derivative is zero. We shall examine the domains on the trigonometric circle in which θ_i may lie. For abbreviation, we denote $D = dy/d\hat{\tau}$ and $Q_j(j = \overline{1,4})$ = the quadrants of the trigonometric circle. Four situations can be differentiated:

$$(a)\ \sin(2\pi k) > 0,\ k > 1\ (a_1 > 0,\ b_1 > 0):$$

$$\begin{aligned} &\theta_i \in Q_1 \Rightarrow D > 0 \text{ for } \theta_i > \theta_{cr}; \\ &\theta_i \in Q_2 \Rightarrow D > 0; \\ &\theta_i \in Q_3 \Rightarrow D > 0 \text{ for } \theta_i < \theta_{cr} + \pi; \\ &\theta_i \in Q_4 \Rightarrow D < 0, \end{aligned} \tag{49}$$

hence $\theta_i \in (\theta_{cr},\ \theta_{cr} + \pi)$.

(b) $\sin(2\pi k) < 0,\ k > 1\ (a_1 > 0,\ b_1 < 0)$:

$$\begin{aligned}&\theta_i \in Q_1 \Rightarrow D > 0;\\ &\theta_i \in Q_2 \Rightarrow D > 0 \text{ for } \theta_i < \theta_{cr} + \pi/2;\\ &\theta_i \in Q_3 \Rightarrow D < 0;\\ &\theta_i \in Q_4 \Rightarrow D > 0 \text{ for } \theta_i > \theta_{cr} - \pi/2,\end{aligned} \tag{50}$$

hence $\theta_i \in (\theta_{cr} - \pi/2,\ \theta_{cr} + \pi/2)$.

(c) $\sin(2\pi k) > 0,\ k < 1\ (a_1 < 0,\ b_1 < 0)$:

$$\begin{aligned}&\theta_i \in Q_1 \Rightarrow D > 0 \text{ for } \theta_i < \theta_{cr};\\ &\theta_i \in Q_2 \Rightarrow D < 0;\\ &\theta_i \in Q_3 \Rightarrow D > 0 \text{ for } \theta_i > \theta_{cr} - \pi;\\ &\theta_i \in Q_4 \Rightarrow D > 0,\end{aligned} \tag{51}$$

hence $\theta_i \in (\theta_{cr} - \pi,\ \theta_{cr})$.

(d) $\sin(2\pi k) < 0,\ k < 1\ (a_1 < 0,\ b_1 > 0)$:

$$\begin{aligned}&\theta_i \in Q_1 \Rightarrow D < 0;\\ &\theta_i \in Q_2 \Rightarrow D > 0 \text{ for } \theta_i > \theta_{cr} + \pi/2;\\ &\theta_i \in Q_3 \Rightarrow D > 0;\\ &\theta_i \in Q_4 \Rightarrow D > 0 \text{ for } \theta_i < \theta_{cr} + 3\pi/2,\end{aligned} \tag{52}$$

hence $\theta_i \in (\theta_{cr} + \pi/2,\ \theta_{cr} + 3\pi/2)$.

In what manner is the unstable orbit moving, for each case, in the permitted domains for θ_i? The second equation (18) yields the direction of the displacement: $dx/d\tau > 0$ indicates a motion outward from the initial orbit, while $dx/d\tau < 0$ indicates a motion inward from the initial orbit.

Table 1 synthesizes these results; the motion outward and inward are indicated by the signs (+) and (—), respectively.

Table 1

Case	Domain	Direction	Domain	Direction
(a)	$(\theta_{cr},\ \pi)$	(+)	$(\pi,\ \theta_{cr} + \pi)$	(—)
(b)	$(\theta_{cr} - \pi/2,\ 0)$	(—)	$(0,\ \theta_{cr} + \pi/2)$	(+)
(c)	$(\theta_{cr} - \pi,\ 0)$	(—)	$(0,\ \theta_{cr})$	(+)
(d)	$(\theta_{cr} + \pi/2,\ \pi)$	(+)	$(\pi,\ \theta_{cr} + 3\pi/2)$	(—)

7. **Characteristic time.** In order to determine the time scale for perturbations to become significant, consider again equation (38) under the restricted form (43). In other words, we consider y small, but not small enough to use equation (47). Taking into account (11), we shall consider the perturbation x as being appreciable for $x = \pm 1/2$, which leads, if we have in view (16), to $y = 1/4$.

Therefore we integrate equation (43) for given θ_i. The integration limits are 0 and 1/4 for y, and 0 and $\hat{\tau}_c$ (characteristic stroboscopic time) for $\hat{\tau}$. One obtains:

$$\hat{\tau}_c = 2(b_2 \cos(2\theta_i) - a_2 \sin(2\theta_i))^{-1} \cdot \ln(1 + (1/2)(b_2 \cos(2\theta_i) - a_2 \sin(2\theta_i))/(a_1 \sin\theta_i - b_1 \cos\theta_i)). \tag{53}$$

If the numerator of the second term under the logarithm tends to zero, the characteristic time becomes:

$$\hat{\tau}_c = 1/(a_1 \sin\theta_i - b_1 \cos\theta_i), \tag{54}$$

result obtained by Saslaw [9] from equation (47); this is according to expectation if we have in view equations (43) and (47).

Obviously, taking into account (41), equation (53) is valid only for non-integral values of k (on which depend a_j and b_j).

8. **Possible application: the radiation pressure case.** The basis of our study is equation (3), in which the gravitational parameter varies periodically according to the law (9). We give further down a possible physical interpretation of such a phenomenon.

Consider the gravitational constant G as being really constant. Also consider a pulsating star of constant mass M, whose luminosity L changes according to the law:

$$L = L_0(1 + a_p \sin(\nu t)), \tag{55}$$

where L_0 is the average luminosity, $0 < a_p < 1$ is the amplitude of the pulsation, while ν is the pulsation frequency. Finally, consider a spherical body of constant mass $m < M$, and constant albedo, moving around the star M on an initially circular orbit. The perturbing acceleration undergone by the body m, due to the radiation pressure, has the expression (in module):

$$F = (AL/(4\pi mc))/r^2, \tag{56}$$

where A is the effective cross-sectional area of the body m, while c is the speed of light.

Thus, taking into account (55) and (56), $\mu = G(M + m) - K$ (see Section 1) will become:

$$\mu = G(M + m) - AL_0/(4\pi mc) - (AL_0/(4\pi mc))a_p \sin(\nu t), \tag{57}$$

having therefore a constant part (the effective gravity consisting of the classical purely gravitational parameter, taken here as constant, and the constant part of the perturbing force):

$$H = G(M + m) - AL_0/(4\pi mc), \tag{58}$$

and a part which changes periodically in time. From (57) and (58), we readily find:

$$\mu = H(1 - (G(M + m)/H - 1)a_p \sin(\nu t)). \tag{59}$$

For having always the satellite m under the gravitational influence of the star M, suppose that $\mu > 0$. This means, from (58), that $G(M + m) > H$, and, from (59), that $G(M + m)/H < 2$. From these inequalities it results immediately:

$$0 < G(M + m)/H - 1 < 1. \qquad (60)$$

Denote $\alpha = a_p(G(M + m)/H - 1)$. One sees immediately that $0 < \alpha < 1$ and, if the pulsation amplitude is sufficiently small, the condition $0 < \alpha \ll 1$ is also fulfilled. Defining in this manner the small parameter α and replacing it in (59), we obtain the variation law (9) for the gravitational parameter. The study of the perturbed motion of the body m may therefore be performed according to the presented mathematical model.

The physical situation imagined in this last section deviates from the cases considered by Duboshin [1] and mentioned in Section 1. We have here $f_G = f_M = f_m = 0$ and $K \neq 0$, $f_K \neq 0$. Moreover, the perturbations are due to a nongravitational central force (repulsive in our case), whose time-dependence makes the gravitational parameter (in its extended meaning considered here by us, namely $\mu = G(M + m) - K$) time-dependent, too.

Of course, if the luminosity of the star M changes according to a law differing from (55), but still periodic, the analytical treatment of the problem remains analogous to the simple case considered in this paper. But if M is not a pulsating star and its luminosity undergoes another kind of variation (secular, stochastic, etc.), the gravitational parameter will change accordingly in time. In this case, the problem can be analytically treated altogether differently [9], or preserving the general manner presented here and changing only the equations in agreement with the considered concrete situation.

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RECENZII

I. V. Skrypnik, **Nonlinear Elliptic Boundary Value Problems**, TEUBNER-TEXTE zur Mathematik, Band 91, Leipzig, 1986, 232 pp.

This monograph deals with the topological methods of investigation of nonlinear elliptic equations. The author gives a systematic exposition of the theory of the topological degree and applications of the topological degree to differential problems. The other direction of investigationsis connected with the study of the behavior of solutions of the Dirichlet problem for quasiliniar elliptic equations near the boundary and on a family of perforated domains.

The book is clearly written, very well organized and we recommend it to all interested in the theory of differential equations.

IOAN A. RUS

B. Hofmann, **Regularization for Applied Inverse and Ill-Posed Problems**, TEUBNER-TEXTE zur Mathematik, Band 85, Leipzig, 1986, 196 pp.

This book deals with the classification, mathematical representation, discretization and numerical solution of inverse and ill-posed problems. The book ends with a list of notations, a very extensive bibliography and an index of notions. We recommend this book to all interested in inverse and ill-posed problems, numerical method for regularizations, integral equations and in partial differential equations.

IOAN A. RUS

J. M. Rassias (ed.), **Mixed Type Equations**, TEUBNER-TEXTE zur Mathematik, Band 90, Leipzig, 1986, 312 pp.

The book contains 23 papers dedicated to F. G. Tricomi. The topics covered by these papers include Tricomi equation, the Lavrentev-Bitsadze equation, applications to the transonic aerodynamics, the maximum principle for the Euler-Tricomi equation, the Goursat problem, the nonlinear boundary value problems, and the correct boundary value problems.

The book may be used by the specialists who work in these topics.

IOAN A. RUS

Adolf Karger, Josef Novák, **Space Kinematics and Lie Groups**, Gordon and Breach Science Publishers, 1985, 422 pp.

It is well-known that today Space Kinematics uses an advanced mathematical apparatus. The aim of this book is to prezent this mathematical apparatus from the point of view of Lie Groups Theory and to show how it can be used in problems of Kinematics.

The book contains three chapters. Chapter I introduces the reader to the mathematical notions needed for the understanding of the parts which follow. In this chapter there are given some notions and results from Algebra, Topology, Differential Geometry and Lie Groups. Chapter II is devoted to the motion on the unit sphere and it contains five sections. The results of this chapter are classic and are well-known. The relations between the different representations of the spherical motion are described, the direct cones of the motion are introduced, the invariants are found and the trajectory of a point is studied. Chapter III contains the theory of space motion and it is the largest part of the book. This chapter includes some special problems which generalize the results of Chapter II. The content is divided into twelve sections concerning the Lie Group of the congruences in E_3, Klein's quadradics, the representations of space motion, invariants, dual vectors and trajectories of lines, orbits, the applications of the average method.

Many examples and exercices which help the reader to understand and to get thoroughly into the presented content of the book are given. A rich and suggestive bibliography is given.

This book can be used in a twofold purpose. The first is for the readers which work in Mechanics helping then to use Lie Groups Theory in the solutions of some problems in Kinematics. The second is for the mathematicians working in Differential Geometry and Lie Groups. The book offers to these readers many interesting applications and attractive examples.

DORIN ANDRICA

J. Stückrad and W. Vogel, **Buchsbaum rings and applications** VEB Deutscher Verlag der Wissenschaften, Berlin 1986 Mathematische Monographien, Band 21.

Let A be a local ring with maximal ideal $\mathfrak{m}$ and M a Noetherian A-module of Krull dimension $d \geqslant 0$. A family $(x_1, \ldots, x_d)$ of elements of $\mathfrak{m}$ is said to be a system of parameters of M if dim $M/(x_1, \ldots, x_d) M = 0$. An ideal $\mathfrak{q}$ of A is called a parameter ideal of M if $\mathfrak{q}$ contains a system of parameters such that $\mathfrak{q} \cdot M = (x_1, \ldots, x_d) \cdot M$. A system of elements $x_1, \ldots, x_r$ from $\mathfrak{m}$ is called a week M-sequence if for each $i = 1, \ldots, r$ $(x_1, \ldots, x_{i-1}) \cdot M : x_i = (x_1, \ldots, x_{i-1}) \cdot M : \mathfrak{m}$. A Noetherian A-module M is called a Buchsbaum module if every system of parameters of M is a weak M-sequence. A is called a Buchsbaum ring if it is a Buchsbaum module over itself.

Open questions of algebraic geometry led to this concept. Indeed, M is a Buchsbaum module iff the difference of the length $1_A(M/\mathfrak{q} \cdot M)$ and the multiplicity $e_0(\mathfrak{q}; M)$ of M is an integer $I(M)$ not depending on the choice of the parameter ideal $\mathfrak{q}$ of M.

We select from first the chapter a cohomological caracterization. If A is a local ring then any ideal $\mathfrak{a}$ of A determines the so called local cohomology functor $\Gamma_{\mathfrak{a}}$ by $\Gamma_{\mathfrak{a}}(M) = \bigcup_{k=1}^{\infty} (0 :_M \mathfrak{a}^k)$, for every A-module M. If i is a positive integer, $H^i_{\mathfrak{a}}(M)$ denotes the module obtained by applying to M the right derived functor of $\Gamma_{\mathfrak{a}}$ and $\lambda^i_M : H^i(\mathfrak{a}; M) \to H^i_{\mathfrak{a}}(M)$ the canonical homomorphism defined by direct limit. If M is a Noetherian A-module of positive dimension then M is a Buchsbaum module iff the canonical maps $\lambda^i_M : H^i(\mathfrak{m}, M) \to H^i_{\mathfrak{m}}(M)$ are surjective for all $i < d$ ($\mathfrak{m}$ maximal in A).

The second chapter develops a new topic, that of face rings of simplicial complexes, introduces Buchsbaum complexes and uses Hochster–Reisner theory for monomial ideals in a beautiful interaction between algebraic geometry and topology together with combinatorics.

Chapter III deals with liaison among curves in projective three space, that is examination of closed subschemes by introducing equivalence relations on them: roughly speaking, two curves are liaison-equivalent if their union is a complete intersection.

Rees modules and associated graded modules of Buchsbaum modules are studied in chapter IV in an effort to extend Hironaka's contributions on desingularization of algebraic varieties over fields of characteristic 0, to more general situations due to Faltings and Broadmann.

The last chapter contains further applicatios and several examples: a Buchsbaum criterion for affine semigroup rings, examples related to problems of Hironaka and Seidenberg, on Buchsbaum rings obtained by glueing, construction of Buchsbaum rings with a given local cohomology and examples of Segre products.

G. CĂLUGĂREANU

Andor Kertész, **Lectures on artinian rings**, Edited by Richard Wiegandt, Disquisitiones Mathematicae Hungaricae 14, Akadémiai Kiadó, Budapest, 1987

This book is the revised, enlarged and completed version of the original German edition „Vorlesungen über artinsche Ringe" published by the author in 1968. We list from the contents the titles of the chapters: I Sets, relations; II General properties of rings; III Modules and algebras; IV The radical; V Artinian rings in general; VI Rings of linear transformations; VII Semi-simple, primary and completely primary rings; VIII Artinian rings as operator domains; IX The additive groups of artinian rings.

The final six chapters are supplied by friends and colleagues after Prof. Andor Kertesz death in 1974.

X Decompositions of artinian rings, and XV Linearly compact rings; by A. Widiger (Halle/Salle GDR), XI Artinian rings of quotients, and XII Group rings, a theorem of Connell, and XIII Quasi-Frobenius rings by G. Betsch (Tübingen, FRG) and XIV Rings with minimum condition on principal right ideals, by the editor, R. Wiegandt.

Numerous recently discovered results are also include, e.g. Kertesz' theorem on noetrerian rings to be artinian, Widiger's decomposition theorem, Ayoub–Van Huyn's theorem that every MHR ring is split and the Litoff–Anh theorem on simple rings with non-zero socle.

N. VORNICESCU

Stephen Barnet, Polynomials and Linear Control Systems, *Pure and Applied Mathematics*, Vol. 77, Marcel Dekker, Inc., New York and Basel, 1983. 452 pp.

This book is concerned with the properties of polynomials and polynomial matrices and their application to the theory of linear control systems. The first chapter: "Polynomials: approaches to greatest common divisor", deals with the properties of polynomials and matrices. A companion matrix C associated with a polynomial $a(\lambda)$ is introduced and the fundamental property of C is that its characteristic polynomial is $a(\lambda)$. A procedure for determination of the greatest common divisor of two polynomials using the companion matrix is described. Appropriate properties of Sylvester's resultant matrix and bezoutian matrix associated with two polynomials are developed. In the second chapter: "Basic properties of control systems", the linear control system $x(t) = Ax(t) + Bu(t)$, with the output $y(t) = Hx(t)$ and the linear difference equations $x(k+1) = Ax(k)+Bu(k)$, with the output $y(k) = Hx(k)$ are considered. The concepts of controllability and observability for continuous case, reachability and reconstructibility for discrete case, are defined. The standard conditions for complete controllability/reachability and observability/reconstructibility are given. Using the Laplace transform, for a linear control systems whose state matrix is in companion form and for a single-input system, is esthablished a connection between controllability and relative primeness of polynomials. Using Z transform, can be obtained corresponding results for discrete time systems. For a single-input systems, the linear state feedback is treated. In chapter 3: "Root location and stability", a variety of types of stability are defined and stability nature of a linear system is investigated in terms of the coefficients of its characteristic polynomial. A number of theorems on determining the location of the roots of polynomials are given. In chapter 4: "Feedback, realization, and polynomial matrices" is extended a number of results which were obtained in chapter 2 for systems with a single input. The cases in which the linear feedback is possible are investigated, and a mathematical link between the notions of controllability and that of linear feedback is obtained. In chapter 5: "Generalized polynomials and polynomial matrices", the polynomials are given in the form $a(\lambda) = \sum_{i=0}^{n} a_i p_{n-i}(\lambda)$, where the sequence $p_i(\lambda)$ is ortogonal with respect to a weight function on an interval and the degree of $p_i(\lambda)$ is $i, i = \overline{0, n}$. Various generalization of the companion matrix are defined. The greatest common divisor to be obtained directly in generalized form. For the linear control system, the generalized controlability matrix relative to the basis $p_i(\lambda)$ is $C(A, B) = [B, p_1(A)B, \ldots, p_{n-1}(A)B]$ and the linear system is completely controllable if and only if $C(A, B)$ has rank n. The text is illustrated by many examples and problems.

N. VORNICESCU

A. Kufner and A. M. Sändig, Some Applications of weighted Sobolev Spaces, Teubner — Texte zur Mathematik, Bd. 100, 268 pp., Leipzig 1987.

This book is a continuation of the book written by the first author, Weighted Sobolev Spaces, Teubner — Texte zur Mathematik, Bd. 31, Leipzig 1980 (a second edition was published by J. Wiley Sons in 1985), where the fundamental properties of weighted Sobolev spaces were established and some possibilities of application to partial differential equations were indicated. It is the aim of this book to apply sistematically this theory to elliptic boundary value problems. The book is divided into two parts, each consisting of three chapters. Part one, written by $A-M$. Sändig deals with the application of weighted Sobolev spaces to elliptic boundary value problems on domains with boundaries having conical corner points and edges. The qualitative properties of the solutions (including regularity) are described and some numerical methods, based on a modification of the finite element method, are presented. Part two, written by A. Kufner, is devoted to degenerate elliptic problems. Two kind of such problems are considered: problems with "bad" right hand sides and problems with "bad" coefficients. Also nonlinear problems with „bad" coefficients are shortly considered. This part of the book includes primarily the results lately obtained by A. Kufner and his colleagues. (We mention that both parts of the book are self-contained and can be readen independently).

The book is clearly written, contains many interesting examples, illustrating the theoretical methods and results and will be of great interest to all working in this very active domain of research.

S. COBZAŞ

CRONICĂ

I. Publicaţii ale seminariilor de cercetare ale catedrelor de Matematică (seria de preprinturi) :

Preprint 1—1987, Seminar on Functional Analysis and Numerical Methods (edited by I. Păvăloiu);

Preprint 2—1987, Seminar on Celestral Mechanics and Space Research (edited by Á. Pál);

Preprint 3—1987, Seminar on Fixed Point Theory (edited by I. A. Rus);

Preprint 4—1987, Seminar on Geometric Theory of Functions (edited by P. T. Mocanu);

Preprint 5—1987, Seminar on models, structures and dates procesing (edited by S. Groze and Gr. Moldovan);

Preprint 6—1987, Itinerant Seminar on Functional Equations, Aproximation and Convexity (edited by E. Popoviciu);

Preprint 7—1987, Seminar on Mathematical Analysis (edited by I. Muntean);

Preprint 8—1987, Seminar on Optimisation Theory (edited by I. Kolumbán);

Preprint 9—1987, Numerical and Statistical Calculus (edited by D. D. Stancu);

Preprint 10—1987, Seminar on Celestral Mechanics and Space Research; Seminar on Stellar Structures and Stellar evolution (Edited by Á. Pál and V. Ureche).

II. Manifestări ştiinţifice organizate de catedrele de matematică în 1987:

1. Şedinţele de comunicări lunare ale catedrelor de matematică;

2. Seminarul itinerant de ecuaţii funcţionale, aproximare şi convexitate (21—23 mai 1987);

3. A XVIII-a conferinţă naţională de geometrie şi topologie (4—7 octombrie 1987).

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