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SUMAR — CONTENTS — SOMMAIRE

Recenzii — Book Reviews — Comptes Rendus

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2

Comptes Rendus

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A PROPERTY OF TORI T^2 AND T^3 IMMERSED IN SPHERES

GHEORGHE PITIŞ*

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REZUMAT. — O proprietate a torurilor T^2 şi T^3 imersate în sfere. Se studiază variaţia secundară a volumului unor subvarietăţi ale unei varietăţi Sasakiene. Consideraţiile făcute se aplică în cazul stabilităţii torurilor T^2 şi T^3, considerate ca subvarietăţi ale unor sfere dotate cu structuri Sasakiene.

Our purpose, in this paper, is to study the second variation of volume for some submanifolds in a Sasakian manifold. We give algebraic conditions for stability (Theorem 3.2). The last section is devoted to the study of stability of tori T^2, T^3, as submanifolds in S^3 and S^5.

1. **Introduction.** Let $\widetilde{M}$ be a $2n+1$-dimensional C^∞-differentiable manifold, endowed with an almost contact metric structure (F, η, ξ, g), where $F \in \mathcal{T}^1_1(\widetilde{M})$, $\eta \in \mathfrak{X}^*(\widetilde{M})$, $\xi \in \mathfrak{X}(\widetilde{M})$ and g is a Riemannian metric, satisfying the conditions

$$F^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1 \tag{1}$$

$$g(FX, FY) = g(X, Y) - \eta(X)\eta(Y) \qquad X, Y \in \mathfrak{X}(\widetilde{M}) \tag{2}$$

Then it is well-known that

$$F\xi = 0 \qquad \eta \circ F = 0 \qquad \eta(X) = g(X, \xi) \tag{3}$$

Now, we suppose that $\widetilde{M}$ is a Sasakian manifold, and then we have

$$(\widetilde{\nabla}_X F)Y = g(X, Y)\xi - \eta(Y)X \tag{4}$$

where $\widetilde{\nabla}$ denotes the Levi-Civita connection on $\widetilde{M}$. In this case we have

$$\widetilde{\nabla}_X \xi = -FX \qquad X \in \mathfrak{X}(\widetilde{M}) \tag{5}$$

2. **Some lemmas.** Let M be a $n+1$ — dimensional submanifold of the Sasakian manifold $\widetilde{M}$. The Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \qquad \widetilde{\nabla}_X u = -A_u X + \nabla^\perp_X u \tag{6}$$

for $X, Y \in \mathfrak{X}(M)$ and $u \in TM^\perp$. Moreover, it is well-known that

$$g(h(X, Y), u) = g(A_u X, Y) \tag{7}$$

Now, suppose

$$(a) \quad TM = D^\perp \otimes \{\xi\} \qquad FD^\perp_x \subseteq T_x M^\perp \qquad \text{for each } x \in M.$$

* University of Braşov, Department of Mathematics, 2200 Braşov, Romania

Obviously M is anti-invariant, [4], [6]. On the other hand, we remark that the distributions $D^{\perp}$ and $\{\xi\}$ being orthogonal with respect to the induced metric, from (3) and (4) it follows

$$(\widetilde{\nabla}_{\xi}F)X = 0 \quad \text{for } X \in D^{\perp} \tag{8}$$

LEMMA 2.1. *For every $X \in D^{\perp}$ and $Y \in TM$ we have*

$$\nabla^{\perp}_{Y}(FX) = F(\nabla_{Y}X), \qquad g(X, Y)\xi + A_{FX}Y + F(h(X, Y)) = 0 \tag{9}$$

$$||\nabla^{\perp}(FX)|| = ||\nabla X|| \tag{10}$$

Proof. Using (4) and (6) we obtain

$$\nabla^{\perp}_{Y}(FX) = g(X, Y)\xi + F(\nabla_{Y}X) + F(h(X, Y)) + A_{FX}Y$$

and by equalizing the components of same type it follows (9). Now, from (3), (5) and (6) we have

$$\eta(\nabla_{e_i}X) = -g(\nabla_{e_i}\xi, X) = 0$$

$\{e_1, \ldots, e_n, e_{n+1} = \xi\}$ being a local orthonormal basis in TM. Applying (2) we deduce

$$g(F(\nabla_{e_i}X), F(\nabla_{e_i}X)) = g(\nabla_{e_i}X, \nabla_{e_i}X)$$

An analogous computation shows that

$$g(F(\nabla_{\xi}X), F(\nabla_{\xi}X)) = g(\nabla_{\xi}X, \nabla_{\xi}X)$$

and then (10) follows from (9). QED

LEMMA 2.2. *For each $X, Y \in D^{\perp}$, $||Y|| = 1$, we have*

$$\widetilde{R}(FX, Y, FX, Y) = \widetilde{R}(X, FY, X, FY) \tag{11}$$

$$\widetilde{R}(\xi, FX, FX, \xi) = \widetilde{R}(\xi, X, X, \xi) = ||X||^2 \tag{12}$$

Proof. Applying (1) and (2) we get

$$\widetilde{R}(FX, Y, FX, Y) = g(F(\widetilde{R}(FX, Y)Y), X) \tag{13}$$

and by a direct computation using (4) and (5) it follows

$$F(\widetilde{R}(FX, Y)Y) = \widetilde{R}(FX, Y)FY - X - \{g(FX, \widetilde{\nabla}_{Y}Y) - g(Y, \widetilde{\nabla}_{FX}Y) - g([FX, Y], Y)\}\xi + \eta(\widetilde{\nabla}_{Y}Y)FX + g(X, Y)Y \tag{14}$$

Now, from (13), (14) follows (11).

$$\widetilde{R}(\xi, FX, FX, \xi) - \widetilde{R}(\xi, X, X, \xi) = -g(F(\widetilde{R}(FX, \xi)\xi) + \widetilde{R}(X, \xi)\xi, X). \tag{15}$$

and applying (1), (4), (5), (8) we obtain

$$F(\widetilde{R}(FX, \xi)\xi) + \widetilde{R}(X, \xi)\xi = F^2[FX, \xi] + F[X, \xi] = 0 \tag{16}$$

From (15), (16) we obtain

$$\widetilde{R}(\xi, FX, FX, \xi) = \widetilde{R}(\xi, X, X, \xi)$$

and taking into account (5), (6), (8), we have

$$\widetilde{R}(X, \xi, \xi, X) = g(F(\nabla_\xi X) + Fh(X, \xi) + Fh([X, \xi], X) = g(Fh(X, \xi), X)$$

But $h(X, \xi) = -FX$ and then $\widetilde{R}(X, \xi, \xi, X) = ||X||^2$. QED

By straightforward computations, we get the

LEMMA 2.3. *For any $X \in D^\perp$ we have*

$$||A_{FX}||^2 = \sum_{i=1}^{n} ||h(X, e_i)||^2 + ||X||^2$$

3. **Stability.** If M is compact and minimal then the second variation of volume with respect to the normal vector field $u \in TM^\perp$ is given by, [5],

$$V''(u) = \int_M \left\{ ||\nabla^\perp u||^2 - \sum_{k=1}^{n+1} \widetilde{R}(u, e_k, e_k, u) - ||A_u||^2 \right\} dV$$

where $\widetilde{R}$ is the Riemann tensor of $\widetilde{M}$ and dV is the volume element of M.

Suppose M compact, without boundary, minimal and satisfying the condition ($\mathfrak{A}$).

THEOREM 3.1. *The second variation of volume of the submanifold M is given by*

$$V''(u) = \int_M \left\{ \frac{1}{2} ||d\lambda||^2 + (\delta\lambda)^2 - \widetilde{S}(Fu, Fu) - ||X||^2 \right\} dV$$

for any normal vector field u.

Proof. Since $\dim M = n + 1$ there exists $X \in D^\perp$ so that $u = FX$. But $\{Fe_1, \ldots, Fe_n\}$ is an orthonormal local basis in $TM^\perp$ and, by applying lemma 2.2, we obtain

$$T(u) = \sum_{i=1}^{n} \widetilde{R}(u, e_i, e_i, u) + \widetilde{R}(u, \xi, \xi, u) = \widetilde{S}(X, X) - \sum_{i=1}^{n} \widetilde{R}(X, e_i, e_i, X) \tag{17}$$

On the other hand, from the formula of Gauss we have

$$R(X, e_i, e_i, X) = R(X, e_i, e_i, X) - g(h(X, X), h(e_i, e_i)) + ||h(X, e_i)||^2$$

and taking into account the lemmas 2.2, 2.3, the equality (17) becomes

$$\begin{aligned} T(u) = \widetilde{S}(X, X) - S(X, X) + ||X||^2 - ||A_{FX}||^2 + \\ + \sum_{i=1}^{n} g(h(X, X), h(e_i, e_i)) + g(h(X, X), h(\xi, \xi)) \end{aligned} \tag{18}$$

By using $h(\xi, \xi) = 0$, (18) becomes

$$T(u) = \tilde{S}(X, X) - S(X, X) + ||X||^2 - ||A_{FX}||^2 \tag{19}$$

Our statement follows from the known equality ([1], pg. 51)

$$\int_M S(X, X)\,dV = \int_M \left\{ \frac{1}{2}\,||d\lambda||^2 + (\delta\lambda)^2 - ||\nabla X||^2 \right\} dV$$

where λ is the 1 — form associated with Fu.

QED

We recall that M is *stable* if $V''(u) \geqslant 0$ for any $u \in TM^{\perp}$ and *unstable* in the other case.

Now, we can state the following

THEOREM 3.2. a) *If M has positive defined Ricci tensor and $H^1(M, \mathbf{R}) \neq 0$ then M is unstable.*

b) *If $\tilde{M}$ is Einsteinian ($S = \alpha g$) then*

(i) *for $\alpha > -1$ and $H^1(M, \mathbf{R}) \neq 0$ M is unstable,*

(ii) *for $\alpha \leqslant -1$ M is stable;*

c) *If $\tilde{M}(c)$ is a Sasakian space form then*

(i) *for $c \leqslant -\dfrac{3n+1}{n+1}$ M is stable,*

(ii) *for $c > -\dfrac{3n+1}{n+1}$ and $H^1(M, \mathbf{R}) \neq 0$ M is unstable.*

Proof. Because $H^1(M, \mathbf{R}) \neq 0$ there exists an harmonic 1 — form λ and denoting by Y the vector field associated with λ, from the theorem 3.1 we have

$$V''(FY) = -\int_M [\tilde{S}(Y, Y) + ||Y||^2]\,dV \tag{20}$$

Now, a), b_i) are consequences of (20).

b_{ii}) follows from the theorem 3.1.

c) Denoting by c the F — sectional curvature of $\tilde{M}$, we have the known equality ([1], pg. 98)

$$\tilde{S}(X, X) = \frac{n(c+3)+c-1}{2}\,||X||^2 - \frac{(n+1)(c-1)}{2}\,(\eta(X))^2$$

QED

4. Stability of tori T^2 and T^3. Let $S^5 = \{z \in C^3 : |z| = 1\}$ be the 5 — dimensional sphere with the standard Sasakian structure (F, η, ξ, g). It is known, [3], that by putting

$$|z^1| = |z^2| = |z^3| = 1/\sqrt{3} \qquad z = (z^1, z^2, z^3)$$

we obtain an imbedding of a 3-dimensional torus T^3 in S^5, wich is minimal. Moreover, ξ is tangent to T^3 and because FX is normal to T^3 for X orthogonal to ξ, it follows that T^3 satisfy condition ($\mathcal{a}$).

The sphere S^5 is Einsteinian with $\alpha = 4$. On the other hand, it is known that $\dim H^1(T^3, \mathbf{R}) = 3$ and then, from theorem 3.2, b) follows the

PROPOSITION 4.1. *The torus T^3 is an unstable submanifold of S^5, endowed with the standard Sasakian structure.*

If (F, η, ξ, g) is the standard Sasakian structure on the unit sphere S^3 then with the structure

$$F^* = F, \ \eta^* = 4\eta, \ \xi^* = \frac{1}{4}\xi, \qquad g^* = 4g + 12\eta \otimes \eta \qquad (21)$$

S^3 is a Sasakian space form of F^* — sectional curvature $c = -2$ (see for example [1], chap. V). By the theorem 3.2, c) it follows that the sphere S^3 admits only stable anti-invariant hypersurfaces. Particularly, $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2})$ is minimal, hence we have the following

PROPOSITION 4.2. *The torus $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2})$ is stable in S^3 with the Sasakian structure* (21).

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STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXIII, 2, 1988

BROWNIAN DIFFUSION IN THE AXIALLY SYMMETRICAL JET MIXING OF AN INCOMPRESSIBLE DUSTY FLUID

N. DATTA* and S. K. DAS*

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REZUMAT. — Difuzia Browniană într-un jet axial simetric format dintr-un amestec de fluid incompresibil. În lucrare se studiază mişcarea laminară a unui amestec format din particule de praf într-un jet axial simetric prin considerarea difuziei Browniene. Ecuaţiile care guvernează această mişcare, cuplate şi neliniare, sînt rezolvate folosind o schemă cu diferenţe finite de tip implicit. Astfel, s-au determinat caracteristicile mişcării şi transferul de căldură în raport cu parametrii de difuzie S_{cp} (numărul lui Schmidt) şi al amestecului de praf α. Se observă că semi-adîncimea jetului descreşte cu creşterea parametrului α iar pentru numere Schmidt mari particulele de fluid se îndepărtează de axa jetului, producînd concentraţii superioare la suprafaţa liberă a jetului.

Laminar mixing of an axially symmetric jet of incompressible dusty fluid has been considered by taking Brownian diffusion into account. The governing equations which are coupled and non-linear have been solved using finite difference scheme of implicit type. The flow and heat transfer characteristics have been studied along with the effect of the dust parameter and the diffusion parameter on them. It is observed that the half width of the jet decreases with the increase of dust parameter α and for large Schmidt number the particles spread away from the jet axis, giving higher concentration at the jet boundary.

1. Introduction. Jet mixing of a fluid containing suspended solid particles is encountered in various fields, viz. performance of rocket plumes, pulverised coal combustors, diesel-engine sprays etc.

The Brownian diffusion in the jet mixing of dusty fluid has been studied by Soo [1], Rhyming [2], Batchelor [3], Datta and Mishra [4]. While Soo [1] studied the circular jet for highly dilute suspension of particles of submicron size, Rhyming [2] studied the plane laminar jet flow of dusty fluids. Batchelor [3] has shown the effect of Brownian motion on the bulkstress in a suspension of spherical particles. Datta and Mishra [4] have studied the plane laminar jet of dusty fluid considering the Stokes' drag as well as the transverse force due to slip shear on the particles. The problem of axially symmetric jet mixing of an incompressible dusty fluid has been studied by Datta and Das [5]. They have also investigated the effect of dust parameter on the flow and heat transfer characteristic of the jet.

The present study considers the axisymmetric jet mixing by taking into account the Brownian diffusion and viscosity of the particle phase. Further,

* *Department of Mathematics, Indian Institute of Technology Kharagpur, India*

the energy equations of the fluid and the particle phase have also been considered to study the temperature distribution in the mixing region of the jet. In all the previous studies mentioned above, either series expansion or perturbation method has been applied to get an approximate solution of the governing equations. The present study uses an implicit finite difference scheme of Crank-Nicholson type to solve the non-linear coupled governing equations.

2. **Mathematical formulation.** Taking z and r as co-ordinates along and perpendicular to the jet axis and the origin at the centre of the nozzle exit, the dimensionless governing equations for axially symmetric steady jet flow of a dusty fluid can be written using Saffman's model as

$$\frac{\partial}{\partial z}(ru) + \frac{\partial}{\partial r}(rv) = 0 \tag{1}$$

$$u\frac{\partial u}{\partial z} + v\frac{\partial u}{\partial r} = \frac{1}{R}\,\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \alpha\rho_p\frac{(u_p - u)}{\Lambda_m} \tag{2}$$

$$u\frac{\partial T}{\partial z} + v\frac{\partial T}{\partial r} = \frac{1}{Pe}\,\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{2}{3}\,\frac{\alpha}{Pr}\,\rho_p\frac{(T_p - T)}{\Lambda_m} \tag{3}$$

$$u_p\frac{\partial \rho_p}{\partial z} + v_p\frac{\partial \rho_p}{\partial r} = \frac{1}{S_{cp}\cdot R_p}\,\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\frac{\partial \rho_p}{\partial r}\right) \tag{4}$$

$$u_p\frac{\partial u_p}{\partial z} + v_p\frac{\partial u_p}{\partial r} = \frac{1}{R_p}\cdot\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\frac{\partial u_p}{\partial r}\right) - \frac{(u_p - u)}{\Lambda_m} \tag{5}$$

$$u_p\frac{\partial v_p}{\partial z} + v_p\frac{\partial v_p}{\partial r} = \frac{1}{R_p}\,\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\frac{\partial v_p}{\partial r}\right) - \frac{(v_p - v)}{\Lambda_m} \tag{6}$$

$$u_p\frac{\partial T_p}{\partial z} + v_p\frac{\partial T_p}{\partial r} = -\frac{2}{3}\,\frac{1}{Pr}\,\frac{c_p}{c_s}\,\frac{(T_p - T)}{\Lambda_m} \tag{7}$$

where the dimensionaless variables are

$$z = \frac{\bar{z}}{L},\quad r = \frac{\bar{r}}{L},\quad u = \frac{\bar{u}}{U},\quad v = \frac{\bar{v}}{U},\quad u_p = \frac{\bar{u}_p}{U},\quad v = \frac{\bar{v}_p}{U},$$

$T = \frac{\bar{T}}{T_0}$, $T_p = \frac{\bar{T}_p}{T_0}$, $\rho_p = \frac{\bar{\rho}_p}{\rho_{p0}}$. The dimensionless parameters are $R = \frac{UL}{\nu}$, the Reynolds number for the fluid, $R_p = \frac{UL}{\nu_p}$, the Reynolds number for particle phase, $Pr = \frac{\mu c_p}{K}$, the Prandtl number, $S_{cp} = \frac{\nu_p}{D_p}$, the Schmidt number, $Pe = Pr \cdot R$, the Peclet number, $\alpha = \frac{\rho_{p0}}{\rho}$, the concentration parameter, $\Lambda_m = \frac{U\tau_m}{L}$, the momentum relaxation time.

The reference velocity U and the temperature T_0 are respectively the undisturbed velocity and temperature at the core of the jet and L is a characteristics length.

The equations (1)—(7) can be solved under the following entry and boundary conditions

$$u(r,\ 0) = u_0,\ u_p(r,\ 0) = u_{p0}$$
$$T(r,\ 0) = T_0,\ T_p(r,\ 0) = T_{p0},\ \rho_p\ (r,\ 0) = \rho_{p0} \tag{8}$$

$$u(r,\ 0) = 0,\ u_p(r,\ 0) = 0$$
$$T(r,\ 0) = 0,\ T_p(r,\ 0) = 0,\ \rho_p(r,\ 0) = 0 \tag{9}$$

$$\frac{\partial u}{\partial z}(0,\ z) = 0,\quad \frac{\partial T}{\partial r}(0,\ z) = 0,\ \frac{\partial \rho_p}{\partial r}\ (0,\ z) = 0,\ \frac{\partial v_p}{\partial r}\ (0,\ z) = 0 \tag{10}$$

$$v(0,\ z) = 0,\ v_p(0,\ z) = 0 \tag{11}$$

$$u(\infty,\ z) = 0,\ u_p(\infty,\ z) = 0,\ T(\infty,\ z) = 0 \tag{12}$$
$$T_p(\infty,\ z) = 0,\ \rho_p(\infty,\ z) = 0$$

3. Numerical solution using finite-difference method. The equations (1)—(7) satisfying the boundary conditions (8)—(12) have been solved numerically using finite difference schemes of Crank-Nicholson's type. We sub-divide the $(r - z)$ plane into a set of rectangular grids of sides Δr and Δz, such that any representative grid point $(r_i,\ z_j)$ is given by $r_i = i \cdot \Delta r,\quad z_j = j \cdot \Delta z$ with $i = 0$ corresponding to the z-axis and $j = 0$ corresponding to the nozzle exit. While formulating the finite difference scheme the differential coefficients have been approximated in such a way that the unknown quantities appear linearly giving a linear difference equation. The various terms and their derivatives have been approximated by the differences of the form given below.

$$u_{i,j+1} \simeq 2u_{i,j} - u_{i,j-1} \tag{13}$$

$$\left(\frac{\partial u}{\partial z}\right)_{i,j+1} \simeq \frac{3u_{i,\,j+1} - 4u_{i,\,j} + u_{i,\,j-1}}{2 \cdot \Delta z} \tag{14}$$

$$\left(\frac{\partial u}{\partial r}\right)_{i,\,j+1} \simeq \frac{u_{i+1,\,j+1} - u_{i-1,\,j+1}}{2\Delta r} \tag{15}$$

$$\left(\frac{\partial^2 u}{\partial r^2}\right)_{i,\,j+1} \simeq \frac{u_{i+1,\,j+1} - 2u_{i,\,j+1} + u_{i-1,\,j+1}}{(\Delta r)^2} \tag{16}$$

Using (13)—(16) and similar expressions for terms involving v, u_p and ρ_p the equation (2) reduces to the following form

$$A_i u_{i-1,\,j+1} + B_i u_{i,\,j+1} + C_i u_{i+1,\,j+1} = D_i,\ i = 1,\ 2,\ \ldots\ N - 1 \tag{17}$$

where A_i, B_i, C_i and D_i are expressions containing entities at the $(i,\ j)$ and $(i,\ j-1)$ grid points (which have been omitted for saving space). The equations (3), (4), (5), (6) and (7) can be transformed into similar difference equations for T, ρ_p, u_p, v_p and T_p, respectively. It may be noted that the equation (17) is not valid for $i = 0$, i.e., at the axis of the jet, since eq. (2) has a singularity at $r = 0$ due to circular symmetry. The term $\frac{1}{r}\ \frac{\partial u}{\partial r}$ assumes indeterminate form at $r = 0$. Taking the limiting value as $r \to 0$, this term has been replaced by using

$$\underset{r \to 0}{Lt} \frac{1}{r} \frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial r^2}\bigg|_{r=0} \tag{18}$$

Using symmetry relation $u_{1,j+1} = u_{-1,j+1}$ at $i = 0$, corresponding form of (17) at $i = 0$ is given by

$$B_0 u_{0,j+1} + C_0 u_{1,j+1} = D_0 \tag{19}$$

where B_0, C_0 and D_0 are known from the conditions at $i = 0$.

The difference equations corresponding to (3), (4) and (5) also have singularities at $r = 0$, which has been removed and corresponding difference equations valid at $i = 0$ grid have also been derived but not shown here. Since at $i = 0$, v and v_p are known from the boundary conditions, therefore, computation of v and v_p at the jet axis becomes insignificant. Instead of that we use the symmetry condition of v_p at the outer edge of the boundary layer.

Now, the finite difference analogue of the entry and boundary conditions (8) to (12) are

$$\left.\begin{array}{l} u_{i,1} = u_0, \; u_{pi,1} = u_{p0}, \; T_{i,1} = T_0 \\ T_{pi,1} = T_{p0}, \; \rho_{pi,1} = \rho_{p0}, \; v_{i,1} = v_{pi,1} = 0 \end{array}\right| \text{at } r_i < 1 \tag{20}$$

$$\left.\begin{array}{l} u_{i,1} = u_{pi,1} = T_{i,1} = T_{pi,1} 1 - \rho_{pi,1} = 0 \\ v_{i,1} = v_{pi,1} = 0 \end{array}\right| \text{at } r_i > 1 \tag{21}$$

at $i = 0$,i.e. at the jet axis

$$\left.\begin{array}{l} u_{1,j+1} = u_{-1,j+1}, \; u_{p1,j+1} = u_{p-1,j+1} \\ \rho_{p1,j+1} = \rho_{p-1,j+1}, \; T_{1,j+1} = T_{-1,j+1} \\ v_{0,j+1} = v_{p0,j+1} = 0 \end{array}\right| \tag{22}$$

at $i = N$, i.e., at the outer edge of the boundary layer we take

$$\left.\begin{array}{l} v_{p\,N-1,j+1} = v_{p\,N+1,j+1} \\ \text{and} \\ u_{N,j+1} = u_{p\,N,j+1} = T_{N,j+1} = T_{p\,N,j+1} = \rho_{pN,j+1} = 0 \end{array}\right| \tag{23}$$

The difference equation (17) for u and similar equations for u_p, T, T_p and ρ_p give rise to systems of simultaneous linear algebraic equations which can be solved for u at the grid points across the boundary layer. Thomas algorithm [6] has been used to solve the unknown pivotal values in terms of the known pivotal values derived from the finite difference analogue of the entry and boundary conditions.

4. Discussion of results. To have an insight into the flow and heat transfer characteristics, computations have been made by taking values of the various parameters and entry conditions as: $Pr = Pc = 0.72$, $R = 1.0$, $R_p = 0.5$, $\Lambda = 1.0$, $\alpha = 0.1$, 0.2, 0.3 and $u_0 = u_{p0} = T_0 = T_{p0} = \rho_{p0} = 0.1$. The results

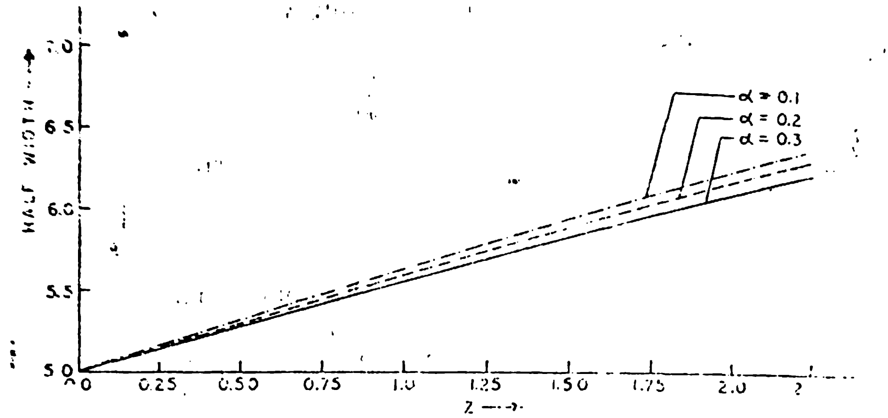

Fig. 1. Profiles of Axial velocity of fluid and of particle ($S_{cp} = 300$)

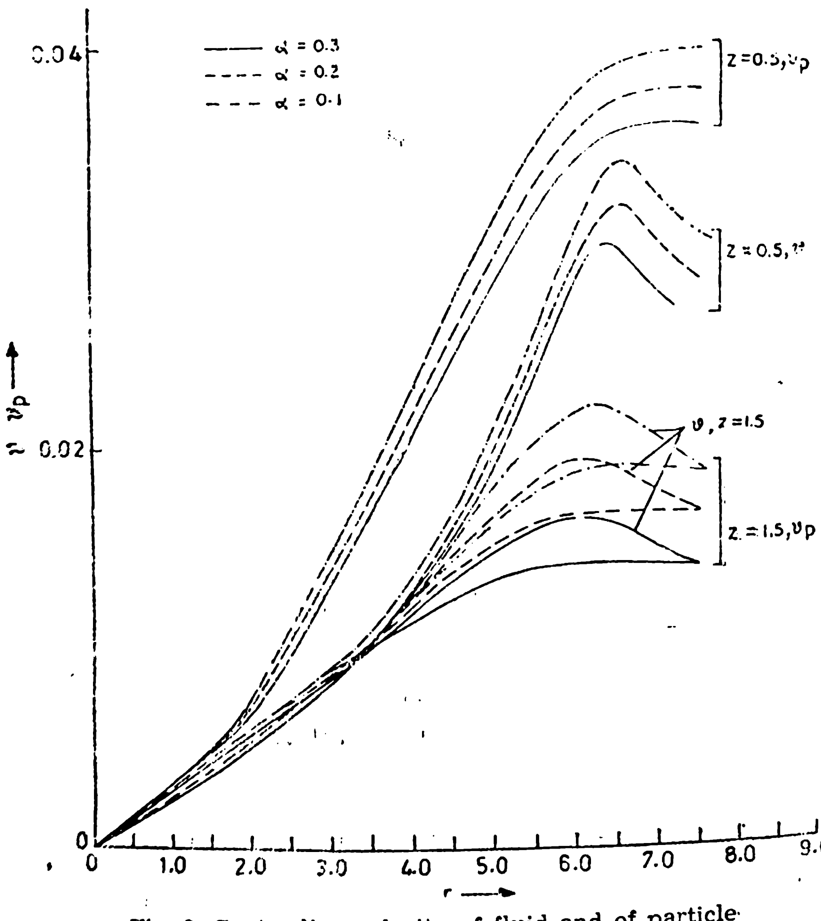

Fig. 2. Centre line velocity of fluid and of particle

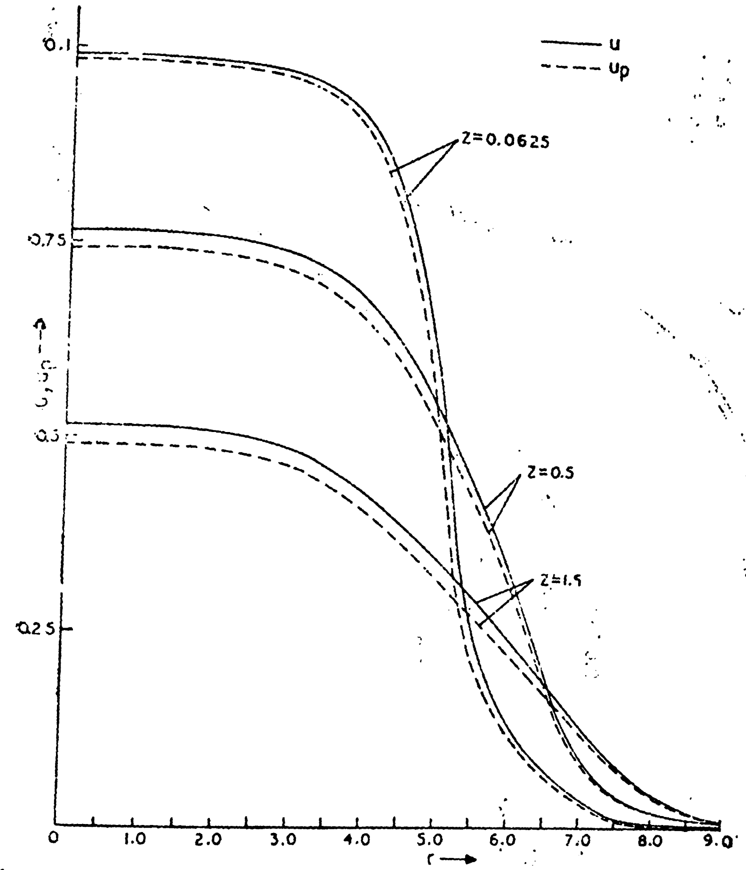

Fig. 3. Variation of jet width with distance fron the nozzle exit.

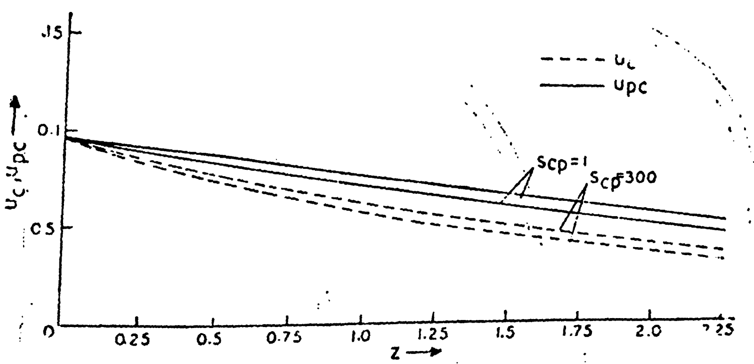

Fig. 4. Profiles of radial fluid and particle velocity

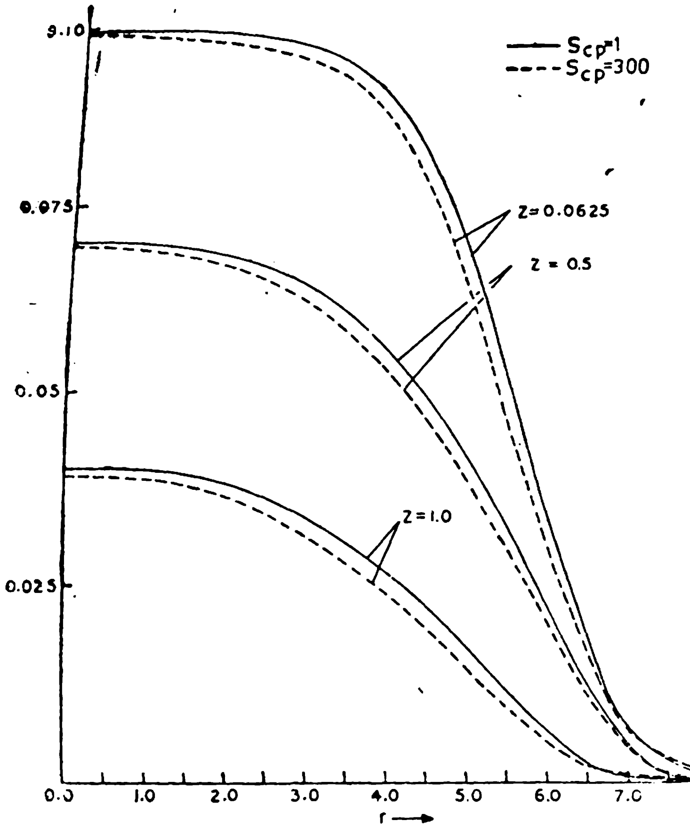

Fig. 6. Profiles of particle density

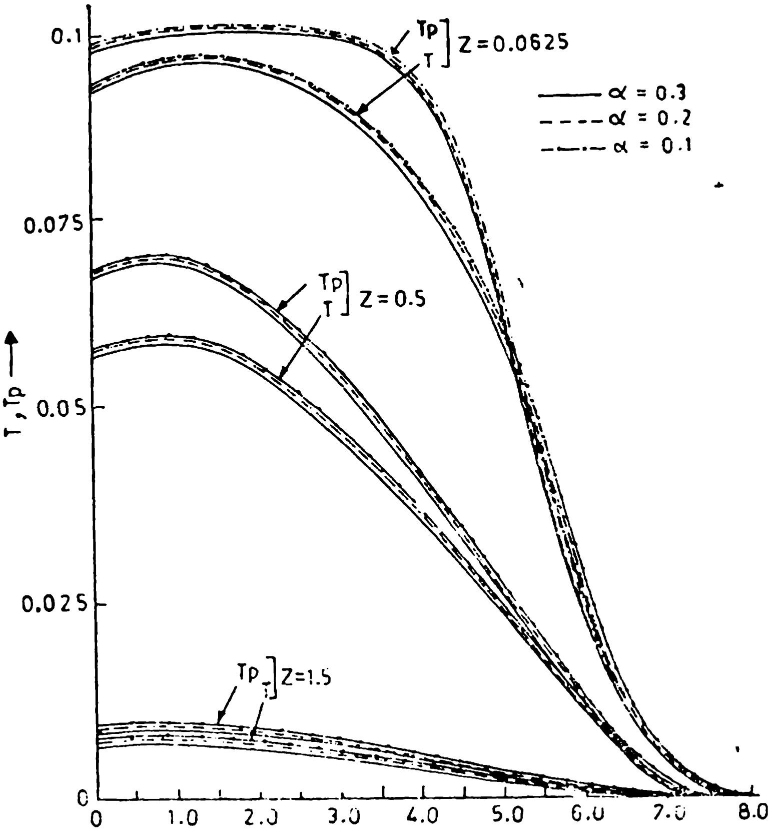

Fig. 5. Profiles of fluid and particle temperature

have been presented in Figs. 1—6. We start with prescribed initial values at $z = 0$ and then the unknowns u, u_p, T, T_p, ρ_p, v and v_p are calculated at $z = \Delta z$ i.e., at $j = 1$. The above procedure is followed recursively for $j = 2$, 3, ... etc. We have assumed that the mesh is sufficiently small so that the co-efficients in the difference equations vary slightly and are considered to be constants. Because of its implicit character the scheme is linearly stable for finite values of p, where $p = \frac{\Delta z}{(\Delta z)^2}$. In particular a constant step size $\Delta z =$ $= 0.625$ and $\Delta r = 0.25$ have been chosen for computation.

Fig. 1 exhibits the profiles of the axial velocity of the carrier fluid and of the dust particles at different sections of the jet. It is seen that the velocity of the carrier fluid is greater than the velocity of the dust particles throughout the mixing region of the jet. It is also observed from Fig. 2 that the centre line velocity of the fluid and particle phase decreases with the increase of large particle Schmidt number (say $S_{cp} = 300$) and for small Schmidt number (say $S_{cp} = 1$) particle velocity is greater then the fluid velocity. The half-width of the jet shown in Fig. 3 vary linearly with respect to the distince from the nozzle exit and it decreases with the increase of dust parameter α.

From Fig. 4, we notice the distribution of radial velocity v of the carrier fluid and that of the dust particles v_p at different sections of the jet. It is clearly seen that near the jet axis v and v_p are same and at large radial distance $v_p > v$ in the mixing region, whereas $v > v_p$ in the far downstream. Both the velocities v and v_p decrease with the increase of concentration parameter α.

Fig. 5 shows the temperature distribution for the fluid and the particle phase. The thermal interaction between the two phases is indicated by the transfer of heat from the particles to the carrier fluid. The fluid and particle temperature decreases gradually along the axial direction and finally both the temperatures attain the temperature of the surrounding stream in the far downstream. Further, T and T_p decrease in both the cases as α increases.

The effect of Brownian diffusion is characterized by Schmidt number S_{cp} which is the ratio of the kinematic viscosity ν_p of the particles and their diffusivity D_p. For $S_{cp} = 1$, the Brownian diffusion of particles is insignificant. It is observed that the particle density for $S_{cp} = 1$ is greater than that for the case $S_{cp} = 300$ when the Brownian diffusion is significant. Fig. 6 shows that for larger values of Schmidt number ($S_{cp} = 300$) the particles spread away from the jet axis, giving higher concentration at the jet boundary.

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THE CONHARMONIC CURVATURE TENSOR AND 4-DIMENSIONAL CATENOIDS

F. DILLEN*, M. PETROVIC-TORGASEV** and L. VERSTRAELEN*

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REZUMAT. — Tensorul de curbură conharmonic şi catenele quatredrice. În secţiunea 2 se prezintă o caracterizare a spaţiilor euclidiene conharmonice din punct de vedere al tensorului conformaţional de curbură a lui Weyl şi al curburii scalare. În secţiunea 3 se dau cîteva aplicaţii ale căror rezultate se combină cu studiul general al hipersuprafeţelor de revoluţie în secţiunea 4, ceea ce ne dă o nouă caracterizare intrinsecă a catenelor quatredrice în spaţii euclidiene pentaedrice.

1. **Introduction.** In Section 2 we give a characterization of the conharmonically Euclidean spaces in terms of the vanishing of Weyl's conformal curvature tensor and of the scalar curvature. As an application, in Section 3, we obtain the specific forms of the shape operators of conharmonically Euclidean hypersurfaces. In Section 4, we make a general study of hypersurfaces of revolution which, combined with the result of Section 3, yields a new intrinsic characterization of the 4-dimensional catenoids in 5-dimensional Euclidean space. In a similar way, a characterization of the 3-dimensional catenoids in 4-dimensional Euclidean space was given in [1], in terms of Weyl's projective curvature tensor and the Ricci endomorphism of the hypersurfaces. Finally, in Section 5, we determine which conformal transformations of any Riemannian manifold preserve the harmonicity of the real functions.

2. **The conharmonic curvature tensor.** Let M^n be a *Riemannian manifold* of dimension $n > 3$, with metric tensor g, Riemann-Christoffel curvature tensor R, Ricci tensor S and scalar curvature τ. Then, Ishii defined the *conharmonic curvature tensor* K by

$$K(X, Y; Z, W) = R(X, Y; Z, W) - \frac{1}{n-2}\{g(X, W)S(Y, Z) + g(Y, Z)S(X, W) - g(X, Z)S(Y, W) - g(Y, W)S(X, Z)\}, \tag{1}$$

where X, Y, Z, W are arbitrary vector fields tangent to M^n; K is invariant under the action of the conformal transformations of M^n which preserve, in a certain sense, real harmonic functions on M^n, and which therefore are called *conharmonic transformations* [2]. M^n is related to the Euclidean space $\mathbf{E}^n$ by a conharmonic transformation, and then is said to be *conharmonically Euclidean*, if and only if $K = 0$, i.e. when K vanishes identically.

* Katholieke Universiteit Leuven, Departement Wiskunde, Celestijnenlaan 200 B, B-3030 Leuven, Belgium,
** University Svetozar „Markovic", Department of Mathematics Radoja Domanovica 12, 34000 Kragujevac, Yugoslavia.

We recall that the *conformal curvature tensor* C of M^n is given by

$$C(X, Y; Z, W) = R(X, Y; Z, W) - \frac{1}{n-2}\{g(X, W)S(Y, Z) + g(Y, Z)S(X, W) - g(X, Z)S(Y, W) - g(Y, W)S(X, Z)\} + \frac{\tau}{(n-1)(n-2)}\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\}. \tag{2}$$

By a theorem of Weyl, M^n is (locally) conformally Euclidean if and only if $C = 0$.

THEOREM 1. $K = 0 \Leftrightarrow C = 0$ & $\tau = 0$, *i.e. M^n is conharmonically Euclidean if and only if M^n is conformally Euclidean and has zero scalar curvature.*

Proof: From (1) and (2) it follows that

$$K(X, Y; Z, W) = C(X, Y; Z, W) - \frac{\tau}{(n-1)(2-n)}\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\}. \tag{3}$$

Consequently, the implication $\Leftarrow$ is trivial. Conversely, assume that $K = 0$. Then (1) implies that

$$R(X, Y; Z, W) = \frac{1}{n-2}\{g(X, W)S(Y, Z) + g(Y, Z)S(X, W) - g(X, Z)S(Y, W) - g(Y, W)S(X, Z)\}. \tag{4}$$

By contracting (4) with respect to the first and the last arguments, we obtain

$$S(Y, Z) = \frac{1}{n-2}\{nS(Y, Z) + \tau g(Y, Z) - S(Y, Z) - S(Y, Z)\}.$$

Thus, $\tau g(Y, Z) = 0$ for all Y and Z, and therefore also $\tau = 0$. Together with (3) this ends the proof.

3. **Conharmonically Euclidean hypersurfaces.** Let M^n be a *hypersurface* of dimension $n > 3$ in a Euclidean space $\mathbf{E}^{n+1}$. The *second fundamental tensor* or *shape operator* of M^n at one of its points p will be denoted by $A(p)$.

PROPOSITION 2. *M^n is a conharmonically Euclidean hypersurface of $\mathbf{E}^{n+1}$ if and only if at each point p of M^n the shape operator has one of the following forms:*

$$A(p) = \begin{bmatrix} 0 & & & 0 \\ & \ddots & & \\ & & 0 & \\ 0 & & & \mu \end{bmatrix}, \quad (\mu \in \mathbf{R}), \tag{5}$$

or

$$A(p) = \begin{bmatrix} \lambda & & & 0 \\ & \ddots & & \\ & & \lambda & \\ 0 & & & \frac{2-n}{2}\lambda \end{bmatrix}, \quad (\lambda \in \mathbf{R}\setminus\{0\}). \tag{6}$$

Proof. By a well-known theorem of Cartan, M^n is conformally Euclidean if and only if it is *quasiumbilical*, i.e. if for each $p \in M^n$, $A(p)$ has an eigenvalue with multiplicity $\geqslant n-1$, or still, if $A(p)$ has the following form:

$$A(p) = \begin{bmatrix} \lambda & & & 0 \\ & \ddots & & \\ & & \lambda & \\ 0 & & & \mu \end{bmatrix}, \quad (\lambda, \mu \in \mathbf{R}). \tag{7}$$

Hence, by Theorem 1, the hypersurfaces of $\mathbf{E}^{n+1}$ for which $K = 0$ are precisely those for which (7) holds together with $\tau = 0$. From (7) it directly follows that the scalar curvature τ of a quasiumbilical hypersurface is given by

$$\tau = (n-1)\lambda[(n-2)\lambda + 2\mu], \tag{8}$$

since, by the equation of Gauss, the sectional curvature at p for a plane section spanned by two eigenvectors E_i and E_j of $A(p)$ with corresponding eigenvalues or principal curvatures λ_i and λ_j is given by $\lambda_i\lambda_j$. Thus $\tau = 0$ if and only if $\lambda = 0$ or $\mu = \frac{n-2}{2}\lambda$, which amounts to the fact that $A(p)$ has the form (5) or (6).

4. **On hypersurfaces of revolution.** PROPOSITION 3. *M^n $(n > 2)$ is (a part of) a hypersurface of revolution in $\mathbf{E}^{n+1}$ whose profile curve is congruent to the graph of a function φ of one real variable which satisfies the differential equation*

$$\varphi\varphi'' + \alpha(1 + \varphi'^2) = 0 \tag{*}$$

if and only if, at each point p of M^n, $A(p)$ has the form

$$\begin{bmatrix} \lambda & & & 0 \\ & \ddots & & \\ & & \lambda & \\ 0 & & & \alpha\lambda \end{bmatrix}, \quad (\lambda \in \mathbf{R}\setminus\{0\}), \tag{**}$$

where α is a real constant. In particular, M^n is (a part of) a hypercatenoid, i.e. a hypersurface of revolution with a catenary as profile curve, if and only if, at each point p of M^n, $A(p)$ has the form

$$\begin{bmatrix} \lambda & & & 0 \\ & \ddots & & \\ & & \lambda & \\ 0 & & & -\lambda \end{bmatrix},$$

where $\lambda \in \mathbf{R}$ is non zero for at least one point p of M^n.

Proof. Let M^n be a hypersurface of revolution in $\mathbf{E}^{n+1}$. Let the axis of revolution be the x_1-axis and let the profile curve be given by $x_2 = \varphi(x_1)$ whereby φ satisfies the differential equation (*) for some real number α. Then M^n is given by the equation

$$[\varphi(x_1)]^2 = x_2^2 + x_3^2 + \ldots + x_{n+1}^2,$$

where $x_1, x_2, \ldots, x_{n+1}$ are Cartesian coordinates in E^{n+1}, and has the following parametrization (with parameters $u, \theta_2, \theta_3, \ldots, \theta_n$):

$$\begin{cases} x_1 = u \\ x_2 = \varphi(u) \cdot \cos\theta_2 \\ x_3 = \varphi(u) \cdot \sin\theta_2 \cdot \cos\theta_3 \\ \vdots \\ x_i = \varphi(u) \cdot \sin\theta_2 \cdot \ldots \cdot \sin\theta_{i-1} \cdot \cos\theta_i \\ \vdots \\ x_n = \theta(u) \cdot \sin\theta_2 \cdot \ldots \cdot \sin\theta_{n-1} \cdot \cos\theta_n \\ x_{n+1} = \theta(u) \cdot \sin\theta_2 \cdot \ldots \cdot \sin\theta_{n-1} \cdot \sin\theta_n. \end{cases}$$

Thus, a basis for the tangent space of M^n at the point with parameter values $u, \theta_2, \theta_3, \ldots, \theta_n$ is given by

$$\begin{cases} \frac{\partial}{\partial u} = (1, \varphi'(u) \cdot \cos\theta_2, \varphi'(u) \cdot \sin\theta_2 \cdot \cos\theta_3, \ldots, \varphi'(u) \cdot \sin\theta_2 \cdot \ldots \cdot \sin\theta_n) \\ \frac{\partial}{\partial\theta_2} = (0, -\varphi(u) \cdot \sin\theta_2, \varphi(u) \cdot \cos\theta_2 \cdot \cos\theta_3, \ldots, \varphi(u) \cdot \cos\theta_2 \cdot \ldots \cdot \sin\theta_n) \\ \vdots \\ \frac{\partial}{\partial\theta_i} = (0, \ldots, 0, -\varphi(u) \cdot \sin\theta_2 \cdot \ldots \cdot \sin\theta_i, \ldots, \varphi(u) \cdot \sin\theta_2 \cdot \ldots \cdot \sin\theta_i \cdot \ldots \cdot \sin\theta_n) \\ \vdots \\ \frac{\partial}{\partial\theta_n} = (0, \ldots, 0, -\varphi(u) \cdot \sin\theta_2 \cdot \ldots \cdot \sin\theta_n, \varphi(u) \cdot \sin\theta_2 \cdot \ldots \cdot \cos\theta_n). \end{cases}$$

Hence, at this point, a unit normal vector to M^n in E^{n+1} is given by

$$\zeta(p) = \left(-\frac{\varphi'(u)}{N}, \frac{1}{N} \cdot \cos\theta_2, \ldots, \frac{1}{N} \cdot \sin\theta_2 \cdot \ldots \cdot \sin\theta_n\right),$$

where $N^2 = 1 + [\varphi'(u)]^2$, and choosing one of the two possible roots for N appropriately, this formula defines a unit normal vector field ζ on M^n in E^{n+1}. Next, we compute the corresponding second fundamental tensor A of M^n:

$$A\left(\frac{\partial}{\partial u}\right) = -D_{\frac{\partial}{\partial u}}\zeta = \frac{\varphi''}{N^3}\frac{\partial}{\partial u},$$

$$A\left(\frac{\partial}{\partial\theta_i}\right) = -D_{\frac{\partial}{\partial\theta_i}}\zeta = -\frac{1}{N\varphi}\frac{\partial}{\partial\theta_i}, \quad (i = 2, \ldots, n).$$

where D is the standard connection of E^{n+1}. Since (*) holds, we find tha

$$\frac{\varphi''}{N^2} = -\frac{\alpha N^2}{\varphi N^2} = \alpha\left(-\frac{1}{N\varphi}\right),$$

which proves that A is of the form (**).

The solutions of the ordinary differential equation (*) are the following.

(i) If $\alpha = 0$, then $\varphi(x) = bx + c$; $b, c \in \mathbf{R}$.

(ii) If $\alpha < 0$, then $\varphi(x) = \pm g_{c,\alpha}(x + b)$, where $b \in \mathbf{R}$, $c \in \mathbf{R}_0^+$ and $g_{c,\alpha}$ is defined by

$$\begin{cases} g_{c,\alpha}(x) = h_{c,\alpha}^{-1}(x), & x > 0, \\ g_{c,\alpha}(0) = \dfrac{1}{c}, & \\ g_{c,\alpha}(x) = h_{c,\alpha}^{-1}(-x), & x < 0 \end{cases}$$

whereby

$$h_{c,\alpha}(x) = \int_{1/c}^{x} \frac{dt}{[(ct)^{-2\alpha} - 1]^{1/2}}, \quad x \in \left]\frac{1}{c}, \infty\right[$$

(iii) If $\alpha > 0$, then $\varphi(x) = \pm g_{c,\alpha}(x + b)$, where $b \in \mathbf{R}$, $c \in \mathbf{R}_0^+$ and $g_{c,\alpha}$ is defined by

$$\begin{cases} g_{c,\alpha} = h_{c,\alpha}^{-1}(x), & x < 0, \\ g_{c,\alpha} = \dfrac{1}{c}, & \\ g_{c,\alpha} = h_{c,\alpha}^{-1}(-x), & x > 0, \end{cases}$$

whereby

$$h_{c,\alpha}(x) = \int_{1/c}^{x} \frac{dt}{[(ct)^{-2\alpha} - 1]^{1/2}}, \quad x \in \left]0, \frac{1}{c}\right[.$$

Note that $g_{c,\alpha}$ is defined on the whole real line $\mathbf{R}$ if and only if $-1 \leqslant \alpha \leqslant 0$ Special cases occur when $\alpha = 1$ or $\alpha = -1$. If $\alpha = -1$, then

$$g_{c,-1}(x) = \frac{1}{c}\cosh cx,$$

i.e. the profile curve is a catenary and M^n is a hypercatenoid. If $\alpha = 1$, then

$$g_{c,1}(x) = \frac{1}{c}[1 - (cx)^2]^{1/2},$$

i.e. the profile curve is the upper half of a circle and M^n is a sphere.

Conversely, if $\alpha = 1$ and A has the form (**), then M^n is totally umbilical, and hence a sphere. Therefore, to prove the converse statement of the previous one, we can assume that $\alpha \neq 1$. So, we assume A has the form (**) every-where. Moreover, without loss of generality, we may take $\lambda > 0$. We consider the distributions T_1 and T_2 defined on M^n by

$$T_1(p) = \{X \in T_pM^n \,|\, AX = \lambda X\},$$
$$T_2(p) = \{X \in T_pM^n \,|\, AX = \alpha\lambda X\}.$$

Since $\dim T_1 = n - 1 > 1$, we can apply Prop. 2.2 and Prop. 2.3 of [3] to conclude that λ determines a differentiable function, that T_1 and T_2 are differentiable and involutive, and that $X \cdot \lambda = 0$ for any $X \in T_1$. By means of the equation of Codazzi, for $X \in T_1$ and $Y \in T_2$, we find that

$$\nabla_Y X \in T_1 \tag{9}$$

and

$$(A - \alpha\lambda)\nabla_X Y + (Y \cdot \lambda)X = 0, \tag{10}$$

where ∇ is the Levi Civita connection on M^n. Since T_1 and T_2 are orthogonal, it follows, for Y and Z belonging to T_2, that $\nabla_Y Z \in T_2$, i.e. that the integral curves of T_2 are geodesics. Next, we choose an orthonormal local frame field $E_1, \ldots, E_n, E_{n+1}$ of $\mathbf{E}^{n+1}$ along some open piece $U \subset M^n$ such that $E_1, \ldots, E_{n-1} \in T_1$, $E_n \in T_2$ and E_{n+1} is normal to M^n. Then

$$\nabla_{E_n} E_n = 0,$$

and because of (10),

$$\nabla_{E_i} E_n = \frac{E_n \cdot (\ln \lambda)}{\alpha - 1} E_i. \tag{11}$$

Let $M_1(p)$ be an integral manifold of T_1 passing through some point $p \in U$. Then the normal bundle of the submanifold $M_1(p)$ of $\mathbf{E}^{n+1}$ is spanned by the vector fields E_n and E_{n+1}. We denote the corresponding second fundamental forms by A_n and A_{n+1}, respectively. From (**) and (11), we obtain that

$$A_{n+1} = \lambda Id$$

and

$$A_n = \left[\frac{E_n \cdot (\ln \lambda)}{1 - \alpha}\right] Id,$$

where Id is the identity transformation. Consequently $M_1(p)$ is totally umbilical in $\mathbf{E}^{n+1}$, and hence a part of a sphere $S^{n-1}(p)$ in a linear subspace $\mathbf{E}^n(p)$ of $\mathbf{E}^{n+1}$ passing through p. The mean curvature vector H of $M_1(p)$ in $\mathbf{E}^{n+1}$ is given by

$$H = \left[\frac{E_n \cdot (\ln \lambda)}{1 - \alpha}\right] E_n + \lambda E_{n+1}.$$

Therefore

$$N_1 = \frac{H}{||H||} = \frac{1}{||H||}\left\{\left[\frac{E_n \cdot (\ln \lambda)}{1 - \alpha}\right] E_n + \lambda E_{n+1}\right\}$$

is the unit inward normal of $S^{n-1}(p)$ in $\mathbf{E}^n(p)$, and

$$N_2 = \frac{1}{||H||}\left\{-\lambda E_n + \left[\frac{E_n \cdot (\ln \lambda)}{1-\alpha}\right] E_{n+1}\right\}$$

is a unit vector normal to $E^n(p)$. The center $m(p)$ of $S^{n-1}(p)$ is given by

$$m(p) = p + \frac{N_1(p)}{||H||}, \tag{12}$$

and the radius of this sphere is $r(p) = \dfrac{1}{||H||}$. If γ_p is an integral curve of T_2 passing through p, then

$$\gamma_p' = E_n,$$
$$\gamma_p'' = D_{E_n} E_n = \alpha\lambda E_{n+1},$$

and

$$\gamma_p''' = D_{E_n}(\alpha\lambda E_{n+1}) = \alpha[E_n \cdot \lambda]E_{n+1} - (\alpha\lambda)^2 E_n.$$

So, we observe that γ_p is a plane curve, lying in the plane $\mathbf{E}^2(p)$ spanned by $E_n(p)$ and $E_{n+1}(p)$. Because of (12), $m(p)$ belongs to this plane $\mathbf{E}^2(p)$. We also know that $\mathbf{E}^2(p)$ and $\mathbf{E}^n(p)$ are orthogonal. For any point $q \in M_1(p)$, different from p and different also from the antipodal point of p on $S^{n-1}(p)$, we have $\mathbf{E}^n(q) = \mathbf{E}^n(p)$, $m(q) = m(p)$, and hence, the intersection of the planes $\mathbf{E}^2(p)$ and $\mathbf{E}^2(q)$ is a line orthogonal to $\mathbf{E}^n(p)$, passing through the point $m(p)$ and going in the direction of $N_2(p)$. We denote this line by $e(p)$; (whatever point q choosen in $M_1(p)$, one always obtains the same line). Next, we choose a chart $x = (x^1, \ldots, x^n) : U(p) \to]-\varepsilon, \varepsilon[^n \subset \mathbf{R}^n$, where $U(p)$ is an open neighbourhood of p in U, such that the integral manifolds in $U(p)$ of T_1 are given by $\{q \in U(p) \,|\, x^n(q) = a^n\}$, where $a^n \in]-\varepsilon, \varepsilon[$, and the integral curves of T_2 by $\{q \in V(p) \,|\, x^1(q) = a^1, \ldots, x^{n-1}(q) = a^{n-1}\}$ where $(a^1, \ldots, a^{n-1}) \in]-\varepsilon, \varepsilon[^{n-1}$, and such that $x(p) = (0, 0, \ldots, 0)$. Let q be a point in $U(p)$, put $x(q) = (c^1, \ldots$ $\ldots, c^n)$, and define $q' = x^{-1}(0, \ldots, 0, c^n)$ and $q'' = x^{-1}(c^1, \ldots, c^{n-1}, 0)$. Then we have $M_1(q) = M_1(q')$, $M_2(q) = M_2(q'')$, $M_1(p) = M_1(q'')$ and $M_2(p) = M_2(q'')$. Therefore $e(p) = \mathbf{E}^2(p) \cap \mathbf{E}^2(q'') = \mathbf{E}^2(q') \cap \mathbf{E}^2(q) = e(q)$, and we conclude that all the spaces $\mathbf{E}^n(q)$ are parallel to $\mathbf{E}^n(p)$. Finally, let γ be defined by

$$\gamma :]-\varepsilon, \varepsilon[\to U(p) : t \to x^{-1}(0, \ldots, 0, t),$$

and let M' be the hypersurface of $\mathbf{E}^{n+1}$ obtained by revolving γ around $e(p)$. We then prove that $M' \subset U(p)$. To do so, put $q = x^{-1}(c^1, \ldots, c^n)$ and $q' =$ $= x^{-1}(0, \ldots, 0, c^n)$. Since $M_1(q) = M_1(q')$ is a part of a sphere S^{n-1} in $\mathbf{E}^n(q)$, orthogonal to $e(p)$ with center $m(q) \in e(p)$, and $q' = \gamma(c^n)$, we see that q is obtained by revolving q' around $e(p)$.

As a consequence of Propositions 2 and 3, we have the following.

THEOREM 4. *A 4-dimensional hypersurface of* $\mathbf{E}^5$ *is conharmonically Euclidean if and only if it is locally a linear subspace, a hypercylinder or a hypercatenoid.*

Remark. Proposition 3 generalizes a result of D. E. Blair [0].

5. **Conformal transformations preserving harmonic functions.** Let M^n be a Riemannian manifold of dimension $n \geq 2$, with metric g and corresponding *Laplace operator* Δ. Let $\tilde{g} = e^{2\sigma}g$ be a conformally related metric on M^n with corresponding Laplace operator $\tilde{\Delta}$. In case $\sigma : M^n \to \mathbf{R}$ is a constant function, the *conformal transformation* $g = e^{2\sigma}g$ is said to be *homothetic*.

Then, for any function $\psi : M^n \to \mathbf{R}$, a straightforward computation yields the following.

LEMMA 5. $\tilde{\Delta}\psi = e^{-2\sigma}[\Delta\psi + (n-2)\sigma^k\psi_k]$.

Here, ψ_k and σ_k are derivations of ψ and σ, respectively, with respect to a local chart $\{x_k\}$, $(k, t = 1, \ldots, n)$, of M^n, $\sigma^l = g_{lt}\sigma_t$, where g^{lt} are the local components of the inverse matrix of the matrix g_{lt} of g with respect to the tangent basis $\left\{\frac{\partial}{\partial x_k}\right\}$, and whereby we use the Einstein summation convention.

From Lemma 5 we immediately derive the following.

THEOREM 6. (i) *On any Riemannian surface $(n = 2)$, every conformal transformation preserves the harmonicity of real functions.*

(ii) *On any Riemannian manifold of dimension $n > 2$, the only conformal transformations which preserves the harmonicity of real functions are the homothetic transformations.*

Here, we say that a conformal transformation $\tilde{g} = e^{2\sigma}g$ *preserves* the *harmonicity* of real functions, if any function $\psi : M^n \to \mathbf{R}$ which is harmonic with respect to the metric $g(\Delta\psi = 0)$, is also harmonic with respect to the metric $\tilde{g}$ $(\tilde{\Delta}\psi = 0)$.

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SUR LES TREILLIS NON-COMMUTATIFS DE TYPE (S)

GH. FĂRCAŞ*

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REZUMAT. — Asupra laticelor necomutative de tip (S). Într-o lucrare anterioară [2] au fost introduse laticile necomutative de tip (S) cu ajutorul unui triplet $(L, \wedge, \vee,)$, unde L este o mulţime nevidă, iar $\wedge$ şi $\vee$ sint două operaţii binare definite în L supuse la anumite condiţii concretizate sub formă de axiome. Observăm că în mulţimea L se pot defini relaţiile binare ρ_1 şi ρ_2 astfel:

$$a\rho_1 b \Leftrightarrow a = a \wedge b$$

$$a\rho_2 b \Leftrightarrow a = b \wedge a$$

care, în lipsa comutativităţii, sint distincte.

În această lucrare se studiază proprietăţile relaţiilor ρ_1 şi ρ_2, constatîndu-se că acestea sint asemănătoare cu cele întîlnite în teoria laticelor. Evident, dacă $(L, \wedge, \vee)$ este latice, atunci ρ_1 şi ρ_2 sint relaţii de ordine în mulţimea L şi $\rho_1 = \rho_2$. Această afirmaţie nu este adevărată în cazul cînd $(L, \wedge, \vee)$ este latice necomutativă de tip (S), deoarece în acest caz, deşi ρ_2 este o relaţie de oridne, ρ_1 este numai o relaţie de preordine şi $\rho_1 \neq \rho_2$.

Le triplet $(L, \wedge, \vee)$, où L est un ensemble et $\wedge$ et $\vee$ sont deux opérations binaires défines en L, s'appelera de treillis non-commutatif de type (S), si pour tout $a, b, c \in L$ vérifie les axiomes:

$$\text{(A)} \quad \begin{cases} (a \wedge b) \wedge c = a \wedge (b \wedge c) \\ (a \vee b) \vee c = a \vee (b \vee c) \end{cases}$$

$$\text{(B)} \quad \begin{cases} a \wedge (a \vee b) = a \\ a \vee (a \wedge b) = a \end{cases}$$

$$\text{(S)} \quad \begin{cases} a \wedge (b \vee c) = a \wedge (c \vee b). \\ a \vee (b \wedge c) = a \vee (c \wedge b). \end{cases}$$

En [2] on montre que, si $(L, \wedge, \vee)$ est treillis non-commutatif de type (S), alors pour tout $a, b, c \in L$ sont vraies les égalités:

$$\text{(i)} \quad \begin{cases} a \wedge a = a \\ a \vee a = a \end{cases}$$

$$\text{(ii)} \quad \begin{cases} a \wedge b = (a \wedge b) \vee (b \wedge a) \\ a \vee b = (a \vee b) \wedge (b \vee a) \end{cases}$$

* L'Institut d'enseignement supérieur de Tîrgu Mureş 4300 Tîrgu Mureş, Roumanie

(iii) $\begin{cases} a \wedge (b \wedge c) = a \wedge (c \wedge b) \\ a \vee (b \vee c) = a \vee (c \vee b) \end{cases}$

(iv) $\begin{cases} a \wedge b \wedge a = a \wedge b \\ a \vee b \vee a = a \vee b. \end{cases}$

Observons que dans l'ensemble L peuvent etre définies les relations binaires ρ_i $(i = 1, 2, \ldots, 8)$ engendrées par les opérations $\wedge$ et $\vee$ à savoir :

$$a\rho_1 b \Leftrightarrow a = a \wedge b \qquad a\rho_5 b \Leftrightarrow a = a \vee b$$
$$a\rho_2 b \Leftrightarrow a = b \wedge a \qquad a\rho_6 b \Leftrightarrow a = b \vee a$$
$$a\rho_3 b \Leftrightarrow b = b \wedge a \qquad a\rho_7 b \Leftrightarrow b = b \vee a$$
$$a\rho_4 b \Leftrightarrow b = a \wedge b \qquad a\rho_8 b \Leftrightarrow b = a \vee b.$$

Dans ce travail on étudie les propriétés de ces relations dans le cas où $(L, \wedge, \vee)$ est treillis non-commutatif de type (S). C'est pourquoi, on observe que $\rho_3 = \rho_1^{-1}$, $\rho_4 = \rho_2^{-1}$ et que ρ_5 est la duale de ρ_1, ρ_6 est la duale de ρ_2, ρ_7 est la duale de ρ_3 et ρ_8 est la duale de ρ_4. Donc, en tenant compte que les systèmes d'axiomes qui définissent le treillis non-commutatif de type (S) sont autoduales, on peut se limiter uniquement à l'étude des propriétés de relations ρ_1 et ρ_2.

(1.1). Si $(L, \wedge, \vee)$ est le treillis non-commutatif de type (S), alors pour tout $a, b, c \in L$ la relation binaire ρ_1 possède les propriétés :

(1) ρ_1 est une relation de préordre en L

(2) $a \wedge b\rho_1 a$ et $a \wedge b\rho_1 b$

(3) $a\rho_1 a \vee b$ et $b\rho_1 a \vee b$

(4) $c\rho_1 a$ et $c\rho_1 b \Rightarrow c\rho_1 a \wedge b$

(5) $c\rho_1 a \Rightarrow c\rho_1 a \vee b$

(6) $b\rho_1 c \Rightarrow a \wedge b\rho_1 c$

(7) $a\rho_1 c$ et $b\rho_1 c \Rightarrow a \vee b\rho_1 c$

(8) $a\rho_1 b \Rightarrow a \wedge c\rho_1 b \wedge c$ et $c \wedge a\rho_1 c \wedge b$

(9) $a\rho_1 b \Rightarrow a \vee c\rho_1 b \vee c$ et $c \vee a\rho_1 c \vee b$.

Démonstration. (1). La propriété de réflexivité de la relation ρ_1 est une conséquence immédiate de la lois de l'idempotence (i). Ensuite, si pour $a, b, c \in L$ on a $a\rho_1 b$ et $b\rho_1 c$ alors $a = a \wedge b$ et $b = b \wedge c$, donc $a = a \wedge b = a \wedge (b \wedge c) = (a \wedge b) \wedge c = a \wedge c$, c'est-à-dire $a\rho_1 c$ et ainsi ρ_1 possède aussi la propriété de transitivité.

(2). En utilisant les propriétés (iv) et (i) on obtient que pour tout $a, b \in L$ nous avons $a \wedge b = a \wedge b \wedge a = (a \wedge b) \wedge a$ et respectivement $a \wedge b = a \wedge (b \wedge b) = (a \wedge b) \wedge b$, donc $a \wedge b\rho_1 a$ et $a \wedge b\rho_1 b$.

(3). En utilisant les axiomes (B) et (S) on obtient $a = a \wedge (a \vee b)$ et $b = b \wedge (b \vee a) = b \wedge (a \vee b)$, donc pour tout $a, b \in L$ nous avons $a\rho_1 a \vee b$ et $b\rho_1 a \vee b$.

(4). Si pour $a, b, c \in L$ on a $c\rho_1 a$ et $c\rho_1 b$, alors $c = c \wedge a$ et $c = c \wedge b$, donc $c = c \wedge b = (c \wedge a) \wedge b = c \wedge (a \wedge b)$, c'est-à-dire $c\rho_1 a \wedge b$.

(5). Si pour $a, c \in L$ on a $c\rho_1 a$, alors $c = c \wedge a$, donc $c = c \wedge a = c \wedge \wedge (a \wedge (a \vee b)) = (c \wedge a) \wedge (a \vee b) = c \wedge (a \vee b)$, c'est-à-dire $c\rho_1 a \vee b$.

(6). Si pour $b, c \in L$ on a $b\rho_1 c$, alors $b = b \wedge c$, donc $a \wedge b = a \wedge (b \wedge c) = (a \wedge b) \wedge c$, c'est-à-dire $a \wedge b\rho_1 c$.

(7). Si pour $a, b, c \in L$ on a $a\rho_1 c$ et $b\rho_1 c$, alors $a = a \wedge c$ et $b = b \wedge c$, donc $c = c \vee (c \wedge a) = c \vee (a \wedge c) = c \vee a$, et $c = c \vee (c \wedge b) = c \vee (b \wedge c) = c \vee b$, c'est-à-dire $a \vee b = (a \vee b) \wedge ((a \vee b) \vee c) = (a \vee b) \wedge (c \vee (a \vee b)) = (a \vee b) \wedge ((c \vee a) \vee b) = (a \vee b) \wedge (c \vee b) = (a \vee b) \wedge c$, en conséquence $a \vee b\rho_1 c$.

(8). Si pour $a, b \in L$ on a $a\rho_1 b$, alors $a = a \wedge b$, donc $a \wedge c = (a \wedge c) \wedge (a \wedge c) = a \wedge c \wedge (a \wedge b) \wedge c = (a \wedge c \wedge a) \wedge (b \wedge c) = (a \wedge c) \wedge (b \wedge c)$ et $c \wedge a = c \wedge (a \wedge b) = (c \wedge a) \wedge b = (c \wedge a \wedge c) \wedge b = (c \wedge a) \wedge (c \wedge b)$, c'est-à-dire $a \wedge c\rho_1 b \wedge c$ et $c \wedge a\rho_1 c \wedge b$.

(9) Si pour $a, b \in L$ on a $a\rho_1 b$, alors $a = a \wedge b$, donc $b = b \vee (b \wedge a) = b \vee (a \wedge b) = b \vee a$, c'est-à-dire $a \vee c = (a \vee c) \wedge ((a \vee c) \vee b) = (a \vee c) \wedge (b \vee (a \vee c)) = (a \vee c) \wedge ((b \vee a) \vee c) = (a \vee c) \wedge (b \vee c)$ et $c \vee a = (c \vee a) \wedge ((c \vee a) \vee b) = (c \vee a) \wedge (c \vee (a \vee b)) = (c \vee a) \wedge (c \vee (b \vee a)) = (c \vee a) \wedge (c \vee b)$, en conséquence $a \vee c\rho_1 b \vee c$ et $c \vee a\rho_1 c \vee b$.

(1.2). Si $(L, \wedge, \vee)$ est le treillis non-commutatif de type (S), alors pour tout $a, b, c \in L$ la relation binaire ρ_2 possède les propriétés :

(1) ρ_2 est une relation d'ordre en L

(2) $a \wedge b\rho_2 a$

(3) $c\rho_2 a$ et $c\rho_2 b \Rightarrow c\rho_2 a \wedge b$

(4) $a\rho_2 c \Rightarrow a \wedge b\rho_2 c$

(5) $a\rho_2 b \Rightarrow a \wedge c\rho_2 b \wedge c$ et $c \wedge a\rho_2 c \wedge b$.

Démonstration. (1). La propriété de réflexivité de la relation ρ_2 est aussi une conséquence de la lois de l'idempotence. Ensuite, si pour $a, b, c \in L$ on a $a\rho_2 b$ et $b\rho_2 c$, alors $a = b \wedge a$ et $b = c \wedge b$, donc $a = b \wedge a = (c \wedge b) \wedge a = c \wedge (b \wedge a) = c \wedge a$, c'est-à-dire $a\rho_2 c$, en conséquence ρ_2 possède la propriété de transitivité. Enfin, si pour $a, b \in L$ on a $a\rho_2 b$ et $b\rho_2 a$, alors $a = b \wedge a$ et $b = a \wedge b$, donc $a = b \wedge a = (a \wedge b) \wedge a = a \wedge b \wedge a = a \wedge b = b$, c'est-à-dire ρ_2 possède la propriété d'antisymétrie.

(2). Pour tout $a, b \in L$ nous avons $a \wedge b = (a \wedge a) \wedge b = a \wedge (a \wedge b)$, donc $a \wedge b\rho_2 a$.

(3). Si $c\rho_2 a$ et $c\rho_2 b$, alors $c = a \wedge c$ et $c = b \wedge c$, donc $c = a \wedge c = a \wedge (b \wedge c) = (a \wedge b) \wedge c$, c'est-à-dire $c\rho_2 a \wedge b$.

(4). Si $a\rho_2 c$, alors $a = c \wedge a$; donc $a \wedge b = (c \wedge a) \wedge b = c \wedge (a \wedge b)$, c'est-à-dire $a \wedge b\rho_2 c$.

(5). Si $a\rho_2 b$, alors $a = b \wedge a$, donc $a \wedge c = (b \wedge a) \wedge c = b \wedge (a \wedge c) = b \wedge (c \wedge a) = b \wedge (c \wedge a \wedge c) = (b \wedge c) \wedge (a \wedge c)$ et $c \wedge a = c \wedge (b \wedge a) =$

$= (c \wedge b) \wedge a = (c \wedge b) \wedge (c \wedge b) \wedge a = (c \wedge b) \wedge c \wedge (b \wedge a) = (c \wedge b) \wedge c \wedge a =$
$= (c \wedge b) \wedge (c \wedge a)$ c'est-à-dire $a \wedge c\rho_2 b \wedge c$ et $c \wedge a\rho_2 c \wedge b$.

(1.3). Si $(L, \wedge, \vee)$ est le treillis non-commutatif de type (S), alors pour tout $a, b, c \in L$ les relations binaires ρ_1 et ρ_2 possèdent les propriétés :

(1) $a\rho_2 b \Rightarrow a\rho_1 b$

(2) $c\rho_2 a$ et $c\rho_1 b \Rightarrow c\rho_2 a \wedge b$.

Démonstration. (1). Si $a\rho_2 b$, alors $a = b \wedge a$, donc $a = b \wedge a = b \wedge (b \wedge a) = b \wedge (a \wedge b) = (b \wedge a) \wedge b = a \wedge b$, c'est-à-dire $a\rho_1 b$.

(2). Si $c\rho_2 a$ et $c\rho_1 b$, alors $c = a \wedge c$ et $c = c \wedge b$, donc $c = c \wedge b = (a \wedge c) \wedge b = a \wedge (c \wedge b) = a \wedge (b \wedge c) = (a \wedge b) \wedge c$, c'est-à-dire $c\rho_2 a \wedge b$.

On constate que ces propriétés sont pareilles à celles obtenues dans le cas où $(L, \wedge, \vee)$ est treillis.

Evidement, si $(L, \wedge, \vee)$ est treillis, alors ρ_1 et ρ_2 sont relations d'ordres en L et $\rho_1 = \rho_2$. Cette affirmation n'est pas vraie dans le cas où $(L, \wedge, \vee)$ est seulement treillis non-commutatif de type (S).

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ON ABELIAN Ω-GROUPS

RODICA COVACI*

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REZUMAT.— Asupra Ω—Grupurilor abeliene. Lucrarea tratează unele aspecte referitoare la noţiunea de „Ω-grup abelian", introdusă de P. J. Higgins în [2]. Se dă o nouă definiţie Ω-grupului abelian, echivalentă cu definiţia din [2] şi se studiază idealele unei sume directe de Ω-grupuri abeliene. Rezultatul principal stabileşte o condiţie necesară şi suficientă pentru ca un ideal al unei sume directe de Ω-grupuri abeliene să fie o sumă directă de ideale ale sumanzilor direcţi.

Preliminaries. In [2], P. J. Higgins introduces a universal algebra called "group with multiple operators" or "Ω-group", generalizing both groups and rings. Some special Ω-groups, like "abelian Ω-groups", are also defined in [2]. It is the aim of this paper to give a novel definition to the notion of "abelian Ω-group". It is shown that our definition is equivalent with that from [2] and some basic properties of abelian Ω-groups are given. Finally, considerations on direct sums of abelian Ω-groups, particularly a theorem giving a necessary and sufficient condition for an ideal of a direct sum of abelian Ω-groups to be a direct sum of ideals of the direct summands are treated.

We give below some definitions from [2], which will be used in the next sections.

DEFINITION 1.1. A triplet $(G, +, \Omega)$ is said to be an *Ω-group* if

(i) $(G, +)$ is a group (let 0 be the zero element);

(ii) Ω is a family of finitary operations on G such that any $\omega \in \Omega$ satisfies

$$\omega(0, \ldots, 0) = 0.$$

The subalgebras of an Ω-group which are subgroups with respect to addition are called *Ω-subgroups*. They generalise both the subgroups and the subrings.

DEFINITION 1.2. A subset A of an Ω-group $(G, +, \Omega)$ is an *Ω-subgroup* if

(i) A is a subgroup of $(G, +)$;

(ii) for any $\omega \in \Omega$, holds

$$a_1, a_2, \ldots, a_n \in A \Rightarrow \omega(a_1, a_2, \ldots, a_n) \in A.$$

The normal subgroups and the ideals can be generalized in Ω-groups by a notion defined in [2] as follows:

* University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania

DEFINITION 1.3. Let $(G, +, \Omega)$ be an Ω-group and $A \subseteq G$. A is said to be an *ideal* of $(G, +, \Omega)$ if

(i) A is a normal subgroup of $(G, +)$;

(ii) for any $\omega \in \Omega$, $a \in A$, $g_j \in G$ $(j = 1, \ldots, n)$, we have

$$-\omega(g_1, \ldots, g_n) + \omega(g_1, \ldots, g_{i-1}, a + g_i, g_{i+1}, \ldots, g_n) \in A,$$
$$i = 1, \ldots, n.$$

It is easy to see that any ideal of an Ω-group is an Ω-subgroup.

We also need:

DEFINITION 1.4. Let $(G, +, \Omega)$ and $(G', +, \Omega)$ be two similar Ω-groups. An *homomorphism* $f: G \to G'$ is a map which is an homomorphism between the corresponding universal algebras, i.e. a map with the following properties:

(i) $\forall g_1, g_2 \in G, f(g_1 + g_2) = f(g_1) + f(g_2)$;

(ii) $\forall \omega \in \Omega$, $\forall g_i \in G$ $(i = 1, \ldots, n)$, we have

$$f(\omega(g_1, \ldots, g_n)) = \omega(f(g_1), \ldots, f(g_n)).$$

2. **Abelian Ω-groups and basic properties.** We first give a novel definition of the abelian Ω-group and prove that it is equivalent with that from [2].

DEFINITION 2.1. A triplet $(G, +, \Omega)$ is said to be an *abelian Ω-group* if

(a) $(G, +)$ is an abelian group;

(b) any operation $\omega \in \Omega$ of the family Ω of finitary operations on G satisfies, for any $g_i, g_i' \in G$ $(i = 1, \ldots, n)$,

$$\omega(g_1 + g_1', g_2 + g_2', \ldots, g_n + g_n') = \omega(g_1, g_2, \ldots, g_n) + \omega(g_1', g_2', \ldots, g_n').$$

Examples 2.2 a) Any abelian group $(G, +)$ is an abelian *Ω-group* with $\Omega = \emptyset$.

b) A ring $(G, +)$ with the multiplication defined by

$$\forall g, h \in G, g \cdot h = 0 \tag{1}$$

is an abelian Ω-group with Ω consisting only of one binary operation, defined by (1).

In the following, let $(G, +, \Omega)$ be an abelian Ω-group. From 2.1., some basic properties can be obtained.

PROPOSITION 2.3. *For any* $\omega \in \Omega$.

$$\omega(0, \ldots, 0) = 0.$$

Proof. By 2.1. (b),

$$\omega(0, \ldots, 0) = \omega(0 + 0, \ldots, 0 + 0) = \omega(0, \ldots, 0) + \omega(0, \ldots, 0).$$

Hence

$$\omega(0, \ldots, 0) = 0.$$

PROPOSITION 2.4. *For any* $\omega \in \Omega$ *and* $g_i, g_i' \in G$ $(i = 1, \ldots, n)$. *holds*

$$\omega(g_1 - g_1', \ldots, g_n - g_n') = \omega(g_1, \ldots, g_n) - \omega(g_1', \ldots, g_n'). \tag{2}$$

Particularly,

$$\omega(-g_1', \ldots, -g_n') = -\omega(g_1', \ldots, g_n'). \tag{3}$$

Proof. Since, by 2.1. (b),

$$\omega(g_1, \ldots, g_n) = \omega(g_1 - g_1' + g_1', \ldots, g_n - g_n' + g_n') =$$
$$= \omega(g_1 - g_1', \ldots, g_n - g_n') + \omega(g_1', \ldots, g_n').$$

it follows (2). Putting in (2) $g_1 = g_2 = \ldots = g_n = 0$ and using 2.3., we deduce (3)

From 2.1.(b) follows obviously

PROPOSITION 2.5. *For any $\omega \in \Omega$ and $g, h, g_j \in G$ $(j = 1, \ldots, n)$ we have*

(i) $\omega(0, \ldots, 0, \underset{i}{g + h}, 0, \ldots, 0) = \omega(0, \ldots, 0, \underset{i}{g}, 0, \ldots, 0) + \omega(0, \ldots, 0, \underset{i}{h}, 0, \ldots, 0),$

$i = 1, \ldots, n$;

(ii) $\omega(g_1, g_2, \ldots, g_n) = \omega(g_1, 0, \ldots, 0) + \omega(0, g_2, 0, \ldots, 0) + \ldots + \omega(0, \ldots, 0, g_n)$.

In [2], the abelian Ω-group is defined as an Ω-group satisfiyng conditions 2.5. (i) and 2.5. (ii). The following theorem shows that definition 2.1. is equivalent with that from [2].

THEOREM 2.6. *Let $(G, +, \Omega)$ be a triplet such that $(G, +)$ is an abelian group and Ω is a family of finitary operations on G. The following two conditions are equivalent:*

(i) *$(G, +, \Omega)$ is an abelian Ω-group;*

(ii) *$(G, +, \Omega)$ is an Ω-group satisfying conditions* 2.5.(i) *and* 2.5.(ii).

Proof. (i) implies (ii). Follows from 2.3. and 2.5.

(ii) implies (i). We verify condition 2.1.(b). Let $\omega \in \Omega$ and $g_i, g_i' \in G$ $(i = 1, \ldots, n)$.

$$\omega(g_1 + g_1', \ldots, g_n + g_n') = \omega(g_1 + g_1', 0, \ldots, 0) + \ldots +$$
$$+ \omega(0, \ldots, 0, g_n + g_n') = [\omega(g_1, 0, \ldots, 0) + \omega(g_1', 0, \ldots, 0)] + \ldots +$$
$$+ [\omega(0, \ldots, 0, g_n) + \omega(0, \ldots, 0, g_n')] = [\omega(g_1, 0, \ldots, 0) + \ldots +$$
$$+ \omega(0, \ldots, 0, g_n)] + [\omega(g_1', 0, \ldots, 0) + \ldots + \omega(0, \ldots, 0, g_n')] =$$
$$= \omega(g_1, \ldots, g_n) + \omega(g_1', \ldots, g_n').$$

PROPOSITION 2.7. *For any $m \in \mathbf{Z}$, $\omega \in \Omega$ and $g \in G$ hold*

$$\omega(0, \ldots, 0, \underset{i}{mg}, 0, \ldots, 0) = m\omega(0, \ldots, 0, \underset{i}{g}, 0, \ldots, 0),$$
$$i = 1, \ldots, n. \tag{4}$$

Proof. If $m > 0$, we have

$$\omega(0, \ldots, 0, mg, 0, \ldots, 0) = \omega(0, \ldots, 0, g + \ldots + g, 0, \ldots, 0) =$$
$$= \omega(0, \ldots, 0, g, 0, \ldots, 0) + \ldots + \omega(0, \ldots, 0, g, 0, \ldots, 0) =$$
$$= m\omega(0, \ldots, 0, g, 0, \ldots, 0).$$

If $m = 0$, we obtain, by 2.3.,

$$\omega(\bar{0}, \ldots, \bar{0}, 0 \cdot g, \bar{0}, \ldots, \bar{0}) = \omega(\bar{0}, \ldots, \bar{0}) = \bar{0} =$$
$$= 0 \cdot \omega(\bar{0}, \ldots, \bar{0}, g, \bar{0}, \ldots, \bar{0}).$$

where $\bar{0}$ denotes, in this case only, the zero element from $(G, +)$.

If $m < 0$, since (4) hold for positive integers and by 2.4., we deduce

$$m\omega(0, \ldots, 0, g, 0, \ldots, 0) = -[(-m)\omega(0, \ldots, 0, g, 0, \ldots, 0)] =$$
$$= -\omega(0, \ldots, 0, (-m)g, 0, \ldots, 0) = -\omega(0, \ldots, 0, -mg, 0, \ldots, 0) =$$
$$= \omega(0, \ldots, 0, mg, 0, \ldots, 0).$$

PROPOSITION 2.8. *For any $m \in \mathbf{Z}$, $\omega \in \Omega$ and $g_j \in G (j = 1, \ldots, n)$,* we have the equalities:

$$\omega(g_1, \ldots, g_{i-1}, mg_i, g_{i+1}, \ldots, g_n) = \omega(g_1, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_n) +$$
$$+ (m-1)\,\omega(0, \ldots, 0, g_i, 0, \ldots, 0). \qquad i = 1, \ldots, n.$$

Proof. Using 2.7., we obtain

$$\omega(g_1, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_n) + (m-1)\,\omega(0, \ldots, 0, g_i, 0, \ldots, 0) =$$
$$= \omega(g_1, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_n) + \omega(0, \ldots, 0, (m-1)g_i, 0, \ldots, 0) =$$
$$= \omega(g_1, \ldots, g_{i-1}, g_i + (m-1)g_i, g_{i+1}, \ldots, g_n) =$$
$$= \omega(g_1, \ldots, g_{i-1}, mg_i, g_{i+1}, \ldots, g_n).$$

Definition 1.3. becomes in the case of abelian Ω-groups as follows:

DEFINITION 2.9. A subset A of an abelian Ω-group $(G, +, \Omega)$ is an *ideal* if

(i) A is a subgroup of $(G, +)$;

(ii) for any $\omega \in \Omega$ and $a \in A$, hold

$$\omega(0, \ldots, 0, a, 0, \ldots, 0) \in A, \qquad i = 1, \ldots, n.$$

PROPOSITION 2.10. *In an abelian Ω-group, the ideals and the Ω-subgroups coincide.*

Proof. In any Ω-group, the ideals are Ω-subgroups. Conversely, let A be an Ω-subgroup of the abelian Ω-group $(G, +, \Omega)$. Condition (i) from 2.9. is obviously verified. Let us prove that (ii) holds. If $\omega \in \Omega$ and $a \in A$, we have, because $0 \in A$ and A is an Ω-subgroup of $(G, +, \Omega)$,

$$\omega(0, \ldots, 0, a, 0, \ldots, 0) \in A, \quad i = 1, \ldots, n.$$

PROPOSITION 2.11. *Let* $(G, +, \Omega)$ *and* $(G', +, \Omega)$ *be two similar* Ω*-groups and* $f: G \to G'$ *an homomorphism. If* $(G, +, \Omega)$ *is an abelian* Ω*-group, then* $(f(G), +, \Omega)$ *is an abelian* Ω*-group.*

Proof. It is clear that $(f(G), +)$ is an abelian group. Let $\omega \in \Omega$ and $f(g_i)$, $f(x_i) \in f(G)$, where $i = 1, \ldots, n$. Then

$$\omega(f(g_1) + f(x_1), \ldots, f(g_n) + f(x_n)) = \omega(f(g_1 + x_1), \ldots, f(g_n + x_n)) =$$
$$= f(\omega(g_1 + x_1, \ldots, g_n + x_n)) = f(\omega(g_1, \ldots, g_n) + \omega(x_1, \ldots, x_n)) =$$
$$= f(\omega(g_1, \ldots, g_n)) + f(\omega(x_1, \ldots, x_n)) =$$
$$= \omega(f(g_1), \ldots, f(g_n)) + \omega(f(x_1), \ldots, f(x_n)).$$

3. **Direct sum of abelian Ω-groups.** The main purpose of this section is to give a necessary and sufficient condition for an ideal of a direct sum of abelian Ω-groups to be a direct sum of ideals of the direct summands.

The notions of "direct product" and "direct sum" of Ω-groups are these from the universal algebra theory.

DEFINITION 3.1. Let $\{(G_i, +, \Omega)\}_{i \in I}$ be a family of similar Ω-groups.

a) We call the *direct product* of the family above, an Ω-group $(\prod_{i \in I} G_i, +, \Omega)$ whose operations are defined by

$$(g_i^1)_{i \in I} + (g_i^2)_{i \in I} = (g_i^1 + g_i^2)_{i \in I};$$
$$\omega((g_i^1)_{i \in I}, \ldots, (g_i^n)_{i \in I}) = (\omega(g_i^1, \ldots, g_i^n))_{i \in I},$$

where $\omega \in \Omega$ and $(g_i^j)_{i \in I} \in \prod_{i \in I} G_i$, $j = 1, \ldots, n$.

b) The Ω-subgroup of the direct product defined by

$$\bigoplus_{i \in I} G_i = \{(g_i)_{i \in I} \in \prod_{i \in I} G_i / \text{supp } (g_i)_{i \in I} \text{ is finite}\},$$

where

$$\text{supp } (g_i)_{i \in I} = \{j \in I / g_j \neq 0\},$$

is said to be the *direct sum* of the above.

PROPOSITION 3.2. *The direct product of a family* $\{(G_i, +, \Omega)\}_{i \in I}$ *of similar abelian* Ω*-groups is an abelian* Ω*-group.*

Proof. Let us denote $G = \prod_{i \in I} G_i$. Obviously, $(G, +)$ is an abelian group. In order to verify 2.1.(b), let $\omega \in \Omega$ and $(g_i^j)_{i \in I}, (g_i'^j)_{i \in I} \in G$, $j = 1, \ldots, n$. We have

$$\omega((g_i^1)_{i \in I} + (g_i'^1)_{i \in I}, \ldots, (g_i^n)_{i \in I} + (g_i'^n)_{i \in I}) =$$
$$= \omega((g_i^1 + g_i'^1)_{i \in I}, \ldots, (g_i^n + g_i'^n)_{i \in I}) = (\omega(g_i^1 + g_i'^1, \ldots, g_i^n + g_i'^n))_{i \in I} =$$
$$= (\omega(g_i^1, \ldots, g_i^n) + \omega(g_i'^1, \ldots, g_i'^n))_{i \in I} =$$
$$= (\omega(g_i^1, \ldots, g_i^n))_{i \in I} + (\omega(g_i'^1, \ldots, g_i'^n))_{i \in I} =$$
$$= \omega((g_i^1)_{i \in I}, \ldots, (g_i^n)_{i \in I}) + \omega((g_i'^1)_{i \in I}, \ldots, (g_i'^n)_{i \in I}).$$

It follows that $(G, +, \Omega)$ is an abelian Ω-group.

COROLLARY 3.3. *The direct sum of a family of abelian Ω-groups is an abelian Ω-group.*

In the following, some results about the ideals of a direct sum of abelian Ω-groups are given.

THEOREM 3.4. *Let $\{(G_i, +, \Omega)\}_{i\in I}$ be a family of similar abelian Ω-groups and $G = \bigoplus_{i\in I} G_i$ the direct sum. If, for any $i \in I$, A_i is an ideal of G_i, then $A = \bigoplus_{i\in I} A_i$ is an ideal of G.*

Proof. Obviously, A is a subgroup of $(G, +)$. Further, condition 2.9. (ii) holds. Indeed, for any $\omega \in \Omega$ and $(a_i)_{i\in I} \in A$, we have

$$\omega((0_i)_{i\in I}, \ldots, (0_i)_{i\in I}, (a_i)_{i\in I}, (0_i)_{i\in I}, \ldots, (0_i)_{i\in I}) =$$
$$= (\omega(0_i, \ldots, 0_i, \underset{j}{a_i}, 0_i, \ldots, 0_i))_{i\in I} \in A, \quad j = 1, \ldots, n.$$

In preparation for the main result, we give:

NOTATION 3.5. *Let $G = \prod_{i\in I} G_i$ and $j \in I$. We denote by*

$$(g_i^{[j]})_{i\in I}$$

the element of G with $g_i^{[j]} = g_j \in G_j$ and $g_i^{[j]} = 0$ for any $i \in I\setminus\{j\}$.

THEOREM 3.6. *Let $\{(G_i, +, \Omega)\}_{i\in I}$ be a family of similar abelian Ω-groups, $G = \bigoplus_{i\in I} G_i$ the direct sum and let A be an ideal of G. The following two conditions are equivalent:*

(a) *$A = \bigoplus_{i\in I} A_i$, where, for any $i \in I$, A_i is an ideal of G_i;*

(b) *$(a_i)_{i\in I} \in A \Rightarrow (a_i^{[j]})_{i\in I} \in A$, $\forall j \in I$.*

Proof. (a) implies (b). Indeed, if

$$(a_i)_{i\in I} \in A = \bigoplus A_i,$$

then, for any $j \in I$, $a_j \in A_j$, hence

$$(a_i^{[j]})_{i\in I} \in \bigoplus_{i\in I} A_i = A.$$

(b) implies (a). We consider for any $j \in I$, the canonical **homomorphism** $p_j : G \to G_j$, $p_j((g_i)_{i\in I}) = g_j$. For any $j \in I$, denote by

$$F_i^j = \begin{cases} G_j \text{ if } i = j; \\ \{0\} \text{ if } i \neq j; \end{cases}$$

$$G_j' = \bigoplus_{i\in I} F_i^j;$$

$$A_j = p_j(A \cap G_j').$$

We notice that $A_j \subset G_j$ and

$$a_j \in A_j \Leftrightarrow (a_i^{[j]})_{i\in I} \in A.$$

Then A_j is an ideal of G_j, for any $j \in I$. Indeed, A_j is a subgroup of $(G_j, +)$ and, for any $\omega \in \Omega$, $a_j \in A_j$ and $k = 1, \ldots, n$, we have $(a_i^{[j]})_{i\in I} \in A$, hence, since A is an ideal of G,

$$((\omega(0_i, \ldots, 0_i, \underset{k}{a_i}, 0_i, \ldots, 0_i))^{[j]})_{i\in I} =$$

$$= \omega((0_i)_{i\in I}, \ldots, (0_i)_{i\in I}, \underset{k}{(a_i^{[j]})_{i\in I}}, (0_i)_{i\in I}, \ldots, (0_i)_{i\in I}) \in A.$$

It follows that

$$\omega(0, \ldots, 0, \underset{k}{a_j}, 0, \ldots, 0) \in A_j, \quad k = 1, \ldots, n.$$

We conclude that A_j is an ideal of G_j.

Since, for any $j \in I$, A_j is an ideal of G_j, we deduce, by 3.4., that $\bigoplus_{i\in I} A_i$ is an ideal of G.

We shall prove that $A = \bigoplus_{i\in I} A_i$. Let $(a_i)_{i\in I} \in A$. By the hypothesis (b), we also have $(a_i^{[j]})_{i\in I} \in A$, $\forall j \in I$, hence $a_j \in A_j$, $\forall j \in I$. Then $(a_i)_{i\in I} \in \bigoplus_{i\in I} A_i$. Conversely, if $(a_i)_{i\in I} \in \bigoplus_{i\in I} A_i$, we have $a_j \in A_j$, $\forall j \in I$ and supp $(a_i)_{i\in I}$ is finite. This implies that, for any $j \in I$, $(a_i^{[j]})_{i\in I} \in A$. It follows that

$$(a_i)_{i\in I} = (\sum_{j\in I'} a_i^{[j]})_{i\in I} = \sum_{j\in I'} (a_i^{[j]})_{i\in I} = \sum_{j\in I'} (a_i^{[j]})_{i\in I} \in A.$$

where $I' = \text{supp}\,(a_i)_{i\in I}$.

THEOREM 3.7. *Let $\{(G_i, +, \Omega)\}_{i\in I}$ be a family of similar abelian Ω-groups and $G = \bigoplus_{i\in I} G$ the direct sum. The following two conditions are equivalent:*

(i) *any ideal A of G satisfies condition* (b) *from* 3.6.;

(ii) *G satisfies:*

$$(a_i)_{i\in I} \in G) \Rightarrow (a_i^{[j]})_{i\in I} \in ((a_i)_{i\in I}), \quad \forall j \in I.$$

where $((a_i)_{i\in I})$ is the principal ideal of G generated by $(a_i)_{i\in I}$.

Proof. (i) implies (ii). Let $(a_i)_{i\in I} \in G$. Clearly, $(a_i)_{i\in I} \in ((a_i)_{i\in I})$. Applying (i), we obtain that

$$(a_i^{[j]})_{i\in I} \in ((a_i)_{i\in I}), \quad \forall j \in I.$$

(ii) implies (i). Let A be an ideal of G and $(a_i)_{i\in I} \in A$. By (ii), we have

$$(a_i^{[j]})_{i\in I} \in ((a_i)_{i\in I}), \quad \forall j \in I.$$

But

$$((a_i)_{i\in I}) \subset A.$$

Hence

$$(a_i^{[j]})_{i\in I} \in A, \quad \forall j \in I.$$

It follows that A satisfies condition 3.6. (b).

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GENERALIZED BERNSTEIN OPERATORS AND CONVEX FUNCTIONS

I. RAŞA*

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REZUMAT. — Operatori Bernstein generalizaţi şi funcţii convexe. În lucrare sînt studiate inegalităţi de tipul $B_n f \geq f$, unde B_n sînt operatori Bernstein generalizaţi iar f este funcţie convexă; sînt precizate condiţiile cînd are loc semnul de egalitate.

1. Let (u_0, u_1) denote an Extended Complete Tchebycheff system in $C[a, b]$ i.e., $u_0 > 0$ and $u_1(t) = u_0(t) \int_a^t w(s)ds$ where $w > 0$ (see [8]).

The subspace of $C[a, b]$ spanned by u_0 and u_1 will be called the space of u — affine functions.

Let $K(u_0, u_1)$ be the set of the functions $f \in C[a, b]$ such that

$$\begin{bmatrix} u_0(x_1) & u_0(x_2) & u_0(x_3) \\ u_1(x_1) & u_1(x_2) & u_1(x_3) \\ f(x_1) & f(x_2) & f(x_3) \end{bmatrix} \geq 0$$

for all $a \leq x_1 < x_2 < x_3 \leq b$. These functions are called convex with respect to (u_0, u_1).

Let Φ be a positive Radon measure on $[a, b]$; denote by $c(\Phi)$ the smallest closed interval which contains supp (Φ). Let e_x be the Dirac measure at x.

The following result is a reformulation of Theorem 1 and Corollary 1 of [8].

THEOREM 1. *Let $\alpha \in [a, b]$ and let Φ be a positive Radon measure on $[a, b]$ such that $\Phi(u_i) = u_i(\alpha)$, $i = 0,1$. Then:*

$$\alpha \in c(\Phi) \tag{1}$$

if α is an end point of $c(\Phi)$, then $\Phi = e_\alpha$. (2)

Let $f \in K(u_0, u_1)$. Then:

$$\Phi(f) \geq f(\alpha) \tag{3}$$

$\Phi(f) = f(\alpha)$ iff f coincides on $c(\Phi)$ with an u — affine function. (4)

Proof. If $\alpha \in \{a, b\}$, then $\Phi = e_\alpha$ (see [8, Corollary 1]). Let $\alpha \in (a, b)$. It is easy to deduce (1) and (2) from [8, Lemma 1]. Let $f \in K(u_0, u_1)$. In the

* *Department of Mathematics, Polytechnic Institute, Cluj-Napoca, 3400 Cluj-Napoca, Romania*

proof of Theorem 1 of [8] an u — affine function l is constructed such that $f \geqslant l$ and $f(\alpha) = l(\alpha)$. Then $\Phi(f) \geqslant \Phi(l) = l(\alpha) = f(\alpha)$.

Suppose that $\Phi(f) = f(\alpha)$. Let $g = f - l$. Then $g \geqslant 0$ and $\Phi(g) = 0$; this implies $g = 0$ on supp (Φ). Let $m = \inf(\text{supp}(\Phi))$, $M = \sup(\text{supp}(\Phi))$. We have $c(\Phi) = [m, M)$ and $g(m) = g(M) = 0$.

If $m = M$, then $\Phi = e_\alpha$. Suppose $m < M$.

Let $m \leqslant x \leqslant M$. Since $g \in K(u_0, u_1)$ we have

$$g(x) \begin{vmatrix} u_0(m) & u_0(M) \\ u_1(m) & u_1(M) \end{vmatrix} \leqslant 0.$$

It follows that $g(x) \leqslant 0$. But $g \geqslant 0$, and so $g = 0$ on $c(\Phi)$.

THEOREM 2 (see [8, Th. 2]). *Let L be a positive linear operator on $C[a, b]$ such that $Lu_0 = u_0$, $Lu_1 = u_1$. Let $f \in K(u_0, u_1)$ and $\alpha \in [a, b]$. Then:*

$Lf \geqslant f$ (7)

$Lf(\alpha) = f(\alpha)$ iff f coincides on $c(e_\alpha \circ L)$ with an (6)

u — affine function.

Proof. It suffices to apply Theorem 1 with $\Phi = e_\alpha \circ L$.

2. Let X be a compact convex set in a locally convex Hausdorff space over $\mathbf{R}$. Let Prob (X) be the space of all probability Radon measures on X. For $\mu \in$ Prob (X) let us denote $c(\mu) = cl$ (conv (supp μ)). Let $r(\mu)$ be the barycenter of μ.

THEOREM 3. *Let $\mu \in$ Prob (X), $b = r(\mu)$. Then:*

$b \in c(\mu)$ [6, *Proposition* 1.2] (7)

if b is an extreme point of $c(\mu)$, then (8)

$\mu = e_b$ [6, Proposition 1.4].

Let $f \in C(X)$ be a convex function. Then:

$\mu(f) \geqslant f(b)$ [6, p. 25] (9)

$\mu(f) = f(b)$ *iff f is affine on $c(\mu)$* (10)

[7, *Theorem* 1]

Let $L : C(X) \to B(X)$ be a positive linear operator such that $L1 = 1$. Let us denote $b_x = r(e_x \circ L)$. In particular, if $La = a$ for all affine continuous function a on X, then $b_x = x$ for all $x \in X$.

THEOREM 4. *Let $f \in C(X)$ be a convex function and $x \in X$. Then:*

$Lf(x) \geqslant f(b_x)$ (11)

$Lf(x) = f(b_x)$ iff f is affine on $c(e_x \circ L)$. (12)

Proof. Apply Theorem 3 with $\mu = e_x \circ L$.

3. Let now $\nu_{xn} \in$ Prob (X) for each $x \in X$ and $n \geqslant 1$. Let $P = \{p_{nj}\}_{n \geqslant 1, j \geqslant 1}$ be a lower triangular stochastic matrix, i.e., an infinite real matrix satisfying:

$p_{nj} \geqslant 0$ for all $n \geqslant 1$ and $j \geqslant 1$, $p_{nj} = 0$ whenever $j > n$ and $\sum_{j \geqslant 1} p_{nj} = 1$ for all $n \geqslant 1$.

Let $P_n : X^n \to X$, $P_n(x_1, \ldots, x_n) = \sum_{j=1}^{n} p_{nj} x_j$.

Let $\lambda_{xn} = \nu_{x1} \otimes \ldots \otimes \nu_{xn}$ be the product measure on X^n and let $\mu_{xn} = P_n(\lambda_{xn})$ be the induced measure on X.

Let $f \in \mathbf{C}(X)$, $x \in X$. Define

$$B_n f(x) = \int_X f d\mu_{xn} \tag{13}$$

Then $e_x \circ B_n = \mu_{xn}$.

Using standard results we deduce that the barycenter of μ_{xn} is $k_{xn} = P_n(r(\nu_{x1}), \ldots, r(\nu_{xn}))$ and $c(\mu_{xn}) = P_n\left(\prod_{j=1}^{n} c(\nu_{xj})\right)$. Thus we have:

COROLLARY 1. *Let $f \in \mathbf{C}(X)$ be a convex function and $x \in X$. Then:*

$$B_n f(x) \geqslant f(k_{xn}) \tag{14}$$

$$B_n f(x) = f(k_{xn}) \text{ iff } f \text{ is affine on } P_n\left(\prod_{j=1}^{n} c(\nu_{xj})\right). \tag{15}$$

Example 1. Let $\nu_x \in \text{Prob}(X)$ with barycenter x.
Let $\nu_{xn} = \nu_x$ for $n \geqslant 1$. Then the operators B_n given by (13) are the Bernstein-Schnabl operators (see [1], [4], [5]). We have $k_{xn} = x$ and $P_n\left(\prod_{j=1}^{n} c(\nu_{xj})\right) = c(\nu_x)$ for all $x \in X$ and $n \geqslant 1$.
Hence for the Bernstein-Schnabl operators the following statements hold for any convex function $f \in \mathbf{C}(X)$ and any $x \in X$:

$$B_n f(x) \geqslant f(x) \tag{16}$$

$$B_n f(x) = f(x) \text{ iff } f \text{ is affine on } c(\nu_x). \tag{17}$$

Example 2. Let $h_n : [0, 1], \to [0,1] n \geqslant 1$. Let

$$\nu_{xn} = (1 - h_n(x))e_0 + h_n(x)e_1, \quad x \in [0, 1], \quad n \geqslant 1.$$

Let $p_{nj} = 1/n$, $j = 1, \ldots, n$. Then the operators B_n given by (13) coincide with the operators constructed in [3]. In this case $k_{xn} = (1/n)\sum_{j=1}^{n} h_j(x)$.

Let $x \in [0, 1]$. Suppose that among the numbers $h_1(x), \ldots, h_n(x)$ z are equal to 0 and u are equal to 1 ($z, u \geqslant 0$, $z + u \leqslant n$). Then

$$P_n\left(\prod_{j=1}^{n} c(\nu_{xj})\right) = [u/n, 1 - (z/n)].$$

Consequently, for each convex function $f \in C[0, 1]$ we have

$$B_n f(x) \geqslant f\left((1/n) \sum_{j=1}^{n} h_j(x)\right) \tag{18}$$

the equality occurs iff f is affine on $[u/n, 1 - (z/n)]$. **(19)**

In particular, if $h_n(x) = x$, $x \in [0,1]$, $n \geqslant 1$, we obtain the classical Bernstein operators.

From (18) and (19) we deduce the following known results (see [2, p. 287]):

$B_n f \geqslant f$ for each convex function $f \in C[0, 1]$ **(20)**

Let $f \in C[0, 1]$ be convex and $x \in (0, 1)$. Then **(21)**

$B_n f(x) = f(x)$ iff f is affine on $[0, 1]$.

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DOBINSKI-TYPE FORMULA FOR BINOMIAL POLYNOMIALS

ALEXANDRU LUPAŞ*

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REZUMAT. — Formule de tip Dobinski pentru polinoame binomiale. Se dă o scurtă descriere a unor probleme de calcul umbral. Pentru detalii, vezi [1, 3].

1. In this section we give a brief description of relevant topics from the umbral calculus. For a more detailed treatment we refer the reader to the references [1]—[3].

Let Π be the algebra, over a field K of characteristics zero, of all polynomials in one variabile. By a *polynomial sequence* $p = (p_n)$, $p_n \in \Pi$, we shall always imply that $\deg(p_n) = n$, $n = 0, 1, \ldots$. A polynomial sequence (p_n) is said to be of *binomial type* if it satisfies the identities

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y), \quad n = 0, 1, \ldots .$$

If $e_k(x) = x^k$, $k = 0, 1, \ldots$, then a *delta operator* $Q: \Pi \to \Pi$ is a linear operator with the properties:

i) Q is shift-invariant, i.e., if $(E^a f)(x) = f(x+a)$, then $QE^a = E^a Q$ for all a in K.

ii) $(Qe_1)(x)$ is a nonzero constant.

Delta operators possess many of the properties of the derivative operator D. For instance $Qe_0 = 0$, and if f is a polynomial of degree n then $\deg(Qf) = n - 1$. Let Q be a delta operator. A polynomial sequence (p_n) is called the sequence of *basic polynomials* for Q if

1) $p_0(x) = 1$,

2) $p_n(0) = 0$, $n = 1, 2, \ldots$,

3) $Qp_n = np_{n-1}$.

In [1]—[3] it is proved the following:

A) Every delta operator has a unique sequence of basic polynomials;

B) If $p = (p_n)$ is a basic sequence for some delta operator Q, then it is a sequence of binomial type;

C) If $p = (p_n)$ is a sequence of polynomials of binomial type, then it is a basic sequence for some delta operator;

*University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania

D) A linear operator $Q:\Pi\to\Pi$ is a delta operator if and only if $Q=f(D)$ where f belongs to the ring of formal power series in one variable over K,

$$f(t)=\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k,$$

and $f(0)=0$, $f'(0)\neq 0$.

We note that to every formal power series $f(t)$ with $f(0)=0$, $f'(0)\neq 0$, there corresponds a unique inverse power series. If $Q=f(D)$ is a delta operator with basic polynomials (p_n), then

$$\sum_{k=0}^{\infty}p_k(x)\frac{t^k}{k!}=e^{xf^{-1}(t)}.$$

Let $p=(p_n)$ be a polynomial sequence; we define the linear operator $\tau_p:\Pi\to\Pi$ by

$$\tau_p e_k=p_k,\quad k=0,1,\ldots.$$

In other words, if

$$f(x)=\sum_{k=0}^{n}c_kx^k$$

then

$$(\tau_p f)(x)=\sum_{k=0}^{n}c_kp_k(x);$$

τ_p will be called the *umbral operator* corresponding to the sequence p. The *umbral composition* of two polynomial sequences $p=(p_n)$, $q=(q_n)$ is defined as the sequence $pq=(\tau_pq_n)$. An important result established by R. Mullin and Gian-Carlo Rota [1] assert that the umbral composition of two binomial sequences is laso a polynomial sequence of binomial type. More precisely, let $p=(p_n)$ be a binomial sequence with the delta operator $Q_1=f_1(D)$, and $q=(q_n)$ another binomial sequence with respect to the delta operator $Q_2=f_2(D)$. If $r_n=\tau_pq_n$, $n=0,1,\ldots$, then $r=(r_n)$ is the sequence of basic polynomials for the delta operator Q, where $Q=f_2(f_1(D))$.

One of the simplest delta operator is the *forward difference* $\Delta_a=E^a-I$, where $a\neq 0$, $(\Delta_af)(x)=f(x+a)-f(x)$. In this case $\Delta_a=h(D)$ with $h(t)=e^{at}-1$. The basic sequence for Δ_a is the *falling factorial sequence* $w_a=(w_n(\cdot\,;a))$, where

$$w_0(x;a)=1,\quad w_n(x:a)=\frac{1}{a^n}x(x-a)(x-2a)\ldots(x-\overline{n-1}a),\ n=1,2,\ldots \tag{1}$$

Let $p=(p_n)$ be a polynomial sequence of binomial type associated for the delta operator $Q=f(D)$,

$$p_n(x)=\sum_{k=0}^{n}c_{kn}x^k.$$

It follows that the sequence $b_a = (b_n)$, where

$$b_n(x) = b_n(x;\, a) = (\tau w_a p_n)(x) = \sum_{k=0}^{n} c_{kn} w_k(x;\, a),$$

is again a binomial sequence relative to $Q_0 = f(\Delta_a)$. The sequence $b_a = (b_n)$ is called the *a-difference analogue* of the sequence $p = (p_n)$.

2. The *exponential polynomials*, $t=(t_n)$, introduced by Steffensen and studied by Touchard, are basic polinomials for the delta operator $T = \ln\,(I + D)$. These polinomials are

$$t_n(x) = \sum_{k=0}^{n} S(n,\, k)x^k, \qquad n = 0,\, 1,\, \ldots,$$

where $S(n,\, k)$, $k = 0,\, 1,\, \ldots,$ denote the Stirling numbers of the second kind, defined by

$$x^n = \sum_{k=0}^{n} S(n,\, k) w_k(x;\, 1)$$

where we have used the notation (1). Let us observe that $S(n,\, k)$ is a divided difference, that is

$$S(n,\, k) = [0,\, 1,\, \ldots,\, k;\, e_n]$$

where

$$[x_0,\, x_1,\, \ldots,\, x_k;\, f] = \sum_{j=0}^{k} \frac{f(x_j)}{w'(x_j)},\quad w(x) = (x - x_0)\,\ldots\,(x - x_k).$$

Therefore

$$t_n(x) = \sum_{k=0}^{n} [0,\, 1,\, \ldots,\, k;\, e_n]x^k.$$

The following remarkable identity is known in literature [1] as *Dobinski formula*

$$t_n(x) = e^{-x} \sum_{k=0}^{\infty} \frac{k^n}{k!}\, x^k.$$

Our aim is to give a general Dobinski-type formula for an arbitrary polynomial sequence of binomial type.

THEOREM. *Let $p = (p_n)$ be a polynomial sequence of binomial type. If $b_a = (b_n)$ is the a-difference analogue of the sequence $p = (p_n)$, $(a \neq 0)$, then*

$$p_n(x) = e^{-x} \sum_{k=0}^{\infty} \frac{b_n(ak)}{k!}\, x^k = \sum_{k=0}^{n} [0,\, a,\, 2a,\, \ldots,\, ka;\, b_n]\,(ax)^k. \tag{2}$$

Proof. Let p be the basic sequence for the delta operator $Q = f(D)$. Further, let $g(t) = f(e^{at} - 1)$ and $b_a = (b_n)$ be the a-difference analogue of $p = (p_n)$. From Newton interpolation formula we have

$$b_n(x) = b_n(x\,;\,a) = \sum_{k=0}^{n} [0,\ a,\ 2a,\ \ldots,\ ka\,;\ b_n(.\,;\ a)]a^k w_k(x\,;\ a) \tag{3}$$

where $w_k(x\,;\,a)$ is defined in (1). Let $q = (q_n)$ be the basic sequence for the delta operator $Q_1 = f_1(D) = \frac{1}{a}\ln(I + D)$. Then $(\tau_q b_n)$ is the binomial sequence associated for the delta operator $Q = g(f_1(D)) = f(D)$. In other words $\tau_q b_n = p_n$. Further, $(\tau_q w_n(.\,;\,a))$ is the basic sequence for the delta operator $Q_2 = h(f_1(D)) = D$ that is $\tau_q w_n(.\,;\,a) = e_n$. Taking into account that τ_q is a linear operator, from (3) we find

$$p_n(x) = \sum_{k=0}^{n} [0,\ a,\ 2a,\ \ldots,\ ka\,;\ b_n(.\,;\ a)]\,(ax)^k. \tag{4}$$

Now

$$[0,\ a,\ \ldots,\ ka\,;\ b_n(.\,;\ a)]a^k = \begin{cases} \displaystyle\sum_{j=0}^{k} \frac{(-1)^{k-j} b_n(ja\,;\ a)}{(k-j)!\,j!}, & k \leqslant n \\ 0, & k > n \end{cases}$$

and (4) enables us to write

$$p_n(x) = \sum_{k=0}^{\infty} x^k \sum_{j=0}^{k} \frac{b_n(ja\,;\ a)}{j!} \cdot \frac{(-1)^{k-j}}{(k-j)!} = e^{-x} \sum_{k=0}^{\infty} \frac{b_n(ak\,;\ a)}{k!} x^k$$

which completes the proof of (2).

Remark. Taking into account that

$$w_j(ka\,;\ a) = w_j(k\,;\ 1) = \begin{cases} j!\dbinom{k}{j}, & j \leqslant k, \\ 0, & j > k, \end{cases}$$

we have

$$b_n(ak\,;\ a) = b_n(k\,;\ 1) = \sum_{j=0}^{\min(k,n)} j!\binom{k}{j} c_{jn}$$

$$p_n(x) = \sum_{j=0}^{n} c_{jn} x^j,$$

and

$$[0,\ a,\ 2a,\ \ldots,\ ka\,;\ b_n(.\,;\ a)]a^k = [0,\ 1,\ \ldots,\ k\,;\ b_n(.\,;\ 1)].$$

We next consider some examples of Dobinski-type formulas. Let us denote by $s(n, j)$, $j = 0, 1, \ldots$, the Stirling numbers of the first kind, that is

$$x(x-1)\ldots(x-n+1) = \sum_{j=0}^{n} s(n, j)x^j.$$

Likewise, t_n, $n = 0, 1, \ldots$, denote the exponential polynomials. Then

$$x(x-1)\ldots(x-n+1) = e^{-x}\sum_{k=1}^{\infty} a_{nk}\frac{x^k}{k!}$$

where

$$a_{nk} = \sum_{j=0}^{\min(k,n)} j!\binom{k}{j}s(n, j).$$

Further

$$\sum_{k=1}^{n} s(n, k)t_k(x) = e^{-x}\sum_{k=0}^{\infty}\frac{k(k-1)\ldots(k-n+1)}{k!}x^k =$$

$$= \sum_{k=0}^{n} [0, 1, \ldots, k; w_n(.; 1)]x^k$$

and

$$\sum_{k=1}^{n}\binom{n-1}{k-1} n^{n-k}t_k(x) = e^{-x}\sum_{k=1}^{\infty}\frac{k(n+k)^{n-1}}{k!}x^k.$$

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NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT USING SPLINE FUNCTIONS

G. MICULA* and H. AKCA**

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REZUMAT. — Rezolvarea numerică a ecuaţiilor diferenţiale cu argument modificat folosind funcţiile spline. În această lucrare se dă o metodă de rezolvare numerică a ecuaţiilor diferenţiale cu argument modificat cu ajutorul funcţiilor spline. Se arată că unele metode spline cunoscute la rezolvarea problemelor cu valori iniţiale obişnuite se pot extinde şi adapta şi la ecuaţiile diferenţiale cu argument modificat.

1. Introduction. A great number of dynamical processes can be described and investigated by differential equations with deviating argument. Such modellings are known, for instance, for the population dynamic, for the optimal control of a chemical reactor, for the behaviour of erytrocyte production by hormonal stimulation after a sudden loss of blood, for hereditary control problems etc. Numerous papers have been devoted to deal with such kind of problems [9] [31] [33]—[36] etc.

In recent years there has been a growing interest in the numerical treatment of differential equations with deviating argument, (see [1]—[16]). Because of the versatility of such equations in the mathematical modelling of processes in various application fields, especially in biology, chemistry, deviated argument differential equations provide the best and sometimes the only realistic simulation of observed phenomena.

In the numerical treatment of differential equations with deviating argument two essential difficulties occur. First, for the evaluation of the right-hand side of the differential equation, an approximation of the deviating argument function is used. Secondly, the solution routine has to pay attention to the jump-discontinuities of the solution in various derivatives, which are inherent for deviating argument differential equations.

First investigations have been made by Myshkis [28]. Stetter [34], Oberle and Pesch [31] suggested the use of Hermite interpolation for the approximation of the deviating argument and also numerical solutions which are based on Runge—Kutta—Fehlberg methods. Feldstein and Goodman [13] studied the propagation of discretization errors for deviating argument differential equations. Bock and Schlöder [9] developed a general approach to the numerical solution of such problems, and numerical results for a variable order, variable step Adams methods are given.

Recently, many authors (Fröhner [14]—[15], Bank, [3], Banks, Rosen and Ito [4]—[5], Lénard and Szekelyhidi [22], Ble-

* University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania
** Erciyes University, Faculty of Sciences, Dept. of Mathematics, 38039 Kayseri, Turkey

yer and Preuss [8], Micula [26], Kemper [21], Nikolova and Bainov [30], Wiggins [37] etc.) have proposed different methods to approximate the solution of the deviating argument differential equations by means of spline functions.

It seems that the spline approximation solutions for such kind of equations, as in the case of usual ordinary differential equations, possess some advantages over other methods.

In this paper we consider a spline approximation method for the numerical solution of differential equations with deviating argument. The purpose of the present study is to extend the results of [23], [24], [14]—[15] from the ordinary case to the deviating argument one. In the same manner we shall construct a spline approximating solution, and also we shall investigate the estimation of error and the convergence of the given procedure.

2. **Description of the spline method.** Let us consider the following initial value problem for differential equation with deviating argument

$$y'(t) = f(t, y(t), y(g(t))), \quad t \in [0, T]$$

$$y(t) = \varphi(t), \ t \in [\alpha, 0], \quad \alpha < 0 \tag{2.1}$$

where f: $[0, T] \times R^2 \to R$, $(t, u, v) \to f(t, u, v)$ is continuous in t, u, v, satisfies the Lipschitz condition

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leqslant L(|u_1 - u_2| + |v_1 - v_2|) \tag{2.2}$$

and $g \in C[0, T]$, $g(t) \leqslant t - a$, $a > 0$, $t \in [0, T]$. We suppose that $\varphi \in C^m[\alpha, 0]$, $m > 1$ and $\exists \beta > 1$ so that: $|v_1 - v_2| < \beta |f(t, u_1, v_1) - f(t, u_2, v_2)|$ These conditions assure the existence and uniqueness of a continuous solution y: $[\alpha, T] \to R$ of (2.1) (see [11], [32]).

For the qualitative behaviour of the solution y, in particular the presence of jump-discontinuities in the higher derivatives caused by the deviating function g, known as primary discontinuity the reader is referred to [6], [31]. Jump-discontinuity can occur in various derivatives of the solution even if f, g, φ are analytic in their arguments. Such jump-discontinuities are caused by the deviating g and propagate from the point ζ_0 as the order of the derivative increases. The jump-discontinuities are the points (ζ_i), which are the roots of the equations $g(\zeta_i) = \zeta_{i-1}$, $\zeta_0 = t_0$ is the jump-discontinuity of φ. For each derivative, the location of the jump-discontinuity ζ_i depends upon g and upon the discontinuities of the lower derivatives.

Since in this paper g does not depend on y (no state-dependent deviating) we shall consider the jump-discontinuities to be known for sufficiently high-order derivatives and they are disposed in the form

$$\zeta_0 < \zeta_1 < \ldots < \zeta_{k-1} < \zeta_k < \ldots < \zeta_M$$

Therefore, we shall construct a spline approximating function s: $[0, T] \to R$, which will be defined in each interval $[\zeta_{k-1}, \zeta_k]$ as a polynomial spline of degree m and continuity class C^{m-1} $(m \geqslant 1)$. For this construction we shall use successively the collocation method as in [23], [24].

Let us, consider the first interval $[\zeta_0, \zeta_1]$, which is $[0, \zeta_1]$, divided by a uniform partition defined by the knots

$$0 = t_0 < t_1 < \ldots < t_{k-1} < t_k < \ldots < t_N = \zeta_1,\ t_j = jh,\ h = \frac{\zeta_1}{N},$$

We shall denote the linear spaces of the polynomial spline function of degree m and continuity class C^{m-1} with the knots t_j by S_m. On the first interval $[t_0, t_1]$ the spline component is defined by

$$s_0(t) = \varphi(0) + \frac{\varphi'(0)}{1!}t + \ldots + \frac{\varphi^{(m-1)}_{(0)}}{(m-1)!}t^{m-1} + \frac{a_0}{m!}t^m,\ 0 \leqslant t \leqslant h \tag{2.3}$$

with the last coefficient undetermined. We now determine a_0 by requiring that s_0 should satisfy equation (2.1) for $t = t_1 = h$. This gives the equation

$$s_0'(t_1) = f(t_1, s_0, (t_1), s_0(g(t_1))) \tag{2.4}$$

to be solved for a_0.

Having determined the polynomial (2.3), on the next interval $[t_1, t_2]$ we define

$$s_0(t) = \sum_{j=0}^{m-1} \frac{s_0^{(j)}(t_1)}{j!}(t - t_1)^j + \frac{a_1}{m!}(t - t_1)^m, \quad t \in [t_1, t_2]$$

where $s_0^{(j)}(t_1)$, $0 \leqslant j \leqslant m-1$ are left-hand limits of derivatives as $t \to t_1$ of the segment of s defined in (2.3) on $[t_0, t_1]$ and a_1 is determined as to satisfy the equation

$$s_0'(t_2) = f(t_2, s_0(t_2), s_0(g(t_2)))$$

Continuing in this manner we obtain a spline function $s_0 : [\zeta_0, \zeta_1] \to R$ of degree m and class C^{m-1} which approximates the solution y of (2.1), and which satisfies the equations:

$$s_0'(t_{k+1}) = f(t_{k+1}, s_0(t_{k+1}), s_0(g(t_{k+1})))\quad k = 0, 1, \ldots, N-1.$$

If we consider now the interval $[\zeta_j, \zeta_{j+1}]$, $(j = \overline{0, M-1})$ which is also divided by a uniform partision with the points

$$t_k = t_0 + kh,\ k = 0, 1, \ldots, N,\ t_0 = \zeta_j,\ t_N = \zeta_{j+1},\ h = \frac{t_k - t_{k-1}}{N},$$

and if we denote by s, $s \in S_m$, the spline function approximating the solution of (2.1), then on the interval $[t_k, t_{k+1}]$ s is defined by

$$s(t) = \sum_{i=0}^{m-1} \frac{s^{(i)}(t_k)}{i!}(t - t_k)^i + \frac{a_k}{m!}(t - t_k)^m, \quad t_k \leqslant t \leqslant t_{k+1} \tag{2.5}$$

where $s^{(i)}(t_k)$, $0 \leq i \leq m-1$ are left-hand limits of the derivatives as $t \to t_k$ of the segment of s defined on $[t_{k-1}, t_k]$, and the parameter a_k is determined such that

$$s'(t_{k+1}) = f(t_{k+1}, s(t_{k+1}), s(g(t_{k+1}))), \tag{2.6}$$

This procedure yields a spline function of degree m and class C^{m-1} over the entire interval $[\zeta_j, \zeta_{j+1}]$ with the knots $\{t_k\}_{k=1}^{N}$.

In what follows, we shall show that for h sufficiently small, the parameter a_k, $0 \leq k \leq N$, can be uniquely determined from (2.6).

THEOREM 1. *If h is small enough and the functions f, φ, g satisfy the assumed conditions, then there exists a unique spline approximating solution of the problem (2.1) given by the above construction.*

Proof. It remains to be proved that a_k can be uniquely determined from (2.6). Replacing s given by (2.5) in (2.6) we obtain

$$a_k = \frac{(m-1)!}{h^{m-1}}\Bigg[f\Bigg(t_{k+1}, \; s(t_k) + \frac{h}{1!}s'(t_k) + \ldots + \frac{h^{m-1}}{(m-1)!}s^{(m-1)}(t_k) + \frac{h^m}{m!}a_k,$$

$$s(t_k) + \frac{s'(t_k)}{1!}(g(t_{k+1}) - t_k) + \ldots + \frac{s^{(m-1)}(t_k)}{(m-1)!}(g(t_{k+1}) - t_k)^{m-1} \tag{2.7}$$

$$+ \frac{a_k}{m!}(g(t_{k+1}) - t_k)^m\Bigg) - s'(t_k) - \ldots - \frac{s^{(m-1)}(t_k)}{(m-1)!}h^{m-2}\Bigg]$$

Because $g(t_{k+1}) \leq t_{k+1}$, and f satisfies (2.2), it is easy to see that the right-hand side of (2.7) is a contraction mapping if $h < \frac{m}{2L}$ and if $|g(t_{k+1}) - t_k| \leq h$. If $|g(t_{k+1}) - t_k| > h$. the proof is similar as in [38].

In order to make a connection between the above spline method and the known linear multistep methods, we present the following theorem which gives the relations holding between the values of a spline and its derivatives at the knots.

THEOREM 2 [23, p. 435]. *If $s \in \mathcal{S}_m$, then there exists a unique linear consistency relation between the quantities $s(t_k)$ and $s'(t_k)$, $k = 0, 1, \ldots, n-1$, namely,*

$$\sum_{k=0}^{m-1} a_k^{(m)} s(t_k) = h \sum_{k=0}^{m-1} b_k^{(m)} s'(t_k)$$

whose coefficients may be written as

$$a_k^{(m)} = (m-1)! \quad (Q_m(k) - Q_m(k+1))$$
$$b_k^{(m)} = (m-1)! \quad Q_{m+1}(k+1)$$

where

$$Q_{m+1}(t) = \frac{1}{m!}\sum_{i=0}^{m+1}(-1)^i\binom{m-1}{i}(t-i)_+^m$$

As in a ussual case, the values $s(t_k)$ of the spline are exactly the values furnished by the discrete multistep method described by the recurrence relation

$$\sum_{k=0}^{m-1} a_k^{(m)} y_{k-m+j+1} = h \sum_{k=0}^{m-1} b_k^{(m)} y'_{k-m+j+1}, \quad j = m-1, \ldots N \tag{2.8}$$

if the starting values $y_0 = s(t_0), \ldots, y_{m-1} = s(t_{m-1})$ are used. Since $s \in C^{m-1}$, we define its m^{th} derivative in the knots t_k by

$$s^{(m)}(t_k) = \frac{1}{2}\left[s^{(m)}\left(t_k - \frac{1}{2}h\right) + s^{(m)}\left(t_k + \frac{1}{2}h\right)\right], \quad k = \overline{1, N-1} \tag{2.9}$$

Our purpose now is to discuss the convergence of spline approximation to the exact solution as $h \to 0$. Let y, φ be the unique solutions of (2.1) and as usual we denote

$$y_k = y(t_k),\ y'_k = y'(t_k),\ \varphi_k = \varphi(t_k),\ \varphi'_k = \varphi'(t_k),$$
$$s_k = s(t_k),\ s'_k = s'(t_k),\ i = 1, 2, \ldots,\ t_k = kh$$

LEMMA 1. *If*

$$| s(t_k) - y(t_k) | < Kh^p,\ | s(g(t_k)) - y(g(t_k)) | < Kh^p$$

where K is a constant and

$$s'(t_k) = f(t_k, s(t_k), s(g(t_k)))$$

then there exists a constant K_1 such that

$$| s(t_k) - y(t_k) | < K_1 h^p,\ | s'(t_k) - y'(t_k) | < K_1 h^p$$

Proof. Applying Lipschitz condition (2.2) we have

$$| s'(t_k) - y'(t_k) | = | f(t_k, s(t_k), s(g(t_k))) - f(t_k, y(t_k), y(g(t_k))) | \leqslant$$
$$\leqslant L[\,|s(t_k) - y(t_k)| + |s(g(t_k)) - y(g(t_k)))|\,] \leqslant$$
$$\leqslant L[Kh^p + Kh^p] = 2\,LKh^p$$

We can take $K_1 = \{K, 2\,LK\}$

LEMMA 2. *Let $y \in C^{m+1}[0, T]$ and $s \in \mathcal{S}$ with knots t_k such that the following conditions hold.*

$$\begin{aligned} | s^{(r)}(t_k) - y^{(r)}(t_k) | &= 0(h^{pr}) & r &= 0, 1, \ldots, m-1 \\ | s^{(r)}(g(t_k)) - y^{(r)}(g(t_k)) &= 0(h^{pr}) & k &= 0, 1, \ldots, N-1 \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} | s^{(m)}(t) - y^{(m)}(t) | &= 0(h) & t_k < t < t_{k+1} \\ | s^{(m)}(g(t)) - y^{(m)}(g(t)) | &= 0(h) \end{aligned} \tag{2.11}$$

Under these assumptions we have

$$|s(t) - y(t)| = 0(h^p), \; s(g(t)) - y(g(t))| = 0(h^p) \tag{2.12}$$

where $p = \min_{r=0,\ldots,m} (r + p_r),\ p_m = 1$

So that

$$|s^{(m)}(t) - y^{(m)}(t)| = 0(h), \; |s^{(m)}g(t)) - y^{(m)}(g(t))| = 0(h). \tag{2.13}$$

The proof is similar as in [23].

The most important cases are $m = 2$ and $m = 3$ that means the quadratic and cubic spline approximations.

3. **Quadratic spline functions and the trapezoidal rule.** For $m = 2$, (2.8) gives

$$y_k - y_{k-1} = \frac{h}{2}[y'_k + y'_{k+1}] =$$

$$= \frac{h}{2}[f(t_k, y(t_k), y(g(t_k))) + f(t_{k-1}, y(t_{k-1}), y(g(t_{k-1})))] \tag{3.1}$$

This is a one step method which furnishes the same values in the knots as the quadratic spline s. The method (3.1) has a degree of exactnes two and y_0 is trivialy the only starting value needed.

THEOREM 3. *If* $f \in C^2([0, T] \times R^2)$, *then there exists a constant* K, *such that for any* h *small enough and* $t \in [0, T]$, *following inequalities hold*

$$|s(t) - y(t) < Kh^2, \; |s'(t) - y'(t)| < Kh^2, \; |s''(t) - y''(t)| < Kh$$

provided that s'' is calculated according to (2.9) for $m = 2$.

Proof. By Theorem 2, the values of the quadratic splines on the knots are the same as the values yielded by the rule (3.1), which is known to be a second order discrete method [18]. So a constant K_2 exists such that

$$|s(t_k) - y(t_k)| < K_2h^2$$

From the Lemma 1, it follows immediatly that (2.10) is satisfied taking $m = p_0 = p_1 = 2$. Expanding by Taylor's theorem $s'_{k+1} = s'(t_{k+1})$ and $y'_{k+1} := y'(t_{k+1})$, gives for any $t \in (t_k, t_{k+1})$

$$s'_{k+1} = s'_k + hs''(t), \; y'_{k+1} = y'_k + hy''(\xi), \; t_k < \xi < t_{k+1},$$

Therefore

$$h|s''(t) - y''(\xi)| \leqslant |s'_k - y'_k| + |s'_{k+1} - y'_{k+1}|.$$

By Lemma 1, $|\xi - t| < h$ and we may write

$$s''(t) = y''(t) + 0(h)$$

Applying the Lemma 2 for $m = p_0 = p_1 = 2$, it follows

$$|s(t) - y(t)| = 0(h^2)$$

Using Lemma 2 once again we get

$$|s'(t) - y'(t)| = 0(h^2)$$

The last inequality results directly from Lemma 1.

4. **Cubic spline functions and the Milne-Simpson method.** From the consistency relation for $m = 3$ we get

$$y_k - y_{k-2} = \frac{h}{3}(y'_k + 4y'_{k-1} + y'_{k-2}) = \frac{h}{3}(f(t_k, y_k, y(g_k)) +$$
$$+ 4f(t_{k-1}, y_{k-1}, y_{k-1}(g_{k-1})) + f(t_{k-2}, y_{k-2}, y(g_{k-2}))) \tag{4.1}$$

which is one way of expressing Simpson's Rule. On the basis of Theorem 2, Simpson's rule yields a discrete solution $s(t_k)$ coinciding with cubic spline provided $y_0 = \varphi(t_0)$ and $y_1 = s(t_0 + h)$ are taken as initial values. The method based on Simpson's rule is of fourth order, providing that the starting values are of the same order. We therefore begin by considering the error in the starting value $s(h) = s_3(h)$.

LEMMA 3. *Let $m = 3$. There exists a constant K such that*

$$|s(t_1) - y(t_1)| < K h^4$$

The proof is identical as in [23] for the ordinary case.

Because the starting values $s(t_0)$ and $s(t_1)$ have the errors $0(h^4)$, we may conclude that

$$|s(t_k) - y(t_k)| = 0(h^4), \quad |s'(t_k) - y'(t_k)| = 0(h^4) \tag{4.2}$$

LEMMA 4. [23] *Let $y \in C^4[0, T]$, and let t_k and $t_{k+1} = t_k + h$ be in $[0, T]$. Suppose P is the unique cubic polynomial that satisfies the Hermite interpolation conditions*

$$P(t_k) = y(t_k), \quad P'(t_k) = y'(t_k)$$
$$P(t_{k+1}) = y(t_{k+1}), \quad P'(t_{k+1}) = y'(t_{k+1})$$

Then there exists a constant K such that

$$|P'''(t_k) - y'''(t_k)| \leq Kh$$

THEOREM 4. *If s is the cubic spline function approximating the solution of problem (2.1) and $f \in C^3([0, T] \times R^2)$, then there exists a constant K, independent of h, such that for all h small enough and $t \in [0, T]$,*

$$|s(t) - y(t)| < Kh^4 \quad |s'(t) - y'(t)| < Kh^3$$
$$|s''(t) - y''(t)| < Kh^2 \quad |s'''(t) - y'''(t)| < Kh$$

provided $s'''(t_k)$ is given by (2.9) with $m = 3$.

Proof. Denote the cubic spline component over $[t_k, t_{k+1}]$ by

$$s(x) = b_k + c_k(t - t_k) + d_k(t - t_k)^2 + e_k(t - t_k)^3, \quad t_k \leqslant t \leqslant t_{k+1}$$

By the Hermite-interpolation conditions from Lemma 4 it follows

$$e_k = \frac{1}{h^3}(2s_k - h\,s'_k - 2s_{k+1} + h\,s'_{k+1})$$

and taking into account the relation (4.2) we have

$$e_k = \frac{1}{h^3}(2y_k + h\,y'_k - 2y_{k+1} + h\,y'_{k+1}) + 0(h) = P_3'''(t_k) + 0(h)$$

where P_3 is the unique cubic polynomial that interpolates the data y_k, y'_k, y_{k+1} and y'_{k+1} taken from y. Because of $s'''(t)' = 6\,e_k$, from Lemma 4 it follows

$$s'''(t) = P_3'''(t_k) + 0(h) = y'''(t_k) + 0(h) = y'''(t) + (t_k - t)y^{(4)}(\zeta) + 0(h)$$

By assumption, $|t_k - t| < h$, we obtain

$$s'''(t) = y'''(t) + 0(h)$$

Also the condition (2.11) is satisfied for $m = 3$. Because the step function s''' is constant on (t_k, t_{k+1}), we may write

$$y(t_{k+1}) = y_k + h\,y'_k + \frac{h^2}{2}y''_k + \frac{h^3}{6}y'''(\zeta), \quad t_k < \zeta < t_{k+1}$$

$$s(t_{k+1}) = s_k + h\,s'_k + \frac{h^2}{2}s'_k + \frac{h^3}{6}s'''(\zeta)$$

and it follows

$$|s(t_{k+1}) - y(t_{k+1})| \leqslant |s_k - y_k| + h|s'_k - y'_k| + \frac{h^2}{2}|s''_k - y''_k|$$

$$+ \frac{h^3}{6}|s'''(\zeta) - y'''(\zeta)|$$

which gives us

$$|s''_k - y''_k| = 0(h^2)$$

Also the conditions of Lemma 2 are satisfied with $m = 3$, $p_0 = 4$, $p_1 = 4$, $p_2 = 2$ and since $f \in C^3([0, T] \times R^2)$, it follows that $y \in C^4$. Assuming the role of s in the Lemma 2 and applying Lemma 2 with s, s' and s'' succesively all the assertions of Theorem 4 are fulfiled.

Exactly as in the case of ordinary differential equations the quadratic and cubic spline methods, considered above present several adventages over the standard classical methods, producing smooth, accurate and global approximations to the solution of (2.1) and its first derivatives. The step size h can be changed at any step, if it is necessary, without additionaled complications. Another advantage over the corresponding discrete variable methods is that the spline method need no starting value.

For the higher degree of spline, as in a usual initial value problems we have the following negative result.

5. **The problem of the stability for splines of degree $\geqslant 4$.** THEOREM 5. *The spline solutions s given by the above construction is divergent as $h \to 0$ for $m \geqslant 4$.* The proof is identicall as in [23], [25]. The divergency of the spline method presented above is coming from the too high smoothing of spline approximate solution, because s has the degree m and belongs to the C^{m-1}. As in a classical initial value problems these conditions can be relaxated. In what follows, we shall propose a method to approximate the solution of (2.1) by a deficient spline function of degree four and class of continuity C^2.

THEOREM 6 [32]. *Let (X, d_1), (Y, d_2) be two complete metric spaces and $f: X \times Y \to X \times Y$, $f = (f_1, f_2)$.* If

i) *There are numbers α, $\beta \in R$, $0 < \alpha < 1$, $\beta > 0$ such that*

$$d_1(f_1(x_1, y_1), f_1(x_2, y_2)) \leqslant \alpha\, d_1(x_1, x_2) + \beta\, d_2(y_1, y_2)$$

$$for\ (x_1, y_1),\ (x_2, y_2) \in X \times Y$$

ii) *There are γ, $\delta \in R$, γ, $\delta > 0$, $\dfrac{\gamma\beta}{1-\alpha} + \delta < 1$*

such that

$$d_2(f_2(x_1, y_1), f_2(x_2, y_2)) \leqslant \gamma d_1(x_1, x_2) + \delta d_2(y_1, y_2),$$

for (x_1, y_1), $(x_2, y_2) \in X \times Y$. Then f has a unique fixed point.

Let the spline approximating solution of (2.1) be on the interval $[\zeta_k, \zeta_{k+1}]$ $k = 0, 1, 2, \ldots, N$, which is divided by a partition

$$\zeta_k = t_0 < t_1 < \ldots < t_j < t_{j+1} < \ldots < t_N = \zeta_{k+1}$$

in the following form:

$$s(t) = s(t_k) + s'(t_k)\frac{(t-t_k)}{1!} + s''(t_k)\frac{(t-t_k)^2}{2!} + a_k\frac{(t-t_k)^3}{3!} + b_k\frac{(t-t_k)^4}{4!} \tag{5.1}$$

where $s(t_k)$, $s'(t_k)$, $s''(t_k)$ are known, and the parameters a_k and b_k are to be determined from the conditions

$$\begin{aligned} s'(t_{k+1}) &= f(t_{k+1}, s(t_{k+1}), s(g(t_{k+1}))) \\ s'\left(t_{k+\frac{1}{2}}\right) &= f\left(t_{k+\frac{1}{2}}, s\left(t_{k+\frac{1}{2}}\right), s\left(g\left(t_{k+\frac{1}{2}}\right)\right)\right) \end{aligned} \tag{5.2}$$

We shall prove that for h small enough, the coefficients a_k and b_k, $k = 0, 1, \ldots$ can be determined uniquely from system (5.2). Under these conditions it is clear that $s \in C^2[0, T]$.

Replacing s given by (5.1) in (5.2) we obtain

$$a_k = \frac{21}{h^2}\left(f\left(t_{k+1},\ s_k + \ldots + \frac{a_k}{3!}\cdot h^3 + \frac{b_k}{4!}h^4,\right.\right.$$

$$\left.\left. s_k + \ldots + \frac{a_k}{3!}(g_{k+1} - t_k)^3 + \frac{b_k}{4!}\left(g_{k+1} - t_k\right)^4\right) - s_k' - h s_k'' - \frac{b_k}{3!}h^3\right), \tag{5.3}$$

$$b_k = \frac{48}{h^3}\left(f\left(t_{k+\frac{1}{2}},\ s_k + \ldots + \frac{a_k}{3!}\,\frac{h^3}{8} + \frac{b_k}{4!}\,\frac{h^4}{16},\ s_k + \ldots + \frac{a_k}{3!}\left(g_{k+\frac{1}{2}} - t_k\right)^3 + \frac{b_k}{4!}\left(g_{k+\frac{1}{2}} - t_k\right)^4\right) - s'_k - \frac{h}{2}s''_k - \frac{a_k}{2!}\,\frac{h^2}{4}\right) \tag{5.4}$$

For brevity we write (5.3) and (5.4) in the form

$$a_k = F(a_k,\ b_k)$$

$$b_k = G(a_k,\ b_k)$$

Now we shall use the Theorem 6 with $X = Y = R$, $d_1 = d_2 = d$. For h small enough all the conditions of Theorem 6 are satisfied. So the system (5.2) in a_k and b_k has a unique solution for $0 \leqslant k \leqslant N$. Hence the approximating spline function is uniquely determinated.

Remark. Instead of the conditions (5.2) we may use the following conditions

$$\begin{aligned} s'(t_{k+1}) &= f(t_{k+1},\ s(t_{k+1}),\ s(g(t_{k+1}))) \\ s''(t_{k+1}) &= \frac{d}{dt} f(t_{k+1},\ s(t_{k+1}),\ s(g(t_{k+1}))) \end{aligned} \tag{5.5}$$

Using these conditions there exists also a consistency relations for the spline functions with deficiency. For instance, if the spline function is of degree four and has a deficiency two, the consistency relation is

$$s_{k+1} - s_k = \frac{h}{2}(s'_{k+1} + s'_k) + \frac{h^2}{12}(s''_k - s''_{k+1}), \quad k = 0,1,\ldots$$

The corresponding multistep method

$$y_{k+1} - y_k = \frac{h}{2}(y'_k + y'_{k+1}) + \frac{h^2}{12}(y''_k - y''_{k+1})$$

has the order of exactness 4, if the starting vlaues are suitably chosen the truncation error is $\frac{h}{720} y^{(5)}(t_n) + 0(h^6)$. (see [14]).

Therefore, we conclude that if s is the spline approximating function for the solution of problem (2.1) we have

$$|s_k(t) - y(t_k)| = 0(h^4)$$

and taking into account the Lemma 1, it follows

$$|s'(t_k) - y'(t_k)| = 0(h^4)$$

In a similar manner as in previous considerations, the following theorem holds.

THEOREM 7. *If $f \in C^4([0, T] \times R^2)$ and s is the spline function of the fourth-degree and deficiency two, approximating the solution y of the problem (2.1), then there exists a constant K such that, for h sufficiently small and $t \in [0, T]$,*

$$|s^{(j)}(t) - y^{(j)}(t)| < K h^{4-j}$$

provided that $s^{(4)}(t_k)$, $s^{(3)}(t_k)$ is calculated by (2.9) for $m = 3.4.$

Remark. Theorem 7 suggests the possibilities to approximate the solution of problem (2.1) by spline function of degree m and deficiency $k(k \leqslant m)$.

6. Numerical exemple. Let us consider the following non-constant deviating argument differential equation:

$$y'(t) = \frac{t-1}{t} y(t) \cdot y(t - \ln t - 1), \quad t \in [1, 6]$$

$$y(t) = 1, \quad 0 \leqslant t \leqslant 1$$

already studied in [6] and [35].

The jump points are given by $\xi_1 = 3.1461932$ and $\xi_2 = 5.9254498$, etc. and we have applied the spline method in $[1, \xi_1]$. The exact solution of this problem in the interval $[1, \xi_2]$ is:

$$y(t) = \begin{cases} \exp(t - \ln t - 1) & 1 \leqslant t \leqslant \xi_1 \\ \exp\left[1 + \int\limits_{\xi_1}^{t} \frac{x-1}{x} e^{(x - \ln(x(x - \ln x - 1)) - 2)} dx\right], & \xi_1 \leqslant t \leqslant \xi_2 \end{cases}$$

Using the cubic spline function s on the interval $[1, \xi_1]$ we have, obtained the following numerical result.

	$h = 0,05$		$h = 0,02$	
k	a_k	$\lvert y(t_k) - s(t_k) \rvert$	a_k	$\lvert y(t_k) - s(t_k) \rvert$
0	-2.45029	0	-7.15681×10^{-1}	0
1	-2.67188×10^{-1}	1.26076×10^{-3}	-2.25184×10^{-1}	2.92301×10^{-4}
2	-7.86021×10^{-1}	1.56188×10^{-3}	-2.84021×10^{-1}	$3.87549 \times 10^{-}$[illegible]
3	-6.30613×10^{-1}	6.16906×10^{-3}	-2.01914×10^{-1}	$5.57828 \times 10^{-}$[illegible]
4	-5.29244×10^{-1}	4.37117×10^{-3}	-3.83506	$9.59849 \times 10^{-}$[illegible]
5	-1.08463	6.43706×10^{-3}	-9.12136	$1.23208 \times 10^{-}$[illegible]
6	-1.91917	9.35316×10^{-3}	-7.45056	$9.52983 \times 10^{-}$[illegible]
7	-3.08169	1.11089×10^{-2}	-2.07414	9.3[illegible]979 $\times 10^{-}$[illegible]
8	-5.46496	1.06704×10^{-2}	-5.28711	7.[illegible]5371 $\times 10^{-}$[illegible]
9	-8.30841	6.06227×10^{-3}	-1.51832	$2.20776 \times 10^{-}$[illegible]
10	-8.29547×10^{-1}	1.06161×10^{-2}	-9.26127	8.[illegible]7340 $\times 10^{-4}$

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DECISIVITY IN PROPOSITIONAL LOGIC

N. BOTH*

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REZUMAT — **Decisivitate în logica propozițiilor.** Se definește noțiunea de subformulă $A(F)$-decisivă a unei formule propoziționale de n variabile. Se demonstrează un criteriu de decisivitate și se indică aplicabilitatea lui la rezolvarea ecuațiilor booleene.

Denote with P_x the set of n-ary bivalent propositional formulae of n (effective or fictitious) arguments $x_1 \ldots, x_n$, and $x = (x_1, \ldots, x_n)$. ([4]).

If φ is a subformula of $\Phi \in P_x$ then we write $\Phi_\varphi = \Phi(x_1, \ldots, x_n, \varphi)$ and $S_\varphi^\pi \Phi_\varphi(x_1, \ldots, x_n, \varphi)$ represents the formula obtained from Φ by substitution of the part φ with the formula $\pi \in P_x$, that is:

$$S_\varphi^\pi \Phi_\varphi(x_1, \ldots, x_n, \varphi) = \Phi_\pi(x_1, \ldots, x_n, \pi).$$

Let Φ_φ^x be the set of formulae

$$\{\Phi_\pi(x_1, \ldots, x_n, \pi) \mid \pi \in P_x\}$$

and τ (K respctively) the set of true (false) formulae. We may consider that $\tau, K \in P_x$.

As $\pi \equiv \theta \Rightarrow S_\varphi^\pi \Phi \equiv S_\varphi^\theta \Phi$, instead of P_x, in the definition of the set Φ_φ^x, we may take a system of distinct representatives in $P_x/\equiv$.

DEFINITION The subformula φ of the formula Φ is called *A-decisive* (*F-de-cisive*) if:

$$\Phi_\varphi^x \cap \tau \neq \emptyset (\Phi_\varphi^x \cap K \neq \emptyset \text{ respectively}).$$

Examples. 1. If Φ is a tautology (a contradiction) then each of its parts are A (F)-decisive.

2. If $\Phi = \Phi' \vee \varphi$ ($\Phi = \Phi' \wedge \varphi$) then φ is A (F)-decisive.

Remark 1. The part φ is A-decisive in Φ if and only if is F-decisive in $\bar{\Phi}$. In fact, if $\Phi_\pi \in \Phi_\varphi^x \cap \tau \neq \emptyset$, that is $\Phi_\pi \in \Phi_\varphi^x$ and $\Phi_\pi \in \tau$, then $\bar{\Phi}_\pi \in \bar{\Phi}_\varphi^x$ and $\bar{\Phi}_\pi \in K$, therefore $\bar{\Phi}_\varphi^x \cap K \neq \emptyset$. The second part of Remark 1 follows analogously.

The formula $\Phi_X = S_\varphi^X \Phi_\varphi(x_1, \ldots, x_n, \varphi) = \Phi_X(x_1, \ldots, x_n, X)$, considered as a propositional formula of $n+1$ bivalent arguments, will be called the φ-extension of the formula Φ.

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* University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania

Next we use the following notations:

$F_{\Phi_X} = \{\alpha \in V^{n+1} \mid \Phi_X(\alpha) = 0\}$

$A_{\Phi_X} = \{\alpha \in V^{n+1} \mid \Phi_X(\alpha) = 1\} = V^{n+1} - F_{\Phi_X}$

$D^*_{n+1} = \{(\alpha, \beta) \in V^{n+1} \times V^{n+1} \mid \alpha_i = \beta_i, \ \alpha_{n+1} \neq \beta_{n+1}, \ i = \overline{1,n}\}$,

where $V = \{0, 1\}$, $\alpha = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$, $\beta = (\beta_1, \ldots, \beta_n, \beta_{n+1})$.

Observe that $D^*_{n+1} = \{(\alpha, \alpha^-) \mid \alpha^- = (\alpha_1, \ldots, \alpha_n, \bar{\alpha}_{n+1})\}$

THEOREM 1. *The part (subformula) φ is A-decisive in Φ if and only if:*

$$D^*_{n+1} \cap F^2_{\Phi_X} = \emptyset.$$

Proof. Suppose that $D^*_{n+1} \cap F^2_{\Phi_X} \neq \emptyset$; this means that there exist α, α^- in V^{n+1} such that $\Phi_X(\alpha) = 0 = \Phi_X(\alpha^-)$, where $\bar{0} = 1$ and $\bar{1} = 0$. Therefore, for anyone $\pi(x_1, \ldots, x_n)$ we have

$$\Phi_\pi(\alpha_1, \ldots, \alpha_n, \pi(\alpha_1, \ldots, \alpha_n)) = 0,$$

and so, $\Phi^x_\varphi \cap \tau = \emptyset$, that is, φ is not A-decisive in Φ.

Conversely, if $D^*_{n+1} \cap F^2_{\Phi_X} = \emptyset$ then there is no α, $\alpha^- \in V^{n+1}$ with the property: $\Phi_X(\alpha) = \Phi_X(\alpha^-) = 0$.
This implies that if $\Phi_X(\alpha) = 0$ then $\Phi_X(\alpha^-) = 1$. So, we may define a formula $\pi \in P_X$, with the only condition:

$$\alpha = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) \in F_{\Phi_X} \Rightarrow \pi(\alpha_1, \ldots, \alpha_n) = \bar{\alpha}_{n+1}.$$

The construction of π, generally, is not unique. So, we obtain

$$\Phi(\alpha^-) = \Phi(\alpha_1, \ldots \alpha_n, \bar{\alpha}_{n+1}) = 1, \text{ where } \bar{\alpha}_{n+1} = \pi(\alpha_1, \ldots, \alpha_n)$$

that is,

$\Phi_\pi(\alpha_1, \ldots, \alpha_n, \pi(\alpha_1, \ldots, \alpha_n)) = 1$ for each $(\alpha_1, \ldots, \alpha_n)$ for which $(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) \in F_{\Phi_X}$.

Now, suppose a $(\beta_1, \ldots, \beta_n) \in V^n$ such that

$$\beta = (\beta_1, \ldots, \beta_n, \beta_{n+1}) \in A_{\Phi_X}, \text{ that is } \Phi(\beta_1, \ldots, \beta_n, \beta_{n+1}) = 1.$$

If also $\Phi(\beta_1, \ldots, \beta_n, \bar{\beta}_{n+1}) = 1$ then for anyone $\pi(x_1, \ldots, x_n)$ we have $\Phi_\pi(\beta_1, \ldots, \beta_n, \pi(\beta_1, \ldots, \beta_n)) = 1$. If, on the contrary, $\Phi(\beta_1, \ldots, \beta_n, \bar{\beta}_{n+1}) = 0$, that is $\beta^- = (\beta_1, \ldots, \beta_n, \bar{\beta}_{n+1}) \in F_{\Phi_X}$ then, by the construction of π, it results that $\pi(\beta_1, \ldots, \beta_n) = \bar{\bar{\beta}}_{n+1} = \beta_{n+1}$, and so $\Phi_\pi(\beta_1, \ldots, \beta_n, \pi(\beta_1 \ldots, \beta_n)) = \Phi(\beta_1, \ldots, \beta_n, \beta_{n+1}) = 1$. Hence $\Phi_x \in \Phi^x_\varphi \cap \tau \neq \emptyset$, what means that φ is A-decisive in Φ.

Remark 2. Generally, if φ is A-decisive in Φ then $|F_{\Phi_X}| \leqslant 2^n$. If $|F_{\Phi_X}| = 2^n$ then the construction of the formula $\pi(x_1, \ldots, x_n)$ is unique; if moreover, $\alpha = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) \in F_{\Phi_X} \Rightarrow \alpha_{n+1} = 0$ then π is a tautology.

Example 3. Consider $\Phi = \Phi(x_1, x_2) = x_1 \wedge x_2 \vee (x_1 \supset x_2)$, $x_1 \supset x_2 = \varphi$, $\Phi_\varphi(x_1, x_2, \varphi) = x_1 \wedge x_2 \vee \varphi(x_1, x_2)$. Verify if φ is A-decisive in Φ and, in the affirmative, construct $\pi(x_1, x_2)$ such that $\Phi_\pi \in \tau$. For the φ-extension $\Phi_X = x_1 \wedge x_2 \vee X$ we have

$$F_{\Phi_X} = \{(1, 0, 0), (0, 1, 0), (0, 0, 0)\}.$$

Since $(1, 0) \neq (0, 1) \neq (0, 0) \neq (1, 0)$, hence φ is A-decisive in Φ. As in the Proof of the Theorem 1, define $\pi(x_1, x_2)$ by:

$$\pi(1, 0) = \pi(0, 1) = \pi(0, 0) = 1 \text{ and } \pi(1, 1) = 0$$

therefore $\pi(x_1, x_2) = x_1 \wedge \bar{x}_2 \vee \bar{x}_1 \wedge x_2 \vee \bar{x}_1 \wedge \bar{x}_2 \equiv \bar{x}_1 \vee \bar{x}_2$. One constate that $\Phi_\pi(x_1, x_2) = x_1 \wedge x_2 \vee \bar{x}_1 \wedge \bar{x}_2$ is a tautology.

Remark 3. The construction of the formula π in the preceding example is not unique; there is a second case, $\pi(1, 1) = 1$, in which π becomes a tautology. (See also the last assertion of Remark 2).

THEOREM 2. *The part φ is F-decisive in Φ if and only if*

$$D^*_{n+1} \cap A^2_{\Phi_X} = \emptyset.$$

Proof. The part φ is F-decisive in Φ if and only if φ is A-decisive in $\bar{\Phi}$ (see Remark 1). Apply Theorem 1 and obtain further that φ is F- decisive in $\bar{\Phi}$ if and only if:

$$D^*_{n+1} \cap F^2_{\bar{\Phi}_X} = \emptyset.$$

But $F_{\bar{\Phi}} = A_\Phi$, and so Theorem 2 follows.

Remark 4. Starting from the Remark 2, we may formulate a necessary condition of decisivity:

$$\varphi \text{ is } A(F)\text{-decisive in } \Phi \Rightarrow \text{Ab } \Phi_X \leqslant \frac{1}{2} \left(\text{Ab } \Phi_X \geqslant \frac{1}{2}\right),$$

where

$$\text{Ab } \Phi_X = \frac{|F_{\Phi_X}|}{2^{n+1}} \text{ (see [1])}.$$

Also, from the abve we have:

$$\text{Ab } \Phi_X = \frac{1}{2^{n+1}} \Rightarrow \varphi \text{ is } A\text{-decisive in } \Phi.$$

A n a p p l i c a t i o n. Consider the boolean equation

$$\Psi(p_1, \ldots, p_n, x) = \Lambda(p_1, \ldots, p_n)$$

where p_i, $i = \overline{1, n}$, are boolean parameters. This has an equivalent formulation: $\Psi \vee \bar{\Lambda} = 1$, or, generally,

$$\Phi(p_1, \ldots p_n, x) = 1.$$

The existence of the solution $x(p_1, \ldots, p_n)$ is given by Theorem 1; the uniqueness is given by Remark 2. The solution herself may be obtained by the construction indicated in the proof of Theorem 1.

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GENERALIZED T-RECURRENT SEMI-SYMMETRIC CONNECTIONS

P. ENGHIŞ*

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REZUMAT — Conexiuni semi-simetrice T-recurente. În lucrare se defineşte T-recurenţa generalizată a spaţiilor A_n cu conexiune semi-simetrică prin relaţia (4). Se stabilesc relaţiile: (6), (8) şi (11) pe care le verifică tensorul de T-recurenţă generalizată. Se arată (propoziţia 1) că din (4) rezultă (5), iar în propoziţia 3 se stabileşte reciproca propoziţiei 1. În spaţiile A_n cu E-conexiune semi-simetrică T-recurentă generalizată se pune în evidenţă sistemul (14) pe care-l verifică covectorii ψ_r şi a_r.

Let A_n be a space with affine connection. In a coordinate system, we denote by Γ^i_{jk} the components of the affine connection, with $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$ the components of the torsion tensor of the connection Γ, and with $T_k = T^i_{ik}$ the components of the torsion vector (the Vrînceanu's vector).

The connection Γ is called semi-symmetric (Schouten) if there is a covariant vector field so that

$$T^i_{jk} = \delta^i_j S_k - \delta^i_k S_j \tag{1}$$

Contracting (1) in i and j and taking it into account we have [2]:

$$T^i_{jk} = \frac{1}{n-1}(\delta^i_j T_k - \delta^i_k T_j) \tag{2}$$

relation which characterizes the semi-symmetric connections.

In [4] we defined the A_n spaces of generalized recurrent torsion, as spaces in which exists a vector ψ_r and a tensor Q^i_{jkr} so that

$$T^i_{jk,r} = \psi_r T^i_{jk} + Q^i_{jkr} \tag{3}$$

where comma denotes the covariant derivation with respect to Γ.

In present paper, let us consider the tensor Q^i_{jkr} degenerate of first order and rank one.

In this case relation (3) will write

$$T^i_{jk,r} = \psi_r T^i_{jk} + a_r Q^i_{jk} \tag{4}$$

* University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania

DEFINITION. The A_n spaces for which there exists the covectors ψ_r, a_r and a tensor Q^i_{jk} so that (4) take place, are called generalized T-recurrent.
Remark: The tensor Q^i_{jk} from (4) is anti-symmetric in j and k.

Contracting (4) in i and j we have

$$T_{k,r} = \psi_r T_k + a_r Q_k \tag{5}$$

where $\Phi_k = \Phi^i_{ik}$.

PROPOSITION 1. *In a generalized T-recurrent A_n space, the Vrînceanu's vector is also generalized T-recurrent with the same T-recurrency vectors ψ and a, and with $Q_k = Q^i_{ik}$.*

If the generalized T-recurent A_n space is with semi-symmetric connection from (2), (4) and (5) it follows

$$Q^i_{jk} = \frac{1}{n-1}(\delta^i_j Q_k - \delta^i_k Q_j) \tag{6}$$

relation of the same kind as (2), therefore:

PROPOSITION 2. *In a generalized T-recurrent A_n space $(n > 1)$ with semi-symmetric connection, the generalized T-recurrency tensor Q^i_{jk} and his contracted Q_k satisfy the relation (6).*

From relations (2) it follows that in an A_n space with semi-symmetric connection, the torsion of the space is complete determined by the torsion vector, therefore in such spaces, proposition 1 has a converse.

Indeed, if in (2) we apply the covariant derivation and take account of (5) it follows (4) with Q^i_{jk} given by (6), therefore:

PROPOSITION 3. *The A_n spaces $n > 1$ with semi-symmetric connection and Vrînceanu's vector, generalized T-recurrent are generalized T-recurrent with the same vectors of generalized T-recurrency ψ_r and a_r and the tensor Q^i_{jk} given by (6).*

Writing the relation of S. Golab [5] for spaces with semi-symmetric connection

$$T^i_{sj} T^s_{kh} + T^i_{sk} T^s_{hj} + T^i_{sh} T^s_{jk} = 0 \tag{7}$$

derivating it covariantly and taking it and (4) into account, we have:

$$Q^i_{sj} T^s_{kh} + Q^s_{kh} T^i_{sj} + Q^i_{sk} T^s_{hj} + Q^s_{hj} T^i_{sk} + Q^i_{sh} T^s_{jk} + Q^s_{jk} T^i_{sh} = 0 \tag{8}$$

we therefore have:

PROPOSITION 4. *In a generalized T-recurrent A_n space, $n > 1$, with semi-symmetric connection (8) take place.*

In an A_n space with semi-symmetric connection, we have [2]:

$$T^i_{jk} T_i = 0 \tag{9}$$

Transvecting in (6) by Q_i it follows

$$Q^i_{jk}Q_i = 0 \tag{10}$$

relation of the same kind as (9), therefore

PROPOSITION 5. *In a generalized T-recurrent A_n space, $n > 1$, with semi-symmetric connection, relation (10) take place.*

Derivating covariantly (9) and taking (4) and (5) into account, we have

$$Q^i_{jk}T_i + T^i_{jk}Q_i = 0 \tag{11}$$

therefore:

PROPOSITION 6. *In an A_n space, $n > 1$, with semi-symmetric connection, between the torsion tensor and the generalized T-recurrency tensor (11) take place.*

If the connection of A_n space is a semi-symmetric E-connection [2]

$$T_{i,j} - T_{j,i} = 0 \tag{12}$$

from vanish of the torsion divergence

$$T^i_{jk,i} = 0 \tag{13}$$

taking (4) into account we have:

$$\psi_i T^i_{jk} + a_i Q^i_{jk} = 0 \tag{14}$$

and therefore

PROPOSITION 7. *The vectors ψ_i and a_i of generalized T-recurency of an A_n space, $n > 1$, with semi-symmetric E-connection, verify the homogeneous system (14).*

In a previous paper [3] we pointed out an application of the semi-symmetric connections in theoretical physics, constructing a remarcable linear connection, the spin connection by which we gave a geometrical interpretation of the spin tensor. The considerations from present paper one extend naturally over these connections too.

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A NOTE ON FUZZY INFORMATION THEORY

D. DUMITRESCU*

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REZUMAT — Notă asupra teoriei informaţiei nuanţate. În lucrări anterioare autorul a introdus noţiunea de entropie a unei partiţii nuanţate. Folosind această noţiune în lucrarea de faţă se definesc informaţia relativă a două partiţii nuanţate şi ciştigul de informaţie. Rezultatele obţinute sînt analoge celor din teoria clasică a informaţiei; putind servi la fundamentarea unei teorii a informaţiei ce admite evenimente imprecise.

Using the concept of entropy introduced in the papers [2], [3], we intend to give some preliminars of a Fuzzy Information Theory.

A fuzzy set A on the universe X is a function $A : X \to [0, 1]$. If A and B are fuzzy sets on X, we define

$$(A \bigcup B)(x) = \min (A(x) + B(x), 1), \qquad (x \in X). \tag{1.a}$$

$$(A \bigcap B)(x) = \max (A(x) + B(x) - 1, 0), \qquad (x \in X). \tag{1.b}$$

$$(A \cdot B)(x) = A(x) \cdot B(x), \qquad (x \in X). \tag{1.c}$$

The family $A_1, \ldots, A_n$, $n \geqslant 2$, of fuzzy sets is said to be *disjoint* [1] iff:

$$\left(\bigcup_{i=1}^{j} A_i\right) \bigcap A_{j+1} = \emptyset, \quad j = 1, \ldots, n - 1. \tag{2}$$

$(A_n)_{n \in N}$ is a *disjoint sequence* of fuzzy sets iff $A_1, \ldots, A_n$ is a disjoint family for every $n \geqslant 2$.

Let A be a fuzzy set. A finite fuzzy partition of A is a family $A_1, \ldots, A_n$ of disjoint fuzzy sets such that $\bigcup_{i=1}^{n} A_i = A$. It is easy to see [1] that this definition is equivalent with the condition $\sum_{i=1}^{n} A_i(x) = A(x)$, for every x from X.

Let $P = \{A_1, \ldots, A_n\}$, $Q = \{B_1, \ldots, B_m\}$ two fuzzy partitions of A. Q is said to be a *refinement* of P (and we write $P \prec Q$) iff every atom of P is an union of some atoms of Q. The *common refinement* $P \vee Q$ of P and Q is defined by

$$P \vee Q = \{A_i \cdot B_j \mid 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m\}, \tag{3}$$

with lexicographic ordering.

* University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania

Let F be a σ-algebra of fuzzy sets [1]. A *fuzzy measure* [1] m over F is a function $m : F \to \mathbf{R}$ such that:

(i) $m(A) \geqslant 0$, $A \in F$, $m(\emptyset) = 0$.

(ii) If $(A_n)_{n \in \mathbf{N}}$ is a disjoint sequence of fuzzy sets and $A_n \in F$, then $m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n)$.

A triple (X, F, m) is a called a *fuzzy measure space*. If $m(X) = 1$ this triple may be called a fuzzy probability space. In this case the elements of F are called fuzzy events and $m(A)$ may be interpreted as the probability of the fuzzy event A.

The conditional probability of the fuzzy event A given the fuzzy event B is

$$m(A \mid B) = \frac{m(AB)}{m(B)} \tag{4}$$

The fuzzy events A and B are *independent* iff

$$m(AB) = m(A)m(B).$$

Let (X, F, m) be a fuzzy probability space. A system $\mathcal{A} = \{A_1, \ldots, A_n\}$, $A_i \in F$, is a *complete system of fuzzy events* iff $\mathcal{A}$ is a fuzzy partition of X. Every fuzzy partition P for which $A_i \in P$ entails $A_i \in F$ is said to be *admissible*. An admissible fuzzy partition if therefore a complete system of fuzzy events.

The entropy of the admissible fuzzy partition $P = \{A_1, \ldots, A_n\}$ is defined [2] by

$$H(P) = -\sum_{i=1}^{n} m(A_i) \log_2 m(A_i). \tag{5}$$

with the condition $0 \log_2 0 = 0$.

If we consider the fuzzy events $A_1, \ldots, A_n$ as the outcomes of an experiment E then $H(P)$ may be interpreted as the information gained (or uncertainty removed) by performing the experiment E.

The conditional entropy of the fuzzy partition $P = \{A_1, \ldots, A_n\}$ given the fuzzy set (event) D is defined by

$$H(P \mid D) = -\sum_{i=1}^{n} m(A_i \mid D) \log_2 m(A_i \mid D).$$

The conditional entropy of P given the fuzzy partition $Q = \{B_1, \ldots, B_p\}$ is

$$H(P \mid Q) = \sum_{j=1}^{p} m(B_j) H(P \mid B_j) \tag{6.a}$$

$$= -\sum_{i,j} m(A_i B_j) \log_2 \frac{m(A_i B_j)}{m(B_j)}. \tag{6.b}$$

$H(P|Q)$ measures the uncertainty about the outcome of the experiment associated with P given the outcome of the experiment associated with Q.

The properties of the introduced fuzzy entropy are summarized by the next

PROPOSITION 1. *If P, Q, R are finite admissible fuzzy partitions of X, then*

(i) $H(P \vee Q | R) = H(P | R) + H(Q | P \vee R)$

(ii) $H(P \vee Q) = H(P) + H(Q | P)$.

(iii) $P \prec Q \Rightarrow H(P | R) \leqslant H(Q | R), \quad \forall R.$

(iv) $P \prec Q \Rightarrow H(P) \leqslant H(Q)$.

(v) $Q \prec R \Rightarrow H(P | Q) \geqslant H(P | R), \quad \forall P.$

(vi) $H(P) \geqslant H(P | R), \quad \forall R.$

(vii) $H(P \vee Q | R) \leqslant H(P | R) + H(Q | R)$.

(viii) $H(P \vee Q) \leqslant H(P) + H(Q)$.

For the proof see [2] and [4].

The *relative information* given by Q about P is defined as

$$I(P, Q) = H(P) - H(P|Q). \tag{7}$$

If P and Q are independent, i.e. $m(A_iB_j) = m(A_i)m(B_j), \ \forall A_i \in P, \ B_j \in Q$ thus

$$I(P, Q) = H(P) - H(P | Q) = H(P) - H(P) = 0.$$

If $P^0 = \{X, \emptyset\}$, we have

$$I(P, P^0) = H(P) - H(P | P^0) = H(P) - H(P) = 0.$$

Using the inequality (vi) from Proposition 1 we have

$$I(P, Q) \geqslant 0,$$

and using (ii) we may write

$$I(P, Q) = I(Q, P) = H(P) + H(Q) - H(P \vee Q). \tag{8}$$

The *gain of information* $H(P, Q)$ resulting by the replacement of the (a priori) fuzzy partition $P = \{A_1, \ldots, A_n\}$ by the (a posteriori) fuzzy partition $Q = \{B_1, \ldots, B_n\}$ is defined as

$$H(Q, P) = -\sum_{i=1}^{n} m(B_i) \log_2 \frac{m(A_i)}{m(B_i)}. \tag{9}$$

PROPOSITION 2. *If P and Q are admissible fuzzy partitions of X then*

$$H(P, Q) \geqslant 0.$$

Proof. Since $\sum_{i=1}^{n} m(B_i) = m\left(\bigcup_{i=1}^{n} B_i\right) = m(X) = 1,$

and $-\log$ is a convexe function, from the Jensen's inequality we have

$$H(P, Q) = \sum_i m(B_i)(-1)\log_2 \frac{m(A_i)}{m(B_i)}$$

$$\geqslant -\log_2\left(\sum_i m(B_i)\frac{m(A_i)}{m(B_i)}\right) = 0.$$

Taking into account the definition of $H(P|D)$ we may also define $H(P|D, P)$ as

$$H(P|D, P) = -\sum_i m(A_i|D)\log_2 \frac{m(A_i)}{m(A_i|D)}$$

$$= -\sum_i \frac{m(A_i D)}{m(D)}\log_2 \frac{m(A_i)m(D)}{m(A_i D)}$$

PROPOSITION 3. If $P = \{A_1, \ldots, A_n\}$, $Q = \{B_1, \ldots, B_n\}$ *are fuzzy admissible partitions of* X, *then*

(i) $I(P, Q) = \sum_i m(B_j)H(P|B_j, P).$

(ii) $\sum_i m(B_j)H(P|B_j, P) = \sum_{j,i} m(B_j)(H(P) - H(P|B_j)).$

Proof. (i) Using the definition of $H(P|B_j, P)$ we may write

$$\sum_j m(B_j)H(P|B_j, P) = -\sum_{i,j} m(B_j)\frac{m(A_i B_j)}{m(B_j)}\log_2 \frac{m(A_i)m(B_j)}{m(A_i B_j)}$$

$$= -\sum_{i,j} m(A_i B_j)\log_2 \frac{m(A_i)m(B_j)}{m(A_i B_j)}$$

$$= -\sum_{i,j} m(A_i B_j)\log_2 m(A_i) + \sum_{i,j} m(A_i B_j)\frac{m(A_i B_j)}{m(B_j)}$$

$$= -\sum_i m(A_i \bigcup_j B_j)\log_2 m(A_j) - H(P|Q)$$

$$= H(P) - H(P|Q) = I(P, Q).$$

(ii) The relative information may be written as

$$I(P, Q) = H(P) - H(P|Q) = H(P)\sum_j m(B_j) - \sum_j m(B_j)H(P|B_j) =$$

$$= \sum_j m(B_j)(H(P) - H(P|B_j)).$$

From this equality and (i) follows (ii).

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PICARD MAPPINGS (I)

IOAN A. RUS*

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REZUMAT — **Aplicații de tip Picard (I).** În această parte a lucrării se dau condiții ce asigură următoarea implicație: (f asimptotic regulară) $\Rightarrow$ (f este o aplicație de tip Picard). Cazul special al aplicațiilor neexpansive va fi analizat în următoarea parte a lucrării.

1. Introduction. Let us consider the following Cauchy problem

$$y' = f(x, y), \; y(a) = y_0. \tag{1}$$

and the corresponding sequence of succesive approximations

$$y_k(x) := y_0 + \int_a^x f(s, y_{k-1}(s))\,ds \tag{2}$$

where f: $[a, b] \times R^n \to R^n$.

The folowing result is given by Dieudonné in 1945:

Dieudonné's theorem ([6]). *Let* $f \in C([a, b] \times R^n, R^n)$. *We suppose that the problem* (1) *has a unique solution. Then there is* $0 < h < b - a$ *such that the successive approximations* (2) *converge to the solution of* (1) *uniformly on* $[a, a + h]$ *if and only if* $(y_{n+1} - y_n)_{n \in N}$ *converges to 0 uniformly on* $[a, a + h]$.

This result was generalized by Wazewski ([20]) for the Cauchy problem in Banach spaces. An abstract form of Wazewski's result is given by Pelczar ([11]) in 1969.

From these results, and many other ([2], [1], [19], ...),, the following problem arise:

Problem 1 (see [16]). *Let* (X, d) *be a complete metric space. Which are the conditions on* f *such that we have:*

(f is asymptotically regular) $\Rightarrow$ *(f is a Picard mapping)*?

The following problems are in connection with the Problem 1:

Problem 1.1. (see [17]). *Which are the conditions on* f *such that we have:*

$$(x^* \text{ is a cluster point of } (f^n_{(x)})) \Rightarrow (x^* \in F_f)?$$

Problem 1.2. (see [17]). *Which are the conditions on* f *such that we have:*

$$(x^* \text{ is a cluster point of } (f^n(x)) \Rightarrow (\lim_{n\to\infty} f^n(x) = x^*)?$$

* *University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania*

Remark 1. Throughout in this paper we follow terminologies and notations in [14] and [15].

In the present paper we give some partial results for these problems.

2. α_{DP}-condensing mapping. We begin with

Definition 1 (see [4] and [9]). Let (X, d) be a metric space. A mapping $\alpha_{DP}: P_b(X) \to R_+$ is a Daneš-Pasicki measure of noncompactness if

(i) $\alpha_{DP}(A) = 0 \Rightarrow \bar{A} \in P_{cp}(X)$, for all $A \in P_b(X)$,

(ii) $\alpha_{DP}(\bar{A}) = \alpha_{DP}(A)$, for all $A \in P_b(X)$,

(iii) $A \subseteq B \Rightarrow \alpha_{DP}(A) \leqslant \alpha_{DP}(B)$, $A, B \in P_b(X)$,

(iv) $\alpha_{DP}(A \cup \{x\}) = \alpha_{DP}(A)$, for all $A \in P_b(X)$, $x \in X$.

Let us illustrate this notion by

Example 1. $\alpha_{DP} = \alpha_K$ (Kuratowskii's measure of non-compactness).

Example 2. $\alpha_{DP} = \alpha_H$ (Hausdorff's measure of non-compactness).

DEFINITION 2. Let (X, d) a metric space. A mapping $f: X \to X$ is α_{DP}-condensing if $(A \in P_b(X),\ f(A) \subseteq A,\ \alpha_{DP}(A) \neq 0) \Rightarrow (\alpha_{DP}(f(A)) < \alpha_{DP}(A))$.

For some examples of α_{DP}-condensing mappings see [15], [14].

3. Problem 1. We begin with

LEMMA 1 (see [18]). *Let (X, d) be a compact metric space and $f: X \to X$ a continuous mapping such that:*

(i) *f has at most a fixed point,*

(ii) *f is asymptotically regular.*

Then f is a Picard mapping.

Proof. Let $x \in X$. The sequence $(f^n(x))$ has a convergent subsequence $(f^{n_i}(x))$. Let $f^{n_i}(x) \to x^*$. By the continuity of f, $f^{n_i+1}(x) \to f(x^*)$. But f is asymptotically regular. This implies $f(x^*) = x^*$. From (i), f is a Picard mapping.

Remark 2. The Lemma 1 will remain true if the compactness of X is replaced by: there exists $n \in N$ such that $f^n(X)$ is compact.

The main result of this paper is the following.

THEOREM 1. *Let (X, d) be a bounded complete metric space and $f: X \to X$ a continuous α_{DP}-condensing mapping. If*

(i) *f has at most a fixed point,*

(ii) *f is asymptotically regular,*

then f is a Picard mapping.

Proof. Let $x \in X$. Let $0(x)$ denote the orbit of x relative to f. We have $0(x) \in I_b(f)(I_b(f) := \{A \in P_b(X) \mid f(A) \subseteq A,\ A \neq \emptyset\})$, and by the continuity of f, $\overline{0(x)} \in I_b(f)$. On the other hand

$$\alpha_{DP}(0(f(x))) = \alpha_{DP}(f(0(x)) < \alpha_{DP}(0(x)) = \alpha_{DP}(0(f(x)).)$$

This implies $\alpha_{DP}(\widetilde{0(x)}) = 0$. Thus we have that $\widetilde{0(x)}$ is compact and $\widetilde{0(x)} \in I(f)$. Now the proof follows from Lemma 1.

From the Theorem 1 we have

THEOREM 2. (see [13]). *Let (X, d) be a bounded complete metric space and $f: X \to X$ a continuous mapping such that*

(i) $\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0$, *for all* $x, y \in X$,

(ii) *f is α_{DP}-condensing.*

Then f is a Picard mapping.

Proof. From (i) we have that card $F_f \leqslant 1$ and f is asymptotically regular. The Theorem 2 follows from the Theorem 1.

Let $F : X \times X \to R$ be a real lower semicontinuous function. The mapping $f: X \to X$ is said to be F-contractive if (see [7]).

$$F(f(x), f(y)) < F(x, y), \text{ for all } x, y \in X, \ x \neq y.$$

By standard proof we have

THEOREM 3. *Let (X, d) be a bounded complete metric space and $f: X \to X$ a continuous α_{DP}-condensing and F-contractive mapping.*

Then f is a Picard mapping.

Remark 3. For $\alpha_{DP} = \alpha_K$ see [7].

THEOREM 4. *Let (X, d) be a bounded complete metric space and $f: X \to X$ such that*

(i) *f is α_{DP}-condensing,*

(ii) *f is of diminishing orbital diameter,*

(iii) *there exists $c > 0$ such that*

$$d(f^k(x), f^k(y)) \leqslant cd(x, y), \text{ for all } x, y \in X, \text{ and each } k \in N.$$

Then f is a weakly Picard mapping.

Proof. From the condition (i) $\overline{0(x)}$ is compact. The Theorem 4 follows from a theorem of Kirk (see [17]).

THEOREM 5. *Let X be a Banach space, $Y \subset X$ a closed bounded subset of X, and $f: Y \to Y$ a continuous mapping. We suppose that:*

(i) *f is a α_{DP}-condensing,*

(ii) $F_f \neq \emptyset$,

(iii) $||f(x) - p|| \leqslant ||x - p||$, *for all* $x \in Y$, $x \notin F_f$, $p \in F_f$.

Then f is a weakly Picard mapping.

Proof. Let $x \in X$. From (i), $\overline{0(x)}$ is compact. Now the proof follows from Corollarry 1.1. in [12].

Remark 4. For various considerations on the Problem 1 see: [2], [10], [11], [12], [17], [19], [20].

Remark 5. For various considerations on the Problem 1.1. and 1.2 see: [1], [2], [3], [5], [8], [17].

Remark 6. In the next part, the Problem 1 in the case of nonexpansive mapping will be discussed.

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GENERALIZED RECURRENCY IN SPACES WITH AFFINE CONNECTION

P. ENGHIŞ* and M. BOER*

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REZUMAT. — Recurenţa generalizată în spaţii cu conexiune afină. În lucrare se introduce prin (1,1) şi (1,4) recurenţa generalizată în spaţii cu conexiune afină. În primul paragraf se dau proporţiile 1.1 şi 1.2 ce decurg imediat din definiţie. În paragraful doi se studiază spaţiile cu conexiune simetrică. Se introduc spaţiile proiectiv recurente generalizate, dîndu-se un răspuns la problemele puse în observaţiile 1.1 şi 2.1. În paragraful trei se studiază spaţiile cu torsiune recurentă generalizată şi se dă şi aci un răspuns la problema pusă în observaţia 1.2 precum şi un caz în care reciproca propoziţiei 1.2 este adevărată.

1. **Generalities.** Let A_n be a space with affine connection. In a coordinates system, we denote by Γ^i_{jk}, the components of affine connection, by Γ^i_{jkh} the components of curvature tensor, by $\Gamma_{jk} = \Gamma^i_{jik}$ the components of Ricci tensor and by $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$ the components of torsion tensor of the connection. Γ.

DEFINITION 1.1. The space A_n is called generalized recurrent curvature space, if there exists a covector φ_r and a tensor A^i_{jkhr} so that

$$\Gamma^i_{jkh,r} = \varphi_r \Gamma^i_{jkh} + A^i_{jkhr} \tag{1.1}$$

where comma denotes the covariant derivation with respect to Γ.

Contracting (1,1) in i and k, we have:

$$\Gamma_{jh,r} = \varphi_r \Gamma_{jh} + A_{jhr} \tag{1.2}$$

where

$$A_{jhr} = A^i_{jihr} \tag{1.3}$$

and therefore:

PROPOSITION 1.1. *The generalized recurrent curvature A_n spaces, have the Ricci tensor also generalized recurrent with the same vector φ_r and tensor A_{jh} given by (1,3).*

Remark 1.1. From (1.1) it follows that beeing given the vector φ_r, the tensor A^i_{jkhr} is determined, having the same properties as the curvature tensor. Conversely, beeing given the tensor A^i_{jkhr}, can we determine a vector φ_r so that (1,1) take place? This problem, together with others, will be solved in a case in second paragraph.

* University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania

DEFINITION 1.2. A space A'_n, is called generalized recurrent torsion space if there exists a covector ψ_r and a tensor Q^i_{jkr} so that

$$T^i_{jk,r} = \psi_r T^i_{jk} + Q^i_{jkr} \tag{1.4}$$

where comma denotes the covariant derivation with respect to Γ.

If in (1,4) we apply a contration in i and j, we obtain

$$T_{k,r} = \psi_r T_k + Q_{kr} \tag{1.5}$$

where $T_k = T^i_{ik}$ is the Vrînceanu vector and $Q_{kr} = Q^i_{ikr}$. We have therefore:

PROPOSITION 1.2. *The generalized recurrent torsion A_n spaces, have th) Vrînceanu vector also generalized recurrent, with the same vector ψ_r and tensor $Q_{kr} = Q^i_{ikr}$.*

Remark 1.2. Here too, one observe that beeing given the vector ψ_r, the tensor Q^i_{jkr} is determined. The converse problem will be studied in the third paragraph.

2. **Generalized recurrent curvature A_n spaces.** In this paragraph, we consider the space A_n with symmetric affine connection, therefore $T^i_{jk} = 0$. We denote by R^i_{jkh} the components of curvature tensor and by R_{jk} the components of Ricci tensor. Relations (1,1) and (1,2) will write

$$R^i_{jkh,r} = \varphi_r R^i_{jkh} + A^i_{jkhr} \tag{2.1}$$

$$R_{jh,r} = \varphi_r R_{jh} + A_{jhr} \tag{2.2}$$

For such a space, taking account of the Bianchi identities of second type, it follows

$$\varphi_r R^i_{jkh} + \varphi_k R^i_{jhr} + \varphi_h R^i_{jrk} + A^i_{jkhr} + A^i_{jhrk} + A^i_{jrkh} = 0 \tag{2.3}$$

If in (2,3) we apply a contration in i and r, we have

$$(R^i_{jkh} + \delta^i_h R_{jk} - \delta^i_k R_{jh})\varphi_i + A^i_{jkhi} + A_{jkh} - A_{jhk} = 0 \tag{2.4}$$

and therefore:

PROPOSITION 2.1. *In a A_n space of recurrent curvature, endowed with a symmetric connection, the vector φ_i is solution of the linear and non-homogeneous system of n^3 equations with n unknowns (2.4).*

Proposition 2.1 ans relation (2.4) give for these spaces, the answer to the remark 1.1 through.

PROPOSITION 2.2. *In an A_n space with symmetric connection, beeing given the tensor A^i_{jkhr} the determination of the vector φ_r which verifies (2.1) depends on the compatibility of the system (2.4).*

Outstanding problems appear when the generalized recurrency tensor A^i_{jkhr} is degenerate. From various degeneration possibilities, we consider here the case of degeneration of first order and first rank, therefore

$$A^i_{jkhr} = a_r A^i_{jkh} \tag{2.5}$$

In this case, relations (2.1) and (2.2) become

$$R^i_{jkh,r} = \varphi_r R^i_{jkh} + a_r A^i_{jkh} \tag{2.6}$$

$$R_{jh,r} = \varphi_r R_{jh} + a_r A_{jh} \tag{2.7}$$

and we have:

DEFINITION 2.1. The A_n spaces that verify (2.6) we call generalized recurrent of vectors φ_r and a_r and tensor A^i_{jkh}.

DEFINITION 2.2. The A_n spaces that verify (2.7) we call generalized Ricci-recurrent.

From the way (2.7) was obtained from (2.6) it follows.:

PROPOSITION 2.3. *A space A_n generalized recurrent of vectors φ_r and a_r, endowed with a symmetric connection is also generalized Ricci-recurrent with the same recurrency vectors.*

Remark 2.1. The converse of assertion 2.3 is generally not true. In this paper we will give also a case in which the converse take place.

Let us consider now in the space A_n with symmetric connection the Weyl tensor

$$W^i_{jkh} = R^i_{jkh} - \frac{n}{n^2-1}(\delta^i_k R_{jh} - \delta^i_h R_{jk}) - \frac{1}{n^2-1}(\delta^i_k R_{hj} - \delta^i_h R_{kj}) - \delta^i_j \frac{R_{hk} - R_{kh}}{n+1} \tag{2.8}$$

also called the projective curvature tensor.

Derivating covariantly (2.8) and taking (2.6) and (2.7) into account we have

$$\begin{aligned} W^i_{jkh,r} = {} & R^i_{jkh,r} - \frac{n}{n^2-1}(\delta^i_k R_{jh,r} - \delta^i_h R_{jk,r}) - \frac{1}{n^2-1}(\delta^i_k R_{hj,r} - \delta^i_h R_{kj,r}) - \\ & - \delta^i_j \frac{R_{hk,r} - R_{kh,r}}{n+1} = \varphi_r \Big[R^i_{jkh} - \frac{n}{n^2-1}(\delta^i_k R_{jh} - \delta^i_h R_{jk}) - \frac{1}{n^2-1}(\delta^i_k R_{hj} - \delta^i_h R_{kj}) - \\ & - \delta^i_j \frac{R_{hk} - R_{kh}}{n+1}\Big] + a_r \Big[A^i_{jkh} - \frac{n}{n^2-1}(\delta^i_k A_{jh} - \delta^i_h A_{jk}) - \\ & \frac{1}{n^2-1}(\delta^i_k A_{hj} - \delta^i_h A_{kj}) - \delta^i_j \frac{A_{hk} - A_{kh}}{n+1}\Big] \end{aligned}$$

where if we note

$$B^i_{jkh} = A^i_{jkh} - \frac{n}{n^2-1}(\delta^i_k A_{jh} - \delta^i_h A_{jk}) - \frac{1}{n^2-1}(\delta^i_k A_{hj} - \delta^i_h A_{kj}) - \delta^i_j \frac{A_{hk} - A_{kh}}{n+1} \tag{2.9}$$

and taking (2.8) into account, we obtain

$$W^i_{jkh,r} = \varphi_r W^i_{jkh} + a_r B^i_{jkh} \tag{2.10}$$

DEFINITION 2.3. The A_n spaces that verify (2.9) are called generalized projectiv recurrent.

From definition 2.3 and relations (2.9) it follows

PROPOSITION 2.4. *The generalized recurrent A_n spaces, endowed with symmetric connection are also generalized projectiv recurrent with the same vectors φ_r and a_r with generalized recurrency tensor given by (2.9).*

Remark 2.2. Generalized projectiv recurrency tensor B^i_{jkh} given by (2.9) is expressed by means of the generalized recurrency tensor A^i_{jkh} and of his contracted A_{jh}, in the same manner as the projectiv curvature tensor W^i_{jkh} is expressed by means of the curvature tensor R^i_{jkh} and of his contracted R_{jh}.

For the converses of assertions 2.3 and 2.4, if in (2.10) we take count of (2.7) and (2.8), we have

$$R^i_{jkh,r} = \varphi_r R^i_{jkh} + a_r \Big[B^i_{jkh} + \frac{n}{n^2-1}(\delta^i_k A_{jh} - \delta^i_h A_{jk}) - \\ - \frac{1}{n^2-1}(\delta^i_k A_{hj} - \delta^i_h A_{kj}) - \delta^i_j \frac{A_{hk} - A_{kh}}{n+1}\Big] \tag{2.11}$$

and the space is generalized recurrent of vectors φ_r and a_r and tensor

$$A^i_{jkh} = B^i_{jkh} + \frac{n}{n^2-1}(\delta^i_k A_{jh} - \delta^i_h A_{jk}) - \frac{1}{n^2-1}(\delta^i_k A_{hj} - \delta^i_h A_{kj}) - \delta^i_j \frac{A_{hk} - A_{kh}}{n+1} \tag{2.12}$$

we have therefore:

PROPOSITION 2.5. *A generalized Ricci-recurrent and generalized projectiv recurrent A_n space with the same vectors φ_r and a_r, endowed with symmetric connection, is generalized recurrent with the same vectors φ_r and a_r and with generalized recurrency tensor given by (2.12).*

PROPOSITION 2.6. *The generalized projectiv recurrent A_n spaces, endowed with symmetric connection, are generalized recurrent if and only if they are generalized Ricci-recurrent.*

3. Generalized recurrent torsion A_n spaces. We consider now a space A_n endowed with a non-symmetric affine connection Γ.

We introduced in definition 1.2 the generalized recurrency of the torsion raising in remark 1.2 the problem of the determination of the vector ψ_r when the tensor Q^i_{jkr} is given. We can give an answer to this question in the case of A_n spaces endowed with a semi-symmetric E-connection [2]. In this case it is known [3] that we have:

$$T^i_{jk,r} + T^i_{kr,j} + T^i_{rj,k} = 0 \tag{3.1}$$

In (3.1) taking count of (1.4) we have:

$$\psi_r T^i_{jk} + \psi_j T^i_{kr} + \psi_k T^i_{rj} + Q^i_{jkr} + Q^i_{krj} + Q^i_{rjk} = 0 \tag{3.2}$$

If in (3.2) we apply a contraction in i and r, and we take count of (1.5) and of the fact that A_n is endowed with a E-connection, it follows

$$\psi_i T^i_{jk} + Q^i_{jki} = 0 \tag{3.3}$$

and so we have:

PROPOSITION 3.1. *In an A_n space endowed with a semi-symmetric E-connection, the vector ψ_i is solution of the system (3.3).*

An answer to the question raised in remark 1.2 can therefore be given in this way:

PROPOSITION 3.2. *The determination of a vector ψ_r that verifies relation (1.4), for a given tensor Q^i_{jkr}, depends on the compatibility of the system (3.3).*

In proposition 1.2 we saw that the generalized recurrent torsion A_n spaces have the Vrînceanu vector generalized recurrent too. Let us give now a case in which the converse of this assertion is true.

Let us consider the space A_n, $n > 1$, endowed with a semi-symmetric connection. In this case we have [2]

$$T^i_{jk} = \frac{1}{n-1}(\delta^i_j T_k - \delta^i_k T_j) \tag{3.4}$$

Derivating covariantly and taking count of (1.5) we have

$$T^i_{jk,r} = \frac{1}{n-1}[\delta^i_j(\psi_r T_k + Q_{kr}) - \delta^i_k(\psi_r T_j + Q_{jr})]$$

or

$$T^i_{jk,r} = \psi_r \frac{1}{n-1}(\delta^i_j T_k - \delta^i_k T_j) + \frac{1}{n-1}(\delta^i_j Q_{kr} - \delta^i_k Q_{jr}) \tag{3.5}$$

Taking count of (3.4), relation (3.5) can write

$$T^i_{jk,r} = \psi_r T^i_{jk} + Q^i_{jkr}$$

where

$$Q^i_{jkr} = \frac{1}{n-1}(\delta^i_j Q_{kr} - \delta^i_k Q_{jr}) \tag{3.6}$$

We have therefore

PROPOSITION 3.3 *The A_n spaces endowed with semi-symmetric connection, with Vrînceanu vector generalized recurrent of vector ψ_r and tensor Q_{kr} are also genera-*

lised recurrent torsion spaces, with the same vector ψ_r and tensor Q^i_{jk} given by (3.6).

Remark: A more detailed study of the spaces A_n endowed with semi-symmetric connection and with generalized recurrent torsion, will be made later.

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RECENZII

J. Dieudonné, **Grundzüge der modernen Analysis**, 9, VEB Deutscher Derlag der Wissenschaften Berlin 1987, 380 Seiten.

Der vorliegende Band ist die Ubersetzung aus dem Französischen des wohlbekannten Buches von J. Dieudonné „Élements d'Analyse", Tome IX, Chapitre XXIV, Bordas, Paris, 1982. Darin verfolgt der Verfasser die wichtigsten Aspekte der Algebraischen Topologie und der Differentialtopologie entsprechend dem Geist der modernen Analysis darzustellen. Aus dem Inhalt erwähnen wir hier: Kohomologie und Kohomologie mit kompakten Trägern einer differentiarbaren Mannigfaltigkeit, die Poincaré Dualität, Homologie der Ströme, die Stokessche Formel, die Sätze von de Rham, die singuläre Kohomologie, Chernsche Klassen, Pontrjaginsche Klassen, Stiefel-Whitneysche Klassen, die Theorie von Hodge, die Formel von Atiyah-Bott-Lefschetz, Kohomologie Liescher Gruppen.

Nach dem Text folgt ein Anhang mit Ergänzungen aus der Algebra (Fortsetzung des Anhangs zu Band 5/6), der die wichtigsten Begriffe und Hilfsmittel der linearen und multilinearen Algebra, also der homologischen Algebra enthält, Z.B. Tensorprodukte von Moduln, exakte Sequenzen, Kohomologie eines graduierten Differentialmoduls, Ergänzungen zu den Vektorräumen, Ergänzungen zu den **Z**—Moduln endlichen Typs.

Der Band schliesst mit einer Liste von Bezeichnungen, zusätzlicher Literatur für die Bände 1—9 und einem Sachverzeichnis.

Dank seines Aufbaus, der originellen Problembehandlung und der Gesamtkonzeption, vermittelt das Werk eine bemerkenswerte Einfürung in die betrachteten Theorien die sowohl dem Studenten der Mathematik, als auch dem forschenden Mathematiker zugänglich ist.

M. ŢARINĂ

Geometric and Algebraic Topology, Banach Center Publications volume 18, Warszawa 1986, 417 p. (H. Torunczyk, S. Jackowski, S. Spiez, eds.)

This volume contains the most important and suggestive papers presented during the Topology Semester which was held at the Stefan Banach International Mathematical Center in the spring of 1984. The content of the book consists of 26 papers divided in 7 sections. In what follows we present the sections and the authors of the papers. 1. *Low-dimensional manifolds* (4 papers): N. V. Ivanov, K. Johannson, D. McCullough, D. Repovš; 2. *Higher-dimensional manifolds* (6 papers): R. J. Daverman, L. Montejano, T. B. Rushing, E. V. Shchepin, J. J. Walsh, D. G. Wright; 3. *Group actions* (3 papers): K. Pawalowski, M. Steinberger P. Traczyk, J. West; 4. *Differential topology and geometry* (3 papers): J. Eichhrn, A. Szücs, Ch. B. Thomas; 5. *Cyclic homology and homology of groups* (3 papers): R. Geoghegan, J. L. Loday, Z. Marciniak; 6. *Shape theory and its homological aspects* (4 papers): J. T. Lisitsa, S. Mardešic, L. D. Mdzinarishvili, J. Segal, T. Watanabe; 7. *Dimension theory and theory of continua* (3 papers): J. Krasinkiewicz, L. G. Oversteegen, L. R. Rubin, E. D. Tymchatyn.

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Lei 35