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SUR L'ENTROPIE D'UNE VARIABLE ALEATOIRE CONTINUE

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Au professeur D. D. Stancu,
pour son 60[e] anniversaire

ABSTRACT. — On the Entropy of a Continuing Aleatory Variable. Some theoretical results on the entropy of a continuing aleatory variable are set forth in the paper, given through three theoremes and some more consequences.

Soit X une variable aléatoire avec la densité de probabilité $f: R \to R$ ainsi que

$$f(x) \geqslant 0, \quad x \in \mathbf{R}$$
$$f(x) = 0, \quad x \notin [a, b]$$
$$\int_a^b f(x)dx = 1 \tag{1}$$

où $[a, b]$ est une intervalle finit de l'axe réele. Dans ces conditions on a

THÉORÈME 1 *Si*

$$\left|\int_a^b f(x) \ln f(x)\, dx\right| < +\infty \tag{2}$$

alors en considérant une division du l'intervalle $[a, b]$ *dans* n *intervalles égales la limite, pour* $n \to \infty$, *de la différence entre l'entropie maxime*

$$H\left(\frac{1}{n}, \ldots, \frac{1}{n}\right) = \ln n$$

et

$$H_f(q_1, \ldots, q_n) = -\sum_{i=1}^{n} q_i \ln q_i$$

ou

$$q_i = \frac{p_i}{P_n}, \quad p_i = \frac{f(\xi_i)}{n}(b - a), \quad i = \overline{1, n}$$
$$x_i = \frac{b-a}{n} i + a, \quad i = \overline{0, n}$$
$$\xi_i \in (x_i, x_{i+1}], \quad i = \overline{0, n-1},$$
$$P_n = \sum_{i=1}^{n} p_i$$

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est

$$\lim_{n\to\infty}(\ln n - H_f(q_1, \ldots, q_n)) = \int_a^b f(x)\ln f(x)dx + ln(b-a) \tag{3}$$

Démonstration Soit l'expression :

$$E = \ln n - H_f(q_1, \ldots, q_n) = \sum_{i=1}^{n} q_i \ln n + \sum_{i=1}^{n} q_i \ln q_i = \sum_{i=1}^{n} q_i \ln (nq_i)$$

Conformément aux notations anterieurs, on a :

$$E = \frac{1}{P_n}\sum_{i=1}^{n}\frac{(b-a)f(\xi_i)}{n}\ln\frac{f(\xi_i)(b-a)}{P_n} \tag{4}$$

En notant

$$\Delta x_i = x_{i+1} - x_i = \frac{b-a}{n}, \; i = \overline{1, n-1}$$

il résulte :

$$\ln n - H_f(q_1, \ldots, q_n) = \sum_{i=1}^{n}\frac{f(\xi_i)}{P_n}\Delta x_i[\ln f(\xi_i) + \ln(b-a) - \ln P_n] =$$

$$= \frac{1}{P_n}\sum_{i=1}^{n} f(\xi_i)\ln f(\xi_i)\Delta x_i + \frac{\ln(b-a)}{P_n}\sum_{i=1}^{n} f(\xi_i)\Delta x_i - \ln P_n \sum_{i=1}^{n} f(\xi_i)\Delta x_i \tag{5}$$

En tenant compte de (1), (2) et de

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_i)\,\Delta x_i = \int_a^b f(x)dx = 1$$

la rélation (5) à la limite pour $n \to \infty$ dévient ;

$$\lim_{n\to\infty}(\ln n - H_f(q_1, \ldots, q_n)) = \int_a^b f(x)\ln f(x)dx + \ln(b-a)$$

donc le théorème 1 est démontré.

Conséquence. Soit X une variable aléatoire qui vérifie les conditions du théorème 1 alors pour n suffisamment grand on a la valeur approximative pour l'entropie

$$H_f(q_1, \ldots, q_n) \approx -\int_a^b f(x)\ln f(x)dx + \ln\frac{n}{b-a} \tag{6}$$

où

$$q_i = \frac{p_i}{P_n}, \quad p_i = \frac{f(\xi_i)}{n}(b-a), \quad i = \overline{1, n}$$

$$\xi_i \in (x_i, \; x_{i+1}], \quad x_i = \frac{b_i - a_i}{n} + a, \quad i = \overline{0, \; n-1}, \quad P_n = \sum_{i=1}^{n} p_i$$

THÉORÈME 2. *Soit* $f: \mathbb{R} \to \mathbb{R}$, $f(x) \geqslant 0$, $x \in R$ *et* $I \subset \mathbb{R}$ *une intervalle ainsi que*

$$\text{mes}\,(I) = +\infty$$

$$f(x) = 0, \; x \notin I$$

$$\int_I f(x)dx = 1$$

$$\left|\int_I f(x)\ln f(x)\,dx\right| < +\infty$$

on considère une suite d'intervalles fini $(I_k)_{k\in\mathbb{N}}$, $I_k \subset I_k$, $k\in\mathbb{N}$ $I \subset I_{k+1}$, $k \in \mathbb{N}$, $\bigcup_{k=1} I_k = I$, *et soit* $f_k : \mathbb{R} \to \mathbb{R}$

$$f_k(x) = \begin{cases} \dfrac{f(x)}{\int_{I_k} f(x)dx}, & x \in I_k \\ 0, & x \notin I_k \end{cases}$$

alors

$$\int_{I_k} f(x)\ln f(x)\,dx = \lim_{k\to\infty}\int_{I_k} f_k(x)\ln f_k(x)dx \tag{7}$$

Demonstration. On considère l'expression ;

$$G_k(x) = f(x)\ln f(x) - f_k(x)\ln f_k(x) = f(x)\ln f(x) -$$
$$- \frac{f(x)}{\int_{I_k} f(x)dx}\ln\frac{f(x)}{\int_{I_k} f(x)dx}, \quad x \in I_k$$

Alors ;

$$\lim_{k\to\infty}\int_{I_k} G_k(x)dx = \lim_{k\to\infty}\int_{I_k}[f(x)\ln f(x) - f_k(x)\ln f_k(x)]\,dx =$$

$$= \lim_{k\to\infty}\int_{I_k} f(x)\ln f(x)\,dx - \lim_{k\to\infty}\int_{I_k} f_k(x)\ln f_k(x)\,dx = \int_{I_k} f(x)\ln f(x)\,dx -$$

$$-\lim_{k\to\infty}\int_{I_k} f_k(x)\ln f_k(x)dx = \int_{I_k} f(x)\ln f(x)dx - \lim_{k\to\infty}\frac{1}{\int_{I_k} f(x)dx}\int_{I_k} f(x)\Big[\ln f(x) -$$

$$-\ln\int_{I_k} f(x)\,dx\Big]dx = \int_I f(x)\ln f(x)\,dx - \int_I f(x)\ln f(x)\,dx = 0$$

d'où il résulte (7).

On remarque que la fonction f est une fonction densité de probabilité et donc

$$\int_{I_k} f_k(x)\ln f_k(x)\,dx \approx -H_{f_k}(q_1, \ldots, q_n) + \ln\frac{n}{\text{mes}\,(I_k)} \tag{8}$$

où

$$q_i = \frac{p_i}{P_n},\ p_i = \frac{f_k(\xi_i)}{n}\text{mes}(I_k),\ i = \overline{1, n}$$

$$\xi_i \in (x, x_{i+1}],\ i = \overline{0, n-1},\ x_i = \frac{b_k - a_k}{n} i + a_k,\ i = \overline{0, n}$$

$$I_k = [a_k, b_k],\ P_n = \sum_{i=1}^{n} p_i.$$

THÉORÈME 3. *Soit X une variable aléatoire et f; $\mathbb{R} \to \mathbb{R}$ la densité de probabilité corespondant de chaque facon*

$$f(x) \neq 0,\ x \in I$$

$$f(x) = 0,\ x \notin I$$

$$\int_I f(x)dx = 1$$

où I est une intervalle de l'axe récle ainsi que

$$\text{mes}\,(I) = +\infty$$

et soit $(I_k)_{k\in\mathbb{N}}$ une suite des intervalles fini $I_k \subset I$, $I_k \subset I_{k+1}$ $k \in \mathbb{N}$ et

$$I = \bigcup_{k\in\mathbb{N}} I_k$$

On considère

$$H_{f_k}(q_1^k, \ldots, q_{n_k}^k) = -\sum_{i=1}^{n_k} q_i^k \ln q_i^k$$

où

$$q_i^k = \frac{p_i^k}{P_{n_k}^k}, \quad p_i^k = \frac{f_k(\xi_i^k)}{n_k}(b_k - a_k), \quad i = \overline{0, n_k - 1}$$

$$\xi_i^k \in (x_i^k, x_{i+1}^k], \ i = \overline{0, n_k - 1}, \ x_i^k = \frac{b_k - a_k}{n_k} i + a_k, \ i = \overline{0, n_k}$$

$$P_n^k = \sum_{i=1}^{n_k} p_i^k$$

et $f_k : \mathbb{R} \to \mathbb{R}$

$$f_k(x) = \begin{cases} \dfrac{f(x)}{\int_{I_k} f(x)dx}, & x \in I_k \\ 0 & , \ x \notin I_k \end{cases}$$

Si

$$\lim_{n_k \to \infty} \frac{n_k}{\text{mes}(I_k)} = L \tag{9}$$

alors

$$\lim_{n_k \to \infty} H_{f_k}(q_1^k, \ldots, q_{n_k}^k) = -\int_I f(x) \ln f(x) dx + \ln L \tag{10}$$

Demonstration. De la relation (7) on a

$$H_{f_k}(q_1^k, \ldots, q_{n_k}^k) = -\int_{I_k} f_k(x) \ln f_k(x)\, dx + \ln \frac{n_k}{\text{mes}(I_k)} + o(1)$$

En tenant compte du théoreme 2 et de la relation (9) on a

$$\lim_{n_k \to \infty} H_{f_k}(q_1^k, \ldots, q_{n_k}^k) = -\lim_{n_k \to \infty} \int_{I_k} f_k(x) \ln f_k(x)\, dx + \lim_{n_k \to \infty} \frac{n_k}{\text{mes}(I_k)} =$$

$$= -\int_I f(x) \ln f(x) dx + L$$

Donc la relation (10) est verifiée.

Remarques. 1. Le théoreme 1 donne une evalution (6) pour n suffisamment grand de l'entroie d'une variable aléatoire avec la densité de probabilité f.

l'expression donnée par (6) représente l'entropie de la variable aléatoire continue avec la densité f *d'ordre d'approximation* $\frac{1}{n}$,

$$H_n(f) = -\int_a^b f(x) \ln f(x)\, dx + \ln \frac{n}{b-a}$$

2. L'expression

$$-\int_a^b f(x) \ln f(x)\, dx \tag{11}$$

qui ne dépende pas de n représente *l'entropie caracteristique* de la variable aléatoire avec la densité f. Cette entropie peut être tantôt positive tantôt negative. Pratiquement l'expression (11) est utilisé dans les applications concretes pour l'entropie d'une variable aléatoire du type continue.

3. Rélativement à la rélation (10) si $L = 1$ on obtient la limite

$$\lim_{n_k \to \infty} H_{f_k}(q_1^k, \ldots, q_{n_k}^k) = -\int_I f(x) \ln f(x)\, dx$$

L'expression

$$-\int_I f(x) \ln f(x)\, dx$$

est l'entropie caracteristique de la variable aléatoire X corespondante.

4. Dans le cas que mes$(I) = +\infty$, soit $\alpha > 0$, il existe l'intervalle I_α, $I_\alpha \subset I$, mes $(I_\alpha) < +\infty$, et la fonction f_α; $\mathbb{R} \to \mathbb{R}$

$$f_\alpha(x) = \begin{cases} \dfrac{f(x)}{\int_{I_\alpha} f(x)\, dx}, & x \in I_\alpha \\ 0 & , \ x \notin I_\alpha \end{cases}$$

de façon que

$$-\int_I f(x) \ln f(x)\, dx = -\int_{I_\alpha} f_\alpha(x) \ln f_\alpha(x)\, dx + \alpha$$

Alors on peut utiliser pour l'entropie H_f, dans les applications, l'entropie H_{f_α}.

Exemples. 1. Soit X une variable aléatoire uniforme sur l'intervalle $[a, b]$ et f; $\mathbb{R} \to \mathbb{R}$ la densité de probabilité

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & x \in [a; b] \\ 0 & , \ x \notin [a, b] \end{cases}$$

alors l'entropie caracteristique est

$$-\int_a^b f(x)\ln f(x)\,dx = \ln(b-a)$$

et l'entropie d'ordre d'approximation $\frac{1}{n}$

$$H_n(f) = \ln n.$$

Donc pour une variable aléatoire qui verifie la loi uniforme, l'entropie $H_n(f)$ est maxime. Cette proprieté, donne pour les variables aléatoires continues, une proprieté analogue de la variable aléatoire discrète

$$X = \begin{pmatrix} x_i \\ \frac{1}{n} \end{pmatrix}_{i=\overline{1,n}}$$

parmi les variables aléatoires

$$X = \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{i=\overline{1,n}}$$

2. Pour une variable aléatoire X qui verifie la loi normale $N(m, \sigma)$ on a

$$-\int_R f(x)\ln f(x)dx = \ln(\sigma\sqrt{2\pi e})$$

Soit $\alpha \in \mathbb{R}$, $0 < \alpha < \frac{1}{2}$ on considère la fonction f_α; $\mathbb{R} \to \mathbb{R}$

$$f_\alpha(x) = \begin{cases} \frac{f(x)}{2\Phi(z_\alpha)}, & x \in [-z_\alpha, z_\alpha] \\ 0, & x \notin [-z_\alpha, z_\alpha] \end{cases}$$

ou Φ est la fonction de Laplace

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_0^z e^{-\frac{x^2}{2}}dx$$

On prend z_α ainsi que

$$-\int_{-z_\alpha}^{z_\alpha} f_\alpha(x)\ln f_\alpha(x)\,dx = \ln\sqrt{2\pi e}\,\sigma - \alpha$$

alors on peut evaluer l'entropie F_f par H_{f_α}.

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ON A BIVARIATE INTEGRAL SPLINE OPERATOR

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Dedicated to Professor D. D Stancu on his 60th anniversary

REZUMAT. — Asupra unui operator spline integral bidimensional. În această lucrare se defineşte operatorul notat $T^{k,l}_{m,n}$, care se aplică unor funcţii de două variabile independente, integrabile pe domeniul $D = [0, 1] \times [0, 1]$ şi care au ca imagini prin acest operator funcţii spline polinomiale de două variabile. Se studiază cîteva proprietăţi ale acestui operator, printre care remarcăm cele relative la convergenţa şirului $\left(T^{k,l}_{m,n}(f)\right)$ la funcţia f, care sînt date prin teorema 3.1, precum şi evaluări ale restului în formula de aproximare construită cu acest operator şi care sînt date în teorema 4.1.

1. In [1, 8] it has been considered an univariate integral spline operator which is a generalization of the Kantorovitch operator [4]. Ih this paper i is constructed a bivariate integral spline operator and there are statied som of this properties.

Let us consider the domain $D = [0, 1] \times [0, 1]$ and the grid realize by the partitions

$$\Delta_m : x_{-k} = \ldots = x_0 = 0 < x_1 < \ldots < x_{m-1} < 1 = x_m = \ldots = x_{m+k}$$

and

$$\widetilde{\Delta}_n : y_{-l} = \ldots = y_0 = 0 < y_1 < \ldots < y_{n-1} < 1 = y_n = \ldots = y_{n+l}$$

with $m, n, k, l \in \mathbf{N}$.

One considers the nodes

$$\xi_{i,k} = \frac{x_{i+1} + \cdots + x_{i+k}}{k}, \quad i = \overline{-k, m-1}$$

respectively

$$\eta_{j,l} = \frac{y_{j+1} + \cdots + y_{j+l}}{l}, \quad j = \overline{-l, n-1}.$$

We have

$$0 = \xi_{-k,k} < \xi_{-k+1,k} < \ldots < \xi_{m-1,k} = 1$$

and

$$0 = \eta_{-l,l} < \eta_{-l+1,l} < \ldots < \eta_{n-1,l} = 1.$$

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One defines the normalized B-splines of the order $k+1$ (or of the degree k)

$$N_{i,k}(x) = \frac{x_{i+k+1} - x_i}{k+1} M_{i,k}(x), \quad i = \overline{-k, m-1}$$

respectively the normalized B-splines of the order $l+1$ (or of the degree l)

$$\widetilde{N}_{j,l}(y) = \frac{y_{j+l+1} - y_j}{l+1} \widetilde{M}_{j,l}(y), \; j = \overline{-l, n-1},$$

where

$$M_{i,k}(x) = [x_i, \ldots, x_{i+k+1}\,;\, (k+1)(s-x)_+^k]$$

is the $(k+1)$-th devided difference of the function $u(s) = (k+1)(s-x)_+^k$. $M_{i,k}$ is called the B-spline of degree k. Similarly, $\widetilde{M_{j,l}}$, defined by

$$\widetilde{M}_{j,l}(y) = [y_j, \ldots, y_{j+l+1}\,;\, (l+1)(t-y)_+^l]$$

is a B-spline of degree l.

Among of the properties of these functions we recall:

$$\sum_{i=-k}^{m-1} N_{i,k}(x) = 1, \; \sum_{j=-l}^{n-1} \widetilde{N}_{j,l}(y) = 1, \; \forall\, x, y \in [0, 1],$$

$$\int_0^1 M_{i,k}(x)dx = 1, \; \int_0^1 \widetilde{M}_{j,l}(y)dy = 1,$$

$$M_{i,k}(x) \geqslant 0, \; \overline{\text{supp}\,(M_{i,k})} = [x_i, x_{i+k+1}],$$

$$\widetilde{M}_{j,l}(y) \geqslant 0, \; \overline{\text{supp}\,(\widetilde{M}_{j,l})} = [y_j, y_{j+l+1}].$$

2. Let us consider the operator $T_{m,n}^{k,l}$:

$$T_{m,n}^{k,l}(f; x, y) = \sum_{i=-k}^{m-1} \sum_{j=-l}^{n-1} M_{i,k}(x) \widetilde{M}_{j,l}(y) \iint_{U_{ij}^{kl}} f(s, t)\, ds\, dt,$$

where $U_{ij}^{kl} = [\xi_{i-1,k+1}, \xi_{i,k+1}] \times [\eta_{j-1,l+1}, \eta_{j,l+1}]$ and f is an intergrable function on D. This operator is a bivariate polynomial spline operator. It associates to an integrable funtion f a bivariate polynominal spline function of the degree k on respect with the first variable and of the degree l on respect to the second variable.

The operator $T^{k,l}_{m,n}$ is a linear and positive operator and $T^{k,l}_{m,n}R_{00}=R_{00}$, where $R_{i,j}(x, y) = x^i y^j$. Indeed, taking into account the properties of the B-spline function, we succesively have

$$T^{k,l}_{m,n}(R_{00}, x, y) = \sum_{i=-k}^{m-1} \sum_{j=-l}^{n-1} M_{i,k}(x)\widetilde{M}_{j,l}(y)(\xi_{i,k+1} - \xi_{i-1,k+1})\cdot(\eta_{j,l+1}-\eta_{j-1,l+1})=$$

$$= \sum_{i=-k}^{m-1} \frac{x_{i+k+1}-x_i}{k+1} M_{i,k}(x) \sum_{j=-l}^{n-1} \frac{y_{j+l+1}-y_j}{l+1} \widetilde{M}_{j,l}(y)$$

$$= \sum_{i=-k}^{m-1} N_{i,k}(x) \sum_{j=-l}^{n-1} \widetilde{N}_{j,l}(y) = 1.$$

The linearity and the positivity are evidently.

3. The operator $T^{k,l}_{m,n}$ can be considered as a linear approximation method on the space $L_p(D)$, $1 \leqslant p \leqslant \infty$, with the usual L_p—norm, $\|\cdot\|_p$.

Theorem 3.1. If $|\Delta_m| = \max\{x_i - x_{i-1}, i=\overline{1, m}\}$, $|\widetilde{\Delta}_n| = \max\{y_j - y_{j-1}, i=\overline{1, n}\}$ *and* $f \in L_p(D)$, $1 \leqslant p \leqslant \infty$, *then*

$$\lim_{|\Delta_m|, |\widetilde{\Delta}_n| \to 0} \|f - T^{k,l}_{m,n}(f)\|_p = 0 \quad (k,\ l \text{ are fixed}) \tag{3.1}$$

$$\lim_{k,l \to 0} \|f - T^{k,l}_{m,n}(f)\|_p = 0 \quad (m,\ n \text{ are fixed}). \tag{3.2}$$

Proof. First, we prove that the sequence of the operators $(T^{k,l}_{m,n})$ is uniform bounded in m, n, k and l. For this, one considers the positive defined function

$$H^{k,l}_{m,n}(x, y;\ s, t) = \sum_{i=-k}^{m-1} \sum_{j=-l}^{n-1} M_{i,k}(x)\widetilde{M}_{j,l}(y) 1_{U^{kl}_{ij}}(s, t).$$

where $1_{U^{kl}_{ij}}$ is the characteristic function of the set U^{kl}_{ij}. It follows that

$$\iint_D H^{k,l}_{mn}(x, y, s, t)\, dsdt = \sum_{i=-k}^{m-1} \sum_{j=-l}^{n-1} M_{i,k}(x)\widetilde{M}_{j,l}(y) \iint_{U^{kl}_{ij}} dsdt = 1$$

and

$$\iint_D H^{k,l}_{mn}(x, y;\ s, t)\, dxdy = \sum_{i=-k}^{m-1} \sum_{j=-l}^{n-1} 1_{U^{kl}_{ij}}(s, t) \iint_D M_{i,k}(x)\widetilde{M}_{j,l}(y) dxdy = 1.$$

Using the function $H^{k,l}_{m,n}$ the operator $T^{k,l}_{m,n}$ can be written in the integral form

$$T^{k,l}_{m,n}(f;\ x, y) = \iint_D H^{k,l}_{m,n}(x, y;\ s,t) f(s, t) dsdt.$$

By Hölder's inequality, for $1 < p < \infty$, one obtains

$$|T^{k,l}_{m,n}(f;\ x,\ y)| \leqslant \left\{\iint_D H^{k,l}_{m,n}(x, y; s, t)dsdt\right\}^{1/q} \left\{\iint_D H^{k,l}_{m,n}(x, y;\ s, t)|f(s, t)|^p\, dsdt\right\}^{1/p} =$$

$$= \left\{\iint_D H^{k,l}_{m,n}(x,\ y;\ s,\ t)|f(s,\ t)|^p\, dsdt\right\}^{1/p}$$

with $1/p + 1/q = 1$. This way, we have succesively

$$\|T^{k,l}_{m,n}(f)\|_p = \left(\iint_D \left|\iint_D H^{k,l}_{m,n}(x,\ y;\ s,\ t) f(s,\ t)dsdt\right|^p dxdy\right)^{1/p} \leqslant$$

$$\leqslant \left(\iint_D \left[\iint_D H^{k,l}_{m,n}(x,\ y;\ s,\ t)|f(s,\ t)|^p\, dsdt\right] dxdy\right)^{1/p} =$$

$$= \left(\iint_D |f(s,\ t)|^p \left[\iint_D H^{k,l}_{m,n}(x, y;\ s,\ t)dxdy\right] dsdt\right)^{1/p} =$$

$$= \left(\iint_D |f(s,\ t)|^p\, dsdt\right)^{1/p} = \|f\|_p.$$

So, $\| T^{k,l}_{m,n}\|_p \leqslant 1$, for $1 < p < \infty$.

For $p = 1$, we have

$$\| T^{k,l}_{m,n}(f)\|_1 \leqslant \iint_D \left| \iint_D H^{k,l}_{m,n}(x, y; s, t) f(s, t)\ dsdt \right| dxdy \leqslant$$

$$\leqslant \iint_D |f(s, t)| \left[\iint_D H^{k,l}_{m,n}(x, y;\ s,\ t)\ dxdy\right] dsdt =$$

$$= \iint_D |f(s, t)|\, dsdt = \|f\|_1,$$

so that $\| T^{k,l}_{m,n}\|_1 \leqslant 1$.

If $p = \infty$, we also have

$$\| T^{k,l}_{m,n}(f)\|_\infty \leqslant \operatorname*{supess}_{(x,y)\in D} \left|\iint_D H^{k,l}_{m,n}(x, y; s, t)\, f(s, t)\ dsdt\right| =$$

$$\leqslant \operatorname*{supess}_{(x,y)\in D} \left(\iint_D H^{k,l}_{m,n}(x, y;\ s,\ t)\ dsdt\right) \|f\|_\infty = \|f\|_\infty.$$

Hence, $\| T^{k,l}_{m,n}\|_p \leqslant 1$ for any p, $1 \leqslant p \leqslant \infty$.

Taking into account that $C(D)$ is a dense subspace of the space $L_p(D)$ and the sequence $(T^{k,l}_{m,n})$ is bounded, it is sufficiently to prove the theorem only for $f \in C(D)$.

Let us consider the bivariate Schoenberg's operator $S^{k,l}_{m,n}$:

$$S^{k,l}_{m,n}(f;\, x,\, y) = \sum_{i=-k}^{m-1} \sum_{j=-l}^{n-1} N_{i,k}(x)\tilde{N}_{j,l}(y)\, f(\xi_{i,k},\, \eta_{j,l}),$$

which was studied in [2]. Then we can write

$$| T^{k,l}_{m,n}(f;\, x,\, y) - S^{k,l}_{m,n}(f;\, x,\, y) | \leqslant \sum_{i=-k}^{m-1} \sum_{j=-l}^{n-1} M_{i,k}(x)\tilde{M}_{j,l}(y) \iint_{U^{kl}_{ij}} |f(s,\, t) -$$

$$- f(\xi_{i,k},\, \eta_{j,l}) \,|\, dsdt.$$

But, using the property of the bidimensional modulus of continuity, we ha

$$|f(s,\, t) - f(\xi_{i,k},\, \eta_{j,l})| \leqslant \omega_\infty(f;\, |s - \xi_{i,k}|,\, |t - \eta_{j,l}|).$$

Because of the inequalities

$$|s - \xi_{i,k}| \leqslant \xi_{i,k+1} - \xi_{i-1,k+1} \leqslant \min\left\{|\Delta_m|,\, \frac{1}{k+1}\right\}$$

$$|t - \eta_{j,l}| \leqslant \eta_{j,l+1} - \eta_{j-1,l+1} \leqslant \min\left\{|\tilde{\Delta}_n|,\, \frac{1}{l+1}\right\}$$

it follows that

$$|f(s,\, t) - f(\xi_{i,k},\, \eta_{j,l})| \leqslant \omega_\infty(f;\, |\Delta_m|,\, |\tilde{\Delta}_n|)$$

respectively

$$|f(s,\, t) - f(\xi_{i,k},\, \eta_{j,l})| \leqslant \omega_\infty\left(f;\, \frac{1}{k+1},\, \frac{1}{l+1}\right).$$

Thus, we have

$$| T^{k,l}_{m,n}(f;\, x,\, y) - S^{k,l}_{m,n}(f;\, x,\, y) | \leqslant \omega_\infty(f;\, |\Delta_m|,\, |\tilde{\Delta}_n|)$$

$$| T^{k,l}_{m,n}(f;\, x,\, y) - S^{k,l}_{m,n}(f;\, x,\, y) | \leqslant \omega_\infty\left(f;\, \frac{1}{k+1},\, \frac{1}{l+1}\right).$$

So,

$$\| T^{k,l}_{m,n}(f) - S^{k,l}_{m,n}(f) \|_\infty \leqslant \omega_\infty(f;\, |\Delta_m|,\, |\tilde{\Delta}_n|) \quad (3$$

respectively

$$\| T^{k,l}_{m,n}(f) - S^{k,l}_{m,n}(f) \|_\infty \leqslant \omega_\infty\left(f;\, \frac{1}{k+1},\, \frac{1}{l+1}\right). \quad (3$$

On the other hand, we have $\|\cdot\|_p \leqslant \|\cdot\|_\infty$, so that, by [2], it follo

$$\|f - S^{k,l}_{m,n}(f)\|_p \to 0 \text{ when } \frac{|\Delta_m|}{k} + \frac{|\tilde{\Delta}_n|}{l} \to 0. \quad (3$$

As,

$$\|f - T^{k,l}_{m,n}(f)\|_p \leqslant \|f - S^{k,l}_{m,n}(f)\|_p + \|T^{k,l}_{m,n}(f) - S^{k,l}_{m,n}(f)\|_p, \tag{3.6}$$

using (3.3), (3.4) and (3.5) it follows (3.1) and (3.2).

4. Let $L^1_p(D)$ be the space of the functions $f \in L_p(D)$ with $f(\cdot, y)$ and $f(x, \cdot)$ absolutely continuous when $(x, y) \in D$ is fixed, and $f_x, f_y \in L_p(D)$.

Theorem 4.1. If $f \in L^1_p(D)$, $1 \leqslant p \leqslant \infty$, then

$$\|f - T^{k,l}_{m,n}(f)\|_p \leqslant \left(1 + \sqrt{\frac{r}{2}}\right)|\Delta_{m,n}|(\|f_x(\cdot, 0)\|_p + \|f_y\|_p) \tag{4.1}$$

and

$$\|f - T^{k,l}_{m,n}(f)\|_p \leqslant \left(1 + \sqrt{\frac{s}{2}}\right)\frac{1}{s}(\|f_x(\cdot, 0)\|_p + \|f_y\|_p), \tag{4.2}$$

where $r = \max\{k, l\}$, $s = \min\{k, l\}$ *and* $|\Delta_{mn}| = \max\{|\Delta_m|, |\tilde{\Delta}_n|\}$.

Proof. Let us denote

$$R^{k,l}_{m,n}(f) = f - T^{k,l}_{m,n}(f),$$

the remainder term of the approximation formula generated by the operator $T^{k,l}_{m,n}$. Using the Taylor formula

$$f(x, y) = f(0, 0) + \int_0^1 (x - s)^0_+ f_x(s, 0)\, ds + \int_0^1 (y - t)^0_+ f_y(x, t) dt,$$

by Peano's theorem, one obtains

$$R^{k,l}_{m,n}(f; x, y) = \int_0^1 K_{10}(x, y; s) f_x(s, 0) ds + \int_0^1 K_{01}(x, y; t) f_y(x, t) dt,$$

where

$$K_{10}(x, y; s) = R^{k,l}_{m,n(x,y)}((x - s)^0_+; x, y) = \varphi_{10}(x, s)$$

and

$$K_{01}(x, y; t) = R^{k,l}_{m,n(x,y)}((y - t)^0_+; x, y) = \varphi_{01}(y, t).$$

So that,

$$\|R^{k,l}_{m,n}(f)\|_p \leqslant \|h_{10}\|_p + \|h_{01}\|_p, \tag{4.3}$$

where

$$h_{10}(x, y) = \int_0^1 K_{10}(x, y; s) f_x(s, 0)\, ds$$

and

$$h_{01}(x, y) = \int_0^1 K_{01}(x, y; t) f_y(x, t)\, dt.$$

In the case $1 < p < \infty$, using the Hölder's inequality, we can write

$$\|h_{10}\|_p = \left(\iint_D \left|\int_0^1 \varphi_{10}(x, s) f_x(s, 0) ds\right|^p dxdy\right)^{1/p}$$

$$\leqslant \left(\iint_D \left[\int_0^1 |\varphi_{10}(x, s)|\, ds\right]^{p/q} \left[\int_0^1 |\varphi_{10}(x, s)|\, |f_x(s, 0)|^p ds\right] dx\, dy\right)^{1/p}.$$

If

$$M_{10} = \sup_{x \in [0,1]} \int_0^1 |\varphi_{10}(x, s)|\, ds, \quad N_{10} = \sup_{s \in [0,1]} \int_0^1 |\varphi_{10}(x, s)|\, dx \tag{4.4}$$

we have succesively

$$\|h_{10}\|_p \leqslant M_{10}^{1/q} \left(\iint_D \left[\int_0^1 |\varphi_{10}(x, s)|\, |f_x(s, 0)|^p ds\right] dxdy\right)^{1/p} =$$

$$= M_{10}^{1/q} \left(\iint_D \left[\int_0^1 |\varphi_{10}(x, s)|\, dx\right] |f_x(s, 0)|^p dsdy\right)^{1/p} \leqslant M_{10}^{1/q} N_{10}^{1/p} \|f_x(\cdot, 0)\|_p. \tag{4.5}$$

On the same way, we obtain

$$\|h_{01}\|_p \leqslant M_{01}^{1/q} N_{01}^{1/p} \|f_y\|_p, \tag{4.6}$$

where

$$M_{01} = \sup_{y \in [0,1]} \int_0^1 |\varphi_{01}(y, t)|\, dt, \quad N_{01} = \sup_{t \in [0,1]} \int_0^1 |\varphi_{01}(y, t)|\, dy.$$

In the case $p = 1$, we have

$$\|h_{10}\|_1 \leqslant N_{10} \|f_x(\cdot, 0)\|_1, \quad \|h_{01}\|_1 \leqslant N_{01} \|f_y\|_1,$$

while, for $p = \infty$, one obtains

$$\|h_{10}\|_\infty \leqslant M_{10} \|f_x(\cdot, 0)\|_\infty, \quad \|h_{01}\|_\infty \leqslant M_{01} \|f_y\|_\infty.$$

Now, we extimate the values M_{10}, M_{01}, N_{10} and N_{01}. We have:

$$\varphi_{10}(x, s) = (x-s)_+^0 - \sum_{i=-k}^{m-1}\sum_{j=-l}^{n-1} M_{i,k}(x)\widetilde{M}_{j,l}(y)\iint_{U_{ij}^{kl}} (u-s)_+^0\,dudv =$$

$$= (x-s)_+^0 - \sum_{j=-l}^{n-1}\widetilde{N}_{j,l}(y)\sum_{i=-k}^{m-1} M_{i,k}(x)\int_{\xi_{i-1,k+1}}^{\xi_{i,k+1}} (u-s)_+^0\,du =$$

$$= (x-s)_+^0 - \sum_{i=-k}^{m-1} M_{i,k}(x)\int_{\xi_{i-1,k+1}}^{\xi_{i,k+1}} (u-s)_+^0\,du.$$

Hence, for $s \leqslant x$, one obtains

$$\varphi_{10}(x, s) = 1 - \sum_{i=-k}^{m-1} M_{i,k}(x)\int_{\xi_{i-1,k+1}}^{\xi_{i,k+1}} (u-s)_+^0\,du =$$

$$= \sum_{i=-k}^{m-1} M_{i,k}(x)\int_{\xi_{i-1,k+1}}^{\xi_{i,k+1}} [1-(u-s)_+^0]du \geqslant 0$$

while, for $s > x$

$$\varphi_{10}(x, s) = -\sum_{i=-k}^{m-1} M_{i,k}(x)\int_{\xi_{i-1,k+1}}^{\xi_{i,k+1}} (u-s)_+^0\,du \leqslant 0.$$

So, we have

$$\int_0^1 |\varphi_{10}(x, s)|ds = \int_0^x \varphi_{10}(x, s)ds - \int_x^1 \varphi_{10}(x, s)ds =$$

$$= \sum_{i=-k}^{m-1} M_{i,k}(x)\int_{\xi_{i-1,k+1}}^{\xi_{i,k+1}} [x-u+2(u-x)_+]du =$$

$$= \sum_{i=-k}^{m-1} M_{i,k}(x)\int_{\xi_{i-1,k+1}}^{\xi_{i,k+1}} |x-u|du.$$

Hence,

$$\int_0^1 |\varphi_{10}(x, s)|ds = T_{m(u)}^k(|x-u|; x), \tag{4.7}$$

where T_m^k is the univariate integral spline operator studied in [1; 8].

As in [1], we have

$$T_m^k(|x-u|;\ x) \leqslant \left(\sqrt{\frac{k}{2}}+1\right)|\Delta_m| \tag{4.8}$$

and

$$T_m^k(|x-u|;\ x) \leqslant \left(\sqrt{\frac{k}{2}}+1\right)\frac{1}{k}. \tag{4.9}$$

From (4.4), (4.7), (4.8) and (4.9), it follows that

$$M_{10} \leqslant \left(\sqrt{\frac{k}{2}}+1\right)|\Delta_m| \quad \text{and} \quad M_{10} \leqslant \left(\sqrt{\frac{k}{2}}+1\right)\frac{1}{k},$$

hence

$$M_{10} \leqslant \left(1+\sqrt{\frac{k}{2}}\right)\min\left\{|\Delta_m|,\ \frac{1}{k}\right\}. \tag{4.10}$$

For the estimation of N_{10}, we succesively have:

$$\int_0^1 |\varphi_{10}(x,\ s)|\,dx = -\int_0^s \varphi_{10}(x,\ s)dx + \int_s^1 \varphi_{10}(x,\ s)dx =$$

$$= \int_0^s \sum_{i=-k}^{m-1} M_{i,k}(x)\,[(\xi_{i,k+1}-s)_+ - (\xi_{i-1,k+1}-s)_+]dx +$$

$$+ \int_s^1 \sum_{i=-k}^{m-1} M_{i,k}(x)\,[(s-\xi_{i-1,k+1})_+ - (s-\xi_{i,k+1})_+]\,dx =$$

$$= \int_0^s \frac{d}{dx} S^k_{m+1(u)}[(u-s)_+;\ x]dx - \int_s^1 \frac{d}{dx} S^k_{m+1(u)}[(s-u)_+;\ x]dx,$$

where S^k_{m+1} is the Schoenberg's operator [7, 9]. Then,

$$\int_0^1 |\varphi_{10}(x;\ s)|\,dx = S^k_{m+1}[(u-s)_+;\ s] - S^k_{m+1}[(u-s)_+;\ 0] -$$

$$- S^k_{m+1}[(s-u)_+;\ 1] + S^k_{m+1}[(s-u)_+;\ s] =$$

$$= S^k_{m+1}\,[(u-s)_+ + (s-u)_+];\ s) = S^k_{m+1}(|u-s|;\ s).$$

It was used the property that S^k_{m+1} interpolates the corresponding function at the edgees of the interval [0, 1] and it is a linear operator.

From [6], we have that

$$S^{k}_{m+1}(|u-s|;s) \leqslant \min\left\{\sqrt{\frac{k+1}{2}}\,|\Delta^{-}_{m}|,\ \sqrt{\frac{1}{2(k+1)}}\right\}, \tag{4.11}$$

hence, from (4.4) and (4.11) it follows that

$$N_{10} \leqslant \min\left\{\sqrt{\frac{k+1}{2}}\,|\Delta_{m}|,\ \sqrt{\frac{1}{2(k+1)}}\right\}. \tag{4.12}$$

Taking into account (4.10) and (4.12), one obtains

$$M^{1/q}_{10}N^{1/p}_{10} \leqslant \left(1+\sqrt{\frac{k}{2}}\right)^{1/q}\left(\sqrt{\frac{k+1}{2}}\right)^{1/p}|\Delta_{m}| \leqslant \left(1+\sqrt{\frac{k}{2}}\right)|\Delta_{m}| \tag{4.13}$$

and

$$M^{1/q}_{10}N^{1/p}_{10} \leqslant \left(\frac{1}{\sqrt{2k}}+\frac{1}{k}\right)^{1/q}\left(\sqrt{\frac{1}{2(k+1)}}\right)^{1/p} \leqslant \frac{1}{k}\left(1+\sqrt{\frac{k}{2}}\right). \tag{4.14}$$

In the same manner, one obtains that

$$M^{1/q}_{01}N^{1/p}_{01} \leqslant \left(1+\sqrt{\frac{l}{2}}\right)|\tilde{\Delta}_{n}| \tag{4.15}$$

respectively

$$M^{1/q}_{01}N^{1/p}_{01} \leqslant \frac{1}{l}\left(1+\sqrt{\frac{l}{2}}\right). \tag{4.16}$$

Finaly, from (4.3), (4.5), (4.6) and (4.13), (4.15), it follows that

$$\|f-T^{k,l}_{m,n}(f)\|_{p} \leqslant \left(1+\sqrt{\frac{k}{2}}\right)|\Delta_{m}|\,\|f_{x}(\cdot,0)\|_{p}+\left(1+\sqrt{\frac{l}{2}}\right)|\tilde{\Delta}_{n}|\ \|f\|_{p}$$

which implies (4.1), respectively from (4.3), (4.5), (4.6) and (4.14), (4.16), one obtains

$$\|f-T^{k,l}_{m,n}(f)\|_{p} \leqslant \frac{1}{k}\left(1+\sqrt{\frac{k}{2}}\right)\|f_{x}(\cdot,0)\|_{p}+\frac{1}{l}\left(1+\sqrt{\frac{l}{2}}\right)\|f_{y}\|_{p},$$

which implies (4.2).

The cases $p=1$ and $p=\infty$ must be considered separately, but the result is the same.

Remark. If one takes an adequate Taylor's formula, we can prove that

$$\|f-T^{k,l}_{m,n}(f)\|_{p} \leqslant \left(1+\sqrt{\frac{k}{2}}\right)|\Delta_{m,n}|(\|f_{x}\|_{p}+\|f_{y}(0,\cdot)\|_{p})$$

and

$$\|f-T^{k,l}_{m,n}(f)\|_{p} \leqslant \frac{1}{s}\left(1+\sqrt{\frac{s}{2}}\right)(\|f_{x}\|_{p}+\|f_{y}(0,\cdot)\|_{p}).$$

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A LINEAR POSITIVE OPERATOR ASSOCIATED WITH THE PEARSON'S — χ^2 DISTRIBUTION

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REZUMAT. — Un operator liniar pozitiv asociat cu distribuţia Pearson — χ^2. În articolul [1] am definit un operator liniar pozitiv folosind o metodă probabilistică [4] asociată unei distribuţii χ^2. În prezenta lucrare se dau unele proprietăţi ale acestui operator.

1. **Introduction.** In our paper [1], we defined a new linear positive operator, using a probabilistic method [4], which was associated with the Pearson's — χ^2 distribution:

$$(L_n f)(x) = E\left[f\left(\frac{1}{n}\sum_{k=1}^{n} X_k^2\right)\right] = \frac{1}{(2x)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\int_0^\infty t^{\frac{n}{2}-1} e^{-t/2x} f\left(\frac{t}{n}\right) dt, \quad X > 0, \tag{1.1}$$

where the sequence of independent random variable $(X_k)_{k\in N}$ having the same normal distribution $N(0, \sqrt{x})$, $x > 0$, $E[X_k] = 0$, $D^2[X_k] = x$, $k \in N$, and f is a real function bounded on $(0, +\infty)$ such that the mean value of the random variable $f\left(\frac{1}{n}\sum_{k=1}^{n} X_k^2\right)$ exists, for any $n \in N$.

2. **Some properties of the linear positive operator.** THEOREM 2.1. *The operator* (1.1) *admitted the following representation*:

$$(L_n f)(x) = \frac{1}{\Gamma\left(\frac{n}{2}\right)}\int_0^\infty u^{\frac{n}{2}-1} e^{-u} f\left(\frac{2x}{n} u\right) du, \quad x > 0 \tag{1.2}$$

Proof: Taking $t/(2x) = u$ in (1.1), the proof is immediate, but the representation (1.2) is very important, because the parameter x passing from the kernel to the argument of function, and so we can demonstrate the following properties.

THEOREM 2.2. *:If f is a real bounded and non-decreasing function on $(0, +\infty)$, then $L_n f$ is a non-decreasing function on $(0, +\infty)$, for any $n \in N$.*

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Proof: Let be $0 < x \leqslant y$. Because $f(x) \leqslant f(y)$ we have

$$(L_n f)(x) = \frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty u^{\frac{n}{2}-1} e^{-u} f\left(\frac{2x}{n} u\right) du \leqslant \frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty u^{\frac{n}{2}-1} e^{-u} f\left(\frac{2y}{n} u\right) du =$$
$$= (L_n f)(y), \quad n \in N.$$

THEOREM 2.3: *If $f \in$ Lip (α) is a Lepschitz function of order α, $0 < \alpha \leqslant 1$, then $L_n f \in$ Lip (α), $0 < \alpha \leqslant 1$, for any $n \in N$.*

Proof: If $f \in$ Lip (α), $0 < \alpha \leqslant 1$, then exists a positive constant $K_f > 0$, so that

$$|f(x) - f(y)| \leqslant K_f |x - y|^\alpha, \ (\forall) x, y > 0.$$

Hence

$$|(L_n f)(x) - (L_n f)(y)| \leqslant \frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty u^{\frac{n}{2}-1} e^{-u} \left| f\left(\frac{2x}{n} u\right) - f\left(\frac{2y}{n} u\right)\right| du \leqslant$$

$$\leqslant \frac{\left(\frac{2}{n}\right)^\alpha}{\Gamma\left(\frac{n}{2}\right)} K_f |x - y|^\alpha \int_0^\infty u^{\alpha + n/2 - 1} e^{-u}\, du = \left(\frac{2}{n}\right)^\alpha \frac{\Gamma\left(\alpha + \frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} K_f |x - y|^\alpha.$$

THEOREM 2.4. *If f is a non-concave (non-convexe) function of first order on $(0, +\infty)$, then $L_n f$ is a non-concave (non-convexe) function of first order on $(0, +\infty)$, for any $n \in N$,*

Proof: Let be $0 < \lambda < 1$, $x, y > 0$ and f a non-concave function:

$$f(\lambda x + (1 - \lambda) y) \leqslant \lambda f(x) + (1 - \lambda) f(y)$$

Then,

$$(L_n f)(\lambda x + (1 - \lambda) y) = \frac{1}{\Gamma(n)} \int_0^\infty u^{\frac{n}{2}-1} e^{-u} f\left(\frac{2(\lambda x + (1 - \lambda) y}{n} u\right) du \leqslant$$

$$\leqslant \frac{1}{\Gamma(n)} \int_0^\infty u^{\frac{n}{2}-1} e^{-u} \left[\lambda f\left(\frac{2x}{n} u\right) + (1 - \lambda) f\left(\frac{2y}{n} u\right)\right] du =$$

$$= \lambda (L_n f)(x) + (1 - \lambda)(L_n f)(y), \ n \in N, \ x, y > 0.$$

3. **Asymptotic estimate of the remainder.** Since, in the paper [1], we have studied the convergence property of the sequence (1.1) for $f \in C(0, a)$, $0 < a < +\infty$ and we have given an estimate of the order of approximation, in this section, we shall deal with the asymptotic estimate of the remainder.

Using our results [2], [3], we obtain:

THEOREM 3.1. *If f is a real bounded function on $(0, +\infty)$ having the second derivative at a point $x \in (0, +\infty)$, then for the sequence $(L_n f)_{n \in N}$ we have:*

$$\lim_{n\to\infty} n\left[f(x) - \frac{1}{(2x)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\int_0^\infty t^{\frac{n}{2}-1} e^{-t/2x} f\left(\frac{t}{n}\right)dt\right] = -x^2 f''(x) \qquad (3.1)$$

Proof: In the case of our operator (1.1),

$(L_n f(x) = E\left[f\left(\frac{1}{n}\sum_{k=1}^{n} X_k^2\right)\right]$, where $(X_k)_{k\in N}$ having the same normal distribution $N(0, \sqrt{x})$, $X > 0$, our results [2]

$$\lim_{n\to+\infty} n\left\{f(x) - E\left[f\left(\frac{1}{n}\sum_{k=1}^{n} X_k\right)\right]\right\} = -\frac{1}{2}f''(x)D^2[X_k],$$

become (3.1), since

$$D^2[X_k^2] = E[X_k^4] - (E[X_k^2])^2 = 3x^2 - x^2 = 2x^2.$$

THEOREM 3.2. *If f is a real bounded function on $(0, +\infty)$, having the derivative $f^{(2k)}$ at a point $x \in (0, +\infty)$ then for the sequence $(L_n f)_{n\in N}$ we have:*

$$\lim_{n\to+\infty} n^k\left[(L_n f)(x) - f(x) - \sum_{\nu=1}^{2k-1}\frac{f^{(\nu)}(x)}{\nu!\,n^\nu}T_{n,\nu}(x)\right] = \frac{x^{2k}}{k!}f^{(2k)}(x) \qquad (3.2)$$

where $T_{n,\nu}(x) = E\left[\left(\sum_{k=1}^{n} X_k^2 - nx\right)^\nu\right]$, $(X_k)_{k\in N}$ having the same normal distribution $N(0, \sqrt{x})$, $x > 0$, with $E[X_k] = 0$ and $D^2[X_k] = x$, $k \in N$.

Proof: With $D^2[X_k^2] = 2x^2$, our results [3]

$$\lim_{n\to\infty} n^k\left[E\left[f\left(\frac{1}{n}\sum_{k=1}^{n} X_k\right)\right] - f(x) - \sum_{\nu=1}^{2k-1}\frac{f^{(\nu)}(x)}{\nu!\,n^\nu}T_{n,\nu}(x)\right] = \frac{1}{2^k k!}(D^2(X_k))^k f^{(2k)}_{(x)}$$

become (3.2), in the case of the positive linear operators (1.1).

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THE REMAINDER OF CERTAIN SHEPARD TYPE INTERPOLATION FORMULAS

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REZUMAT. — Restul unor formule de interpolare de tip Shepard. În lucrare se studiază termenul rest al unor formule de aproximare generate de operatori liniari şi pozitivi de tip Shepard, pentru funcţii de o variabilă respectiv de două variabile. Rezultatele obţinute sînt prezentate în teoremele 1—5.

1. **Introduction.** On of the preferable field of Professor D. D. Stancu is the theory of linear and positive operators. In many of his interesting papers, using such operators, Professor D. D. Stancu has studied the uniform approximation of the functions of one and several variables and the remainder of such approximation formulas (i.e. [6], [7], [9], [11], [12], [13]). It must be also remarked the linear and positive operators obtained by Professor D. D. Stancu, using probabilistic methods (i.e. [8], [10]).

Inspired by the works of Professor D. D. Stancu, in this paper, I shall study the remainder of some approximation formulas generated by Shepard's type linear and positive operators, for functions of one and two variables.

First, let us remind Shepard's interpolation function [5]. One considers a function f defined on the real plane $\mathbf{R}^2$, and let P_i, $P_i = (x_i, y_i)$, $i = \overline{0, n}$ be distinct points in $\mathbf{R}^2$. Let also ρ be a metric in $\mathbf{R}^2$. For a given point P, $P = (x, y)$, let r_i, $r_i = r_i(x, y)$, be the distance between the points P and P_i, i.e. $r_i(x, y) = \rho((x, y), (x_i, y_i))$. Then, Shepard's interpolation function is defined by

$$F(x, y) = \sum_{i=0}^{n} A_i(x, y) f(x_i, y_i), \tag{1}$$

where

$$A_i(x, y) = \prod_{\substack{j=0\\ j\neq i}}^{n} [r_j(x, y)]^{\mu} \Bigg/ \left(\sum_{j=0}^{n} \prod_{\substack{k=0\\ k\neq j}}^{n} [r_k(x, y)]^{\mu}\right) \tag{2}$$

and μ is a real number, $0 < \mu < \infty$.

The functions A_i, $i = \overline{0, n}$, have the cardinality property:

$$A_i(x_j, y_j) = \delta_{ij}, \; i, j = \overline{0, n},$$

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i.e.

$$F(x_i, y_i) = f(x_i, y_i), \quad i = \overline{0, n}$$

and

$$A_i(x, y) \geqslant 0 \quad \text{for all} \quad (x, y) \in \mathbf{R}^2.$$

Shepard himself [5] has pointed out some of the properties and applications of the interpolating function F. Such a very important property is that $\min\limits_{0 \leqslant i \leqslant n} f(x_i, y_i) \leqslant F(x, y) \leqslant \max\limits_{0 \leqslant i \leqslant n} f(x_i, y_i)$ for all $(x, y) \in \mathbf{R}^2$. Next, in the papers by Gordon and Wixon [1], Schumaker [4], as in the thesis by Poeppelmeir [2], mentioned in the provious papers, were remarked new and interesting properties of this function.

As it is remarked in [1], the interpolation function F can be viewed as a projection of f onto the finite-dimensional linear space spanned by the functions A_i, $i = \overline{0, n}$. Let us denoted the corresponding projector by P. So, P is a linear and positive ($A_i \geqslant 0$, $i = \overline{0, n}$) operator.

Next, we study the remainder of the interpolation formula generated by Shepard's projector, in both one and two-dimensional cases.

1. *One-dimensional case.* Let f be a real-valued function on an interval $[a, b]$ and $x_i \in [a, b]$, $i = \overline{0, n}$. Shepard's interpolation formula is

$$f = P_1 f + R_1 f$$

where

$$(P_1 f)(x) = \sum_{i=0}^{n} A_i(x) f(x_i),$$

with

$$A_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} |x - x_j|^{\mu} \Big/ \Big(\sum_{k=0}^{n} \prod_{\substack{j=0 \\ j \neq k}}^{n} |x - x_j|^{\mu} \Big), \tag{3}$$

and $R_1 f$ is the remainder term. The operator P_1 is a projector, i.e. P_1 is linear and idempotent. For the remainder term we have:

THEOREM 1. *If f is absolutely continuous on $[a, b]$ then*

$$(R_1 f)(x) = \int_a^b \varphi_1(s\,;\, s) f'(s)\, ds, \tag{4}$$

where

$$\varphi_1(x\,;\, s) = (x - s)_+^0 - \sum_{i=0}^{n} A_i(x)(x_i - s)_+^0,$$

and

$$|(R_1 f)(x)| \leqslant K(x) M_1 f,$$

where

$$K(x) = x - \sum_{i=0}^{n} x_i A_i(x) + 2 \sum_{i=0}^{n} A_i(x)(x_i - x)_+$$

and

$$M_1 f = \sup_{a \leqslant x \leqslant b} |f'(x)|.$$

Proof. Taking into account that $R_1 e_0 = e_0$, where $e_k(x) = x^k$, formula (4) follows by Peano's theorem. Now, let us suppose that $x \in (x_{k-1}, x_k)$ and $\varphi_1^j(x;\cdot) = \varphi_1(x;\cdot)|_{[x_{j-1}, x_j]}$, for $j = \overline{1, n}$.

Then, we have

$$\varphi_1^j(x;s) = (x-s)_+^0 - \sum_{i=j}^{n} A_i(x).$$

Using the properties that $A_0(x) + \ldots + A_n(x) = 1$ and $A_i(x) \geqslant 0$, $i = \overline{0, n}$, one obtains

$$\varphi_1^j(x;s) = \begin{cases} \sum_{i=0}^{j-1} A_i(x), & \text{for } s \leqslant x \\ -\sum_{i=j}^{n} A_i(x), & \text{for } s > x. \end{cases}$$

Hence, $\varphi_1^j(x;s) \geqslant 0$ for $s \leqslant x$ and $\varphi_1^j(x;s) \leqslant 0$ for $s > x$.
As, $x \in (x_{k-1}, x_k)$, it follows that $\varphi_1^j(x;s) \geqslant 0$ for $j = \overline{0, k-1}$, $\varphi_1^j(x;s) \leqslant 0$ for $j = \overline{k+1, n}$ and $\varphi_1^k(x;s) \geqslant 0$ for $s \leqslant x$ and $\varphi_1^k(x;s) \leqslant 0$ for $s > x$. This way, one obtains that $\varphi_1(x;s) \geqslant 0$ for $s \leqslant x$ and $\varphi_1(x;s) \leqslant 0$ for $s > x$. From (4), we have

$$|(R_1 f)(x)| \leqslant M_1 f \int_a^b |\varphi_1(x;s)|\, ds.$$

But,

$$\int_a^b |\varphi_1(x;s)|\, ds = \int_a^x \varphi_1(x;s)\, ds - \int_x^b \varphi_1(x;s) ds = K(x),$$

and the theorem is proven.

Now, let us suppose that at each point x_i, $i = \overline{0, n}$ there also exists the derivative $f'(x_i)$. We define the operator P_2 as follows:

$$(P_2 f)(x) = \sum_{i=0}^{n} A_i(x)[f(x_i) + (x - x_i) f'(x_i)].$$

THEOREM 2. *If* $\mu > 1$ *then* $(P_2f)^{(k)}(x_i) = f^{(k)}(x_i)$, $i = \overline{0, n}$, *for* $k = 0, 1$.

Proof. It is obviously that $(P_2)f(x_i) = f(x_i)$, $i = \overline{0, n}$. For the first derivative, we have

$$(P_2f)'(x) = \sum_{i=0}^{n} \{A_i(x)f'(x_i) + A_i'(x)[f(x_i) + (x - x_i)f'(x_i)]\},$$

and, for $\mu > 1$, using the formula (3), it is easily to check that $A_i'(x_p) = 0$ for any i, $p = \overline{0, n}$. Indeed, if A_i is written in the form $A_i = g_i/h_i$, where

$$g_i(x) = \prod_{\substack{j=0\\ j\neq i}}^{n} |x - x_j|^{\mu}, \quad h_i(x) = \sum_{k=0}^{n} \prod_{\substack{j=0\\ j\neq k}}^{n} |x - x_j|^{\mu}$$

then $A_i' = (g_i' h_i - g_i h_i')/h_i^2$. But, $\mu > 1$ implies $g_i(x_k) = g_i'(x_k) = 0$ for any $k = \overline{0, n}$, $k \neq i$, $h_i(x_i) = g_i(x_i)$, $h_i'(x_i) = g_i'(x_i)$, i.e.
$A_i'(x_k) = 0$ for any k, $i = \overline{0, n}$. Hence

$$(P_2f)'(x_k) = \sum_{i=0}^{n} A_i(x_k)\, f'(x_i).$$

As, $A_i(x_k) = \delta_{ik}$, it follows that $(P_2f)'(x_k) = f'(x_k)$, $k = \overline{0, n}$. So, the operator P_2 interpolates either the function f and its derivative f' at the points x_k, $k = \overline{0, n}$.

LEMMA 1. *P_2 is a linear operator and $P_2e_k = e_k$, for $k = 0,1$.*

The proof is a straightforward computation.

Let

$$f = P_2f + R_2f$$

be the interpolation formula generated by the operator P_2.

THEOREM 3. *If $f \in H^2[a, b]$ then*

$$(R_2f)(x) = \int_a^b \varphi_2(x\,;\,s)\, f''(s)\, ds \tag{6}$$

where

$$\varphi_2(x\,;\,s) = (x - s)_+ - \sum_{i=0}^{n} A_i(x)[(x_i - s)_+ + (x - x_i)(x_i - s)_+^0].$$

If, more than that, f'' is continuous on the interval $[a, b]$ then

$$(R_2f)(x) = \left[-\frac{x^2}{2} + \frac{1}{2}\sum_{i=0}^{n} A_i(x)x_i^2\right]f''(\xi), \quad a \leqslant \xi \leqslant b. \tag{7}$$

Proof. The integral representation (6) follows by the lemma 1 and b Peano's theorem. Now, $\varphi_2^k(x;\cdot) = \varphi_2(x;\cdot)|_{[x_{k-1}, x_k]}$ has the expression

$$\varphi_2^k(x;s) = (x-s)_+ - (x-s)\sum_{i=k}^{n} A_i(x).$$

So,

$$\varphi_2^k(x;s) = \begin{cases} (x-s)\sum_{i=0}^{k-1} A_i(x), & \text{for } s \leqslant x \\ -(x-s)\sum_{i=k}^{n} A_i(x), & \text{for } s > x, \end{cases}$$

i.e. $\varphi_2^k(x;s) \geqslant 0$ for any $k = \overline{1, n}$. This way, it follows that $\varphi_2(x;s) \geqslant 0$ f $x, s \in [a, b]$, and by the mean theorem (f'' is continuous), one obtains t representation (7).

2. *Bivariate case.* Let f be a bivariate real-valued function defined on t rectangle D, $D = [a, b] \times [c, d]$ and (x_i, y_i), $i = \overline{0, n}$ distinct pouints in l One considers Shepard's interpolation formula

$$f = P_{11}f + R_{11}f,$$

where $P_{11}f$ is given by (1), i.e.

$$(P_{11}f)(x, y) = \sum_{i=0}^{n} A_i(x, y)\, f(x_i, y_i),$$

and $R_{11}f$ is the remainder term. P_{11} will be called Shepard's bivariate oper tor. It is known that P_{11} is a linear and positive operator [1].

Next, we shall study the remainder term $R_{11}f$.

THEOREM 4. *If $f \in C^{11}(D)$ then*

$$(R_{11}f)(x, y) = \int_a^b \varphi_{10}(x, y; s) f^{(1,0)}(s, c)\, ds + \int_c^d \varphi_{01}(x, y; t) f^{(0,1)}(a, t)\, dt + \\ + \iint_D \varphi_{11}(x, y; s, t) f^{(1,1)}(s, t)\, ds\, dt, \tag{8}$$

where

$$\varphi_{10}(x, y; s) = (x-s)_+^0 - \sum_{i=0}^{n} A_i(x, y)(x_i - s)_+^0$$

$$\varphi_{01}(x, y; t) = (y-t)_+^0 - \sum_{i=0}^{n} A_i(x, y)(y_i - t)_+^0$$

$$\varphi_{11}(x, y; s, t) = (x-s)_+^0 (y-t)_+^0 - \sum_{i=0}^{n} A_i(x, y)(x_i - s)_+^0 (y_i - t)_+^0,$$

and

$$|(R_{11}f)(x, y)| \leqslant H_{10}(x, y)M_{10}f + H_{01}(x, y)M_{01}f + H_{11}(x, y)M_{11}f, \quad (9)$$

with

$$H_{10}(x, y) = x - \sum_{i=0}^{n} x_i A_i(x, y) + 2\sum_{i=0}^{n} A_i(x, y)(x_i - x)_+$$

$$H_{01}(x, y) = y - \sum_{i=0}^{n} y_i A_i(x, y) + 2\sum_{i=0}^{n} A_i(x, y)(y_i - y)_+$$

$$H_{11}(x, y) = (x - a)(y - c) - \sum_{i=0}^{n} A_i(x, y)(x_i - a)(y_i - c) +$$

$$+ 2\sum_{i=0}^{n} A_i(x, y)[(x_i - a)(y_i - y)_+ + (x_i - x)_+(y_i - c) - (x_i - x)_+(y_i - y)_+ ,$$

where

$$M_{10}f = \sup_{a \leqslant x \leqslant b} |f^{(1,0)}(x, c)|, \; M_{01}f = \sup_{c \leqslant y \leqslant d} |f^{(0,1)}(y, a)|,$$

$$M_{11}f = \sup_{D} |f^{(1,1)}(x, y)|.$$

Proof. As $f \in C^{11}(D)$, we have [6]

$$f(x, y) = f(a, c) + (Rf)(x, y),$$

with

$$(Rf)(x, y) = \int_a^b (x - s)_+^0 f^{(1,0)}(s, c)\, ds + \int_c^d (y - t)_+^0 f^{(0,1)}(a, t)\, dt +$$

$$+ \iint_D (x - s)_+^0 (y - t)_+^0 f^{(1,1)}(s, t)\, ds\, dt.$$

Taking into account that $R_{11}e_{00} = e_{00}$, where $e_{kj}(x, y) = x^k y^j$, the formula (8) follows immediately.

To prove the second part of the theorem, we remark that:

$$\varphi_{10}(x, y; s) \geqslant 0 \text{ for } s \leqslant x \text{ and } \varphi_{10}(x, y; s) \leqslant 0 \text{ for } s > x;$$

$$\varphi_{01}(x, y; t) \geqslant 0 \text{ for } t \leqslant y \text{ and } \varphi_{01}(x, y; t) \leqslant 0 \text{ for } t > y$$

respectively $\varphi_{11}(x, y; s, t) \geqslant 0$ for $(s, t) \in [a, x] \times [c, y)$ and $\varphi_{11}(x, y; s, t) \leqslant 0$

for $(s, t) \in D/[a, x] \times [c, y]$. Thus,

$$\int_a^b \varphi_{10}(x, y; s)\, ds = H_{10}(x, y), \quad \int_c^d \varphi_{01}(x, y; t)\, dt = H_{01}(x, y)$$

and

$$\iint_D \varphi_{11}(x, y; s, t)\, ds\, dt = H_{11}(x, y).$$

But, from (8) we obtain

$$|(R_{11}f)(x,y)| \leq M_{10}f \int_a^b |\varphi_{10}(x, y; s)|\, ds + M_{01}f \int_c^d |\varphi_{01}(x, y; t)|\,dt +$$

$$+ M_{11}f \iint_D |\varphi_{11}(x, y; s, t)|\, ds\, dt$$

and the theorem is proven.

An interesting interpolation formula is generated by Shepard's type operator P_{22} [4], defined by

$$(P_{22}f)(x, y) = \sum_{i=0}^{n} A_i(x, y)[f(x_i, y_i) + (x - x_i)f^{(1,0)}(x_i, y_i) + (y - y_i)f^{(0,1)}(x_i, y_i)].$$

LEMMA 2. *P_{22} is a linear and positive operator which interpolates the function f and its first partial derivatives at the points (x_i, y_i), $i = \overline{0, n}$, and $P_{22}e_{kj} = e_{kj}$ for $(k, j) \in \{(0,0), (1,0), (0,1)\}$.*

Proof. The positivity follows by the relations $A_i \geq 0$, $i = \overline{0, n}$. The interpolation properties: $(P_{22}f)^{(j,k)}(x_i, y_i) = f^{(j,k)}(x_i, y_i)$, $i = \overline{0, n}$, $(j, k) \in \{(0,0), (1,0), (0,1)\}$, can be checked by a direct computation.
Also, it is easy to check that $P_{22}e_{jk} = e_{jk}$ for $(j, k) \in \{(0,0), (1,0), (0,1)\}$ and the lemma is proven.

For the remainder term of the interpolation formula

$$f = P_{22}f + R_{22}f$$

we have:

THEOREM 5. *If $f \in B_{1,1}(a, c)$ [3] then*

$$(R_{22}f)(x, y) = \int_a^b \varphi_{20}(x, y; s)f^{(2,0)}(s, c)\, ds + \int_c^d \varphi_{02}(x, y; t)f^{(0,2)}(a, t)\, dt +$$

$$+ \iint_D \varphi_{11}(x, y; s, t)f^{(1,1)}(s, t)\, ds\, dt \tag{10}$$

where

$$\varphi_{20}(x, y; s) = (x - s)_+ - \sum_{i=0}^{n} A_i(x, y)[(x_i - s)_+ + (x - x_i)(x_i - s)_+^0]$$

$$\varphi_{02}(x, y; t) = (y - t)_+ - \sum_{i=0}^{n} A_i(x, y)[(y_i - t)_+ + (y_i - y)(y_i - t)_+^0]$$

$$\varphi_{11}(x, y; s, t) = (x - s)_+^0 (y - t)_+^0 - \sum_{i=0}^{n} A_i(x, y)(x_i - s)_+^0 (y_j - t)_+^0,$$

and

$$|(R_{22}f)(x, y)| \leqslant H_{20}(x, y)M_{20}f + H_{02}(x, y)M_{02}f + H_{11}(x, y)M_{11}f \tag{11}$$

with

$$H_{20}(x, y) = -\frac{x^2}{2} + \frac{1}{2}\sum_{i=0}^{n} x_i^2 A_i(x, y)$$

$$H_{02}(x, y) = -\frac{y^2}{2} + \frac{1}{2}\sum_{i=0}^{n} y_i^2 A_i(x, y)$$

H_{11} is given in (9) and

$$M_{20}f = \sup_{a \leqslant x \leqslant b} |f^{(2,0)}(x, c)|,$$

$$M_{02}f = \sup_{c \leqslant y \leqslant d} |f^{(0,2)}(a, y)|.$$

Proof. The integral representation (10) follows by the lemma 2 and by the corresponding Peano's theorem [3]. To prove the second part of the theorem, first, we observe that $\varphi_{20} \geqslant 0$. Indeed, if $\varphi_{20}^k(x, y; .) = \varphi_{20}(x, y; .)|_{[x_{k-1}, x_k]}$ then

$$\varphi_{20}^k(x, y; s) = (x - s)_+ - \sum_{i=k}^{n} A_i(x, y)(x - s),\ s \in [x_{k-1}, x_k],$$

or

$$\varphi_{20}^k(x, y; s) = \begin{cases} (x - s)\left[1 - \sum_{i=k}^{n} A_i(x, y)\right], & \text{for } s \leqslant x \\ (s - x)\sum_{i=k}^{n} A_i(x, y), & \text{for } s > x \end{cases}$$

As, $A_i(x, y) \geqslant 0$ for any $(x, y) \in D$ and $A_0(x, y) + \ldots + A_n(x, y) = 1$, it follows that $\varphi_{20}^k(x, y; s) \geqslant 0$ for $s \in [x_{k-1}, x_k]$. Hence, $\varphi_{20}(x, y; s) \geqslant 0$ for any $s \in [a, b]$ and $(x, y) \in D$. In the same way, one obtains that $\varphi_{02}(x, y; t) \geq 0$ for $t \in [c, d]$ and $(x, y) \in D$.

Now

$$|(R_{22}f)(x, y)| \leq M_{20}f\int_a^b \varphi_{20}(x, y; s)\, ds + M_{02}f\int_c^d \varphi_{02}(x, y; t)\, dt +$$

$$+ M_{11}f\iint_D |\varphi_{11}(x, y; s, t)|\, ds\, dt.$$

But,

$$\int_a^b \varphi_{20}(x, y; s)\, ds = H_{20}(x, y), \quad \int_c^d \varphi_{02}(x, y; t) = H_{02}(x, y)$$

respectively

$$\iint_D |\varphi_{11}(x, y; s, t)|\, ds\, dt = H_{11}f$$

and the theorem is proven.

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A METHOD ANALOGOUS TO THE CHEBYSHEV METHOD FOR THE SOLVING OF THE OPERATOR EQUATIONS DEFINITED IN FRÉCHET SPACES

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REZUMAT. — O metodă analogă cu metoda lui Cebişev de rezolvare a ecuaţiilor operatoriale definite în spaţii Fréchet. În lucrare se studiază problema rezolvării unei ecuaţii operatoriale $P(x) = \Theta$ $P : X \to Y$, X şi Y fiind spaţii Fréchet folosind metoda iterativă $x_{n-1} = x_n - \Lambda_n P(x_n) - \Lambda_n [x_n, x_{n-1}, x_{n-2} P] \Lambda_{n-1} P(x_{n-1}) \Lambda_n \bar{P}(x_n)$ unde $\Lambda_n = [x_n, x_{n-1}; P]^{-1}$ şi $\bar{\Lambda}_n = [x_n, x_{n-2}, P]^{-1}$. Spre deosebire de studiul similar făcut în 1, în lucrare se evită uniform mărginirea normei generalizate a operatorului $\Lambda = [x', x''; P^{-1}]$, presupunindu-se numai existenţa lui.

1. Let be the equation

$$P(x) = 0 \tag{1}$$

where $P : X \to Y$ is a nonlinear continuous mapping, X and Y Fréchet spaces and θ the null element of the space Y. Suppose also that P admites divided differences up to the third order, inclusively.

The existence and unicity of the solution of the equation (1), using the iterative method

$$x_{n+1} = x_n - \Lambda_n P(x_n) - \Lambda [x_n, x_{n-1}, x_{n-2}; P] \Lambda_{n-1} P(x_{n-1}) \bar{\Lambda}_n P(x_n) \tag{2}$$

where $\Lambda_n = [x_n, x_{n-1}; P]^{-1}$ and $\bar{\Lambda}_n = [x_n, x_{n-2}; P]^{-1}$, was studied in [2], proving the following theorem:

THEOREM A. 1°. *If it exists* $\Lambda = [x', x''; P]^{-1}$, $\forall x', x'' \in S(x_0, R)$, *and* $)|\Lambda|(\leq B$, *by* $)|\cdot|($ *we denote the cvasinorm of an element of a Fréchet space* [1].

2°. $)|P(x_i)|(\leq \eta_i$, $i = -2, -1, 0$ *and* $\eta_0 \leq \eta_{-1} \leq \eta_{-2}$

3°. *For any* $x', x'', x''' \in S$, *we have*

$$)|[x', x'', x'''; P]|(\leq M, \quad)|[x', x'', x''', x^{IV}; P]|(\leq N$$

4°. $h_{-2}E_{-2} < 1$, *where* $h_{-2} = B^2M\eta_{-2}$,

$$E^2_{-2} = \frac{1}{H_{-2}}\left\{3 + \left(\frac{N}{BM^2} + 1\right)(1 - h_{-2})^3\right\} \quad \text{and} \quad H_{-2} = 1 - h_{-2}(1 + h_{-2}) > 0$$

then the equation (1) *has, in* $S(x_0, R)$ *with* $R = \frac{2B\eta_{-1}}{1 - (h_{-2}E_{-2})^2}$, *a solution* x^* *and*

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only one, which is the limit of the sequence (2), *the convergence order being given by the inequality*

$$\|x^* - x_n\| \leqslant \frac{2B\eta_{-2}}{1-(h_{-2}E_{-2})^2}(h_{-2}E_{-2})^{2\sum_{i=2}^{n-1} t_i} \leqslant R(h_{-2}E_{-2})^{2\sum_{i=2}^{n-1} t_i}$$

where $t_i = t_{i-3} + t_{i-2} + t_{i-1}$ $(i = 2, 3, \ldots)$ and $t_{-1} = -1$, $t_0 = t_1 = 1$.

2. In the following we will change the condition 1° of the theorem A removing the uniform bounded of the mapping, proving:

THEOREM 1. *Suppose that for some initial approximations* $x_{-2}, x_{-1}, x_0 \in S(x_0, R)$ *the following conditions are satisfied:*

1. *The mapping* $\Lambda_0 = [x_0, x_{-1}; P]$ *exists;*
2. *There exist* η_i, $i = -2, -1, 0$ *such that*
$\|\Lambda_0 P(x_i)\| \leqslant \eta_i$ *and* $\eta_0 \leqslant \eta_{-1} \leqslant \eta_{-2}$;
3. $\|\Lambda_0[x', x'', x'''; P]\| \leqslant \widetilde{M}$, $\|\Lambda_0[x', x'', x''', x^{IV}; P]\| \leqslant \widetilde{N}$ $\forall x', x'', x''', x^{IV} \in S$;
4. $\widetilde{h}_{-2}\widetilde{E}_{-2} < 1$, *where* $\widetilde{h}_{-2} = \widetilde{M}\eta_{-2}$

$$\widetilde{E}^2_{-2} = \frac{1}{\widetilde{H}_{-2}}\left\{3 + \left(\frac{\widetilde{N}}{\widetilde{M}^2} + 1\right)(1 - \widetilde{h}_{-2})^3\right\} \text{ and } \widetilde{H}_{-2} = 1 - \widetilde{h}_{-2}(1 + \widetilde{h}_{-2}) > 0.$$

Then the equation (1) *has, in* $S(x_0, \widetilde{R})$ *with* $\widetilde{R} = \frac{2\eta_{-2}}{1-(\widetilde{h}_{-2}\widetilde{E}_{-2})^2}$, *a solution* x^* *and only one, which is the limit of the sequence generated by* (2), *the convergence order being given by*

$$\|x^* - x_n\| \leqslant \widetilde{R}(\widetilde{h}_{-2}\widetilde{E}_{-2})^{2\sum_{i=2}^{n-1} t_i} \tag{4}$$

where $t_{-1} = -1$, $t_0 = 1$, $t_i = t_{i-3} + t_{i-2} + t_{i-1}$ $(i = 2, 3, \ldots)$.

Proof. We will show that in the hypothesis of theorem 1, the hypotesis of the theorem A are satisfied for equation

$$\widetilde{P}(x) \equiv \Lambda_0 P(x) = \widetilde{\theta} \tag{1'}$$

where $\widetilde{\theta}$ is a null element of the space X, equation equivalent with (1).

To approximate the solution of the equation (1′) we use the algorithm

$$\widetilde{x}_{n+1} = \widetilde{x}_n - \widetilde{\Lambda}_n\widetilde{P}(\widetilde{x}_n) - \widetilde{\Lambda}_n[\widetilde{x}_n, \widetilde{x}_{n-1}, \widetilde{x}_{n-2}; \widetilde{P}]\widetilde{\Lambda}_{n-1}\widetilde{P}(\widetilde{x}_{n-1})\widetilde{\Lambda}_n\widetilde{P}(\widetilde{x}_n). \tag{2'}$$

For $\widetilde{x}_{-2} = x_{-2}$, $\widetilde{x}_{-1} = x_{-1}$, $\widetilde{x}_0 = x_0$ the sequence $(\widetilde{x}_n)$ generated by (2′) is identicaly with the sequence generated by (2), which is easily proved, using the induction.

We show that the hypothesis of the theorem A are satisfied.

1) $\Lambda_0 = [x_0, x_{-1}; \widetilde{P}]^{-1} = \Lambda_0[x_0, x_{-1}; P]^{-1} = I$.

hence $\widetilde{\Lambda}_0$ exists and $)|\Lambda_0|(= 1 = B_0$

2) $)|\widetilde{P}(x_i)|(=)|\Lambda_0 P(x_i)|(\leqslant \eta_{-i}, i = -2, -1, 0$ and $\eta_0 \leqslant \eta_{-1} \leqslant \eta_{-2}$

3) $)|[x', x'', x'''; \widetilde{P}]|(=)|\Lambda_0[x', x'', x'''; P]|(\leqslant \widetilde{M}$

$)|[x', x'', x''', x^{IV}; \widetilde{P}]|(=)|\Lambda_0[x', x'', x''', x^{IV}; P]|(\leqslant \widetilde{N}$

$\forall x', x'', x''', x^{IV} \in S$

4) $\widetilde{h}_{-2}\widetilde{E}_{-2} < 1$, where $\widetilde{h}_{-2} = \widetilde{B}^2\widetilde{M}\eta_{-2} = \widetilde{M}\eta_{-2}$

$$\widetilde{E}_{-2} = \frac{1}{\widetilde{H}_{-2}}\left\{3 + \left(\frac{\widetilde{N}}{\widetilde{M}^2} + 1\right)(1 - \widetilde{h}_{-2})^3\right\} \text{ and } \widetilde{H}_{-2} = 1 - \widetilde{h}_{-2}(1 + \widetilde{h}_{-2}) > 0$$

in $S(x_0, \widetilde{R})$ with $\widetilde{R} = \dfrac{2\eta_{-2}}{1 - (\widetilde{h}_{-2}\widetilde{E}_{-2})^2}$.

It results that hypothesis of theorem A are satisfied by $\widetilde{P}$, hence equation (1′) has a solution $x^* \in S$ and only one, which is the limit of sequence generated with algorithm (2) or (2′), rapidity of convergence being given by (4).

Remark. The theorem above improve the condition 1° and 4° from theorem A. So, the bounding of inverse operator Λ is not suppose and, also, inequality $\widetilde{h}_{-2} \leqslant h_{-2}$ takes place. The last relation results from the following property of divided differences:

$$[x', x'', x'''; \Lambda_0 P] = \Lambda_0[x', x'', x'''; P]$$

which causes $\widetilde{M} \leqslant B_0 M$; $\widetilde{N} \leqslant B_0 N$.

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ON THE SOLVING OF OPERATORIAL EQUATIONS DEFINED IN FRÉCHET SPACES BY A MODIFIED CHORD METHOD

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Dedicated to Professor D. D. Stancu on his 60th anniversary

REZUMAT. — Aszpra rezolvării ecuaţiilor operaţionale definite în spaţii Fréchet printr-o metodă modificată a coardei. Folosind metoda coardei în rezolvarea ecuaţiilor operatoriale definite în spaţii Fréchet, lucrarea îşi propune slăbirea condiţiilor de existenţă ale soluţiilor ei date în lucrările [1] şi [2] prin renunţarea la existenţa diferenţelor divizate de ordinul doi ale operatorului $P : X \to Y$, X şi Y fiind spaţii Fréchet şi la mărginire lui $\Lambda_0 = [x_0, x_{-1}; P]^{-1}$

1. The chord method for approximatively solving of an operatorial equation, given by algorithm

$$x_{n+1} = x_n - [x_n, x_{n-1} ; P]^{-1} P(x_n)$$

has many disadvantages in practical applications, because, in real calculus of approximates of solution of given equation, in every step of iteration operator $[x', x'' ; P]^{-1}$ must be calculated. To eliminate this desadvantage, to generate the sequence of approximations of solution, the modified chord method can be used, based on the algorithm:

$$x_{n+1} = x_n - [x_0, x_{-1}; P]^{-1} P(x_n), \tag{1}$$

where $[x_0, x_{-1} ; P]^{-1}$ operator is calculated only once, for initial approximations x_0, x_{-1}.

In paper [1], enough conditions concerning convergence of modified chord method to a solution of a given equation are given.

This paper proposes to study the convergence of this method, establishing conditions given in [1].

2. Let be the operatorial equation

$$P(x) = \theta, \tag{2}$$

where $P: X \to Y$ is a continous nonlinear operator, X and Y are Fréchet spaces and θ, the null element of Y.

Concerning the existence of a solution of (2), calculated by iterative method (1), we prove:

THEOREM 1. *If for the initial approximations $x_0, x_{-1} \in S \subset X$ the following conditions are satisfied:*

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1. *There exist* $\Lambda_0 = [x_0, x_{-1}; P]^{-1}$ *and* $)|\Lambda_0|(\leqslant B_0$.

2. *There exist* η_0 *and* η_{-1} *which for*

$$)|\Lambda_0 P(x_0)|(\leqslant \eta_0 \quad and \quad)|x_0 - x_{-1}|(\leqslant \eta_{-1}; \quad \eta_0 \leqslant \eta_{-1}$$

3. $)|[x' x''; P] - [x'', x'''; P]|(\leqslant K)|x' - x''|($ *for every* x', x'', x''' *from* $S(x_0, 2\eta_0)$

4. $h_0 = B_0 K(\eta_0 + \eta_{-1}) < \frac{1}{4}$

then:

i) *sequence* (x_n) *generated by* (1) *is convergente;*

ii) $\lim_{n\to\infty} x_n = x^* \in S(x_0, 2\eta_0)$ *is a solution of equation* (2);

iii) *rapidity of convergence is given by*

$$)|x^* - x_n|(\leqslant K^{n-1})|x_1 - x^*|(, \text{ where } K = 3h_0 < 1.$$

Proof: Conditions 1—4 of theorem are the same with those of theorem presented in [2]; it results that the sequence $(\tilde{x}_n)$ of approximations, generated by initial chord method is convergent to a solution of equation (2), rapidity of convergence being given by

$$)|x^* - \tilde{x}_n|(\leqslant 2^{1-s_n} q^{s_{n-1}}(4h_0)^{s_n}\eta_0$$

where $q = \frac{8}{9}$ and $s_n = \sum_{i=1}^{n} u_i = u_{n+2} - 1$, u_i being the terms of a Fibonacci sequence.

We show that sequence (x_n) generated by method (1) has the same limit x^*, hence

$$)|x_n - x^*|(\to 0 \text{ for } n \to \infty.$$

Notice that $x_0 = \tilde{x}_0$ and $x_{-1} = \tilde{x}_{-1}$ then $x_1 = \tilde{x}_1$.

Consider the set $\mathfrak{M} = S \cap S^*$, where $S^*(x^*,)|x_1 - x^*|($ $)$, hence $)|x - x^*|(\leqslant)|x^* - x_1|($.

Let $x \in \mathfrak{M}$ be some element and the operator $A: X \to X$, so that, for $a \in X$ we get

$$A(a) = a - \Lambda_0 P(a).$$

It has the properties:

$$A(x_n) = x_{n+1}; \; A(x^*) = x^*$$

$$[u, v; A] = I - \Lambda_0[u, v; P], \quad \forall u\, v \in X$$

$$[x_0\, x_{-1}; A] = \theta_1.$$

Let $\bar{x} = A(x)$ we prove that $\bar{x} \in \mathfrak{M}$, hence it belongs both to S and to S^* hence $\|\bar{x} - x_0\| \leqslant 2\eta_0$ and $\|\bar{x} - x_0\| \leqslant \|x^* - x_1\|$.
Indeed, we have

$$\|\bar{x} - x^*\| = \|A(x) - A(x^*)\| = \|[x, x^*; A]\| \cdot \|x - x^*\|.$$

To evaluated $\|[x, x^*; A]\|$ consider the evidently inequality

$$\|[x,x^*; A]\| - \|[x,x_0; A]\| \leqslant \|[x,x^*; A] - [x,x_0; A]\|$$

which, taking account of operator A properties and condition 3 leads to

$$\|[x, x^*; A]\| \leqslant \|[x, x^*; A] - [x, x_0; A]\| + \|[x,x_0; A]\| =$$
$$= \|[x, x^*; A] - [x, x_0; A]\| + \|[x, x_0; A] - [x_0, x_{-1}; A]\| \leqslant$$
$$\leqslant B_0K(\|x_0 - x^*\| + \|x - x_{-1}\|).$$

Because

$$\|x^* - x_0\| \leqslant 2\eta_0 < 3\eta_0$$
$$\|x - x_{-1}\| \leqslant \|x - x_0\| + \|x_0 - x_{-1}\| \leqslant 2\eta_0 + \eta_{-1} < 3\eta_{-1}$$

it results that

$$\|\bar{x} - x^*\| < 3B_0K(\eta_0 + \eta_{-1}) \|x - x^*\| = K\|x - x^*\|.$$

Because $K < 1$ and $\|x - x^*\| \leqslant \|x^* - x_1\|$ results that $\bar{x} \in S^*$. We must show only that $\bar{x} \in S$. For this, consider

$$\|\bar{x} - x_0\| \leqslant \|A(x) - x_1\| + \|x_1 - x_0\| \leqslant \|A(x) - A(x_0)\| + \|x_1 - x_0\| =$$
$$= \|[x, x_0; A](x - x_0)\| + \eta_0 = \|([x, x_0; A] - [x_0, x_{-1}; A])(x - x_0)\| + \eta_0 \leqslant$$
$$\leqslant B_0\|([x_0, x_{-1}; P] - [x, x_0; P])(x - x_0)\| + \eta_0 \leqslant$$
$$\leqslant B_0K \|x - x_{-1}\| \cdot \|x - x_0\| + \eta_0 \leqslant$$
$$\leqslant 6B_0K\eta_0\eta_{-1} + \eta_0 = 6B_0K(\eta_0 + \eta_{-1})\eta_0 = 6h_0\eta_0 < 2\eta_0.$$

From those presented above results that operator A lets invariantes the elements of set $\mathfrak{M}$.

Returning to (3), taking account of properties of operator A and taking $x = x_1$,

$$\| x_2 - x^* \| < K\| x_1 - x^* \| \text{ is found.}$$

By induction, one deduces

$$\| x_n - x^* \| \leqslant K^{n-1}\| x_1 - x^* \|$$

and because $K < 1$ we have for $n \to \infty$, $\|x_n - x^*\| \to 0$, hence

$$\lim_{n\to\infty} x_n = x^*.$$

So the theorem is proved.

The convergence of modified chord method to a solution of equation (2) may be proved by weaking the conditions of previous theorem, for instance by giving up to the bounding of operator Λ_0. So we have

THEOREM 2. *If for initial approximations* $x_0, x_{-1} \in S$ *we have the conditions*

1. *There exists* $\Lambda_0 = [x_0, x_{-1}; P]^{-1}$
2. $)| \Lambda_0 P(x_0) |(\leqslant \eta_0, | x_0 - x_{-1} |(\leqslant \eta_{-1}, \ (\eta_0 \leqslant \eta_{-1})$
3. $)| ([x', x''; \Lambda_0 P] - [x'', x'''; \Lambda_0 P]) |(\leqslant \widetilde{K} \cdot (x' - x'')$ *for every* $x', x'', x''' \in S(x_0, 2\eta_0)$
4. $\widetilde{h}_0 = \widetilde{K}(\eta_0 + \eta_1) < \frac{1}{4}$

then the sequence generated by (1) *is convergente to a solution* x^* *of equation* (2) *the rapidity of convergence being given by relation*

$$)| x^* - x_n |(\leqslant \widetilde{K})| x_1 - x^* |(, \qquad \widetilde{K} = 3h_0 < 1$$

Proof. Consider the ecuation

$$\widetilde{P}(x) \equiv \Lambda_0 P(x) = \theta \tag{2'}$$

equivalente with (2). We show that from the conditions of theoremo 2, results the conditions of theorem 1 for equation (2').

To solve approximatively the equation (2'), we consider the algorithm

$$\widetilde{x}_{n+1} = \widetilde{x}_n - \widetilde{\Lambda}_0 \widetilde{P}(\widetilde{x}_n) \tag{1'}$$

where $\widetilde{\Lambda}_0 = [\widetilde{x}_0, \widetilde{x}_{-11}; \widetilde{P}]^{-1}$

If we suppose that $\widetilde{x}_0 = x_0$ and $\widetilde{x}_{-1} = x_-$, based on (1'), we can deduce that

$$\widetilde{x_1} = x_0 - [x_0, x_{-1}; \widetilde{P}]^{-1}\widetilde{P}(x_0) = x_0 - (\Lambda_0[x_0, x_{-1}; P])^{-1}\Lambda_0 P(x_0) =$$
$$= x_0 - \Lambda_0 P(x_0) = x_1.$$

Repeating the argument, we deduce that sequence $(\widetilde{x}_n)$ generated by (1') is the same with sequence (x_n) generated by (1).

We verify now the performing of conditions 1—4 of theorem 1.

1. $\widetilde{\Lambda}_0 = [x_0, x_{-1}; \widetilde{P}]^{-1} = (\Lambda_0[x_0, x_{-1}; P])^{-1} = I^{-1} = I$
   and so there exists $\widetilde{\Lambda}_0$, and $)| \Lambda_0 |(= \widetilde{B}_0 = 1$
2. $)| \widetilde{\Lambda}_0 \widetilde{P}(x_0) |(\leqslant)| \widetilde{\Lambda}_0 |(\cdot)| \widetilde{P}(x_0) |(=)| \Lambda_0 P(x_0) |(\leqslant \eta_0$
   and evidently $)| x_0 - x_{-1} |(\leqslant \eta_{-1}$
3. $)| [x', x''; P] - [x'', x'''; P] |(=)| \Lambda_0([x', x''; P] - [x'', x'''; P]) |(\leqslant$
   $\leqslant \widetilde{K}(x' - x''')$
4. $\widetilde{h}_0 = \widetilde{K}(\eta_0 + \eta_{-1}) < \frac{1}{4}$

From those above, result that equation (2′) has a solution $x^* \in S(x_0 ; 2\eta_0)$ which is the limit of sequence generated by (1) or (1′), rapidity of convergence being characterized by

$$\|x^* - x_n\| \leqslant K^{n-1}\|x_1 - x^*\|, \quad K = 3h_0 < 1$$

Because (2′) is equivalent with (2), the statement results.

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ON THE CONSTRUCTION OF A ROTOTRANSLATION—PERFORMING PROFILE IN A FLUID BY AN INVERSE BOUNDARY PROBLEM

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Dedicated to Professor D. D. Stancu on his 60th anniversary

REZUMAT. — Asupra construcţiei unui profil executînd o rototranslaţie într-un fluid printr-o problemă la limită inversă. În prezenta lucrare se caută să se rezolve problema construirii „a posteriori" a unui profil de aripă, care efectuează o mişcare de rototranslaţie într-un fluid ideal, cunoscînd distribuţia „a priori" a mărimii vitezei fluidului în toate punctele conturului necunoscut al profilului. Presupunînd că fluidul este animat la mari distanţe de o viteză constantă dată, iar mişcarea fluidului păstrează caracterul incompresibil potenţial şi plan, profilul avînd şi un bord de fugă unghiular, problema va fi rezolvată în cadrul unei probleme la limită inverse ataşate operatorului lui Laplace.

In what follows we shall try to solve the problem of an „a posteriori" construction of a wing profile (c) performing a rototranslation in an inviscid fluid by „a priori" knowing the distribution of the velocity magnitude

$$v = f(s, t), \qquad 0 \leqslant s \leqslant L, \qquad 0 < t_0 \leqslant t,$$

in all the points of the contour c_t of (c). (We denoted here by s, the natural parameter on the contour c_t, by t and t_0 the current and the initial moment respectively and by L the contour perimeter supposed given with the problem). One supposes that the fluid flow having a constant velocity $\vec{V}_\infty$ at great distances, is incompresible potential and plane. Denoting then by $l(t)$, $m(t)$, $\omega(t)$ the parameters of the rototranslation of the profile (c), the complex potential $f(z, t)$ of the fluid flow in the physical plane (z) — resulting by superposition on the uniform stream with the velocity $\vec{V}_\infty$ of the fluid flow induced by displacement of (c) and estimated in a mobile system of coordinates Oxy originated in the center of the profile have the form:

$$f(z, t) = V_\infty e^{i\theta} z + (l - V_\infty \cos\theta) g^{(1)}(z) + (m + V_\infty \sin\theta) g^{(2)}(z) + \omega g^{(3)}(z),$$

where $g^{(1)}$, $g^{(2)}$, $g^{(3)}$ are holomorphic functions in the whole domain of the flow, $\theta = \theta_0 + \int_{t_0}^{t} \omega(t) dt$, θ_0 being the angle between the fixed axis Ox_1 (oriented as $\vec{V}_\infty$) and the mobile axis Ox, at the initial moment t_0.

If the profile has an angular point at the trailing adge $z_F(F)$, as we suppose, then admitting that the semi-tangents angle in this point is $\pi - \mu\pi$ $(-1 \leqslant$

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$\leqslant \mu < 0$), the fluid flow has to be considered with a circulation Γ chosen so that one has the boundress of the velocity in this point. Precisely [1] we must take

$$\Gamma = 4\pi a(m + V_\infty \sin \theta + \omega\Omega).$$

where a is the radius of an origin-centrating disk (C) of the plane (Z) on whose exterior one maps conformally the exterior of the profile (c) by a canonical conformal mapping $z = H(Z)$; here Ω is the real quantity [1]

$$\Omega = \frac{1}{4\pi i}\int_C \frac{dr^2}{\xi - a} = \frac{1}{8\pi a}\int_C \operatorname{ctg}\frac{\nu}{2}\, dr^2,$$

where

$$r^2 = z_1 \bar{z}_1|_{c_1} = H(\zeta)\,\overline{H(\zeta)},$$

$$\zeta = a e^{i\nu}, \quad (0 \leqslant \nu \leqslant 2\pi),$$

$Z = a$ being the image of the angular point $z_F \in c_1$. In that case to the complex potential $f(z, t)$ one must add a term of the type

$$\frac{\Gamma}{2\pi i} g^{(4)}(z),$$

where $g^{(4)}$ is an holomorphic function in all the points at a finite distance of the domain flow, having a logarithmic singularity at infinity and whose imaginary part ψ_4 vanishes on c_1.

If the velocity magnitude of the fluid is known on the contour c_1, and at the same time the circulation Γ is given, then the equality

$$\Gamma = \int_{c_1} V(r)\, dr = -\int_0^{s_A} V(s)ds + \int_{s_A}^{L} V(s)ds,$$

where s_A is the natural coordinate of the attack edge, establishes a relation between a, m, ω, V_∞ and $H(\zeta)$ on Ω. In the sequel we admit that the mobile system of axis Oxy, with the origin in the center of the profile, has the real axis passing through the trailing edge $z_F(F)$ whose distance to the center of the profile is known, together with the date of the problem, as we shall see at once. At the same time the natural coordinate of the angular point (trailing edge) will always be constant in zero ($s_F = 0$), while the same coordinate for the point A, variable in time, will have the form $s_A = s_A(t)$, function which is „a priori" unknown. Concerning the function $V = f(s, t)$, the magnitude of the absolute fluid velocity in the points of the unknown contour c_1, this is a positive, uniform and continuous function, for $(s, t \in [0, L) \times [t_0, \infty)$ satisfying with regard to $s \in [0, L)$ a Hölder condition. From the obvious conditions connected with the vanishing of the relative velocity in the attack and trailing edges, respectively, we also get

$$f^2(s_A, t) = l^2 + m^2 - 2l\omega y_A + 2m\omega x_A + \omega^2 r_A^2\,; \qquad /r_A^2 = x_A^2 + y_A^2/$$

and

$$f^2(0, t) \equiv f^2(L, t) = l^2 + m^2 + 2m\,\omega\, x_F + \omega^2 x_F^2$$

the last relation precising, as we have already anticipated, the fixed position of the trailing edge, F, with respect to that of the center of the profile.

Concerning the function $\varphi(s, t)$ the velocities potential of the absolute fluid flow, it will be obviously, defined by

$$U(s, t) = \begin{cases} \displaystyle\int_0^s -f(s, t)\, ds & \text{for } 0 < s \leqslant s_A \text{ (the lower surface)} \\ \displaystyle\varphi_A + \varphi_H + \int_{s_A}^{s} f(s, t) ds, & \text{for } s_A \leqslant s < L \text{ (the upper surface)} \\ 0 & \text{for } s = 0 \\ \varphi_A + \varphi_H \equiv \Gamma & \text{for } s = L; \end{cases}$$

where we denoted by

$$\varphi_A = \int_0^{s_A} -f(s, t)\, ds$$

and by

$$\varphi_H = \int_{s_A}^{L} f(s, t)\, ds$$

We remark that the function $\varphi(s, t)$ is a continuous and with respect to s a multiforme function of period Γ, derivable on $(0, L) \times [t_0, \infty)$ where $\dot\varphi(s, t) = -f(s, t)$ for $0 < s \leqslant s_A$, $t_0 \leqslant t < \infty$ and $\dot\varphi(s, t) = f(s, t)$ for $s_A \leqslant s < L$, $t_0 \leqslant t < \infty$, while it does not admit derivates in 0 and L; precisely,

$$\dot\varphi(0+, t) = -f(0, t) \text{ and } \dot\varphi(0-, t) = f(0, t)$$

/analogously for $s = L$/.

Concerning the stream function $\psi(s, t)$ the known slippery conditions, lead to

$$\psi(s, t) = l\,y(s) - m\,x(s) - \frac{\omega}{2}\left(x^2(s) + y^2(s)\right) + \mathfrak{K}(t)\,;$$
$$(s, t) \in [0, L] \times [t_0, \infty)$$

where $x = x(s)$ and $y = y(s)$ are the equations of the unknown curve c_i and $\mathcal{K}(t)$ an arbitrary frunction of time. This last function could be precised by the condition that

$$\psi(0,\ t) = -\ m\ \ x_F - \frac{\omega}{2}\, x_F^2 + \mathcal{K}(t) = 0.$$

As the contour c_i is necessarily closed i.e. $x(0) = x(L)$ and $y(0) = y(L)$ we also have $\psi(0, t) = \psi(L, t)$, while the nonderivability of the functions $x(s)$ and $y(s)$ in $s = 0$ or $s = L$ implies the nonderivability of $\psi(s, t)$ in the same points.

We also mention that the existence of a sharp trailing edge in $s = 0$ leads to a behaviour of the complex potential $w(z, t)$ of the type $0\left|(z - z_F)^{\frac{1}{1-\mu}}\right|$ and, correspondingly, of one of the type $0\left(s^{\frac{1}{1-\mu}}\right)$ for the functions $\varphi(s, t)$ and $\psi(s, t)$. Immediately $\dot{\varphi}(s, t)$ and $\dot{\psi}(s, t)$ behave in the $V(0)$ like $s^{\frac{\mu}{1-\mu}}$, result which will be used later on.

Once precised the date of the problem let us formulate it exactly. Mathematically speaking, the boundary inverse problem involved here is an external one, of the Dirichlet type for the complex potential $w(z, t) = \varphi(x, y, t) + i\psi(x, y, t)$ which admits at infinity a simple pole and a logarithmic singularity and satisfies on the unknown contour c_i the condition $w|_{c_i} = \varphi(s, t) + i\psi(s, t)$.

To solve this problem we are considering, for the beginning, the canonical conformal mapping $z = H(Z)$ which conformally represents the exterior of the airfoil (c) of the physical plane (z) on the exterior of the disk $(C)/|Z| \leqslant a/$ of the plane (Z) correspondence of the infinity points of these two planes being assured. But we know that the complex velocity of the fluid flow past a circular obstacle which is performing the same rototranslation, with the same circulation Γ, is of the type

$$W(Z,\ t) = V_\infty\, e^{i\theta}\, Z - [1 - V_\infty \cos\theta + i(m - V_\infty \sin\theta)]\, \frac{a^2}{Z} - \\ - \frac{\omega i}{2}\, a^2 + \frac{\Gamma}{2\pi i}\, \mathrm{Log}\, \frac{Z}{a} + k\,;$$

Here the real constant a and the function of time k are fixed by the condition of vanishing of the relative fluid velocity in the images of the critical points A and F, i.e. in $Z_A = ae^{i\gamma_1}$ and $Z_F = ae^{i\gamma_2} \equiv a$ $(\gamma_2 = 0)$ where the values of the potential velocity Φ are also known. More exactly on the circumference $|Z| = a$ we have

$$\Phi = aV_\infty[\cos(\theta + \gamma) + \cos(\theta - \gamma)] - a(l\cos\gamma + m\sin\gamma) + \\ + \frac{\Gamma}{2\pi}\gamma + k \equiv 2aV_\infty \cos\theta\, \cos\gamma - a\,(l\, \cos\, \gamma + m\, \sin\, \gamma) + \frac{\Gamma}{2\pi}\gamma + k\,;$$

while the magnitude of absolute fluid velocity in the points of the same circumference C will be given by

$$V_C = \frac{1}{a}\frac{\partial\Phi}{\partial\gamma} = -2aV_\infty \cos\theta \sin\gamma + l\sin\gamma - m\cos\gamma + \frac{\Gamma}{2\pi a}.$$

By imposing now the vanishing conditions of the relative velocity in the images of the points A and E, the values of the velocity potential being preserved in the homologous points, the real constants or functions of time γ_1 (or s_A)[1], a and k satisfy the system:

$$2aV_\infty \cos\theta - al + k = 0; \quad (1)$$

$$2aV_\infty \cos\theta \cos\gamma_1 - a(l\cos\gamma_1 + m\sin\gamma_1) + \frac{\Gamma}{2\pi}\gamma_1 + k = \varphi_A; \quad (2)$$

$$\left(-m + \frac{\Gamma}{2\pi a}\right)^2 = 2m\omega a + \omega^2 a^2;\ ^{2)} \quad (3)$$

$$\left(-2V_\infty \cos\theta \sin\gamma_1 + l\sin\gamma_1 - m\cos\gamma_1 + \frac{\Gamma}{2\pi a}\right)^2 =$$
$$= l^2 + m^2 - 2l\omega a\sin\gamma_1 + 2m\omega a\cos\gamma_1 + \omega^2 a^2 \quad (4)$$

We now observe that the equation (3) allows to determine a what leads through the definition of Γ to Ω. The equation (1) fixes then k, while (4) precises γ_1; the relation between s_A and γ_1 is obtained by the equation (2).

Once precised the quantities a, k and γ_1 we could furnish a complete solution of the problem. More exactly, from the equality $\varphi(s) \equiv \Phi(\gamma)$, we determine the dependence $s = s(\gamma)$ through which it immediately obtains [3]

$$r = H(Z) = \int_a \left(\exp - \frac{1}{2\pi}\int_0^{2\pi} \ln\frac{ds}{d\gamma}\frac{e^{i\gamma}+Z}{e^{i\gamma}-Z}d\gamma\right)dZ$$

function by which we determine the image of the circumference $|Z| = a$, i.e. the „apriori" unknown contour c_t, and also the loocked for complex potential $w(z, t) = W(H^{-1}(z, t))$.

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1) From $\varphi_A = -\int_0^{s_A} V(s)ds = \int_0^{\gamma_1} V_C(\gamma)a\,d\gamma$, for example, we could directly obtain the functional dependence $\gamma_1 = \gamma_1(s_A)$.

2) From the obvious expression of the components of the relative velocity (U_r, V_r) i.e.

$$U_r = U - l + \omega Y,$$
$$V_r = V - m - \omega X,$$

we get in the points of relative stagnation that

$$U^2 + V^2 = l^2 + m^2 - 2l\omega Y + 2m\omega X + \omega^2 a^2.$$

We remark that the determined function $z = H(Z)$ fullfils the demand $\lim_{|z|\to\infty} \frac{dz}{dZ} = 1$, accordingly to the canonical feature of the conformal mapping. At the same time the multiformity of the restriction of $w(z, t)$ on c_t /due to the period Γ of the function $\varphi(s, t)$/ is satisfied, correspondingly, by the existence of the same period of multformity Γ for the function $W(Z, t)$. It is also well known that in order to have a profile with a sharp trailing edge in z_F, as image of the disk $|Z| = a$, one must admit for the function $z = H(Z)$, in the neighbourhood of $Z = a$, a development of the type

$$z - z_F = A(Z - a)^{1-\mu} + \dots$$

But from the behaviour of $\dot{\varphi}(s, t)$ in $\overline{V}(0)$ one has that in the same neighbourhood $\frac{d\gamma}{ds} = 0\left(s^{\frac{\mu}{1-\mu}}\right)$. i.e. $\text{Re}\left\{\text{Log}\, \frac{dz}{dZ}\right\} = \ln \frac{ds}{d\gamma}$ behaves like $O(|Z - a|^{-\mu}) \equiv 0 \left(|z - z_F|^{-\frac{\mu}{1-\mu}}\right)$ which finally leads for the solution $z = H(Z)$ to a behaviour of the type

$$z = H(Z) = A(Z - a)^{1-\mu} + \dots \qquad \text{q.e.d.}$$

Finally to avoid the multiple points of the contour c_t it is sufficient that the conformal mapping $z = H(Z)$ would be univalente. Univalence criteria for such functions defined outside a disk could be found, for example, in [4].

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ON SOME SPLINE-TYPE OPERATORS OF APPROXIMATION

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REZUMAT. — Asupra unor operatori de aproximare de tip spline. În această lucrare autorul ia în discuţie noi cazuri importante ale unui operator liniar pozitiv, de tip spline, care depinde de doi parametri ne-negativi, introdus în lucrarea [14], reprezentînd o generalizare a operatorului lui I. J. Schoenberg [10]. De asemenea se consideră un operator de tip spline introdus anterior de T. Popoviciu [8] şi studiat în continuare în lucrarea [9].

1. In an earlier paper [14] we have introduced and studied a spline-type linear positive operator, depending on two non-negative parameters, generalizing the operator discovered in 1965 by I. J. Schoenberg [10], investigated in detail in 1966 in a joint paper by M. J. Marsden and I. J. Schoenberg [5] and later in two papers of M. J. Marsden [6], [7].

In the present paper we discuss new important cases of this operator and also a polynomial spline-type approximating operator introduced in 1942 by T. Popoviciu [8] in the case of equally spaced knots.

Let m $(m \geqslant 1)$ and n $(n \geqslant 0)$ be two integers and denote by α and β two real parameters satisfying the condition $0 \leqslant \alpha \leqslant \beta$.

The Schoenberg-type linear positive spline operator which we have introduced in [14] is defined, for any function $f: [0,1] \to \mathbf{R}$, by the following formula

$$(S_{m,n}^{\alpha,\beta} f)(x) := \sum_{j=0}^{m+n} N_{m,n,j}(x) f(\xi_{m,j}^{\alpha,\beta}), \tag{1}$$

where

$$0 = \underbrace{x_{-m} = \ldots = x_{-1} = x_0}_{m+1} < x_1 < \ldots < x_n < \underbrace{x_{n+1} = \ldots = x_{n+m+1}}_{m+1} = 1,$$

$$\xi_{m,n}^{\alpha,\beta} := \frac{1}{m+\beta}(x_{-m+1+j} + \ldots + x_{-1} + x_0 + x_1 + \ldots + x_j + \alpha), \tag{2}$$

$$N_{m,n,j}(x) := (x_{j+1} - x_{-m+j})[x_{-m+j}, \ldots, x_{-1}, x_0, x_1, \ldots, x_{j+1}; (t-x)_+^m], \tag{3}$$

the brackets representing the symbol for divided differences.

The abscissas $x_1, x_2, \ldots, x_n$ are the *knots*, $\xi_{m,n}^{\alpha,\beta}$ are the *nodes* and $N_{m,n,j}$ are the *fundamental spline functions* representing *normalized B-splines*.

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It is seen that

$$\xi^{\alpha,\beta}_{m,0} = \frac{\alpha}{m+\beta}, \quad \xi^{\alpha,\beta}_{m,1} = \frac{x_1+\alpha}{m+\beta}, \quad \xi^{\alpha,\beta}_{m,2} = \frac{x_1+x_2+\alpha}{m+\beta},$$

$$\ldots, \quad \xi^{\alpha,\beta}_{m,m+n-1} = \frac{x_n+m-1+\alpha}{m+\beta}, \quad \xi^{\alpha,\beta}_{m,m+n} = \frac{m+\alpha}{m+\beta}$$

and we have

$$0 \leqslant \xi^{\alpha,\beta}_{m,0} < \xi^{\alpha,\beta}_{m,1} < \ldots < \xi^{\alpha,\beta}_{m,m+n-1} < \xi^{\alpha,\beta}_{m,m+n} \leqslant 1.$$

If $\alpha = \beta = 0$ then we obtain the Schoenberg original operator: $S_{m,n} = S^{0,0}_{m,n}$, which is interpolatory at both sides of the interval $[0,1]$.

One can distinguish two cases:

$$\text{(i)} \quad m \geqslant n \geqslant 0 \quad \text{and} \quad \text{(ii)} \quad n > m > 0.$$

In our paper [14] we have investigated the first case (i) and we observed that in the special case $n = 0$ we have no knots and we obtain

$$(S^{\alpha,\beta}_{m,n}f)(x) = \sum_{j=0}^{m} [\underbrace{0, \ldots, 0}_{m+1-j}, \underbrace{1, \ldots, 1}_{j+1}; (t-x)^m_+]\, f\left(\frac{j+\alpha}{m+\beta}\right) =$$

$$= \sum_{j=0}^{m} \binom{m}{j} x^j (1-x)^{m-j} f\left(\frac{j+\alpha}{m+\beta}\right),$$

which is the Bernstein-type polynomial $B^{\alpha,\beta}_m f$ on x, depending on the parameters α and β, which has been introduced and investigated in the papers [12], [13] and [3].

2. In the second case (ii) we can write

$$(S^{\alpha,\beta}_{m,n}f)(x) = \sum_{j=0}^{m} x_{j+1} [\underbrace{0, \ldots, 0}_{m+1-j}, x_1, \ldots, x_{j+1}; (t-x)^m_+]\, f\left(\frac{x_1+\ldots+x_j+\alpha}{m+\beta}\right) +$$

$$+ \sum_{j=m+1}^{n} (x_{j+1} - x_{-m+j}) [x_{-m+j}, \ldots, x_{j+1}; (t-x)^m_+]\, f\left(\frac{x_{-m+j+1}+\ldots+x_j+\alpha}{m+\beta}\right) +$$

$$+ \sum_{j=n+1}^{n+m} (1 - x_{-m+j}) [x_{-m+j}, \ldots, x_n, \underbrace{1, \ldots, 1}_{j+1-n}; (t-x)^m_+] \cdot$$

$$\cdot f\left(\frac{x_{-m+j+1}+\ldots+x_n+j-n+\alpha}{m+\beta}\right).$$

By making use of the following decomposition formula for divided differences

$$[a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_n; f(t)] = \left[a_0, a_1, \ldots, a_m; \frac{f(t)}{(t-b_0)\ldots(t-b_n)}\right] +$$

$$+ \left[b_0, b_1, \ldots, b_n; \frac{f(t)}{(t-a_0)\ldots(t-a_m)}\right]$$

and the relation

$$(t - x)_+^m = (t - x)^m + (-1)^{m+1}(x - t)_+^m,$$

we can write the preceding formula in the following more compact form:

$$(S_{m,n}^{\alpha,\beta} f)(x) = \sum_{j=0}^{m} x_{j+1}\left[x_1, \ldots, x_{j+1}; \frac{(t-x)_+^m}{t^{m+1-j}}\right] f\left(\frac{x_1 + \cdots + x_j + \alpha}{m+\beta}\right) +$$

$$+ \sum_{j=m+1}^{n} (x_{j+1} - x_{-m+j})[x_{-m+j}, \ldots, x_{j+1}; (t-x)_+^m] f\left(\frac{x_{-m+j+1} + \cdots + x_j + \alpha}{m+\beta}\right) +$$

$$+ \sum_{j=n+1}^{n+m} (-1)^{m+n-j}(1 - x_{-m+j})\left[x_{-m+j}, \ldots, x_n; \frac{(t-x)_+^m}{(1-t)^{j+1-n}}\right] \cdot$$

$$\cdot f\left(\frac{x_{-m+j+1} + \cdots + x_j + j - n + \alpha}{m+\beta}\right). \tag{4}$$

Remarks. 1°. The Marsden [6] condition for uniform convergence: $\frac{1}{m} \|\Delta\| \to 0$, where $\|\Delta\|$ is the norm of the partition of the interval [0,1] by the knots x_j, when m is bounded, is assured by $\|\Delta\| \to 0$, which implies $n \to \infty$.

2°. It should be observed that in the case (ii) we cannot obtain the Bernstein polynomial as a special case of $S_{m,n}^{\alpha,\beta} f$, since always we have knots $x_j (j \geqslant 1)$.

As Marsden [6] has pointed out, in practice it is preferable to choose this case, when n is large and m small.

3°. Since $n > m$, there follows that there are no divided differences containing all three types of points: 0, 0, ...; $x_1, x_2, \ldots$; 1, 1, ...

Now let us consider the case $m = 1$ of the formula (4):

$$(S_{1,n}^{\alpha,\beta} f)(x) = \frac{(x_1 - x)_+}{x_1} f\left(\frac{\alpha}{\beta+1}\right) +$$

$$+ \sum_{j=1}^{n} (x_{j+1} - x_{j-1})[x_{j-1}, x_j, x_{j+1}; (t-x)_+] f\left(\frac{x_j + \alpha}{\beta+1}\right) + \frac{(x - x_n)_+}{1 - x_m} f\left(\frac{\alpha+1}{\beta+1}\right).$$

If we set $\alpha = \beta = 0$ then it becomes

$$(S_{1,n} f)(x) = \frac{(x_1 - x)_+}{x_1} f(0) +$$

$$+ \sum_{j=1}^{n} (x_{j+1} - x_{j-1})[x_{j-1}, x_j, x_{j+1}; (t-x)_+] f(x_j) + \frac{(x - x_n)_+}{1 - x_n} f(1).$$

3. We further note that if we seek to find the equation of the interpolatory polygonal line for the points: $0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = 1$, under the form

$$P_n^f(x) = A + \sum_{j=0}^{n} A_j (x - x_j)_+, \tag{5}$$

then we can obtain directly that

$$P_n^f(x) = f(0) + [0, x_1; f]x + \sum_{j=1} (x_{j+1} - x_{j-1}) [x_{j-1}, x_j, x_{j+1}; f](x - x_j)_+ . \quad (6)$$

So we have two distinct representations of the same function, i.e. $S_{1,n} f \equiv \equiv P_n^f$.

4. Let us further derive the formula for the polygonal line P_n^f, interpolating the function f on the distinct points $x_0, x_1, \ldots, x_{n+1}$, where $x_0 = a$, $x_{n+1} = = b$.

If we write that $P_n^f(x_k) = f(x_k)$ $(k = \overline{0, n+1})$, then we obtain the linear lower triangular system

$$A + \sum_{j=0}^{k} A_j (x_k - x_j) = f(x_k) \quad (k = \overline{0, n+1}),$$

that is

$$\begin{aligned} A &= f(x_0) \\ A + A_0(x_1 - x_0) &= f(x_1) \\ A + A_0(x_2 - x_0) + A_1(x_2 - x_1) &= f(x_2) \\ &\ldots \\ A + A_0(x_{n+1} - x_0) + A_1(x_{n+1} - x_1) + \ldots + A_n(x_{n+1} - x_n) &= f(x_{n+1}). \end{aligned}$$

From the first two equations find

$$A = f(x_0) = [x_0; f], \quad A_0 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = [x_0, x_1; f].$$

Let $j \geq 1$. If from the equation

$$A + \sum_{k=0}^{j+1} A_k (x_{j+1} - x_k) = f(x_{j+1})$$

we substract

$$A + \sum_{k=0}^{j} A_k (x_j - x_k) = f(x_j),$$

then we get

$$\sum_{k=0}^{j} A_k = [x_j, x_{j+1}; f].$$

If we next substract from this equation

$$\sum_{k=0}^{j-1} A_k = [x_{j-1}, x_j; f],$$

we finally obtain:

$$A_j = (x_{j+1} - x_{j-1})\ [x_{j-1},\ x_j,\ x_{j+1};\ f].$$

Thus, in the general case of broken lines interpolation we have

$$P_n^f(x) = f(x_0) + [x_0, x_1; f](x - x_0)_+ + \sum_{j=1}^{n} (x_{j+1} - x_{j-1})[x_{j-1},\ x_j,\ x_{j+1};\ f](x - x_j)_+, \tag{7}$$

where : $a = x_0 < x_1 < \ldots < x_n < x_{n+1} = b$ and $f\colon [a,\ b] \to \mathbf{R}$.

Remark. It should be noticed that in an earlier paper [11] H. Schwerdtfeger has also derived an expression using divided differences for P_n^f, but it is incorrect; in place of second-order divided differences he found (wrongly!) divided differences of higher orders: 0, 1, 2, ..., n.

At (7) we have a "Newtonian form" of the broken line interpolant.

The "Lagrangian form" of such interpolant is

$$P_n^f(x) = \sum_{j=0}^{n+1} \lambda_j(x) f(x_j),$$

where

$$\lambda_j(x) = (x_{j+1} - x_{j-1})[x_{j-1},\ x_j,\ x_{j+1};\ (t - x)_+] = \begin{cases} \dfrac{x - x_{j-1}}{x_j - x_{j-1}}, & x \in [x_{j-1},\ x_j] \\[2mm] \dfrac{x_{j+1} - x}{x_{j+1} - x_j}, & x \in [x_j,\ x_{j+1}]. \end{cases}$$

The functions λ_j represent the so called "hat functions" and they have the following properties

$$\lambda_j(x_k) = \delta_{jk}, \quad \sum_{j=0}^{n+1} \lambda_j(x) = 1, \qquad x \in [a,\ b].$$

These functions form a basis for the linear space of all continuous broken lines on $[a,\ b]$ with breaks at $x_1,\ x_2,\ \ldots,\ x_n$ (see, e.g. [1]).

It is known that if we take $x_j = a + jh$ $(j = \overline{0, n+1})$, $h = (b - a)/(n + 1)$, then if $f \in C[a,\ b]$, we have

$$\|f - P_n^f\|_C \leqslant 2\omega\left(\frac{b - a}{n + 1}\right)$$

where ω is the modulus of continuity of the function f.

5. Since cubic spline functions appear as the logical next step after piecewise linear ("poligonal") functions ($m = 1$), let us consider in (4) $m = 3$. By taking $\alpha = \beta = 0$ we obtain

$$(S_{3,n}f)(x) = \frac{(x_1 - x)_+^3}{x_1^3} f(0) + x_2\left[x_1, x_2 ; \frac{(t-x)_+^3}{t^3}\right] f\left(\frac{x_1}{3}\right) +$$

$$+ x_3\left[x_1, x_2, x_3 ; \frac{(t-x)_+^3}{t^2}\right] f\left(\frac{x_1+x_2}{3}\right) + x_4\left[x_1, x_2, x_3, x_4 ; \frac{(t-x)_+^3}{t}\right] f\left(\frac{x_1+x_2+x_3}{3}\right) +$$

$$+ \sum_{j=4}^{n} (x_{j+1} - x_{j-3})[x_{j-3}, x_{j-2}, x_{j-1}, x_j, x_{j+1} ; (t-x)_+^3] f\left(\frac{x_{j-2}+x_{j-1}+x_j}{3}\right) +$$

$$+ (1 - x_{n-2})\left[x_{n-2}, x_{n-1}, x_n ; \frac{(x_t - t)_+^3}{(1-t)^3}\right] f\left(\frac{x_n+2}{3}\right) + \frac{(x-x_n)_+^3}{(1-x_n)^3} f(1).$$

It is known (see [10]) that $S_{3,n}f$, in the case of equally spaced knots gives an approximation of order $\omega\left(\frac{1}{n+1}\right)$.

Concerning the remainder of the approximation formula $f(x) = (S_{m,n}f)(x) + (R_{m,n}f)(x)$, which has the degree of exactness equal with one, assuming that f has a continuous second derivative, by applying a known theorem of Peano it can be expressed in an integral form (see D. Leviatan [4] and Gh. Coman [2]).

6. Let m and n be two natural numbers auch that $n > 2m - 1$ and f a function defined and continuous on the bounded and closed interval $[a, b]$ of the real-axis. Consider a partition of this interval by the points x_j ($j = \overline{0, n+1}$) : $a = x_0 < x_1 < \ldots < x_n < x_{n+1} = b$.

We define now a linear spline operator $P_{m,n} : C[a, b] \to C[a, b]$ by the formula

$$(P_{m,n}f)(x) := (Q_m f(x) +$$
$$+ \sum_{j=1}^{n+1-m} (x_{j+m} - x_{j-1})[x_{j-1}, x_j, \ldots, x_{j+m} ; f](x - x_{j+m-1})_+^m,$$

where $Q_m f$ is a polynomial defined by

$$(Q_m f)(x) := \sum_{k=0}^{m} \frac{1}{\binom{m}{k}} [x_0, x_1, \ldots, x_k ; f] [x_{k-1}, x_k, \ldots, x_{m-1} ; (x-t)^m].$$

One observes that $P_{m,n}f$ represents a polynomial spline function of degree m, having the knots x_j.

For $m = 1$ we have

$$(Q_1 f)(x) = f(x_0) + (x - x_0)[x_0, x_1 ; f] = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1),$$

— the Lagrange two-points interpolation polynomial.

The corresponding spline operator is

$$(P_{1,n}f)(x) = f(x_0) + (x - x_0)[x_0, x_1; f] + \\ + \sum_{j=1}^{n} (x_{j+1} - x_{j-1})[x_{j-1}, x_j, x_{j+1}; f](x - x_j)_+,$$

which is just the broken line interpolatory operator.

It is easy to see that $P_{m,n}$ reproduces the linear functions.

The operator $P_{m,n}$ in the case when $x_j = a + j(b - a)/(n + 1)$ $(j = \overline{0, n+1})$, has been considered first, already in 1942, by T. Popoviciu [8]. He calls "elementary function of order m" a function which now is known under the name of "polynomial spline function of order m".

In [8] has been proved that for $n \to \infty$ we have $\lim P_{m,n}f = f$, uniformly on $[a, b]$ and we have

$$\|f - P_{m,n}f\| \leqslant \frac{C_n}{m!}[1 + (b - a)m]\,\omega\left(\frac{1}{n+1}\right),$$

where C_n is a number independent of m.

T. Popoviciu has found that when $a = 0$, $b = 1$ and $x_j = j/(n + 1)$ $(j = \overline{0, n+1})$, then we have

$$(Q_mf)(x) = \frac{1}{m!\,h^m}\left\{\sum_{k=0}^{m} f(x_k)\left[\sum_{j=0}^{m-k}(-1)^j\binom{m+1}{j}(x - x_{k+j-1})^m\right]\right\} =$$

$$\frac{1}{m!}\sum_{k=0}^{m}\left\{k![x_0, x_1, \ldots, x_k; f]\cdot\frac{1}{h^{m-k}}\sum_{j=0}^{m-k}(-1)^j\binom{m-k}{j}(x - x_{k+j-1})^m\right\},$$

where $h = 1/(n + 1)$.

In 1959 he has proved in [9] that if f has on [0,1] a continuous derivative of order k $(0 \leqslant k \leqslant m - 1)$, then the sequence $(P_{m,n}f)^{(k)}$ converges to $f^{(k)}$ uniformly on [0,1] when $n \to \infty$. Moreover, he has proved that

$$\|f^{(k)} - P_{m,n}f)^{(k)}\| \leqslant (m + 1 - k)\,M\,\omega_k\left(\frac{m}{n+1}\right),$$

where ω_k is the modulus of continuity of $\frac{1}{k!}f^{(k)}$, while

$$M = \frac{k!}{(m-k)!}(2m - k)^{m-k}(2^{m+1-k} - 1).$$

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A POLYNOMIAL SPLINE APPROXIMATION METHOD FOR SOLVING SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

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Dedicated to Professor D. D. Stancu on his 60[th] anniversary

REZUMAT. — O metodă de rezolvare numerică a sistemului neliniar de ecuaţii diferenţiale. În această lucrare se prezintă o nouă metodă de rezolvare numerică a sistemului neliniar de ecuaţii diferenţiale $y' = f_1(x, y, z)$, $z' = f_2(x, y, z)$, $y(x_0) = y_0$, $z(x_0) = z_0$ cu ajutorul funcţiilor spline polinomiale. Metoda propusa este o metodă cu un pas de ordinul $0(h^{\alpha+r})$ pentru soluţiile exacte y şi z şi de ordinul $0(h^{\alpha+r-q+1})$ pentru derivatele $y^{(q)}$ şi $z^{(q)}$ ale acestora, $q = 1(1)r+1$, şi $0 < \alpha \leq 1$, în ipoteza ca $f_1, f_2 \in C^r$.

Description of the method. Consider the system of ordinary differential equations

$$y' = f_1(x, y, z),\ y(x_0) = y_0, \tag{1}$$

$$z' = f_2(x, y, z),\ z(x_0) = z_0, \tag{2}$$

where $f_1, f_2 \in C^r([0, 1] \times R^2)$.

Let Δ be the partition

$$\Delta : 0 = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_n = 1$$

where $x_{k+1} - x_k = h < 1$ and $k = 0(1)n - 1$.

Let L_1 and L_2 be the Lipschitz constants satisfied by the functions $f_1^{(q)}$ and $f_2^{(q)}$ respectively, i.e.,

$$|f_1^{(q)}(x, y_1, z_1) - f_1^{(q)}(x, y_2, z_2)| \leq L_1\{|y_1 - y_2| + |z_1 - z_2|\}, \tag{3}$$

$$|f_2^{(q)}(x, y_1, z_1) - f_2^{(q)}(x, y_2, z_2)| \leq L_2\{|y_1 - y_2| + |z_1 - z_2|\} \tag{4}$$

for all (x, y_1, z_1) and (x, y_2, z_2) in the domain of definition of f_1 and f_2 and all $q = 0(1)r$.

The functions $f_1^{(q)}, f_2^{(q)}, q = 1(1)r$ are functions of x, y and z only and they are defined by the following algorithm:

$$f_1^{(0)} = f_1,\ f_2^{(0)} = f_2$$

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and if $f_1^{(q-1)}$, $f_2^{(q-1)}$ are defined, then

$$f_1^{(q)} = \frac{\partial f_1^{(q-1)}}{\partial x} + \frac{\partial f_1^{(q-1)}}{\partial y} f_1 + \frac{\partial f_1^{(q-1)}}{\partial z} f_2$$

and

$$f_2^{(q)} = \frac{\partial f_2^{(q-1)}}{\partial x} + \frac{\partial f_2^{(q-1)}}{\partial y} f_1 + \frac{\partial f_2^{(q-1)}}{\partial z} f_2$$

Then, we define the spline functions approximating $y(x)$ and $z(x)$ by $S_\Delta(x)$ and $\bar{S}_\Delta(x)$ where

$$S_\Delta(x) = S_k(x) = S_{k-1}(x_k) + \sum_{j=0}^{r} f_1^{(j)}[x_k,\ S_{k-1}(x_k),\ \bar{S}_{k-1}(x_k)] \frac{(x - x_k)^{j+1}}{(j+1)!} \tag{5}$$

and

$$\bar{S}_\Delta(x) \equiv \bar{S}_k(x) = \bar{S}_{k-1}(x_k) + \sum_{j=0}^{r} f_2^{(j)}[x_k,\ S_{k-1}(x_k),\ \bar{S}_{k-1}(x_k)] \frac{(x - x_k)^{j+1}}{(j+1)!} \tag{6}$$

where $x_k \leqslant x < x_{k+1}$, $k = 0(1)n - 1$, $S_{-1}(x_0) = y_0$ and $\bar{S}_{-1}(x_0) = z_0$.

By construction, it is clear that $S_\Delta(x)$ and $\bar{S}_\Delta(x) \in C[0, 1]$.

Error estimations. For all $x \in [x_k, x_{k+1}]$, $k = 0(1)n - 1$ the exact solution can be written — by Taylor's expansion — in the following forms:

$$y(x) = \sum_{j=0}^{r} \frac{y_k^{(j)}}{j!} (x - x_k)^j + \frac{y^{(r+1)}(\xi_k)}{(r+1)!} (x - x_k)^{r+1} \tag{7}$$

and

$$z(x) = \sum_{j=0}^{r} \frac{z_k^{(j)}}{j!} (x - x_k)^j + \frac{z^{(r+1)}(\eta_k)}{(r+1)!} (x - x_k)^{r+1} \tag{8}$$

where $\xi_k,\ \eta_k \in (x_k,\ x_{k+1})$ and $k = 0(1)n - 1$.

We now estimate $|y(x) - S_k(x)|$ where $x_k \leqslant x \leqslant x_{k+1}$ and $k = 0(1)n - 1$. For this purpose, we state first the following notations:

$$\left.\begin{aligned} e(x) &= |y(x) - S_\Delta(x)|, \\ \bar{e}(x) &= |z(x) - \bar{S}_\Delta(x)|, \\ e_k &= |y_k - S_\Delta(x_k)| \\ \text{and} \\ \bar{e}_k &= |z_k - \bar{S}_\Delta(x_k)|. \end{aligned}\right\} \tag{9}$$

Now, using (5), (7) and the Lipschitz condition (3), we get:

$$|y(x) - S_k(x)| = \Bigg| \sum_{j=0}^{r} \frac{y_k^{(j)}}{j!} (x - x_k)^j + \frac{y^{(r+1)}(\xi_k)}{(r+1)!} (x - x_k)^{r+1} -$$

$$- S_{k-1}(x_k) - \sum_{j=0}^{r} f_1^{(j)}[x_k,\ S_{k-1}(x_k),\ \bar{S}_{k-1}(x_k)] \cdot \frac{(x - x_k)^{j+1}}{(j+1)!} \Bigg|$$

i.e.,

$$|y(x) - S_k(x)| \leqslant |y_k - S_{k-1}(x_k)| + \sum_{j=0}^{r-1} |y_k^{(j+1)} - f_1^{(j)}[x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)]| \cdot$$

$$\cdot \frac{|x - x_k|^{j+1}}{(j+1)!} + |y^{(r+1)}(\xi_k) - f_1^{(r)}[x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)]| \cdot \frac{|x - x_k|^{r+1}}{(r+1)!} \quad (10)$$

Now let,

$$T = |y_k^{(j+1)} - f_1^{(j)}[x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)]|$$

Then, using the Lipschitz condition (3), we get:

$$T \leqslant L_1\{|y_k - S_{k-1}(x_k)| + |z_k - \bar{S}_{k-1}(x_k)|\} \quad (11)$$

Using the fact that $S_\Delta(x) \in C[0, 1]$ and $\bar{S}_\Delta(x) \in C[0, 1]$ and the notations (9), inequality (11) becomes:

$$T \leqslant L_1(e_k + \bar{e}_k) \quad (12)$$

Similarly, if we let

$$V = |y^{(r+1)}(\xi_k) - f_1^{(r)}[x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)]|$$

Then using the Lipschitz condition (3) and the notations (9), we get:

$$V \leqslant |Y^{(r+1)}(\xi_k) - Y_k^{(r+1)}| + |f_1^{(r)}(x_k, y_k, z_k) - f_1^{(r)}[x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)]| \leqslant$$

$$\leqslant \omega(y^{(r+1)}, h) + L_1(x_k + \bar{e}_k) \quad (13)$$

Using (9) — (13), we get:

$$e(x) \leqslant e_k + L_1(e_k + \bar{e}_k)\sum_{j=0}^{r-1} \frac{h^{j+1}}{(j+1)!} + \frac{h^{r+1}}{(r+1)!}\{L_1(e_k + \bar{e}_k) + \omega(y^{(r+1)}, h)\} \quad (14)$$

Noting that

$$\sum_{j=0}^{r-1} \frac{h^j}{(j+1)!} \leqslant e^h < e$$

Then, inequality (14) yields:

$$e(x) \leqslant e_k(1 + c_0 h) + c_0 h \bar{e}_k + \frac{h^{r+1}}{(r+1)!}\omega(y^{(r+1)}, h) \quad (15)$$

where $c_0 = L_1\left(e + \frac{1}{(r+1)!}\right)$, is a constant independent of h and $h < 1$.

Similarly, utilizing (6), (8) and the Lipschitz condition (4), it can be shown that

$$\bar{e}(x) \equiv |z(x) - \bar{S}_\Delta(x)| \leqslant c_1 h e_k + (1 + c_1 h)\bar{e}_k + \frac{h^{r+1}}{(r+1)!}\,\omega(z^{(r+1)}, h) \tag{16}$$

where $c_1 = L_2\left(e + \frac{1}{(r+1)!}\right)$, is a constant independent of h and $h < 1$.

To complete the proof of the convergence we recall the definition of the matrix inequality.

DEFINITION (1). Let $A = [a_{ij}]$, $B = [b_{ij}]$ be two matrices of the same order, then we say that $A \leqslant B$ iff

(i) a_{ij} and b_{ij} are nonnegative,

(ii) $a_{ij} \leqslant b_{ij}$ $\forall i, j$.

According to this definition, and if we use the matrix notations

$$E(x) = (e(x)\ \bar{e}(x))^T,$$

$$E_k = (e_k\ \bar{e}_k)^T,\ k = 0(1)n - 1$$

then, we can write the estimations (15) and (16) in the form:

$$E(x) \leqslant (I + hA)E_k + \frac{h^{r+1}}{(r+1)!}\,\omega(h) B \tag{17}$$

where $A = \begin{pmatrix} c_0 & c_0 \\ c_1 & c_1 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\omega(h) = \max\{\omega(Y^{(r+1)}, h),\ \omega(z^{(r+1)}, h)\}$,

$\omega(Y^{(r+1)}, h)$ and $\omega(z^{(r+1)}, h)$ are the modulii of continuity of the functions $Y^{(r+1)}$ and $z^{(r+1)}$ respectively, and I is the identity matrix of order 2.

Then, we give the following definition of the matrix norm.

DEFINITION (2): Let $T = [t_{ij}]$ be an $m \times n$ matrix, then we define

$$\| T \| = \max_i \sum_{j=1}^{n} | t_{ij} |$$

Using this definition, we get:

$$\| E(x) \| = \max(e(x), \bar{e}(x)) \tag{18}$$

Since (17) is valid for all $x \in [x_k, x_{k+1}]$, $k = 0(1)n - 1$, then the following inequalities hold true:

$$\| E(x) \| \leqslant (1 + h \| A \|) \| E_k \| + \frac{h^{r+1}}{(r+1)!}\,\omega(h)$$

$$(1 + h \| A \|) \| E_k \| \leqslant (1 + h \| A \|)^2 \| E_{k-1} \| + \frac{h^{r+1}}{(r+1)!}\,\omega(h)(1 + h \| A \|)$$

$$(1 + h \| A \|)^2 \| E_{k-1} \| \leqslant (1 + h \| A \|)^3 \| E_{k-2} \| + \frac{h^{r+1}}{(r+1)!}\,\omega(h)(1 + h \| A \|)^2$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$(1 + h \| A \|)^k \| E_1 \| \leqslant (1 + h \| A \|)^{k+1} \| E_0 \| + \frac{h^{r+1}}{(r+1)!}\,\omega(h)(1 + h \| A \|)^k$$

Adding L.H.S. and R.H.S. of these inequalities and noting that $\|E_0\| = 0$, we get:

$$\|E(x)\| \leqslant c_2 h^r \omega(h) \tag{19}$$

where $c_2 = \dfrac{e^{\|A\|}}{(r+1)!}$, is a constant independent of h.

Thus, applying definition (18), we get:

$$e(x) \leqslant c_2 h^r \omega(h) = 0(h^{\alpha+r}) \tag{20}$$

and

$$\bar{e}(x) \leqslant c_2 h^r \omega(h) = 0(h^{\alpha+r}) \tag{21}$$

We are going to estimate $|y^{(q)}(x) - S_k^{(q)}(x)|$ where $q = 1(1)r$. Using (5), (7) and the Lipschitz condition (3), we get:

$$| y^{(q)}(x) - S_k^{(q)}(x) | \leqslant \sum_{j=q}^{r} | y_k^{(j)} - f_1^{(j-1)}[x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)] | \cdot$$

$$\cdot \frac{|x - x_k|^{j-q}}{(j-q)!} + | y^{(r+1)}(\xi_k) - f_1^{(r)}[x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)] | \cdot$$

$$\cdot \frac{|x - x_k|^{r+1-q}}{(r+1-q)!} \leqslant$$

$$\leqslant L_1(e_k + \bar{e}_k) \sum_{j=q}^{r} \frac{h^{j-q}}{(j-q)!} + \frac{h^{r+1-q}}{(r+1-q)!} [\omega(y^{(r+1)}, h) + L_1(e_k + \bar{e}_k)] \tag{22}$$

Using (20) and (21), inequality (22) yields:

$$| y^{(q)}(x) - S_k^{(q)}(x) | \leqslant c_3 h^{r+1-q} \omega(h) = 0(h^{\alpha+r+1-q}) \tag{23}$$

where c_3 is a constant independent of h and $h < 1$.

Now, for the case $q = r + 1$, we have:

$$\begin{aligned} |y^{(r+1)}(x) - S_k^{(r+1)}(x)| &= |f^{(r)}[x, y(x), z(x)] - f^{(r)}[x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)]| \leqslant \\ &\leqslant |f^{(r)}[x, y(x), z(x)] - f^{(r)}[x_k, y_k, z_k]| + |f^{(r)}[x_k, y_k, z_k] - \\ &- f^{(r)}[x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)]| \leqslant \omega(Y^{(r+1)}, h) + L_1(e_k + \bar{e}_k) \leqslant \\ &\leqslant c_4 \omega(h) = 0(h^{\alpha}) \end{aligned} \tag{24}$$

where c_4 is a constant independent of h and $h < 1$.

Similarly, using (4), (6), (8), (20) and (21), it can be easily shown that:

$$| z^{(q)}(x) - \bar{S}_k^{(q)}(x) | \leqslant c_5 h^{r+1-q} \omega(h) = 0(h^{\alpha+r+1-q}) \tag{25}$$

where $q = 1(1)r + 1$ and c is a constant independent of h and $h < 1$.

Thus, we have proved the following theorem:

THEOREM: *Let* $y(x)$ *and* $z(x)$ *be the exact solutions to the problem* (1)–(2). *If* $S_\Delta(x)$ *and* $\bar{S}_\Delta(x)$, *given in* (5) *and* (6), *are the approximate solutions and* $f_1, f_2 \in C^r([x_0, x_n] \times R^2)$, *then for all* $x \in [x_k, x_{k+1}]$, $k = 0(1)n-1$, *we have:*

$$|y(x) - S_\Delta(x)| \leqslant ch^r\omega(h),$$

$$|y^{(i)}(x) - S_\Delta^{(i)}(x)| \leqslant c^* h^{r-i+1}\omega(h),\ i = 1(1)r+1$$

$$|z(x) - \bar{S}_\Delta(x)| \leqslant Kh^r\omega(h)$$

and

$$|z^{(i)}(x) - \bar{S}_\Delta^{(i)}(x)| \leqslant K^* h^{r-i+1}\omega(h),\ i = 1(1)r+1$$

where c, c^*, K and K^* are constant independent of h.

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POLYNOMIALS OF BINOMIAL TYPE AND APPROXIMATION OPERATORS

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REZUMAT. — Polinoame de tip pozitiv şi operatori de aproximare. Folosind un şir de bază (p_n) pentru un delta operator Q (în sensul precizat în lucrarea [10]), autorii introduc şi studiază un şir de operatori liniari polinomiali L_n^Q : $C[0, 1] \to C[0, 1]$, definiţi prin formula (8). Considerînd cazul cînd aceşti operatori sînt de tip pozitiv, prin aplicarea criteriului de convergenţă al lui Popoviciu — Korovkin, se demonstrează o teoremă de uniform convergenţă a şirului $(L_n^Q f)$ către funcţia $f \in C[0, 1]$ şi se evaluează ordinul de aproximare corespunzător. Se menţionează mai multe cazuri speciale, cunoscute de operatori liniari ozzitivi, arătînd modul de obţinere ale acestora din operatorii introduşi la (8).

1. During the past three decades a number of classes and sequences of linear positive operators both summation type and those defined by certain integrals, have been introduced and studied by various authors. Among the persons interested in the field of this kind of approximation operators, we mention with partidular indebtedness Professor D. D. STANCU and the group of mathematicians around him ([7], [8], [9], [19]).

By a *polynomial sequence* we mean a sequence of polynomials (p_n), $n = 0, 1, \ldots$, where p_n is exactly of degree n, for all n. A polynomial sequence is said to be of *binomial type* ([10]) if it satisfies the identity

$$p_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)\, p_{n-k}(y). \tag{1}$$

These special polynomial sequences occur in combinatorics and in analysis. The enumeration properties of these polynomials have been studied lately by Gian-Carlo Rota and others in a series of papers on the foundations of umbral calculus ([10], see, the references from [14]).

Our aim is to construct sequences of approximation operators $L_n : C(I) \to C(I)$, $n = 0, 1, \ldots$, $I = [0, 1]$, of the following form

$$(L_n f)(x) = \frac{1}{p_n(n)} \sum_{k=0}^{n} \binom{n}{k} p_k(nx) p_{n-k}(n - nx) f\left(\frac{k}{n}\right)$$

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Operators $A_n : C(I) \to C(I)$, $n = 0, 1, \ldots$, of the form

$$(A_n)(x) = \frac{1}{p_n(1)} \sum_{k=0}^{n} \binom{n}{k} p_k(x)\, p_{n-k}(1-x)\, f\left(\frac{k}{n}\right)$$

have been studied in [1], [2], [8], [9], [11], [15], [17].

2. This section contains basic facts needed in the subsequent analysis. Let $\mathfrak{A}$ be the algebra of all polynomials in one variable, with real coefficients. By Π_n we denote the linear space of all polynomials of the degree at most n. All operators we consider will be tacitly assumed to be linear. The shift operator $E^a : \mathfrak{A} \to \mathfrak{A}$ is defined by $(E^a p)(x) = p(x + a)$, $p \in \mathfrak{A}$. An operator $T : \mathfrak{A} \to \mathfrak{A}$ for which $E^a T = TE^a$, for all $a \in R$, is called a *shift-invariant operator*.

In the following let us denote $e_n(t) = t^n$, $n = 0, 1, \ldots$ A *delta operator* $Q : \mathfrak{A} \to \mathfrak{A}$ is defined as a shift-invariant operator for which Qe_1 is a non-zero constant. Delta operators possess many of the properties of the derivative operator D (see [10]). For instance

i) $Qe_0 = 0$

ii) $Qp \in \Pi_{n-1}$ whenever $p \in \Pi_n$.

A polynomial sequence (p_n), $n = 0, 1, \ldots$, is called the sequence of *basic polynomials* for a delta operator Q if:

$$1)\ p_0(x) = 1,\quad 2)\ p_n(0) = 0,\ n \geq 1,\quad 3)\ Qp_n = np_{n-1}. \tag{1}$$

It is known that every delta operator has an unique sequence of basic polynomials associated with it. Moreover, if $K : \mathfrak{A} \to \mathfrak{A}$ is the operator $(Kp)(x) = xp(x)$ then $T' = TK - KT$ is the so-called *Pincherle-derivative of an operator* $T : \mathfrak{A} \to \mathfrak{A}$. If Q is a delta operator, then Q' is shift-invariant and moreover $(Q')^{-1}$ exists. Further, shift-invariant operator commute. Ronald Mullin and Gian-Carlo Rota prove the following beautiful result:

THEOREM A ([10]) a) *If (p_n) is a basic sequence for some delta operator Q, then it is of binomial type;*

b) *If (p_n) is a sequence of polynomials of binomial type, then it is a basic sequence for some delta operator;*

c) *$Q : \mathfrak{A} \to \mathfrak{A}$ is a delta operator if and only if $Q = DP$ for some shift-invariant operator P, where P^{-1} exists and $Dp = p'$;*

d) *Let (p_n) be the sequence of basic polynomials for the delta operator $Q = DP$. Then*

i) $p_n = Q'P^{-n-1}e_n$,

ii) $p_n = P^{-n}e_n - (P^{-n})'e_{n-1}$, (2)

iii) $p_n = KP^{-n}e_{n-1}$,

iv) $p_n = K(Q')^{-1}p_{n-1}$.

3. If (p_n), $p_n(n) \neq 0$, is a basic sequence for the delta operator Q let us denote

$$w_n(Q) = 1 - \frac{n(n-1)}{p_n(n)} (Q'^{-2} p_{n-2})(n) \tag{3}$$

DEFINITION. A linear operator $Q : \mathcal{A} \to \mathcal{A}$ belongs to *the class W* if and only if the following conditions are satisfied

(i) Q is a delta operator with the basic sequence (p_n);
(ii) $p_n'(0) \geqslant 0$ for $n = 1, 2, \ldots$;
(iii) $\lim\limits_{n \to \infty} w_n(Q) = 0$ where $w_n(Q)$ is defined in (3).

Let us consider the following examples:

I) *The differentiation operator* D with the basic sequence $c_n(x) = x^n$, $n = 0, 1, \ldots$; then $D' = J$, J being the *identity operator*, und

$$w_n(D) = \frac{1}{n}. \tag{4}$$

II) *Backward difference operator* $\nabla = J - E^{-1}$, with the basic sequence $(x)_n = x(x+1) \ldots (x+n-1)$, $(x)_0 = 1$. In this case

$$\nabla' = E^{-1}, \quad \nabla'^{-2} = E^2$$

and

$$w_n(\nabla) = \frac{2}{n+1}. \tag{5}$$

III) *The Touchard-operator* $T : \mathcal{A} \to \mathcal{A}$, $T = \ln(J + D)$, is a delta operator. Its basic sequence (t_n) is the so-called system of *exponential polynomials* (or Touchard polynomials), where

$$t_n(x) = \sum_{k=0}^{n} [0, 1, \ldots, k; c_n] x^k =$$

$$= \sum_{k=0}^{n} \left[0, \frac{1}{n}, \ldots, \frac{k}{n}; c_n\right] n^{n-k} x^k = e^{-x} \sum_{k=0}^{\infty} \frac{k^n x^k}{k!}. \tag{6}$$

By the symbol $[x_0, x_1, \ldots, x_k; f]$ we have denoted the divided difference of a function f on a system of distinct points $x_0, x_1, \ldots, x_k$, from its domain. For the operator T one has

$$T'^{-1} = J + D, \quad T'^{-2} = J + 2D + D^2.$$

From theorem A (see (2)), we may write $t_n = K(J + D)t_{n-1}$ that is

$$t_n(x) = x(t_{n-1}(x) + t'_{n-1}(x)).$$

Taking into account that the Stirling numbers of the second kind $S(n, k) = [0, 1, \ldots, k; c_n]$, $k = 0, 1, \ldots$, are non-negative, we have

$$0 < \frac{t_{n-1}(x)}{t'_n(x)} \leqslant \frac{1}{n}, \quad x > 0.$$

Now

$$(T'^{-2}t_{n-2})(x) = \frac{t_n(x) - t_{n-1}(x)}{x^2}$$

and

$$w_n(T) = \frac{1}{n} + \frac{n-1}{n}\,\frac{t_{n-1}(n)}{t_n(n)}\,.$$

Therefore

$$\frac{1}{n} < w_n(T) \leqslant \frac{2}{n}$$

and in conclusion $T \in W$.

IV) *The Laguerre operator* $L : \mathcal{A} \to \mathcal{A}$, $L = \dfrac{D}{J + D}$. The basic sequence for this delta operator is (ℓ_n), where $\ell_0(x) = 1$, and

$$\ell_n(x) = \sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1} x^k, \qquad n \geqslant 1.$$

If

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}, \qquad n = 0, 1, \ldots,$$

are the Laguerre polynomials (of order α), then $\ell_n(x) = n!\, L_n^{(-1)}(-x)$. In this case

$$L' = \frac{J}{(J+D)^2}, \ (L')^{-2} = (J + D)^4$$

and from $L\ell_n = n\ell_{n-1}$ or $D\ell_n = n(J + D)\ell_{n-1}$, we find

$$(L'^{-2}\ell_{n-2})(n) = \frac{1}{n(n^2-1)(n+2)}\,\ell_{n+2}^{(4)}(n).$$

From

$$\begin{aligned} &x^3\ell_{n+2}^{(4)}(x) = n(n+1)(n+2)\ \left[(n+1)(x+2)\,\ell_n(x) - 2\,\ell_{n+1}(x)\right] \\ &\ell_{n+1}(x) = (2n + x)\,\ell_n(x) - n(n-1)\,\ell_{n-1}(x) \\ &x\ell'_{n+1}(x) = (n+1)\,[\ell_{n+1}(x) - n\ell_n(x)] \end{aligned} \tag{7}$$

one obtains

$$w_n(L) = -\frac{3n+2}{n^2} + \frac{2}{n^2}\,\frac{\ell_{n+1}(n)}{\ell_n(n)}\,.$$

Using the inequalities $\ell'_{n+1}(x) > 0$, $x^3\ell_{n+2}^{(4)}(x) > 0$, $x > 0$, from (7) we find

$$n\ell_n(n) < \ell_{n+1}(n) < 3n\,\ell_n(n).$$

In this manner

$$-\frac{3}{n} < w_n(L) < \frac{3}{n}$$

which enables us to assert that $L \in W$.

4. If (p_n) is the basic sequence for a delta operator Q, then we define the sequence of linear polynomial operators $L_n^Q : C(I) \to C(I)$, $n = 1, 2, \ldots$, by

$$(L_n^Q)(x) = \frac{1}{p_n(n)} \sum_{k=0}^{n} \binom{n}{k} p_k(nx)\, p_{n-k}(n - nx) f\left(\frac{k}{n}\right) \tag{8}$$

where $f \in C(I)$, $x \in I = [0, 1]$, $p_n(n) \neq 0$.
Using theorem A we observe that

$$L_n^Q e_0 = e_0, \qquad L_n^Q e_1 = e_1 \tag{9}$$
$$(L_n^Q e_2)(x) = e_2(x) + x(1 - x) w_n(Q).$$

Likewise, in a similar way as in the proof of lemma 4.2.2 from [8], we have proved that

$$(L_n^Q e_s)(x) = \frac{1}{n^s\, p_n(n)} \sum_{k=1}^{s} k! \binom{n}{k} [0, 1, \ldots, k; e_s] c_{k,s}(nx, n - nx) \tag{10}$$

where $c_{k,s}(x, y) = (P^k E^y p_{n-k})(x)$, $P : \mathcal{A} \to \mathcal{A}$ being the linear operator $P = KQ'^{-1}$, i.e., $(Ph)(x) = x(Q'^{-1}h)(x)$.

LEMMA 1. *If L_n^Q, $n = 1, 2, \ldots$, are positive operators then the sequence $(w_n(Q))_{n=1}^{\infty}$ is bounded. More precisely*

$$0 \leqslant w_n(Q) \leqslant 1, \ n = 1, 2, \ldots .$$

Proof. For $t \in I$, $t^2 - t + \frac{1}{4} \geqslant 0$; therefore

$$L_n^Q e_2 - L_n^Q e_1 + \frac{1}{4} L_n^Q e_0 \geqslant 0.$$

From (9)

$$x(1 - x) w_n(Q) + \frac{1}{4} - x(1 - x) \geqslant 0, \quad x \in I.$$

If we select $x = \frac{1}{2}$ we obtain $w_n(Q) \geqslant 0$. Likewise, from $t(1 - t) \geqslant 0$, $t \in I$, one finds

$$L_n^Q e_1 - L_n^Q e_2 \geqslant 0$$

that is $x(1 - x)(1 - w_n(Q)) \geqslant 0$, $x \in I$, which implies $w_n(Q) \leqslant 1$.

LEMMA 2. *If $Q \in W$ then L_n^Q, $n = 1, 2, \ldots$, are positive operators.*
Proof. From the identity (see [14])

$$p_n(x) = x \sum_{k=0}^{n-1} p_{n-1-k}(x)\, p'_{k+1}(0) \binom{n-1}{k} \tag{11}$$

it follows that p_n is non-negative on $[0, \infty)$. Indeed

$$p_0(x) = 1, \quad p_1(x) = x p_1'(0) \geqslant 0, \quad x \in [0, \infty),$$

and if we suppose $p_j(x) \geqslant 0$ for $x \in [0, \infty)$, $j = 0, 1, \ldots, n-1$, then (11) furnishes us the inequality $p_n(x) \geqslant 0$ on $[0, \infty)$.

THEOREM 1. *If $Q \in W$ and $f \in C(I)$, then*

$$\lim_{n\to\infty} \| f - L_n^Q f \| = 0$$

where $\|f\| = \max_{t \in I} |f(t)|$.

Proof. According to the Popoviciu-Korovkin theorem ([13], [3]) it is sufficient to show that

$$\lim_{n\to\infty} \| e_k - L_n^Q e_k \| = 0, \quad k = 0, 1, 2.$$

In our case, from (9)

$$\| e_j - L_n^Q e_j \| = 0, \; j = 0, 1, \quad \| e_2 - L_n^Q e_2 \| \leqslant \frac{1}{4} w_n(Q)$$

where $\lim_{n\to\infty} w_n(Q) = 0$.

THEOREM 2. *Let $(L_n^Q)_{n=1}^{\infty}$, $Q \in W$, be the sequence of linear positive operators defined in* (8).

i) *If $h \in C(I)$ is convex on I, then $h(x) \leqslant (L_n^Q h)(x)$, $x \in I$;*

ii) *If $f \in C^{(2)}(I)$, $m_f = \min_{x \in I} f''(x)$,* (12)

$M_f = \max_{x \in I} f''(x)$, *then for $x \in I$*

$$\frac{1}{2} m_f \cdot w_n(Q)\, x(1 - x) \leqslant (L_n^Q f)(x) - f(x) \leqslant \frac{1}{2} M_f \cdot w_n(Q) x(1-x). \tag{13}$$

Proof. If $c_0, c_1, \ldots, c_n$ are non-negative numbers with $c_0 + c_1 + \ldots + c_n = 1$, then for every system of points $x_0, x_1, \ldots, x_n$ from I

$$h\left(\sum_{k=0}^{n} c_k x_k\right) \leqslant \sum_{k=0}^{n} c_k h(x_k). \tag{14}$$

Let us consider

$$c_k = \frac{1}{p_n(n)} \binom{n}{k} p_k(nx)\, p_{n-k}(n - nx), \qquad x_k = \frac{n}{k},$$

x being arbitrary in I. Then

$$\sum_{k=0}^{n} c_k\, x_k = (L_n^Q e_1)(x) = x$$

and (14) is the same with (12). Let $f \in C^{(2)}(I)$; the functions

$$h_1(x) = \frac{1}{2} M_f \cdot x^2 - f(x), \quad h_2(x) = f(x) - \frac{1}{2} m_f \cdot x^2$$

are convex on I. From $h_1 \leqslant L_n^Q h_1$, $h_2 \leqslant L_n^Q h_2$ on I, we conclude with (13).

THEOREM 3. *Let $Q \in W$, $f \in C(I)$, and denote by $\omega(f;\ \delta)$, the modulus of continuity of the function f. If $x \in I$, then*

$$|f(x) - (L_n^Q f)(x)| \leqslant 2\omega(f; \sqrt{x(1-x) w_n(Q)})$$

$$\|f - L_n^Q f\| \leqslant \frac{5}{4}\, \omega(f; \sqrt{w_n(Q)}).$$

Proof. If $L : C(I) \to C(I)$, $Le_0 = e_0$, is a linear positive operator, then (see for instance theorems 4.2 and 4.5 from [7])

$$|f(x) - (Lf)(x)| \leqslant 2\omega(f; \sqrt{(L\Omega_2)(x)})$$

and

$$\|f - Lf\| \leqslant \omega(f;\ \delta) \inf_{m=1,2,\ldots} \{1 + \delta^{-m} \| L\Omega_m \|\},\ \delta > 0, \tag{15}$$

where $\Omega_j(t,\ x) = |t - x|^j$. Because

$$(L_n^Q \Omega_2)(x) = x(1-x)\, w_n(Q),$$

choosing $m = 2$ in (15) the theorem is established.

5. Now we observe that L_n^D is the Bernstein operator

$$(L_n^D f)(x) = (B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

At the same time

$$(L_n^\nabla f)(x) = \frac{2(n!)}{(2n)!} \sum_{k=0}^{n} \binom{n}{k} (nx)_k (n - nx)_{n-k} f\left(\frac{k}{n}\right), \qquad n = 1, 2, \ldots.$$

This sequence of approximation operators may be obtained from the class of operators introduced by D. D. Stancu [15] — [17]. By means of a betatransform (see [4]), the operator L_n^∇ admits the representation

$$(L_n^\nabla f)(x) = \frac{1}{B(nx, n - nx)} \int_0^1 t^{nx-1}(1-t)^{n-nx-1}(B_n f)(t)\,dt, \quad x \in (0, 1). \tag{16}$$

It is easily deduced that in this case

$$L_n^\nabla e_k = e_k, \quad k = 0, 1, \quad (L_n^\nabla e_2)(x) = e_2(x) + \frac{2x(1-x)}{n+1}$$

$$(L_n^\nabla e_3)(x) = e_3(x) + \frac{6x(1-x)}{(n+1)(n+2)} + \frac{6nx^2(1-x)}{(n+1)(n+2)},$$

$$(L_n^\nabla e_4)(x) = e_4(x) + \frac{12(n^2+1)x^3(1-x)}{(n+1)(n+2)(n+3)} + \frac{(36n-12)x^2(1-x)}{(n+1)(n+2)(n+3)} +$$

$$+ \frac{(26n-2)x(1-x)}{n(n+1)(n+2)(n+3)}, \quad n \geqslant 4.$$

Consequently

$$(L_n^\nabla \Omega_2)(x) = \frac{2x(1-x)}{n+1}, \quad (L_n^\nabla \Omega_4)(x) = \frac{2x(1-x)\,[6n(n-7)x(1-x) + 13n - 1]}{n(n+1)(n+2)(n+3)}.$$

Using the inequality

$$\omega(f; \delta\lambda) \leqslant (1 + [\lambda])\omega(f; \delta), \quad \lambda > 0, \quad \delta > 0,$$

the first estimation of theorem 3 gives

$$|f(x) - (L_n^\nabla f)(x)| \leqslant 4\omega\left(f; \sqrt{\frac{x(1-x)}{n+1}}\right), \; f \in C(I), \qquad x \in I.$$

As a consequence of the equalities

$$\lim_{n\to\infty} n[e_2(x) - (L_n^\nabla e_2)(x)] = -2x(1-x), \; \lim_{n\to\infty} n(L_n^\nabla \Omega_4)(x) = 0,$$

the following asymptotic approximation formula is true (see [7], theorem 4.II): if $f \in C(I)$ and if f is twice differentiable at a point x_0, $x_0 \in I$, then

$$\lim_{n\to\infty} n[f(x_0) - (L_n^\nabla f)(x)_0] = -x_0(1-x_0)f''(x_0).$$

We note that (16) enables us to write

$$(L_n^\nabla f)(x) = \sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{n+k-1}{k} n^k} \left[0, \frac{1}{n}, \ldots, \frac{k}{n}; f\right] (nx)_k.$$

In the case of Touchard-operator, the corresponding operators are defined by

$$(L_n^T f)(x) = \frac{1}{t_n(n)} \sum_{k=0}^{n} \binom{n}{k} t_k(nx)\, t_{n-k}(n - nx) f\left(\frac{k}{n}\right) =$$
$$= \frac{e^{-n}}{t_n(n)} \sum_{k=0}^{\infty} \frac{k^n n^k}{k!} (B_k B_n f)(x).$$

In view of the convexity-preserving properties of the Bernstein operators B_n, $n = 0, 1, \ldots$, we conclude that if f is convex of order s on I, then the polynomial $L_n^T f$ is also a convex function of the same order s on the interval I.

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STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXII, 4. 1987

AN AUTOMORPHISM PROPERTY OF A CLASS OF POLYNOMIAL-TYPE POSITIVE CONVOLUTIVE OPERATORS

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Dedicated to Professor D. D. Stancu on gis 60th anniversary

REZUMAT. — O proprietate de automorfism a unei clase de operatori convolutivi pozitivi. În această notă se generalizează un rezultat al autrului, din lucrarea [2,] pentru funcţii de mai multe variabile.

1. **Introduction.** Bernstein's operators were and are object of study for many mathematicians; as a consequence, many and various results in this domain have been published. One gave different generalizations, firstly for functions of several variables, then, using probabilistic and non-probabilistic methods, new classes of linear and positive Bernstein-type operators were constructed. We must particularly mention the important results obtained in this domain by D. D. Stancu [4, 5], and by many other authors as well. The positive convolutive operators also constitute generalizations of both Bernstein's operators and other operators of this kind.

Most of the studies concerning the Bernstein-type operators refer to properties of uniform convergence to a function to which one associates Bernstein's operators.

In this note we deal with an algebraic property of the positive convolutive operators for functions of several variables. More exactly, the author's result obtained in [2] is generalized for functions of several variables.

2. **Polynomial convolutive operators.** We shall consider the positive convolutive operator:

$$L_n(f;\ x) = A_n(x) \sum_{k=0}^{n} f(x_k^{(n)}) \binom{n}{k} P_k(x)\ P_{n-k}(1-x), \tag{1}$$

where:

$$A_n(x) = [P_n(1)]^{-1},\quad n \in N^n;\quad x,\ x_k^{(n)} \in X \subset R^m,$$

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and

$$x_k^{(n)} = \left(x_{k_1}^{(n)},\ x_{k_2}^{(n)},\ \ldots,\ x_{k_m}^{(n)}\right);\ k_i = \overline{0,\ n_i};\ n_i \in N;\ i = \overline{1,\ m};$$

$$x = (x_1,\ x_2,\ \ldots,\ x_m);$$

$$f: X \to R,\ X = [0,\ 1]^m;$$

$$P_s(z) = P_{s_1}(z_1) \cdot P_{s_2}(z_2) \ldots P_{s_m}(z_m);$$

$$\sum_{k=0}^{n} = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \cdots \sum_{k_m=0}^{n_m};\ x_k^{(n)} = \left(\frac{k_1}{n_1},\ \frac{k_2}{n_2},\ \ldots,\ \frac{k_m}{n_m}\right).$$

This operator is associated to a relation of convolution, as the author showed in [3].

The positive convolutive operator (1) is a polynomial-type one if $P_s(z)$ are polynomials; we shall consider it as being defined on $C[X]$ and with values in the same set of the continuous functions on the compactum $X = [0,\ 1]^m$.

THEOREM. *Let be the restriction of the operator L_n, defined by (1), for the set of polynomials $\mathfrak{L}_r$, $r = (r_1,\ r_2,\ \ldots,\ r_m)$ and $0 \leqslant r_i \leqslant n_i$, $i = \overline{1,\ m}$. This linear polynomial operator achieves an automorphism in the set $\mathfrak{L}_r$, namely*

$$L_n\mathfrak{L}_r = \mathfrak{L}_r.$$

Proof. The similar result for polynomial-type positive convolutive operators relating to functions of only one variable was established by us in the paper [2]. For the convolutive operators defined by (1), this property may be naturally generalized by performing a tensor product of some relations for the unidimensional case. Hence, the stated theorem holds.

EXAMPLES. 1° *Bernstein's operators.* If we consider in (1)

$$P_k(z) = z_1^{k_1} z_2^{k_2} \ldots z_m^{k_m},$$

then these operators become Bernstein's operators B_n of the first kind relating to functions of several variables; they are known and we have for them:

$$B_n\mathfrak{L}_r = \mathfrak{L}_r.$$

2° *D. D. Stancu's operators.* If we consider in (1)

$$P_{k_i}(z) = z(z-1) \ldots (z-k+1),$$

then we obtain Stancu's operators S_n, defined in [4], for which we have therefore:

$$S_n\mathfrak{L}_r = \mathfrak{L}_r.$$

It goes without saying that these operators have many other interesting and important properties as well.

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MECANICA NEWTONIANĂ ŞI LANSAREA PRIMULUI SATELIT ARTIFICIAL AL PĂMÎNTULUI[1]

ÁRPÁD PÁL*

Aniversăm în acest an împlinirea a 300 de ani de la apariţia monumentalei opere a lui Isaac Newton *Philosophiae Naturalis Principia Mathematica*, ce reprezintă o piatră de hotar în istoria gîndirii omeneşti, şi — printr-o coincidenţă fericită — 30 de ani de la lansarea primului satelit artificial al Pămîntului — *Sputnik 1* —, care a deschis o nouă eră ştiinţifică şi tehnică denumită — pe drept cuvînt — era cosmică.

Publicarea lucrării lui Newton, sub auspiciile vestitei Societăţi Regale din Londra (Royal Society) a reprezentat o cotitură decisivă în dezvoltarea gîndirii omeneşti, în general, şi a mecanicii (ştiinţa mişcării) — una dintre cele mai vechi ştiinţe ale naturii — în special. Apariţia *Principiilor* — cum i se spune pe scurt operei lui Newton — a însemnat pentru mecanică ceea ce a reprezentat opera lui Euclid *Elementele* pentru geometrie — matematica mileniilor anterioare.

În *Prefaţa la ediţia I*, Newton expune scopul şi programa lucrării. Astfel, în concepţia sa, rostul cercetării naturii este „să reducă fenomenele naturale la legi matematice". Deci scopul cărţii este „cultivarea matematicii, întrucît ea priveşte filozofia". De aceea el spune că nu se ocupă cu cele cinci puteri (maşini simple): pîrghia, scripetele, roata cu sul, planul înclinat şi şurubul, care formau preocupările celor vechi, ci cu forţele naturii, cum sînt: gravitatea, forţa elastică, rezistenţa fluidelor şi în general forţele de atracţie şi de respingere. Acestui mod de tratare a mecanicii îi dă numele de *principiile matematice ale filozofici naturale*, numire care justifică şi titlul cărţii.

Deci Newton concepe *mecanica* de două feluri: cea *practică*, ce include „toate artele manuale care se ocupă îndeosebi de corpurile în mişcare", de la care s-a împrumutat chiar numele de mecanică, şi cea *raţională*, care procedează prin enunţuri şi demonstraţii precise privind „mişcările ce rezultă din forţe oarecare şi forţele necesare unor mişcări oarecare".

Deoarece — spune Newton — „orice dificultate a filozofiei constă în a găsi din fenomenele de mişcare forţele naturii şi din aceste forţe să demonstrăm celelalte fenomene", el şi-a propus rezolvarea acestei probleme în *Cartea I* şi *a II-a*, iar în *Cartea a III-a* să descrie „un exemplu al acestui procedeu prin explicarea sistemului lumii". „Căci acolo, din fenomenele cereşti, cu ajutorul propoziţiilor matematice demonstrate în Cărţile precedente se deduc forţele gravităţii după care corpurile tind spre Soare şi spre diversele planete. Apoi, din aceste forţe, iarăşi prin propoziţii matematice, se deduc legile mişcării planetelor, cometelor, Lunii şi mării" — scrie Newton şi adaugă: „De s-ar putea ca toate celelalte feno-

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[1] Prezentat la Simpozionul „300 de ani de la publicarea operei lui Isaac Newton *Philosophiae Naturalis Principia Mathematica* şi 30 de ani de la lansarea primului satelit artificial al Pămîntului", 15 octombrie 1987.

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mene ale naturii să se deducă din principiile mecanice prin acelaşi fel de raţionament", încheind cu modestia-i cunoscută: „Sper însă că principiile stabilite aici vor aduce oarecare lumină fie acestui mod de a filozofa, fie altuia mai adevărat".

Deosebit de importantă este şi concepţia lui Newton potrivit căreia „geometria se întemeiază pe practica mecanică şi nu este nimic altceva decît acea parte a mecanicii universale care propune şi demonstrează arta de a măsura precis". Reluînd această idee profundă, *Albert Einstein* şi-a construit mecanica relativistă sub formă de geometrie a unui spaţiu-timp curbat de materie, care avea să schimbe din temelii reprezentările noastre despre materie, mişcare, spaţiu şi timp, producînd o revoluţie în ştiinţă ce a fost pregătită, din punct de vedere matematic, de o revoluţie similară ce a avut loc în matematică o dată cu apariţia geometriilor neeuclidiene, în prima jumătate a secolului XIX, la Gauss, Lobacevski şi Bolyai.

Studiind cu atenţie opera unor geniali şi atenţi observatori ai naturii ca *Leonardo da Vinci, Nicolai Kopernik, Johann Kepler, Francis Bacon, Galileo Galilei*, care, eliberîndu-se de dogmatismul Evului Mediu, au pus la dispoziţia ştiinţei fapte mecanice de o expresivitate deosebită, Newton este primul care intuieşte necesitatea definirii unui *spaţiu absolut* — modelat matematic cu spaţiul euclidian cu trei dimensiuni (E_3) — şi a unui *timp absolut* — modelat matematic cu axa numerelor reale (**R**) —, a unor *noţiuni fundamentale ale mecanicii* — ca acelea de spaţiu, timp şi masă — şi a formulat *axiomele* sau *legile fundamentale ale mecanicii* care îi poartă numele. Acestea sînt prezentate la începutul lucrării lui Newton la care ne referim.

Mecanica clasică se bazează pe aceste noţiuni şi principii, se dezvoltă în cadrul acestora ca o ştiinţă teoretică şi aplicată, confruntîndu-se şi confruntîndu-şi mereu construcţiile cu realitatea (terestră şi cosmică), cu observaţia şi experimentul.

Ca o culme a virtuozităţii sale matematice, Newton îşi expune rezultatele sub formă geometrică, pentru a le face mai accesibile contemporanilor săi. Preocupat de problemele de mişcare puse de mecanică, el îşi făureşte însă, ca instrument de lucru, o nouă doctrină matematică — *teoria fluenţilor şi fluxiunilor*, adică calculul diferenţial şi integral, cu denumirea de astăzi. Se ştie că, în aceeaşi perioadă, *Leibniz* punea, sub o altă formă, bazele aceluiaşi calcul diferenţial şi integral, fiind continuat în această realizare de marii matematicieni *Iacob Bernoulli* şi *Ioan Bernoulli*.

De acum o perspectivă imensă de dezvoltare i se deschide mecanicii. Vastul edificiu, ale cărei baze solide au fost puse de Newton, va fi supraetajat prin opera unor demni urmaşi, dintre care amintim pe:

— *Mac-Laurin*, care va studia formele de echilibru relativ ale corpurilor cereşti în rotaţie în jurul axei lor;

— *D'Alembert*, care va dezvolta teoria mişcărilor de precesie şi nutaţie ale Pămîntului;

— *Daniel Bernoulli* şi *Leonhard Euler*, care vor desăvîrşi opera lui Newton prin stabilirea teoremelor generale ale mecanicii;

— *Lagrange* şi *Laplace*, care vor crea mecanica analitică şi mecanica cerească;

— *Gauss*, care va elabora metode de determinare a orbitelor planetelor, cometelor etc. din observaţii astronomice;

— *Poisson, Hamilton, Jacobi, Le Verrier, Poincaré, Birkhoff, Liapunov, Sundman, Chazy, Hill, Brown, Delaunay* etc., care vor elabora noi metode matematice de rezolvare a problemelor de mecanică, elaborînd totodată şi teorii de mişcare a corpurilor.

Opera lui Newton va continua şi în secolele următoare, pînă în zilele noastre, cînd noi concepte şi doctrine matematice încep să pătrundă în mecanică; de la ele se aşteaptă rezolvarea unor probleme specifice mecanicii, rămase încă deschise.

Mecanica lui Newton a înregistrat succese epocale, printre care se cuvine a menţiona descoperirea numai prin calcul a unor noi planete, ca Neptun şi Pluto, dezvoltarea aviaţiei subsonice, transsonice şi supersonice, lansarea sateliţilor artificiali şi a navelor interplanetare (*Sputnik 1* — în U.R.S.S., la 4 octombrie 1957, *Explorer 1* — în S.U.A., la 1 februarie 1958), primul zbor al omului în cosmos (*Vostok 1* —12 aprilie 1961), aselenizarea omului (*Apollo 11* — iulie 1969) etc. Dar întreaga tehnică modernă este tributară mecanicii, care stă la baza tuturor disciplinelor inginereşti.

Se poate afirma cu certitudine că, fără cunoaşterea legilor de mişcare a planetelor şi sateliţilor, şi fără contribuţia tehnicii rachetelor, realizarea satelitului artificial al Pămîntului ar fi fost cu neputinţă.

Privită acum în perspectiva celor trei sute de ani de la crearea sa, opera fundamentală a lui Newton trezeşte — aşa cum arată acad. *Caius Iacob* — aceleaşi sentimente de admiraţie pe care le-a produs şi la contemporanii săi, matematizarea mecanicii servind şi continuînd să servească drept model pentru dezvoltarea şi a celorlalte ştiinţe fundamentale ale naturii, care urmează un proces de matematizare analog.

Universitatea noastră se poate mîndri — pe bună dreptate — cu faptul că traducerea în limba română a valoroasei opere a lui Newton a fost realizată în acest lăcaş al ştiinţei, învăţămîntului şi culturii, de un distins profesor, *Victor Marian*, fost şef de catedră în cadrul Facultăţii de Matematică şi Fizică, textul definitiv al traducerii fiind stabilit împreună cu profesorul *Victor Vâlcovici* de la Universitatea din Bucureşti; lucrarea a fost publicată la Editura Academiei R.P. Române, în anul 1956. Textul original după care s-a făcut traducerea este acela al ediţiei a III-a a *Principiilor*, din 1726 — ediţia cea mai completă şi apărută în timpul cît trăia încă Newton, avînd şi aprobarea lui. La sfîrşitul traducerii, profesorul Victor Marian a plasat o instructivă prezentare a vieţii şi activităţii lui Newton, însoţită de bibliografia cronologică a operelor savantului (pag. 421—441), precum şi *Adnotările* personale (pag. 443—483), care — în intenţia profesorului — „au drept scop să ajute pe cititor la înţelegerea chestiunilor fundamentale tratate, mai ales în ceea ce priveşte aspectul lor filozofic".

Această carte, cu principiile şi teoriile stabilite în ea, a adus — cum anticipa autorul ei — *multă lumină* studiului fenomenelor naturii, care s-a constituit într-un impozant edificiu numit astăzi *mecanică clasică* sau *mecanică newtoniană*, deschizînd totodată perspectivă „unui alt mod de a filozofa, mai adevărat". Într-adevăr, revoluţia produsă în fizică de *teoria relativităţii* s-a reflectat imediat şi în celelalte ştiinţe ale naturii, fie direct — în măsura în care intrau în discuţie categoriile fundamentale amintite mai înainte —, fie indirect — invitînd cercetătorii la a privi critic şi cu îndrăzneală orice concept şi teorie, oricît de justificată ar fi ea din punct de vedere intuitiv.

Dar, de fapt, nici teoria relativităţii, nici mecanica cuantică sau cea ondulatorie n-au înlăturat mecanica lui Newton, n-au contrazis-o, ci doar au completat-o. Pentru fenomenele macroscopice, fenomenele obişnuite din natură, cu care ne întilnim la fiecare pas, fizica lui Newton îşi păstrează întreaga valabilitate. Iată de ce opera lui Newton *Philosophiae Naturalis Principia Mathematica* este eternă şi nu-şi va pierde niciodată marea ei valoare.

RECENZII

Y. Choquet-Bruhat, B. Coll, R. Kerner, A. Lichnerowicz, **Geometrie et physique, Journees relativistes de Marseille-Luminy**, avril 1985, éd. Hermann, Collection Travaux en cours, Paris, 1987, 211 pages

Le volume édité par l'équipe de mécanique relativiste de l'Université Paris VI rassemble les travaux présentés au „Journées relativistes 1985" organisées sous le patronage de la Société Mathématique de France. Les contributions des participants à cette reunion s'exprime par 22 textes des conférences et communications faites autant sur les problèmes classiques de la Relativité générale que sur des nouveaux problèmes concernant les théories relativistes modèrnes, la théorie des jauges et les théories de supergravité. Quelques exposés relevent aussi des questions théoriques avancées de Géométrie différentielle et de Topologie algébrique posées par les nouvelles théories de la Rélativité.

M. ŢARINĂ

Javier Barja Perez, **Sucesiones exactes de Ocho terminos en Homologia de Grupos. II (G, Z)**, Universidad de Santiago de Compostela, Departamento de Algebra, Nr. 45, 1986

Prezenta monografie este al 45-lea volum din colecţia „Algebra", colecţie realizată de departamentul de Algebră al Universităţii din Santiago (Spania). În lucrare se studiază şirul exact de omologie, cu coeficienţii în Z, asociat unui epimorfism din categoria grupurilor. Recomandăm această monografie studenţilor şi cercetătorilor cu preocupări în Teoria categoriilor de Algebră omologică.

IOAN A. RUS

Julij A. Dubinskij, ***Sobolev Spaces of Infinite Order and Differential Equations***, Teubner Texte zur Mathematik, Band 87, Leipzig 1986, 161 pp.

For a complex valued infinitely differentiable function u defined on a region $G \subseteq R^n$, put

$$\rho(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha u\|_{r_\alpha}^{p_\alpha} \text{ where } a_\alpha \geqslant 0,\ p_\alpha \geqslant 1,$$

$r_\alpha \geqslant 1$ are given sequences and $\|\cdot\|_r$ denotes the Lebesgue norm in $L_r(G)$. The Sobolev space of infinite order $W^\infty(a_\alpha, p_\alpha)$ is formed of all functions u for which $\rho(u)$ is finite. In contrast to the finite order Sobolev spaces the first question which arises in the infinite order case is the nontriviality of the space. The nontriviality of the space $W^\infty(a_\alpha, p_\alpha)$ depends on the parameters a_α, p_α and on the region G (the values of r_α are immaterial in the question of nontriviality and so a not used in the notation of the corresponding spaces). In the first chapter the author solves this problem (giving necessary and sufficient conditions) in the cases of a bounded region, of all space R^n, of the tours and of a strip. The theory of Sobolev spaces of infinite order is closely related to the boundary problems for differential equations of infinite order — the nontriviality of the energy space $W^\infty(a_\alpha, p_\alpha)$ is, essentially, equivalent to the correctness of the corresponding boundary problem. Other topics treated in the book are: elliptic boundary value problems, trace theory and inhomogeneous Dirichlet problems, imbedding theory, nonstationary boundary value problems (all the considered boundary problems are of infinite order).

The book will be of interest for those doing research in Sobolev spaces and differential equations.

S. COBZAŞ

H. Triebel, **Analysis and Mathematical Physics**, BSB B. G. Teubner Verlagsgesellschaft, Leipzig 1986, 456 pp.

The book is based on a ten term course of lectures read by the author at the F. Schiller University, Jena on various topics in analysis and mathematical physics. The book is only a skeleton of these lectures since the proofs are largely omitted. The book is adressèd to mathematicians and physicists. The mathematicians will find the principles of classical and modern physics presented in a language familiar to them and physicists will find concise descriptions of the mathematical theories on which these principles are based on. As the author points out in the preface "True to Hilbert's ideal, mathematical theories are carefully separated from their physical interpretations". The mathematical part of the book contains: differential and integral calculus, measure theory and Lebesgue integration in R^n, complex function theory, elements of functional analysis in Banach and Hilbert spaces, ordinary and partial differential equations, calculus of variations, distribution theory, geometry in R^3 and geometry on manifolds, singularity theory and catastrophe theory. The physical theories presented in the book are: classical machanics, hydrodinamics, classical field theory, special relativity and electrodynamics, quantum mechanics, general theory of relativity. Some applications of catastrophe theory to biology and to other domains (e.g. there is a section suggestively entitled "Dogs and mathematicians") are also included.

The book will be very useful for all interested in mathematics and mathematical physics.

S. COBZAŞ

Ivo Marek, Karel Žitný, **Matrix Analysis for Applied Sciences**, Teubner Texte zur Mathematik, vol. I, Band 60, Leipzig 1983, vol. II, Band 84, Leipzig 1986, 196+149 pp.

The authors present the theory of matrices as representing linear operators acting on finite dimensional vector spaces. The connection of the matrix with the linear operators facilitates the application of the functional — analytical methods to the matrix calculus. The book is fairly self-contained — the reader is assumed only familiar with elementary matrix calculus (including determinants) and with fundaments of mathematical analysis. The first volume contains: algebraic preliminaries, metric spaces, elements of mathematical and functional analysis, spectral properties and Riesz operator calculus. The topics treated in the second volume are: operator with rational solvent functions, complexification, Jordan reprezentations, variational principles, Drazin's and Moore-Penrose's pseudoinverse operators.

By the use of rather deep concepts and results of functional analysis the proofs are presented in a most natural and transparent manner and the book links these to fields of research: functional analysis and linear algebra.

The book will be useful to all interested in matrix theory and its applications (e.g. in numerical methods, differentiall equations).

S. COBZAŞ

F. Klein, **Riemannsche Flächen**, Teubner Archiv zur Mathematik, Leipzig 1986, 284 pp.

The book contains a reprint of the originals lectures on Riemann surfaces taught by F. Klein in Göttingen in the winter semester 1891/92 and the summer semester 1892. The ideas of F. Klein of Riemann surface had and still have a great influence on various fields of investigation as theory of functions of one and several complex variables, algebraic geometry, number theory. The lectures are edited and commented by G. Eisenreich and W. Purkert. The book contains also a list of books and papers which are referred toto in the original text and also a list of contemporary books on Riemann surfaces and related topics. A sketch on the live and mathematical work of Felix Klein is also included.

S. COBZAŞ

Ch. Posthoff, D. Bochmann, K. Haubold, **Diskrete mathematik**, Teubner Verlag, Leipzig, 1986 (Mathematisch-Naturwissenschaftliche Bibliotek, Band 70), 248 pp. 74 fig.

A book for students interested in discret mathematics, with the following chapters: Logic and sets, Numbers and number systems, Algebraic structures, Binary functions, Theory of graphs, Applications of graphs, Theory of automata, a reference list of 55 titles. Well written, with a lot of examples and figures and with references to computer science, this book is greatly recommended to all beginners who are interested in these subjects.

Z. KÁSA

Reprezentation of Lie Groups and Lie Algebras, (A. A. Kirillov, editor) Akadémiai Kiadó, Budapest, 1985, 225 p.

The book presents the completed versions of the lectures for beginners of the 1971 Summer School organized by the János Bolyai Mathematical Society in Budapest. The volume covers the basis of representation theory as well as its most up-to-date results. The lectures are as follows: Introduction to the representation theory of finite and compact groups (A. A. Kirillov), Representations of contragradient Lie algebras, and the Kac-MacDonald identities (B. L. Feigin A. V. Zelevinsky), On Gelfand—Zetlin bases for classical Lie algebras (D. P. Zhelobenko), Representation of $SL(2, F_q)$ (S. Tanaka), Models of representations of current groups (I. M. Gelfand, M. I. Graev, A. M. Vershik), Unitary representations of the infinite symmetric group: a semigroup approach (G. I. Olshansky), On the applications of induced representations to quantum mechanics (G. W. Mackey)

A. B. NÉMETH

E. Griepentrop, R. Marz, **Differential—Algebraic Equations and Their Numerical Treatment**, Teubner-texte zur Mathematik, Band 88, Leipzig, 1986, 220 pp.

This monograph deals with the theoretical statements on differential-algebraic equations and their numerical treatment. The chapters of the book are the following: 1. Analysis of differential-algebraic equations; 2. Integration methods; 3. Difference methods for boundary value problems. We recommend this book to all interested in differential equations.

IOAN A. RUS

T. M. Rassias, **Foundations of Global Nonlinear Analysis**, Treubner-texte zur Mathematic, Leipzig, 1986, 218 pp.

The aim of the book is to present the theory of minimal surfaces and the problem of Plateau from the point of view of Morse—Palais—Smale critical point theory and Morse index theorem. We recommend this book to all interested in global nonlinear analysis and its applications in analysis, geometry and physics.

IOAN A. RUS

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