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SUR UNE FORMULE DU CUBATURE DE GAUSS-TURAN

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REZUMAT. — O formulă de cubatură a lui Gauss — Turan. În lucrare se studiază formulele de cubatură (10), relative la un dreptunghiu, formule ce constituie o extensiune a formulelor de cuadratură de tip Gauss—Turan (6).

1. On sait que pour une formule de quadrature avec les noeuds fixes $x_1, x_2, \ldots, x_n$, où $a < x_1 < \ldots < x_n < b$ de la forme

$$\int_a^b f(x)dx = \sum_{i=1}^n A_i^{(1)} f(x_i) + \sum_{i=1}^n A_i^{(2)} f'(x_i) + \sum_{i=1}^n A_i^{(3)} f''(x_i) + R[f] \tag{1}$$

on peut déterminer les coefficients $A_i^{(s)}$; $i = 1, 2, \ldots, n$; $s = 1, 2, 3$ de manière que $R[x^i] = 0$; $i = 0, 1, \ldots, 3n - 1$. Alors si $f \in C^{3n}[a, b]$ on peut mettre le reste $R[f]$ sous la forme

$$R[f] = (-1)^{3n} \int_a^b \psi(x) f^{(3n)}(x) dx \tag{2}$$

où

$$\psi(x) = \frac{(x-a)_+^{3n}}{n!} + \sum_{i=1}^{3} \sum_{j=0}^{n-1} (-1)^i A_{j+1}^{(i)} \frac{(x - x_{j+1})_+^{3n-i}}{(3n-i)!} \tag{3}$$

On sait également que si l'on regarde dans la formule (1) les noeuds $x_1, x_2, \ldots, x_n$ comme nedéterminés, on peut les déterminer de manière que $R[x^i] = 0$, pour $i = 0,1 \ldots, 4n - 1$. Alors la formule (1) devient la formule de quadrature de G a u s s-T u r a n [9], dont les noeuds sont les zéros du pôlynome de degré n, qui realise le minimum de la fonctionelle

$$I(\omega) = \int_a^b |\omega(x)|^4 dx \tag{4}$$

sur la classe des polynômes

$$\omega(x) = x^n + a_1 x^{n-1} + \ldots + a_n$$

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Si $f \in C^{4n}$ $[a, b]$, on peut mettre son reste sous la forme

$$R[f] = \int_a^b \psi(x) f^{(4n)}(x).dx \tag{5}$$

où la fonction ψ est donnée par la formule

$$\psi(x) = \frac{(x-a)^{4n}}{(4n)!} - \sum_{i=1}^{3} \sum_{j=0}^{n-1} (-1)^i A_{j+1}^{(i)} \frac{(x - x_{j+1})_+^{4n-i}}{(4n-i)!}.$$

La fonction ψ est positive sur l'intervalle [a, b], [7].

Dans la cas $n = 1$, la formule de quadrature de Gauss-Turan est

$$\int_{x_0}^{x_2} f(x)dx = A_1^{(1)} f(x_1) + A_1^{(2)} f'(x_1) + A_1^{(3)} f''(x_1) + R[f] \tag{6}$$

où

$$x_1 = \frac{x_0 + x_2}{2}, \quad A_1^{(1)} = x_2 - x_0, \quad A_1^{(1)} = 0, \quad A_1^{(3)} = \frac{(x_2 - x_0)^3}{24} \tag{7}$$

La fonction ψ est donné par la formule

$$\Psi(x) = \frac{(x-x_0)^4}{4!} + (x_2 - x_0) \frac{(x - x_1)_+^3}{3!} + \frac{(x_2 - x_0)^3}{24} \frac{(x - x_1)_+}{1!} \tag{8}$$

et nous avons pour le rest $R[f]$

$$|R[f]| \leqslant 2 \frac{(x_2 - x_0)^5}{2^5 \cdot 5!} f^{(4)}(\xi); \quad \xi \in [x_0, x_2] \tag{9}$$

2. Dans la présente note nous donnons une extension de la formule de quadrature de Gauss-Turan (6) à des formule de cubature de la forme:

$$\iint_D f(x, y)dxdy = A_1^{(1)} \int_{y_0}^{y_2} f(x_1, y)\, dy + A_1^{(3)} \int_{y_0}^{y_2} \left[\frac{(y - y_0)_+^2}{2!} + (y_0 - y_2) \frac{(y - y_1)_+}{1!} \right] \times$$

$$\times \frac{\partial^4 f(x_1, y)}{\partial x^2 \partial y^2} dy + B_1^{(1)} \int_{x_0}^{x_2} f(x, y_1)\, dx + B_1^{(3)} \int_{x_0}^{x_2} \left[\frac{(x - x_0)_+^2}{2!} + (x_0 - x_2) \frac{(x - x_1)_+}{1!} \right] \times$$

$$\times \frac{\partial^4 f(x, y_1)}{\partial x^2 \partial y^2} dx - C_1^{(1)} f(x_1, y_1) - C_1^{(3)} \frac{\partial^4 f(x_1, y_1)}{\partial x^2 \partial y^2} + R[f] \tag{10}$$

3. Considérons un rectangle

$$\Delta = \{x_1 \leqslant x \leqslant x_2;\ y_1 \leqslant y \leqslant y_2\}$$

et

$f, \varphi : \Delta \to \mathbf{R}$ deux fonctions continues sur Δ, avec leurs dérivées partielles qui intervient dans nos considerations.

On a lieu la formule genèrale d'integration par partie pour les integrales doubles [4]

$$\iint\limits_{\Delta} \varphi \frac{\partial^8 f}{\partial x^4 \partial y^4} dxdy = \sum_{r=0}^{3} \left[\left(\frac{\partial^{2(3-r)}\varphi}{\partial x^{3-r} \partial y^{3-r}} \cdot \frac{\partial^{2r} f}{\partial x^r \partial y^r} \right)(x_2, y_2) - \right.$$

$$- \left(\frac{\partial^{2(3-r)}\varphi}{\partial x^{3-r} \partial y^{3-r}} \frac{\partial^{2r} f}{\partial x^r \partial y^r} \right)(x_2, y_1) - \left(\frac{\partial^{2(3-r)}\varphi}{\partial x^{3-r} \partial y^{3-r}} \frac{\partial^{2r} f}{\partial x^r \partial y^r} \right)(x_1, y_2) +$$

$$\left. - \left(\frac{\partial^{2(3-r)}\varphi}{\partial x^{3-r} \partial y^{3-r}} \frac{\partial^{2r} f}{\partial x^r \partial y^r} \right)(x_1, y_1)) \right] + \tag{11}$$

$$+ \sum_{r=0}^{3} \int_{x_1}^{x_2} \left[\left(\frac{\partial^{2(4-r)-1}\varphi}{\partial x^{4-r} \partial y^{3-r}} \cdot \frac{\partial^{2r} f}{\partial x^r \partial y^r} \right)(x, y_1) - \left(\frac{\partial^{2(4-r)-1}\varphi}{\partial x^{4-r} \partial y^{3-r}} \frac{\partial^{2r} f}{\partial x^r \partial y^r} \right)(x, y_2) \right] dx +$$

$$+ \sum_{r=0}^{3} \int_{y_1}^{y_2} \left[\left(\frac{\partial^{2(4-r)-1}\varphi}{\partial x^{3-r} \partial y^{4-r}} \frac{\partial^{2r} f}{\partial x^r \partial y^r} \right)(x_1, y) - \left(\frac{\partial^{2(4-r)-1}\varphi}{\partial x^{3-r} \partial y^{4-r}} \cdot \frac{\partial^{2r} f}{\partial x^r \partial y^r} \right)(x_2, y) \right] dy +$$

$$+ \iint\limits_{\Delta} \frac{\partial^8 \varphi}{\partial x^4 \partial y^4} f \, dx \, dy$$

4. Soit le rectangle

$$D = \{x_0 \leqslant x \leqslant x_2 ; y_0 \leqslant y \leqslant y_2\}$$

et nous ne proposons de determiner le noeud (x_1, y_1) et les coefficients $A_1^{(s)}$, $B_1^{(s)}$, $C_1^{(s)}$; $s = 1, 2, 3$ de manière que la formule de cubature, avec son reste soit de la forme (10).

Nons désignons par

$$D_{ik} = \{x_i \leqslant x \leqslant x_{i+1} ; \ y_k \leqslant y \leqslant y_{k+1}\} ; \ i, k = 0, 1$$

et attachons aux rectangles. D_{ik} les fonctions φ_{ik} continues sur D_{ik}, avec leurs dérivées partielles

$$\frac{\partial^{2s-1}\varphi_{ik}}{\partial x^s \partial y^{s-1}}, \ \frac{\partial^{2s-1}\varphi_{ik}}{\partial x^{s-1} \partial y^s}, \ \frac{\partial^{2s}\varphi_{ik}}{\partial x^s \partial y^s}; \quad s = 1, 2, 3, 4.$$

En appliquant la formule génèrale d'integration par partie pour l'intégrales doubles (11) a chaque couple de fonctions f, φ_{ik} sur D_{ik}, et en faisant la some des toutes ces formules, on obtient la formule suivante:

$$\iint\limits_{D} \varphi \frac{\partial^8 f}{\partial x^4 \partial y^4} dxdy = \sum_{r=0}^{3} \left[\left(\frac{\partial^{2(3-r)}\varphi_{11}}{\partial x^{3-r} \partial y^{3-r}} \frac{\partial^{2r} f}{\partial x^r \partial y^r} \right)(x_2, y_2) - \right.$$

$$-\left(\frac{\partial^{2(3-r)}\varphi_{10}}{\partial x^{3-r}\partial y^{3-r}}\cdot\frac{\partial^{2r}f}{\partial x^{r}\partial y^{r}}\right)(x_2, y_0)-\left(\frac{\partial^{2(3-r)}\varphi_{01}}{\partial x^{3-r}\partial y^{3-r}}\cdot\frac{\partial^{2r}f}{\partial x^{r}\partial y^{r}}\right)(x_0, y_2)+$$

$$+\left(\frac{\partial^{2(3-r)}\varphi_{00}}{\partial x^{3-r}\partial y^{3-r}}\frac{\partial^{2r}f}{\partial x^{r}\partial y^{r}}\right)(x_0, y_0)\Bigg]+\sum_{r=0}^{3}\Bigg[\left(\frac{\partial^{2(3-r)}(\varphi_{10}-\varphi_{00})}{\partial x^{3-r}\partial y^{3-r}}\frac{\partial^{2r}f}{\partial x^{r}\partial y^{r}}\right)(x_1, y_0)-$$

$$-\left(\frac{\partial^{2(3-r)}(\varphi_{11}-\varphi_{01})}{\partial x^{3-r}\partial y^{3-r}}\frac{\partial^{2r}f}{\partial x^{r}\partial y^{r}}\right)(x_1, y_2)\Bigg]+$$

$$+\sum_{r=0}^{3}\Bigg[\left(\frac{\partial^{2(3-r)}(\varphi_{01}-\varphi_{00})}{\partial x^{3-r}\partial y^{3-r}}\frac{\partial^{2r}f}{\partial x^{r}\partial y^{r}}\right)(x_0\cdot y_1)-$$

$$-\left(\frac{\partial^{2(3-r)}(\varphi_{11}-\varphi_{10})}{\partial x^{3-r}\partial y^{3-r}}\frac{\partial^{2r}f}{\partial x^{r}\partial y^{r}}\right)(x_2, y_1)\Bigg]+$$

$$+\sum_{r=0}^{3}\left(\frac{\partial^{2(3-r)}(\varphi_{11}+\varphi_{00}-\varphi_{10}-\varphi_{01})}{\partial x^{3-r}\partial y^{3-r}}\frac{\partial^{2r}f}{\partial x^{r}\partial y^{r}}\right)(x_1, y_1)+ \qquad (12)$$

$$+\sum_{r=0}^{3}\sum_{i=0}^{1}\int_{x_i}^{x_{i+1}}\Bigg[\left(\frac{\partial^{2(4-r)-1}\varphi_{i0}}{\partial x^{4-r}\partial y^{3-r}}\frac{\partial^{2r}f}{\partial x^{r}\partial y^{r}}\right)(x, y_0)-\left(\frac{\partial^{2(4-r)-1}\varphi_{i1}}{\partial x^{4-r}\partial y^{3-r}}\frac{\partial^{2r}f}{\partial x^{r}\partial y^{r}}\right)(x, y_2)\Bigg]dx+$$

$$+\sum_{r=0}^{3}\sum_{i=0}^{1}\int_{x_i}^{x_{i+1}}\left(\frac{\partial^{2(4-r)-1}(\varphi_{i1}-\varphi_{i0})}{\partial x^{4-r}\partial y^{3-r}}\frac{\partial^{2r}f}{\partial x^{r}\partial y^{r}}\right)(x, y_1)\,dx+$$

$$+\sum_{r=0}^{3}\sum_{k=0}^{1}\int_{y_k}^{y_{k+1}}\Bigg[\left(\frac{\partial^{2(4-r)-1}\varphi_{0k}}{\partial x^{3-r}\partial y^{4-r}}\frac{\partial^{2r}f}{\partial x^{r}\partial y^{r}}\right)(x_0, y)-$$

$$-\left(\frac{\partial^{2(4-r)-1}\varphi_{1k}}{\partial x^{3-r}\partial y^{4-r}}\frac{\partial^{2r}f}{\partial x^{r}\partial y^{r}}\right)(x_2, y)\Bigg]dy+$$

$$+\sum_{r=0}^{3}\sum_{k=0}^{1}\int_{y_k}^{y_{k+1}}\left(\frac{\partial^{2(4-r)-1}(\varphi_{1k}-\varphi_{0k})}{\partial x^{3-r}\partial y^{4-r}}\frac{\partial^{2r}f}{\partial x^{r}\partial y^{r}}\right)(x_1, y)\,dy+$$

$$+\sum_{i=0}^{1}\sum_{k=0}^{1}\iint_{D_{ik}}\frac{\partial^{8}\varphi_{ik}}{\partial x^{4}\partial y^{4}}f\,dx\,dy.$$

5. Relativement à la formule (12) nous traiterons les problèmes aux limites suivantes:

Problème 1. Déterminer la solution de l'équation aux dérivées partielles

$$\frac{\partial^8 \varphi_{00}}{\partial x^4 \partial y^4} = 1 \tag{13}$$

satisfaisant aux conditions aux limites

$$\begin{aligned}
&\left.\frac{\partial^{2(4-r)-1}\varphi_{00}}{\partial x^{3-r}\partial y^{4-r}}\right|_{(x_0,\, y)} = 0 \qquad r = 0, 1, 2, 3\\
&\left.\frac{\partial^{2(4-r)-1}\varphi_{00}}{\partial x^{4-r}\partial y^{3-r}}\right|_{(x,\, y_0)} = 0 \qquad r = 0, 1, 2, 3\\
&\left.\frac{\partial^{2(3-r)}\varphi_{00}}{\partial x^{3-r}\partial y^{3-r}}\right|_{(x_0,\, y_0)} = 0 \qquad r = 0, 1, 2, 3
\end{aligned} \tag{14}$$

Problème 2. Déterminer la solution de l'équation aux dérivées partielles

$$\frac{\partial^8 \varphi_{10}}{\partial x^4 \partial y^4} = 1 \tag{15}$$

satisfaisant aux conditions aux limites

$$\begin{aligned}
&\left.\frac{\partial^{2(4-r)-1}\varphi_{10}}{\partial x^{4-r}\partial y^{3-r}}\right|_{(x_1,\, y_0)} = 0 \qquad r = 0, 1, 2, 3\\
&\left.\frac{\partial^{2(3-r)}}{\partial x^{3-r}\partial y^{3-r}}(\varphi_{10} - \varphi_{00})\right|_{(x_1,\, y_0)} = 0 \qquad r = 0, 1, 2, 3\\
&\left.\frac{\partial^{2(4-r)-1}}{\partial x^{3-r}\partial y^{4-r}}(\varphi_{00} - \varphi_{10})\right|_{(x_1,\, y)} = \begin{cases} A_1^{(1)} & r = 0\\ 0 & r = 1\\ A_1^{(3)}\dfrac{(y-y_0)^2}{2!} & r = 2\\ 0 & r = 3\end{cases}\\
&\left.\frac{\partial^{2(3-r)}\varphi_{10}}{\partial x^{3-r}\partial y^{3-r}}\right|_{(x_2,\, y_0)} = 0 \qquad r = 0, 1, 2, 3\\
&\left.\frac{\partial^{2(4-r)-1}\varphi_{10}}{\partial x^{3-r}\partial y^{4-r}}\right|_{(x_2,\, y)} = 0 \qquad r = 0, 1, 2, 3.
\end{aligned} \tag{16}$$

Problème 3. Déterminer la solution de l'équation aux dérivées partielles

$$\frac{\partial^8 \varphi_{01}}{\partial x^4 \partial y^4} = 1 \tag{17}$$

satisfaisant aux conditions aux limites

$$\frac{\partial^{2(3-r)}}{\partial x^{3-r}\,\partial y^{3-r}}(\varphi_{01}-\varphi_{00})\Big|_{(x_0,\,y_1)}=0 \qquad r=0,1,2,3$$

$$\frac{\partial^{2(4-r)-1}\varphi_{01}}{\partial x^{3-r}\,\partial y^{4-r}}\Big|_{(x_0,\,y)}=0 \qquad r=0,1,2,3$$

$$\frac{\partial^{2(4-r)-1}}{\partial x^{4-r}\,\partial y^{3-r}}(\varphi_{00}-\varphi_{01})\Big|_{(x,\,y_1)}=\begin{cases}B_1^{(1)} & r=0\\ 0 & r=1\\ B_1^{(3)}\dfrac{(x-x_0)^2}{2!} & r=2\\ 0 & r=3\end{cases} \tag{18}$$

$$\frac{\partial^{2(3-r)}\varphi_{01}}{\partial x^{3-r}\,\partial y^{3-r}}\Big|_{(x_0,\,y_2)}=0 \qquad r=0,1,2,3$$

$$\frac{\partial^{2(4-r)-1}\varphi_{01}}{\partial x^{4-r}\,\partial y^{3-r}}\Big|_{(x,\,y_2)}=0 \qquad r=0,1,2,3$$

Problème 4. Déterminer la solution de l'équation aux dérivées partielles

$$\frac{\partial^8\varphi_{11}}{\partial x^4\,\partial y^4}=1 \tag{19}$$

satisfaisant aux conditions aux limites

$$\frac{\partial^{2(4-r)-1}}{\partial x^{4-r}\,\partial y^{3-r}}(\varphi_{10}-\varphi_{11})\Big|_{(x,\,y_1)}=\begin{cases}B_1^{(1)} & r=0\\ 0 & r=1\\ B_1^{(3)}\left[\dfrac{(x-x_0)^2}{2!}+(x_0-x_2)\dfrac{(x-x_1)_+}{1!}\right] & r=2\\ 0 & r=3\end{cases}$$

$$\frac{\partial^{2(4-r)-1}}{\partial x^{3-r}\,\partial y^{4-r}}(\varphi_{01}-\varphi_{11})\Big|_{(x_1,\,y)}=\begin{cases}A_1^{(1)} & r=0\\ 0 & r=1\\ A_1^{(3)}\left[\dfrac{(y-y_0)^2}{2!}+(y_0-y_2)\dfrac{(y-y_1)_+}{1!}\right] & r=2\\ 0 & r=3\end{cases} \tag{20}$$

$$\frac{\partial^{2(3-r)}}{\partial x^{3-r}\,\partial y^{3-r}}(\varphi_{10}+\varphi_{01}-\varphi_{11}-\varphi_{00})\Big|_{(x_1,\,y_1)}=\begin{cases}c_{11}^{(r+1)} & r=0,1,2\\ 0 & r=3\end{cases}$$

En plus il faut que φ_{ik}; $i, k = 0$, vérifient les conditions supplémentaires

$$\left.\frac{\partial^{2(3-r)}\varphi_{11}}{\partial x^{3-r}\,\partial y^{3-r}}\right|_{(x_1,\,y_1)} = 0\,; \qquad \left.\frac{\partial^{2(4-r)-1}\varphi_{11}}{\partial x^{3-r}\,\partial y^{4-r}}\right|_{(x_1,\,y)} = 0\,;$$

$$\left.\frac{\partial^{2(4-r)-1}\varphi_{11}}{\partial x^{4-r}\,\partial y^{3-r}}\right|_{(x,\,y_1)} = 0\,; \quad \ldots \quad \left.\frac{\partial^{2(3-r)}}{\partial x^{3-r}\,\partial y^{3-r}}(\varphi_{01}-\varphi_{11})\right|_{(x_1,\,y_1)} = 0\,; \tag{21}$$

$$\left.\frac{\partial^{2(3-r)}}{\partial x^{3-r}\,\partial y^{3-r}}(\varphi_{10}-\varphi_{11})\right|_{(x_1,\,y_1)} = 0.$$

6. Les solutions de ces problèmes sont :

$$\varphi_{00}(x, y) = \frac{(x-x_0)^4}{4!}\,\frac{(y-y_0)^4}{4!}$$

$$\varphi_{10}(x, y) = \frac{(y-y_0)^4}{4!}\left[\frac{(x-x_0)^4}{4!} - A_1^{(1)}\frac{(x-x_1)^3}{3!} - A_1^{(3)}\frac{(x-x_1)}{1!}\right]$$

$$\varphi_{01}(x, y) = \frac{(x-x_0)^4}{4!}\left[\frac{(y-y_0)^4}{4!} - B_1^{(1)}\frac{(y-y_1)^3}{3!} - B_1^{(3)}\frac{y-y_1}{1!}\right] \tag{22}$$

$$\varphi_{11}(x, y) = \left[\frac{(x-x_0)^4}{4} - A_1^{(1)}\frac{(x-x_1)^3}{3!} - A_1^{(3)}\frac{x-x_1}{1!}\right]\cdot$$

$$\cdot\left[\frac{(y-y_0)^4}{4!} - B_1^{(1)}\frac{(y-y_1)^3}{3!} - B_1^{(3)}\frac{y-y_1}{1!}\right]$$

où

1°. $A_1^{(s)}$; $s = 1, 2, 3$ et x_1, sont les coefficients et le noeud de la formule de quadrature de Gauss-Turan (6).

2°, De même, les $B_1^{(s)}$, $s = 1, 2, 3$ et y_1, sont les coefficients et le noeud de la formule de quadrature de Gauss — Turan,

$$\int_{y_1}^{y_2} f(y)dy = (y_2 - y_0)f\left(\frac{y_0+y_2}{2}\right) + \frac{(y_2-y_0)^3}{24}f''\left(\frac{y_0+y_2}{2}\right) + \int_{y_0}^{y_2}\Phi(y)f^{(4)}(y)\,dy \tag{6'}$$

3°. Les coefficients $C_1^{(s)}$; $s = 1, 2, 3$ de la formule (10) sont donnés par les formules

$$C_1^{(1)} = -A_1^{(1)}B_1^{(1)}\,;\; C_1^{(2)} = 0\,;\; C_1^{(3)} = -A_1^{(3)}B_1^{(3)}$$

4°. Les fonctions ψ, Φ et φ coïncident sur les intervalles $[x_i, x_{i+1}]$, $[y_k, y_{k+1}]$ et sur le rectangle $D_{ik} = \{x_i \leqslant x \leqslant x_{i+1}\,;\, y_k \leqslant y \leqslant y_{k+1}\}$ avec les fonctions ψ_i, Φ_k, φ_{ik} et nous avons démontré qu'on a

$$\varphi_{ik}(x, y) = \psi_i(x),\,\Phi_k(y)\,;\; i, k = 0,1 \tag{23}$$

Les formules (23) sont importantes et elles démontrent que dans la formule de cubature (10) de Gauss–Turan la fonction φ est positive sur le rectangle ouvert D.

5°. Le reste de la formule de cubature (10) est donné par la formule

$$R[f] = \iint_D \varphi(x, y) \frac{\partial^8 f(x, y)}{\partial x^4 \partial y^4} dx\, dy \tag{24}$$

où la fonction φ coïncide sur les rectangles D_{ik} avec les fonctions φ_{ik}, données par les formules (22).

7. En changeant dans la formule (6) respectivement (6') la notation, et mettant

$$x_0 = y_1 - h, \quad x_1 = x_1, \quad x_2 = x_1 + h$$
$$y_0 = y_1 - k, \quad y_1 = y_1, \quad y_2 = y_1 + k$$

la formule de quadrature de Gauss–Turan ($n = 1$) devient

$$\int_{x_1-h}^{x_1+h} f(x) dx = 2h\, f(x_1) + \frac{h^3}{3} f''(x_1) + \int_{x_1-h}^{x_1+h} \psi(x) f^{(4)}(x)\, dx \tag{25}$$

$$\int_{y_1-k}^{y_1+k} g(y)\, dy = 2k\, g(y_1) + \frac{k^3}{3} g''(y_1) + \int_{y_1-k}^{y_1+k} \Phi(y)\, g^{(4)}(y)\, dy \tag{26}$$

Aussi la formule (10) devient

$$\iint_D f(x, y)\, \mathrm{d}x\, dy = -\, 4hk\, f(x_1, y_1) - \frac{h^3 k^3}{9} \frac{\partial^4 f}{\partial x^2 \partial y^2}(x_1, y_1) +$$

$$+ 2h \int_{y_1-k}^{y_1+k} f(x_1, y) dy + \frac{h^3}{3} \int_{y_1-k}^{y_1+k} \left[\frac{(y - y_1 + k)^2}{2!} - 2k \frac{(y - y_1)_+}{1!}\right] \frac{\partial^4 f(x_1, y)}{\partial x^2 \partial y^2} dy + \tag{10'}$$

$$+ 2k \int_{x_1-h}^{x_1+h} f(x, y_1) \mathrm{d}x + \frac{k^3}{3} \int_{x_1-h}^{x_1+h} \left[\frac{(x - x_1 + h)^2}{2!} - 2h \frac{(x - x_1)_+}{1!}\right] \frac{\partial^4 f(x, y_1)}{\partial x^2 \partial y^2} dx + R[f]$$

En appliquant les formules (25), (26), la formule du Simpson et la formule

$$\int_{x_1}^{x_1+h} f(x) dx = \frac{h}{2} (f(x_1) + f(x_1 + h)) + \int_{x_1}^{x_1+h} \theta(x) f''(x)\, dx$$

où

$$\left| \int_{x_1}^{x_1+h} \theta(x) f''(x)\, dx \right| \leqslant \frac{h^3}{12} f''(\xi)$$

a chaque intègrale de la formule (10′), on est conduit à la formule de cubature de Gauss—Turan

$$\iint_D f(x, y)\, dx\, dy = 4hk\, f(x_1, y_1) + \frac{2}{3} hk\left[k^2 \frac{\partial^2 f}{\partial y^2} + h^2 \frac{\partial^2 f}{\partial x^2}\right](x_1, y_1) +$$

$$+ \frac{7}{9} h^3 k^3 \frac{\partial^4 f}{\partial x^2\, \partial y^2}(x_1, y_1) + \frac{2}{9} h^3 k^3 \left[k \frac{\partial^5 f}{\partial x^2\, \partial y^3} + h \frac{\partial^5 f}{\partial x^3\, \partial y^2}\right](x_1, y_1) +$$

$$+ \frac{h^3k^3}{18}\left[k^2 \frac{\partial^6 f}{\partial x^2\, \partial y^4} + h^2 \frac{\partial^6 f}{\partial x^4\, \partial y^2}\right](x_1, y_1) - \frac{h^3k^3}{2}\left[\frac{\partial^4 f}{\partial x^2\, \partial y^2}(x_1 + h, y_1) +\right.$$

$$\left. + \frac{\partial^4 f}{\partial x^2\, \partial y^2}(x_1, y_1 + k)\right] + R[f],$$

avec la reste

$$|R[f]| \leqslant \frac{4}{5!} hk[k^4 M_{04} + h^4 M_{40}] + h^3k^3 \left\{\frac{4}{(5!)^2} h^2k^2 M_{44} + \frac{4}{45}(h^2 M_{42} + k^2 M_{24}) +\right.$$

$$\left. + \frac{2}{45}(k^3 M_{25} + h^3 M_{52}) + \frac{1}{90}(k^4 M_{26} + h^4 M_{62}) + \frac{1}{9}(M_{23} + M_{32})\right\}$$

où

$$M_{ik} = \sup_D \left|\frac{\partial^{i+k} f(x, y)}{\partial x^i\, \partial y^k}\right|$$

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ESPACES K_n^* DOUÉS D'UNE D-E CONNEXION

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REZUMAT. – **Spaţii K_n^* înzestrate cu o conexiune D–E.** În prezenta notă prin şase definiţii, trei propoziţii şi patru observaţii aducem citeva precizări şi completări la lucrarea „Problèmes de récurrence pour des connexions semi-symétriques métriques" [3].

Si L_n est une variété différentiable à n dimensions douée d'une métrique riemannienne g de composantes g_{ij} dans une carte locale, on note avec ∇ la connexion de Levi-Civita qui lui correspond de composantes $\left\{ {i \atop jk} \right\}$ dans la carte locale considérée. On associe à la connexion ∇ une connexion semi-symétrique métrique [1], [6] donnée par:

$$\Gamma^h_{jk} = \left\{ {h \atop jk} \right\} + \omega_j \delta^h_k - g_{jk}\omega^h \quad (\omega^h = g^{hi}\omega_i) \tag{1}$$

DÉFINITION 1. La connexion (1) sera nommée *D*-connexion.

On a

$$T^h_{jk} = \omega_j \delta^h_k - \omega_k \delta^h_j \qquad (T^h_{jk} = \Gamma^h_{jk} - \Gamma^h_{kj}) \tag{2}$$

$$g_{ij;k} = 0 \tag{3}$$

où on a noté par ; la dérivée covariante par rapport à D.

Si dans (2) on applique une contraction dans h et j on obtient

$$T^h_{hk} = T_k = (1-n)\omega_k \tag{4}$$

et si dans (2) on tient compte de (4) on a

$$T^h_{jk} = \frac{1}{1-n}\left(T_j \delta^h_k - T_k \delta^h_j\right) \tag{5}$$

donc:

PROPOSITION 1. *La torsion des variétés L_n douées d'une D-connexion est déterminée par le vecteur de torsion T_k (le vecteur de Vranceanu).*

Pour la connexion D, on note avec R^h_{ijk} les composantes du tenseur de courbure, avec $R_{jk} = R^i_{jik}$ les composantes du tenseur de Ricci et avec $R = g^{jk}R_{jk}$ la courbure scalaire.

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DÉFINITION 2. On dira que la variété L_n douée d'une D-connexion est D-symétrique Cartan si on a:

$$\bar{R}^h_{ijk,r} = 0 \tag{6}$$

DÉFINITION 3. On dira que la variété L_n douée d'une D-connexion est D-récurrente, s'il existe un champ vectorial covariant φ_r ainsi qu'on ait:

$$\bar{R}^h_{ijk/r} = \varphi_r \bar{R}^h_{ijk} \tag{7}$$

Si dans (7) on applique une contraction dans h et j on obtient:

$$\bar{R}_{ik} = \varphi_r \bar{R}_{ik} \tag{8}$$

DÉFINITION 4. La variété L_n douée d'une D-connexion qui vérifie (8) est nommée D-Ricci récurrente.

Observation 1. Une variété L_n D-récurrente est aussi D-Ricci récurrente, la réciproque n'est pas en général vraie.

On dit [5] qu'une variété riemannienne L_n est un K_n^* espace, si elle est récurrente, c'est-à-dire:

$$R^h_{ijk,r} = \varphi_r R^h_{ijk} \tag{9}$$

ou si elle est symétrique Cartan c'est-à-dire:

$$R^h_{ijk,r} = 0 \tag{10}$$

et s'il existe un champ vectorial φ_r ainsi que:

$$\varphi_r R^h_{ijk} + \varphi_j R^h_{ikr} + \varphi_k R^h_{irj} = 0 \tag{11}$$

où R^h_{ijk} sont les composantes du tenseur de courbure de la connexion ∇ et par la virgule on a noté la dérivée covariante par rapport à ∇. Les variétés récurrentes (9) vérifient elles-aussi les relations (11) d'après ce qui résulte des identités de Bianchi de deuxième espèce. Les variétés symétriques Cartan qui vérifient (11) et celles récurrentes, ont des propriétés communes en ce qui concerne les champs parallèles, formes canoniques, etc. d'après ce qui a été mis en évidence par A. G. Walker [5] qui les a nommées des espaces K_n^*.

Dans ce qui suit on met en évidence des espaces K_n^* doués d'une D-connexion.

Une connexion semi-symétrique est nommée spéciale si ω_i est gradient. Une telle connexion est caractérisée par:

$$T_{i/j} = T_{j/i} \tag{12}$$

nommée aussi E-connexion [3].

DÉFINITION 5. Une D-connexion qui vérifie (12) sera nommée D-E-connexion.

PROPOSITION 2. *Dans une D-E-connexion le vecteur de Vrănceanu est gradient.*

Observation 2. Dans une D-E-connexion le tenseur de Ricci est symétrique $R_{ij} = R_{ji}$ [3].

Observation 3. On vérifie aisément que si une connexion est D-E-connexion, alors les identités de Bianchi de première espèce ont lieu :

$$R^s_{ijk} + R^s_{jki} + R^s_{kij} = 0 \tag{13}$$

Supposons maintenant la variété L_n douée d'une D-E-connexion. Dans [4] on a montré que dans une telle connexion on a :

$$R^s_{ijk|r} + R^s_{ikr|j} + R^s_{irj|k} = 2(\omega_r R^s_{ijk} + \omega_j R^s_{ikr} + \omega_k R^s_{irj}) \tag{14}$$

De (14) il résulte dans le cas de la D-récurrence (7) ou dans le cas D-symétrique Cartan (6) que dans une telle connexion ou lieu les relations du type (11). On a donc :

PROPOSITION 3. *Dans une D-E-connexion D-récurrente ou D-symétrique Cartan ont lieu les relations (11).*

De cette proposition on a :

DÉFINITION 6. Une variété L_n douée d'une D-E-connexion est nommée D-E-K_n^* si elle est D-récurrente à vecteur ω_r ou D-symétrique Cartan.

Observation 4. Pour les variétés D-E-K_n^* sont valables tous les résultats de A. G. Walker [5] ainsi que ceux obtenus dans [2] pour le vecteur de récurrence ω_i.

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MULTIPLIER RULES FOR OPTIMIZATION PROBLEMS WITH A FINITE NUMBER OF CONSTRAINTS

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REZUMAT. — **Reguli cu multiplicatori pentru probleme de optimizare cu un număr finit de restricţii.** Continuîndu-se cercetările lui M. R. Hestenes [3] şi A. Gittleman [2] legate de mulţimile derivate, se arată în lucrare că soluţiile locale ale problemelor de optimizare cu un număr finit de restricţii satisfac o regulă generală a multiplicatorilor (teorema 4.3). Prin particularizarea acestei reguli pentru anumite clase de probleme de optimizare în spaţii liniare topologice şi în spaţii liniare normate se obţin apoi în mod unitar criterii necesare concrete pentru soluţiile acestor probleme.

1. Introduction. By an *optimization problem with a finite number of constraints* we understand a problem of the following form: minimize $f_0(x)$ subject to

$$x \in X,\ f_i(x) \leqslant 0\ \ (i \in \{1,\ldots, m\}),\ g_j(x) = 0\ \ (j \in \{1,\ldots, n\}),$$

where X is a non-empty set and $f_0, f_1, \ldots, f_m, g_1, \ldots, g_n$ are real-valued functions defined on X.

M. R. Hestenes [3], [4] has shown that the solutions of such a problem satisfy a certain multiplier rule which generalizes the classical Lagrange multiplier rule concerning minimization of real-valued functions subject to equality constraints. His investigations have made use of so-called derived sets which suitably approximate the image of X under the mapping

$$x \in X \mapsto (f_0(x), f_1(x), \ldots, f_m(x), g_1(x), \ldots, g_n(x)) \in \mathbf{R}^{1+m+n}$$

near the image of the solution of the optimization problem.

Hestenes' multiplier rule is applicable to diverse variational problems and optimal control problems. However, it is not usable for such optimization problems in which instead of differentiability only semidifferentiability (see L. W. Neustadt [7]) of the involved functions is assured, not even when these functions are convex. That is why Hestenes could not derive the multiplier rule for convex optimization problems (Theorem 4.2 on pp. 336—337 in [5]) as a corollary to his generalized multiplier rule (Theorem 10.1 on pp. 375—376 in [5]), but had to use another independent procedure.

Generalizing Hestenes' concept of a derived set, A. Gittleman [2] has succeeded in obtaining a multiplier rule which has not the above-mentioned disadvantages of the multiplier rule stated by Hestenes.

By using a slightly modified definition of the derived set and a new approach, we prove in the present paper a local variant of Gittleman's general multi-

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plier rule (Theorem 2.1 in [2]), i. e. a multiplier rule which holds even for local solutions of optimization problems with a finite number of constraints. Furthermore, we point out a few multiplier rules for some special optimization problems in topological linear spaces and normed linear spaces, respectively, which may be of interest in practice since their hypotheses are much easier to verify than those of the general multiplier rule. These multiplier rules, in part already known, but until now proved only by other techniques, will be stated all by applying the general multiplier rule. By proceeding in this way, we clearly emphasize the unifying power of the concept of a derived set.

2. **Notations.** Throughout this paper $\mathbf{R}$ is the set of all real numbers, while $\mathbf{R}^n$ is, for every positive integer n, the usual n-dimensional Euclidean space of all n-tuples $\lambda = (\lambda_1, \ldots, \lambda_n)$ of real numbers. 0_n denotes the zero-vector in $\mathbf{R}^n$, and 1_n the vector whose components are all equal to 1. The inner product of two vectors λ and μ in $\mathbf{R}^n$ is expressed by $\langle \lambda, \mu \rangle$.

As customary, if $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ belong to $\mathbf{R}^n$, we write $\lambda \leqslant \mu$ if $\lambda_j \leqslant \mu_j$ for all $j \in \{1, \ldots, n\}$. The subset of $\mathbf{R}^n$ consisting of all vectors λ such that $\lambda \geqslant 0_n$ is denoted by $\mathbf{R}^n_+$. In particular, $\mathbf{R}_+$ designates the set of all non-negative real numbers.

If M is a subset of a topological linear space, we denote by int M its interior, by conv M its convex hull, by lin M its linear hull, and by cone M its conical hull.

Given a point x_0 in a normed linear space and a number $r > 0$, $B(x_0, r)$ denotes the open and $\bar{B}(x_0, r)$ the closed ball centered at x_0 with radius r. For the intersection of the open ball $B(0_n, r)$ with $\mathbf{R}^n_+$ we use the symbol $B_+(0_n, r)$.

If f is a function from a non-empty set X into $\mathbf{R}^n$ having the components $f_1, \ldots, f_n$, we write $f = (f_1, \ldots, f_n) : X \to \mathbf{R}^n$.

3. **Derived Sets.** In this section let X be a non-empty subset of a topological space X_0, let f_0 be a real-valued function defined on X, f a function from X into $\mathbf{R}^m$, and g a function from X into $\mathbf{R}^n$. Denote by x_0 any point in X.

A subset Γ of $\mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^n$ is said to be a *derived set* for (f_0, f, g) at x_0 with respect to X if it is not empty and if, for every non-empty finite subset

$$\{(\lambda_0^1, \lambda^1, \mu^1), \ldots, (\lambda_0^N, \lambda^N, \mu^N)\} \tag{3.1}$$

of Γ and every number $\varepsilon > 0$, there exist a number $\delta > 0$ and a function $\bar{x} : B_+(0_N, \delta) \to X$ with the following properties:

(DS$_1$) $\bar{x}(0_N) = x_0$ and $\bar{x}$ is continuous at 0_N;

(DS$_2$) the function $a \in B_+(0_N, \delta) \mapsto g(\bar{x}(a)) \in \mathbf{R}^n$ is continuous on $B_+(0_N, \delta)$;

(DS$_3$) the inequalities

$$\frac{1}{\|a\|}\left[f_0(\bar{x}(a)) - f_0(x_0) - \sum_{j=1}^{N} a_j \lambda_0^j\right] \leqslant \varepsilon,$$

$$\frac{1}{\|a\|}\left[f(\bar{x}(a)) - f(x_0) - \sum_{j=1}^{N} a_j \lambda^j\right] \leqslant \varepsilon 1_m$$

hold for all vectors $a = (a_1, \ldots, a_N) \in B_+(0_N, \delta)\backslash\{0_N\}$;

(DS$_4$) $$\lim_{a \to 0_N} \frac{1}{\|a\|}\left[g(\bar{x}(a)) - g(x_0) - \sum_{j=1}^{N} a_j \mu^j\right] = 0_n.$$

A simple result concerning derived sets is given by the proposition which follows and in which instead of (DS$_3$) another condition occurs.

PROPOSITION 3.1. *Let* Γ *be a non-empty subset of* $\mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^n$ *such that for every non-empty finite subset* (3.1) *of* Γ *there exist a number* $\delta > 0$ *and a function* $\bar{x} : B_+(0_N, \delta) \to X$ *satisfying* (DS$_1$), (DS$_2$) *and* (DS$_4$) *as well as*

$$\limsup_{a \to 0_N} \frac{1}{\|a\|}\left[f_0(\bar{x}(a)) - f_0(x_0) - \sum_{j=1}^{N} a_j \lambda_0^j\right] \leqslant 0, \tag{3.2}$$

$$\limsup_{a \to 0_N} \frac{1}{\|a\|}\left[f(\bar{x}(a)) - f(x_0) - \sum_{j=1}^{N} a_j \lambda^j\right] \leqslant 0_m. \tag{3.3}$$

Then Γ *is a derived set for* (f_0, f, g) *at* x_0 *with respect to* X.

Proof. Let (3.1) be any non-empty finite subset of Γ, and let $\varepsilon > 0$ be arbitrary. By the hypotheses of the proposition there exist a number $\delta_0 > 0$ and a function $\bar{x}_0 : B_+(0_N, \delta_0) \to X$ satisfying the following conditions:

(a) $\bar{x}_0(0_N) = x_0$ and $\bar{x}_0$ is continuous at 0_N;

(b) the function $a \in B_+(0_N, \delta_0) \mapsto g(\bar{x}_0(a)) \in \mathbf{R}^n$ is continuous on $B_+(0_N, \delta_0)$;

(c) $\displaystyle\limsup_{a \to 0_N} \frac{1}{\|a\|}\left[f_0(\bar{x}_0(a)) - f_0(x_0) - \sum_{j=1}^{N} a_j \lambda_0^j\right] \leqslant 0$;

(d) $\displaystyle\limsup_{a \to 0_N} \frac{1}{\|a\|}\left[f(\bar{x}_0(a)) - f(x_0) - \sum_{j=1}^{N} a_j \lambda^j\right] \leqslant 0_m$;

(e) $\displaystyle\lim_{a \to 0_N} \frac{1}{\|a\|}\left[g(\bar{x}_0(a)) - g(x_0) - \sum_{j=1}^{N} a_j \mu^j\right] = 0_n$.

In virtue of (c) and (d) we can choose a number $\delta \in [0, \delta_0]$ such that the inequalities

$$\frac{1}{\|a\|}\left[f_0(\bar{x}_0(a)) - f_0(x_0) - \sum_{j=1}^{N} a_j \lambda_0^j\right] \leqslant \varepsilon, \tag{3.4}$$

$$\frac{1}{\|a\|}\left[f(\bar{x}_0(a)) - f(x_0) - \sum_{j=1}^{N} a_j \lambda^j\right] \leqslant \varepsilon 1_m \tag{3.5}$$

hold for all vectors $a = (a_1, \ldots, a_N) \in B_+(0_N, \delta)\backslash\{0_N\}$. Denoting by $\bar{x}$ the restriction of $\bar{x}_0$ to $B_+(0_N, \delta)$, it follows from (a), (b), (3.4), (3.5) and (e) that δ and $\bar{x}$ satisfy all the conditions (DS$_1$) – (DS$_4$). Hence Γ is a derived set for (f_0, f, g) at x_0 with respect to X.

In general a derived set is neither convex nor a cone. But, as the next theorem shows, when necessary one can always assume, without loss of generality, that it has both these properties.

THEOREM 3.2. *Let $\Gamma_0 \subseteq \mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^n$ be a derived set for (f_0, f, g) at x_0 with respect to X, and let Γ be the convex cone generated by Γ_0. Then Γ is a derived set for (f_0, f, g) at x_0 with respect to X.*

Proof. Let $\{\gamma^1, \ldots, \gamma^N\}$ be any non-empty finite subset of Γ, and let $\varepsilon > 0$ be any number. Inasmuch as Γ is generated by Γ_0, we can select a finite number of vectors

$$\gamma_0^1 = (\nu_0^1, \nu^1, \xi^1), \ldots, \gamma_0^K = (\nu_0^K, \nu^K, \xi^K)$$

in Γ_0 as well as numbers $c_{jk} \in \mathbf{R}_+$ ($j \in \{1, \ldots, N\}$, $k \in \{1, \ldots, K\}$) such that

$$\gamma^j = \sum_{k=1}^{K} c_{jk}\gamma_0^k \text{ for all } j \in \{1, \ldots, N\}. \tag{3.6}$$

Representing each vector γ^j ($j \in \{1, \ldots, N\}$) by the triple $(\lambda_0^j, \lambda^j, \mu^j)$, we obtain from (3.6) for every $j \in \{1, \ldots, N\}$ the following equalities:

$$\lambda_0^j = \sum_{k=1}^{K} c_{jk}\nu_0^k, \quad \lambda^j = \sum_{k=1}^{K} c_{jk}\nu^k, \tag{3.7}$$

$$\mu^j = \sum_{k=1}^{K} c_{jk}\xi^k. \tag{3.8}$$

Put

$$\theta = 1 + \sum_{j=1}^{N}\sum_{k=1}^{K} c_{jk}.$$

Since Γ_0 is a derived set for (f_0, f, g) at x_0 with respect to X, there exist a number $\delta_0 > 0$ and a function $\tilde{x}_0 : B_+(0_K, \delta_0) \to X$ satisfying the following conditions:

(a) $\tilde{x}_0(0_K) = x_0$ and $\tilde{x}_0$ is continuous at 0_K;

(b) the function $a \in B_+(0_K, \delta_0) \mapsto g(\tilde{x}_0(a)) \in \mathbf{R}^n$ is continuous on $B_+(0_K, \delta_0)$;

(c) the inequalities

$$f_0(\tilde{x}_0(a)) - f_0(x_0) - \sum_{k=1}^{K} a_k\nu_0^k \leqslant \frac{\varepsilon}{\theta}||a||,$$

$$f(\tilde{x}_0(a)) - f(x_0) - \sum_{k=1}^{K} a_k\nu^k \leqslant \frac{\varepsilon}{\theta}||a||1_m$$

hold for all vectors $a = (a_1, \ldots, a_K) \in B_+(0_K, \delta_0)$;

(d) $\lim_{a \to 0_K} \frac{1}{||a||}\left[g(\tilde{x}_0(a)) - g(x_0) - \sum_{k=1}^{K} a_k\xi^k\right] = 0_n.$

Now, consider the mapping $A = (A_1, \ldots, A_K) : \mathbf{R}^N \to \mathbf{R}^K$ whose components $A_K : \mathbf{R}^N \to \mathbf{R}$ $(k \in \{1, \ldots, K\})$ are defined by

$$A_k a = \sum_{j=1}^{N} a_j c_{jk} \text{ for all } a = (a_1, \ldots, a_N) \in \mathbf{R}^N.$$

Note that

$$Aa \in \mathbf{R}_+^K \text{ and } ||Aa|| \leqslant \sum_{k=1}^{K} \left(\sum_{j=1}^{N} a_j c_{jk} \right) \leqslant \theta ||a|| \tag{3.9}$$

for all $a = (a_1, \ldots, a_N) \in \mathbf{R}_+^N$. Set $\delta = \delta_0/\theta$. In view of (3.9) it follows that A maps $B_+(0_N, \delta)$ into $B_+(0_K, \delta_0)$. Consequently, we can define the function $\bar{x} : B_+(0_N, \delta) \to X$ by

$$\bar{x}(a) = \bar{x}_0(Aa) \text{ for all } a \in B_+(0_N, \delta).$$

This function satisfies all the conditions $(\mathrm{DS}_1) - (\mathrm{DS}_4)$.

Indeed, taking into account that $A0_N = 0_K$ and that A is continuous at 0_N, we see that (a) implies (DS_1). From (b) we conclude that (DS_2) holds, since A is continuous on $B_+(0_N, \delta)$. Next we prove (DS_3). Let $a = (a_1, \ldots, a_N)$ be any vector belonging to $B_+(0_N, \delta)\backslash\{0_N\}$. By using (3.7), (c) and (3.9), it results that

$$f_0(\bar{x}(a)) - f_0(x_0) - \sum_{j=1}^{N} a_j \lambda_0^j =$$

$$= f_0(\bar{x}_0(Aa)) - f_0(x_0) - \sum_{k=1}^{K} (A_k a) \nu_0^k \leqslant \frac{\varepsilon}{\theta} ||Aa|| \leqslant \varepsilon ||a||,$$

and so that

$$\frac{1}{||a||} \left[f_0(\bar{x}(a)) - f_0(x_0) - \sum_{j=1}^{N} a_j \lambda_0^j \right] \leqslant \varepsilon$$

holds. Similarly one shows that

$$\frac{1}{||a||} \left[f(\bar{x}(a)) - f(x_0) - \sum_{j=1}^{N} a_j \lambda^j \right] \leqslant \varepsilon 1_m$$

holds too.

Finally, we prove (DS_4). Let $\varepsilon' > 0$ be arbitrary. From (d) it follows that there is a number $\delta_0' \in]0, \delta_0]$ such that

$$\left|\left| g(\bar{x}_0(a)) - g(x_0) - \sum_{k=1}^{K} a_k \xi^k \right|\right| \leqslant \frac{\varepsilon'}{\theta} ||a|| \tag{3.10}$$

for all $a = (a_1, \ldots, a_K) \in B_+(0_K, \delta_0')$. But in view of the continuity of A at 0_N there is a number $\delta' \in]0, \delta]$ such that

$$Aa \in B_+(0_K, \delta_0') \text{ for all } a \in B_+(0_N, \delta').$$

Given $a = (a_1, \ldots, a_N)$ in $B_+(0_N, \delta')\setminus\{0_N\}$, it follows therefore from (3.10) that

$$\left\| g(\bar{x}_0(Aa)) - g(x_0) - \sum_{k=1}^{K} (A_k a)\xi^k \right\| \leq \frac{\varepsilon'}{\theta} || Aa || \leq \varepsilon' || a ||,$$

which yields by (3.8) the inequality

$$\frac{1}{||a||} \left\| g(\bar{x}(a)) - g(x_0) - \sum_{j=1}^{N} a_j \mu^j \right\| \leq \varepsilon'.$$

Since $\varepsilon' > 0$ was arbitrarily chosen, (DS_4) holds.

The convex cone Γ is accordingly a derived set for (f_0, f, g) at x_0 with respect to X.

4. **The General Multiplier Rule.** In this section we are concerned with the following optimization problem:

(OP) Minimize $f_0(x)$ subject to $x \in X$, $f(x) \leq 0_m$, $g(x) = 0_n$.

Here X is a non-empty subset of a topological space X_0, and f_0, $f = (f_1, \ldots, f_m)$, $g = (g_1, \ldots, g_n)$ are functions defined on X and taking values in $\mathbf{R}$, $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively.

A point $x_0 \in X_0$ is called a *feasible solution* to (OP) if it belongs to X and satisfies the constraints $f(x) \leq 0_m$, $g(x) = 0_n$. Let S denote the set of all feasible solutions to (OP).

A point $x_0 \in X_0$ is said to be a *local solution* to (OP) if it is a feasible solution to (OP) and there exists a neighbourhood U_0 of x_0 such that $f_0(x_0) \leq f_0(x)$ for all $x \in U_0 \cap S$.

Our goal is to state in terms of derived sets a necessary condition for the local solutions to (OP). We begin by proving two auxiliary lemmas.

LEMMA 4.1. *Let $A \subseteq \mathbf{R}^p$ be a neighbourhood of 0_p, and let*

$$G = (G_1, \ldots, G_n) : A \to \mathbf{R}^n$$

be a function satisfying the following conditions:

(L_1) *$G(0_p) = 0_n$ and G is continuous on A;*

(L_2) *G is differentiable at 0_p and the derivative $G'(0_p)$ is surjective.*

Then, whenever a point $c \in \mathbf{R}^p$ satisfies $G'(0_p)c = 0_n$, there exist a number $t_0 > 0$ and a function $\bar{a} : [0, t_0] \to \mathbf{R}^p$ with the following properties:

(P_1) *$\bar{a}(0) = 0_p$ and $\bar{a}'(0) = c$;*

(P_2) *$\bar{a}(t) \in A$ and $G(\bar{a}(t)) = 0_n$ for all $t \in [0, t_0]$.*

Proof. By the surjectivity of $G'(0_p)$ it follows that $p \geq n$ and that the matrix

$$\begin{bmatrix} \frac{\partial G_1}{\partial a_1}(0_p) & \ldots & \frac{\partial G_1}{\partial a_n}(0_p) & \ldots & \frac{\partial G_1}{\partial a_p}(0_p) \\ \ldots & & \ldots & & \ldots \\ \frac{\partial G_n}{\partial a_1}(0_p) & \ldots & \frac{\partial G_n}{\partial a_n}(0_p) & \ldots & \frac{\partial G_p}{\partial a_p}(0_p) \end{bmatrix},$$

has rank n. Without loss of generality we can therefore assume that

$$\det \begin{bmatrix} \frac{\partial G_1}{\partial a_1}(0_p) & \cdots & \frac{\partial G_1}{\partial a_n}(0_p) \\ \cdots & \cdots & \cdots \\ \frac{\partial G_n}{\partial a_1}(0_p) & \cdots & \frac{\partial G_n}{\partial a_n}(0_p) \end{bmatrix} \neq 0.$$

Let $c = (c_1, \ldots, c_p) \in \mathbf{R}^p$ satisfy $G'(0_p)c = 0_n$. Since A is a neighbourhood of 0_p and the function $(t, a) \in \mathbf{R} \times \mathbf{R}^p \mapsto tc + a \in \mathbf{R}^p$ is continuous at $(0, 0_p)$, we can choose some number $r > 0$ such that $tc + a \in A$ for all $(t, a) \in \in [-r, r] \times [-r, r]^p$.

Put $D = [-r, r]$ and $E = [-r, r]^n$. Define the function $F: D \times E \to \mathbf{R}^n$ by

$$F(t, y) = G(tc_1 + y_1, \ldots, tc_n + y_n, tc_{n+1}, \ldots, tc_p)$$

for all $t \in D$, $y = (y_1, \ldots, y_n) \in E$. It is easy to see that F satisfies the following conditions:

(a) F is continuous on $D \times E$;

(b) F is differentiable at $(0, 0_n)$;

(c) $F(0, 0_n) = 0_n$ and $F_t'(0, 0_n) = 0_n$;

(d) $F_y'(0, 0_n)$ is invertible.

By a known implicit function theorem (see B. N. Pšeničnyj [8, Satz 4.7, pp. 87—88]) there exist a number $t_0 > 0$ and a function

$$\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n) : [0, t_0] \to \mathbf{R}^n$$

such that $\bar{y}(0) = 0_n$, $\bar{y}'(0) = 0_n$, $\bar{y}(t) \in E$ and $F(t, \bar{y}(t)) = 0_n$ for all $t \in [0, t_0]$. Define now $\bar{a}: [0, t_0] \to \mathbf{R}^p$ by

$$\bar{a}(t) = (tc_1 + \bar{y}_1(t), \ldots, tc_n + \bar{y}_n(t), tc_{n+1}, \ldots, tc_p)$$

for all $t \in [0, t_0]$. It follows immediately that $\bar{a}$ has the desired properties (P_1) and (P_2).

Remark. Lemma 4.1 can be expressed in terms of tangent vectors. To this end set

$$A_0 = \{a \in A : G(a) = 0_n\},$$

and recall that a vector $c \in \mathbf{R}^p$ is said to be a *tangent vector* of A_0 at 0_p if there are a number $t_0 > 0$ and a function $\bar{a}: [0, t_0] \to A_0$ such that $\bar{a}(0) = 0_p$ and $\bar{a}'(0) = c$. According to this definition, Lemma 4.1 asserts that every vector $c \in \mathbf{R}^p$ satisfying $G'(0_p)c = 0_n$ is a tangent vector of A_0 at 0_p. Note that the converse is always true. Indeed, if $c \in \mathbf{R}^p$ is a tangent vector of A_0 at 0_p, and $\bar{a}: [0, t_0] \to A_0$ is a function such that $\bar{a}(0) = 0_p$ and $\bar{a}'(0) = c$, then we have

$$G(\bar{a}(t)) = 0_n \text{ for all } t \in [0, t_0].$$

Hence the derivative of the function $t \in [0, t_0] \mapsto G(\bar{a}(t)) \in \mathbf{R}^n$ at 0 must be 0_n, i. e. $G'(0_p)c = 0_n$. Consequently, under the assumptions of Lemma 4.1, the set of all tangent vectors of A_0 at 0_p coincides with the kernel of $G'(0_p)$.

LEMMA 4.2. *If the convex hull of a non-empty subset M of the space $\mathbf{R}^p$ does not contain 0_p as an interior point, then there exists a vector $\lambda^* \in \mathbf{R}^p\setminus\{0_p\}$ such that*

$$\inf\{\langle\lambda, \lambda^*\rangle : \lambda \in M\} \geqslant 0. \tag{4.1}$$

Proof. By a well-known version of the separation theorem for convex sets (see W. W. Breckner [1, Propoziţia 9.4.2, p. 174]) there is a vector λ^* in $\mathbf{R}^p\setminus\{0_p\}$ such that

$$\inf\{\langle\lambda, \lambda^*\rangle : \lambda \in \operatorname{conv} M\} \geqslant 0.$$

Obviously this inequality implies (4.1).

We are now in a position to establish the main theorem of this paper, which will be referred to as the *general multiplier rule* for (OP).

THEOREM 4.3. *If $x_0 \in X_0'$ is a local solution to* (OP) *and $\Gamma \subseteq \mathbf{R}\times\mathbf{R}^m\times\mathbf{R}^n$ is a derived set for (f_0, f, g) at x_0 with respect to X, then there exists $(\lambda_0^*, \lambda^*, \mu^*) \in \mathbf{R}_+\times\mathbf{R}_+^m\times\mathbf{R}^n\setminus\{(0, 0_m, 0_n)\}$ such that*

$$\langle f(x_0), \lambda^*\rangle = 0; \tag{4.2}$$

$$\inf\{\lambda_0\lambda_0^* + \langle\lambda, \lambda^*\rangle + \langle\mu, \mu^*\rangle : (\lambda_0, \lambda, \mu) \in \Gamma\} \geqslant 0. \tag{4.3}$$

Proof. Let M_0 denote the set of those μ in the space $\mathbf{R}^n$ for which there exists a point $(\lambda_0, \lambda) \in \mathbf{R}\times\mathbf{R}^m$ such that $(\lambda_0, \lambda, \mu) \in \Gamma$. If 0_n is not in the convex hull of M_0, then by Lemma 4.2 we can find a vector μ^* in $\mathbf{R}^n\setminus\{0_n\}$ satisfying

$$\inf\{\langle\mu, \mu^*\rangle : \mu \in M_0\} \geqslant 0.$$

By choosing $\lambda_0^* = 0$ and $\lambda^* = 0_m$, it follows that the triple $(\lambda_0^*, \lambda^*, \mu^*)$ is in $\mathbf{R}_+\times\mathbf{R}_+^m\times\mathbf{R}^n\setminus\{(0, 0_m, 0_n)\}$ as well as that (4.2) and (4.3) hold. Consequently, in this case the theorem is proved.

It still remains to consider the case when 0_n belongs to the interior of conv M_0. In this event there is a number $r_0 > 0$ such that $\bar{B}(0_n, r_0) \subseteq M_0$. Now, let $\{e^1, \ldots, e^n\}$ be the standard basis for the space $\mathbf{R}^n$. All the points $y^i = r_0e^i$ and $y^{n+i} = -r_0e^i$, where $i \in \{1, \ldots, n\}$, are then in conv M_0 and satisfy the following equalities:

$$y^1 + \ldots + y^{2n} = 0_n;\ \operatorname{lin}\{y^1, \ldots, y^{2n}\} = \mathbf{R}^n. \tag{4.4}$$

For the sake of simplicity put $2n = p$. Taking into account that each y^i $(i \in \{1, \ldots, p\})$ is expressible in the form

$$y^i = \sum_{j=1}^{m_i} c_{ij}\mu^{ij},$$

where $c_{i1} > 0, \ldots, c_{im_i} > 0$, while

$$(\lambda_0^{i1}, \lambda^{i1}, \mu^{i1}), \ldots, (\lambda_0^{im_i}, \lambda^{im_i}, \mu^{im_i}) \tag{4.5}$$

are points selected from Γ, it is seen that (4.4) implies

$$\sum_{i=1}^{p} \sum_{j=1}^{m_i} c_{ij} \mu^{ij} = 0_n \tag{4.6}$$

and

$$\text{lin} \{\mu^{11}, \ldots, \mu^{1m_1}, \ldots, \mu^{p1}, \ldots, \mu^{pm_p}\} = \mathbf{R}^n.$$

Next let $I(x_0)$ be the set of active indices at x_0, that is, the set of indices k in $\{1, \ldots, m\}$ for which $f_k(x_0) = 0$. Two cases must be distinguished.

Case 1: $I(x_0)$ is not empty. Then denote by K the number of indices belonging to $I(x_0)$. For convenience we can assume that the functions $f_1, \ldots, f_m$ are in such a manner numbered that $I(x_0) = \{1, \ldots, K\}$.

Define now M to be the set consisting of all points $(\alpha_0, \alpha, \beta)$ in $\mathbf{R} \times \mathbf{R}^K \times \times \mathbf{R}^n$ for which there exists a point $(\lambda_0, \lambda, \mu) \in \Gamma$ such that $\lambda_k < \alpha_k$ $(k \in \in \{0, 1, \ldots, K\})$ and $\mu = \beta$. We claim that the origin of the space $\mathbf{R} \times \mathbf{R}^K \times \times \mathbf{R}^n$ is not in the convex hull of M. Suppose that the contrary holds. Then we can find numbers $c_{01} > 0, \ldots, c_{0m_0} > 0$ as well as points

$$(\lambda_0^{01}, \lambda^{01}, \mu^{01}), \ldots, (\lambda_0^{0m_0}, \lambda^{0m_0}, \mu^{0m_0}) \tag{4.7}$$

in Γ such that

$$\sum_{j=1}^{m_0} c_{0j} \lambda_k^{0j} < 0 \ (k \in \{0, 1, \ldots, K\}) \text{ and } \sum_{j=1}^{m_0} c_{0j} \mu^{0j} = 0_n. \tag{4.8}$$

According to (4.8) and (4.6), it follows that

$$\sum_{j=1}^{m_0} c_{0j} \lambda_k^{0j} + \theta_0 \sum_{i=1}^{p} \sum_{j=1}^{m_i} c_{ij} \lambda_k^{ij} < 0 \ (k \in \{0, 1, \ldots, K\}), \tag{4.9}$$

$$\sum_{j=1}^{m_0} c_{0j} \mu^{0j} + \theta_0 \sum_{i=1}^{p} \sum_{j=1}^{m_i} c_{ij} \mu^{ij} = 0_n, \tag{4.10}$$

where $\theta_0 > 0$ is a suitably chosen number.

Let N be the number of elements of the subset of Γ built up from the points (4.5) and (4.7). In the sequel the elements of this subset will be denoted simply by $(\lambda_0^j, \lambda^j, \mu^j)$ $(j \in \{1, \ldots, N\})$. Corresponding to this modification we shall also change the notation of the coefficients in (4.9) and (4.10). Thus the conclusion we have arrived at by the above-made reflections may be formulated as follows: there are N points

$$\gamma^1 = (\lambda_0^1, \lambda^1, \mu^1), \ldots, \gamma^N = (\lambda_0^N, \lambda^N, \mu^N)$$

in Γ and N numbers $c_1 > 0, \ldots, c_N > 0$ such that

$$\sum_{j=1}^{N} c_j \lambda_k^j < 0 \quad (k \in \{0, 1, \ldots, K\}); \tag{4.11}$$

$$\sum_{j=1}^{N} c_j \mu^j = 0_n; \tag{4.12}$$

$$\operatorname{lin}\{\mu^1, \ldots, \mu^N\} = \mathbf{R}^n. \tag{4.13}$$

Put $(c_1, \ldots, c_N) = c$.

In view of (4.11) and the inequality $f_k(x_0) < 0$ which holds for every $k \in \{1, \ldots, m\} \setminus I(x_0)$, we can choose a number $\theta > 0$ satisfying the following inequalities:

$$\sum_{j=1}^{N} c_j \lambda_k^j \leqslant -2\theta \text{ for all } k \in \{0, 1, \ldots, K\}; \tag{4.14}$$

$$f_k(x_0) \leqslant -\theta \text{ for all } k \in \{1, \ldots, m\} \setminus I(x_0). \tag{4.15}$$

Since Γ is a derived set for (f_0, f, g) at x_0 with respect to X, there exist for the subset $\{\gamma^1, \ldots, \gamma^N\}$ of Γ which was singled out above and for $\varepsilon = \theta/(1 + \|c\|)$ a number $\delta > 0$ and a function $\bar{x} : B_+(0_N, \delta) \to X$ having the properties $(DS_1)-(DS_4)$.

For each $a = (a_1, \ldots, a_N) \in \mathbf{R}^N$ let now $a^+ = (a_1^+, \ldots, a_N^+)$ be the point whose coordinates are defined by

$$a_j^+ = (|a_j| + a_j)/2 \quad (j \in \{1, \ldots, N\}).$$

Note that

$$a^+ \in \mathbf{R}_+^N \quad \text{and} \quad \|a^+\| \leqslant \sum_{j=1}^{N} |a_j| \leqslant N\|a\|$$

for all $a = (a_1, \ldots, a_N) \in \mathbf{R}^N$. It follows that

$$a^+ \in B_+(0_N, \delta) \text{ for every } a \in B(0_N, r), \tag{4.16}$$

where $r > 0$ is a number satisfying $Nr \leqslant \delta$. Consequently, we are able to define the function $G : B0_N, r) \to \mathbf{R}^n$ by

$$G(a) = g(\bar{x}(a^+)) + \sum_{j=1}^{N} (a_j - a_j^+)\mu^j$$

for all $a = (a_1, \ldots, a_N) \in B(0_N, r)$.

Remark that $G(0_N) = 0_n$, since $\bar{x}(0_N) = x_0$ and $g(x_0) = 0_n$. Taking into account that the functions

$$a \in B(0_N, r) \mapsto a^+ \in B_+(0_N, \delta), \; a \in B_+(0_N, \delta) \mapsto g(\bar{x}(a)) \in \mathbf{R}^n$$

are continuous on $B(0_N, r)$ and $B_+(0_N, \delta)$ respectively, it results that G is continuous on $B(0_N, r)$. Finally, (DS_4) implies the differentiability of G at 0_N and that

$$G'(0_N)a = \sum_{j=1}^{N} a_j \mu^j \text{ for all } a = (a_1, \ldots, a_N) \in \mathbf{R}^N.$$

Hence, by (4.13) $G'(0_N)$ is surjective, while (4.12) yields $G'(0_N)c = 0_n$. So all the conditions in Lemma 4.1 are satisfied. By applying this lemma, we conclude that there exist a number $t_0 > 0$ and a function $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_N):$ $[0, t_0] \to \mathbf{R}^N$ with the following properties:

$$\bar{a}(0) = 0_N \text{ and } \bar{a}'(0) = c; \tag{4.17}$$

$$\bar{a}(t) \in B(0_N, r) \text{ for all } t \in [0, t_0]; \tag{4.18}$$

$$G(\bar{a}(t)) = 0_n \text{ for all } t \in [0, t_0]. \tag{4.19}$$

From (4.17) we obtain

$$\lim_{\substack{t \to 0 \\ t > 0}} \bar{a}(t) = 0_N, \tag{4.20}$$

$$\lim_{\substack{t \to 0 \\ t > 0}} \frac{1}{t} \bar{a}(t) = c. \tag{4.21}$$

Making use of (4.21) and of the fact that c lies in the interior of $\mathbf{R}^N_+$, we see that there exists a number $t_1 \in]0, t_0]$ such that

$$\bar{a}(t) \in \operatorname{int} \mathbf{R}^N_+ \text{ for all } t \in]0, t_1].$$

In view of (4.18) and (4.16) we have therefore

$$\bar{a}(t) \in B_+(0_N, \delta) \setminus \{0_N\} \text{ for all } t \in]0, t_1].$$

Put now $\bar{x}_0(t) = \bar{x}(\bar{a}(t))$ for every $t \in [0, t_1]$. We shall prove that $\bar{x}_0(t)$ is for sufficiently small $t>0$ a feasible solution to (OP) and satisfies $f_0(\bar{x}_0(t)) < f_0(x_0)$.

First note that

$$\bar{x}_0(t) \in X \text{ for all } t \in [0, t_1]; \tag{4.22}$$

$$g(\bar{x}_0(t)) = 0_n \text{ for all } t \in [0, t_1]. \tag{4.23}$$

Next observe that (4.21) and (4.14) imply

$$\lim_{\substack{t \to 0 \\ t > 0}} \sum_{j=1}^{N} \frac{1}{t} \bar{a}_j(t) \lambda_k^j = \sum_{j=1}^{N} c_j \lambda_k^j \leq -2\theta$$

for all $k \in \{0, 1, \ldots, K\}$. Besides, (4.21) furnishes

$$\lim_{\substack{t \to 0 \\ t \to 0}} \left\| \frac{1}{t} \bar{a}(t) \right\| = \| c \|.$$

So a number $t_2 \in]0, t_1]$ can be chosen such that

$$\sum_{j=1}^{N} \frac{1}{t} \bar{a}_j(t)\lambda_k^j \leqslant -\theta \text{ for all } k \in \{0, 1, \ldots, K\},$$

$$\left\|\frac{1}{t}\bar{a}(t)\right\| < 1 + ||c||$$

for every $t \in]0, t_2]$. For each $t \in]0, t_2]$ these inequalities may be written as follows:

$$\sum_{j=1}^{N} \bar{a}_j(t)\lambda_k^j \leqslant -\theta t \text{ for all } k \in \{0, 1, \ldots, K\},$$

$$||\bar{a}(t)|| < (1 + ||c||)t.$$

Together with (DS_3) they imply

$$f_k(\bar{x}_0(t)) = f_k(x_0) + \left[f_k(\bar{x}_0(t)) - f_k(x_0) - \sum_{j=1}^{N} \bar{a}_j(t)\lambda_k^j\right] +$$

$$+ \sum_{j=1}^{N} \bar{a}_j(t)\lambda_k^j \leqslant f_k(x_0) + \varepsilon||\bar{a}(t)|| + \sum_{j=1}^{N} \bar{a}_j(t)\lambda_k^j < f_k(x_0) +$$

$$+ t[\varepsilon(1 + ||c||) - \theta] = f_k(x_0)$$

for every $k \in \{0, 1, \ldots, K\}$ and every $t \in]0, t_2]$. Consequently, we have shown that

$$f_0(\bar{x}_0(t)) < f_0(x_0) \text{ for all } t \in]0, t_2]; \tag{4.24}$$

$$f_k(\bar{x}_0(t)) < 0 \text{ for all } k \in \{1, \ldots, K\} \text{ and all } t \in]0, t_2]. \tag{4.25}$$

In the end choose a number $t_3 \in]0, t_2]$ such that

$$||\bar{a}(t)|| \leqslant \frac{\theta}{\varepsilon + ||\lambda^1|| + \ldots + ||\lambda^N||} \text{ for all } t \in]0, t_3].$$

In view of (4.20) such a number exists. By (4.15) and (DS_3) we have then

$$f_k(\bar{x}_0(t)) = f_k(x_0) + \left[f_k(\bar{x}_0(t)) - f_k(x_0) - \sum_{j=1}^{N} \bar{a}_j(t)\lambda_k^j\right] +$$

$$+ \sum_{j=1}^{N} \bar{a}_j(t)\lambda_k^j \leqslant -\theta + \varepsilon||\bar{a}(t)|| + \left(\sum_{j=1}^{N} ||\lambda^j||\right)||\bar{a}(t)|| \leqslant 0$$

for every $k \in \{1, \ldots, m\} \setminus I(x_0)$ and every $t \in]0, t_3]$. Together with (4.25) this result expresses that

$$f(\bar{x}_0(t)) \leqslant 0_m \text{ for all } t \in]0, t_3]. \tag{4.26}$$

From (4.22), (4.26), (4.23) and (4.24) we conclude that $\bar{x}_0(t)$ is a feasible solution to (OP) and satisfies $f_0(\bar{x}_0(t)) < f_0(x_0)$ whenever $t \in]0, t_3]$. But, on the other hand, (4.20) and (DS_1) imply

$$\lim_{\substack{t\to 0\\ t>0}} \bar{x}_0(t) = x_0.$$

Thus x_0 cannot be a local solution to (OP), which contradicts our hypothesis. This proves our assertion that the origin of the space $\mathbf{R} \times \mathbf{R}^K \times \mathbf{R}^n$ is not in the convex hull of M.

Consequently, there exists, by virtue of Lemma 4.2, a vector

$$(\lambda_0^*, \nu^*, \mu^*) \in \mathbf{R} \times \mathbf{R}^K \times \mathbf{R}^n \setminus \{(0, 0_K, 0_n)\}$$

such that

$$\inf \{\alpha_0\lambda_0^* + \langle \alpha, \nu^* \rangle + \langle \beta, \mu^* \rangle : (\alpha_0, \alpha, \beta) \in M\} \geqslant 0. \tag{4.27}$$

We claim that $\lambda_0^* \in \mathbf{R}_+$ and $\nu^* \in \mathbf{R}_+^K$. To prove this fix any point $(\alpha_0, \alpha, \beta)$ in M. For all $t \in \mathbf{R}_+$ we have $(\alpha_0 + t, \alpha, \beta) \in M$. By (4.27) it results that

$$(\alpha_0 + t)\lambda_0^* + \langle \alpha, \nu^* \rangle + \langle \beta, \mu^* \rangle \geqslant 0 \text{ for all } t > 0,$$

which yields

$$\lambda_0^* \geqslant -\frac{1}{t}(\alpha_0\lambda_0^* + \langle \alpha, \nu^* \rangle + \langle \beta, \mu^* \rangle) \text{ for all } t > 0.$$

Letting $t \to +\infty$, we obtain $\lambda_0^* \geqslant 0$. Now, let $\{E^1, \ldots, E^K\}$ be the standard basis of the space $\mathbf{R}^K$, and let k be any index belonging to $\{1, \ldots, K\}$. For all $t \in \mathbf{R}_+$ we have $(\alpha_0, \alpha + tE^k, \beta) \in M$. Hence it follows from (4.27) that

$$\alpha_0\lambda_0^* + \langle \alpha + tE^k, \nu^* \rangle + \langle \beta, \mu^* \rangle \geqslant 0 \text{ for all } t > 0.$$

This inequality can be put in the alternative form

$$\langle E^k, \nu^* \rangle \geqslant -\frac{1}{t}(\alpha_0\lambda_0^* + \langle \alpha, \nu^* \rangle + \langle \beta, \mu^* \rangle) \text{ for all } t > 0.$$

Letting $t \to +\infty$, we get $\nu_k^* \geqslant 0$. Since k was arbitrarily chosen in $\{1, \ldots, K\}$, we have $\nu^* \in \mathbf{R}_+^K$.

Define $\lambda^* = (\lambda_1^*, \ldots, \lambda_m^*)$ to be the vector in $\mathbf{R}^m$ whose coordinates are

$$\lambda_k^* = \begin{cases} \nu_k^* & \text{if } k \in I(x_0) \\ 0 & \text{if } k \in \{1, \ldots, m\} \setminus I(x_0). \end{cases}$$

Obviously λ^* belongs to $\mathbf{R}_+^m$ and satisfies (4.2). Moreover, (4.3) will hold because of (4.27).

Case 2: $I(x_0)$ is empty. In this case we proceed as follows. Let M be the set composed of all points $(\alpha_0, \beta) \in \mathbf{R} \times \mathbf{R}^n$ for which there exists a point $(\lambda_0, \lambda, \mu) \in \Gamma$ such that $\lambda_0 < \alpha_0$ and $\mu = \beta$. Similar arguments to those used in the preceding case show that the origin of the space $\mathbf{R} \times \mathbf{R}^n$ is not in the convex hull of M. Applying once more Lemma 4.2 provides a vector $(\lambda_0^*, \mu^*) \in \mathbf{R} \times \mathbf{R}^n \setminus \{(0, 0_n)\}$ such that

$$\inf \{\alpha_0 \lambda_0^* + \langle \beta, \mu^* \rangle : (\alpha_0, \beta) \in M\} \geqslant 0.$$

As before, by the use of this inequality, it is proved that $\lambda_0^* \geqslant 0$. By choosing $\lambda^* = 0_m$, it follows that the triple $(\lambda_0^*, \lambda^*, \mu^*)$ is in $\mathbf{R}_+ \times \mathbf{R}_+^m \times \mathbf{R}^n \setminus \{(0, 0_m, 0_n)\}$ as well as that (4.2) and (4.3) hold.

5. **Multiplier Rules in Topological Linear Spaces.** We continue to investigate the local solutions to (OP) but unlike the previous section we shall impose now conditions on the space X_0 as well as on the involved functions f_0, f and g. In this way we shall obtain some special forms of the general multiplier rule.

We start by assuming in the present section that X_0 is a real or complex topological linear space. This assumption will remain unchanged throughout the whole section without further specification.

In this as well as in the next section we need some special types of functions. For completeness we recall here their definitions.

Let Y be a non-empty subset of X_0. A function $F : Y \to \mathbf{R}$ is said to be

(i) *convex* if

$$F((1 - \alpha)y_1 + \alpha y_2) \leqslant (1 - \alpha)F(y_1) + \alpha F(y_2)$$

for all $\alpha \in]0, 1[$ and all $y_1, y_2 \in Y$ such that $(1 - \alpha)y_1 + \alpha y_2 \in Y$;

(ii) *affine* if

$$F((1 - \alpha)y_1 + \alpha y_2) = (1 - \alpha)F(y_1) + \alpha F(y_2)$$

for all $\alpha \in]0, 1[$ and all $y_1, y_2 \in Y$ such that $(1 - \alpha)y_1 + \alpha y_2 \in Y$;

(iii) *sublinear* if

$$F(\alpha y_1 + \beta y_2) \leqslant \alpha F(y_1) + \beta F(y_2)$$

for all $\alpha, \beta \in \mathbf{R}_+$ and all $y_1, y_2 \in Y$ such that $\alpha y_1 + \beta y_2 \in Y$;

(iv) *linear* if

$$F(\alpha y_1 + \beta y_2) = \alpha F(y_1) + \beta F(y_2)$$

for all $\alpha, \beta \in \mathbf{R}$ and all $y_1, y_2 \in Y$ such that $\alpha y_1 + \beta y_2 \in Y$.

According to the above definitions a vector-valued function

$$G = (G_1, \ldots, G_p) : Y \to \mathbf{R}^p$$

is said to be *convex* (resp. *affine, sublinear, linear*) if all its components $G_1, \ldots, G_p$ are convex (resp. affine, sublinear, linear).

After these preliminaries we are now able to prove our first specialization of the general multiplier rule.

THEOREM 5.1. *Let* $x_0 \in X_0$ *be a local solution to* (OP), *and suppose that* Y *is a subset of* X_0 *satisfying the following conditions*:

(A_1) Y *is not empty and convex*;

(A_2) *for every non-empty finite subset* $\{y_1, \ldots, y_N\}$ *of* Y *there exists a number* $\delta > 0$ *such that*

$$x_0 + \sum_{j=1}^{N} a_j y_j \in X \tag{5.1}$$

whenever $(a_1, \ldots, a_N) \in B_+(0_N, \delta)$, *and for which the function*

$$(a_1, \ldots, a_N) \in B_+(0_N, \delta) \mapsto g\left(x_0 + \sum_{j=1}^{N} a_j y_j\right) \in \mathbf{R}^n \tag{5.2}$$

is continuous on $B_+(0_N, \delta)$;

(A_3) *there exist convex functions*

$$F_0 : Y \to \mathbf{R} \text{ and } F = (F_1, \ldots, F_m) : Y \to \mathbf{R}^m$$

such that

$$\limsup_{\substack{t \to 0 \\ t > 0}} \frac{1}{t} [f_0(x_0 + ty) - f_0(x_0)] \leqslant F_0(y), \tag{5.3}$$

$$\limsup_{\substack{t \to 0 \\ t > 0}} \frac{1}{t} [f(x_0 + ty) - f(x_0)] \leqslant F(y) \tag{5.4}$$

for all $y \in Y$, *and for every convex polytope* $P \subseteq Y$ *the convergence in these inequalities is uniform with respect to* $y \in P$;

(A_4) *there exists an affine function*

$$G = (G_1, \ldots, G_m) : Y \to \mathbf{R}^n$$

such that

$$\lim_{\substack{t \to 0 \\ t > 0}} \frac{1}{t} [g(x_0 + ty) - g(x_0)] = G(y) \tag{5.5}$$

for all $y \in Y$, *and for every convex polytope* $P \subseteq Y$ *the convergence in this equality is uniform with respect to* $y \in P$.

Then there is a vector

$$(\lambda_0^*, \lambda^*, \mu^*) \in \mathbf{R}_+ \times \mathbf{R}_+^m \times \mathbf{R}^n \setminus \{(0, 0_m, 0_n)\}$$

such that

$$\langle f(x_0), \lambda^* \rangle = 0, \tag{5.6}$$

$$\inf \{F_0(y)\lambda_0^* + \langle F(y), \lambda^* \rangle + \langle G(y), \mu^* \rangle : y \in Y\} \geqslant 0. \tag{5.7}$$

Proof. It suffices to show that $\Gamma = \{(F_0(y), F(y), G(y)) : y \in Y\}$ is a derived set for (f_0, f, g) at x_0 with respect to X.

Let $\{(F_0(y_1), F(y_1), G(y_1)), \ldots, (F_0(y_N), F(y_N), G(y_N))\}$ be any non-empty finite subset of Γ, where $y_1, \ldots, y_N$ are points taken from Y. By condition (A_2) we can select a number $\delta > 0$ such that (5.1) holds whenever $(a_1, \ldots, a_N) \in B_+(0_N, \delta)$, and for which the function (5.2) is continuous on $B_+(0_N, \delta)$. Define now $\bar{x} : B_+(0_N, \delta) \to X$ by

$$\bar{x}(a) = x_0 + \sum_{j=1}^{N} a_j y_j \quad \text{for all } a = (a_1, \ldots, a_N) \in B_+(0_N, \delta).$$

Obviously this function $\bar{x}$ satisfies (DS_1) and (DS_2).

Next we show that (3.2), (3.3) and (DS_4) hold with $(\lambda_0^j, \lambda^j, \mu^j)$ replaced by $(F_0(y_j), F(y_j), G(y_j))$ $(j \in \{1, \ldots, N\})$. To this end denote by P the convex hull of the points $y_1, \ldots, y_N$. Since Y is convex, we have $P \subseteq Y$. Fix now two indices $i \in \{0, 1, \ldots, m\}$ and $k \in \{1, \ldots, n\}$. Let $\varepsilon > 0$ be arbitrary. By the conditions (A_3) and (A_4) there exists a number $t_0 \in]0, \delta]$ such that

$$f_i(x_0 + ty) - f_i(x_0) < t\left[F_i(y) + \frac{\varepsilon}{N}\right], \tag{5.8}$$

$$|g_k(x_0 + ty) - g_k(x_0) - tG_k(y)| < t\frac{\varepsilon}{N} \tag{5.9}$$

for all $t \in]0, t_0[$ and all $y \in P$. Let $a = (a_1, \ldots, a_N)$ be any point in $B_+(0_N, t_0/N) \setminus \{0_N\}$. Put

$$t = \sum_{j=1}^{N} a_j \quad \text{and} \quad y = \sum_{j=1}^{N} \frac{a_j}{t} y_j.$$

Then we have $0 < t \leqslant N||a|| < t_0$ and $y \in P$. Taking into account that F_i is convex and G_k affine, (5.8) and (5.9) imply

$$f_i\left(x_0 + \sum_{j=1}^{N} a_j y_j\right) - f_i(x_0) < t\left[F_i\left(\sum_{j=1}^{N} \frac{a_j}{t} y_j\right) + \frac{\varepsilon}{N}\right] \leqslant$$

$$\leqslant \sum_{j=1}^{N} a_j F_i(y_j) + t\frac{\varepsilon}{N} \leqslant \sum_{j=1}^{N} a_j F_i(y_j) + \varepsilon||a||,$$

$$\left|g_k\left(x_0 + \sum_{j=1}^{N} a_j y_j\right) - g_k(x_0) - \sum_{j=1}^{N} a_j G_k(y_j)\right| =$$

$$= \left|g_k\left(x_0 + \sum_{j=1}^{N} a_j y_j\right) - g_k(x_0) - tG_k\left(\sum_{j=1}^{N} \frac{a_j}{t} y_j\right)\right| < t\frac{\varepsilon}{N} \leqslant \varepsilon||a||.$$

Hence we have shown that

$$\frac{1}{||a||}\left[f_i(\bar{x}(a)) - f_i(x_0) - \sum_{j=1}^{N} a_j F_i(y_j)\right] < \varepsilon,$$

$$\left|\frac{1}{||a||}\left[g_k(\bar{x}(a)) - g_k(x_0) - \sum_{j=1}^{N} a_j G_k(y_j)\right]\right| < \varepsilon$$

for all $a \in B_+(0_N, t_0/N)\setminus\{0_N\}$. Since $\varepsilon > 0$ was arbitrarily chosen, it follows that

$$\limsup_{a \to 0_N} \frac{1}{||a||}\left[f_i(\bar{x}(a)) - f_i(x_0) - \sum_{j=1}^{N} a_j F_i(y_j)\right] \leqslant 0,$$

$$\lim_{a \to 0_N} \frac{1}{||a||}\left[g_k(\bar{x}(a)) - g_k(x_0) - \sum_{j=1}^{N} a_j G_k(y_j)\right] = 0.$$

But $i \in \{0, 1, \ldots, m\}$ and $k \in \{1, \ldots, n\}$ have been also arbitrarily chosen. Therefore (3.2), (3.3) and (DS_4) hold, where $\lambda_0^j = F_0(y_j)$, $\lambda^j = F(y_j)$ and $\mu^j = G(y_j)$ $(j \in \{1, \ldots, N\})$.

In virtue of Proposition 3.1 it follows that Γ is a derived set for (f_0, f, g) at x_0 with respect to X. Consequently, we can appeal to Theorem 4.3 and conclude that the assertion of our theorem is true.

As a corollary to Theorem 5.1 we derive the following well-known multiplier rule for convex optimization problems.

COROLLARY 5.2. *Let X be a non-empty convex subset of X_0, let $f_0 : X \to \mathbf{R}$ and $f = (f_1, \ldots, f_m) : X \to \mathbf{R}^m$ be convex functions, and let $g = (g_1, \ldots, g_n) : X \to \mathbf{R}^n$ be an affine function. Suppose that $x_0 \in X_0$ is a solution to* (OP). *Then there exists a vector*

$$(\lambda_0^*, \lambda^*, \mu^*) \in \mathbf{R}_+ \times \mathbf{R}_+^m \times \mathbf{R}^n\setminus\{(0, 0_m, 0_n)\}$$

such that

$$\min\{f_0(x)\lambda_0^* + \langle f(x), \lambda^*\rangle + \langle g(x), \mu^*\rangle : x \in X\} = f_0(x_0)\lambda_0^*. \quad (5.10)$$

Proof. We put $Y = X - x_0$ and show that the conditions (A_1) — (A_4) in Theorem 5.1 are satisfied.

Obviously Y is not empty and convex. In order to prove (A_2) let $\{y_1, \ldots, y_N\}$ be any non-empty finite subset of Y. For each $j \in \{1, \ldots, N\}$ we can choose a point $x_j \in X$ such that $y_j = x_j - x_0$. Since X is convex, we have then

$$x_0 + \sum_{j=1}^{N} a_j y_j = \left(1 - \sum_{j=1}^{N} a_j\right) x_0 + \sum_{j=1}^{N} a_j x_j \in X$$

for all $(a_1, \ldots, a_N) \in B_+(0_N, 1/N)$. Moreover, taking into account that

$$g\left(x_0 + \sum_{j=1}^{N} a_j y_j\right) = g(x_0) + \sum_{j=1}^{N} a_j [g(x_j) - g(x_0)],$$

for all $(a_1, \ldots, a_N) \in B_+(0_N, 1/N)$, it follows that the function

$$(a_1, \ldots, a_N) \in B_+(0_N, 1/N) \mapsto g\left(x_0 + \sum_{j=1}^{N} a_j y_j\right) \in \mathbf{R}^n$$

is continuous on $B_+(0_N, 1/N)$. Thus (A_2) holds.

Since f_0, f are convex and g is affine, we have

$$f_0(x_0 + ty) = f_0((1-t)x_0 + t(x_0 + y)) \leqslant (1-t)f_0(x_0) + tf_0(x_0 + y),$$
$$f(x_0 + ty) = f((1-t)x_0 + t(x_0 + y)) \leqslant (1-t)f(x_0) + tf(x_0 + y),$$
$$g(x_0 + ty) = g((1-t)x_0 + t(x_0 + y)) = (1-t)g(x_0) + tg(x_0 + y),$$

i. e.

$$\frac{1}{t}\,[f_0(x_0 + ty) - f_0(x_0)] \leqslant f_0(x_0 + y) - f_0(x_0),$$

$$\frac{1}{t}\,[f(x_0 + ty) - f(x_0)] \leqslant f(x_0 + y) - f(x_0),$$

$$\frac{1}{t}\,[g(x_0 + ty) - g(x_0)] = g(x_0 + y) - g(x_0)$$

for all $t \in]0, 1]$ and all $y \in Y$. Defining now $F_0 : Y \to \mathbf{R}$, $F : Y \to \mathbf{R}^m$ and $G : Y \to \mathbf{R}^n$ by $F_0(y) = f_0(x_0 + y) - f_0(x_0)$, $F(y) = f(x_0 + y) - f(x_0)$, $G(y) = g(x_0 + y) - g(x_0)$ for all $y \in Y$, it results that F_0, F are convex, G is affine, and that (5.3), (5.4), (5.5) hold, as well as that the convergence in (5.3), (5.4), (5.5) is uniform with respect to $y \in Y$. Consequently, the conditions (A_3) and (A_4) are satisfied.

By Theorem 5.1 there is then a vector

$$(\lambda_0^*, \lambda^*, \mu^*) \in \mathbf{R}_+ \times R_+^m \times \mathbf{R}^n \setminus \{(0, 0_m, 0_n)\}$$

for which (5.6) and (5.7) hold. In virtue of the definitions of the functions F_0, F and G, we obtain from (5.6) and (5.7) the equality (5.10).

From Theorem 5.1 we derive also the next result which is related to another already known multiplier rule (see L. W. Neustadt [7, Theorem 4, p. 65]).

COROLLARY 5.3. *Let $x_0 \in X_0$ be a local solution to* (OP), *and suppose that the following conditions are satisfied:*

(B_1) *for every non-empty finite subset $\{z_1, \ldots, z_N\}$ of $X - x_0$ there exists a number $\delta > 0$ such that*

$$x_0 + \sum_{j=1}^{N} a_j z_j \in X \tag{5.11}$$

whenever $(a_1, \ldots, a_N) \in B_+(0_N, \delta)$, and for which the function

$$(a_1, \ldots, a_N) \in B_+(0_N, \delta) \mapsto g\left(x_0 + \sum_{j=1}^{N} a_j z_j\right) \in \mathbf{R}^n \tag{5.12}$$

is continuous on $B_+(0_N, \delta)$;

(B_2) *there exist convex functions*

$$F_0 : \text{cone}\,(X - x_0) \to \mathbf{R} \text{ and } F : \text{cone}\,(X - x_0) \to \mathbf{R}^m$$

such that the inequalities (5.3) *and* (5.4) *hold for all* $y \in \text{cone}(X - x_0)$, *and for every convex polytope* $P \subseteq \text{cone}\,(X - x_0)$ *the convergence in these inequalities is uniform with respect to* $y \in P$;

(B_3) *there exists an affine function*

$$G : \text{cone}\,(X - x_0) \to \mathbf{R}^n$$

such that equality (5.5) *holds for all* $y \in \text{cone}\,(X - x_0)$, *and for every convex polytope* $P \subseteq \text{cone}\,(X - x_0)$ *the convergence in this equality is uniform with respect to* $y \in P$.

Then there is a vector

$$(\lambda_0^*,\ \lambda^*,\ \mu^*) \in \mathbf{R}_+ \times \mathbf{R}_+^m \times \mathbf{R}^n \setminus \{(0,\ 0_m,\ 0_n)\}$$

satisfying

$$\langle f(x_0),\ \lambda^* \rangle = 0,$$

$$\inf\,\{F_0(y)\lambda_0^* + \langle F(y),\ \lambda^* \rangle + \langle G(y),\ \mu^* \rangle : y \in \text{cone}\,(X - x_0)\} \geqslant 0.$$

Proof. Take in Theorem 5.1 as Y the set cone $(X - x_0)$. Then all the hypotheses of Theorem 5.1 are satisfied. Indeed, note first that the conditions (A_3) and (A_4) hold, since in conformity with our choice of Y, they coincide with (B_2) and (B_3), respectively. Next we prove (A_2).

Let $\{y_1, \ldots, y_N\}$ be a non-empty finite subset of Y. Then there exist N numbers $\alpha_1 > 0, \ldots, \alpha_N > 0$ and N points $z_1, \ldots, z_N$ in $X - x_0$ such that $y_j = \alpha_j z_j$ for every $j \in \{1, \ldots, N\}$. Set $\alpha_0 = \max\,\{\alpha_1, \ldots, \alpha_N\}$. By condition ($B_1$) we can select a number $\delta_0 > 0$ such that (5.11) holds whenever $(a_1, \ldots, a_N) \in B_+(0_N,\ \delta_0)$, and for which the function (5.12) is continuous on $B_+(0_N,\ \delta_0)$. Choosing $\delta = \delta_0/\alpha_0$, it follows that (5.1) holds for all $(a_1, \ldots, a_N) \in B_+(0_N,\ \delta)$, and that the function (5.2) is continuous on $B_+(0_N,\ \delta)$. Thus (A_2) holds.

In particular, for every pair of points y' and y'' belonging to Y there exists in virtue of (A_2) a numbre $\delta > 0$ such that

$$a_1 y' + a_2 y'' \in X - x_0 \text{ for all } (a_1,\ a_2) \in B_+(0_2,\ \delta). \tag{5.13}$$

Now let y denote a convex combination of y' and y''. From (5.13) it follows that $ty \in X - x_0$ for all $t \in [0, \delta[$. Therefore we have $y \in \frac{2}{\delta}\,(X - x_0)$, which shows that y belongs to Y. Hence Y is convex. With this it is proved that all the conditions (A_1)–(A_4) are satisfied.

By applying Theorem 5.1, it follows that the assertion of the corollary is true.

6. Multiplier Rules in Normed Linear Spaces. In this section we state multiplier rules under the assumption that X_0 is a real or complex normed linear space.

THEOREM 6.1. *Let $x_0 \in X_0$ be a local solution to (OP), and suppose that condition* (B_1) *as well as the following conditions are satisfied:*

(C_1) *there exist sublinear functions*

$$F_0 : \text{cone}\,(X - x_0) \to \mathbf{R},$$

$$F = (F_1, \ldots, F_m) : \text{cone}\,(X - x_0) \to \mathbf{R}^m$$

such that

$$\limsup_{x \to x_0} \frac{1}{\|x - x_0\|} [f_0(x) - f_0(x_0) - F_0(x - x_0)] \leq 0,$$

$$\limsup_{x \to x_0} \frac{1}{\|x - x_0\|} [f(x) - f(x_0) - F(x - x_0)] \leq 0_m;$$

(C_2) *there exists a linear function*

$$G = (G_1, \ldots, G_n) : \text{cone}\,(X - x_0) \to \mathbf{R}^n$$

such that

$$\lim_{x \to x_0} \frac{1}{\|x - x_0\|} [g(x) - g(x_0) - G(x - x_0)] = 0_n.$$

Then there is a vector

$$(\lambda_0^*, \lambda^*, \mu^*) \in \mathbf{R}_+ \times \mathbf{R}_+^m \times \mathbf{R}^n \setminus \{(0, 0_m, 0_n)\}$$

satisfying

$$\langle f(x_0), \lambda^* \rangle = 0,$$

$$\inf \{F_0(x - x_0)\lambda_0^* + \langle F(x - x_0), \lambda^* \rangle + \langle G(x - x_0), \mu^* \rangle : x \in X\} \geq 0.$$

Proof. We show that the conditions (B_2) and (B_3) in Corollary 5.3 are satisfied. Let P be a convex polytope, say

$$P = \text{conv}\,\{y_1, \ldots, y_N\},$$

contained in cone $(X - x_0)$. Then there exist N numbers $\alpha_1 > 0, \ldots, \alpha_N > 0$ and N points $z_1, \ldots, z_N$ in $X - x_0$ such that $y_j = \alpha_j z_j$ for every $j \in \{1, \ldots, N\}$. Set $\alpha_0 = \max\{\alpha_1, \ldots, \alpha_N\}$. By condition (B_1) we can select a number $\delta_0 > 0$ such that (5.11) holds whenever $(a_1, \ldots, a_N) \in B_+(0_N, \delta_0)$. Choosing $\delta = \delta_0/\alpha_0$, it follows that

$$x_0 + \sum_{j=1}^{N} a_j y_j \in X \text{ for all } (a_1, \ldots, a_N) \in B_+(0_N, \delta).$$

Thus we have $x_0 + tP \subseteq X$ for all $t \in [0, \delta[$.

Next we choose a number r_0 for which $\max\{\|y_1\|, \ldots, \|y_N\|\} \leq r_0$. Obviously we have $\|y\| \leq r_0$ for all $y \in P$.

Fix now two indices $i \in \{0, 1, \ldots, m\}$ and $k \in \{1, \ldots, n\}$. Given $\varepsilon > 0$, we can find in view of (C_1) and (C_2) a number $r < 0$ such that

$$f_i(x) - f_i(x_0) \leqslant F_i(x - x_0) + \frac{\varepsilon}{r_0} ||x - x_0||, \tag{6.1}$$

$$|g_k(x) - g_k(x_0) - G_k(x - x_0)| \leqslant \frac{\varepsilon}{r_0} ||x - x_0|| \tag{6.2}$$

for all $x \in B(x_0, r) \cap X$. Put $t_0 = \min\{\delta, r/r_0\}$. Then we have $x_0 + ty \in B(x_0, r) \cap X$ for all $t \in [0, t_0[$ and all $y \in P$. Therefore (6.1) and (6.2) imply

$$f_i(x_0 + ty) - f_i(x_0) \leqslant tF_i(y) + t\frac{\varepsilon}{r_0} ||y|| \leqslant t[F_i(y) + \varepsilon],$$

$$|g_k(x_0 + ty) - g_k(x_0) - tG_k(y)| \leqslant t\frac{\varepsilon}{r_0} ||y|| \leqslant \varepsilon t,$$

i.e.

$$\frac{1}{t}[f_i(x_0 + ty) - f_i(x_0)] \leqslant F_i(y) + \varepsilon,$$

$$\left| \frac{1}{t}[g_k(x_0 + ty) - g(x_0)] - G_k(y) \right| \leqslant \varepsilon$$

for all $t \in]0, t_0[$ and all $y \in P$. Since $\varepsilon > 0$ was arbitrarily chosen, it follows that

$$\limsup_{\substack{t \to 0 \\ t > 0}} \frac{1}{t}[f_i(x_0 + ty) - f_i(x_0)] \leqslant F_i(y),$$

$$\lim_{\substack{t \to 0 \\ t > 0}} \frac{1}{t}[g_k(x_0 + ty) - g_k(x_0)] = G_k(y)$$

for all $y \in P$, and that the convergence in this relations is uniform with respect to $y \in P$. Because i and k were arbitrary indices from $\{0, 1, \ldots, m\}$ and $\{1, \ldots, n\}$, respectively, the conditions (B_2) and (B_3) hold, as claimed.

By applying Corollary 5.3, we obtain the assertion of our theorem.

In the end we derive the well-known multiplier rule for optimization problems involving Fréchet differentiable functions:

COROLLARY 6.2. *Let $x_0 \in X_0$ be a local solution to* (OP) *satisfying the following conditions:*

(D_1) *x_0 is interior to X:*

(D_2) *f_0, f and g are Fréchet differentiable at x_0;*

(D_3) *there exists a number $r_0 > 0$ such that g is continuous on $B(x_0, r_0) \cap X$.*
Then there is a vector

$$(\lambda_0^*, \lambda^*, \mu^*) \in \mathbf{R}_+ \times \mathbf{R}_+^m \times \mathbf{R}^n \setminus \{(0, 0_m, 0_n)\}$$

satisfying

$$\langle f(x_0), \lambda^* \rangle = 0, \tag{6.3}$$

$$\sum_{i=0}^{m} \lambda_i^* f_i'(x_0) + \sum_{j=1}^{n} \mu_j^* g_j'(x_0) = 0. \tag{6.4}$$

Proof. The conditions (B_1), (C_1) and (C_2) are satisfied. Indeed, since x_0 is an interior point of X, there exists a number $r \in]0, r_0]$ such that $\bar{B}(x_0, r) \subseteq X$. Given a non-empty finite subset $\{z_1, \ldots, z_N\}$ of $X - x_0$, put $\delta = r/(1 + ||z_1|| + \ldots + ||z_N||)$. Then we have

$$\left|\left|\sum_{j=1}^{N} a_j z_j\right|\right| \leqslant \sum_{j=1}^{N} a_j ||z_j|| \leqslant ||a|| \sum_{j=1}^{N} ||z_j|| \leqslant \delta \sum_{j=1}^{N} ||z_j|| < r$$

for all $a = (a_1, \ldots, a_N) \in B_+(0_N, \delta)$. From this follows

$$x_0 + \sum_{j=1}^{N} a_j z_j \in \bar{B}(x_0, r) \text{ whenever } (a_1, \ldots, a_N) \in B_+(0_N, \delta).$$

Therefore (5.11) holds for all $(a_1, \ldots, a_N) \in B_+(0_N, \delta)$, and the function (5.12) is continuous on $B_+(0_N, \delta)$. In other words, condition (B_1) is satisfied. The conditions (C_1) and (C_2) are also satisfed, in virtue of (D_2).

By applying Theorem 6.1, we conclude that there exists a vector

$$(\lambda_0^*, \lambda^*, \mu^*) \in \mathbf{R}_+ \times \mathbf{R}_+^m \times \mathbf{R}^n \setminus \{(0, 0_m, 0_n)\}$$

such that (6.3) holds as well as

$$\inf \left\{ \sum_{i=0}^{m} \lambda_i^* f_i'(x_0)(x - x_0) + \sum_{j=1}^{n} \mu_j^* g_j'(x_0)(x - x_0) : x \in X \right\} \geqslant 0. \tag{6.5}$$

Consider now an arbitrary point $u \in X_0$ and choose a number $t_0 > 0$ such that $x_0 + tu \in X$ for all $t \in [-t_0, t_0]$. According to (6.5), we have then

$$t\left[\sum_{i=0}^{m} \lambda_i^* f_i'(x_0)u + \sum_{j=1}^{n} \mu_j^* g_j'(x_0)u\right] \geqslant 0$$

for all $t \in [-t_0, t_0]$, and hence

$$\sum_{i=0}^{m} \lambda_i^* f_i'(x_0)u + \sum_{j=1}^{n} \mu_j^* g_j'(x_0)u = 0.$$

Since u was arbitrarily chosen in X_0, equality (6.4) must hold.

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THE DISTRIBUTION OF THE LIMIT CYCLES FOR A GENERALIZED DYNAMICAL SYSTEM

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REZUMAT. — Distribuţia ciclurilor limită pentru un sistem dinamic generalizat. În lucrare se generalizează sistemele de tip Liénard studiate în [4—6] şi se dau condiţii de existenţă a unui număr fixat de cicluri limită care conţin un număr impar de puncte singulare.

1. Introduction. The number of limit cycles for dynamical systems is of practical interest, because these represent the periodic solutions in the phase plane for the considered systems. The use of Poincaré-Bendixon theorem allows to determine the domains in which there exist or not limit cycles for second order autonomous systems. In this case there were generally studied Liénard-type systems, which have rich and various technical and scientific applications. One usually considers that the given system has a single equilibrium position (singular point,) which coincides with just the origin of the coordinate frame. There have been given in this case conditions for the existence and uniqueness of the limit cycles in a certain domain. If the system has a certain peculiar form, then in [1], [3] there have been given existence conditions for exactly n limit cycles. In [2] there have been given the conditions in which a Liénard-type system admits a limit cycle containing several singular points. Also, in [6], [7] the existence of a fixed number of limit cycles surrounding $2n+1$ singular points is studied. These results have been generalized in [4] and [5]. We shall give further another generalization for Liénard-type systems, in the case when these admit an odd number of singular points.

2. Existence conditions for limit cycles. Let be the system:

$$\begin{cases} \dot{x} = h(x)\bar{y} - F(x) \\ \dot{\bar{y}} = -g(x) + h(x)\bar{y} \end{cases} \tag{1}$$

where the functions

$$g(x),\ h(x),\ F(x) = \int_0^x f(s)\,ds$$

are continuous and fulfil all the conditions which ensure the existence and uniqueness of any Cauchy problem formulated upon the system (1).

THEOREM 1. *If* $g(\alpha_i) = 0$, $i = -1, 0, 1, \ldots, 2n-1$, $g(x) > 0$ $\forall x \in (\alpha_{2i-3}, \alpha_{2i-2})$ and $x \in (\alpha_{2n-1}, \infty)$; $g(x) < 0$ $\forall x \in (\alpha_{2i-2}, \alpha_{2i-1})$ *and* $x \in (-\infty,$

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$\alpha_{-1})$, $i = \overline{1, n}$, $F(\alpha_i) = 0$; $f(\alpha_i) = 0$. *If there exist the functions* $\varphi_k(x)$, $k \geq 2$ *and the systems of numbers:*

$$x_{-k} < x_{-k+1} < \ldots < x_{-1} < \alpha_{-1}; \ \alpha_{2n-1} < x_1 < \ldots < x_k$$

such that we have $\varphi_i(0) = 0$, $\varphi_i(x) \neq x$, $h(x) \cdot [\varphi_i'(x) - 1](-1)^i < 0 \ \forall x \in (x_{-i}, x_i)$, $i = \overline{1, k}$, $h(x) > 0$ *and*

$$F_i(x_{-i})(-1)^{i+1} < \Phi_i(x)(-1)^{i+1} < F_i(x_i)(-1)^{i+1}$$
$$F_i(x_{-i})(-1)^{i+1} \leq (-1)^{i+1} F_i(x) \leq F_i(x_i)(-1)^{i+1},$$
$$\forall x \in [x_{-i}, x_i], \ i = \overline{1, k}$$

where

$$F_i(x) = F(x) - h(x)\varphi_i(x); \ \Phi_i(x) = \frac{F(x)\varphi_i'(x)}{h(x)[\varphi_i'(x) - 1]} - \varphi_i(x) - \frac{g(x)}{h(x)[\varphi_i'(x) - 1]},$$

then the system (1) *admits* $k - 1$ *limit cycles which contain inside* $2n + 1$ *singular points, in every domain* $[x_{-i}, x_i]$, $i = \overline{1, k}$ *existing at most* $i - 1$ *limit cycles, out of which* $[i/2]$ *are unstable and* $[(i - 1)/2]$ *are stable.*

Proof. We prove the theorem for $k = 2$, by using a method analogous to Zhilevich's [7] one. In the system (1) we make the substitution:

$$\bar{y} = y + \varphi_1(x) \tag{2}$$

obtaining the system:

$$\begin{cases} \dot{x} = h(x)y - F_1(x) \\ \dot{y} = -h(x) \cdot [\varphi_1'(x) - 1] \cdot (y - \Phi(x)) \end{cases} \tag{3}$$

In order to apply the Poincaré-Bendixon method, we construct a ring domain as follows: let C_1 be a rectangle having the legs parallel to the coordinate axes and the vertices in the points $B_1(x_1, F_1(x_1))$ and $B_{-1}(x_{-1}, F_1(x_{-1}))$. From the conditions of Theorem 1 follows easily that all the trajectories of the system (3), for increasing t, cross the legs of this rectangle, coming inside it. Then we make the following substitution in the system (3):

$$\bar{\bar{y}} = y + \varphi_1(x) - \varphi_2(x) \tag{4}$$

obtaining the system:

$$\begin{cases} \dot{x} = h(x)\bar{\bar{y}} - F_2(x) \\ \dot{\bar{\bar{y}}} = -h(x) \cdot [\varphi_2'(x) - 1] \cdot (\bar{\bar{y}} - \Phi_2(x)) \end{cases} \tag{5}$$

We denote by $\bar{C}_1$ the closed curve in which the contour C_1 is transformed through (4). The trajectories of the system (5) cross $\bar{C}_1$ for increasing t and come inside it. We construct then the rectangle C_2 with the vertices in the points $B_2(x_2, F_2(x_2))$, $B_{-2}(x_{-2}, F_2(x_{-2}))$ and with the legs parallel to the coordinate axes. In the conditions of the theorem, the trajectories of the system (5) cross the legs of this one, getting of it, and $C_1 \subset C_2$. On the basis of Poincaré-Bendixon theorem, there exists in the ring domain bounded by $\overline{C}_1$ an C_2 at least an unstable limit cycle ; then the theorem is proved for $k = 2$. For $k \geqslant 3$ the proof is analogous.

THEOREM 2. *If the conditions of Theorem* 1 *are fulfilled for* $k \geqslant 1$ *and if there are fulfilled the conditions:* $f(x) < 0$,

$$F(x)g(x)/h(x) < 0; \quad \forall x \in (\beta_{-1}, \beta_1), \quad x \neq \alpha_i, \ f(\beta_{-1}) =$$
$$= f(\beta_1), \ \beta_{-1} < \alpha_{-1}, \ \beta_1 > \alpha_{2n-1}, \ F(\beta_{-1}) = F(\beta_1) = 0$$

and

$$\int_0^{\beta_i} g(x)/h(x)dx \geqslant 0, \qquad i = -1, 1;$$

$$\int_0^{t} g(x)/h(x)dx \leqslant \min_{i=-1,1} \int_0^{\beta_i} g(x)/h(x)dx, \qquad \forall t \in (\alpha_1, \alpha_{2n-2}),$$

then the system (1) *has at least* k *limit cycles surrounding* $2n + 1$ *singular points, in each domain* (x_{-i}, x_i) *existing at least* i *limit cycles, out of which* $[(i + 1)/2]$ *are stable and* $[i/2]$ *are unstable.*

Proof. One proceeds analogous to the case of Theorem 1, but a frontier of the first ring domain is the curve

$$\frac{(y + \varphi_1(x))^2}{2} + \int_0^x g(t)/h(t)dt = C_0$$

where

$$C_0 = \min_{i=-1;1} \int_0^{\beta_i} g(s)/h(s)ds.$$

It follows that inside this curve there are no closed trajectories and cycles, all these getting outside.

REMARKS 1°. By means of a proceeding analogous to Theorem 3 of [4], one can prove for $i = 1$ that all the limit cycles, if they exist, are distributed into (x_{-1}, x_1).

2°. If the conditions of Theorem 1 are adequately modified, then the nature of the cycle stability is inverted.

3°. The system (1) generalized the system (1) of [4].

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INTERPOLATION FUNCTION WITH CONTROL OF DERIVATIVES ON INTERPOLATION POINTS

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REZUMAT. — Funcţie interpolatoare cu control al derivatelor pe punctele de interpolare. În lucrare se defineşte funcţia interpolatoare de la (3) care interpolează funcţia dată $f: [a, b] \to R$ pe punctele diviziunii Δ definită la (1). Cu ajutorul vectorilor de ajustare definiţi la (2) se pot controla derivatele funcţiei G (teoremele 1 şi 2), ceea ce constituie un avantaj în trasarea curbelor cu ajutorul calculatorului.

Let $f: [a, b] \to \mathbf{R}$ be a bounded function and Δ an arbitrary partition of finite interval $[a, b]$,

$$\Delta: \{a = x_0 < x_1 < x_2 < \ldots x_{m-1} < x_m = b\}. \tag{1}$$

To each interval $I_i = [x_i, x_{i+1}]$, $i = \overline{1, m-1}$, we consider the „fitting vectors"

$$Y_i = (y_{i,0}, y_{i,1}, \ldots, y_{i,n_i-1}, y_{i,n_i}) \in \mathbf{R}^{n_i+1}, \tag{2}$$

where

$y_{ij} (i = \overline{0, m-1}, j = \overline{1, n_i - 1})$ are given real numbers;

$y_{i,0} = f(x_i)$, $i = \overline{1, m-1}$, and $y_{m-1,n_{m-1}} = f(x_m)$.

Relatively to function f, partition Δ and vector Y_i we consider the function $G: [a, b] \to \mathbf{R}$,

$$G(x) = \begin{cases} \sum_{i=0}^{m-1} \sum_{j=0}^{n_i} J_{n_i,j}(x) y_{ij}, & \text{if } x \neq x_i \\ f(x_i) & , \text{if } x = x_i \end{cases} \quad ; \quad i = \overline{1, m-1} \tag{3}$$

where

$$J_{n_i,j}(x) = C_{n_i}^j \frac{(x_{i+1} - x)_+^{n_i - j} (x - x_i)_+^j}{(x_{i+1} - x_i)^{n_i}}. \tag{4}$$

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As usual, by $(z-c)_+^p$ one means:

$$(z-c)_+^p = \begin{cases} (z-c)^p, & \text{if } z \geqslant c \\ 0 \quad , & \text{if } z < c \end{cases} \tag{5}$$

LEMMA. *The function G defined by* (3) *has a Bernstein basis on each interval* I_i, $i = \overline{0,\ m-1}$.

The proof is immediately. Indeed from (4) and (5) we have, for $x \in I_i$,

$$J_{n_i,j}(x) = C_{n_i}^i \frac{(x_{i+1}-x)^{n_i-j}(x-x_i)^j}{(x_{i+1}-x_i)^{n_i}} \tag{6}$$

After the transformation

$$x = (x_{i+1}-x_i)t + x_i, \tag{7}$$

we obtain

$$J_{n_i,j}(x(t)) = C_{n_i}^j (1-t)^{n_i-j}\ t^j,\ t \in [0, 1]. \tag{8}$$

THEOREM 1. *If* $a_i^{(p)}$, $(i = \overline{1, m};\ p = \overline{0, q})$, *and* $b_i^{(r)}$, $(i = \overline{0,\ m-1},\ r = \overline{0,\ s},$ $q + s \leqslant n_i - 2)$ *are given real numbers with* $a_i^0 = b_i^0 = f(x_i)$, *then we can determine the vectors* Y_{i-1} *and* Y_i *defined by* (2) *so that*

$$G^{(p)}(x_i - 0) = a_i^{(p)} \text{ and } G^{(r)}(x_i + 0) = b_i^{(r)} \tag{9}$$

Proof. Using formulas (5.70) and (5.71), from [1] page 142, we deduce that

$$\left(\sum_{j=0}^{n_{i-1}} C_{n_{i-1}}^j (1-t)^{n_{i-1}-j}\ t^j\ y_{i-1,j}\right)_{t=1}^{(p)} = \frac{n_{i-1}!}{(n_{i-1}-p)!} \sum_{k=0}^{p} (-1)^k C_p^k\, y_{i-1,j-k} \tag{10}$$

and

$$\left(\sum_{j=0}^{n_i} C_{n_i}^j (1-t)^{n_i-j}\ t^j\ y_{i,j}\right)_{t=0}^{(k)} = \frac{n_i!}{(n_i-r)!} \sum_{k=0}^{r} (-1)^{r-k} C_r^k\, y_{i,j} \tag{11}$$

Taking into account by (6), (7), (8) and the operator

$$\frac{d}{dx} = \frac{1}{x_s - x_{s-1}} \frac{d}{dt},\quad (s = \overline{1,\ m})$$

we obtaine

$$G^{(p)}(x_i - 0) = \frac{n_{i-1}!}{(x_i - x_{i-1})^p (n_{i-1}-p)!} \sum_{k=0}^{p} (-1)^k C_p^k\, y_{i-1,n_{i-1}-k} \tag{12}$$

and respectively

$$G^{(r)}(x_i + 0) = \frac{n_i!}{(x_{i+1}-x_i)^r\ (n_i - r)!} \sum_{k=0}^{r} (-1)^{r-k} C_r^k\, y_{i,k}. \tag{13}$$

From (9), (12) and (13) follows the system of equations

$$\sum_{k=0}^{p}(-1)^k C_p^k y_{i-1,n_{i-1}-k} = \frac{(x_i - x_{i-1})^p (n_{i-1}-p)!}{n_{i-1}!} a_i^p, \; p = \overline{0, q} \tag{14}$$

$$\sum_{k=0}^{r}(-1)^{r-k} C_r^k y_{i,k} = \frac{(x_{i+1} - x_i)^r (n_i - r)!}{n_i!} b_i^r, \; (r = \overline{0, s}), \tag{15}$$

which gives the components $y_{i-1,n_{i-1}-k}$ and $y_{i,k}$ of fitting vectors Y_{i-1}, respectively Y_i and Theorem 1 is proved.

THEOREM 2. *Let $C_i^{(s)}$ be given real numbers. If*

$$\frac{n_{i-1}!}{(x_i - x_{i-1})^s (n_{i-1}-s)!}\sum_{k=0}^{s}(-1)^{s-k} C_s^k y_{i-1,n_{i-1}-k} =$$

$$= \frac{n_i!}{(x_{i+1}-x_i)^s (n_i - s)!}\sum_{k=0}^{s}(-1)^{s-k} C_s^k y_{i,k} = C_i^{(s)}, \tag{16}$$

then

$$G^{(s)}(x_i) = C_i^{(s)}, \; (i = \overline{1, m-1}) \tag{17}$$

and for $i = 0$, $i = m$ we have

$$G^{(s)}(a+0) = \frac{n_0!}{(x_1 - a)^s (n_0 - s)!}\sum_{k=0}^{s}(-1)^{s-k} C_s^k y_{0,k} \tag{18}$$

respectively

$$G^{(s)}(b-0) = \frac{n_{m-1}!}{(b - x_{m-1})^s (n_{m-1}-s)!}\sum_{k=0}^{s}(-1)^k C_s^k y_{m-1,n_{m-1}-k}. \tag{19}$$

The proof follows from theorem 1, if we take in (9) $p = r = s$ and $a_i^{(s)} = b_i^{(s)} = C_i^{(s)}$.

Remarks:

1. If $y_{i-1,n_{i-1}-1} = y_{i-1,n_{i-1}-2} = \ldots = y_{i-1,n_{i-1}-p}$ and $y_{i,1} = y_{i,2} = \ldots = y_{i,r}$ the formulas (10) and (11) become

$$\left(\sum_{j=0}^{n_{i-1}} C_{n_i}^j (1-t)^{n_{i-1}-j} t^j y_{i-1,j}\right)_{t=1}^{(p)} = \frac{n_{i-1}!}{(n_{i-1}-p)!}(y_{i-1,n_{i-1}} - y_{i-1,n_{i-1}-1}) \tag{10'}$$

respectively

$$\left(\sum_{j=0}^{n_i} C_{n_i}^j (1-t)^{n_i-j} t^j y_{i,j}\right)_{t=0}^{(r)} = \frac{n_i!}{(n_i-r)!}(-1)^r (y_{i,0} - y_{i,1}) \tag{11'}$$

Here we have used formulas

$$\sum_{i=1}^{p} (-1)^i C_p^i = -1 \text{ and } \sum_{i=1}^{r} (-1)^{r-i} C_r^i = (-1)^{r+1}.$$

Consequently the formulas (12) and (13) will have following forms:

$$G^{(p)}(x_i - 0) = \frac{n_{i-1}!}{(x_i - x_{i-1})(n_{i-1} - p)!} (y_{i-1,n_{i-1}} - y_{i-1,n_{i-1}-1}) \tag{12'}$$

respectively

$$G^{(r)}(x_i + 0) = \frac{n_i!}{(x_{i+1} - x)^r (n_i - r)!} (-1)^r (y_{i,0} - y_{i,1}) \tag{13'}$$

2. If the partition Δ defined at (1) is uniformly, that is $x_i - x_{i-1} = (b - - a)/m$, $i = 1, m$, then all formulas which containe $x_i - x_{i-1}$ will have simplier forms.

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A MODIFICATION OF THE INITIAL SPHERE CHOICE FOR KHACHIYAN'S ALGORITHM

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REZUMAT. — O modificare a alegerii sferei iniţiale în algoritmul lui Khachiyan. Lucrarea prezintă un nou mod de calculare a razei sferei iniţiale şi de alegere a centrului acesteia, in ipoteza că sistemul de inegalităţi $Ax \leqslant b$ ($A \in Z^m \times Z^n$, $b \in Z^m$), căruia i se testează consistenţa prin algoritmul lui Khachiyan, conţine şi inegalitatea $x \geqslant 0$. Se obţine o diminuare considerabilă a razei (şi volumului) sferei iniţiale, ducînd la o substanţială reducere a numărului de iteraţii.

1. Introduction. The algorithm proposed by Khachiyan [1] tests the consistency of the system

$$Ax \leqslant b \tag{1}$$

of linear inequalities, where $A \in \mathbf{Z}^m \times \mathbf{Z}^n$ and $b \in \mathbf{Z}^m$. This algorithm builds a finite sequence of ellipsoids (being called for this reason the ellipsoid algorithm) which contain solutions of the system (1) if this one is consistent.

The first ellipsoid is chosen as being a sphere, with the centre O, containing solutions of the system (1) if (1) is consistent.

After this algorithm was published, in 1979 [1], one noticed that the radius of the initial sphere can be diminished; such proposals were made in [3], [4] and [5]. We give here a new manner of calculating this radius, as well as a new choice for the centre of the initial sphere, into the frame of the assumption that one adds to the system (1) the inequality

$$x \geqslant 0, \tag{2}$$

which usually holds in the linear programming problems where Khackiyan's algorithm is applied.

2. The ellipsoid algorithm. Khachiyan's algorithm uses the quantity

$$L = \left[\sum_i \sum_j \log_2(|a_{ij}| + 1) + \sum_i \log_2(|b_i| + 1) + \log_2(mn)\right] + 1 \tag{3}$$

which represents the minimum lenght needed for storing the input informations in binary code.

The idea of the algorithm is very simple. In order to solve (1), the ellipsoids

$$E_k = \{y \mid y = x_k + B_k z, \ \ ||z|| \leqslant 1\} \sim (x_k, B_k), \tag{4}$$

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which contain the solutions of (1) from the initial sphere, are built step by step until the centre x_k of E_k is an admissible point, or a beforehand fixed number of iterations is performed. If an inequality of (1) is not verified by x_k, for instance $A_i x_k > b_k$, where $A_i = (a_{i1}, a_{i2}, \ldots, a_{in})$, this inequality induces the semispace $\{y \mid A_i(y - x_k) \leqslant 0\}$ and, with this, a semiellipsoid

$$E_k/2 = \{y \in E_k \mid A_i(y - x_k) \leqslant 0\}.$$

One builds then an ellipsoid $E'_k \sim (x'_k, B'_k)$ which contains $E_k/2$. Since the calculations are performed with a finite accuracy, one will calculate the ellipsoid E_{k+1}, which is the ellipsoid E'_k enlarged with the factor $1 + 1/(16\, n^2)$, and the inclusion $E_k/2 \subset E_{k+1}$ is verified.

The ellipsoid algorithm consists of:

Input: $n, m, a_{ij}, b_i, i = \overline{1, m}, j = \overline{1, n}$.

Output: (1) is compatible or not.

Step 1 (initialization): Let $k := 0$, $x_0 := 0$, $B_0 := \text{diag}\,(2^L, \ldots, 2^L)$, $\theta_0 := 2^L$.

Step 2 (test): Compute $\theta(x_k) := \max_i \{A_i x_k - b_i\}$.

If $\theta(x_k) \leqslant 2^{-L}$, then (1) is compatible and if $\theta(x_k) \leqslant 0$ then x_k is an exact solution of this one; else, x_k is an approximate solution stop.

If $k > 6n^2L$, then (1) is not compatible, stop.

Step 3 (iteration): Let i_k be the index for which $A_{i_k} x_k - b_{i_k} = \theta(x_k)$. Let $\theta_{k+1} := \min \{\theta_k, \theta(x_k)\}$ and compute $\eta_k = B_k A_{i_k}^T$; $\|\eta_k\| = \sqrt{\eta_k^T \eta_k}$; $\bar{\eta}_k = \eta_k/\|\eta_k\|$; $x_{k+1} := x_k - B_k \bar{\eta}_k/(n+1)$; $B_{k+1} := (1 + 1/(16\, n^2))(1/\sqrt{n^2 - 1})(B_k + (\sqrt{(n-1)/(n+1)} - 1)\bar{\eta}_k \bar{\eta}_k^T)$. Put $k := k + 1$ and take again with the step 2.

Hence the initial sphere has the radius $R_1 = 2^L$.

For a theoretical substantiation of the algorithm, see [2].

3. **Modification of the first step.** Khachiyan shows in [3] that, instead of the value of L, which is large, one can use the value

$$L' = [\log_2(2\Delta n)] + 1, \qquad (5)$$

the radius of the initial sphere being $R_2 = \Delta\sqrt{n}$, where Δ is a constant which majorizes the modules of all the possible minors of the extended matrix of the system (1).

An estimate of Δ was given in [5] under the form

THEOREME 1. *Given the system (1), the following inequality holds:*

$$\Delta \leqslant \alpha := \prod_{i=1}^{p} \|a_i\|, \qquad (6)$$

where $p = \min\{n + 1, m\}$, while a_i is the vector in $\mathbf{R}^{n+1}$ of the coefficients of the i-th inequation, together with the absolute term. The inequations are assumed to be decreasingly ordered, according to the values of the norms of the vectors a_i.

With this theorem, one obtains the radius $R_2' = \alpha\sqrt{n}$.

Another choice for the radius of the initial sphere:

$$R_3 = \beta\sqrt{n}, \tag{7}$$

where β is an edge for the absolute values of the components of the solutions of the system (1), was proposed in [4].

It is easy to ascertain that $R_2' \leqslant R_3$, the equality holding when all the coefficients are equal.

Since Khachiyan's algorithm is used to solve the linear programming problem, which generally contains the inequality (2) amongst its restrictions, the initialization step of the algorithm can be modified as it results from:

LEMMA. 1. *If the system obtained from* (1) *and* (2) *is compatible, then it has a solution into the sphere of radius*

$$R_4 = \alpha\sqrt{n}/2 \tag{8}$$

and centre $x^0 = (\alpha/2, \alpha/2, \ldots, \alpha/2)$.

Proof. Since the system of inequations contains the inequation (2) too, it results that its solutions lie into the first octant. On the other hand, these ones lie into the sperhe of radius R_2' and centre O, too. Therefore, they will be lying into the hypercube having a vortex in O, the edges on the coordinate axes, and the side R_2'. The sphere circumscribed to this hypercube has the radius equal to the half of the diagonal of the hypercube. So, the lemma is proved.

This result is an improvement of that obtained in [6].

While calculating the radii R_2', R_3, R_4, one has determined a point in $\mathbf{R}^n$ whose coordinates are majorants for the values of the components of the eventual solutions of the system of inequalities, and then one has found the Euclidean distance from the origin (which is considered the centre of the sphere) to this point.

If the system of inequalities (1) also contains (2), then majorants for certain components of the eventual solutions of the system of inequations can be obtained from those inequations $A_i x \leqslant b_i$ which have the property that $A_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \geqslant 0$, because if $b_i < 0$ the system is incompatible, otherwise for every $a_{ij} \neq 0$ the j-th component, x_j, of all the solutions must fulfil

$$x_j \leqslant b_i/a_{ij} = \gamma_j. \tag{9}$$

So, the point $x^s = (x_1^s, x_2^s, \ldots, x_n^s)$, where some components of x^s have the values γ_i calculated according (9), while the other components have the value α, is a majorant for all the solutions of the considered system.

Similarly, for the system (1) which also contains the inequalities (2) one can determine minorants for certain components of the solutions of the system, by considering the inequations $A_i x \leqslant b_i$ which have the property that $A_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \leqslant 0$ and $b_i < 0$; for every $a_{ij} \neq 0$ the j-th component of all the solutions must fulfil

$$x_i \geqslant b_i/a_{ij} = \delta_i. \tag{10}$$

So, the point $x^i = (x_1^i, x_2^i, \ldots, x_n^i)$, where some components of x^i were calculated with (10), while the other components have the calue *0*, is a minorant for all the solutions of the considered system.

On the basis of the above presented considerations, the following lemma obviously holds:

LEMMA 2. *If the system* (1) *which also contains the inequations* (2) *is compatible then it has a solution in the sphere having the centre*

$$x^0 = (x^i + x^s)/2 \tag{11}$$

and the radius

$$R_5 = || x^s - x^0 ||. \tag{12}$$

As one ascertains easily, the radius of the initial sphere was very much diminished, and its volume too. In this manner, one obtains a substantial reduction of the number of iterations, as well as an enlargement of the error admitted in the calculation.

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GENERALIZED STIRLING NUMBERS

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ABSTRACT. — Giving a new interpretation of Stirling's numbers defined by L. Comtet in [3], we obtain a further generalization: Stirling numbers associated to a double infinite matrix.

1. Introduction. Starting from a given sequence $a = (a_k)_{k\geq 0}$, in the pape[r] [3], L. Comtet has given the following generalization of classical Stirli[ng] numbers: the Stirling's numbers of the first kind (associated to the sequen[ce] a), denoted by $s_a(n, k)$, are defined by:

$$\prod_{k=0}^{n-1} (x - a_k) = \sum_{k=0}^{n} s_a(n, k) \cdot x^k$$

and the Stirling numbers of the second kind, denoted by $S_a(n, k)$, are define[d] by:

$$x^n = \sum_{k=0}^{n} S_a(n, k)\,(x - a_0)\,\ldots\,(x - a_{k-1})$$

Among the examples given there are the binomial coefficients, obtaine[d] for $a_k = 1$, and the usual Stirling numbers which correspond to $a_k = k$.

In what follows we give an interpretation of $s_a(n, k)$ and $S_a(n, k)$ in term[s] of generalized finite differences, which allows a further generalization.

2. Generalized finite differences. As it is stated in [5] I. Aldanond[o] has given in [1] the following generalization of finite differences:

Let $a = (a_k)_{k\geq 0}$ be a given sequence. One define the finite differences of sequence $x = (x_m)_{m\geq 0}$ (with respect to the sequence a) by:

$$D_a^0 x_m = x_m, \quad D_a^{n+1} x_m = D_a^n x_{m+1} - a_n \cdot D_a^n x_m \qquad [1]$$

Introducing the shift operator E and the identity operator I, defined re[s]pectively by:

$$E x_m = x_{m+1},\; I x_m = x_m,\; (\text{but } E a_k = a_k E \;\; I a_k = a_k I)$$

we get:

$$D_a^n x_m = \sum_{k=0}^{n-1} (E - a_k I)\, x_m$$

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or

$$D_a^n x_m = \sum_{k=0}^{n} s_a(n, k) \cdot x_{m+k} \tag{2}$$

that is we can obtain on this way the Stirling's numbers of the first kind. If we invert the relation (2), we get the Stirling's numbers of the second kind:

$$x_{m+n} = \sum_{k=0}^{n} S_a(n, k) D_a^k x_m \tag{3}$$

The finite differences defined by (1) contain those given by D. S. Mitrinović in [4] (where $a_k = a + b \cdot k$) or those defined by a Markov systems (see [6]). But, in [2] we have proposed a further generalization which we want now to use.

3. **Generalized Stirling numbers.** Let us consider the double infinite matrix $A = (a_{n,m})_{n,m \geqslant 0}$.

We define the finite differences (in respect to A) by:

$$D_A^0 x_m = x_m, \; D_A^{n+1} x_m = D_A^n x_{m+1} - a_{n,m} \cdot D_A^n x_m ; \; n, m \geqslant 0 \tag{4}$$

Taking $a_{n,m} = a_n$, we get (1).

Now we can generalize the Stirling numbers of the two kinds defining $s_A(n, m, k)$ and $S_A(n, m, k)$ by:

$$D_A^n x_m = \sum_{k=0}^{n} s_A(n, m, k) \cdot x_{m+k} \tag{5}$$

respectively:

$$x_{m+n} = \sum_{k=0}^{n} S_A(n, m, k) D_A^k x_m. \tag{6}$$

If we consider that

$$E \cdot (a_{n,m} I) = a_{n,m+1} E \text{ and } (a_{n,m} I) \cdot E = a_{n,m} E$$

the relation (4) gives:

$$D_A^n x_m = (E - a_{n-1,m} I) \ldots \cdot (E - a_{1,m} I)(E - a_{0,m} I) x_m$$

that is:

$$s_A(n, m, n) = 1, \; s_A(n, m, n-1) = -\sum_{k=1}^{n} a_{n-k,m+k-1}, \ldots$$

$$\ldots, \; s_A(n, m, 0) = (-1)^n a_{n-1,m} \ldots a_{0,m}.$$

Also from (4) we get the triangular recurrence relations:

$$s_A(n+1, m, k) = s_A(n, m+1, k-1) - a_{n,m} \cdot s_A(n, m, k) \tag{7}$$

with $s_A(n, m, k) = 0$ for $k < 0$ or $k > n$.

Step by step, (7) gives the vertical recurrence relation :

$$s_A(n+1, m, k) = \sum_{j=k-1}^{n} (-1)^{n-j} \cdot s_A(j, m+1, k-1) \prod_{i=j+1}^{n} a_{i,m}$$

with the convention : $\prod_{i=n+1}^{n} a_{i,m} = 1$.

The relation (8) allows to pass from a k to another. For example it

$$s_A(n, m, 1) = (-1)^{n-1} \sum_{j=0}^{n-1} \prod_{i=0}^{j-1} a_{i,m+1} \prod_{i=j+1}^{n-1} a_{i,m}.$$

For the numbers of the second kind, we have :

$$x_{m+n+1} = \sum_{k=0}^{n+1} S_A(n+1, m, k) D_A^k x_m = \sum_{k=0}^{n} S_A(n, m+1, k) D_A^k x_{m+1} =$$

$$= \sum_{k=0}^{n} S_A(n, m+1, k)[D_A^{k+1} x_m + a_{k,m} D_A^k x_m]$$

which gives the triangular recurrence relation :

$$S_A(n+1, m, k) = S_A(n, m+1, k-1) + a_{k,m} S_A(n, m+1, k)$$

As $S_A(k, m, k) = 1$, this gives also the vertical recurrence relation :

$$S_A(n+1, m, k) = \sum_{j=k-1}^{n} S_A(j, m+n-j+1, k-1) \prod_{i=m}^{m+n-j-1} a_{k,i}.$$

Form (5) and (6) we get also :

$$\sum_{k=j}^{n} s_A(n, m, k) S_A(k, m, j) = \delta_{n,j}$$

and

$$\sum_{k=j}^{n} S_A(n, m, k) s_A(k, m, j) = \delta_{n,j}$$

that is the matrices

$$s_A(m) = (s_A(n, m, k))_{n,k\geq 0} \text{ and } S_A(m) = (S_A(n, m, k))_{n,k};$$

verify :

$$s_A(m) \cdot S_A(m) = S_A(m) \cdot s_A(m) = I.$$

Of course, all the relations, generalize known results.

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SUR L'APPROXIMATION DE L'ENTROPIE D'UNE VARIABLE ALÉATOIRE BIDIMENSIONNELLE CONTINUE

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REZUMAT. — Asupra aproximării entropiei unei variabile aleatoare bidimensionale continue. În prezenta lucrare se consideră aproximarea entropiei unei variabile aleatoare bidimensionale continue. Se dau patru teoreme şi se definesc entropia caracteristică şi entropia de ordinul $\frac{1}{n}$ corespunzătoare unei variabile aleatoare continue.

Introduction. Dans l'article [4] on a considéré le cas d'une variable aléatoire continue et on a établi plusieurs théorèmes d'approximation de l'entropie à l'aide de l'entropie maxime d'une variable aléatoire de type discret. Dans cet article on considère un problème analogue, pour une variable aléatoire bidimensionnelle de type continue.

Soit (X, Y) une variable aléatoire bidimensionnelle de type continue, et $f; \mathbf{R}^2 \to \mathbf{R}$ la densitée de probabilité correspondente, ainsi que

$$f(x, y) \geqslant 0, \quad x, y \in \mathbf{R}$$
$$f(x, y) > 0, \quad (x, y) \in [a, b] \times [c, d]$$
$$f(x, y) = 0, \quad (x, y) \notin [a, b] \times [c, d]$$
$$\int_a^b \int_c^d f(x, y)\, dx\, dy = 1 \tag{1}$$

où $[a, b]$, $[c, d]$ sont des intervalles finis d'axe réele. Dans ces conditions a lieu le théorème

THÉORÈME 1. *Si*

$$\left| \int_a^b \int_c^d f(x, y) \log_2 f(x, y)\, dx dy \right| < +\infty \tag{2}$$

alors, en considérant des divisions des intervalles $[a, b]$, $[c, d]$ *en* n, *respectivement* m *intervalles égales, la limite pour* $n \to \infty$, $m \to \infty$ *de la différence entre l'entropie maxime*

$$H\left(\frac{1}{nm}, \frac{1}{nm}, \ldots, \frac{1}{nm}\right) = \log_2 n + \log_2 m \tag{3}$$

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et

$$H_f(q_{11}, q_{12}, \ldots, q_{nm}) = -\sum_{i=1}^{n}\sum_{j=1}^{m} q_{ij}\log_2 q_{ij} \tag{4}$$

où

$$q_{ij} = \frac{p_{ij}}{P_{nm}}, \; p_{ij} = \frac{f(\xi_i, \eta_j)}{nm}(b-a)(d-c), \; i = \overline{1,n}, \; j = \overline{1,m} \tag{5}$$

$$x_0 = a, \; x_i = \frac{b-a}{n} i + a, \; i = \overline{1,n}; \; y_0 = c, \; y_j = \frac{d-c}{m} j + c, \; j = \overline{1,m}$$

$$\xi_i \in (x_{i-1}, x_i), \; i = \overline{1,n}$$

$$\eta_j \in (y_{j-1}, y_j), \; j = \overline{1,m}$$

$$P_{nm} = \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij} > 0$$

est

$$\lim_{\substack{n\to\infty \\ m\to\infty}} [(\log_2(nm) - H_f(q_{11}, q_{12}, \ldots q_{nm})] =$$

$$= \int_a^b\int_c^d f(x,y)\log_2 f(x,y)\,dxdy + \log_2[(b-a)(d-c)] \tag{6}$$

Démonstration. On remarque que

$$q_{ij} \geqslant 0, \; i = \overline{1,n}, \; j = \overline{1,m}$$

et

$$\sum_{i=1}^{n}\sum_{j=1}^{m} q_{ij} = \frac{1}{P_{nm}}\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij} = 1$$

ainsi que $H_f(q_{11}, q_{12}, \ldots, q_{nm})$ donné par (4) constitue un schéma de probabilités.

Soit l'expression

$$E = \log_2(nm) - H_f(q_{11}, q_{12}, \ldots, q_{nm}) =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} q_{ij}\log_2(nm) + \sum_{i=1}^{n}\sum_{j=1}^{m} q_{ij}\log_2 q_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{m} q_{ij}\log_2(nmq_{ij})$$

Conformément aux notations antérieurs, on a;

$$E = \frac{1}{P_{nm}} \sum_{i=1}^{n} \sum_{j=1}^{m} f(\xi_i, \eta_j) \frac{(b-a)(d-c)}{nm} \log_2 \frac{f(\xi_i, \eta_j)(b-a)(d-c)}{P_{nm}} =$$

$$= \frac{1}{P_{nm}} \sum_{i=1}^{n} \sum_{j=1}^{m} f(\xi_i, \eta_j) \Delta x_i \Delta y_j \{\log_2 f(\xi_i, \eta_j) + \log_2 [(b-a)(d-c)] -$$

$$- \log_2 P_{nm}\} = \frac{1}{P_{nm}} \sum_{i=1}^{n} \sum_{j=1}^{m} f(\xi_i, \eta_j) \log_2 f(\xi_i, \eta_j) \Delta x_i \Delta y_j +$$

$$+ \frac{\log_2 [(b-a)(d-c)]}{P_{nm}} \sum_{i=1}^{n} \sum_{j=1}^{m} f(\xi_i, \eta_j) \Delta x_i \Delta y_j - \frac{1}{P_{nm}} \log_2 P_{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} f(\xi_i, \eta_j) \Delta x_i \Delta y_j$$

où

$$\Delta x_i = x_i - x_{i-1} = \frac{b-a}{n}, \quad i = \overline{1, n}$$

$$\Delta y_j = y_j - y_{j-1} = \frac{d-c}{m}, \quad j = \overline{1, m}$$

Mais

$$\lim_{\substack{n \to \infty \\ m \to \infty}} P_{nm} = \lim_{\substack{n \to \infty \\ m \to \infty}} \sum_{i=1}^{n} \sum_{j=1}^{m} f(\xi_i, \eta_j) \Delta x_i \Delta y_j = \int_a^b \int_c^d f(x, y) dx dy =$$

ainsi que

$$\lim_{\substack{n \to \infty \\ m \to \infty}} [\log_2(nm) - H_f(q_{11}, q_{12}, \ldots, q_{nm})] =$$

$$= \int_a^b \int_c^d f(x, y) \log_2 f(x, y) dx dy + \log_2 [(b-a)(d-c)]$$

Conséquence. On déduit de la relation (6) pour l'entropie d'une variable aléatoire bidimensionnelle continue la valeur de l'approximation

$$H_f(q_{11}, q_{12}, \ldots, q_{nm}) = - \int_a^b \int_c^d f(x, y) \log_2 f(x, y) \, dx dy + \log_2 \frac{nm}{(b-a)(d-c)} + \varepsilon_{nm}$$

où q_{ij} sont donnés par (5).

L'expression

$$H_f^0 = - \int_a^b \int_c^d f(x, y) \log_2 f(x, y) \, dx dy$$

qui ne dépende pas de n et m représente *l'entropie caractéristique* de la variable aléatoire avec la densité de probabilité f. Cette entropie peut être tant positive que négative.

L'expression

$$H_f^{nm}(q_{11}, q_{12}, \ldots, q_{nm}) = H_f^0 + \log_2 \frac{nm}{(b-a)(d-c)}$$

représente l'entropie de la variable aléatoire continue avec la densité de probabilité f *d'ordre d'approximation* $\frac{1}{nm}$.

En considerant les densités de probabilité

$$h : \mathbf{R} \to \mathbf{R}_+, \quad g_x : \mathbf{R} \to \mathbf{R}_+$$

ainsi que

$$\int_{-\infty}^{x} h(t)dt = P(X < x)$$

$$\int_{-\infty}^{y} g_x(t)dt = P(Y < y | X = x)$$

nous avons

$$f(x, y) = h(x)g_x(y), \quad x, y \in \mathbf{R}$$

THÉORÈME 2. *Si la relation* (2) *est verifiée alors*

$$\lim_{\substack{n \to \infty \\ m \to \infty}} [\log_2(nm) - H_f(q_{11}, q_{12}, \ldots, q_{nm})] =$$

$$= - H(X) - H_X(Y) + \log_2[(b-a)(d-c)] \quad (7)$$

$$H(X) = -\int_a^b h(x) \log_2 h(x)\, dx$$

$$H_x(Y) = -\int_c^d g_x(y) \log_2 g_x(y)\, dy$$

$$H_X(Y) = \int_a^b h(x)\, H_x(Y)\, dx$$

Démonstration. On considère

$$x_0 = a,\ x_i = \frac{b-a}{n} i + a,\ \xi_i \in (x_{i-1}, x_i),\ i = \overline{1, n},$$

$$y_0 = c,\ y_j = \frac{d-c}{m} j + c,\ \eta_j \in (y_{j-1}, y_j),\ j = \overline{1, m}$$

$$p_{ij} = \frac{h(\xi_i) g_{\xi_i}(\eta_j)}{nm} (b-a)(d-c)$$

$$P_{nm} = \sum_{i=1}^{n} \sum_{j=1}^{m} p_{ij};\quad q_{ij} = \frac{p_{ij}}{P_{nm}} \tag{8}$$

Dans ce cas, en mode analogue avec les calculs du théorème 1, nous avons

$$\lim_{\substack{n\to\infty \\ m\to\infty}} [\log_2(nm) - H_f(q_{11}, q_{12}, \ldots, q_{nm})] =$$

$$= \int_a^b \int_c^d h(x) g_x(y) [\log_2 h(x) + \log_2 g_x(y)]\, dx dy + \log_2[(b-a)(d-c)] =$$

$$= \int_a^b h(x) \log_2 h(x) dx \int_c^d g_x(y) dy + \int_a^b h(x) \int_c^d g_x(y) \log_2 g_x(y) dy +$$

$$+ \log_2[(b-a)(d-c)] = -H(X) - H_Y(X) + \log_2[(b-a)(d-c)]$$

parce que

$$\int_c^d g_x(y)\, dy = P(Y < d \,|\, X = x) = \frac{P((Y < d) \cap (X = x))}{P(X = x)} = \frac{P(\Omega \cap (X = x))}{P(X = x)} = 1$$

Conséquence. 1. Il résulte de la relation (7) que

$$H_f(q_{11}, q_{12}, \ldots, q_{nm}) = H(X) + H_X(Y) + \log_2 \frac{nm}{(b-a)(d-c)} + \varepsilon_{nm} \tag{9}$$

où q_{ij} sont donnés par (8).

2. On déduit de même manière l'approximation

$$H_f(q_{11}, q_{12}, \ldots, q_{nm}) = H(Y) + H_Y(X) + \log_2 \frac{nm}{(b-a)(d-c)} + \varepsilon_{nm} \tag{10}$$

3. Dans le cas quand les variables X et Y sont des variables indépendentes on déduit

$$H_f(q_{11}, q_{12}, \ldots, q_{nm}) = H(Y) + H_y(X) + \log_2 \frac{nm}{(b-a)(d-c)} + \varepsilon_{nm}$$

4. Si les intervalles $[a, b]$, $[c, d]$ se divisent dans le même nombres des sousintervalles de (9) et (10) on déduit

$$H(X) + H_X(Y) = H(Y) + H_Y(X) + \delta_{nm}$$

où

$$H(X) - H_Y(X) = H(Y) - H_X(Y) + \delta_{nm}$$

THÉORÈME 3. *Soit* $f: \mathbf{R}^2 \to \mathbf{R}$,

$$f(x, y) \geqslant 0, \ x, y \in \mathbf{R}$$

et $D = [a, b] \times [c, d]$ *ainsi que*

$$\text{mes } (D) = +\infty$$

$$f(x, y) = 0, \quad (x, y) \notin D$$

$$\iint f(x, y) dxdy = 1$$

$$\left| \iint_D f(x, y) \log_2 f(x, y) dxdy \right| < +\infty$$

et $D_k = [a_k, b_k] \times [c_k, d_k]$, $D_k \subset D$, $D_k \subset D_{k+1}$, $k \in N$

$$\bigcup_{k \in N} D_k = D$$

on considère

$$f_k: \mathbf{R}^2 \to \mathbf{R} \tag{11}$$

$$f_k(x, y) = \begin{cases} \dfrac{f(x, y)}{\iint_{D_k} f(x, y) \, dxdy}, & (x, y) \in D_k \\ 0, & (x, y) \notin D_k \end{cases}$$

alors

$$\iint_D f(x, y) \log_2 f(x, y) \, dxdy = \lim_{k \to \infty} \iint_{D_k} f_k(x, y) \log_2 f_k(x, y) \, dxdy \tag{12}$$

Démonstration. On considère

$$G_k(x, y) = f(x, y) \log_2 f(x, y) - f_k(x, y) \log_2 f_k(x, y)$$

alors pour $(x, y) \in D$

$$G_k(x, y) = f(x, y) \log_2 f(x, y) - \frac{f(x, y)}{\iint_{D_k} f(x, y) dxdy} \log_2 \frac{f(x, y)}{\iint_{D_k} f(x, y) d\,xdy}$$

et donc

$$\lim_{k\to\infty}\iint_{D_k} G_k(x, y)dxdy = \lim_{k\to\infty}\iint_{D_k} f(x, y)\log f(x, y)dx\,dy -$$

$$-\lim_{k\to\infty}\frac{1}{\iint_{D_k} f(x, y)dxdy}\iint_{D_k} f(x, y)\left[\log_2 f(x, y) - \log_2\iint_{D_k} f(x, y)dxdy\right]dxdy =$$

$$=\iint_D f(x, y)\log_2 f(x, y)\,dxdy - \iint_D f(x, y)\log_2 f(x, y)dxdy = 0$$

d'où il résulte (12).

On observe que f_k est une fonction densité de probabilité et donc

$$\iint_{D_k} f_k(x, y)\log_2 f_k(x, y)dxdy = -H_{f_k}(q_{11}, q_{12}, \ldots, q_{nm}) +$$

$$\log_2\frac{nm}{\text{mes }(D_k)} + \varepsilon_{nm}$$

où q_{ij} sont donnés par (5).

THÉORÈME 4. *Soit* (X,Y) *une variable aléatoire et* $f: \mathbf{R}^2 \to \mathbf{R}$ *la densité de probabilité correspondante de façon que*

$$f(x, y) \neq 0,\ (x, y) \in D$$

$$f(x, y) = 0,\ (x, y) \notin D$$

$$\iint_D f(x, y)\,dxdy = 1$$

$$\left|\iint_D f(x, y)\log_2 f(x, y)dxdy\right| < +\infty$$

où $D = [a, b] \times [c, d]$ *est un domaine ainsi que*

$$\text{mes }(D) = +\infty$$

et soit $(D_k)_{k\in N}$ *de façon que* $D_k = [a_k, b_k] \times [c_k, d_k] \subset D,\ D_k \subset D_{k+1}$ $k \in N$ *et*

$$D = \bigcup_{k\in N} D_k$$

On considère

$$H_{f_k}(q_{11}^k, q_{12}^k, \ldots, q_{n_k m_k}^k) = -\sum_{i=1}^{n_k}\sum_{j=1}^{m_k} q_{ij}^k \log_2 q_{ij}^k$$

où

$$q_{ij}^k = \frac{p_{ij}^k}{P_{n_k m_k}^k}, \quad p_{ij}^k = \frac{f_k(\xi_i^k, \eta_j^k)}{n_k m_k}(b_k - a_k)(d_k - c_k), \quad i = \overline{1, n_k}, \; j = \overline{1, m_k}$$

$$x_0^k = a_k, \; x_i^k = \frac{b_k - a_k}{n_k} i + a_k, \; i = \overline{1, n_k}; \; y_0^k = c_k, \; y_j^k = \frac{d_k - c_k}{m_k} j + c_k, \; j = \overline{1, m_k}$$

$$\xi_i^k \in (x_{i-1}^k, x_i^k), \; i = \overline{1, n_k}$$

$$\eta_j^k \in (y_{j-1}^k, y_j^k), \; j = \overline{1, m_k}$$

$$P_{n_k m_k} = \sum_{i=1}^{n_k}\sum_{j=1}^{m_k} p_{ij}^k > 0$$

et $f_k : R^2 \to \mathbf{R}$ de façon que

$$f_k(x, y) = \begin{cases} \dfrac{f(x, y)}{\iint\limits_{D_k} f(x, y)\,dx\,dy}, & (x, y) \in D_k \\ 0 \quad , & (x, y) \notin D_k \end{cases}$$

Si

$$\lim_{k\to\infty} \frac{n_k m_k}{\text{mes}\,(D_k)} = L \tag{13}$$

alors

$$\lim_{k\to\infty} H_{f_k}(q_{11}^k, q_{12}^k, \ldots, q_{n_k m_k}^k) = -\iint\limits_{D} f(x, y)\log_2 f(x, y)\,dx\,dy + \log_2 L \tag{14}$$

Démonstration. On observe que

$$H_{f_k}(q_{11}^k, q_{12}^k, \ldots, q_{n_k m_k}^k) = -\iint\limits_{D_k} f_k(x, y)\log_2 f_k(x, y)\,dx\,dy +$$

$$+ \log_2 \frac{n_k m_k}{\text{mes}\,(D_k)} + (1) \tag{15}$$

ainsi que de (12) et (13) il résulte

$$\lim_{k\to\infty} H_{f_k}(q_{11}^k, q_{12}^k, \ldots, q_{n_k m_k}^k) = -\lim_{k\to\infty} \iint_{D_k} f_k(x, y) \log_2 f_k(x, y)\, dxdy +$$

$$+ \lim_{k\to\infty} \frac{n_k m_k}{\text{mes}(D_k)} = -\iint f(x, y) \log_2 f(x, y)\, dxdy + \log_2 L$$

et donc la relation (14) este verifiée.

Dans le cas quand $L = 1$ nous avons

$$\lim_{k\to\infty} H_{f_k}(q_{11}^k, q_{12}^k, \ldots, q_{n_k m_k}^k) = -\iint_D f(x, y) \log_2 f(x, y)\, dxdy$$

L'expression

$$H_f^0 = -\iint_D f(x, y) \log_2 f(x, y)\, dxdy$$

est l'entropie caractéristique de la variable aléatoire (X, Y).

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A SEMIGROUP APPROACH TO HAMILTON-JACOBI EQUATION IN HILBERT SPACE

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REZUMAT. — O abordare semigrupală a ecuaţiei lui Hamilton-Jacobi în spaţii Hilbert. În prezenta lucrare se demonstrează că semigrupul asociat ecuaţiei lui Hamilton-Jacobi într-un spaţiu Hilbert este generat de un operator univoc *m*-disipativ în spaţiul funcţiilor uniform continue şi mărginite.

1. **Introduction.** We will study here the Hamilton-Jacobi equation

$$\varphi_t(t, x) + F(-D\varphi_x(t, x)) + (Ax, \varphi_x(t, x)) = g(x), \quad \text{in } \mathbf{R}^+ \times \mathbf{H} \tag{1.1}$$
$$\varphi(0,x) = \varphi_0(x),$$

in a real Hilbert space $\mathbf{H}$ with the norm $||\cdot||$ and scalar product $(.,.)$. The unknown function φ is a real valued function defined on $[0, +\infty) \times \mathbf{H}$ and φ_0, g are given functions on $\mathbf{H}$.

As regard the function F and the operators D and A we will assume that

(i) $F : \mathbf{U} \to \overline{\mathbf{R}} = (-\infty, +\infty]$ is a convex continuous function on a real Hilbert space $\mathbf{U}$; D is a linear continuous operator from $\mathbf{H}$ to $\mathbf{U}$.

(ii) A is a maximal monotone (multivalued) operator from $\mathbf{H}$ to itself.

We will denote by $||\cdot||$ and $(.,.)_U$ the norm and respectively the scalar product of $\mathbf{U}$. Given a metric space $\mathbf{X}$ denote by $BUC(\mathbf{X})$ the space of bounded uniformly continuous real valued functions on $\mathbf{X}$ endowed with the norm

$$||f||_b = \sup \{|f(x)| : x \in \mathbf{X}\}. \tag{1.2}$$

By Lip $(\mathbf{X})$ we denote the space of all Lipschitz functions $f : \mathbf{X} \to \mathbf{R} = (-\infty, +\infty)$. In the following $\mathbf{X}$ will be the closure $\overline{D(A)}$ of the domain $D(A)$ of A. It is well known that Eq. (1.1) is related to optimal control problems governed by the equation

$$y'(t) + Ay(t) \ni D^*u, \; t \geq 0 \tag{1.3}$$
$$y(0) = x$$

and with the pay-off

$$\int_0^t (g(y(s)) + h(u(s)))\, ds + \varphi_0(y(t)). \tag{1.4}$$

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Recall that for every $x \in D(A)$ and $u \in L^1(0, T; \mathbf{U})$ problem (1.3), admits a unique integral solution $y = y(t, x, u)$ in the sense of Bénilan (see e.g. [1] p. 124). Moreover, we have

$$|y(t, x_0, u) - y(t, z_0, v)| \leqslant |y_0 - z_0| + \int_0^t |D^*u - D^*v| \, ds \qquad \forall t \in [0, T]. \tag{1.5}$$

Here $h : \mathbf{U} \to \bar{\mathbf{R}}$ is the conjugate of F, i.e.,

$$h(u) = \sup \{(p, u)_U - F(p) : p \in \mathbf{U}\} \qquad \forall u \in \mathbf{U}. \tag{1.6}$$

More precisely, the function

$$\varphi(t, x) = (S(t)\, \varphi_0)(x) = \inf \left\{ \int_0 (g(y(s)) + h(u(s)))ds + \right.$$

$$\left. + \varphi_0(y(t)) : u \in L^1(0, t; \mathbf{U}),\ y(s) = y(s, x, u) \right\} \quad t \geqslant 0,\ x \in \overline{D(A)} \tag{1.7}$$

can be viewed as a generalized solution to Eq. (1.1) and in fact it is a viscosity solution in the sense of M. G. Crandall and P. L. Lions ([4], [5]).

Here we shall prove (see Theorem 1 below) that for φ_0 and g in a certain subset of $BUC\,(\overline{D(A)})$ this function is the solution to nonlinear evolution equation of the form

$$\frac{d\varphi}{dt} = \mathcal{A}_0 \varphi \quad \text{in } \mathbf{R}^+ \tag{1.8}$$

$$\varphi(0) = \varphi_0$$

where $\mathcal{A}_0$ is a m-dissipative single valued operator on $BUC\,(\overline{D(A)})$. More precisely φ is given by the Crandall-Liggett exponential formula

$$\varphi(t) = \lim_{n \to \infty} \left(I - \frac{t}{n} \mathcal{A}_0\right)^{-n} \varphi_0 \qquad \forall t \geqslant 0. \tag{1.9}$$

This result improves and precises Theorem 4 in [2].

The main ingredient of the proof is the following nonlinear version of the Chernoff theorem (Brézis and Pazy [3]).

PROPOSITION 1. *Let C be a closed convex subset of a Banach space $\mathbf{Y}$ and let $\mathcal{A}_0$ be a m-dissipative subset of $\mathbf{Y} \times \mathbf{Y}$. Let $\{\Gamma(t) ; t \geqslant 0\}$ be a family of nonexpansive mappings from C into itself such that*

$$\lim_{\rho \downarrow 0} \left(I - \lambda \frac{\Gamma(\rho) - 1}{\rho}\right)^{-1} x = (I - \lambda \mathcal{A}_0)^{-1} x \tag{1.10}$$

for all $x \in \overline{D(\mathcal{A}_0)} \cap C$ and $\lambda > 0$. Then

$$\lim_{n \to \infty} \left(\Gamma\left(\frac{t}{n}\right)\right)^n = e^{\mathcal{A}_0 t} x \quad \forall t \geqslant 0, \qquad x \in \overline{D(\mathcal{A}_0)} \cap C \tag{1.11}$$

where $e^{\mathcal{A}_0 t}$ is the semigroup generated by $\mathcal{A}_0$.

2. The construction of the generator. Note first that since F is everywhere finite, the conjugate h has the growth property

$$\lim_{||u||\to+\infty} h(u)/||u|| = \infty. \tag{2.1}$$

For any $f \in \mathrm{BUC}(\overline{D(A)})$ and $\lambda > 0$ define the function

$$(R(\lambda)f)(x) = \inf\left\{\int_0^\infty e^{-\lambda t}(f(y(t)) + h(u(t)))dt\,;\right.$$

$$\left. u \in L^1_{\mathrm{loc}}(\mathrm{R}^+;\ \mathrm{U}) : y' + Ay \ni D^*u \text{ in } \mathrm{R}^+,\ y(0) = x\right\} \tag{2.2}$$

where D^* is the adjoint of D.

LEMMA 1. *For all* $\lambda > 0$ *and* $f \in BUC(\overline{D(A)})$, $R(\lambda) f \in BUC\ (\overline{D(A)})$ *and*

$$R(\lambda) f = R(\mu)((\mu - \lambda)R(\lambda)f + f) \text{ if } 0 < \lambda \leqslant \mu < \infty. \tag{2.3}$$

Moreover, if $f \in \mathrm{Lip}\ (\overline{D(A)})$ *then* $R(\lambda) f \in \mathrm{Lip}\ (\overline{D(A)})$.

Proof. The proof is essentially the same as that of Lemma 11 in [2]. However we outline it for reader's convenience.

Let $u_0 \in \mathrm{U}$ be such that $h(u_0) < +\infty$ and let $y_0(t) = y(t,\ x,\ u_0)$ be the solution to (1.3) where $u = u_0$ and $x \in \overline{D(A)}$.

We have the obvious inequality

$$(R(\lambda)f)(x) \leqslant \int_0^\infty e^{-\lambda t}(f(y(t,\ x,\ u_0)) + h(u(t)))dt \leqslant C \text{ for all } x \in \overline{D(A)}.$$

Hence $\sup\{(R(\lambda)f)(x) :\ x \in \overline{D(A)}\} < +\infty$. Note also that $\inf\{(R(\lambda)f)(x) : x \in \overline{D(A)}\} > -\infty$ because otherwise it would exist the sequences $\{x_n\}$, $\{u_n\}$ such that

$$\int_0^\infty e^{-\lambda t}(f(y(t,\ x_n,\ u_n)) + h(u_n))dt \to -\infty.$$

This would imply that $\int_0^\infty e^{-\lambda t}h(u_n)dt \to -\infty$ for $n \to \infty$ which would contradict (2.1). Thus $R(\lambda) f$ is bounded on $\overline{D(A)}$. To show that $R(\lambda)f$ is uniformly continuous consider two arbitrary points $x,\ \tilde{x}$ in $D(A)$. For every $\varepsilon > 0$ there are

u_ε and $\tilde{u}_\varepsilon$ such that

$$(R(\lambda)f)(x) \geq \int_0^\infty e^{-\lambda t}(f(y_\varepsilon(t)) + h(u_\varepsilon(t)))dt - \varepsilon$$

$$(R(\lambda)f)(\tilde{x}) \geq \int_0^\infty e^{-\lambda t}(f(\tilde{y}_\varepsilon(t)) + h(\tilde{u}_\varepsilon(t)))dt - \varepsilon$$

where $y_\varepsilon = y(t, x, u_\varepsilon)$, $\tilde{y}_\varepsilon = y(t, \tilde{x}, \tilde{u}_\varepsilon)$. We have

$$(R(\lambda)f)(x) - (R(\lambda)f)(\tilde{x}) \leq \int_0^\infty e^{-\lambda t}(f(\tilde{z}_\varepsilon(t)) - f(\tilde{y}_\varepsilon(t)))dt + \varepsilon$$

$$(R(\lambda)f)(\tilde{x}) - (R(\lambda)f)(x) \leq \int_0^\infty e^{-\lambda t}(f(z_\varepsilon(t)) - f(y_\varepsilon(t)))dt + \varepsilon$$

where $z_\varepsilon = y(t, \tilde{x}, u_\varepsilon)$ and $\tilde{z}_\varepsilon = y(t, x, \tilde{u}_\varepsilon)$.

Inasmuch as by (1.5), $|\tilde{z}_\varepsilon(t) - \tilde{y}_\varepsilon(t)| \leq |x - \tilde{x}|$ and $|y_\varepsilon(t) - z_\varepsilon(t)| \leq |x - \tilde{x}|$ we infer that

$$|(R(\lambda)f)(x) - (R(\lambda)f)(\tilde{x})| \leq \delta(\varepsilon) \text{ if } |x - \tilde{x}| \leq \varepsilon.$$

By a similar argument it follows that $R(\lambda)f \in \mathrm{Lip}\,(\overline{D(A)})$ if $f \in \mathrm{Lip}\,(\overline{D(A)})$. Next by definition of $R(\lambda)$ we have for all $\lambda, \mu > 0$

$$R(\mu)((\mu - \lambda)R(\lambda)f + f)(x) = \inf\Big\{\int_0^\infty e^{-\mu t}(f(y(t)) + h(u(t)) +$$

$$+ (\mu - \lambda)\inf\Big(\int_0^\infty e^{-\lambda s}(f(z(s)) + h(v(s)))ds\,;\, z' + Az \ni D^*v,$$

$$z(0) = y(t))\,dt : y' + Ay \ni D^*u,\ y(0) = x\Big\}$$

and therefore

$$R(\mu)((\mu - \lambda)R(\lambda)f + f)(x) \leq \int_0^\infty e^{-\mu t}(f(y(t)) + h(u(t)) +$$

$$+ (\mu - \lambda)\int_0^\infty e^{-\lambda s}(f(y(t + s)) + h(u(t + s)))ds)dt$$

for all (y, u) satisfying Eq. (1.3).

This yields

$$R(\mu)((\mu-\lambda)R(\lambda)f+f)(x) \leqslant \int_0^\infty e^{-\lambda t}\,(f(y(t))+h(u(t))dt.$$

In other words

$$R(\mu)((\mu-\lambda)R(\lambda)f+f)(x) \leqslant (R(\lambda)f)(x) \quad \forall\, x \in D(A),\ \lambda,\mu \geqslant 0.$$

For opposite inequality we start with the inequality

$$(R(\lambda)f)(x) \leqslant \int_0^t e^{-\lambda s}\,(f(y^*(s)) + h(u^*(s)))\,ds + \tag{2.4}$$
$$+ \int_t^\infty e^{-\lambda s}\,(f(z^*(s-t)) + h(v^*(s-t)))ds$$

where (y^*, u^*), (z^*, v^*) satisfy Eq. (1.3) with initial conditions $y^*(0) = x$, $z^*(0) = y^*(t)$ and chosen in a such a way that

$$R(\mu)((\mu-\lambda)R(\lambda)f+f) \geqslant \int_0^\infty e^{-\mu t}\,(f(y^*(t)) + h(u^*(t)))dt +$$
$$+ (\mu-\lambda)\int_0^\infty e^{-\mu t}\,dt + \int_0^\infty e^{-\lambda s}\,(f(z^*(s)) + h(v^*(s)))ds - \varepsilon$$

where $\varepsilon > 0$ is arbitrary.

Multiplying inequality (2.4) by $e^{-(\mu-\lambda)t}$ and integrating on $\mathbf{R}^+$ we find after some calculation that for $\mu \geqslant \lambda$

$$(R(\lambda)f)(x) \leqslant R(\mu)((\mu-\lambda)R(\lambda)f+f)(x) + \varepsilon \qquad \forall \varepsilon > 0.$$

This completes the proof.

Now let $\mathcal{A} : D(\mathcal{A}) \subset BUC\,(\overline{D(A)}) \to BUC\,(\overline{D(A)})$ be the operator (eventually multivalued) defined by

$$\mathcal{A}R(1)f = R(1)f - f \qquad \forall f \in BUC\,(\overline{D(A)}) \tag{2.5}$$

with the domain

$$D(\mathcal{A}) = \{\varphi = R(1)f : f \in BUC\,(\overline{D(A)})\}. \tag{2.6}$$

We will denote by $\mathbf{Y}$ the space $BUC\,(\overline{D(A)})$.

LEMMA 2. *The operator $\mathcal{A}$ is m-dissipative in $\mathbf{Y} \times \mathbf{Y}$ and*

$$(\lambda I - \mathcal{A})^{-1} = R(\lambda) \quad \textit{for all } \lambda > 0. \tag{2.7}$$

(I is the identity operator in $\mathbf{Y}$).

Proof. By Eq. (2.3) and the definition of $\mathcal{A}$ we see that

$$(\lambda I - \mathcal{A})^{-1} = R(\lambda) \text{ for } 0 < \lambda \leqslant 1, \tag{2.8}$$

whilist by (2.2)

$$\| R(\lambda)f - R(\lambda)g \|_b \leqslant \lambda^{-1} \| f - g \|_b \quad \forall \lambda > 0; f, g \in \mathbf{Y}.$$

Hence $\mathcal{A}$ is m-dissipative (see e.g. [1], p. 73). Then by (2.5), (2.8) and the obvious resolvent equation

$$(\lambda I - \mathcal{A})^{-1}f = (\mu I - \mathcal{A})^{-1}((\mu - \lambda)(\lambda I - \mathcal{A})^{-1}f + f) \qquad \forall \lambda, \mu > 0, \ f \in \mathbf{Y},$$

we infer that (2.7) holds for all $\lambda > 0$.

According to the Crandall-Liggett theorem (see for instance [1], p. 104) for all $\varphi_0 \in \overline{D(\mathcal{A})}$ (the closure of $D(\mathcal{A})$ in $\mathbf{Y}$) and $g \in \mathbf{Y}$ the Cauchy problem

$$\frac{d\varphi}{dt} \in \mathcal{A}\varphi + g \text{ in } \mathbf{R}^+ \tag{2.9}$$

$$\varphi(0) = \varphi_0$$

has a unique weak continuous solution $\varphi : \mathbf{R}^+ \to \mathbf{Y}$ defined by the exponential formula

$$\varphi(t) = \lim_{n \to \infty} J^n\left(\frac{t}{n}\right) \varphi_0 \ \ \forall\, t \geqslant 0 \tag{2.10}$$

where $J(\lambda)\varphi_0 = (I - \lambda\mathcal{A})^{-1}(\lambda g + \varphi_0) = (I - \lambda\mathcal{A}_0)^{-1}\varphi_0$ ($\mathcal{A}_0$ is the operator $\varphi \to \mathcal{A}\varphi + g$).

Equivalently, $\varphi = \lim_{\varepsilon \downarrow 0} \varphi_\varepsilon$ where φ_ε is the solution to difference equation

$$\varepsilon^{-1}(\varphi_\varepsilon(t) - \varphi_\varepsilon(t - \varepsilon)) \in \mathcal{A}\varphi_\varepsilon(t) + g \text{ for } t \geqslant \varepsilon \tag{2.11}$$

$$\varphi_\varepsilon(t) = \varphi_0 \qquad\qquad \text{for } -\varepsilon \leqslant t \leqslant 0.$$

The map $T(t) : \overline{D(\mathcal{A})} \to \overline{D(\mathcal{A})}$ defined by

$$\varphi(t) = T(t)\varphi_0 \qquad \forall t \geqslant 0 \tag{2.12}$$

is a continuous semigroup of nonlinear contractions on $\overline{D(\mathcal{A})}$.

Coming back to formula (1.7) it is readilly seen that $S(t)$ maps $BUC(\overline{D(A)})$ into itself and

$$S(t)S(s)\varphi_0 = S(t+s)\varphi_0 \ \ \forall\, t, s \geqslant 0, \ \varphi_0 \in BUC\,(\overline{D(A)}) \tag{2.13}$$

$$\| S(t)\varphi_0 - S(t)\tilde{\varphi}_0 \|_b \leqslant \| \varphi_0 - \tilde{\varphi}_0 \|_b \ \forall\, t \geqslant 0; \ \varphi_0, \tilde{\varphi}_0 \in BUC(\overline{D(A)}). \tag{2.14}$$

In other words, $S(t)$ is a semigroup of nonlinear contractions on $BUC\,(\overline{D(A)})$. The main result is

THEOREM 1. *Assume that* $g \in BUC\,(\overline{D(A)})$. *Then*

$$S(t)\,\varphi_0 = T(t)\varphi_0 \text{ for all } t \geqslant 0 \text{ and } \varphi_0 \in \overline{D(\mathcal{A})}. \tag{2.15}$$

Moreover the operator $\mathcal{A}$ *is single valued and for all* $\varphi_0 \in D(\mathcal{A})$

$$\lim_{t \downarrow 0} ((S(t)\varphi_0)(x) - \varphi_0(x))t^{-1} = \mathcal{A}\varphi_0(x) + g(x), \ \forall \ x \in D(A) \tag{2.16}$$

in the strong topology of $\mathbf{H}$.

In few words Theorem 1 amounts to saying that the operator $\mathcal{A}$ is the generator of the semigroup $S(t)$ (see [1], p. 98). As regards the set $\overline{D(\mathcal{A})}$ it is precised by Proposition 1 below.

PROPOSITION 2. *Under assumptions* (i), (ii), *we have* $\overline{D(\mathcal{A})} = \mathfrak{D}$ *where*

$$\mathfrak{D} = \Big\{ \varphi \in BUC\,(\overline{D(A)}) \cap \text{Lip}\,(\overline{D(A)}) : |\varphi(y(t,\ x,\ u)) - \varphi(x)| \leqslant$$

$$\leqslant L_\varphi \left(t + \int_0^t ||u(s)||\,ds\right), \ \forall\, t \in [0,\ 1],\ u \in L^1(0,1;\ \mathbf{U})\Big\}. \tag{2.17}$$

Here $y\,(t,\ x,\ u)$ is the solution to Eq. (1.3).

If A is linear the domain $D(\mathcal{A})$ can be described more precisely (see [2]).

The representation of Eq. (1.1) as a nonlinear dissipative evolution equation allows to apply the theory of nonlinear semigroup of contractions and in particular the approximation and perturbation theory to Hamilton-Jacobi equations of this form. For instance if $\mu : \mathbf{R} \to \mathbf{R}$ is a continuous and monotone increasing function then the operator

$$(\mathcal{B}\varphi)(x) = -\,\mu(\varphi(x)) \ \ \forall \ x \in \overline{D(A)}$$

is continuous and dissipative on $BUC\,(\overline{D(A)})$. Thus $\mathcal{A} + \mathcal{B}$ is m-dissipative (see [1], p. 158) and so the evolution equation

$$\frac{d}{dt}\,\varphi = \mathcal{A}\varphi + \mathcal{B}\varphi + g,\ \varphi(0) = \varphi_0$$

has a unique weak solution which may be viewed as a generalized solution to Hamilton-Jacobi equation

$$\varphi_t + F(-\,D\varphi_x) + (Ax,\ \varphi_x) + \mu(\varphi) = g \text{ in } \mathbf{R}^+ \times \overline{D(A)}$$
$$\varphi(0,\ x) = \varphi_0(x). \tag{2.18}$$

3. Proofs.

Proof of Proposition 2. We will prove first that

$$(I - \mu\mathcal{A})^{-1}f \in \mathfrak{D} \text{ for all } \mu > 0 \text{ and all} \tag{3.1}$$
$$f \in BUC\,(\overline{D(A)}) \cap \text{Lip}\,(\overline{D(A)}).$$

Let $\mu > 0$ and $f \in BUC\,(\overline{D(A)}) \cap \text{Lip}\,(\overline{D(A)})$ be arbitrary but fixed. We set

$\varphi = (1-\mu\mathcal{A})^{-1} f = R(\mu^{-1})(\mu^{-1}f)$. For every $\varepsilon>0$ there are $(y_\varepsilon, u_\varepsilon) \in C([0, T]; \mathbf{H}) \times L^1(0, T; \mathbf{U})$ for any $T>0$, such that $y_\varepsilon' + Ay_\varepsilon \ni D^* u_\varepsilon$ in R^+, $y_\varepsilon(0) = x$ and

$$\left|\varphi(x) - \int_0^\infty e^{-\mu^{-1}t}(f(y_\varepsilon(t))\mu^{-1} + h(u_\varepsilon(t))dt\right| \leqslant \varepsilon \tag{3.2}$$

Let (y, u) be any pair satisfying Eq. (1.3). Then by (1.5) we have

$$|y(t) - y_\varepsilon(t)| \leqslant \|D^*\| \int_0^t (\|u_\varepsilon(s)\| + \|u(s)\|)\,ds \quad \forall t \geqslant 0$$

and therefore

$$|\varphi(y(t)) - \varphi(y_\varepsilon(t))| \leqslant C \int_0^t (\|u_\varepsilon(s)\| + \|u(s)\|)\,ds \quad \forall t \geqslant 0 \tag{3.3}$$

because by Lemma 1 φ is Lipschitz.
(We shall denote by C several positive constants independent of ε, u and t).

On the other hand, from the well known optimality principle we have for all $t>0$

$$(R(\lambda)f)(x) = \inf\left\{\int_0^t e^{-\lambda s}(f(y(s)) + h(u(s)))ds + \right.$$

$$\left. + e^{-\lambda t}(R(\lambda)f)(y(t)) : y' + Ay \ni D^*u,\ y(0) = x\right\}. \tag{3.4}$$

Along with (3.2) the latter yields

$$|\varphi(x) - e^{-\mu^{-1}t}\varphi(y_\varepsilon(t)) - \int_0^t e^{-\mu^{-1}s}(\mu^{-1}f(y_\varepsilon(s)) + h(u_\varepsilon(s)))ds\ | \leqslant \varepsilon.$$

Hence

$$|\varphi(x) - e^{-\mu^{-1}t}\varphi(y_\varepsilon(t))| \leqslant C(t + |\int_0^t h(u_\varepsilon(s))ds|) + \varepsilon \quad \forall t \geqslant 0. \tag{3.5}$$

On the other hand, by (3.4) we have

$$\int_0^t e^{-\mu^{-1}s}(\mu^{-1}f(y_\varepsilon(s)) + h(u_\varepsilon(s)))ds + e^{-\mu^{-1}t}\varphi(y_\varepsilon(t)) \leqslant$$

$$\leqslant \int_0^t e^{-\mu^{-1}s}(\mu^{-1}f(y_0) + h(u_0))ds + e^{-\mu^{-1}t}\varphi(y_0(t))$$

where $u_0 \in \mathbf{U}$ is such that $h(u_0) < +\infty$ and $y_0 = y(t, x, u_0)$. The latter yields

$$\int_0^t h(u_\varepsilon(s))ds \leqslant C(t + \int_0^t ||u_\varepsilon(s)||ds) + \varepsilon \quad \forall t \geqslant 0$$

because h is bounded from below by an affine function. Then by a little calculation involving (2.1) we obtain

$$\left|\int_0^t h(u_\varepsilon(s))\,ds\right| + \int_0^t \|u_\varepsilon(s)\|\,ds \leqslant Ct + \varepsilon \quad \forall t \geqslant 0. \tag{3.6}$$

Substituting the latter estimate in (3.3) and using (3.5) we get

$$|\varphi(y(t)) - \varphi(x)| \leqslant C\left(t + \int_0^t ||u(s)||\,ds\right) + \varepsilon \quad \forall \varepsilon > 0,\ t \geqslant 0$$

where C is independent of ε, u and t. Hence $\varphi \in \mathfrak{D}$.

Since the space $BUC(\overline{D(A)}) \cap \mathrm{Lip}(\overline{D(A)})$ is dense in $BUC(\overline{D(A)})$ (see e.g. [6]) and the operator $(I - \mu\mathcal{A})^{-1}$ is nonexpansive in $BUC(\overline{D(A)})$ we conclude that $(I - \mu\mathcal{A})^{-1}f \in \overline{\mathfrak{D}}$ for all $f \in BUC(\overline{D(A)})$. On the other hand, we have (see for instance [1], p. 73),

$$\lim_{\mu \downarrow 0} (I - \mu\mathcal{A})^{-1}f = f \text{ in } BUC(\overline{D(A)}),$$

for all $f \in \overline{D(\mathcal{A})}$. Along with (3.1) the latter implies that $\overline{D(\mathcal{A})} \subset \overline{\mathfrak{D}}$.

To prove the converse inclusion relation consider f an arbitrary element of $\mathfrak{D}$ and set

$$f_\mu = (I - \mu\mathcal{A})^{-1}f = R(\mu^{-1})(\mu^{-1}f),\ \mu > 0.$$

We will prove that $f_\mu \to f$ in $BUC(\overline{D(A)})$ as $\mu \to 0$. Indeed for every $\mu > 0$ and all x in $\overline{D(A)}$ there are y_μ, u_μ satisfying Eq. (1.3) and such that

$$f_\mu(x) \geqslant \int_0^\infty e^{-\mu^{-1}t}(\mu^{-1}f(y_\mu(t)) + h(u_\mu(t)))dt - \mu \tag{3.7}$$

and

$$f_\mu(x) \leqslant \int_0^\infty e^{-\mu^{-1}t}(\mu^{-1}f(y(t)) + h(u(t)))\,dt \tag{3.8}$$

where (y, u) is an arbitrary solution to (1.3).

Hence for $u = u_0$ constant and $y_0 = y(t, x, u_0)$ we have

$$f_\mu(x) - f(x) \leqslant \mu^{-1}\int_0^\infty e^{-\mu^{-1}t}(f(y_0(t)) - f(x)) + \mu h(u_0))\,dt$$

$$\leqslant Ce^{-\mu^{-1}\delta} + |h(u_0)|\delta + C(1 - e^{-\mu^{-1}\delta}) \qquad \forall\, \delta > 0 \tag{3.9}$$

because $f \in \mathfrak{D}$. Similarly, by (3.7) we have

$$f(x) - f_\mu(x) \leqslant \mu + \mu^{-1}\int_0^\infty e^{-\mu^{-1}t}(f(x) - f(y_\mu(t)))dt -$$

$$- \int_0^\infty e^{-\mu^{-1}t} h(u_\mu(t))dt \leqslant \mu + Ce^{-\mu^{-1}\delta} +$$

$$+ C\mu^{-1}\int_0^\delta e^{-\mu^{-1}t}\left(t + \int_0^t ||u_\mu(s)||ds\right)dt + \tag{3.10}$$

$$+ C\int_0^\infty e^{-\mu^{-1}t}||u_\mu(t)||\,dt$$

where C is independent of μ and δ.

Finally, again by (3.7), (3.8) we obtain

$$\int_0^\infty e^{-\mu^{-1}t} h(u_\mu(t))dt \leqslant C(\mu + \mu^{-1}\int_0^\infty e^{-\mu^{-1}t}\left(t + \int_0^t ||u_\mu(s)||ds\right)dt \leqslant$$

$$\leqslant \nu(\mu) + C\int_0^\infty e^{-\mu^{-1}t}||u_\mu(t)||dt$$

where $\nu(\mu) \to 0$ for $\mu \to 0$, because $f \in \mathrm{Lip}\,(\overline{D(A)})$ and by (1.5)

$$|y_\mu(t) - y_0(t)| \leqslant \int_0^t ||D^*(u_\mu(s) - u_0)||\,ds \leqslant C\left(t + \int_0^t ||u_\mu(s)||ds\right).$$

Using once again (2.1) we obtain the estimate

$$\int_0^\infty e^{-\mu^{-1}t}|h(u_\mu(t))|\,dt + \int_0^\infty e^{-\mu^{-1}t}||u_\mu(t)||dt \to 0$$

as $\mu \to 0$ and by (3.10) we have

$$f(x) - f_\mu(x) \leqslant C\left(\omega(\mu) + e^{-\mu^{-1}\delta} + \int_0^\delta e^{-\mu^{-1}t}\|u_\mu(t)\|\,dt\right)$$

$$\forall\, \mu, \delta > 0 \tag{3.11}$$

where $\omega(\mu) \to 0$ as $\mu \to 0$. Since δ is arbitrary we infer by (3.10) and (3.11) that for $\mu \to 0, f_\mu \to f$ in $BUC\,(\overline{D(A)})$ as claimed. Since $f_\mu \in D(\mathcal{A})$ we conclude that $f \in \overline{D(\mathcal{A})}$. Hence $\mathfrak{D} \subset \overline{D(\mathcal{A})}$ and the proof of Proposition 2 is complete

Proof of Theorem 1. Let us prove first that $S(t)$ maps $\overline{D(\mathcal{A})}$ into itself. To this end we fix φ_0 in $\mathfrak{D}$. By the optimality principle we have for all $0 \leqslant s \leqslant t$

$$(S(t)\,\varphi_0)\,(x) = \inf\left\{\int_0^s (g(y(\tau)) + h(u(\tau)))d\tau + (S(t-s)\varphi_0)(y(s)) :\right.$$

$$y' + Ay \ni D^*u \text{ in } (0, s)\,;\ y(0) = x,$$

$$\left. u \in L^1\,(0, t\,;\,\mathbf{U})\right\}$$

Hence

$$(S(t)\varphi_0)(x) - (S(t)\,\varphi_0)(y_0(s)) \leqslant \int_0^s (g(y_0(\tau)) + h(u_0))d\tau +$$

$$+ \|S(s)\varphi_0 - \varphi_0\|_b \tag{3.12}$$

where $u_0 \equiv$ constant and $y_0\,(\tau) = y(\tau, x, u_0)$. (Here we have used Eqs. (2.13), (2.14).)

Similarly, there exist $(y_\varepsilon, u_\varepsilon)$ satisfying (1.2) such that

$$(S(t)\varphi_0)(x) - (S(t)\varphi_0)(y_\varepsilon(s)) \geqslant \int_0^s (g(y_\varepsilon(\tau)) + h(u_\varepsilon(\tau)))\,d\tau -$$

$$- \|S(s)\varphi_0 - \varphi_0\|_b - \varepsilon. \tag{3.13}$$

As a matter of fact we choose $(y_\varepsilon, u_\varepsilon)$ such that

$$\int_0^s (g(y_\varepsilon(\tau)) + h(u_\varepsilon(\tau)))d\tau + (S(t-s)\varphi_0)(y_\varepsilon(s)) \leqslant (S(t)\varphi_0)(x) + \varepsilon$$

$$\leqslant \int_0^s (g(y_0(\tau)) + h\,(u_0))d\tau + (S(t-s)\varphi_0)(y_0(s)) + \varepsilon$$

and this yields

$$\int_0^s h(u_\varepsilon(\tau))d\tau \leqslant C\left(\varepsilon + s + \int_0^s ||u_\varepsilon(\tau)||d\tau\right) \qquad 0 \leqslant s \leqslant t.$$

Taking $\varepsilon = s$ and using (2.1) we get

$$\int_0^s |h(u_\varepsilon(\tau)|d\tau + \int_0^s ||u_\varepsilon(\tau)||d\tau \leqslant Cs \qquad \forall s \in (0, t).$$

Coming back to (3.12), (3.13) we obtain

$$|(S(t)\varphi_0)(x) - (S(t)\varphi_0)(y(s))| \leqslant C\left(s + \int_0^s ||u(\tau)||d\tau\right) +$$

$$+ ||S(s)\varphi_0 - \varphi_0||_b \qquad \forall s \in (0, t) \tag{3.14}$$

where (y, u) is any pair satisfying Eq. (1.3). (To get (3.14) we have used estimate (1.5) and the obvious fact that $S(t)\varphi_0 \in \mathrm{Lip}\ (\overline{D(A)})$).

Finally, by definition of $S(t)$ it follows by a similar argument that

$$|(S(s)\varphi_0)(x) - \varphi_0(x)| \leqslant Cs \qquad \forall s > 0,\ x \in \overline{D(A)}$$

and by (3.14) we conclude that $S(t)\varphi_0 \in \mathfrak{D}$.

To conclude the proof we will apply Proposition 1 where $C = \overline{\mathfrak{D}}$, $\Gamma(\rho) = S(\rho)$ and $\mathcal{A}_0\varphi = \mathcal{A}\varphi + g$ for $\varphi \in D(\mathcal{A})$. To this end we will prove that for all $\mu > 0$,

$$\lim_{\varepsilon \downarrow 0} (I - \mu\varepsilon^{-1}(S(\varepsilon) - I))^{-1}\varphi_0 = (I - \mu\mathcal{A})^{-1}(\varphi_0 + \mu g)$$
$$\forall \varphi_0 \in \mathfrak{D}. \tag{3.15}$$

Since $(I - \mu\mathcal{A})^{-1}$ and $(I - \mu\varepsilon^{-1}(S(\varepsilon) - I))^{-1}$ are non expansive on $\overline{\mathfrak{D}} = \overline{D(\mathcal{A})}$ it suffices to prove (3.15) for $\varphi_0 \in \mathfrak{D}$. We set

$$\varphi_\varepsilon = (I - \mu\varepsilon^{-1}(S(\varepsilon) - I))^{-1}\varphi_0,\ \varphi = (I - \mu\mathcal{A})^{-1}(\varphi_0 + \mu g).$$

We have

$$\varphi_\varepsilon = \frac{\varepsilon}{\varepsilon + \mu}\varphi_0 + \frac{\mu}{\varepsilon + \mu} S(\varepsilon)\varphi_\varepsilon \tag{3.16}$$

or equivalently

$$\varphi_\varepsilon(x) = \frac{\varepsilon}{\varepsilon + \mu}\varphi_0(x) + \frac{\mu}{\mu + \varepsilon}\inf\left\{\int_0^\varepsilon (g(y(t)) + h(u(t)))dt + \right.$$

$$\left. + \varphi_\varepsilon(y(\varepsilon))\,;\ y' + Ay \ni D^*u,\ y(0) = x\right\}. \tag{3.17}$$

In the following we shall denote by $||\cdot||_{\mathrm{Lip}(\overline{D(A)})}$ the Lipschitz norm on Lip $(\overline{D(A)})$.

By definition of $S(\varepsilon)$ we see that

$$||S(\varepsilon)\varphi_0||_{\mathrm{Lip}(\overline{D(A)})} \leqslant ||g||_{\mathrm{Lip}(\overline{D(A)})} + ||\varphi_0||_{\mathrm{Lip}(\overline{D(A)})}$$

$$||S(\varepsilon)\varphi_0||_b \leqslant \varepsilon(||g||_b + C) + ||\varphi_0||_b.$$

By (3.16) this yields

$$||\varphi_\varepsilon||_{\mathrm{Lip}(\overline{D(A)})} \leqslant ||\varphi_0||_{\mathrm{Lip}(\overline{D(A)})} + \mu||g||_{\mathrm{Lip}(\overline{D(A)})}. \tag{3.18}$$

$$||\varphi_\varepsilon||_b \leqslant C \qquad \forall\, \varepsilon > 0. \tag{3.19}$$

Now let $(y_\varepsilon, u_\varepsilon)$ be such that $y_\varepsilon' + Ay_\varepsilon \ni D^*u_\varepsilon$ in $(0, \varepsilon)$, $y_\varepsilon(0) = x$ and

$$\frac{\mu}{\varepsilon+\mu}\left(\int_0^\varepsilon (g(y_\varepsilon) + h(u_\varepsilon))\,dt + \varphi_\varepsilon(y_\varepsilon(\varepsilon))\right) + \frac{\varepsilon}{\varepsilon+\mu}\varphi_0(x) \leqslant \varphi_\varepsilon(x) + \varepsilon^2. \tag{3.20}$$

This yields

$$\frac{\mu}{\varepsilon+\mu}\int_0^\varepsilon h(u_\varepsilon)dt \leqslant \varphi_\varepsilon(y(\varepsilon)) - \varphi_\varepsilon(y_\varepsilon(\varepsilon)) + \frac{\mu}{\varepsilon+\mu}\int_0^\varepsilon (g(y_\varepsilon(t)) -$$

$$- g(y(t))dt + h(u_0)\varepsilon + \varepsilon$$

where $y' + Ay \ni Bu_0$, $y(0) = x$ and by (3.20) we infer that

$$\int_0^\varepsilon h(u_\varepsilon(t))dt \leqslant C\left(\varepsilon + \int_0^\varepsilon ||u_\varepsilon(t)||\,dt\right) \qquad \forall\, \varepsilon > 0.$$

Then by (2.1) we see that

$$\int_0^\varepsilon |h(u_\varepsilon(t))|dt + \int_0^\varepsilon ||u_\varepsilon(t)||dt \leqslant C\varepsilon \qquad \forall\, \varepsilon > 0 \tag{3.21}$$

where C is independent of ε. Since $\varphi_0 \in \mathfrak{D}$ the latter yields

$$|\varphi_0(y(t)) - \varphi_0(x)| \leqslant C\varepsilon \qquad \forall x \in \overline{D(A)}, \quad \forall\, t \in (0,\varepsilon). \tag{3.22}$$

On the other hand, we have (see (3.4))

$$\varphi(x) = \inf\left\{\int_0^\varepsilon e^{-\mu^{-1}t}(g(y(t)) + \mu^{-1}\varphi_0(y(t)) + h(u(t)))dt + \right.$$

$$\left. + e^{-\mu^{-1}\varepsilon}\varphi(y(\varepsilon))\,;\; y' + Ay \ni D^*u \text{ in } (0, \varepsilon),\; y(0) = x\right\} \tag{3.23}$$

and therefore there exist a pair $(z_\varepsilon, v_\varepsilon)$ such that
$z'_\varepsilon + Az_\varepsilon \ni D^* v_\varepsilon$ in $[0, \varepsilon]$, $z_\varepsilon(0) = x$ and

$$\varphi(x) \geq \int_0^\varepsilon e^{-\mu^{-1}t}(g(z_\varepsilon(t)) + \mu^{-1}\varphi_0(z_\varepsilon(t)) + h(v_\varepsilon(t)))\,dt + e^{-\mu^{-1}\varepsilon}\,\varphi(z_\varepsilon(\varepsilon)) - \varepsilon^2. \quad (3.24)$$

Arguing as above we see that

$$\int_0^\varepsilon |h(v_\varepsilon(t))|\,dt + \int_0^\varepsilon ||v_\varepsilon(t)||\,dt \leq C\varepsilon \qquad \forall\, \varepsilon > 0. \quad (3.25)$$

and therefore

$$|\varphi_0(z_\varepsilon(t)) - \varphi_0(x)| \leq C \qquad \forall t \in (0, \varepsilon),\ x \in \overline{D(A)}. \quad (3.26)$$

By (3.23) and (3.24) we see that

$$\varphi_\varepsilon(x) - \varphi(x) \leq \frac{\varepsilon}{\varepsilon+\mu}\varphi_0(x) + \frac{\mu}{\varepsilon+\mu}\int_0^\varepsilon (g(z_\varepsilon) + h(v_\varepsilon))\,dt + \frac{\mu}{\varepsilon+\mu}\varphi_\varepsilon(z_\varepsilon(\varepsilon)) -$$

$$- e^{-\mu^{-1}\varepsilon}\varphi(z_\varepsilon(\varepsilon)) - \int_0^\varepsilon e^{-\mu^{-1}t}(g(z_\varepsilon(t)) + \mu^{-1}\varphi_0(z_\varepsilon(t)) + h(v_\varepsilon(t)))\,dt + \varepsilon \quad (3.27)$$

whilist by (3.17) and (3.20) we have

$$\varphi(x) - \varphi_\varepsilon(x) \leq \int_0^\varepsilon e^{-\mu^{-1}t}(g(y_\varepsilon) + \mu^{-1}\varphi_0(y_\varepsilon) + h(u_\varepsilon))\,dt + e^{-\mu^{-1}\varepsilon}\varphi(y_\varepsilon(\varepsilon)) -$$

$$- \frac{\varepsilon}{\varepsilon+\mu}\int_0^\varepsilon (g(y_\varepsilon) + h(u_\varepsilon))\,dt - \frac{\varepsilon}{\varepsilon+\mu}\varphi_0(x) + \varepsilon. \quad (3.28)$$

Taking in account estimates (3.18), (3.19), (3.21), (3.22), (3.25) and (3.26) we see after some calculation involving (3.27), (3.28) that

$$||\varphi_\varepsilon - \varphi||_b \leq \delta(\varepsilon) \to 0 \text{ as } \varepsilon \to 0$$

and (3.15) follows.

Then by Proposition 1 we have

$$T(t)\varphi_0 = \lim_{n\to\infty}\left(S\left(\frac{t}{n}\right)\right)^n\varphi_0 = S(t)\varphi_0 \qquad \forall \varphi_0 \in \bar{\mathfrak{D}},\ t \geq 0. \quad (3.29)$$

We will prove now (2.16). By definition of $S(t)$ we have

$$(S(t)\varphi_0)(x) - \varphi_0(x) \leq \int_0^t (g(y(s)) + h(u(s)))\,ds + \varphi_0(y(t)) - \varphi_0(x) \quad (3.30)$$

for all (y, u) satisfying (1.3) Since $\varphi_0 \in D(\mathcal{A})$ there is $f \in BUC(\overline{D(A)})$ such that

$$\varphi_0(x) = (R(1)f)(x) = \inf\left\{\int_0^\infty e^{-s}(f(y(s)) + h(u(s)))ds\,;\right.$$

$$\left. y' + Ay \ni D^*u,\ y(0) = x\right\} = \inf\left\{\int_0^t e^{-s}(f(y(s)) + h(u(s)))ds + \right.$$

$$\left. + e^{-t}\varphi_0(y(t))\,;\ y' + Ay \ni D^*u,\ y(0) = x\right\}.$$

Then by (3.30) for every $\varepsilon > 0$ there exist y_ε, u_ε such that $y_\varepsilon' + Ay_\varepsilon \ni D^*u_\varepsilon$, $y_\varepsilon(0) = x$ and

$$(S(t)\varphi_0)(x) - \varphi_0(x) \leqslant \int_0^t (1 - e^{-s})h(u_\varepsilon(s))ds + \int_0^t g(y_\varepsilon(s))ds -$$
$$- (e^{-t} - 1)\,\varphi_0(y_\varepsilon(t)) - \int_0^t e^{-s}f(y_\varepsilon(s))ds + \varepsilon. \tag{3.31}$$

On the other hand, we have

$$\int_0^t e^{-s}h(u_\varepsilon(s))ds + \int_0^t e^{-s}f(y_\varepsilon(s))ds + e^{-t}\varphi_0(y_\varepsilon(t)) \leqslant$$

$$\int_0^t e^{-s}(f(y_0(s)) + h(u_0))ds + e^{-t}\varphi_0(y_0(t))$$

where $y_0' + Ay_0 \ni D^*u_0$, $y_0(0) = x$. This yields

$$\int_0^t e^{-s}h(u_\varepsilon(s))ds \leqslant C\left(t + \int_0^t \|u_\varepsilon(s)\|ds\right) \qquad \forall\, t \geqslant 0$$

and arguing as above we obtain

$$\int_0^t (|h(u_\varepsilon(s))| + \|u_\varepsilon(s)\|)ds \leqslant Ct \qquad \forall\, t \in [0, 1]$$

where C is independent of ε.

Noting that

$$|y_\varepsilon(t) - x| \leqslant t|Ax| + \int_0^t \|D^*u_\varepsilon(s)\|ds \qquad \forall\, t \geqslant 0$$

we conclude by (3.30) where $\varepsilon = t^2$ that

$$\limsup_{t \downarrow 0} t^{-1}((S(t)\varphi_0)(x) - \varphi_0(x)) \leqslant g(x) + \varphi_0(x) - f(x) \quad \text{for all } x \in D(A). \tag{3.32}$$

Similarly, we have

$$\varphi_0(x) - (S(t)\varphi_0)(x) \leqslant \int_0^t e^{-s}(f(z_\varepsilon(s) + h(v_\varepsilon(s)))ds + e^{-t}\varphi_0(z_\varepsilon(t)) -$$

$$- \int_0^t (g(z_\varepsilon(s)) + h(v_\varepsilon))\, ds - \varphi_0(z_\varepsilon(t)) + \varepsilon, \tag{3.33}$$

where $z_\varepsilon' + Az_\varepsilon \ni D^* v_\varepsilon$, $z_\varepsilon(0) = x$.

Arguing as above we find that for $x \in D(A)$

$$\limsup_{t \downarrow 0} (\varphi_0(x) - (S(t)\varphi_0)(x))t^{-1} \leqslant f(x) - \varphi_0(x) - g(x). \tag{3.34}$$

Combining (3.32) and (3.34) we see that

$$\lim_{t \downarrow 0} t^{-1}((S(t)\varphi_0)(x) - \varphi_0(x)) = \varphi_0(x) - f(x) + g(x) \qquad \forall\, x \in D(A)$$

for all $\varphi_0 = R(1)f$ and $f \in BUC(\overline{D(A)})$. Hence $\mathcal{A}$ is single valued and (2.16) follows. The proof of Theorem is complete.

REMARK 1. We see by (3.31) and (3.33) that if $g, f \in \mathfrak{D}$ or if A is bounded on H then

$$\lim_{t \downarrow 0} t^{-1}(S(t)\varphi_0 - \varphi_0) = \mathcal{A}\varphi_0 + g$$

uniformly in x, i.e., in $BUC(\overline{D(A)})$. However, in general we do not know whether $\varphi \to \mathcal{A}\varphi + g$ is the strong generator of $S(t)$.

Note also that Theorem 1 and the previous treatment remain valid if $A - \omega I$ is m-dissipative for some ω real.

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RECENZII

A. Pietsch, Eigenvalues and s-Numbers, Akademische Verlagsgeselschaft Geest aud Portig K.-G., Leipzig 1987, 360 pp.

The book may be considered as a supplement to author's well known monograph „Operator ideals", VEB, Berlin 1987, North-Holland, Amsterdam 1980 and Russian translation Moscow, 1982, but it is fairly selfcontained and can be readen independently. The theory of eigenvalues for operator originated with the famous paper of F. Riesz published in 1918. Since then the theory developed, first in the context of integral operators and then in abstract Hilbert and Banach spaces, by the contribution of great mathematicians as J. Schur, H. Weyl, J. von Neumann, A. Grothendieck, H. König and others. The theory of operator ideals permitted to obtain many striking results on eigenvalue distributions for operators, proving how tools from modern operator theory yield new results and solve old problems in classical analysis. The aim of this monograph is to give an extensive and systematical presentation of thes problems. Written by an eminent specialist in the field, the book is very well organized. Its contents is: Preliminaries, I. Absolutely summing operators, 2. s-Numbers, 3. Eigenvalues, 4. Traces and determinants, 5. Matrix operators, 6. Integral operators, 7. Historical survey. A list of open problems, stimulating the reader to do his own research, is also included.

The book is a valuable contribution to operator theory and its applications and will be of interest to all working in this branch of functional analysis.

S. COBZAŞ

Numerical Treatment of Differential Equations (Proceedings of the Third Seminar held in Halle, 1985), Edited by Karl Strehmel, Teubner-Texte zur Mathematik, Bd. 82, Leipzig, 1985, 204 pag.

The papers of the present issue tackle a widw range of actual looks concerning the development of the methods of numerical integration of the differential and partial differential equations. The issue consists of 30 papers whose topics can be dicided into the following categories:

(i) Numerical analysis of some differential equations which describe certain models taken from nature and science.

(ii) Numerical methods for differential equations.

(iii) Numerical methods in the theory of partial differential equations.

(iv) Qualitative theory, optimal control and differential inclusions.

The first category contains five papers which may certainly be classified as belonging to the other categories too, according to their subjects. We mention for instance M. Schleiff's paper entitled *A Mathematical Model of Sedimentation*, in which the phenomenon is modelled by means of partial differential equations.

In the second group of problems one can include nine papers; various looks concerning the Runge-Kutta-type methods (six papers), equations with retarded argument, stability of different numerical schemes are developed. We mention here the bidimensional methods of Runge-Kutta-type (M. Fritsche, p. 19; H. Toparkus, p. 175) in which Lie series are used.

The third category, concerning the numerical methods for partial differential equations, constitutes the widest part by both the number of papers (eleven) and their topics. One deals here with finite element-type methods (R. Lehmann, p. 102), Galerkin's method for the convection-diffusion equation (M. Fröhner, p. 25), discretization of hyperbolic differential equations with periodic solutions (P. J. van der Houwen, p. 75), as well as problems concerning the convergency of different methods applied to solving the partial differential equations.

The fourth category of problems is tackled by rather few papers; this fact can be explained by the topics of the Seminar. These papers study the periodic solutions (M. Holodnick et al., p. 72), the bifurcation of the periodic solutions (M. Kubicek and M. Holodniok, p. 94), or deal with the optimal control (F. V. Lubyshev, p. 106) or inclusions (H. D. Niepage, p. 121).

The papers of the presented issue constitute a real success for the Seminar, by their high scientific level and value (both theoretical and practical), by the applicability of their results to the most various domains of the science and engineering.

NICOLAE LUNGU

Hans Sachs Ebene, **Isotrope Geometrie**, v. Friedr. Vieweg and Sohn Braunschweig/Wiesbaden, 1987, 198 Seiten, 54 Figuren

Das Buch ist eine Darstellung der isotropen Geometrie begründet auf einer systematische gruppentheoretische Methode. Nach der Meinung des Verfasser er hat eine zweifache Ausgabe: einerseits diese interessante ebene Geometrie auf einem elementaren Weg darbieten und andererseits die nötigen Grundlagen für das Studium der isotropen Raumgeometrien zu vorbereiten. Das Struktur der Arbeit ist in dem Inhaltverzeichnis wiedergespiegelt: §1. Cayley-Kleinsche Geometrien und Erlangen Programm. §2. Ebene isotrope Geometrien und ihre Invarianten. §3. Elementar Geometrie der isotropen Ebene. §4. Lineare Kreismannigfaltigkeiten der isotropen Ebene. §5. Kurven 2. Ordnung in der isotropen Ebene. Metrische Dualität in den isotropen Ebene. 7. Die Kurventheorie der isotropen Ebene bezüglich der Gruppe 3. 8. Verallgemeinerte komplexe Zahlen: euklidische, pseudoeuklidische und isotrope Geometrie. Möbiusgeometrie. 9. Die Kurventheorie der isotropen Ebene bezüglich anderer Gruppen. 10. Ergänzungen.

Die Literaturverzeichnis berücksichtigt Originalarbeiten bis zum Jahre 1986, viele in Deutsche und Russische Sprache.

M. ŢARINĂ

Hagen Meltzer, **The Structure of Indecomposable Modules**, Teubner-texte zut Mathematik, Band 83, Leipzig, 1986

The book presents the elements of representation theory of finite-dimensional algebras, investigating the inner structure of indecomposable modules over an algebra. It contains a historical survey of classical prototypes of indecomposable modules as studied by Köthe, Nakayama and Tachikawa, together with recent resulst concerning new classes of indecomposable modules.

The main pupose of the monograph is to present a characterization of finite-dimensional algebras over a perfect field for which every indecomposable module has a core (part of the doctoral thesis of the author). More precisely, this is the theorem 6.1:

For a finite-dimensional algebra A over a perfect field K the following statements are equivalent:

(I) Every indecomposable A-module has a core

(II) The core of any indecomposable nonlocal A-module is colocal

(III) A satisfies the following conditions

(1) The radical of any nonuniserial local A-module is a sum of two uniserial submodules

(2) The radical of any local noncolocal A-module L is a sum of two uniserial submodules L_1 and L_2, where $E(L_1)/soc(L_1)$ or $E(L_2)/soc(L_2)$ is colocal

(3) Any colocal nonlocal A-module L is a sum of a uniserial submodule L_1 and a local submodule L_2 such that $L_1 \cap L_2$ is simple or, if L_i is uniserial, then $L_i/L_1 \cap L_2$ is simple and $P(L_i)$ is uniserial.

(IV) A is isomorphic to a bounden species algebra $(T\Sigma)/R$, where the bounden species (Σ, R) satisfies twelve technical conditions (which we avoid).

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