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ASUPRA TOPOLOGIEI ATAŞATE UNEI n-METRICI

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ABSTRACT. — On the Topology Attached to an n-Metrics. Some properties of the analysis situs attached to an n-metrics, defined in [2], are presented in this paper.

1. În 1928, K. Menger [5] defineşte spaţiile n-metrice, care pot fi privite ca analogul n-dimensional al spaţiilor metrice, fără a fi preocupat de topologizarea acestor spaţii. În 1963, S. Gähler [3] înzestrează un spaţiu 2-metric oarecare cu o structură topologică, intim legată de 2-metrică şi face un amplu studiu al proprietăţilor topologiei ataşate unei 2-metrici. Sugerează în acelaşi timp, posibilitatea eventuală a extinderii unora dintre rezultatele obţinute, la cazul general al spaţiilor n-metrice. Demn de semnalat este faptul că, încă în 1958, deci cu cinci ani mai devreme, matematicianul român Al. Froda, publică o notă intitulată „Espaces p-métrique et leur topologie" [2], în care asociază unei p-metrici oarecare o topologie, care coincide cu cea introdusă de S. Gähler în cazul $p = 2$, şi stabileşte anumite rezultate privind această topologie. Unele dintre acestea (de exemplu normalitatea unui spaţiu p-metric) nu le regăsim în studiul întreprins de S. Gähler, în cazul $p = 2$.

Lucrarea de faţă, prezintă unele proprietăţi ale topologiei ataşate unei n-metrici (n număr natural, $n \geqslant 2$).

2. Facem următoarele convenţii de notaţie: X fiind o mulţime arbitrară vom folosi simbolurile: $\mathfrak{L}(X)$, pentru mulţimea părţilor lui X, $\mathfrak{L}_0(X)$ pentru mulţimea părţilor finite, nevide ale mulţimii X, iar pentru i număr natural, $\mathfrak{L}^{(i)}(X)$ pentru mulţimea sistemelor (neordonate) cu i elemente (nu neapărat distincte) din X. Convenim de asemenea că dacă $M \in \mathfrak{L}^{(i)}(X)$ şi $N \in \mathfrak{L}^{(j)}(X)$ atunci $M \cup N \in \mathfrak{L}^{(i+j)}(X)$, iar cînd în plus $N \leqslant M$, să avem $M \setminus N \in \mathfrak{L}^{(i-j)}(X)$.

DEFINIŢIE (2.1) [2]. Fie n un număr natural dat ($n \geqslant 1$) şi X o mulţime arbitrară cu cel puţin $n + 1$ elemente. Prin n-metrică pe X înţelegem o aplicaţie $\rho : X^{n+1} \to \mathbf{R}$, satisfăcînd următoarele axiome (care extind axiomele clasice ale 1-metricii):

(ρ_1) $\rho(x_1, x_2, \ldots, x_{n+1})$ este invariantă faţă de orice transpoziţie (x_i, x_j), unde $(x_1, x_2, \ldots, x_{n+1}) \in X^{n+1}$;

(ρ_2) oricare ar fi $(x_1, \ldots, x_{n+1}) \in X^{n+1}$ şi $x \in X$

$$\rho(x_1, x_2, \ldots, x_{n+1}) \leqslant \rho(x_1, \ldots, x_n, x) + \rho(x_1, \ldots, x_{n-1}, x, x_{n+1}) + \\ + \ldots + \rho(x, x_2, \ldots, x_{n+1});$$

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(ρ_3) $\rho(x_1, \ldots, x_{n+1}) = 0$ dacă şi şi njumai dacă există

$i, j \in \{1, \ldots, h + 1\}$, astfel ca $i \neq j$ şi $x_i = x_j$.

$\rho(x_1, \ldots, x_{n+1})$ se numeşte n-distanţa dintre cele $n + 1$ puncte din X. În baza axiomei (ρ_1), ea este independentă de ordinea celor $n + 1$ puncte. Dacă $M \in \mathfrak{L}^{(n+1)}(X)$ deci dacă M este un sistem (neordonat de $n + 1$ puncte (distincte sau nu) din X, convenim să notăm cu $\rho(M)$ n-distanţa punctelor sistemului M. Dacă

$M = (x_i \mid i = \overline{1, n + 1}) \in \mathfrak{L}^{(n+1)}(X)$ (ordinea punctelor în sistemul M este indiferentă) şi $x \in X$, axioma (ρ_2) se poate scrie:

$$\rho(M) \leqslant \sum_{i=1}^{n+1} \rho((M \setminus \{x_i\}) \cup \{x\})$$

Se observă cu uşurinţă că $\rho(M) \geqslant 0$, pentru orice $M \in \mathfrak{L}^{(n+1)}(X)$. Cuplul (X, ρ). în care X este o mulţime şi ρ o n-metrică pe X, se numeşte spaţiu n-metric.

Într-un spaţiu n-metric se defineşte noţiunea de n-bilă în modul următor:

DEFINIŢIE (2.2) [2]. Fie $\rho : X^{n+1} \to \mathbf{R}_+$ o n-metrică pe X, $M \in \mathfrak{L}^{(n)}(X)$ şi $r > 0$. Numim n-bilă de n-centru M şi de rază r, mulţimea $B_\rho(M ; r) = \{x \in X \mid \rho(M \cup \{x\}) < r\}$; cînd nu este pericol de confuzie, vom folosi notaţia $B(M ; r)$, (fără a mai pune în evidenţă n-metrica ρ).

DEFINIŢIE (2.3) [2]. Topologia care admite ca subbază familia

$$\mathcal{S} = \{B\rho(M ; r) \mid M \in \mathfrak{L}^{(n)}(X),\ r > 0\} \subseteq \mathfrak{L}(X)$$

se notează cu $\mathcal{T}_\rho$ şi se numeşte topologia ataşată n-metricii ρ.

Observaţie (2.1). În cazul $n = 2$, topologia $\mathcal{T}_\rho$ coincide cu aşa numite topologie naturală ataşată de S. G ä h l e r [3], unei 2-metrici.

În cele ce urmează vom stabili unele proprietăţi ale acestei topologii.

3. Din definiţia (2.3) urmează că familia $\mathcal{B}$ a mulţimilor de forma $\bigcap_{i=1}^{p} B_\rho(M_i ; r_i)$ (intersecţie a unui număr finit de elemente din $\mathcal{S}$) este o bază a topologiei $\mathcal{T}_\rho$. Prin urmare familia tuturor mulţimilor de această formă (deci elemente ale bazei $\mathcal{B}$) care conţin un punct dat $x \in X$, formează o bază de vecinătăţi pentru punctul x. Teorema care urmează. indică o bază de vecinătăţi mult mai avantajoasă.

TEOREMA (3.1). *Fie (X, ρ) un spaţiu n-metric şi $x \in X$. Familia de părţi ale lui X.*

$$\mathcal{V}_x^0 = \{W_\Sigma(M; r) = \bigcap_{M \in \Sigma} B_\rho(M \cup \{x\} ; r) \mid \Sigma \in \mathfrak{L}_0(\mathfrak{L}^{(n-1)}(X)),\ r > 0\}$$

este o bază de vecinătăţi pentru x.

Demonstraţie. Orice element al mulţimii $\mathcal{V}_x^0$ este o intersecţie finită de n-globuri, ale căror n-centre conţin pe x, şi avînd toate o aceeaşi rază r. Să observăm că x aparţine oricărui n-glob, al cărui n-centru îl conţine şi cum orice

n-glob este o mulţime deschisă în topologia $\mathcal{T}_\rho$ (ca element al subbazei), urmează că orice element din $\mathcal{V}_x^0$ este o mulţime deschisă care conţine punctul x, deci o vecinătate a lui x. Notînd cu $\mathcal{V}_x$, sistemul tuturor vecinătăţilor lui x avem, prin urmare, $\mathcal{V}_x^0 \subseteq \mathcal{V}_x$. Notînd acum cu $\mathcal{V}_x^*$ sistemul elementelor bazei $\mathcal{B}$, care conţin punctul x, deci

$\mathcal{V}_x^* = \{V^* \in \mathcal{B} \mid x \in V^*\}$ rămîne să arătăm că $\forall V^* \in \mathcal{V}_x^* \exists V_0 \in \mathcal{V}_x^0 : V^0 \subseteq V^*$.

Fie $x \in V^* = \bigcap_{i=1}^{p} B(M_i ; r_i) \in \mathcal{B}$, unde $M_i \in \mathfrak{X}^{(n)}(X)$ şi $r_i > 0$, pentru $i \in \{1, 2, \ldots, p\}$. Urmează că $\rho(M_i \cup \{x\}) < r_i$, pentru $i \in \{1, 2, \ldots, p\}$. Fie $M_i = (x_1^i, x_2^i, \ldots x_n^i) \in \mathfrak{X}^{(n)}(X)$ şi notăm $\rho(M_i \cup \{x\}) = \varepsilon_i$; presupunem că $x \notin M_i$, deci $\varepsilon_i > 0$, pentru fiecare $i \in \{1, \ldots, p\}$. Notăm acum $\Sigma_i = \{M_i \setminus \{x_j^i\} \mid j = \overline{1, n}\}$ şi fie

$$W_{\Sigma_i}\left(x ; \frac{\varepsilon_i}{n+1}\right) = \bigcap_{j=1}^{n} B\left((M_i \setminus \{x_j^i\}) \cup \{x\}) ; \frac{\varepsilon_i}{n+1}\right)$$

Demonstrăm că $W_{\Sigma_i}\left(x ; \frac{\varepsilon_i}{n+1}\right) \subseteq B(M_i ; r_i)$, pentru $i = \overline{1, p}$.

Fie $x^* \in W_{\Sigma_i}\left(x ; \frac{\varepsilon_i}{n+1}\right)$; avem atunci $\rho((M_i \setminus \{x_j^i\}) \cup \{x, x^*\}) < \frac{\varepsilon_i}{n+1}$ pentru orice $j \in \{1, \ldots, n\}$. Conform axiomei (ρ_2), avem:

$$\rho(M_i \cup \{x^*\}) = \rho(x_1^i, x_2^i, \ldots, x_n^i, x^*) \leqslant \rho(x_1^i, \ldots, x_n^i, x) +$$
$$+ \ldots + \rho(x, x_2^i, \ldots, x_n^i, x^*) < \frac{\varepsilon_i}{n+1}(n+1) = \varepsilon_i < r_i ;\text{ prin urmare}$$

$x^* \in B(M_i ; r_i)$. Considerăm acum $\Sigma = \bigcup_{i=1}^{p} \Sigma_i \in \mathfrak{X}_0(\mathfrak{X}^{(n-1)}(X))$. Cu notaţiile adoptate putem scrie, în definitiv,

$$W_\Sigma(x ; r) = \bigcap_{i=1}^{p} W_{\Sigma_i}(x ; r), \text{ unde } r = \min\left\{\frac{\varepsilon_i}{n+1} \,\middle|\, i = \overline{1, p}\right\}.$$

Dacă pentru un anumit $i \in \{1, \ldots, p\}$, $x \in M_i$, atunci in consideraţiile de mai sus, în locul mulţimii $W_{\Sigma_i}\left(x ; \frac{\varepsilon_i}{n+1}\right)$ se ia mulţimea $B(M_i ; r_i)$ (adică $\Sigma_i = M_i \setminus \{x\}$, şi în loc de $\frac{\varepsilon_i}{n+1}$ (care este 0) se ia r_i). În final, putem scrie:

$$x \in W_\Sigma(x ; r) = \bigcap_{i=1}^{p} W_{\Sigma_i}(x ; r) \subseteq \bigcap_{i=1}^{p} W_{\Sigma_i}\left(x ; \frac{\varepsilon_i}{n+1}\right) \subseteq \bigcap_{i=1}^{p} B(M_i, r_i) = V^*.$$

Cu aceasta teorema este demonstrată.

Observaţie (3.1). Sistemul de părţi ale lui X

$$\left\{W_\Sigma\left(x;\ \frac{1}{k}\right) = \bigcap_{M\in\Sigma} B\left(M \cup \{x\};\ \frac{1}{k}\right) \mid \Sigma \in \mathfrak{L}_0(\mathfrak{L}^{(n-1)}(X)),\ k \in \mathbb{N}\right\}$$

este de asemenea o bază de vecinătăţi pentru x. Evident, în general nu va fi vorba despre o bază de vecinătăţi numărabilă. Privind existenţa unei baze numerabile de vecinătăţi pentru un punct se poate extinde la cazul n-metricii, un rezultat stabllit de S. G ä h l e r [3], în cazul particular $n = 2$.

În prealabil introducem pentru un spaţiu n-metric Proprietatea (A).

DEFINIŢIE (3.1) Spunem că spaţiul n-metric (X, ρ) posedă Proprietatea (A) într-un punct $x \in X$, relativ la o mulţime $M \in \mathfrak{L}^{(n)}(X\setminus\{x\})$, dacă oricare ar fi o secvenţă $(x_i)_{i\in\mathbb{N}}$ de puncte din X, are loc implicaţia:

$$\forall y \in M : \rho((M\setminus\{y\}) \cup \{x,\ x_i\}) \to 0 \text{ pentru } i \to \infty \Rightarrow$$

$$\Rightarrow \forall M^* \in \mathfrak{L}^{(n-1)}(X) : \rho(M^* \cup \{x,\ x_i\}) \to 0 \text{ pentru } i \to \infty.$$

TEOREMA (3.2). *Fie* (X, ρ) *un spaţiu n-metric,* $x \in X$ *şi* $M = (\mathfrak{z}_l \mid l = \overline{1, n}) \in \mathfrak{L}^{(n)}(X \setminus \{x\})$. *Atunci familia*

$$\left\{W_k(x) = \bigcap_{l=1}^{n} B\left((M \setminus \{y_l\}) \cup \{x\};\ \frac{1}{k}\right) \mid k \in \mathbb{N}\right\}$$

este o bază (evident numărabilă) de vecinătăţi pentru punctul x, dacă şi numai dacă, spaţiul (X, ρ) *posedă proprietatea (A) în x, relativ la mulţimea M.*

Demonstraţie. (a). Necesitatea. Presupunem, prin absurd că $\{W_k(x) \mid k\in\mathbb{N}\}$ este o bază de vecinătăţi în punctul x şi că (X, ρ) nu posedă proprietatea (A) în x, relativ la M. Există atunci o secvenţă $(x_i)_{i\in\mathbb{N}}$ cu proprietatea că, oricare ar fi $y \in M$, $\rho((M \setminus \{y\}) \cup \{x,\ x_i\}) \to 0$, pentru $i \to \infty$ şi există $M^* \in \mathfrak{L}^{(n-1)}(X)$ cu $\rho(M^* \cup \{x,\ x_i\}) \not\to 0$ pentru $i \to \infty$; prin urmare, există $\varepsilon > 0$, astfel ca, oricare ar fi $i \in \mathbb{N}$ să existe $i' \in \mathbb{N}$ cu $i' > i$ şi $\rho(M^* \cup \{x,\ x_{i'}\}) \geq \varepsilon$. Mulţimea $B(M^* \cup \{x\};\ \varepsilon)$ este o vecinătate a punctului x. În baza ipotezei făcute, există $k_0 \in \mathbb{N}$, astfel ca $W_{k_0}(x) \subseteq B(M^* \cup \{x\};\ \varepsilon)$, deci

$\bigcap_{l=1}^{n} B\left((M \setminus \{y_l\}) \cup \{x\};\ \frac{1}{k_0}\right) \subseteq B(M^*\cup\{x\};\ \varepsilon)$. Cum am presupus că $\rho((M\setminus\{y\})\cup \{x,\ x_i\}) \to 0$, pentru $i \to \infty$ şi y arbitrar din M, iar mulţimea M este finită, urmează că, pentru $K_0 \in \mathbb{N}$ pus în evidenţă mai sus, există $i_0 \in \mathbb{N}$ astfel ca $i \geq i_0$ să implice $\rho((M \setminus \{y\}) \cup \{x,\ x_i\}) < \frac{1}{k_0}$, oricare ar fi $y \in M$. Prin urmare $x_i \in W_{k_0}(x) \subseteq B(M^* \cup \{x\};\ \varepsilon)$, pentru $i \geq i_0$, în contradicţie cu cele stabilite anterior, că pentru orice $i_0 \in \mathbb{N}$ există $i_0' \in \mathbb{N}$ cu $i_0' > i_0$ şi $\rho(M^* \cup \{x,\ x_{i_0'}\}) \geq \varepsilon$.

(b) Suficienţa. Raţionăm din nou prin reducere la absurd. Admitem deci că (X, ρ) posedă proprietatea (A) pentru X, relativ la $M = (\mathfrak{z}_l \mid l = \overline{1, n})$ şi că sistemul $\{W_k(x) \mid k \in \mathbb{N}\}$ nu formează o bază de vecinătăţi pentru x. Există atunci, ţinînd seama de teorema (3.1), un $\Sigma = \{M_1,\ M_2,\ \ldots,\ M_t\} \in \mathfrak{L}_0(\mathfrak{L}^{(n-1)}(X))$

(deci $M_j \in \mathfrak{A}^{(n-1)}(X)$ pentru $j = \overline{1, l}$ şi un $\varepsilon > 0$, astfel ca $W_\Sigma(x; \varepsilon) = = \bigcap_{j=1}^{l} B(M_j \cup \{x\}; \varepsilon) \not\supset W_k(x)$, oricare ar fi $k \in \mathbb{N}$. Urmează că pentru orice $k \in \mathbb{N}$, există $a_k \in X$, cu $a_k \in W_k(x)$ şi $a_k \notin W_\Sigma(x; \varepsilon)$, adică $\rho((M \setminus \{y\}) \cup \{x, a_k\}) < \frac{1}{k}$ pentru fiecare $y \in M$ şi există $j \in (1, 2, \ldots, l\}$ astfel ca $\rho(M_j \cup \{x, a_k\} \geqslant \varepsilon$. Prin urmare, fiecărui $k \in \mathbb{N}$ îi corespunde un $j \in \{1, 2, \ldots, l\}$, astfel încît condiţia de mai sus să fie satisfăcută. Cum j poate să ia numai un număr finit de valori, iar k parcurge mulţimea numerelor naturale, deducem că există cel puţin un $j \in \{1, 2, \ldots, l\}$ care corespunde unei infinităţi de termeni ai şirului $(a_k)_{k \in \mathbb{N}}$; prin urmare, există $j \in \{1, \ldots, l\}$ şi există subşirul $(a_{k_i})_{k \in \mathbb{N}}$ al şirului $(a_k)_{k \in \mathbb{N}}$, astfel încît $\rho(M_j \cup \{x, a_{k_i}\}) \geqslant \varepsilon$, pentru orice $i \in \mathbb{N}$. Punînd acum $a_{k_i} = x_i$ pentru $i \in \mathbb{N}$, avem în definitiv $\lim_{i\to\infty} \rho(M \setminus \{y\}) \cup \{x_i, x\}) = 0$, pentru fiecare $y \in M$ şi $\rho(M_j \cup \{x, x_i\}) \geqslant \varepsilon$, pentru orice $i \in \mathbb{N}$, în contradicţie cu ipoteza că spaţiul posedă proprietatea (A) pentru x, relativ la M.

Observaţie. (3.2). Dacă ρ este o n-metrică pe X şi pentru fiecare punct $x \in X$ există $M \in \mathfrak{A}^{(n)}(X \setminus \{x\})$, astfel încît condiţia (A) pentru x, relativ la M, să fie satisfăcută atunci spaţiul (X, ρ) satisface prima axiomă de numărabilitate.

TEOREMA (3.3). *Fie (X, ρ) un spaţiu n-metric şi $M \in \mathfrak{A}^{(n)}(X)$ fixat. Atunci funcţia $f: X \to \mathbb{R}$, definită prin $f(x) = \rho(M \cup \{x\})$, pentru orice $x \in X$, este continuă pe X (relativ la topologia $\mathfrak{T}_\rho$ pe X şi la cea naturală pe $\mathbb{R}$).*

Demonstraţie. Fie $M = (y_i \mid i = 1, n) \in \mathfrak{A}^{(n)}(X)$, $\varepsilon > 0$ şi $x \in X$. În baza axiomei (ρ_2) putem scrie:

$$f(x) = \rho(M \cup \{x\}) \leqslant \rho(M \cup \{x'\}) + \sum_{i=1}^{n} \rho((M \setminus \{y_i\}) \cup \{x, x'\})$$

pentru orice $x' \in X$. Pentru $\varepsilon > 0$ ales, să considerăm acum vecinătatea punctului x.

$$W(x) = \bigcap_{i=1}^{n} B\left((M \setminus \{y_i\}) \cup \{x\}; \frac{\varepsilon}{n}\right) \text{ şi fie } x' \in W(x)$$

Avem atunci $\rho(M \setminus \{y_i\}) \cup \{x, x'\}) < \frac{\varepsilon}{n}$ pentru $i \in \{1, 2, \ldots, n\}$ şi conform celor precedente $f(x) < f'(x') + n \cdot \frac{\varepsilon}{n}$, deci $f(x) - f(x') < \varepsilon$. Pentru $x' \in W(x)$, avem de asemenea $f(x') = \rho(M \cup \{x'\}) \leqslant \rho(M \cup \{x\}) + \sum_{i=1}^{n} \rho((M \setminus \{y_i\}) \cup \{x, x'\}) < < f(x) + \varepsilon$, adică $f(x') - f(x) < \varepsilon$.

În definitiv, am arătat că fiecărui $\varepsilon > 0$, îi corespunde o vecinătate a punctului x, încît pentru orice x' din această vecinătate să avem $|f(x) - f(x')| < \varepsilon$, adică f este continuă în punctul x.

Observaţie (3.3). Teorema (3.3) stabileşte în esenţă continuitatea parţială a funcţiei $\rho : X^{n+1} \to \mathbb{R}$, în raport cu fiecare dintre cele $n + 1$ variabile. În general a n-metrică nu este continuă în raport cu ansamblul variabilelor, nici măcar în cazul $n = 2$. (S. Gähler a dat exemple de 2-metrici care nu sînt continue în raport cu ansamblul variabilelor).

4. Dacă $\rho : X^{n+1} \to \mathbb{R}_+$ este o n-metrică pe X ($\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geqslant 0\}$) şi $f : \mathbb{R}_+ \to \mathbb{R}_+$ se poate pune problema în ce condiţii funcţia $f \circ \rho : X^{n+1} \to \mathbb{R}_+$ este de asemenea o n-metrică. În această ordine de idei teoremele care urmează oferă condiţii suficiente. Ele extind rezultate cunoscute în cazul 1-metricii.

TEOREMA (4.1). *Fie (X, ρ) un spaţiu n-metric şi $f : \mathbb{R}_+ \to \mathbb{R}_+$ cu proprietăţile:*

(i) *f funcţie nedescrescătoare;*

(ii) $f(a) = 0 \Leftrightarrow a = 0$

(iii) $\{a_i \mid_i = \overline{1, n+1}\} \subseteq \mathbb{R}_+ \Rightarrow f\left(\sum_{i=1}^{n+1} a_i\right) \leqslant \sum_{i=1}^{n+1} f(a_i)$

Atunci funcţia $\varphi = f \circ \rho : X^{n+1} \to \mathbb{R}_+$ este o n-metrică pe X.

Demonstraţie. Se verifică cu uşurinţă axiomele n-metricii pentru funcţia φ.

Fie $(x_1, \ldots, x_{n+1}) \in X^{n+1}$. Avem $\varphi(x_1, \ldots, x_{n+1}) = f[\rho(x_1, \ldots, x_{n+1})]$ şi cum $\rho(x_1, \ldots, x_{n+1})$ este invariantă faţă de orice transpoziţie (x_i, x_j), urmează că φ satisface axiome (ρ_1). Fie acum $M = (x_i \mid i = \overline{1, n+1}) \in \mathfrak{L}^{(n+1)}(X)$. Avem $\rho(M) \leqslant \sum_{i=1}^{n+1} \rho(M \setminus \{x_i\} \cup \{x\})$ pentru orice $x \in X$. Folosind (i) şi (iii) obţinem:

$$\varphi(M) = f[\rho(M)] \leqslant f\left[\sum_{i=1}^{n+1} \rho((M \setminus \{x_i\}) \cup \{x\})\right] \leqslant$$

$$\leqslant \sum_{i=1}^{n+1} f[\rho(M \setminus \{x_i\}) \cup \{x\})] = \sum_{i=1}^{n+1} \rho((M \setminus \{x_i\}) \cup \{x\})$$

Prin urmare φ satisface şi axioma (ρ_2).
În sfîrşit, $\varphi(x_1, \ldots, x_{n+1}) = 0 \Leftrightarrow f[\rho(x_1, \ldots, x_{n+1})] = 0 \Leftrightarrow \rho(x_1, x_2, \ldots, x_{n+1}) = 0 \Leftrightarrow$ $\Leftrightarrow \exists i, j \in \{1, \ldots, (n+1)\}$ cu $i \neq j$ şi $x_i = x_j$.
Funcţia φ satisfăcînd deci şi (ρ_3) este o n-metrică.

TEOREMA (4.2). *Dacă (X, ρ) este un spaţiu n-metric şi $f : \mathbb{R}_+ \Rightarrow \mathbb{R}_+$ cu proprietăţile:*

(i) $f(0) = 0$;

(ii) $\exists \alpha \in \mathbb{R}_+ \setminus \{0\}\ \forall t \in \mathbb{R}_+ \setminus \{0\} : f(t) \in [\alpha, (n+1)\alpha]$

atunci $\varphi = f \circ \rho : X^{n+1} \to \mathbb{R}_+$ este o n-metrică.

Demonstraţie. Axioma (ρ_1) se verifică întocmai ca mai sus. Fie acum $(x_1, \ldots, x_{n+1}) \in X^{n+1}$. Dacă există $i, j \in \{1, 2, \ldots, n+1\}$ cu $i \neq j$ şi $x_i = x_j$ avem $\rho(x_1, \ldots, x_{n+1}) = 0$ şi în baza condiţiei (i) $\varphi(x_1, \ldots, x_{n+1}) = f(0) = 0$.

Pe de altă parte, cum în baza condiţiei (ii) există un $\alpha > 0$ astfel ca $f(t) \geqslant \alpha$, pentru orice $t > 0$, deducem că
$\varphi(x_1, \ldots, x_{n+1}) = 0 \Leftrightarrow f(\rho(x_1, \ldots, x_{n+1})) = 0 \Rightarrow \rho(x_1, \ldots, x_{n+1}) = 0 \Rightarrow$
$\Rightarrow \exists i, j \in \{1, \ldots, n+1\}$ cu $i \neq j$ şi $x_i = x_j$. Funcţia φ verifică aşa dar şi axioma (ρ_3).

Fie acum $M = (x_1, \ldots, x_{n+1}) \in \mathfrak{X}^{(n+1)}(X)$. Dacă nu toate punctele x_i sînt distincte, atunci $\varphi(M) = 0$ şi inegalitatea care intervine în (ρ_2) este evident satisfăcută. Dacă $x \in X$ coincide cu unul dintre punctele x_i, în (ρ_2) este evident verificată egalitatea. Să presupunem acum că $\rho(x_1, \ldots, x_{n+1}) > 0$ (deci punctele x_i sînt distincte două cîte două) şi că $x \neq x_i$ pentru $i \in \{1, \ldots, n+1\}$. În baza condiţiei (ii) există $\alpha > 0$ astfel ca oricare ar fi $t > 0$ să avem $f(t) \in [\alpha, (n+1)\alpha]$; putem deci scrie:

$$\varphi(M) = f[\rho(M)] \leqslant (n+1)\alpha = \overbrace{\alpha + \alpha + \ldots + \alpha}^{n+1 \text{ termeni}} \leqslant$$

$$\leqslant \sum_{i=1}^{n+1} f[\rho((M \setminus \{x_i\}) \cup \{x\})] = \sum_{i=1}^{n+1} \rho((M \setminus \{x_i\}) \cup \{x\})$$

Prin urmare φ satisface şi axioma (ρ_2).

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CONSIDERAŢII PROBABILISTICE PRIVIND FUNCŢIILE B-SPLINE

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ABSTRACT. — Probabilistic Considerations on B-Spline Functions. In this note some probabilistic aspects on B-spline functions are presented, taking into account that these can be considered as probability densities for some random variables. Computing formulas of the initial moments for these random variables are given. If the used knots from the definition of the B-spline function are equidistant the generating function of the moments is obtained, and using this function the moments are computed in the case when the knots are symmetrical in respect to origin.

1. *Funcţii B-spline.* Se consideră nodurile $x_1 \leqslant \ldots \leqslant x_{n+1}$, cu proprietatea că $x_1 < x_{n+1}$.

Se numeşte funcţie B-spline de ordin n, relativă la nodurile $x_1, \ldots, x_{n+1}$, funcţia definită cu diferenţa divizată de ordinul n

$$B_n(x) = [x_1, \ldots, x_{n+1}; \; n(t-x)_+^{n-1}], \; x \in \mathbb{R},$$

unde s-a folosit notaţia $u_+ = \max(u, 0)$.

Dintre proprietăţile remarcabile ale funcţiilor B-spline amintim:

(i) $B_n(x) \geqslant 0$, cu egalitatea dacă $x \notin (x_1, x_{n+1})$,

(ii) $\displaystyle\int_{x_1}^{x_{n+1}} B_n(x)dx = 1$,

(iii) $\displaystyle [x_1, \ldots, x_{n+1}; f] = \frac{1}{n!}\int_{x_1}^{x_{n+1}} B_n(x) f^{(n)}(x)dx$,

dacă $f \in C^n[x_1, x_{n+1}]$.

2. *Momentele unei funcţii B-spline.* Pe baza proprietăţilor (i) şi (ii) rezultă că o funcţie B-spline poate fi considerată ca funcţia densitate de probabilitate pentru o anumită variabilă aleatoare X.

Momentele iniţiale, ν_k, $k \in \mathbb{N}$, corespunzătoare densităţii de probabilitate $B_n(x)$ se pot calcula cu ajutorul formulei

$$\nu_k = \binom{n+k}{k}^{-1} \sum_{k_1+\ldots+k_{n+1}=k} x_1^{k_1} \ldots x_{n+1}^{k_{n+1}}. \tag{1}$$

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Formula (1) rezultă din proprietatea (iii) dacă se consideră funcţia $f(x) = x^{n+k}$.

Din formula (1) se obţine că momentul de ordinul unu sau valoarea medie are expresia

$$\nu_1 = \frac{x_1 + \ldots + x_{n+1}}{n+1}, \tag{2}$$

iar momentul de ordinul doi se poate scrie sub forma

$$\nu_2 = \frac{2}{(n+1)(n+2)} \sum_{\substack{j,k=1 \\ j \leq k}}^{n+1} x_j x_k. \tag{3}$$

Folosind relaţiile (2) şi (3) rezultă dispersia

$$\sigma^2 = \frac{1}{(n+1)^2(n+2)} \sum_{j<k} (x_j - x_k)^2.$$

Urmînd o altă metodă, valoarea medie şi dispersia au fost calculate în [3].

3. *Funcţia generatoare a momentelor în cazul nodurilor echidistante.* Fie nodurile funcţiei B-spline echidistante, adică $x_k = a + (k-1)h$, $k = \overline{1, n+1}$, unde $h = (x_{n+1} - x_1)/n$.

În formula care exprimă proprietatea (iii) se alege $f(x) = e^{tx}$ şi se obţine

$$[x_1, \ldots, x_{n+k}; e^{tx}] = \frac{t^n}{n!} \int_{-\infty}^{+\infty} B_n(x) e^{tx} dx.$$

Integrala din membrul drept reprezintă funcţia generatoare a momentelor, iar diferenţa divizată din membrul stîng se exprimă succesiv

$$[x_1, \ldots, x_{n+1}; e^{tx}] = \frac{\Delta_h^n e^{tx}|_{x=a}}{h^n \cdot n!} = \frac{e^{at}(e^{ht} - 1)^n}{h^n \cdot n!}. \tag{4}$$

Astfel funcţia generatoare a momentelor are expresia

$$g_n(t) = \frac{e^{at}(e^{ht} - 1)^n}{(ht)^n}. \tag{5}$$

Să considerăm în continuare că nodurile sînt echidistante şi simetrice faţă de origine.

Dacă originea se află printre noduri, numărul acestora este impar, deci $n = 2p$, iar $a = -ph$. Din relaţia (5) se obţine

$$g_{2p}(t) = \left(\frac{\mathrm{sh}\, \frac{ht}{2}}{\frac{ht}{2}} \right)^{2p}. \tag{6}$$

Cînd originea nu se află printre noduri, numărul acestora este par, adică $n = 2p - 1$, iar $a = -(2p - 1)h/2$. De asemenea din (5) rezultă

$$g_{2p-1}(t) = \left(\frac{\operatorname{sh}\frac{ht}{2}}{\frac{ht}{2}}\right)^{2p-1} \tag{7}$$

4. *Calculul momentelor cu ajutorul funcției generatoare.* Folosind formulele (6) și (7) vom da expresii pentru momentele funcțiilor B-spline în cazul nodurilor echidistante și simetrice față de origine.

Variabila aleatoare X care are funcția generatoare de la (6) se poate scrie ca suma a p variabile aleatoare independente $X_1, \ldots, X_p$, fiecare avînd aceeași funcție generatoare a momentelor

$$g(t) = \left(\frac{\operatorname{sh}\frac{ht}{2}}{\frac{ht}{2}}\right)^2, \tag{8}$$

care corespunde legii de probabilitate a lui Simpson pe intervalul $[-h, h]$.

Folosind funcția generatoare de la (8) se obține că

$$M(X_i^k) = \begin{cases} 0, \text{ pentru } k \text{ impar,} \\ \dfrac{2h^k}{(k+1)(k+2)}, \text{ pentru } k \text{ par, } i = \overline{1, p}. \end{cases} \tag{9}$$

Pe de altă parte se poate scrie

$$\nu_k = M[(X_1 + \ldots + X_p)^k] = \sum_{k_1 + \ldots + k_p = k} \frac{k!}{k_1! \ldots k_p!} M(X_1^{k_1}) \ldots M(X_p^{k_p}).$$

Pentru k impar există cel puțin un termen în suma $k_1 + \ldots + k_p = k$, care să fie impar. Avînd în vedere relațiile de la (9) rezultă că în acest caz momentul inițial de ordin k este zero.

Cînd k este par $(k = 2r)$ se obține pe baza relațiilor (9)

$$\nu_{2r} = \sum_{\substack{k_1+\ldots+k_p=2r \\ k_j \text{ par}}} \frac{(2r)!}{k_1! \ldots k_p!} \cdot \frac{2^p h^{2r}}{(k_1+1) \ldots (k_p+1)(k_1+2) \ldots (k_p+2)}.$$

Dacă se face notația $k_j = 2r_j$ atunci

$$\nu_{2r} = \sum_{r_1+\ldots+r_p=r} \frac{(2r)!\,h^{2r}}{(r_1+1) \ldots (r_p+1)(2r_1+1)! \ldots (2r_p+1)!}. \tag{10}$$

Funcţia generatoare de la (7) este puterea $2p-1$ a funcţiei generatoare

$$g(t) = \frac{\operatorname{sh}\frac{ht}{2}}{\frac{ht}{2}},$$

ce corespunde unei variabile aleatoare ce urmează legea uniformă pe intervalul $\left[-\frac{h}{2}, \frac{h}{2}\right]$.

Aşadar variabila aleatoare X se poate scrie ca suma variabilelor aleatoare independente $X_1, \ldots, X_{2p-1}$, fiecare urmînd legea uniformă pe intervalul $\left[-\frac{h}{2}, \frac{h}{2}\right]$.

Se ştie că momentele acestor variabile aleatoare se calculează cu formula

$$M(X_i^k) = \begin{cases} 0, \text{ pentru } k \text{ impar}, \\ \frac{1}{k+1}\left(\frac{h}{2}\right)^k, \text{ pentru } k \text{ par}, \ i = \overline{1, 2p-1}. \end{cases} \tag{11}$$

Ca mai înainte se obţine că $\nu_k = 0$ pentru k impar, iar pentru k par ($k = 2r$) avem

$$\nu_{2r} = \sum_{k_1+\ldots+k_{2p-1}=2r} \frac{(2r)!}{k_1! \ldots k_{2p-1}!} M(X_1^{k_1}) \ldots M(X_{2p-1}^{k_{2p-1}}).$$

Dacă se ţine seama de (11) rezultă

$$\nu_{2r} = \left(\frac{h}{2}\right)^{2r} \sum_{r_1+\ldots+r_{2p-1}=r} \frac{(2r)!}{(2r_1+1)! \ldots (2r_{2p-1}+1)!}.$$

Faptul că momentele de ordin impar sînt zero, cînd nodurile sînt echidistante şi simetrice faţă de origine, rezultă şi din proprietatea funcţiilor *B*-spline de a fi funcţii pare, atunci cînd nodurile sînt simetrice faţă de origine.

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ON SOME INVERSION-LIKE TRANSFORMATIONS OF DEGENERATE HYPERBOLIC PLANE

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ABSTRACT. — In the paper we define reflections in equidistant curves of degenerate hyperbolic plane and describe analytica and geometrica groups generated by them.

Introduction. In this paper we introduce and study some transformation of the degenerate hyperbolic plane i.e. of affine halfplane. One can consider them as an analogue of (Euclidean) inversions in the sense that they transform the set of all equidistant curves onto itself — but they may not transform lines onto lines. Investigations presented here have two origins. Degenerate hyperbolic planes are in many points similar to usual hyperbolic planes (comp. [5], [3]). In [4] it was proposed to study reflections in equidistant curves on hyperbolic planes (similar constructions were made in [3], though with completely different ideas). It is natural to ask about analogous constructions on degenerate hyperbolic plane; the resulting transformations and groups generated by them are considered in the paper.

Secondly, in [3] there were some incidence structures discussed, among them there was a system consisting of all equidistant curves. It turns out that equidistant reflections generate a collineation group of this incidence system.

The degenerate hyperbolic plane and symmetrics of this plane Let $F = \langle F, +, \cdot, 0, 1, \langle\, \rangle$ be an ordered field. Further we shall require F is Euclidean. We denote by $A(F) = \langle F, L' \rangle$ the affine plane with lines coordinatized by F and we denote by $P(F) = \langle P, L'' \rangle$ the projective plane over F such that $A(F) \subseteq P(F)$. Let us denote by M_1 the improper („infinite") line in $P(F)$, $M_0 = \{\langle x, y\rangle : x = 0\}$, let ω be the direction of M_0. Next we put $W = \{\langle x, y\rangle \in F^2 : x > 0\}$. Denote by L the set $L = \{K \cap W : K \in L' \wedge K \cap W \neq \emptyset\}$. Finally we define $B(F) = \langle W, L\rangle$ and we call $B(F)$ a degenerate hyperbolic plane over F. Elements of L are called lines of $B(F)$. Clearly every line $K \in L$ can be extended to a unique line $K'' \in L''$. For every $a, b \in P$, $a \neq b$ there is a unique line, denoted by $\overline{ab}$, joining a, b; and the same holds in $B(F)$. If $K \in L$, $\omega \in K''$ then K will be called isotropic.

$$I = \{K \in L : \omega \in K''\} = \{K \in L : K' \subseteq W\}.$$

Let H denote the relation of harmonic coniugacy. Denote by $C(B)$ the group of collineations of $B(F)$. If $\alpha \in C(B)$ then α can be uniquely extended to a collineation α' of P. If $\beta \in C(P)$, $\beta(W) = W$ then $\beta | EW \in C(B)$. From the

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above $\alpha \in C(B)$ is involutive iff α' is an involution. Any $\alpha \in C(B)$ satisfies: $\alpha(\omega) = \omega$, $\alpha'(\{M_0, M_1\}) = \{M_0, M_1\}$. Therefore every symmetry in $C(B)$ must be a restriction of some harmonic homology σ_K^a with axis K and centre a, and any such a symmetry of degenerate hyperbolic plane must belong to one of the following three families:

$\Pi(B) = \Pi = \{\sigma_K^a : a \in W \wedge H(M_0, M_1; K, \overline{a\omega})\}$ — central symmetries,

(If $\sigma_K^a \in \Pi$ then K is determined by a, we simply write $\sigma_a \overset{\bullet}{=} \sigma_K^a$);

$$\left.\begin{array}{l}\Sigma(B) = \Sigma = \{\sigma_K^\omega : K \in L \setminus I\},\\ \Gamma(B) = \Gamma = \{\sigma_K^a : K \in I \wedge H(M_0, M_1; K, \overline{a\omega})\}\end{array}\right\} \text{ — line symmetries.}$$

These symmetries are described analytically by the following formulas.

THEOREM 1. *If* $\sigma \in \Pi$ *then* $\sigma(x, y) = \left(\frac{c^2}{x}, \frac{(y - tx - b)(-c) + tc^2 + bx}{x}\right)$ *where* $t, b \in F$, $c \in F^+$.

If $\sigma \in \Sigma$ *then* $\sigma(x, y) = (x, 2(tx + b) - y)$ *where* $t, b \in F$.

If $\sigma \in \Gamma$ *then* $\sigma(x, y) = \left(\frac{c^2}{x}, \frac{(y - tx - b)c + tc^2 + bx}{x}\right)$ *where* $t, b \in F$, $c \in F^+$.

Below we shall introduce some notations. If G is a family of transformations of a set X, $a \in X$ then $G[a] := \{f(a) : f \in G\}$.
For $\alpha, \beta \in C(B)$, let $\alpha | \beta$ denote $\alpha\beta = \beta\alpha \wedge \alpha \neq \beta$. We recall that $\sigma_A^a | \sigma_B^b \Leftrightarrow$ $\Leftrightarrow a \in B \wedge b \in A$.

FACT. 2. $\alpha | \beta \wedge \alpha, \beta \in \Sigma \cup \Gamma \Rightarrow (\alpha \in \Sigma \Leftrightarrow \beta \in \Gamma)$

DEFINITION 3. $Z(\alpha) := \{\beta : \beta | \alpha \wedge \beta \in \Sigma \cup \Gamma\}$

COROLLARY 4. $\alpha \in \Gamma \Rightarrow Z(\alpha) \subseteq \Sigma$, $\alpha \in \Sigma \Rightarrow Z(\alpha) \subseteq \Gamma$.

DEFINITION 5. $E_K[a] := Z(\sigma_K^\omega)[a]$.

We call $E_K[a]$ an equidistant curve of degenerate hyperbolic plane. We call K an axis (or a base) of equidistant curve $E_K[a]$. Let $E = \{E_K[a]! : K \in$ $\in L \setminus I$, $a \in W\}$.
Trivially we get

THEOREM 6. *If* $K \in L \setminus I$, $a \in K$ *then* $E_K[a] = K$.

THEOREM 7. *If* $K \in L \setminus I$, $a \notin K$, $a \in W$ *then* $E_K[a]$ *is this branch of some parabola* $P \subseteq W$ *tangent to line* M_0, *with axis* K, *which passes through* a.

Proof. Let K_1 be a line described by conditions: $y = 0$, $x > 0$, and let $a = (x_1, y_1)$. Then $E_K[a] = \left\{(u_1, u_2) \in W : u_1 = \frac{c^2}{x_1}, u_2 = \frac{cy_1}{x_1}, c \in F^+\right\}$.
Thus $E_K[a]$ is a branch of a parabola with equation $x = by^2$ where $b = \frac{x_1}{y_1^2} \in F^+$.

From a deeper analysis we get that $E_{K_1}[a]$ is this branch of parabola which passes through a, because we can simply show that if $a = (x_1, y_1)$ and $y_1 > 0$ then

$$\forall v = (v_1, v_2)\left(v_2 > 0 \wedge v_1 = \frac{x_1}{y_1^2} v_2^2 \Rightarrow v \in E_{K_1}[a]\right).$$

Let $K \in L \setminus I$ — be a line. We take $\Psi \in C(B)$ such that $\Psi(x, y) = (x, y + tx + b)$ and $\Psi(K_1) = K$. Then $E_K[a] = \Psi(E_{K_1}[C])$ where $c = \Psi^{-1}(a)$. Because Ψ is an affine transformation the thesis arise from previous calculations.

COROLLARY 8. *E consists of lines $K \in L \setminus I$ as well as of branches of parabolas P contained in W and tangent to M_c.*

In particular $E \cap I = \emptyset$. Durectly from Corollary 8 it follows also that affine transformations belonging to $C(B)$ transform E onto itself.

DEFINITION 9. $IE_\gamma[a] := Z(\gamma)[a]$, for $\gamma \in \Gamma$.

THEOREM 10. *For every $a \in W$ and $\gamma \in \Gamma$, $IE_\gamma[a] \in I$ and $a \in IE_\gamma[a]$.*

Proof. If $\gamma \in \Gamma$, then $\gamma = \sigma_K^a$, $K \in I$. We have $Z(\gamma) = \{\sigma_L^\omega : a \in L\}$. Now the thesis is trivial.
Now we shall define a reflection in an equidistant curve which is not an isotropic line. In the sequel we shall study these symmetries.

DEFINITION 11. If $E \in E$ then $\sigma_E(p) = q \Leftrightarrow q = \sigma_{\bar{p}}(p) \wedge \bar{p} \in E \wedge I(p\bar{p})$.

THEOREM 12. *If $E \in E \setminus L$ then there are $\gamma, \beta \in F$ and $\alpha \in F \setminus \{0\}$ scuh that $\sigma_E(x, y) = (\alpha, \alpha\sqrt{x} + \gamma x - y + \beta)$.*

Proof. If $E_1 \in E$ is an equidistant curve with axis K_1 described by equation $y = 0$, then E_1 is described by equation $x = ay^2$ where $a \in F^+$. Then from Definition 11 we infer that $\sigma_{E_1}(x, y) = (x, \alpha\sqrt{x} - y)$ where $\alpha = \frac{2}{\sqrt{a}} \in F \setminus \{0\}$.
Let E be an equidistant curve with some axis K. There exists an affinity Ψ such that $\Psi(K_1) = K$. Then $\Psi^{-1}(E) = E_1$ where E_1 is some equidistant curve axis K_1. Let us take such an affinity $\Psi \in C(B)$, $\Psi(x, y) = (x, y + tx + b)$, let $\Psi(E_1) = E$. Then $\sigma_E(x, y) = \sigma_{\Psi(E_1)}(x, y) = \Psi(\sigma_{E_1})(x, y) = \Psi\sigma_E\Psi^{-1}(x, y) = (x, \alpha\sqrt{x} + \gamma\alpha - y + \beta)$ where $\alpha \in F \setminus \{0\}$, $\gamma = 2t$, $\beta = 2b$ are elements of F. From Theorem 1 and Theorem 12 it follows in general.

COROLLARY 13. *If $E \in E$ then there exist $\alpha, \beta, \gamma \in F$ such that*

$$\sigma_E(x, y) = (x, \alpha\sqrt{x} + \gamma x - y + \beta).$$

Directly from the previous calculations we have also that if $f(x, y) = (x, \alpha\sqrt{x} + \gamma x - y + \beta)$ then there exists $E \in E$ such that $f = \sigma_E$. Immediately from Corollary 13 we obtain

THEOREM 14. *If $E \in E$ and $K \in I$ then $\sigma_E(K) = K$.*

Proving the next theorem shall show that every reflection σ_E with $E \in E$ transforms the set of equidistant curves which are not isotropis onto itself.

THEOREM 15. *If $E,\ E' \in E$ then $\sigma_E(E') \in E$.*

Proof. Let $E,\ E' \in E$. If $E \in L$, then the thesis is trivial by Theorem 1. Let $E \in E \setminus L$. Let us assume that $E' \in L$. Let $E_{a_1} \in E$ and E_{a_1} be the set of points satisfying conditions

$$(1): x = a_1 y^2,\ a_1 \in F^+$$

$$(2): y > 0 \text{ or } (2'): y < 0.$$

Let $E' \in L$. Then E' can be described by equation $y = ax + b$ where $a,\ b \in F$. Then $\sigma_{E_{a_1}}(E')$ satisfies in $A(F)$ the equation $y = \alpha\sqrt{x} - ax - b$ with $\alpha = \pm \frac{2}{\sqrt{a_1}}$. The projective closure of this set has an equation in $P(F)$:

$$x_2^2 + a^2x_1^2 + 2ax_1x_2 + 2bx_0x_2 + (2ab - \alpha^2)x_0x_1 + b^2x_0^2 = 0.$$

Analysing this equation, from the affine classification of conics (see [1]) we infer that $\sigma_{E_{a_1}}(E') \in E \setminus L$. Let $E \in E \setminus L$ be an equidistant curve. There exists an affinity $\Psi \in C(B)$, $\Psi(x,\ y) = (x,\ y + a_2x + b_2)$ such that $\Psi(E_{a_1}) = E$ for suitable $a_2,\ b_2 \in F$, $a_1 \in F^+$. Then $\Psi^{-1}(E') \in L \setminus I$ and $\Psi^{-1}(\sigma_E(E')) = \sigma_{E_{a_1}}(\Psi^{-1}(E')) \in E \setminus L$. Therefore $\sigma_E(E') \in E \setminus L$.

Let us assume now that $E' \in E \setminus L$. As previously we consider $E_a \in E$ consisting of all points satisfying

$$(1):\ x = ay^2,\ a \in F^+,$$

$$(2):\ y < 0 \text{ or } (2'):\ y < 0.$$

Then $\sigma_F(E_a)$ satisfies the equation $x = a\left(\frac{y - \gamma x - \beta}{\alpha\sqrt{a} - 1}\right)^2$.

The projective closure of this set has an equation in $P(F)$:

$$-x_2^2 - \gamma^2x_1^2 + 2\gamma x_1x_2 + \left(\alpha^2 - \frac{2\alpha}{\sqrt{a}} + \frac{1}{a} - 2\beta\gamma\right) \cdot x_0x_1 + 2\beta x_0x_2 - \beta^2x_0^2 = 0.$$

Affine classification of conics gives now that $\sigma_E(E_a) \in E$. Given arbitrary $E' \in E \setminus L$ we consider $\Psi \in C(B)$, $\Psi(x,\ y) = (x,\ y + a_1x + b_1)$ such that $\Psi(E_a) = E'$ for some $a_1,\ b_1 \in F$, $a \in F^+$. Then we get $\Psi^{-1}(\sigma_E(E')) = \sigma_{\Psi^{-1}(E)}(E_a)$. Thus $\sigma_E(E') \in E$.

Let us introduce the following notations:

$$\Lambda := \{\sigma_E :\ E \in E \setminus L\},$$

$$\Omega := \Sigma \cup \Gamma \cup \Lambda.$$

We shall find analytical description of the groups generated by our reflections, i.e. groups $G(\Lambda)$ and $G(\Omega)$.

DEFINITION 16. $D^+ := \{f : f(x,\ y) = (x,\ \alpha\sqrt{x} + \gamma x + y + \beta),\ \alpha,\ \beta,\ \gamma \in F\}$;

$D^- := \{f : f(x, y) = (x,\ \alpha\sqrt{x} + \gamma x - y + \beta),\ \alpha,\ \beta,\ \gamma \in F\{$;

$D := D^+ \cup D^-$.

THEOREM 17. $G(\Lambda) = D$.

Proof. It is easy to shaw that D is a group of transformations. From Theorems 1, 12 it follows that $\Lambda \subseteq D$, therefore $G(\Lambda) \subseteq D$. It remains to prove the converse inclusion.

If $f \in D^-$, $\alpha \neq 0$ then $f \in \Lambda$. If $f \in D^-$, $\alpha = 0$ then $f = \sigma_1\sigma_2\sigma_3$ where

$$\sigma_1(x, y) = (x, \sqrt{x} + \gamma x - y + \beta), \quad \sigma_2(x, y) = (x, 2\sqrt{x} + x - y + 1),$$
$$\sigma_3(x, y) = (x, \sqrt{x} + x - y + 1). \text{ Clearly } \sigma_1, \sigma_2, \sigma_3 \in \Lambda.$$

If $f \in D^+$, $\alpha \neq 0$ then $f = \sigma_1\sigma_2$ where $\sigma_1(x, y) = (x, 2\alpha\sqrt{x} + \gamma x - y + \beta)$, $\sigma_2(x, y) = (x, \alpha\sqrt{x} - y)$. If $f \in D^+$, $\alpha = 0$ then $f = \sigma_1\sigma_2$ where $\sigma_1(x, y) = (x, \sqrt{x} + \gamma x - y + \beta)$, $\sigma_2(x, y) = (x, \sqrt{x} - y)$ In both cases $\sigma_1, \sigma_2 \in \Lambda$. Thus $D \subseteq G(\Lambda)$.

REMARK 18. From the bove calculations it follows that $G(\Lambda) = \Lambda\Lambda \cup \Lambda\Lambda\Lambda$.

DEFINITION 19. $D^+ := \{f : f(x, y) = (\delta^2 x, \gamma\sqrt{x} + \beta x + \delta xy + b), \gamma, \beta, b, \delta \in F, \delta \neq 0\}$,

$$D_1^- := \left\{f : f(x, y) = \left(\frac{\delta^2}{x}, \frac{\gamma\sqrt{x} + \beta x + \delta y + b}{x}\right), \gamma, \beta, b, \delta \in F, \delta \neq 0\right\},$$

$$D_1 := D_1^+ \cup D_1^-.$$

THEOREM 20. $D_1 = G(\Omega)$.

Proof is analogous to the proof of Theorem 17. We easily show that D_1 is a transformation group. From Theorems 1, 12 we obtain $\Omega \subseteq D_1$, therefore $G(\Omega) \subseteq D_1$. To prove the converse inclusion we notice first that $\Pi \subseteq \Gamma\Sigma \subseteq G(\Omega)$. Let $f \in D_1^+$. If $\beta, b \neq 0$ then $f = \sigma_1, \sigma_2\sigma_3$ where $\sigma_1(x, y) = \left(\frac{b^2\delta^2/\beta^2}{x}, \frac{b\delta y/\beta}{x}\right)$, $\sigma_2(x, y) = \left(\frac{(-b/\beta)^2}{x}, \frac{-b(y - b)/\delta)/\beta}{x} + \frac{b}{\delta}\right)$, $\sigma_3(x, y) = \left(x, \frac{-\gamma}{\delta}\sqrt{x} - y\right)$. Clearly $\sigma_1, \sigma_2 \in \Gamma \cup \Pi$, $\sigma_3 \in \Sigma \cup \Lambda$. If $\beta = 0$ or $b = 0$ then $f = \sigma_1\sigma_2\sigma_3$ for $\sigma_1(x, y) = \left(\frac{\delta^2}{x}, \frac{\delta y}{x}\right)$, $\sigma_2(x, y) = \left(\frac{1}{x}, \frac{-y}{x}\right)$, $\sigma_3(x, y) = \left(x, \frac{-\gamma}{\delta}\sqrt{x} - \frac{\beta}{\delta}x - y - \frac{b}{\delta}\right)$; again $\sigma_1, \sigma_2 \in \Gamma \cup \Pi$, $\sigma_3 \in \Sigma \cup \Lambda$.

Let $f \in D^-$. We consider $\sigma_1(x, y) = \left(\frac{\delta^2}{x}, \frac{-\delta y}{x}\right)$, $\sigma_2(x, y) = \left(x, \frac{-\gamma}{\delta}\sqrt{x} + \frac{-\beta}{\delta}x - y - \frac{b}{\delta}\right)$. Then $f = \sigma_1\sigma_2$ and $\sigma_1 \in \Gamma \cup \Pi$, $\sigma_2 \in \Lambda \cup \Sigma$. Thus finally $D_1 \subseteq G(\Omega)$.

In the sequel we shall try to characterise the groups $G(\Lambda)$ and $G(\Omega)$ geometrically.

DEFINITION 21. $ab \equiv {}_1cd \Leftrightarrow (\exists f \in G(\Sigma))[f(a) = c \wedge f(b) = d] \wedge I(ab)$.

DEFINITION 22. $ab \equiv {}_2cd \Leftrightarrow (\exists f \in G(\Sigma \cup \Gamma))[f(a) = c \wedge f(b) = d] \wedge I(ab)$.

DEFINITION 23. Let $K_1, K_2, K_3, K_4 \in I$.

$$K_1K_2 \equiv {}_3K_3K_4 \Leftrightarrow (\exists f \in G(\Gamma))[f(K_1) = K_3 \wedge f(K_2) = K_4].$$

Immediately from the definitions we obtain the following facts (comp. [3]).

FACT 24. If $K \in I$, $a, b, c, d \in K$ then $ab \equiv {}_1cd \Leftrightarrow ab \equiv {}_2cd$.

FACT 25. Let $a = (a_1, a_2)$, $b = (b_1, b_2)$, $c = (c_1, c_2)$, $d = (d_1, d_2)$. If $I(ab)$, $I(ac)$, $I(ad)$ (*i.e.* if $a_1 = b_1 = c_1 = d_1$) then

$$ab \equiv {}_1cd \Leftrightarrow |a_2 - b_2| = |c_2 - d_2|.$$

FACT 26. If lines K_i have equations $c_i = x$, $i = 1, \ldots, 4$ then

$$K_1K_2 \equiv {}_3K_3K_4 \Leftrightarrow \frac{c_1}{c_2} = \frac{c_3}{c_4} \vee \frac{c_1}{c_2} = \left(\frac{c_3}{c_4}\right)^{-1}.$$

DEFINITION 27. $G := \{f : f : W \Rightarrow W \wedge (\forall K \in I) f(K) =$
$= K \wedge (\forall E \in \mathbf{E}) f(E) \in \mathbf{E} \wedge (\forall ab)[I(ab) \Rightarrow ab \equiv {}_2f(a)f(b)]$.

LEMMA 28. *G is a group.*

Proof. Simply G is the group of automorphisms of the structure

$$\langle W;\ \mathbf{E},\ \{K\}_{K \in I},\ \{[a, b]_{\equiv_2} : I(ab)\}\rangle.$$

LEMMA 29. *For every point a and every nonisotropic line K there exists exactly one $E \in \mathbf{E}$ with axis K such that $a \in E$.*

LEMMA 30. *If $h \in G$, $h | E = id_E$, $h(q) = q$ for some $q \notin E$, $E \in \mathbf{E}$ then $h = id$.*

Proof. Let $E = E_K[a]$. Assume $h(q) = q$ for some $q \notin E$. Let $\tilde{E} = E_K[q]$, then $q \in h(\tilde{E})$. We shall show that $h | \tilde{E} = id_{\tilde{E}}$. Let $x \in \tilde{E}$, $y \in E$, $I(xy)$. Since $h \in G$ and $h(y) = y$ we obtain $xy \equiv {}_2h(x)y$, thus by Facts 24, 25 we get $h(x) = \sigma_y(x)$ or $h(x) = x$. But $h(x) = \sigma_y(x)$ implies $h(\tilde{E}) \cap h(E) \neq \emptyset$ (see Fig. 1). It is impossible because $E \cap \tilde{E} = \emptyset$. Therefore $h(x) = x$ and in general $h | \tilde{E} = id_{\tilde{E}}$.

Let z be arbitrary point, $z \notin E \cup \tilde{E}$. Let us consider $z_1 \in E$, $z_2 \in \tilde{E}$ such that $I(zz_1)$, $I(zz_2)$. Then $h(z_1) = z_1$, $h(z_2) = z_2$. Again Facts 24, 25 imply $h(z) = z$.

THEOREM 31. $G = G(\Lambda)$.

Proof. Theorems 15, 17 imply that $G(\Lambda) \subseteq G$. Let $f \in G$. We consider arbitrary $E \in \mathbf{E} \setminus L$ with axis K and denote $\alpha = K \cap M_0$. Let $E' = f(E)$, then $E' \in \mathbf{E}$; we denote its axis by K', $\alpha' = K' \cap M_0$. There exists τ — a translation of $B(F)$

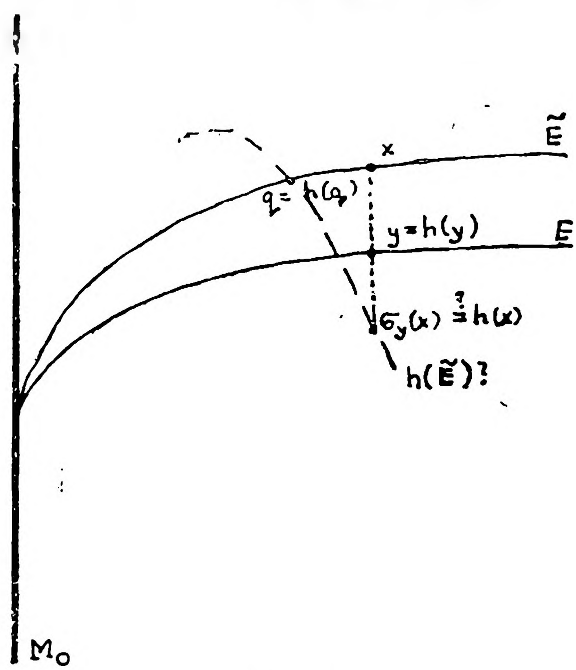

Fig. 1

with $\tau(\alpha') = \alpha$; let $\tau(E') = E''$, $(K') = K''$. There exists g — a shear with axis M_0 mapping K'' onto K, we put $E''' = g(E'')$. Then E and E''' have a common axis K.
Consider $K_1 \in I$, let $(0, 1) \in K_1$. We denote $p = E \cap K_1$, $p''' = E''' \cap K_1$. There exists $H \in E$ with axis K such that $\sigma_H(p''') = p$. Let $E_0 = \sigma_H(E''')$. We get that E_0 has axis K and contains p, therefore $E_0 = E$, i.e. $\sigma_H(E''') = E$.
Let $h = \sigma_H g \tau f$.
We have $h(E) = E$, $h \in G$ (since σ_H, g, $\tau \in G(\Lambda) \subseteq G$), and clearly $h|E = id_G$. If there is $q \notin E$ such that $h(q) = q$ then $h = id$, thus $h \in G(\Lambda)$. If $h \neq id$ then we consider $h' = \sigma_E h$. Now h' satisfy assumptions of Lemma 30, thus $h' = id$, $h = \sigma_E = G(\Lambda)$. From this we obtain $f = \tau^{-1} g^{-1} \sigma_H h \in G(\Lambda)$. Thus $G \subseteq G(\Lambda)$.

LEMMA 32. *If* $f \in \Gamma$ *then*

(i) $(\forall K \in I) f(K) \in I$

(ii) $(\forall E \in E) f(E) \in E$.

Proof. follows immediately from the analytical description of elements of Γ.

THEOREM 33. *The following properties form a minimal system of characteristic invariants of the group* $G(\Lambda)$:

1) $(\forall K \in I) f(K) = K$

2) $(\forall E \in E(f(E) \in E$

3) $I(ab) \Rightarrow ab \equiv_2 f(a)f(b)$.

Proof. From Theorem 31 properties 1), 2), 3), are characteristic for $G(\Lambda)$. Let us consider transformations Ψ_1, Ψ_2, Ψ_3 defined as follows: $\Psi_1(x, y) = (x, 2y)$, $\Psi_2(x, y) = (x, 2x^2 - y)$, $\Psi_3 = \sigma_K^p \in \Gamma$ with $K \in I$. Just from the definitions Ψ_1 satisfies 1), 2); Ψ_2 satisfies 1), 3) and Ψ_3 satisfies 2), 3). But $\Psi_1, \Psi_2, \Psi_3 \notin G(\Lambda)$, thus 1), 2), 3) is a minimal system of properties characterizing $G(\Lambda)$.

REMARK 34. Consider 3') — a condition which results from 3) by substituting $\equiv_1$ instead of $\equiv_2$ i.e.
3'): $I(ab) \Rightarrow ab \equiv_1 f(a)f(b)$.
Then 3') implies 1), 3) and 3') follows from 1), 3).

DEFINITION 35. Let G_1 be the group of all the bijections $f: W \to W$ satisfying:

1') $(\forall K \in I)(f(K) \in I$

2) $(\forall E \in E) f(E) \in E$

4) $K_1 K_2 \in I \Rightarrow K_1 K_2 \equiv_3 f(K_1) f(K_2)$

3) $I(ab) \Rightarrow ab \equiv_2 f(a)f(b)$.

Analogously as in Theorem 28 we get that G_1 is a group.

THEOREM 36. $G_1 = G(\Omega)$.

Proof. From Lemma 32, Facts 24, 25, Theorem 20 we obtain $G(\Omega) \subseteq G_1$. Let $f \in G_1$ be arbitrary. Lte $K_1, K_2 \in I$ have equations $x = c_1$, $x = c_2$, $c_1 \neq c_2$.

Then $K_1K_2 \equiv_3 f(K_1)f(K_2)$ and there exists $g \in G(\Gamma)$ such that $g(K_1) = f(K_1)$, $g(K_2) = f(K_2)$. Let $h = g^{-1}f$, we shall prove that $h \in G(\Lambda)$, clearly $h \in G_1$. First we show that h satisfies 1), i.e. $(\forall K \in I)\ h(K) = K$. Let $K_3 \in I$, $K_3 \neq K$, K_2; let K_3 has equation $x = c_3$. Assume that $h(K_3)$ has equation $x = c_3'$. Since $h \in G_1$, $h(K_1) = K_1$, $h(K_2) = K_2$ we obtain $K_1K_3 \equiv_3 K_1h(K_3)$ and $K_2K_3 \equiv_3 K_2h(K_3)$ Fact 26 implies $c = c_3$ i.e. $h(K_3) = K_3$. Function h satisfies 2), 3) just from the definition of G_1. Therefore $h \in G(\Lambda)$. We have $f = gh \in G(\Gamma)G(\Lambda) \subseteq G(\Omega)$ i.e. $G_1 \subseteq G(\Omega)$. The system 1'), 2), 3), 4) is not a minimal system of chatacteristic invariants of $G(\Omega)$ since 4) implies 1'). However we don't know if {2), 3), 4)} forms a minimal system.

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COLLISION AND ESCAPE GEOMETRY IN THE 2-BODY PROBLEM

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ABSTRACT. — A partition into components of the constant enefgy level is done, the corresponding time intervals are described and properties of the collision or escape solution of the 2-body problem are proved. The idea of continuing the motion after collision on a different energy level is, finally, briefly exposed.

1. **Introduction.** Giving a classification of the final motion in the 2-body problem, C h a z y ([1]), states that, relative to the constant of energy h, the motion could be elliptic ($h < 0$), parabolic ($h = 0$) or hyperbolic ($h > 0$) for unbounded time, representing it as in Fig. 1.

The question leading to this paper is whether such a representation has a real significance. In the following we give some geometrical meaning to this, in the collinear case, studying also the cases when the motion is not defined for all time.

2. **The Equations of Motion.** Consider two material points of masses m and M, and chosoe some inertial frame $Mxyz$ centered at M. Thus the position of m is described by the vector $\vec{r} = (x, y, z)$, and the motion of the two particles by the system:

$$\ddot{\vec{r}} = -\mu \vec{r}/r^3 \tag{1}$$

where $r^2 = x^2 + y^2 + z^2$ and $\mu = G(m + M)$, G being the gravitational constant.
The energy integral is

$$H(\vec{r}, \dot{\vec{r}}) = T(\dot{\vec{r}}) - U(\vec{r}) \tag{2}$$

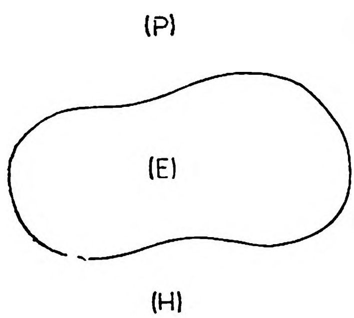

Fig. 1.

where $T : R^3 \to R$, $T(x, y, z) = (1/2) \cdot m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ called the kinetic energy, and $-U : R^3 - \{0\} \to R$, $-U(x, y, z) = -m\mu/r$ called the potential energy. H is the Hamiltonian of the system of particles.

For given initial conditions $(\vec{r}(0), \dot{\vec{r}}(0))$, with $\vec{r}(0) \neq 0$, we have, for the corresponding $(\vec{r}, \dot{\vec{r}})$ solution,

$$H(\vec{r}, \dot{\vec{r}}) = h \tag{3}$$

where h is the enerbgy constant.

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It is known (see [9]) that in the 2-body problem collisions are possible only in the collinear motion. Considering that this one takes place on the Mx axis, (1), (2), (3) become respectively:

$$\ddot{x} = -\mu x/|x|^3 \tag{4}$$

$$H(x, \dot{x}) = (1/2)\dot{x}^2 - \mu/|x| \tag{5}$$

$$H(x, \dot{x}) = h \tag{6}$$

The constant of the angular momentum is zero in the collinear case.

3. The Time Interval. The standard results of the theory of differential equations ensure, for given initial conditions, the existence and uniqueness of a solution for Eq. (4), on some maximal interval (t_1, t_2), $-\infty \leqslant t_1 < 0 < t_2 \leqslant +\infty$. If $-\infty < t_1$ or $t_2 < +\infty$ we say that the solution is singular at t_1, respectively t_2.

It is very well known that for the general 3-body problem all the singularities are due to collisions. Moreover, for the collinear n-body problem, the same statement is true (see [6]).

It is also proved (see [2]) that there is no motion in the collinear n-body problem for $t \in (-\infty, +\infty)$.

We conclude that

THEOREM 3.1 *Any solution of Eq.* (4), *with initial conditions* $(x(0), \dot{x}(0))$, $x(0) \neq 0$, *is defined on some interval* I *having the form*:

(i) $I = (t_1, t_2)$, $-\infty < t_1 < 0 < t_2 < +\infty$,

(ii) $I = (-\infty, t_2)$, $0 < t_2 < +\infty$,

(iii) $I = (t_1, +\infty)$, $-\infty < t_1 < 0$,

and $\lim_{t \searrow t_1} x(t) = \lim_{t \nearrow t_1} x(t) = 0$.

4. Constant Energy Level Geometry. Consider $x > 0$ if m lies on the positive half of Mx and $x < 0$ if m lies on the negative half of Mx. Then $\dot{x} > 0$ corresponds to the motion of m in the same sense as the orientation of Mx and $\dot{x} < 0$ for the opposite sense.

Denote $I^+ = I \cap \,]0, +\infty)$

$$h_+^+ = \{(x, \dot{x}) \,|\, x > 0,\ \dot{x} > 0,\ H(x, \dot{x}) = h\}$$

$$h_-^+ = \{(x, \dot{x}) \,|\, x < 0,\ \dot{x} > 0,\ H(x, \dot{x}) = h\}$$

$$h_-^- = \{(x, \dot{x}) \,|\, x < 0,\ \dot{x} < 0,\ H(x, \dot{x}) = h\}$$

$$h_+^- = \{(x, \dot{x}) \,|\, x > 0,\ \dot{x} < 0,\ H(x, \dot{x}) = h\}$$

Fig. 2 shows the relative motion in different quadrants of the phase space and Figs. 3, 4 and 5 represent the before defined sets for $h = 0$, $h = -1$ and $h = 1$.

Observe that $\mathcal{H}(h) = \{(x, \dot{x}) \,|\, H(x, x) = h\}$ is an invariant set for Eq. (4) and that h_+^+, h_-^+, h_-^-, h_+^- form a partition of $\mathcal{H}(h)$; therefore call them the *components* of $\mathcal{H}(h)$.

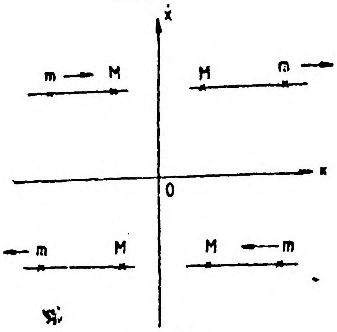

Fig. 2.

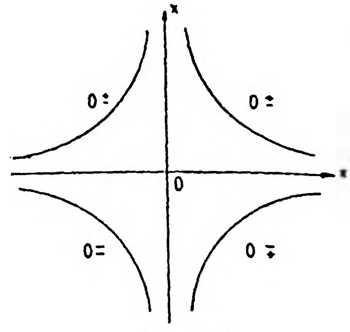

Fig. 3.

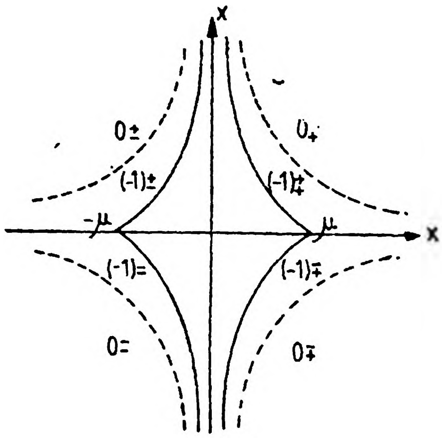

Fig. 4.

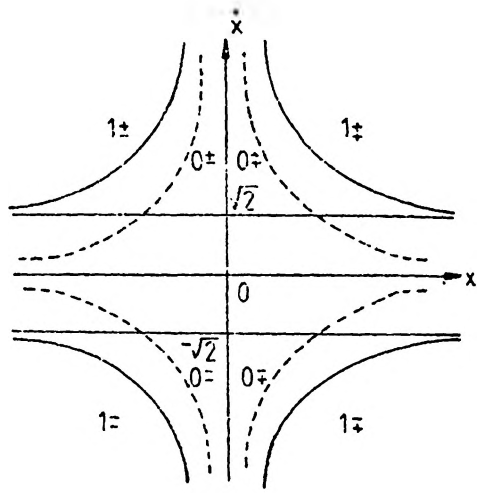

Fig. 5.

5. **Solutions and their Intervals for Initial Conditions on Components.** We first give some estimates of time intervales for arbitrary h.

LEMMA 5.1. *If the initial conditions of Eq.* (4) *are* $(x(0), \dot{x}(0)) \in h_{+}^{-}$ *(or* h_{-}^{+}*) then the solution is defined* $x: [0, t_2) \to R$, *for every* h *where* $t_2 < +\infty$.

Proof: $x(0) > 0$, $\dot{x}(0) < 0$ and (4) imply $\ddot{x}(0) < 0$. Suppose $x: [0, \infty) \to \to R$. If there is $t_0 \in (0, +\infty)$ such that $x(t_0) = 0$, then t_0 is a singularity, therefore $x(t) > 0$ for every $t \in [0, +\infty)$ and corresponding $\ddot{x}(t) < 0$. Thus $\dot{x}$ is a decreasing function and since $\dot{x}(0) < 0$ it follows $\dot{x}(t) < 0$ for every $t \in \in [0, +\infty)$. Therefore x is decreasing and concave down. Using the average theorem of Lagrange it follows easily that $\lim_{t \to +\infty} x(t) = -\infty$, contradiction.

Remark 5.2. We used the estimate $[0, t_2)$ because this interval could have the form (t_1, t_2) with t_1 finite or the form $(-\infty, t_2)$.

An analogus proof and remark could be done for the following statement:

LEMMA 5.3. *If the initial conditions of Eq.* (4) *are* $(x(0), \dot{x}(0)) \in h_+^+$ *(or h_-^-) then the solution is defined* $x: (t_1, 0] \to R$, *for every* h, *where* $-\infty < t_1$.

A better estimate has the below stated case:

LEMMA 5.4 *If the initial conditions for Eq.* (4) *are* $(x(0), \dot{x}(0)) \in h_+^+$ *(respectively h_-^-) with* $h \geqslant 0$, *then* $x: (t_1, +\infty) \to R$.

Proof: Suppose $x: (t_1, t_2) \to R$, $-\infty < t_1 < 0 < t_2 < +\infty$. As $h \geqslant 0$ and since it is not possible that $\lim_{t \nearrow t_1} x(t) = +\infty$ (see [2]), we have $\dot{x}^2 \geqslant 2\mu/|x| >$ $>$ const. > 0, for all $t \in (0, t_2)$. If there would exist $t' \in (0, t_2)$ with $\dot{x}(t') < 0$ then there would exist $t'' \in (0, t')$ such that $x(t'') = 0$, contradiction with the last inequality, and therefore $\dot{x}(t) > 0$, $\forall t \in (t_1, t_2)$. But $x: (t_1, t_2) \to R$ implies $\lim_{t \nearrow t_1} x(t) = 0$ and since $x(0) > 0$, there is t''' with $\dot{x}(t''') < 0$, contradiction.

We could analogously prove:

LEMMA 5.5 *If the initial conditions for Eq.* (4) *are* $(x(0), \dot{x}(0)) \in h_-^+$ *(respectively h_+^-) with* $h \geqslant 0$, *then* $x: (-\infty, t_2) \to R$.

6. **Transition between Components.** The representations of $\mathfrak{X}(h)$ suggest the following results.

PROPOSITION 6.1 *If the initial conditions for Eq.* (4) *are* $(x(0), \dot{x}(0)) \in h_+^+$ *(respectively h_-^-) and the solution* $x: I \to R$ *has the property that there is* $t_0 \in I^+$ *such that* $(x(t_0), \dot{x}(t_0)) \in h_+^-$ *(respectively h_-^+), then* $h < 0$.

Proof: Observe that, by 5.3, the interval I may have the form (t_1, t_2), $-\infty < t_1 < 0 < t_2 < +\infty$ or $(t_1, +\infty)$.

Let be $x(0) > 0$, $\dot{x}(0) > 0$ and suppose there is $t_0 \in I^+$ such that $x(t_0) > 0$ and $\dot{x}(t_c) < 0$. Since x is continuous on $(0, t_c)$ it follows that the remust be some $t' \in (0, t_2)$ such that $\dot{x}(t') = 0$. From (5), (6) we have $x(t') = -\mu/h$. If $h > 0$ then $x(t') < 0$. As $x(0) > 0$ and x continuous there is $t'' \in (0, t')$ such that $x(t'') = 0$, which means that for t'' Eq. (4) is not defined, contradiction.

If $h = 0$, (5), (6) imply that $-\mu/|x(t')| = 0$ which is impossible because t' is not singularity.

The converse is true only with some restriction.

PROPOSITION 6.2. *If the initial conditions for Eq.* (4) *are* $(x(0), \dot{x}(0)) \in h_+^+$ *(respectively h_-^-), the solution is* $x: (t_1, t_2) \to R$, $-\infty < t_1 < 0 < t_2 < +\infty$ *and* $h < 0$ *then there is* $t_c \in (0, t_2)$ *such that* $(x(t_c), \dot{x}(t_c)) \in h^-$ *(respectively h^+).*

Proof.: Let be $x(0) > 0$, $\dot{x}(0) > 0$ and $h < 0$. From (5), (6) we have $|x| \leqslant$ $\leqslant -\mu/h$, therefore $x: (t_1, t_2) \to R$ is bounded. If $\dot{x}(t) \geqslant 0$, $\forall t \in (0, t_2)$, since $x(0) > 0$ it follows $x(t) > 0$, $\forall t \in (0, t_2)$. Since $x(0) > 0$, there exists $c > 0$ (constant) such that $x(0) > c > 0$ and so $x(t) > c$, $\forall t \in (0, t_2)$. Them $\lim_{t \nearrow t_1} x(t) \geqslant$ $\geqslant c > 0$ which contradicts 3.1.

7. **Escape and Capture.** PROPOSITION 7.1 *If the initial conditions for the Eq.* (4) *are* $(x(0), \dot{x}(0)) \in h_+^+$ *(respectively h_-^-) with* $h \geqslant 0$, *then* $\lim_{t \to +\infty} x(t) = \pm\infty$.

Proof: By 5.4 it makes sense to speak about $t \to +\infty$. As in the proof of 5.4 we will show that $\dot{x}^2 > 0$ on $(0, +\infty)$ and since $\dot{x}(0) > 0$ it follows that $\dot{x} > 0$ on $(0, +\infty)$. Therefore x is increasing and since $x(0) > 0$, it is true that $x > 0$ on $(0, +\infty)$. Thus, by (4), $\ddot{x} < 0$ and consequently $\dot{x}$ is a decreasing function.

Since $\dot{x}$ is continuous, $\lim_{t \to +\infty} \dot{x}(t)$ exists. It is obviously finite from (5), (6) and the fact that $\dot{x} > 0$. Therefore its limit could be zero or positive. If $\lim_{t\to+\infty} \dot{x}(t) = 0$, since $(1/2)\,\dot{x}^2 - \mu/|x| = h \geqslant 0$ it follows $|x| \geqslant 2\mu/\dot{x}^2$ and from $\lim_{t\to+\infty} |x(t)| = +\infty$, the conclusion. If $\lim_{t\to+\infty} \dot{x}(t) > 0$, by Lagrange theorem on the interval $[0, t]$, $x(t) = \dot{x}(t_\xi)t + x(0)$ and thus $\lim_{t\to+\infty} x(t) = \pm\infty$.

Beyond this escape statement we may analogously prove a capture one.

PROPOSITION 7.2 *If the initial conditions for Eq (4) are* $(x(0), \dot{x}(0)) \in h^{\pm}_{-}$ *(respectively* h^{-}_{+}*) with* $h \geqslant 0$*, then* $\lim_{t\to-\infty} x(t) = +\infty$.

8. **Transition on different Energy Levels.** We briefly expose the idea of continuouing the motion after collision on different energy levels. Regularizations of the motion as that of S u n d m a n ([7]) or L e v i - C i v i t a ([5]) are done relative to time, on the same energy level, respectively, relative to the initial conditions and are meant only in a mathemtical sense. Another qualitative way to treat the problem, which extends that of L e v i - C i v i t a is due to E a s t o n ([4]). Since we do not know which is going to be the physical behavior of the particles after collision, it is of interest, especialy for space research, to give a better description of the motion in the neighborhood of the collision. In a general case this is done by the author ([3]).

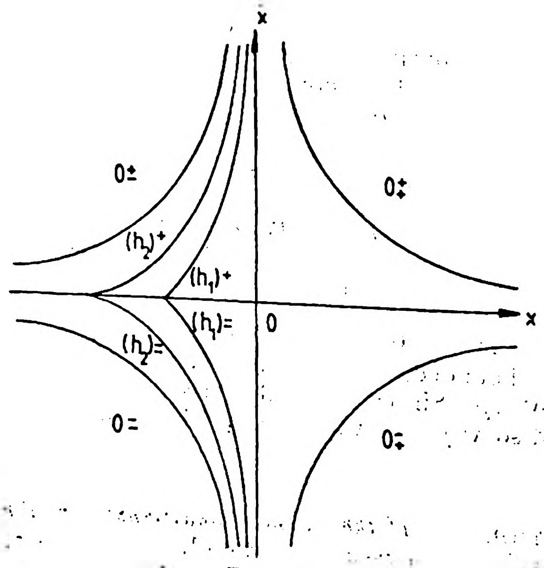

Fig. 6.

In the case of a physical elastic bounce of two masses ($h < 0$), after collision the mechanical system of particles has different initial conditions and, possible, a different constant of energy. Thus, if the continuation of the motion is possible, then it is natural to study it on different energy levels.

If, for example, initial conditions $(x_1(t_{12}), \dot{x}_1(t_{12})) \in (h_1)^{-}_{-}$ with $h_1 < 0$, where $t_{12} \in (t_1, t_2)$, are given, then, by 6.2, a transition to $(h_1)^{+}_{-}$ takes place which leads to a collision at t_2 instant. The motion could be then continued on some h_2 energy level, $h_2 \neq h_1$, considering the motion defined on (t_2, t_3), having the initial condition $(x_2(t_{23}), \dot{x}_2(t_{23})) \in (h_2)^{-}_{+}$, where $t_{23} \in (t_2, t_3)$, t_3 finite or infinite.

It is, for example, the case of the jump of an elastic ball on the Earth which was vertically thrown, neglecting other forces, except the gravitational one.

If $h_2 < h_1$ the transition is like in Fig. 6. If $h_2 \geqslant 0$ the motion is without collision. Therefore for any sequence wich describes the change of the energy constant the motion is indefinitely continued.

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TUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXII, 2, 1987

COMMON FIXED POINTS OF FAMILY OF MAPPINGS IN MENGER SPACES[1]

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ABSTRACT. – The results established in this paper extend and unify several known results regarding common fixed points of a family of mappings in metric and probabilistic metric spaces.

Introduction. The notion of contraction mapping on a probabilistic metric space (PM-space) was introduced by Sehgal [7]. Fixed and common fixed point theorems under general contraction conditions on a PM-space have been proved in [1], [2], [4], [10] etc. (for an extensive bibliography refer to [9]). On the other hand recently Tiwari and Singh [11] have established a common fixed point theorem for a family of mappings and have indicated the possibility of the generalization of a number of results. The intent of the present paper is to extend their result to PM-spaces. Our result generalizes some significant results on metric and PM-spaces.

Preliminaries. A PM-space is an ordered pair $(X, \mathcal{F})$ where $\mathcal{F}$ is a mapping from $X \times X$ to L, the collection of distribution functions. The value of $\mathcal{F}$ at $(u, v) \in X \times X$ is represented by $F_{u,v}$ and $F_{u,v}$ are assumed to satisfy the following conditions:

(a) $F_{u,v}(x) = 1$ for all $x > 0$ iff $u = v$;

(b) $F_{u,v}(0) = 0$;

(c) $F_{u,v} = F_{v,u}$;

(d) if $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x + y) = 1$.

A mapping t: $[0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm if it satisfies:

(e) $t(a, 1) = a$, $t(0, 0) = 0$;

(f) $t(c, d) \geqslant (t(a, b)$ for $c \geqslant a$, $d \geqslant b$;

(g) $t(a, b) = t(b, a)$;

(h) $t(t(a, b,), c) = t(a, t(b, c))$;

for all a, b, c, d in $[0, 1]$.

A Menger space is a triplet $(X, \mathcal{F}, t)$, where $(X, \mathcal{F})$ is a PM-space and t-norm t is such that the inequality

(d′) $F_{u,w}(x + y) \geqslant t\{F_{u,v}(x), F_{v,w}(y)\}$

holds for all $u, v, w \in X$ and all $x \geqslant 0$, $y \geqslant 0$.

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Note that among a number of possible choices for t, $t(a, b) = \min\{a, b\}$ or simply "$t = \min$" is the strongest possible universal (cf. [6], page 318). Moreover, if t satisfies $t(x, x) \geqslant x$ for every $x \in [0, 1]$ then, (f) and (d') imply

(d'') $F_{u,w}(x + y) \geqslant \min\{F_{u,v}(x), F_{v,w}(y)\}$

for all $u, v, w \in X$ and all $x \geqslant 0$, $y \geqslant 0$. Due to the simplicity and universality of "$t = \min$", (d'') willbe used frequently.

The (ε, λ)-topology in $(X, \mathcal{F}, t)$ is introduced by the family of neighbourhoods $\{U_v(\varepsilon, \lambda) : \varepsilon > 0, \lambda \in (0, 1)\}$ of each $v \in X$, where $U_v(\varepsilon, \lambda) = \{u : F_{u,v}(\varepsilon) > 1 - \lambda\}$. If the t-norm is continuous then X is a Hausdorff topological space under this topology.

Results. THEOREM 1. *Let $(X, \mathcal{F}, t)$ be a Menger space, where t is continuous and satisfies $t(x, x) \geqslant x$ for every $x \in [0, 1]$. Suppose T and S_n $(n = 1, 2, \ldots)$ are mappings from X to itself such that*

(1) $S_nT = TS_n$, $n = 1, 2, \ldots$;

(2) $S_n(X) \subseteq T(X)$, $n = 1, 2, \ldots$;

(3) $T(X)$ *is a complete subspaces of* X

and for every $u, v \in X$ and for every pair i, j.

(4) $$F_{S_iu, S_jv}(qx) \geqslant \min\{F_{S_iu, Tu}(x), F_{S_jv, Tv}(x), F_{S_iu, Tv}(2x), F_{S_jv, Tu}(2x), F_{Tu, Tv}(x)\}$$

for all $x > 0$, where $q \in (0, 1)$. Then T and S_n, for each $n = 1, 2, \ldots$, have a unique common fixed point.

The proof of this theorem is prefaced by the following lemma [10].

LEMMA 2. *Let $\{y_n\}$ be a sequence in a Menger space $(X, \mathcal{F}, t)$, where t is continuous and satisfies $t(x, x) \geqslant x$ for every $x \in [0, 1]$. If there is a $q \in (0, 1)$ such that*

$$F_{y_n, y_{n+1}}(qx) \geqslant F_{y_{n-1}, y_n}(x), \quad n = 1, 2, \ldots,$$

for all $x > 0$, then $\{y_n\}$ is a Cauchy sequence in X.

Proof of theorem 1. Pick u_0 in X. We construct a sequence $\{u_n\}$ in X in the following way:

$$Tu_n = S_nu_{n-1}, \quad n = 1, 2, \ldots$$

This can be done since (2) holds.
By (4),

$$F_{Tu_1, Tu_2}(qx) = F_{S_1u_0, S_2u_1}(qx) \geqslant$$
$$\geqslant \min\{F_{Tu_1, Tu_0}(x), F_{Tu_2, Tu_1}(x), F_{Tu_1, Tu_1}(2x), F_{Tu_2, Tu_0}(2x), F_{Tu_0, Tu_1}(x)\}.$$

Since

$$F_{Tu_2, Tu_0}(2x) \geqslant \min\{F_{Tu_2, Tu_1}(x), F_{Tu_1, Tu_0}(x)\},$$

we get,

$$F_{Tu_1, Tu_2}(qx) \geqslant F_{Tu_0, Tu_1}(x).$$

Similarly

$$F_{Tu_2, Tu_3}(qx) \geqslant F_{Tu_1, Tu_2}(x).$$

In general

$$F_{Tu_n, Tu_{n+1}}(qx) \geqslant F_{Tu_{n-1}, Tu_n}(x).$$

So, in view of Lemma 2, $\{Tu_n\}$ is a Cauchy sequence and has a limit in $T(X)$. Call it p. Hence there exists a point z in X such that $Tz = p$,

For $\varepsilon > 0$, $\lambda > 0$, let $U_{S_nz}(\varepsilon, \lambda)$ be a neighbourhood of S_nz. Since $Tu_n \to Tz$, there exists an integer $N(\varepsilon, \lambda)$ such that

(5) $m \geqslant N$ implies $F_{Tu_m, Tz}\left(\frac{1-q}{2q}\varepsilon\right) > 1 - \lambda$ and $F_{Tu_{m+1}, Tz}\left(\frac{1-q}{2q}\varepsilon\right) > 1 - \lambda.$

By (4)

$$F_{Tu_{m+1}, S_nz}(\varepsilon) = F_{S_mu_m, S_nz}(\varepsilon) \geqslant$$

$$\geqslant \min \{F_{Tu_{m+1}, Tu_m}(\varepsilon/q),\ F_{S_nz, Tz}(\varepsilon/q),\ F_{Tu_{m+1}, Tz}(2\varepsilon/q),$$

$$F_{S_nz, Tu_m}(2\varepsilon/q),\ F_{Tu_m, Tz}(\varepsilon/q)\} \geqslant$$

$$\geqslant \min \{F_{Tu_{m+1}, Tu_m}(\varepsilon/q),\ F_{S_nz, Tz}(\varepsilon/q),\ F_{Tu_m, Tz}(\varepsilon/q)\} \geqslant$$

$$\geqslant \min\left\{F_{Tu_{m+1}, Tz}\left(\frac{1-q}{2q}\varepsilon\right),\ F_{Tz, Tu_m}\left(\frac{1+q}{2q}\varepsilon\right),\right.$$

$$\left.F_{S_nz, Tu_{m+1}}\left(\frac{1+q}{2q}\varepsilon\right),\ F_{Tu_{m+1}, Tz}\left(\frac{1-q}{2q}\varepsilon\right),\ F_{Tu_m, Tz}\left(\frac{1-q}{2q}\varepsilon\right)\right\} =$$

$$= \min\left\{F_{Tu_{m+1}, Tz}\left(\frac{1-q}{2q}\varepsilon\right),\ F_{Tu_m, Tz}\left(\frac{1-q}{2q}\varepsilon\right)\right\} > 1 - \lambda \text{ for all } m \geqslant N.$$

Consequently $Tz = S_nz$.

Noting that $TTz = T(S_nz) = S_n(Tz)$, we get by (4),

$$F_{Tz, TTz}(\varepsilon) = F_{S_nz, S_n(Tz)}(\varepsilon) \geqslant$$

$$\geqslant \min \{F_{S_nz, Tz}(\varepsilon/q),\ F_{S_n(Tz), TTz}(\varepsilon/q),\ F_{S_nz, TTz}(2\varepsilon/q),$$

$$F_{S_nTz, Tz}(2\varepsilon/q),\ F_{Tz, TTz}(\varepsilon/q)\} =$$

$$= \min \{1,\ 1,\ F_{Tz, TTz}(2\varepsilon/q),\ F_{TTz, Tz}(2\varepsilon/q),\ F_{Tz, TTz}(\varepsilon/q)\} = F_{Tz, TTz}(\varepsilon/q),$$

proving $TTz = Tz$. Moreover

$$S_n(Tz) = T(S_nz) = T(Tz) = Tz.$$

Thus Tz is a common fixed point of T and S.

To establish the uniqueness of the common fixed point, let for $u \neq v$, $Tu = u = S_n u$ and $Tv = S_n v = v$ $(n = 1, 2, \ldots)$, then by (4) we can easily have

$$F_{u,v}(q\varepsilon) \geqslant F_{u,v}(\varepsilon), \text{ proving } u = v.$$

COROLLARY 3. *If T is taken to be an identity mapping and for each n, $S_n = S$ in Theorem 1, then we obtain a result due to* Ćirić [1].

Theorem 1 has the following metric analogue:

COROLLARY 4. *Let M be a complete metric space, $S_n (n = 1, 2, \ldots)$ a sequence of mappings of M into itself. Let T be a mapping of M into itself such that $S_n T = TS_n$ $(n = 1, 2, \ldots)$, and (2) and (3) hold with $X = M$. If there exists a q such that $0 < q < 1$ and*

$$d(S_i u, S_j v) \leqslant q \max \Big\{ d(S_i u, Tu), d(S_j v, Tv), d(Tu, Tv), \frac{1}{2} d(S_i u, Tv), \frac{1}{2} d(S_j v, Tu) \Big\}$$

for all u, v in X. Then T and S_n have a unique common fixed point.

The above corollary is an improvement over Theorem 1 of Iséki [3]. For other special cases refer to Rhoades [5], Singh [8] and Tiwari and Singh [11].

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STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXII, 2, 1987

UNIQUENESS CLOSURES AND KOROVKIN CLOSURES OF SOME FUNCTION SPACES

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REZUMAT. — Inchideri de unicitate şi inchideri Korovkin ale unor spaţii de funcţii. În lucrare se demonstrează o teoremă generală care, combinată cu anumite teoreme din [1] şi [3], permite regăsirea unor rezultate din teoria Korovkin, conţinut în [2], [4], [5], [6].

1. Let X be a compact Hausdorff space and let K be a compact convex subset of a locally convex Hausdorff space over R. Let $\varphi : X \to K$ be a continuous map; denote $Y = \varphi(X)$.

Let $F \subset C(K)$ be a family of concave functions and let $A \subset C(K)$ be a family of affine functions which separates K and contains the constant function 1. Let H be the linear subspace of $C(K)$ spanned by A and F.

Denote by Prob (X) the set of all probability Radon measures on X. Let e_x be the Dirac measure at $x \in X$.

Let $H\varphi = \{h \circ \varphi : h \in H\}$. Denote by $E(H\varphi)$ the uniqueness closure of $H\varphi$, i.e., the set of all $f \in C(X)$ that satisfy $\mu(f) = f(x)$ for every $\mu \in$ Prob (X) and every $x \in X$ with $\mu = e$ on $H\varphi$.

Various kinds of Korovkin closures of $H\varphi$ coincide with $E(H\varphi)$; see [1], [3].

In this note we are concerned with the determination of $E(H\varphi)$; from the preceding remark and from Theorem 1 below we can deduce some known results about Korovkin closures, contained in [2], [4], [5], [6].

2. If $\mu \in$ Prob (K) let $r(\mu)$ be the barycenter of μ and let $c(\mu) = cl$ (conv (supp μ)).

Let J be the set of all $j \in C(K)$ that are affine on each compact convex subset of K on which every $f \in F$ is affine.

Let G be the set of all $g \in C(Y)$ with the following property: if $\nu \in$ Prob (Y), $r(\nu) \in Y$ and every $f \in F$ is affine on $c(\nu)$, then $\nu(g) = g(r(\nu))$.

Let $S(F)$ be the set of all $x \in K$ such that if $y,\ z \in K,\ y \neq z$, $x = 1(/2)(y + z)$, then there exists $f \in F$ with $f(x) > (1/2)(f(y) + f(z))$.

THEOREM 1. a) *$E(H\varphi)$ coincides with $G\varphi$.*

b) *If $Y \subset S(F)$, then $E(H\varphi) = C(Y)\varphi = \{f \in C(X);$ if $x,\ y \in X$ and $\varphi(x) = \varphi(y)$, then $f(x) = f(y)\}$.*

c) *If $Y = K$, then $E(H\varphi) = J\varphi$.*

d) *If $Y \subset S(F)$ and φ is one-to-one, then $E(H\varphi) = C(X)$.*

Proof. a) Let $\mu \in$ Prob (X), $x \in X$, $\mu = e_x$ on $H\varphi$. If $u \in C(Y)$ let us denote $\nu(u) = \mu(u \circ \varphi)$; then $\nu \in$ Prob (Y). We have $\nu(h) = h(\varphi(x))$ for all $h \in H$; this

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yields $r(\nu) = \varphi(x)$. Using Theorem 1 of [6] we deduce that every $f \in F$ is affine on $c(\nu)$

If $g \in G$, then $\nu(g) = g(r(\nu))$, i.e., $\mu(g \circ \varphi) = (g \circ \varphi)(x)$. It follows that $G\varphi \subset E(H\varphi)$.

Let now $s \in E(H\varphi)$. If $y \in Y$, let $a \in X$ with $\varphi(a) = y$. For each $b \in X$ with $\varphi(b) = y$ we have $e_b = e_a$ on $H\varphi$, hence $s(b) = s(a)$. Thus, if we denote $g(y) = s(a)$, we have $g \in C(Y)$ and $s = g \circ \varphi$. Let us show that $g \in G$.

Let $\nu \in \text{Prob}\,(Y)$ such that $r(\nu) \in Y$ and every $f \in F$ is affine on $c(\nu)$; we have to prove that $\nu(g) = g(r(\nu))$.

Let $u \in C(Y)$. Denote $\mu(u \circ \varphi) = \nu(u)$; then μ is a positive linear functional on $C(Y)\varphi$, $\|\mu\| = \mu(1) = 1$. Using the Hahn–Banach Theorem we obtain a $\lambda \in C(X)'$, $\lambda = \mu$ on $C(Y)\varphi$, $\|\lambda\| = 1$. We have $\|\lambda\| = \lambda(1) = 1$, hence $\lambda \in \text{Prob}(X)$.

Let $x \in X$, $r(\nu) = \varphi(x)$. If $h \in H$, then $\lambda(h \circ \varphi) = \mu(h \circ \varphi) = \nu(h) = h(r(\nu)) = (h \circ \varphi)(x)$. Thus $\lambda = e_x$ on $H\varphi$.

Since $s \in E(H\varphi)$, we have $\lambda(s) = s(x)$. This yields $\lambda(g \circ \varphi) = g(\varphi(x))$, hence $\mu(g \circ \varphi) = g(\varphi(x))$. Finally $\nu(g) = g(r(\nu))$, which completes the proof of a).

b) Let $Y \subset S(F)$. Let $\nu \in \text{Prob}\,(Y)$ having $y \in Y$ as barycenter. If every $f \in F$ is affine on $c(\nu)$, then y is an extreme point of $c(\nu)$, hence $\nu = e_y$. It follows that $G = C(Y)$.

The assertions c) and d) are consequences of a) and b).

3. Let P be a closed linear subspace of $C(X)$ which contains the constant functions and separates X. Let K be the compact convex set $K = \{p \in P';\ p \text{ positive}, p(1) = 1\}$ in the topological dual P' of P endowed with the weak*-topology.

Let $\varphi : X \to K$ be the canonical embedding and let A be the space of all affine functions in $C(K)$. If F is the empty set, then $H\varphi = P$ (see [2]).

Since $E(H\varphi) = G\varphi$, we obtain the following corollary (see also [2])Lemma 1:

COROLLARY 2. *The uniqueness closure (and hence the Korovkin closures mentioned in § 1) of P is the space*
$\{g \circ \varphi : g \in C(Y),\ \nu(g) = g(y)$ *for* all $y \in Y$ *and all* $\nu \in \text{Prob}\,(Y)$ *having y as barycenter*$\}$.

4. Let $X = K$, $\varphi(x) = x$ for all $x \in X$. Applying Theorem 1 we obtain the following result (see also [2], [4], [5], [6]):

COROLLARY 3. *The uniqueness closure (and hance the above mentioned Korovkin closures) of H is J. $E(H)$ coincides with $C(K)$ if and only if $S(F) = K$.*

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THE CONNECTION BETWEEN THE n-DIMENSIONAL STUDENT DISTRIBUTION AND THE INVERTED DIRICHLET DISTRIBUTION

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ABSTRACT. — In this paper we present the connection which exists between the n-dimensional Student distribution and the inverted Dirichlet distribution. This connection was used for an informational characterization of the n-dimensional Student distribution [5].

Let $Z^{(n)} = (Z_1, \ldots, Z_n)$ be an n-dimensional random vector.

DEFINITION 1. [3] *The random vector $Z^{(n)}$ follows a general n-dimensional beta distribution if its probability density function has the form*

$$\beta_n(z_1, \ldots, z_n; a_1, \ldots, a_{n+1}) =$$

$$= \frac{1}{B_n(a_1, \ldots, a_{n+1})} \cdot \prod_{i=1}^{n} \frac{\left(z_i - b_1^{(i)}\right)^{a_i - 1}}{\left(b_2^{(i)} - b_1^{(i)}\right)^{a_i}} \left(1 - \sum_{i=1}^{n} \frac{z_i - b_1^{(i)}}{b_2^{(i)} - b_1^{(i)}}\right)^{a_{n+1}-1}, \tag{1}$$

if $Z^{(n)} \in D_n$, where

$$D_n = \left\{(z_1, \ldots, z_n) \mid 0 < b_1^{(i)} < z_i < b_2^{(i)};\ i = \overline{1, n};\ 1 - \sum_{i=1}^{n} \frac{z_i - b_1^{(i)}}{b_2^{(i)} - b_1^{(i)}} > 0\right\},$$

and

$$a_i > 0,\ i = \overline{1, n+1}, \tag{1b}$$

$$B_n(a_1, \ldots, a_{n+1}) = \frac{\prod_{i=1}^{n+1} \Gamma(a_i)}{\Gamma\left(\sum_{i=1}^{n+1} a_i\right)}, \tag{1c}$$

is the Dirichlet function [2].

DEFINITION 2. [6] *The random vector $X^{(n)} = (X_1, \ldots, X_n)$ follows the inverted Dirichlet distribution if the probability density function has the following form*

$$\bar{\beta}_n(x_1, \ldots, x_n; a_1, \ldots, a_{n+1}) =$$

$$= \frac{1}{B_n(a_1, \ldots, a_{n+1})} \cdot \prod_{i=1}^{n} x_i^{a_i - 1} \left(1 + \sum_{i=1}^{n} x_i\right)^{-\sum_{i=1}^{n+1} a_i}, \tag{2}$$

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if $X^{(n)} \in \bar{D}_n$, *where*

$$\bar{D}_n = \{(x_1, \ldots, x_n) \mid x_i > 0,\ i = \overline{1, n}\}. \tag{2a}$$

$$a_i > 0,\ i = \overline{1, n+1}, \tag{2b}$$

and $B_n(a_1, \ldots, a_{n+1})$ *is the Dirichlet function.*

Remark 1. The general n-dimensional beta distribution, defined explicitly by the probability density function (1), is a distribution which belong to the class of Liouville distributions, namely to the Liouville distribution of the second kind [4]. The Dirichlet distribution, the inverted Dirichlet distribution, the ordered Dirichelt distribution and the n-dimensonal Student dittribution belong to the same class [4].

The connection between the general n-dimensional beta distribution and the inverted Dirichlet distribution is presented in the following theorem.

THEOREM 1. *Applying to the probability density function* (1), *defined in the domain* D_n, *the transformation*

$$z_i = b_1^{(i)} + \frac{x_i}{1 + \sum_{i=1}^{n} x_i}\left(b_2^{(i)} - b_1^{(i)}\right),\ i = \overline{1, n}, \tag{3}$$

we obtain the probability density function (2), *defined in the domain* $\bar{D}_n$, *corresponding to the inverted Dirichlet distribution and the Jacobian of this transformation has the following form*

$$J = \frac{D(z_1, \ldots, z_n)}{D(x_1, \ldots, x_n)} = \frac{1}{\left(1 + \sum_{i=1}^{n} x_i\right)^{n+1}} \cdot \prod_{i=1}^{n}\left(b_2^{(i)} - b_1^{(i)}\right). \tag{3a}$$

Proof. The transformation (3) inwolves the following relation

$$1 - \sum_{i=1}^{n} \frac{z_i - b_1^{(i)}}{b_2^{(i)} - b_1^{(i)}} = \frac{1}{1 + \sum_{i=1}^{n} x_i}, \tag{3b}$$

and the conditions from the relation of definition (1a) of the domain D_n become

$$x_i > 0,\ i = \overline{1, n}, \tag{3c}$$

i.e. just the conditions (2a) which define the domain $\bar{D}_n$.

Because

$$\frac{\partial z_i}{\partial x_i} = \left(1 + \sum_{i=1}^{n} x_i - x_i\right)\left(b_2^{(i)} - b_1^{(i)}\right)\left(1 + \sum_{i=1}^{n} x_i\right)^{-2},\ i = \overline{1, n} \tag{3d}$$

$$\frac{\partial z_j}{\partial x_j} = -\left(b_2^{(i)} - b_1^{(i)}\right) x_i\left(1 + \sum_{i=1}^{n} x_i\right)^{-2},\ i = \overline{1, n};\ j = \overline{1, n};\ i \neq j \tag{3e}$$

the Jacobian of the transformation (3) will have just the form (3a).

The transformation (3), together with the above specification, reduce the probability density function (1) to the probability density function defined by (2).

DEFINITION 3. [1] *The random vector $t = (t_1, \ldots, t_n)$ follows an n-dimensional Student distribution, with s degrees of freedom, if its probability density function has the form*

$$f(t\,;\,s) = \frac{\Gamma\left(\frac{n+s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \cdot \frac{\sqrt{\det A}}{(s\pi)^{\frac{n}{2}}} \left(1 + \frac{Q(t)}{s}\right)^{-\frac{n+s}{2}}, \tag{4}$$

if $t \in \Delta_n$, where

$$\Delta_n = \{(t_1, \ldots, t_n) \mid t_i \in R,\ i = \overline{1, n}\} \tag{4a}$$

$$m = M(t) = (m_1, \ldots, m_n)',\ m_i = M(t_i),\ i = \overline{1, n}\,; \tag{4b}$$

$$A = (a_{ij}),\ i, j = \overline{1, n}\,;\ \ a_{ij} = a_{ji}\,;\ \det A \neq 0, \tag{4c}$$

(A is a symmetric and nonsingular square matrix of order n)

$$s \in Z_+,\ s > 2, \tag{4d}$$

$$Q(t) = (t - m)' \cdot A \cdot (t - m) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(t_i - m_i)(t_j - m_j) \tag{4e}$$

($Q(t)$ is a positive definite quadratic form).

THEOREM 2. *If $t = (t_1, \ldots, t_n)$ is a random vector which follows an n-dimensional Student distribution, then with the help of the transformation*

$$T = T_2^{-1} \cdot T_1 \tag{5}$$

in which

$$T_1 : t = m + C \cdot U \tag{5a}$$

$$T_2 : X = \frac{1}{s} U^2, \tag{5b}$$

where C is a square matrix of order n defined by relation

$$C' \cdot A \cdot C = E\,;\ E \text{ — a unit matrix of order } n, \tag{5c}$$

and

$$|J_T| = \frac{s^{\frac{n}{2}}}{\sqrt{\det A}} \cdot \prod_{i=1}^{n} x_i^{-\frac{1}{2}}, \tag{5d}$$

then we obtain the probability density function (2) associated to the inverted Dirichlet distribution whose parameters a_i take the following particular values

$$a_i = \frac{1}{2},\ i = \overline{1, n}\,;\ a_{n+1} = \frac{s}{2}. \tag{5e}$$

PROOF. Indeed, if we consider the transformation (5) then the corresponding Jacobian will be

$$J_T = J_{T_1} \cdot J_{T_2}^{-1}, \tag{6}$$

respectively,

$$|J_T| = |J_{T_1}| \cdot |J_{T_2}^{-1}|. \tag{6a}$$

If we have in view the explicitly form of the transformation T_1

$$T_1 : t_i + m_i + \sum_{j=1}^{n} c_{ij} u_j, \quad i = \overline{1, n}, \tag{7}$$

then

$$J_T = \frac{D(t_1, \ldots, t_n)}{D(u_1, \ldots, u_n)} = \det C. \tag{7a}$$

The relation (5c) permits us to obtain the following form of the matrix A

$$A = (C \cdot C')^{-1}, \tag{8}$$

and, hence, the following relations

$$A^{-1} = C \cdot C'; \quad A^{-1} \cdot A = C \cdot C' \cdot A = E; \ \det C = \frac{1}{\sqrt{\det A}} \cdot \tag{8a}$$

From (7a) and (8a) we get

$$J_{T_1} = |J_{T_1}| = \frac{1}{\sqrt{\det A}} \cdot \tag{9}$$

Now we make use of the fact that the inverse transformation T_2^{-1} has the form

$$T_2^{-1} : U = \begin{cases} -\sqrt{s \cdot X} & \text{(10a)} \\ +\sqrt{s \cdot X} & \text{(10b)} \end{cases} \tag{10}$$

respectively, the explicitly form

$$T_2^{-1} : u_i = \begin{cases} -\sqrt{s \cdot x_i} & \text{(11a)} \\ +\sqrt{s \cdot x} & \text{(11b)} \end{cases}; \quad i = \overline{1, n}, \tag{11}$$

where

$$X = X^{(n)} \in \bar{D}_n = \{(x_1, \ldots, x_n) \mid x_i > 0, \ i = \overline{1, n}\}. \tag{12}$$

Such we can see that to the domain $\bar{D}_n$ it corresponds, by applying the inverse transformation T_2^{-1}, a number of 2^n distinct domains $\Delta_n(k, n-k)$ whose points $(u_1, \ldots, u_n)$ contain k components of the type (11a), respectively, $(n-k)$ components of the type (11b), $k = \overline{0, n}$. For a such domain fixed, for instance, for the domain $\Delta_n(0, n)$ the transformation which correspond $T_{(0,n)}$ and which leads us from the domain $\bar{D}_n$ to the domain $\Delta_n(0, n)$, will be

$$T_{(0,n)} : u_i = +\sqrt{s \cdot x_i}, \; i = \overline{1, n}. \tag{13}$$

The Jacobian of this transformation (13) will have the following value

$$J_{T_{(0,n)}} = 2^{-n} \cdot s^{\frac{n}{2}} \cdot \prod_{i=1}^{n} x_i^{-\frac{1}{2}} = |J_{T_{(0,n)}}|. \tag{13a}$$

It is easy to see that all the others $2^n - 1$ transformations of the type (11) have the Jacobians with the same absolute values as the absolute value of the Jacobian (13a).

Taking into account this last specification, the absolute value of the Jacobian T_2^{-1} can be represented as follows

$$|J_{T_2^{-1}}| = 2^n \cdot |J_{T_{(0,n)}}| = s^{\frac{n}{2}} \cdot \prod_{i=1}^{n} x_i^{-\frac{1}{2}}. \tag{14}$$

Then, making use of the relations (9) and (14) we get for the absolute value of Jacobian of the transformation (5) just the form (5d). Also, using the same transformation (5), the quadratic form $Q(t)$ can be expressed as follows

$$Q(t) = (t - m)' \cdot A \cdot (t - m) = (C \cdot U)' \cdot A \cdot (C \cdot U) =$$
$$= U' \cdot (C' \cdot A \,.\, C) \cdot U = U' \cdot U$$

i.e.

$$Q(t) \to Q_1(U) = \sum_{i=1}^{n} u_i^2, \tag{15}$$

where $Q_1(U)$ is a cannonical form of the quadratic form $Q(t)$.

This last result together with the explicitly form (11) of the inverse transformation T_2^{-1} implies the following relation

$$1 + \frac{1}{s} Q(t) = 1 + \sum_{i=1}^{n} x_i. \tag{16}$$

In conclusion, if we apply to the probability density function (4) a transformation of the form (5) then it receives the form

$$f(t;\, s) \to f(X;\, s) = f(X^{(n)};\, s) = \beta_n(X^{(n)};\, s). \tag{17}$$

where

$$f(X;\, s) = \frac{\Gamma\left(\frac{n+s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \cdot \frac{\sqrt{\det A}}{(s\pi)^{n/2}} \left(1 + \sum_{i=1}^{n} x_i\right)^{-\frac{n+s}{2}} \cdot |J_T|, \tag{18}$$

respectively, the particular form

$$f(X\,;\,s)=\frac{\Gamma\left(\frac{n+s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\cdot\frac{1}{\Gamma\left(\frac{1}{2}\right)^{n}}\cdot\prod_{i=1}^{n}x_i^{-\frac{1}{2}}\left(1+\sum_{i=1}^{n}x_i\right)^{-\frac{n+s}{2}}, \tag{19}$$

if $X = X^{(n)} \in \bar{D}_n$ and $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$.

This last form (19), of the probability density function $f(X\,;\,s)$ is obtained from the relation (2), which represents the probability density function of the inverted Dirichlet distribution, if we select for the parameterss a_i, $i=\overline{1, n+1}$, the particular values (5e).

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LA REDONDANCE INFORMATIONELLE MULTIPLE

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ABSTRACT. – **Informational Multiple Redundance.** An informational multiple redundance indicator is presented for three or more aleatory variables, as well as its characteristics and some practical applications in mathematical statistics: statistical independency and correspondency.

L'article présente un indicateur informational pour la dépendance aléatoire des trois ou plusieurs variables aléatoires — caractéristiques statistiques avec ses proprietés, en generalisant la redondance pour deux variables aléatoires.

On sait [1] que la redondance des variables aléatoires

$$X\begin{pmatrix} x_i \\ p_i \end{pmatrix}_{i=\overline{1,n}}, \quad Y\begin{pmatrix} y_j \\ q_j \end{pmatrix}_{j=\overline{1,m}}$$

est

$$R(X\,|\,Y) = H(X) - H(X\,|\,Y) \tag{1}$$

ou $H(X)$ est l'entropie de Shannon:

$$H(X) = -\Sigma_i p_i \log p_i$$

$$H(X\,|\,Y) = \Sigma_j q_j H(X\,|\,Y = Y_j)$$

$$H(X\,|Y = y_j) = -\Sigma_i \frac{p_{ij\cdot}}{q_j} \log \frac{p_{ij\cdot}}{q_j}$$

et

$$p_{ij\cdot} = P(X = x_i \cap Y = y_j), \; i = \overline{1,\, n}, \; j = \overline{1,\, m} \;\; \Sigma_{ij} p_{ij\cdot} = 1,$$

caractérisent la réparatition bidimensionelle du vecteur (X, Y). On a aussi $p_i = \Sigma_j p_{ij\cdot}$, $i = \overline{1,\, n}$, $q_j = \Sigma_i\, p_{ij\cdot}$, $j = \overline{1,\, m}$.

Définition. Soit les variables aléatoires X, Y considérées antérieur et Z

$$Z\begin{pmatrix} z_k \\ v_k \end{pmatrix}_{k=\overline{1,r}}.$$

La redondance de X et Y Z est:

$$R(X\,|Y \cap Z) = H(X) - H(X\,|Y \cap Z) \tag{2}$$

$$H(X\,|Y \cap Z) = \Sigma_{j,k} p_{\cdot jk} H(X\,|Y = y_j \cap Z = z_k)$$

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$$H(X|Y = y_j \cap Z = z_k) = -\Sigma_i \frac{p_{ijk}}{p_{.jk}} \log \frac{p_{ijk}}{p_{.jk}}$$

$$p_{ijk} = P((X = x_i \cap Y = y_j \cap Z = z_k), \ \Sigma_{i,j,k} p_{ijk} = 1,$$

$$\Sigma_i p_{ijk} = p_{.jk}, \quad p_{.jk} = P(Y = y_j \cap Z = z_k).$$

On sait [1] que:

$$H(X|Y \cap Z) = H(X \cap Y \cap Z) - H(Y \cap Z) \tag{3}$$

donc il résulte

$$R(X|Y \cap Z) = H(X) + H(Y \cap Z) - H(X \cap Y \cap Z) \tag{4}$$

Aussi de (1) et (2) on a:

$$R(X|Y \cap Z) = R(X|Y) + H(X|Y) - H(X|Y \cap Z) \tag{5}$$

Proprietés. I. La symétrie: $R(X|Y \cap Z) = R(Y \cap Z|X)$ est immédiatement. II. Si X, Y, Z sont indépendantes dans le sens suivant: $\forall\, x, y, z \in R$ la fonction de répartition réspectivement du vecteur (X, Y, Z): F_{XYZ}, de X, F_X et de (Y, Z): F_{YZ}, vérifie la rélation

$$F_{XYZ}(x, y, z) = F_X(x) \cdot F_{YZ}(y, z),$$

alors

$$R(X|Y \cap Z) = 0,$$

et réciproquement.

Démonstration. Si X et $Y \cap Z$ sont indépendantes on a [2]

$$H(X \cap Y \cap Z) = H(X) + H(Y \cap Z)$$

donc de (4) il résulte

$$R(X|Y \cap Z) = 0$$

et réciproquement.

III. Si $X = Y \cap Z$ dans le sens: $p_i = p_{.jk}$, $i = j = k = \overline{1, n}$, on a

$$R(X|Y \cap Z) = H(X)$$

et réciproquement.

IV.

$$0 \leqslant R(X|Y \cap Z) \leqslant H(X).$$

Démonstration. En utilisant l'inégalité de Shannon [3] pour X et $Y \cap Z$ on a: $H(X) \geqslant H(X|Y \cap Z)$, donc

$$R(X|Y \cap Z) = H(X) - H(X|Y \cap Z) \geqslant 0,$$ et aussi $R(X|Y \cap Z) \geqslant H(X)$.

Remarques. 1. On voit immédiatement que III a lieu aussi si $X = Y = Z$ dans le sens considéré.

2. Si $Y = Z$ il résulte $R(X|Y \cap Z) = R(X|Y)$.

3. La redondance normée est

$$\bar{R}(X|Y \cap Z) = 1 - \frac{H(X|Y \cap Z)}{H(X)}, \quad X \neq \text{const.}, \; H(X) \neq 0.$$

4. La redondance de $Y \cap Z$ et X est :

$$R(Y \cap Z|X) = R(X|Y \cap Z).$$

5. Si $Y = f(X)$, $Z = g(X)$, $f, g: R \to R$ ainsi que $f(X)$, $g(X)$ sont des variables aléatoires on a :

$$R(X|f(X) \cap g(X)) = H(X).$$

C'est à dire cet indicateur donne un indication seulement sur la dépendance aléatoire entre X et $Y \cap Z$ et non pas pour l'indépendance déterministe ; il est un indicateur de corrélation.

6. Pratiquement cet indicateur est utilisé dans le cas des caractéristiques statistiques.

7. La redondance pour quatre variables aléatoires est :

$$R(X|Y \cap Z \cap T) = \begin{cases} H(X) - H(X|Y \cap Z \cap T) \\ H(X) + H(Y \cap Z \cap T) - H(X \cap Y \cap Z \cap T) \end{cases} \tag{6}$$

$$R(X \cap Y|Z \cap T) = \begin{cases} H(X \cap Y) - H(X \cap Y|Z \cap T) \\ H(X \cap Y) + H(Z \cap T) - H(X \cap Y \cap Z \cap T) \end{cases} \tag{7}$$

$R(X \cap Y \cap Z|T) = R(T|X \cap Y \cap Z)$ à cause de la symétrie se réduit à (6).

Similairement on peut construire la redondance pour plusieurs que quatre variables aléatoires.

Applications. On peut utiliser la redondance pour établir la dépendance entre trois variables aléatoires et aussi dans le problème de la corespondance statistique.

1. De proprieté II il résulte un criterium pour vérifier l'indépendance des variables aléatoires X, Y, Z. On calcule $R(X|Y \cap Z)$ et $R(Y|Z)$, si

$$R(X|Y \cap Z) = 0 \Rightarrow X, \; Y \cap Z \text{ indépendantes}, \; F_{XYZ}(x, y, z) = F_X(x) F_{YZ}(y, z)$$

et

$$R(Y|Z) = 0 \Rightarrow Y, \; Z \text{ independantes}, \; F_{YZ}(y, z) = F_Y(y) F_Z(z).$$

Donc $F_{XYZ}(x, y, z) = F_X(x) F_Y(y) F_Z(z)$, $x, y, z \in R$. Alors X, Y, Z sont indépendantes.

2. De la proprieté III il résulte le criterium pour la correspondance statistique entre X, Y, Z. On calcule $R(X|Y \cap Z)$ et $R(Y|Z)$, si

$$\left.\begin{array}{l} R(X|Y \cap Z) = H(X) \Rightarrow X = Y \cap Z \\ R(Y|Z) = H(Y) \Rightarrow Y = Z \end{array}\right\} \Rightarrow X = Y = Z \text{ dans le sens des répartitions de probabilités.}$$

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TREILLIS NON-COMMUTATIFS DE TYPE (S)

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ABSTRACT. — **(S)-Type Non-commutative Lattices.** A special class of non-communtative lattices — which we also call the class of the (S)-type non-commutative lattices — is defined. Too, the properties of such a special class of non-commutative lattices are examined in the report. This class is defined in both the language of binary operations and of binary relations, the logical equivalency of these two definitions being demonstrated.

Les treillis non-commutatifs ont été introduits pour la première fois par P. Jordan [4]. Ensuite, en [1], [2] et [3] ont été introduits d'autres classes de treillis non-commutatifs.

Dans ce travail on définie et on etudie les propriétés d'une classe spéciale de treillis non-commutatifs, dénommée par nous la classe des treillis non-commutatifs de type (S). On définie cette classe dans le langage des opérations binaires et dans le langage des relations binaires, à la fois.

1. Considérons donc un ensemble L doté de deux opérations binaires $\wedge$ et $\vee$, qui pour tout $a, b, c \in L$ vérifie les axiomes:

$$(A).\ \begin{cases} (a \wedge b) \wedge c = a \wedge (b \wedge c) \\ (a \vee b) \vee c = a \vee (b \vee c) \end{cases}$$

$$(B).\ \begin{cases} a \wedge (a \vee b) = a \\ a \vee (a \wedge b) = a \end{cases}$$

$$(S).\ \begin{cases} a \wedge (b \vee c) = a \wedge (c \vee b) \\ a \vee (b \wedge c) = a \vee (c \wedge b). \end{cases}$$

Le triplet $(L, \wedge, \vee)$, où L est un ensemble et $\wedge$ et $\vee$ sont deux opérations binaires définies en L qui vérifie les axiomes (A), (B) et (S) s'appelera de treillis non-commutatif de type (S).

On obtient un exemple de treillis non-commutatif de type (S) si dans le produit cartésien $\mathfrak{P}(M) \times \mathfrak{P}(M) = \{(A, B)/A \subseteq M \text{ et } B \subseteq M\}$, où M est un ensemble non-vide, on définie les opérations binaires $\wedge$ et $\vee$ ainsi:

$$(A_1, B_1) \wedge (A_2, B_2) = \begin{cases} (A_1, B_1) \text{ si } A_1 \cap A_2 \neq \emptyset \\ (A_1 \cap A_2, B_1 \cap B_2) \text{ si } A_1 \cap A_2 = \emptyset. \end{cases}$$

$$(A_1, B_1) \vee (A_2, B_2) = \begin{cases} (A_1, B_1) \text{ si } A_1 \cap A_2 \neq \emptyset \\ (A_1 \cup A_2, B_1 \cup B_2) \text{ si } A_1 \cup A_2 = \emptyset \end{cases}$$

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(1.1). *Si* $(L, \wedge, \vee)$ *est le traillis non-commutatif de type* (S), *alors pour tout* $a, b, c \in L$ *sont vraies les égalités*:

(i). $\begin{cases} a \wedge a = a \\ a \vee a = a \end{cases}$

(ii). $\begin{cases} a \wedge b = (a \wedge b) \vee b \wedge a) \\ a \vee b = (a \vee b) \wedge (b \vee a) \end{cases}$

(iii). $\begin{cases} a \wedge (b \wedge c) = a \wedge (c \wedge b) \\ a \vee (b \vee c) = a(\vee (c \vee b) \end{cases}$

(iv). $\begin{cases} a \wedge b \wedge a = a \wedge b \\ a \vee b \vee a = a \vee b. \end{cases}$

Démonstration. (i). En utilisant les axiomes (B) de la définition du treillis non-commutatifs de type (S) on obtient $a \wedge a = a \wedge (a \vee (a \wedge b)) = a$ et $a \vee a = a \vee (a \wedge (a \vee b)) = a$, donc en $(L, \wedge, \vee)$ sont vraies les égalités (i), c'est-à-dire les lois de l'idempotence.

(ii). En utilisant la lois de l'indempotence et les axiomes (S) on apperçoit que pour tout $a, b \in L$ on a $a \wedge b = (a \wedge b) \vee (a \wedge b) = (a \wedge b) \vee (b \wedge a)$ et $a \vee b = (a \vee b) \wedge (a \vee b) = (a \vee b) \wedge (b \vee a)$.

(iii). En utilisant les propriétés (ii) et de nouveau les axiomes (S) on obtient que pour tout $a, b, c \in L$ nous avons $a \wedge (b \wedge c) = a \wedge ((b \wedge c) \vee (c \wedge b)) = a \wedge ((c \wedge b) \vee (b \wedge c)) = a \wedge (c \wedge b)$ et $a \vee (b \vee c) = a \vee ((b \vee c) \wedge (c \vee b)) = a \vee ((c \vee b) \wedge (b \vee c)) = a \vee (c \vee b)$.

(iv). En utilisant les axiomes (A) et les propriétés (i) et (iii) on apperçoit que pour tout $a, b, c \in L$ on a $a \wedge b \wedge a = a \wedge (b \wedge a) = a \wedge (a \wedge b) = (a \wedge a) \wedge b = a \wedge b$ et $a \vee b \vee a = a \vee (b \vee a) = a \vee (a \vee b) = (a \vee a) \vee b = a \vee b$.

Observons que si $(L, \wedge, \vee)$ est treillis non-commutatif de type (S), alors dans l'ensemble L peuvent etre définies les relations binaires ρ_1, ρ_2 et ρ_3 engendrées par les opérations $\wedge$ et $\vee$ à savoir:

$$a \rho_1 b \Leftrightarrow a = a \wedge b$$

$$a \rho_2 b \Leftrightarrow a = b \wedge a$$

$$a \rho_3 b \Leftrightarrow b = a \vee b.$$

(1.2). *Si* $(L, \wedge, \vee)$ *est le treillis non-commutatif de type* (S), *alors les relations binaires* ρ_1, ρ_2 *et* ρ_3 *possèdent les propriétés*:

(i). ρ_1 *est une relation de préordre en* L.

(ii). ρ_2 *et* ρ_3 *sont relations d'ordre en* L.

Démonstration. (i). La propriété de reflexivité de la relation ρ_1 est une conséquence immédiate de la lois de l'idempotence démontrée en (1.1). Ensuite, si pour $a, b, c \in L$ on a $a \rho_1 b$ et $b \rho_1 c$, alors $a = a \wedge b$ et $b = b \wedge c$, donc $a = a \wedge b = a \wedge (b \wedge c) = (a \wedge b) \wedge c = a \wedge c$, c'est-à-dire $a \rho_1 c$ et ainsi ρ_1 possède aussi la propriété de transitivite.

(ii). La propriété de réflexivité de la relations ρ_2 et ρ_3 est, aussi, une conséquence de la lois de l'idempotence. Ensuite, si pour $a, b, c \in L$ on a $a \rho_2 b$ et $b \rho_2 c$, alors $a = b \wedge a$ et $b = c \wedge b$, donc $a = b \wedge a = (c \wedge b) \wedge a = c \wedge (b \wedge a) = c \wedge a$, c'est-à-dire $a \rho_2 c$, mais si pour $a, b \in L$ on a $a \rho_2 b$ et $b \rho_2 a$, alors $a = b \wedge a$ et $b = a \wedge b$, donc $a = b \wedge a = (a \wedge b) \wedge a = a \wedge b \wedge a = a \wedge b = b$, c'est-à-dire ρ_2 possède les propriétés de transitivité et d'antisymétrie. Enfin, si pour $a, b, c \in L$ on a $a \rho_3 b$ et $b \rho_3 c$, alors $b = a \vee b$ et $c = b \vee c$, donc $c = b \vee c = (a \vee b) \vee c = a \vee (b \vee c) = a \vee c$, c'est-à-dire $a \rho_3 c$, mais si pour $a, b \in L$ on a $a \rho_3 b$ et $b \rho_3 a$, alors $b = a \vee b$ et $a = b \vee a$, donc $a = b \vee a = (a \vee b) \vee a = a \vee b \vee a = a \vee b = b$, c'est-à-dire ρ_3 possède aussi les propriétés de transitivité et d'antisymétrie.

(1.3). *Si* $(L, \wedge, \vee)$ *est treillis non-commutatif de type* (S), *alors pour tout* $a, b, c \in L$ *les relations binaires* ρ_1, ρ_2 *et* ρ_3 *possèdent les propriétés:*

(i). $a \wedge b \, \rho_2 \, a$ *et* $a \wedge b \, \rho_1 \, b$

(ii). $a \, \rho_3 \, a \vee b$ *et* $b \, \rho_1 \, a \vee b$

(iii). $a \rho_2 b \Rightarrow a \rho_1 b$ *et* $a \rho_3 b \Rightarrow a \rho_1 b$

(iv). $c \rho_1 a$ *et* $c \rho_1 b \Rightarrow c \rho_1 a \wedge b$

(v). $c \varphi_2 a$ *et* $c \rho_1 b \Rightarrow c \rho_2 a \wedge b$

(vi). $a \rho_1 c$ *et* $b \rho_1 c \Rightarrow a \vee b \, \rho_1 c$

(vii). $a \rho_3 c$ *et* $b \rho_1 c \Rightarrow a \vee b \, \rho_3 c$.

Démonstration. (i). En utilisant la lois de l'idempotence et les axiomes (A) on obtient que pour tout $a, b \in L$ nous avons $a \wedge b = (a \wedge a) \wedge b = a \wedge (a \wedge b)$ et $a \wedge b = a \wedge (b \wedge b) = (a \wedge b) \wedge b$, donc $a \wedge b \, \rho_2 \, a$ et $a \wedge b \, \rho_1 b$

(ii). En utilisant de nouveau la lois de l'idempotence et les axiomes (A), (B) et (S) on obtient $a \vee b = (a \vee a) \vee b = a \vee (a \vee b)$ et $b = b \wedge (b \vee a) = b \wedge (a \vee b)$, donc $a \rho_3 a \vee b$ et $b \rho_1 a \vee b$.

(iii). Si pour $a, b \in L$ on a $a \rho_2 b$, alors $a = b \wedge a$, donc $a = b \wedge a = b \wedge a \wedge b = (b \wedge a) \wedge b = a \wedge b$, c'est-à-dire $a \rho_1 b$. Ensuite, si pour $a, b \in L$ on a $a \rho_3 b$, alors $b = a \vee b$, donc $a = a \wedge (a \vee b) = a \wedge b$, c'est-à-dire $a \rho_1 b$.

(iv). Si pour $a, b, c \in L$ on a $c \rho_1 a$ et $c \rho_1 b$, alors $c = c \wedge a$ et $c = c \wedge b$, donc $c = c \wedge b = (c \wedge a) \wedge b = c \wedge (a \wedge b)$, c'est-à-dire $c \rho_1 a \wedge b$.

(v). Si pour $a, b, c \in L$ on a $c \rho_2 a$ et $c \rho_1 b$, alors $c = a \wedge c$ et $c = c \wedge b$, donc $c = a \wedge c = a \wedge (c \wedge b) = a \wedge (b \wedge c) = (a \wedge b) \wedge c$, c'est-à-dire $c \rho_2 a \wedge b$.

(vi). Si pour $a, b, c \in L$ on a $a \rho_1 c$ et $b \varphi_1 c$, alors $a = a \wedge c$ et $b = b \wedge c$, donc $c \vee (a \wedge c) = c \vee a$ et $c = c \vee (b \wedge c) = c \vee b$, c'est-à-dire $a \vee b = (a \vee b) \wedge (c \vee a \vee b) = (a \vee b) \wedge (c \vee b) = (a \vee b) \wedge c$, en conséquence $a \vee b \, \rho_1 c$.

(vii). Si pour $a, b, c \in L$ on a $a \rho_3 c$ et $b \rho_1 c$, alors $c = a \vee c$ et $b = b \wedge c$, donc $c = a \vee c = a \vee (c \vee (b \wedge c)) = a \vee (c \vee b) = a \vee (b \vee c) = (a \vee b) \vee c$, c'est-à-dire $a \vee b \, \rho_3 c$.

Pour les éléments a, $b \in L$ définissons les notions de infimum et de supremum par rapport aux relations ρ_1, ρ_2 et ρ_3 ainsi :

$$i \in \inf_{(\rho_1,\rho_2)} (a, b) \Leftrightarrow \begin{cases} i \ \rho_2 \ a \ \text{et} \ i \ \rho_1 \ b \\ i' \ \rho_2 \ a \ \text{et} \ i' \ \rho_1 \ b \Rightarrow i' \ \rho_2 \ i \quad (i' \in \varphi L) \\ i' \ \rho_1 \ a \ \text{et} \ i' \ \rho_1 \ b \Rightarrow i' \ \rho_1 \ i \end{cases}$$

$$s \in \sup_{(\rho_1,\rho_3)} (a, b) \Leftrightarrow \begin{cases} a \ \rho_3 \ s \ \text{et} \ b \ \rho_1 \ s \\ a \ \rho_3 \ s' \ \text{et} \ b \ \rho_1 \ s' \Rightarrow s \ \rho_3 \ s' \quad (s' \in L) \\ a \ \rho_1 \ s' \ \text{et} \ b \ \rho_1 \ s' \Rightarrow s \ \rho_1 \ s' \end{cases}$$

(1.4). *Si* $(L, \wedge, \vee)$ *est le treillis non-commutatif de type* (S), *alors pour tout* a, $b \in L$ *on a* $\inf_{(\rho_1,\rho_2)} (a, b) \neq \emptyset$ *et* $\sup_{(\rho_1,\rho_3)} (a, b) \neq \emptyset$.

Démonstration. En utilisant les propriétés des relations ρ_1, ρ_2 et ρ_3 démontrées dans les théorèmes précédents on constante que $a \wedge b \in \inf_{(\rho_1,\rho_2)} (a, b)$ et respectivement $a \vee b \in \sup_{(\rho_1,\rho_3)} (a, b)$.

En retenant les propriétés fondamentales des relations ρ_1, ρ_2 et ρ_3 on peut formuler le suivant théorème conclusif :

(1.5). *Si* $(L, \wedge, \vee)$ *est le treillis non-commutatif de type* (S), *alors dans l'ensemble* L *on peut définir trois relations binaires* ρ_1, ρ_2 et ρ_3 *de manuère que pour tout* a, $b \in L$ *le systèms* $(L, \rho_1, \rho_2, \rho_3)$ *possède les propriétés* :

(1). ρ_1 *est une relation de préordre en* L ;

(2). ρ_2 *et* ρ_3 *sont des relations d'ordre en* L ;

(3). *Pour tout* a, $b \in L$, $a \ \rho_2 \ b \Rightarrow a \ \rho_1 \ b$ *et* $a \ \rho_3 \ b \Rightarrow a \ \rho_1 \ b$;

(4) *Pour tout* a, $b \in L$, $\inf_{(\rho_1,\rho_2)} (a, b) \neq \emptyset$ *et* $\sup_{(\rho_1,\rho_3)} (a, b) \neq \emptyset$.

2. On peut démontrer que l'affirmation inverse du théorème (1.5) est vraie aussi.

(2.1). *Soit* L *un ensemble non vide doté de trois relations binaires* ρ_1, ρ_2 *et* ρ_3. *Si le système* $(L, \rho_1, \rho_2, \rho_3)$ *possède les propriétés* (1)—(4) *mentionnées en* (1.5), *alors dans l'ensemble* L *on peut définir deux opérations binaires* $\wedge$ *et* $\vee$, *de manière que* $(L, \wedge, \vee)$ *devienne treillis non-commutatif de type* (S).

Démonstration. Si i_1, $i_2 \in \inf_{(\rho_1,\rho_2)} (a, b)$, alors de la définition de l'infimum on reçoit $i_1 \ \rho_2 \ i_2$ respectivement $i_2 \ \rho_2 \ i_1$, donc étant donné que ρ_2 est la rleation d'ordre en L résulte que $i_1 = i_2$, par conséquent $\inf_{(\rho_1,\rho_2)} (a, b)$ contient un élément et uniquement un. Par dualité on obtient que $\sup_{(\rho_1,\rho_3)} (a, b)$ aussi contient juste un élément. Cette constatation nous permet que pour tout a, $b \in L$ définissions les opérations binaires $\wedge$ et $\vee$ ainsi :

$$a \wedge b = \inf_{(\rho_1,\rho_2)} (a, b)$$

$$a \vee b = \sup_{(\rho_1,\rho_3)} (a, b).$$

On peut facilement démontrer que les opérations binaires ainsi définies en L vérifient les axiomes (A), (B) et (S), c'est-à-dire $(L, \wedge, \vee)$ devient treillis non-commutatif de type (S).

On constate ensuite que, si au treillis non-commutatif $(L, \wedge, \vee)$ de type (S) on associe par le procédé ci-dessus le système $(L, \rho_1, \rho_2, \rho_3)$ et, au même système on associe, par le procédé indique en (2.1), le treillis non-commutatif de type (S) adéquat, alors les deux treillis non-commutatifs coincident.

Remarquons que si $(L, \wedge, \vee)$ est treillis, alors $\rho_1 = \rho_2 = \rho_3$ et le théorème (2.1) se transforme dans le théorème bien connu de caractérisation des treillis à l'aide d'un ensemble ordoné.

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ON A CLOSE-TO-CONVEXITY PRESERVING INTEGRAL OPERATOR

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ABSTRACT. — Let c be a complex number, with $\operatorname{Re} c > 0$ and let g be an analytic function in the unit disk, with $g(0) = 0$, $g'(0) \neq 0$ and $g(z) \neq 0$, for $0 < |z| < 1$. It is shown that if g satisfies condition (2) and if the integral operator I, defined by (1), preserves the convexity, then I also preserves the close-to-convexity.

1. Introduction. Let $\mathcal{A}$ be the class of all analytic functions f in the unit disk U. Let c be a complex number, with $\operatorname{Re} c > 0$ and let $g \in \mathcal{A}$, with $g(0)=0$, $g'(0) \neq 0$ and $g(z) \neq 0$, for $0 < |z| < 1$. Consider the integral operator $I : \mathcal{A} \to \mathcal{A}$, defined by $F = I(f)$, where

$$F(z) = \frac{1}{[g(z)]^c} \int_0^z f(w)[g(w)]^{c-1} g'(w)dw, \; z \in U, \; f \in \mathcal{A} \tag{1}$$

Making the substitution $w = tz$, (1) can be rewritten

$$F(z) = \left[\frac{z}{g(z)}\right]^c \int_0^1 f(tz) \left[\frac{g(tz)}{tz}\right]^{c-1} g'(tz)t^{c-1}dt,$$

where all powers are the principale ones. This shows that the integral operator I is well defined.

It is well-known that in the particular case $g(z) = z$ and $c = 1$, R. Libera proved that the operator I preserves the starlikeness, the convexity and the close-to-convexity [3]. This remarkable result was extended by many other authors (see, for example, [1], [2], [4], [7]—[13]).

In this paper we show that if g satisfies the condition $\operatorname{Re}[czg'(z)/g(z)] > 0$, in U and if the integral operator I preserves the convexity, then I also preserves the close-to-convexity.

In the case $c = 1$, sufficient conditions on the function g such that I is a convexity-preserving were given in [10]. By using the same technique it is possible to extend this result for all $c > 0$.

2. Preliminaries. We will need the following well-known lemmas to prove our main result.

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LEMMA 1. *If I is an analytic function in U, with* Re $P(0) > 0$ *and if P satisfies*

$$\mathrm{Re}\left[P(z) + \frac{zP'(z)}{P(z)}\right] > 0, \qquad z \in U,$$

then Re $P(z) > 0$ *in U.*

LEMMA 2. *Let P be a complex function, with* Re $P(z) > 0$ *in U. If p is an analytic function in U, with* Re $p(0) > 0$ *and if p satisfies*

$$\mathrm{Re}\left[p(z) + \frac{zp'(z)}{P(z)}\right] > 0, \qquad z \in U,$$

then Re $p(z) > 0$ *in U.*

The above two lemmas are particular cases of some more general results on differential inequalities and subordinations [5], [6].

The function $f \in \mathcal{A}$ is called *convex* if f is univalent and $f(U)$ is a convex domain. Let denote by $\mathcal{K}$ the class of all convex functions in U. It is well-known that

$$f \in \mathcal{K} \Leftrightarrow f'(0) \neq 0 \text{ and } \mathrm{Re}\,\frac{zf''(z)}{f'(z)} + 1 > 0, \; z \in U.$$

The function $f \in \mathcal{A}$ is called *close-to-convex* if there exists a function $\varphi \in \mathcal{K}$ such that

$$\mathrm{Re}\,\frac{f'(z)}{\varphi'(z)} > 0, \qquad z \in U$$

We denote by $\mathcal{C}$ the class of all close-to-convex functions in U. It is well-knowen that each function in $\mathcal{C}$ is univalent.

3. **Main result.** THEOREM. *Let I be the integral operator defined by* (1) *and suppose that*

$$\mathrm{Re}\left[c\,\frac{g'(z)}{g(z)}\right] > 0, \qquad z \in U \tag{2}$$

and

$$I(\mathcal{K}) \subset \mathcal{K}. \tag{3}$$

Then $I(\mathcal{C}) \subset \mathcal{C}$.

Proof. If we let $G(z) = g(z)/[zg'(z)]$, then the condition (2) implies $G \in \mathcal{A}$ and $G(z) \neq 0$ in U. From (1) we obtain

$$zF'(z)G(z) + cF(z) = f(z)$$

and

$$zF''(z)G(z) + [zG'(z) + G(z) + c]F'(z) = f'(z). \tag{4}$$

Let $f \in \mathcal{C}$ and let $\varphi \in \mathcal{K}$, such that

$$\operatorname{Re}\frac{f'(z)}{\varphi'(z)} > 0, \qquad z \in U. \tag{5}$$

If we denote $\Phi = I(\varphi)$, then from (3) we deduce $\Phi \in \mathcal{K}$. We also have

$$z\Phi''(z)G(z) + [zG'(z) + G(z) + c]\Phi'(z) = \varphi'(z). \tag{6}$$

If we let

$$p(z) = \frac{F'(z)}{\Phi'(z)},$$

then (4) can be rewritten in the following form

$$\Phi'(z)G(z)zp'(z) + \{z\Phi''(z)G(z) + [zG'(z) + G(z) + c]\Phi'(z)\}p(z) = f'(z). \tag{7}$$

If we denote

$$P(z) = \frac{z\Phi''(z)}{\Phi'(z)} + 1 + \frac{zG'(z)}{G(z)} + \frac{c}{G(z)}, \tag{8}$$

then from (6) and (7) we obtain

$$p(z) + \frac{zp'(z)}{P(z)} = \frac{f'(z)}{\varphi'(z)}. \tag{9}$$

Letting $z = 0$ in (9), we deduce $\operatorname{Re} p(0) > 0$. From (5) and (9) we obtain

$$\operatorname{Re}\left[p(z) + \frac{zp'(z)}{P(z)}\right] > 0, \qquad z \in U. \tag{10}$$

Since (6) can be written

$$\Phi'(z)G(z)P(z) = \varphi'(z),$$

we easily obtain

$$P(z) + \frac{zP'(z)}{P(z)} = \frac{z\varphi''(z)}{\varphi'(z)} + 1 + c\frac{zg'(z)}{g(z)}.$$

Since φ is convex and g satisfies (2), we deduce

$$\operatorname{Re}\left[P(z) + \frac{zP'(z)}{P(z)}\right] > 0, \qquad z \in U. \tag{11}$$

From (8) we have $\operatorname{Re} P(0) > 0$. Hence, by Lemma 1, the inequality (11), implies $\operatorname{Re} P(z) > 0$ in U and from (10), by using Lemma 2, we conclude that $\operatorname{Re} p(z) > 0$ in U, i.e.

$$\operatorname{Re}\frac{F'(z)}{\Phi'(z)} > 0, \qquad z \in U,$$

which shows that $F \in \mathcal{C}$.

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MAXIMUM PRINCIPLES FOR SOME NONLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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ABSTRACT. — Sôme results given in [7] are generalized for the nonlinear case. Therefore, maximum principles and minimum principles are set forth: (i) the solution of a nonlinear second order differential equation with deviating argument; (ii) the difference of two soltions of such an equation; (iii) the solutions of a nonlinear system of second order differential equations with deviating argument. Some of these principles are applied to the theory of boundary value problems.

1. **Introduction.** Let $a, a_1, b, b_1 \in \mathbb{R}$ be such that $a_1 \leqslant a < b \leqslant b_1$.

DEFINITION 1. (see [7]. A function $y \in C[a_1, b_1] \cap C^2 [a, b]$ is said to satisfy the maximum principle if

$$(\max_{x \in [a_1,b_1]} y(x) = M \geqslant 0 \quad \text{and} \quad y(x_0) = M) \text{ implies } (x_0 \in [a_1, a] \cup [b, b_1]).$$

DEFINITION 2. (see [7]). A function $y \in C [a_1, b_1] \cap C^2 [a, b]$ is said to satisfy the minimum principle if

$$(\min_{x \in [a_1,b_1]} y(x) = m < 0 \text{ and } y(x_0) = m) \text{ implies } (x_0 \in [a_1, a] \cup [b, b_1]).$$

Let us consider the following nonlinear second order differential operator with deviating arguments

$$L(y)(x) : = y''(x) + f(x, y'(x), y(x), y(g_1(x)), \ldots, y(g_m(x))),$$
$$x \in [a, b],$$

where $a_1 \leqslant g_i(x) \leqslant b_1$, $x \in [a, b]$, $i = \overline{1, m}$ and $f: [a, b] \times \mathbb{R}^{m+2} \to \mathbb{R}$.

The object of this paper is to establish maximum principles and minimum principles for the solutions of the following differential inequalities with deviating arguments:

$$L(y) \geqslant 0 \quad (1); \qquad L(y) > 0 \quad (2); \qquad L(y) \leqslant 0 \quad (3);$$
$$L(y) < 0 \quad (4); \qquad L(y) = 0 \quad (5)$$

We follow terminologies and notations in [2], [6] and [7].

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2. **Maximum and minimum principles.** We have

THEOREM 1. *Let* $y \in C[a_1, b_1] \cap C^2[a, b]$ *be a solution of* (1). *We suppose*

(i) $f(x, 0, r, \ldots, r) < 0$, *for all* $r > 0$, $x \in]a, b[$, (6)

(ii) $(t, s \in \mathbb{R}^m, t \leqslant s)$ *implies* $f(x, 0, r, t) \leqslant f(x, 0, r, s)$, (7)

for all $x \in]a, b[$ *and* $r > 0$.

Then y *satisfies the maximum principle.*

Proof. Let $\max\limits_{x \in [a_1, b_1]} y(x) = M > 0$, and $y(x_0) = M$. We suppose that $x_0 \in]a, b[$. We shall show this leads to a contradiction. We obtain

$$L(y)(x_0) = y''(x_0) + f(x_0, 0, M, y(g_1(x_0)), \ldots, y(g_m(x_0))) \leqslant$$
$$\leqslant f(x_0, 0, M, M, \ldots, M) < 0.$$

THEOREM 2. *Let* $y \in C[a_1, b_1] \cap C^2[a, b]$ *be a solution of* (3). *We suppose*

(i) $f(x, 0, r, \ldots, r) > 0$, *for all* $r < 0$, $x \in]a, b[$, (8)

(ii) $(t, s \in \mathbb{R}^m, t \leqslant s)$ *implies* $f(x, 0, r, t) \leqslant f(x, 0, r, s)$, (9)

for all $x \in]a, b[$ *and* $r < 0$.

Then y *satisfies the minimum principle.*

Proof. Let $\min\limits_{x \in [a_1, b_1]} y(x) = m < 0$, and $y(x_0) = m$.

We suppose that $x_0 \in]a. b[$. We have

$$L(y)(x_0) = y''(x_0) + f(x_0, 0, m, y(g_1(x_0)), \ldots, y(g_m(x_0))) \geqslant$$
$$\geqslant f(x_0, 0, m, m, \ldots, m) > 0.$$

We thereby reach a contradiction.

From the Theorem 1 and Theorem 2, we have,

THEOREM 3. *Let* $y \in C[a_1, b_1] \cap C^2[a, b]$ *be a solution of* (5). *We suppose that* f *satisfies* (6), (7), (8) *and* (9). *Then* y *satisfies the maximum and minimum principle.*

By similar arguments we have:

THEOREM 4. *Let* $y \in C[a_1, b_1] \cap C^2[a, b]$ *be a solution of* (2). *If* f *satisfies* (7) *and*

$$f(x, 0, r, \ldots, r) \leqslant 0, \text{ for all } r > 0, \ x \in]a, b[, \tag{10}$$

then y *satisfies the maximum principle.*

THEOREM 5. *Let* $y \in C[a_1, b_1] \cap C^2[a, b]$ *be a solution of* (4). *If* f *satisfies* (9) *and*

$$f(x, 0, r, \ldots, r) \geqslant 0, \text{ for all } r < 0, \ x \in]a, b[, \tag{11}$$

then y *satisfies the minimum principle.*

Remark 1. *For* $f(x, y'(x), y(x), y(g_1(x)), \ldots, y(g_m(x))) = p(x)y'(x) + q(x)y(x) + \sum_{i=1}^{m} q_i(x)y(g_i(x))$, *from the Theorem* 1 — *Theorem* 5, *above, we have Theorem* 1 — *Theorem* 5 *in* [7].

Example 1. If $y \in C[-1, 3] \cap C^2[0, 2]$ is a solution of

$$y''(x) + (1 + \sin x)y'(x) - 3y(x) - e^{-y(x-1)} - e^{-y(x+1)} \geqslant 0 \quad x \in [0, 2],$$

then y satisfies the maximum principle.

3. Maximum and minimum principles for the difference of two solutions. Consider the following nonlinear second order differential equation with deviating arguments

$$L(y)(x) := y''(x) + h(x, y(x), y'(x)) + q(x)y(x) + \sum_{i=1}^{m} q_i(x)y(g_i(x)) = t(x) \quad (12)$$

where

$$x \in [a, b], \; h: [a, b] \times \mathbb{R}^2 \to \mathbb{R}, \; q, q_i, g_i, t: [a, b] \to \mathbb{R}.$$

and $a_1 \leqslant g_i(x) \leqslant b_1$, for all $x \in [a, b]$ and $i = \overline{1, m}$.

We have

THEOREM 6. *We suppose*

(i) $q_i(x) \geqslant 0$, *and* $q(x) + \sum_{i=1}^{m} q_i(x) \leqslant 0$, *for all* $x \in [a, b]$,

(ii) $(s_1 < s_2)$ *implies* $h(x, s_1, r) > h(x, s_2, r)$, *for all* $x \in [a, b]$ *and* $r \in \mathbb{R}$.

If y *and* z *are solutions of* (12), *then the function* $u \in C[a_1, b_1] \cap C^2[a, b]$, $u = y - z$ *satisfies the maximum and the minimum principle.*

Proof. Let $\max_{x \in [a_1, b_1]} u(x) = M > 0$, and $u(x_0) = M$.

We suppose $x_0 \in]a, b[$. We have

$$y(x_0) - z(x_0) > 0, \; y'(x_0) = z'(x_0), \; y''(x_0) - z''(x_0) \leqslant 0.$$

Since y and z satisfy (12) in x_0, hence

$$0 = y''(x_0) \; z''(x_0) + h(x_0, y(x_0), y'(x_0)) - h(x_0, z(x_0), z'(x_0)) +$$

$$+ q(x_0)u(x_0) + \sum_{i=1}^{m} q_i(x_0)u(g_i(x_0)) \leqslant$$

$$\leqslant h(x_0, y(x_0), y'(x_0)) - h(x_0, z(x_0), z'(x_0)) +$$

$$+ (q(x_0) + \sum_{i=1}^{m} q_i(x_0))M \leqslant h(x_0, y(x_0), y'(x_0)) - h(x_0, z(x_0), z'(x_0)) < 0.$$

By a similar argument we prove that the function $u \in C[a_1, b_1] \cap C^2[a, b]$ satisfies the minimum principle.

Remark 2. *If* $q = q_i = 0$, $i = \overline{1, m}$, *then from the Theorem* 6 *we have a result given in* [1] (see [4] pp. 47—49).

Remark 3. *Let us consider the following boundary value problem*

$$L(y) = h,\ y \in C[a_1, b_1] \cap C^2[a, b] \tag{12'}$$

$$y|_{[a_1, a]} = \varphi,\quad y|_{[b, b_1]} = \psi \tag{13}$$

From the Theorem 6, *we have*

THEOREM 7. *Let* L *be as in the Theorem* 6. *Then the problem* (12′) + (13) *has at most one solution.*

Example 2. Consider the following boundary value problem

$$y''(x) - 3y(x) + \sin y'(x) + y(x-1) + y(x+1)) = h,\quad x \in [0, 4] \tag{14}$$

$$y|_{[-1, 0]} = \varphi,\quad y|_{[4, 5]} = \psi \tag{15}$$

By the Theorem 7, the problem (14) + (15) has at most one solution.

4. Generalizations. We begin with: DEFINITION 3. (see [7]). A function $y \in C([a_1, b_1], \mathbb{R}^n) \cap C^2([a, b], \mathbb{R}^n)$ satisfies the maximum principle if there exists a component y_k of y with the following properties.

(i) $\max\limits_{x \in [a_1, b_1]} y_k(x) = M > 0$,

(ii) $y \leqslant M$,

(iii) $\{x \in [a_1, b_1] \mid y_k(x) = M\} \subset [a_1, a] \cup [b, b_1]$.

DEFINITION 4. (see [7]). A function $y \in C([a_1, b_1], \mathbb{R}^n) \cap C^2([a, b], \mathbb{R}^n)$ satisfies the minimum principle if there exists a component y_k of y with the following properties

i) $\min\limits_{x \in [a_1, b_1]} y_k(x) = m < 0$,

(ii) $y \geqslant m$,

(iii) $\{x \in [a_1, b_1] \mid y_k(x) = m\} \subset [a_1, a] \cup [b, b_1]$

Consider the following nonlinear second order differential operators with deviating arguments

$$\begin{aligned} L_k(y)(x) := y_k''(x) + f_k(x, y_k'(x), y_k(x);\ & y_1(g_{k,1,1}(x)), \ldots, y_1(g_{k,1,m}(x)); \\ & y_2(g_{k,2,1}(x)), \ldots, y_2(g_{k,2,m}(x)); \ldots; y_n(g_{k,n,1}(x)), \ldots, y_n(g_{k,n,m}(x))), \end{aligned} \tag{16}$$

where $x \in [a, b]$, $a_1 \leqslant g_{k,j,i}(x) \leqslant b_1$, $a_1 \leqslant a < b \leqslant b_1$, $k = \overline{1, n}$ and $f_k \colon [a, b] \times \mathbb{R}^{nm+2} \to \mathbb{R}$.

By a similar arguments in § 2, we have

THEOREM 8. *Let* $y \in C([a_1, b_1], \mathbb{R}^n) \cap C^2([a, b)], \mathbb{R}^n)$, $y \not\leqslant 0$, *be a solution of the following system of differential inequalities.*

$$L_k(y)(x) \geqslant 0,\ \textit{for all}\ x \in [a, b],\ k = \overline{1, n}$$

We suppose

(i) $f_k(x, 0, r, \ldots, r) < 0$, *for all* $r > 0$, $x \in [a, b]$, $k = \overline{1, n}$ (17)

(ii) $(t, s \in \mathbb{R}^{nm}, t \leqslant s)$ *implies* $f_k(x, 0, r, t) \leqslant f_k(x, 0, r, s)$, (18)

for all $x \in [a, b]$, $r > 0$, *and* $k = \overline{1, n}$.

Then y *satisfies the maximum principle.*

THEOREM 9. *Let* $y \in C([a_1, b_1], \mathbb{R}^n) \cap C^2([a, b], \mathbb{R}^n)$, $y \not> 0$ *be a solution of the following system of differential inequalities*

$$L_k(y)(x) \leqslant 0, \text{ for all } x \in [a, b] \text{ and } k = \overline{1, n}.$$

We suppose

(i) $f_k(x, 0, r, \ldots, r) \leqslant 0$, *for all* $r < 0$, *and* $x \in]a, b[$, $k = \overline{1, n}$, (19)

(ii) $(t, s \in \mathbb{R}^{nm}, t \leqslant s)$ *implies*

$$f_k(x, 0, r, t) \leqslant f_k(x, 0, r, s), \tag{20}$$

for all $x \in [a, b]$, $r < 0$, *and* $k = \overline{1, n}$.

Then y *satisfies the minimum principle.*

THEOREM 10. *Let* $y \in C([a_1, b_1], \mathbb{R}^n) \cap C^2([a, b], \mathbb{R}^n)$, $y \neq 0$, $y \geqslant 0$, *be a solution of the following system of differential equations*

$$L_k(y)(x) = 0, \text{ for all } x \in [a, b] \text{ and } k = \overline{1, n}.$$

If f_k, $k = \overline{1, n}$, *satisfy* (17), (18), (19) *and* (20), *then* y *satisfies the maximum and the minimum principle.*

Remark 4. *From Theorem* 8, *Theorem* 9 *and Theorem* 10, *we have, Theorem* 1, *Theorem* 2 *and Theorem* 3 *respectively.*

Remark 5. *From Theorem* 8, *Theorem* 9 *and Theorem* 10, *we have, Remark* 5, *Remark* 6 *and Theorem* 7 *in* [7] *respectively.*

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THEOREMS ON COMMUTING MAPPINGS SATISFYING A RATIONAL INEQUALITY

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ABSTRACT. – We prove two fixed point theorems for commuting mappings satisfying a rational inequality which general,es Theorem 3 from [1] and Theorem 1 from [3].

In [1] B. Fisher gave the following theorem

THEOREM 1. *Let S and T be mappings of the complete metric space into itself such that for all x, y in X either*

$$d(Sx, Ty) \leqslant \frac{c\, d(x, Sx) \cdot d(y, Ty) + bd(x, Ty) \cdot d(y, Sx)}{d(x, Sx) + d(y, Ty)} \tag{1}$$

if $d(x, Sx) + d(y, Ty) \neq 0$ *where* $b \geqslant 0$ *and* $1 < 0 < 2$, *or*

$$d(Sx, Ty) = 0$$

otherwise. Then each of S and T has a fixed point and these points coincide.

Let (X, d) be a metric space. We denote by $CB(X)$ the set of all nonempty closed bounded subsets of (X, d) and by H the Hausdorff-Pompeiu metric on $CB(X)$.

Let $A, B \in CB(X)$ and $k < 1$. In what follows, the following wellknown fact will be used: For each $a \in A$, there is a $b \in B$ such that

$$d(a, b) \leqslant k\, H(A, B).$$

Let (X, d) be a metric space, we denote

$$\delta(A, B) = \sup \{d(a, b) : a \in A \text{ and } b \in B\}$$

where $A, B \in CB(X)$. If A consists of a single point „a" we write $\delta(A, B) = \delta(a, B)$. If $\delta(A, B) = 0$ then $A = B = \{a\}$ (Lemma 1 [2]). In a recent paper [3] H. Kaneko proved the following theorem .

THEOREM 2. *Let* $T : X \to CB(X)$ *and let* $f : X \to Y$ *and* $g : X \to X$ *be continuous mappings for which* $T(X) \subset f(X) \cap g(X)$. *Suppose that* T *commutes with* f *and* g *and there exists* $h \in (0, 1)$ *such that*

$$H(Tx, Ty) \leqslant h\, d\,(fx, fy)$$

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for each $x, y \leqslant X$. *Moreover, assume that one of the following holds:*

either: (i) $f(x) \neq f^2(x)$ *implies* $f(x) \cap T(x) = \emptyset$

or (ii) $g(x) \neq g^2(x)$ *implies* $g(x) \cap T(x) = \emptyset$.

Then T *has a fixed point in* X, *which is also a fixed point of* f *or* g.

Theorems 3,4 of this paper extends theorems 1,2 for commuting functions and multifunctions.

THEOREM 3. *Let* (X, d) *be a complete metric space,* f, g *continuous mappings from* X *into itself and* S, T *continuous multifunctions from* X *into* $CB(X)$. *Suppose*

(1) *Each of* f, g *commutes with the three others and* $S(X) \cup T(X) \subset fg(X)$

$$H(Sx, Ty) \leqslant \frac{c \cdot d(fx, Sx) \cdot d(gy, Ty) + b\, d(fx, Ty) \cdot d(gy, Sx)}{\delta(fx, Sx) + \delta(gy, Ty)} \tag{2}$$

or all $x, y \in X$ *for which* $\delta(fx, Sx) + \delta(gy, Ty) \neq 0$ *where* $b \geqslant 0$ *and* $1 < c < 2$.

Moreover, assume that one of the following holds:

either: (a) $f(x) \neq f^2(x)$ *implies* $f(x) \cap S(x) = \emptyset$

or (*b*) $g(x) \neq g^2(x)$ *implies* $g(x) \cap T(x) = \emptyset$.

Then S *and* f *or* T *and* g *have a common fixed point.*

Proof. Chosse a real number k with

$$1 < k < 2/c. \tag{3}$$

Take $x_0 \in X$ and denote $u = fg$, using condition (1) we construct a sequence $\{x_n\}$ such that

$$ux_{2n+1} \in Sg\, x_{2n}\,;\ ux_{2n+2} \in Tf\, x_{2n+1}$$

$$d(u\, x_{2n+1},\ u\, x_{2n+2}) \leqslant kH\,(Sg\, x_{2n},\ Tf\, x_{2n+1})$$

and

$$d(ux_{2n+1},\ ux_{2n}) \leqslant kH(Sg\, x_{2n},\ Tf\, x_{2n-1}).$$

Suppose first of all that

$$\delta(fg\, x_{2n},\ Sg\, x_{2n}) + \delta(gf\, x_{2n+1},\ Tf\, x_{2n+1}) = 0$$

for some n. Then $fg\, x_{2n} = \{Sg\, x_{2n}\} = gf\, x_{2n+1} = \{Tf\, x_{2n+1}\}$.
By (b) we have

$$g(fx_{2n+1}) = g^2(fx_{2n+1}) = g\,(Tf\, x_{2n+1}) = T\,(fg\, x_{2n+1}) = T\,(gf\, x_{2n+1})$$

and $gf\, x_{2n+1}$ is a common fixed point for g and T.

Similarly

$$\delta(fgx_{2n}, Sgx_{2n}) + \delta(gfx_{2n-1}, Tfx_{2n-1}) = 0$$

for some n implies that fgx_{2n} is a common fixed point for f and S.

Now suppose that

$$\delta(fgx_{2n}, Sgx_{2n}) + \delta(gfx_{2n+1}, Tfx_{2n+1}) \neq 0$$

for $n = 1, 2, \ldots$. Then by (2) we have succesively

$$d(ux_{2n+1}, ux_{2n+2}) \leqslant kH(Sgx_{2n}, Tfx_{2n+1}) \leqslant$$

$$\leqslant k \cdot \frac{cd(fgx_{2n}, Sgx_{2n})\, d(gfx_{2n+1}, Tfx_{2n+1}) + bd(fgx_{2n}, Tfx_{2n+1})\, d(gfx_{2n+1}, Sgx_{2n})}{\delta(fgx_{2n}, Sgx_{2n}) + \delta(gfx_{2n+1}, Tfx_{2n+1})} \leqslant$$

$$\leqslant k \cdot \frac{cd(u, x_{2n}, ux_{2n+1}) \cdot d(ux_{2n+1}, ux_{2n+2})}{d(ux_{2n}, ux_{2n+1}) + d(ux_{2n+1}, ux_{2n+2})}.$$

If $d(ux_{2n+1}, ux_{2n+2}) = 0$ then $ux_{2n+1} = ux_{2n+2} \leqslant Sgx_{2n}$,

$ux_{2n+1} = ux_{2n+2} \in Tfx_{2n+1}$. By (b) and $gfx_{2n+1} \in Tfx_{2n+1}$ we have

$$g(fx_{2n+1}) = g^2(fx_{2n+1}) = g(gfx_{2n+1}) \in g(Tfx_{2n+1}) = T(gfx_{2n+1})$$

and gfx_{2n+1} is a common fixed point for g and T.

If $d(ux_{2n+1}, ux_{2n+2}) \neq 0$ then

$$d(ux_{2n+1}, ux_{2n+2}) \leqslant (ck - 1) \cdot d(ux_{2n}, ux_{2n+1}).$$

Similarly we have

$$d(ux_{2n}, ux_{2n+1}) \leqslant (ck - 1) \cdot d(ux_{2n-1}, ux_{2n}).$$

Repeating the above argument, we obtained

$$d(ux_n, ux_{n+2}) \leqslant ck - 1)^n d(ux_0, ux_1) \text{ for } n = 0, 1, 2, \ldots.$$

Since $0 < ck - 1 < 1$ by (3), then by a routine calculation one can show that $\{ux_n\}$ is a Cauchy sequence and since X is complete, we have $\lim ux_n = v$ for some $v \in X$.

Then $fgx_{2n+1} \in Sgx_{2n}$ implies

$$f(ux_{2n+1}) \in f(Sgx_{2n}) = S(fgx_{2n}) = Su(x_{2n}).$$

Taking the limit as $n \to \infty$, $f(v) \in Sv$. Similarly, $g(v) \in Tv$. Hence in the event that (a) holds we get

$$f(v) = f^2(v) \in f(Sv) = Sf(v)$$

thus $f(v)$ is a common fixed point of f and S. If (b) holds then $g(v)$ must be a common fixed point of g and T.

THEOREM 4. *Let (X, d) be a complete metric space, f, g continuous mappings from X into itself and S, T continuous multifunctions from X into $CB(X)$. Suppose that*

(1) *Each of f, g commutes with the three othsers and* $S(X) \cup T(X) \subset fg(X)$

(2) $H^p(Sx, Ty) \leqslant \frac{cd(x,Sx) \cdot d^p(gy, Ty) + bd(fx, Ty) \cdot d^p(gy, Sx)}{\delta(fx, Sx) + \delta(gy, Ty)}$

for all $x, y \in X$ *for which* $\delta(fx, Sx) + \delta(gy, Ty) \neq 0$, *where* $p \geqslant 1$, $b \geqslant 0$ *and* $1 < c < 2$. *Moreover, assume that one of the following holds:*

either: (a) $f(x) \neq f^2(x)$ *implies* $f(x) \cap Sx = \emptyset$

or (b) $g(x) \neq g^2(x)$ *implies* $g(x) \cap Tx = \emptyset$.

Then S and f or T and g have a common fixed point.

Proof. It is similar to the proof of the Theorem 3 and Theorem 2 by [4].

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SOLUTION OF CAUCHY'S PROBLEM BY DEFICIENT LACUNARY SPLINE INTERPOLANTS

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ABSTRACT. – The aim of this paper is to obtain approximate solution of Cauchy's problem for second order differential equation by deficient quartic splines which interpolate a given data. The convergence of the approximate solution to the exact solution is studied.

1. Introduction. Let us consider the Cauchy's initial value problem

$$y'' = f(x, y, y'), \qquad x \in [0, 1] \tag{1.1}$$

$$y(0) = y_0, \; y'(0) = y'_0.$$

Here we assume that $f[x, y(x), y'(x)] \in C^r[0, 1]$, $r \geqslant 0$ and that it satisfies the Lipschitz condition

$$|f^{(q)}(x, y_1, y'_1) - f^{(q)}(x, y_2, y'_0)| \leqslant$$

$$\leqslant L[|y_1 - y_2| + |y'_1 - y'_2|] \; (q = 0(1)r) \tag{1.2}$$

for all $x \in [0, 1]$ and all reals y_1, y_2, y'_1, y'_2. These conditions ensure the existence of unique solution of the problem (1.1).

In the last several years various authors have used spline functions for the approximate solution of initial value problems including (1.1). It is known (see for example [1] and [4]) that the spline functions of full continuity do not converge to exact solution for arbitrary degrees of the spline. For this reason the continuity conditions are relaxed. M i c u l a [6] has constructed a deficient spline function to approximate the solution of (1.1). F a w z y [2] has obtained the set of approximate values

$$\bar{Y}^{(q)} : \bar{y}_0^{(q)}, \bar{y}_1^{(q)}, \ldots, \bar{y}_n^{(q)} \qquad q = 0(1)(r + 2)$$

which are approximations to the exact values

$$\bar{Y}^{(q)} : y_0^{(q)}, y_1^{(q)}, \ldots, y_n^{(q)} \qquad q = 0(1)(r + 2)$$

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On the basis of these approximate values he has constructed a spline function S_Δ which interpolates to the set $\bar{Y}$ on the mesh Δ and approximates the solution $(y(x)$ of (1.1). The set $\bar{Y}^{(q)}$ is defined as:

$$\bar{y}_0 = y_0,\ \bar{y}'_0 = y'_0,\quad \bar{y}_0^{(2+q)} = f^{(q)}(x_0, y_0, y'_0)\ (q = 0(1)r),$$

$$\bar{y}_{k+1} = \bar{y}_k + h\bar{y}'_k + \int_{x_k}^{x_{k+1}} \int_{x_k}^{t} f[u,\ y_k^*(u),\ y_k^{**\prime}(u)]\, du\, dt,$$

$$\bar{y}'_{k+1} = \bar{y}'_k + \int_{x_k}^{x_{k+1}} f[t,\ y_k^*,\ y_k^{**\prime}(t)]\, dt,$$

$$\bar{y}_k^{(2+q)} = f^{(q)}(x_{k+1},\ \bar{y}_{k+1},\ \bar{y}'_{k+1})\ (q = 0(1)r),\ k = 0(1)(n-1),$$

where

$$y_k^{**\prime}(x) = \bar{y}'_k + \int_{x_k}^{x} f[t,\ y_k^*(t),\ y_k^*(t)]dt.$$

Here

$$x_k = \frac{h}{n},\quad k = 0(1)n,\quad h = \frac{1}{n}$$

and for $x_k \leqslant x \leqslant x_{k+1}$

$$y_k^*(x) = \sum_{j=0}^{r+2} \frac{\bar{y}_k^{(j)}}{j!}(x - x_k)^j,$$

$$y_k^{*\prime}(x) = \sum_{j=0}^{r+1} \frac{\bar{y}_k^{(j+1)}}{j!}(x - x_k)^j.$$

The error of the approximate values $\bar{y}_{k+1}^{(j)}$ are estimated by the inequalities

$$|\, y_{k+1}^{(j)} - \bar{y}_{k+1}^{(j)} \,| \leqslant c_j\, \omega_{r+2}(h) h^{r+2}, \tag{1.3}$$

where

$$k = 0(1)(n-1) \text{ and } j = 0(1)(r+2).$$

For the values $\bar{y}_1$ and $\bar{y}'_1$ even sharper estimates

$$|y_1 - \bar{y}_1| \leqslant c_0 \omega_{r+2}(h) h^{r+4},\quad |y'_1 - \bar{y}'_1| \leqslant c_1\, \omega_{r+2}(h)\, h^{r+3} \tag{1.4}$$

are valid (see [2] lemma 2.2.1 and 2.2.3). Here c_js denote different constants and $\omega_{r+2}(h)$ is the modulus of continuity of $y^{(r+2)}(x)$.

The aim of this paper is to obtain approximate solution of (1.1) by deficient quartic splines which interpolate a given lacunary data. We also discuss the convergence of the approximate solution to the exact solution. In fact, taking the Fawzy's definitions of $\bar{y}_k^{(q)}$'s we construct spline functions $S(x)$, $G(x)$ and $H(x)$ which are described in section 2. In section 3 we shall consider the convergence of the approximating spline functions.

2. The spline functions $S(x)$, $G(x)$ **and** $H(x)$. Let

$$\Delta : 0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$$

be a uniform partition of the interval $I = [0, 1]$ with $x_{k+1} - x_k = \frac{1}{n} = h$, $k = 0(1)(n-1)$. We denote by $S_{n,4}^{(2)}$ the class of splines $S(x)$ such that $S(x) \in C^2(I)$ and $S(x)$ is a quartic in each $[x_k, x_{k+1}]$, $k = 0(1)(n-1)$. Suppose $\bar{y}_0^{(q)}, \bar{y}_1^{(q)} \ldots, \bar{y}_n^{(q)}$, $q = 0, 1, 2$ are given real numbers. Then for given Δ there exist unique spline functions $\bar{S}_\Delta(x)$, $\bar{G}_\Delta(x)$ and $\bar{H}_\Delta(x) \in S_{n,4}^{(2)}$ satisfying respectively the conditions:

$$\text{(A)}\qquad \begin{aligned} &\bar{S}_\Delta(x_k) = \bar{y}_k \\ &\bar{S}''_\Delta(x_k) = \bar{y}''_k, \quad k = 0(1)n \\ &\bar{S}'_\Delta(x_0) = \bar{y}'_0; \end{aligned}$$

$$\text{(B)}\qquad \begin{aligned} &\bar{G}_\Delta(x) = \bar{y}'_k \\ &\bar{G}''_\Delta(x_k) = \bar{y}''_k, \quad k = 0(1)\,n \\ &\bar{G}_\Delta(x_0) = \bar{y}_0; \end{aligned}$$

$$\text{(C)}\qquad \begin{aligned} &\bar{H}_\Delta(x_k) = \bar{y}_k \\ &\bar{H}'_\Delta(x_k) = \bar{y}'_k, \quad k = 0(1)\,n \\ &\bar{H}''_\Delta(x_0) = \bar{y}''_0. \end{aligned}$$

The existence and uniqueness of these spline functions have been shown in [7]. For the explicit expression we mention only two of them, namely $S_\Delta(x)$ and $G_\Delta(x)$.

$$\bar{S}_\Delta(x) = \begin{cases} \bar{S}_0(x) \text{ when } x_0 \leqslant x \leqslant x_1 \\ \bar{S}_k(x) \text{ when } x_k \leqslant x \leqslant x_{k+1},\ k = 1(1)(n-1), \end{cases} \tag{2.1}$$

where

$$\bar{S}_0(x) = \bar{y}_0 + (x - x_0)\bar{y}'_0 + \frac{1}{2!}(x - x_0)^2\bar{y}''_2 + \frac{1}{3!}(x - x_0)^3\bar{a}_{0,3} + \frac{1}{4!}(x - x_0)^4\bar{a}_{0,4} \tag{2.2}$$

and

$$\bar{S}_k(x) = \bar{y}_k + (x - x_k)\bar{a}_{k,1} + \frac{1}{2!}\bar{y}''_k + \frac{1}{3!}(x - x_k)^3\bar{a}_{k,3} + \frac{1}{4!}(x - x_k)^4\bar{a}_{k,4}. \qquad (2.3)$$

Here

$$\bar{a}_{0,3} = \frac{12}{h^2}\left(\bar{y}_1 - \bar{y}_0 - h\bar{y}'_0 - \frac{h^2}{2}\bar{y}''_0\right) - \frac{1}{h}(\bar{y}''_1 - \bar{y}''_0)$$

$$\bar{a}_{0,4} = -\frac{24}{h^4}\left(\bar{y}_1 - \bar{y}_0 - h\bar{y}'_0 - \frac{h^2}{2}\bar{y}''_0\right) + \frac{4}{h^2}(\bar{y}''_1 - \bar{y}''_0),$$

$$\bar{a}_{k,3} = -\frac{12}{h}\bar{a}_{k,1} + \frac{12}{h}\left(\bar{y}_{k+1} - \bar{y}_k - \frac{h^2}{2}\bar{y}''_k\right) - \frac{1}{h}(\bar{y}''_{k+1} - \bar{y}''_k)$$

$$\bar{a}_{k,4} = \frac{24}{h}\bar{a}_{k,1} - \frac{24}{h}\left(\bar{y}_{k+1} - \bar{y}_k - \frac{h^2}{2}\bar{y}''_k\right) + \frac{4}{h}(\bar{y}''_{k+1} - \bar{y}''_k),$$

$$\bar{a}_{1,1} = \bar{y}'_0 + h\bar{y}''_0 + \frac{2}{h}\left(\bar{y}_1 - \bar{y}_0 - h\bar{y}'_0 - \frac{h^2}{2}\bar{y}''_0\right) + \frac{h}{6}(\bar{y}''_1 - \bar{y}''_0)$$

$$\bar{a}_{k+1,1} + \bar{a}_{k,1} = \frac{2}{h}(\bar{y}_{k+1} - \bar{y}_k) + \frac{h}{6}(\bar{y}''_{k+1} - \bar{y}''_k), \quad k = 1(1)(n-2)$$

$$\bar{G}_\Delta(x) = \begin{cases} \bar{G}_0(x) \text{ when } x_0 \leqslant x \leqslant x_1 \\ \bar{G}_k(x) \text{ when } x_k \leqslant x \leqslant x_{k+1}, \ k = 1(1)(n-1), \end{cases} \qquad (2.4)$$

where

$$\bar{G}_0(x) = \bar{y}_0 + (x - x_0)\bar{y}'_0 + \frac{1}{2!}(x - x_0)^2\bar{y}''_0 + \frac{1}{3!}(x - x_0)^3\bar{b}_{0,3} + + \frac{1}{4!}(x - x_0)^4\bar{b}_{0,4} \qquad (2.5)$$

$$\bar{G}_k(x) = \bar{b}_{k,0} + (x - x_k)\bar{y}'_k + \frac{1}{2!}(x - x_k)^2\bar{y}''_k + \frac{1}{3!}(x - x_k)^3\bar{b}_{k,3} + + \frac{1}{4!}(x - x_k)^4\bar{b}_{k,4}. \qquad (2.6)$$

Here

$$\bar{b}_{0,3} = \frac{6}{h}(\bar{y}'_1 - \bar{y}'_0 - h\bar{y}''_0) - \frac{2}{h}(\bar{y}''_1 - \bar{y}''_0)$$

$$\bar{b}_{0,4} = -\frac{12}{h}(\bar{y}'_1 - \bar{y}'_0 - h\bar{y}''_0) + \frac{6}{h}(\bar{y}''_1 - \bar{y}''_0),$$

$$\bar{b}_{k,3} = -\frac{12}{h}(\bar{y}'_{k+1} - \bar{y}'_k - h\bar{y}''_k) - \frac{2}{h}(\bar{y}''_{k+1} - \bar{y}''_k)$$

$$\bar{b}_{k,4} = -\frac{12}{h}(\bar{y}'_{k+1} - \bar{y}'_k - h\bar{y}''_k) + \frac{6}{h}(\bar{y}''_{k+1} - \bar{y}''_k),$$

$$\bar{b}_{1,0} = \bar{y}_0 + h\bar{y}'_0 + \frac{h^2}{2}\bar{y}''_0 + \frac{h}{2}(\bar{y}'_1 - \bar{y}'_0 - h\bar{y}''_0) - \frac{h^2}{12}(\bar{y}''_1 - \bar{y}''_0)$$

$$\bar{b}_{k+1,0} - \bar{b}_{k,0} = h\bar{y}'_k + \frac{h^2}{2}\bar{y}''_k + \frac{h}{2}(\bar{y}'_{k+1} - \bar{y}'_k - h\bar{y}''_k) - \frac{h}{12}(\bar{y}''_{k+1} - \bar{y}''_k),$$

$$k = 1(1)(n-1).$$

3. Convergence of spline functions to the solution. In this section we will be concerned with the convergence of our spline functions $S_\Delta(x)$ and $G_\Delta(x)$ given in section 2 to the exact solution of (1.1). We prove that they satisfy the differential equation as $n \to \infty$.

Let $\bar{S}_\Delta(x)$ (respectively $\bar{G}_\Delta(x)$) be the spline function corresponding to the approximate values $\bar{y}_k$ and let $S_\Delta(x)$ (respectively $G_\Delta(x)$) be the spline function corresponding to the exact values y_k of (1.1). Then we have

THEOREM 3.1.

$$|S_0^{(q)}(x) - \bar{S}_0^{(q)}(x)| \leqslant k_q\,\omega_4(h)h^{6-q}, \quad q = 0(1)4, \tag{3.1}$$

$$|S_k^{(q)}(x) - \bar{S}_k^{(q)}(x)| \leqslant c_q\omega_4(h)h^{3-q}, \quad q = 0(1)3, \quad k = 1(1)(n-1). \tag{3.2}$$

Here and onward k_q, c_q, λ_q, ... denote different constants independent of h.

THEOREM 3.2.

$$|G_0^{(q)}(x) - \bar{G}_k^{(q)}(x)| \leqslant \lambda_q\,\omega_4(h)h^{6-q}, \quad q = 0(1)4 \tag{3.3}$$

$$|G_k(x) - \bar{G}_k(x)| \leqslant \mu_0\omega_4(h)h^3 \tag{3.4}$$

$$|G_k^{(q)}(x) - \bar{G}_k^{q}(x)| \leqslant \mu_q\omega_4(h)h^{5-q}, \quad q = 1(1)4. \tag{3.5}$$

Proof. of theorem 3.1.

We have owing to (2.2)

$$S_0(x) - \bar{S}_0(x) = \frac{(x-x_0)^3}{3!}(a_{0,3} - \bar{a}_{0,3}) + \frac{(x-x_0)^4}{4!}(a_{0,4} - \bar{a}_{0,4}).$$

Now

$$a_{0,3} - \bar{a}_{0,3} = \frac{12}{h^3}(y_1 - \bar{y}_1) - \frac{1}{h}(y''_1 - \bar{y}''_1)$$

and

$$a_{0,4} - \bar{a}_{0,4} = -\frac{24}{h^4}(y_1 - \bar{y}_1) + \frac{4}{h^2}(y''_1 - \bar{y}''_1).$$

Using (1.4) we have

$$|a_{0,3} - \bar{a}_{0,3}| \leqslant 12c_0h^3\omega_4(h) + c_2h^3\omega_4(h) \leqslant c_5h^3\omega_4(h)$$

and

$$|a_{0,3} - \bar{a}_{0,4}| \leqslant 24c_0h^2\omega_4(h) + 4c_2h^2\omega_4(h) \leqslant c_6h^2\omega_4(h).$$

Hence

$$|S_0(x) - \bar{S}_0(x)| \leqslant k_0 h^6 \omega_4(h)$$

and by succesive differentiation

$$|S_0^{(q)}(x) - \bar{S}_0^{(q)}(x)| \leqslant k_h h^{6-q} \omega_4(h), \; q = 1(1)4.$$

This proves (3.1). Further owing to (2.3)

$$S_k(x) - \bar{S}_k(x) = (y_k - \bar{y}_k) + (x - x_k)(a_{k,1} - \bar{a}_{k,1}) + \frac{(x - x_k)^2}{2!}(y_k'' - \bar{y}_k'') +$$
$$+ \frac{(x - x_k)^3}{3!}(a_{k,3} - \bar{a}_{k,3}) + \frac{(x - x_k)^4}{4!}(a_{k,4} - \bar{a}_{k,4}).$$

We shall first prove

$$|a_{k,1} - \bar{a}_{k,1}| \leqslant c_7 h^2 \omega_4(h). \tag{3.6}$$

We have owing to (1.3) and lemma 1 in [7]

$$|a_{k,1} - \bar{a}_{k,1}| \leqslant |a_{k,1} - y_k'| + |\bar{a}_{k,1} - \bar{y}_k'| + |y_k' - \bar{y}_k'| \leqslant$$
$$\leqslant \frac{h^2}{3}\omega_4(h) + c_1 h^4 \omega_4(h) \leqslant c_7 h^2 \omega_4(h).$$

Now

$$a_{k,3} - \bar{a}_{k,3} = -\frac{12}{h^2}(a_{k,1} - \bar{a}_{k,1}) + \frac{12}{h^3}(y_{k+1} - \bar{y}_{k+1}) + -\frac{12}{h}(y_k - \bar{y}_k) -$$
$$-\frac{5}{h}(y_k'' - \bar{y}_k'') - \frac{1}{h}(y_{k+1}'' - \bar{y}_{k+1}'')$$

and

$$a_{k,4} - \bar{a}_{k,4} = \frac{24}{h^3}(a_{k,1} - \bar{a}_{k,1}) - \frac{12}{h^4}(y_{k+1} - \bar{y}_{k+1}) + \frac{24}{h^4}(y_k - \bar{y}_k) +$$
$$+ \frac{8}{h^2}(y_k'' - \bar{y}_k'') + \frac{4}{h^2}(y_{k+1}'' - \bar{y}_{k+1}'')$$

which on using (3.6) and (1.3) gives

$$|a_{k,3} - \bar{a}_{k,3}| \leqslant c_8 \omega_4(h) \text{ and } h|a_{k,4} - \bar{a}_{k,4}| \leqslant c_9 \omega_4(h)$$

Hence

$$|S_k(x) - \bar{S}_k(x)| \leqslant c_{10} h^3 \omega_4(h)$$

and by differentiation process

$$|S_k^{(q)} - \bar{S}_k^{(q)}(x)| \leqslant c_q h^{3-q} \omega_4(h), \; q = 1(1)3.$$

This completes the proof of the theorem.

Proof of theorem 3.2. From (2.5) we have

$$G_0(x) - \bar{G}_0(x) = \frac{(x-x_0)^3}{3!}(b_{0,3} - \bar{b}_{0,3}) + \frac{(x-x_0)^4}{4!}(b_{0,4} - \bar{b}_{0,4}).$$

Here

$$b_{0,3} - \bar{b}_{0,3} = \frac{6}{h^2}(y_1' - \bar{y}_1') - \frac{2}{h}(y_1'' - \bar{y}_1'')$$

and

$$b_{0,4} - \bar{b}_{0,4} = -\frac{12}{h^3}(y_1'' - \bar{y}_1'') + \frac{6}{h^2}(y_1'' - \bar{y}_1'').$$

Using (1.3) and (1.4) we atonce have

$$|b_{0,3} - \bar{b}_{0,3}| \leqslant c_{11}\omega_4(h)\, h^{3}, \quad |b_{0,4} - \bar{b}_{0,4}| \leqslant c_{12}\omega_4(h)\, h^2$$

which in turn gives

$$|G_0(x) - \bar{G}_0(x)| \leqslant \lambda_0\omega_4(h)\, h^6.$$

By differentiation we arrive at (3.3).

To prove (3.4) and (3.5) we have from (2.6)

$$G_k(x) - G_k(x) = (b_{k,0} - \bar{b}_{k,0}) + (x - x_k)(y_k' - \bar{y}_k') + \frac{(x-x_k)^2}{2!}(y_k'' - \bar{y}_k'') +$$

$$+ \frac{(x-x_k)^3}{3!}(b_{k,3} - \bar{b}_{k,3}) + \frac{(x-x_k)^4}{4!}(b_{k,4} - \bar{b}_{k,4}).$$

We first estimate $|b_{k,0} - \bar{b}_{k,0}|$. We write

$$|b_{k,0} - \bar{b}_{k,0}| \leqslant |b_{k,0} - y_k| + |\bar{b}_{k,0} - \bar{y}_k| + |y_k - \bar{y}_k|.$$

Now Using (1.3) and the lemma 2 in [7] we have

$$|b_{k,0} - \bar{b}_{k,0}| \leqslant c_{13}\omega_4(h)\, h^3. \tag{3.7}$$

Now

$$b_{k,3} - \bar{b}_{k,3} = \frac{6}{h^2}(y_{k+1}' - \bar{y}_{k+1}') - \frac{6}{h^2}(y_k' - \bar{y}_k') - \frac{2}{h}(y_{k+1}'' - \bar{y}_{k+1}'') - \frac{4}{h}(y_k'' - \bar{y}_k'')$$

and

$$b_{k,4} - \bar{b}_{k,4} = -\frac{12}{h^3}(y_{k+1}' - \bar{y}_{k+1}') + \frac{12}{h^3}(y_k' - \bar{y}_k') + \frac{6}{h^2}(y_{k+1}'' - \bar{y}_{k+1}'') +$$

$$+ \frac{6}{h^2}(y_k'' - \bar{y}_k'').$$

Using (1.3) we get

$$|b_{k,3} - \bar{b}_{k,3}| \leqslant c_{14}\omega_4(h)\, h^2 \tag{3.8}$$

$$|b_{k,4} - \bar{b}_{k,4}| \leqslant c_{15}\omega_4(h)\, h^3. \tag{3.9}$$

Hence

$$|G_k(x) - \bar{G}_k(x)| \leqslant \omega_4(h)\, h^3.$$

This proves (3.4). Again

$$G_k' - \bar{G}_k'(x) = (y_k' - \bar{y}_k') + (x - x_k)(y_k'' - \bar{y}_k'') + \frac{(x - x_k)^2}{2!}(b_{k,3} - \bar{b}_{k,3}) +$$

$$+ \frac{(x - x_k)^3}{3!}(b_{k,4} - \bar{b}_{k,4}).$$

Owing to (1.3), (3.8) and (3.9) we have $|G_k'(x) - \bar{G}_k'(x) \leqslant \omega_4(h)\, h^4$. On further difference

$$|G_k^{(q)}(x) - \bar{G}_k^{(q)}(x)| \leqslant {}_q\omega_4(h)\, h^{5-q},\ q = 2,\ 3,\ 4.$$

This completes the proof of theorem 3.2.

The following theorems give error estimates between the approximating splines $\bar{S}_\Delta(x)$ & $\bar{G}_\Delta(x)$ and the exact solution of (1.1).

THEOREM 3.3. *Let* $y(x)$ *be the exact solution of* (1.1). *Then*

$$|y^{(q)}(x) - \bar{S}_\Delta^{(q)}(x)| \leqslant {}_q\omega_4(h)\, h^{3-q},\ q = 0(1)3. \tag{3.6}$$

THEOREM 3.4. *The following estimates are valid.*

$$|y(x) - \bar{G}_\Delta(x)| \leqslant \omega_4(h)\, h^3 \tag{3.7}$$

$$|y^{(q)} - \bar{G}_\Delta^{(q)}(x)| \leqslant \omega_4(h)\, h^{4-q},\ q = 1(1)4. \tag{3.8}$$

Proof of theorem 3.4. We write

$$|y(x) - \bar{G}_\Delta(x)| \leqslant |y(x) - G_\Delta(x)| + |(G_\Delta(x) - \bar{G}_\Delta(x)|.$$

From theorem 5 in [7] the following estimates are valid:

$$|y^{(q)}(x) - G_\Delta^{(q)}(x)| \leqslant 3h^{4-q}\omega_4(h),\ q = 1(1)4$$

and

$$|y(x) - G_\Delta(x)| \leqslant h^3\omega_4(h).$$

Using these estimates and the estimates in theorem 3.2, we have

$$|y^{(q)}(x) - \bar{G}_\Delta^{(q)}(x)| \leqslant |y^{(q)}(x) - G_\Delta^{(q)}(x)| + |G_\Delta^{(q)}(x) - \bar{G}_\Delta^{(q)}(x)| \leqslant$$

$$\leqslant 3h^{4-q}\omega_4(h) + {}_q\omega_4(h)h^{5-q} \leqslant \omega_4(h)\, h^{4-q},\ q = 1(1)4.$$

which proves (3.8). Similarly we obtain (3.7).

The following theorems show that $\bar{S}_\Delta(x)$ and $\bar{G}_\Delta(x)$ satisfy the differential equation (1.1) as $n \to \infty$.

THOEREM 3.5. *We have*

$$|\bar{S}''_\Delta(x) - f(x, \bar{S}_\Delta(x), \bar{S}'_\Delta(x))| \leqslant c_{14} h \omega_4(h), \tag{3.9}$$

$$|\bar{G}''_\Delta(X) - f(x, \bar{G}_\Delta(x), \bar{G}'_\Delta(x))| \leqslant h^2 \omega_4(h). \tag{3.10}$$

Proof. We write

$$\bar{G}''_\Delta(x) - f(x, \bar{G}_\Delta(x), \bar{G}'_\Delta(x)) = \bar{G}''_\Delta(x) - y''(x) + y''(x) -$$
$$- f(x, \bar{G}_\Delta(x), \bar{G}'_\Delta(x)) = [\bar{G}''_\Delta(x) - y''(x)] +$$
$$+ [f(x, G_\Delta(x), G'_\Delta(x)) - f(x, \bar{G}_\Delta(x), \bar{G}'_\Delta(x))].$$

Therefore owing to Lipschitz condition

$$|\bar{G}''_\Delta(x) - f(x, \bar{G}_\Delta(x), \bar{G}'_\Delta(x))| \leqslant |\bar{G}''_\Delta(x) - y''(x)| +$$
$$+ L[|y(x) - \bar{G}_\Delta(x)| + |y'(x) - \bar{G}'_\Delta(x)|]$$

which on using theorem 3.4 for $q = 0, 1, 2$ atonce gives (3.10). Similarly we have (3.9).

Remark. We find that the approximating spline $\bar{G}_\Delta(x)$ offers better approximation to the solution of the differential equation (1.1) than the spline function $\bar{S}_\Delta(x)$.

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APPLICATIONS DE FORMES DIFFÉRENTIELLES EXTÉRIEURES EN DYNAMIQUE DU POINT À MASSE VARIABLE

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ABSTRACT. — Applications of the External Differentiation Forms in the Dynamics of a Variable Mass Point. The movement of a variable mass point is given by differential equations (1). In the paper, some geometrisation is given to the equations of movement, and a couple of consequencies are deduced.

Le mouvement d'un point de masse variable est décrit par des équations différentielles de Méchtcherski-Levi-Civita [2] de la forme:

$$dq - vdt = 0, \quad mdv - Fdt - (u - v)\, dm = 0, \text{ où } m = m(t). \tag{1}$$

Nous proposons de donner aux équations de mouvement, de la forme (1), une géométrisation et d'en déduire certaines conséquences.

1. Soient un domaine $V_3 \subset \mathbb{R}^3$ et $M = T(V_3) \times \mathbb{R}^2$

$$u : M \to \mathbb{R}^3 \text{ et } F : T(V_3) \times \mathbb{R} \to \mathbb{R}^3.$$

Les équations de la forme (1) (sur M) reçoivent une signification précise.

DÉFINITION. On appelle système géométrique, associé au point principal, l'ensemble (M, F, u) formé par l'espace M et les fonctions F et u.

Au système (M, F, u) on associe cannoniquement

$$\Omega_r = mdv^i \wedge dq^i + (F_i dq^i - mv^i dv^i) \wedge dt + (u^i - v^i)(dq^i - v^i dt) \wedge dm; \tag{2}$$

appelée forme géométrique réduite du système géométrique considéré. On a:

PROPOSITION 1. *Dand les points ou $m \neq 0$, le rang de la forme Ω_r est 6.*

PROPOSITION 2. *La forme (2) est invariante à l'action du groupe de Galilée.*

PROPOSITION 3. *La différentielle extérieure de la forme géométrique réduite ne s'annule jamais.*

Etant donné que la forme Ω_r ne peut pas être fermée, on peut trouver une autre forme géométrique „complète" qui admette les mêmes caractéristiques que Ω_r, mais qui soit, en plus, fermée.

On déduit la 2-forme:

$$\Omega_c = mdv^i \wedge dq^i + (E_i dq^i + mP^i dv^i) \wedge dt + B^i dq^j \wedge dq^k + \\ + m^2 Q_i dv^j \wedge dv^k + (u^i - v^i)(dq^i + P^i dt) \wedge dm - [(u - v) \times Q]^i dv^i \wedge dm, \tag{3}$$

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avec les relations de liaison

$$F = E + v \times B, \quad -v = P + F \times Q. \tag{4}$$

PROPOSITION 4. *Le rang de la forme* (3) *est* 6.

PROPOSITION 5. *La classe de formes géométriques complètes* $\{\Omega_c\}$ *est invariante à l'action du groupe de Galilée.*

On obtient pour les coefficients les valeurs:

$$\bar{E} = A(E) + A(B) \times b, \quad \bar{B} = A(B), \quad A \in SO(3).$$
$$\bar{P} = A(P) - b, \quad \bar{Q} = A(Q), \tag{5}$$

La forme différentielle Ω_c contient 15 fonctions: E_i, B^i, P^i, Q_i et u^i, reliées entre elles par les relations (4), en nombre de six; B, Q et u étant arbitraires il résulte E et P. La détermination des fonctions B, Q et u se fait en imposant la condition $d\Omega_c = 0$, nomée le principe de Maxwell.

2. **Systèmes dynamiques.** Considérons sur $M = T(V_3) \times \mathbb{R}$ une fonction $m: (q^i, v^i, t) \to m$, à valeurs réelles (positives).

L'image réciproque de la 2-forme géométrique Ω, est:

$$\Omega^* = \left[m\delta_{ij} - (u^i - v^i)\frac{\partial m}{\partial v^j}\right] dv^j \wedge dq^i +$$
$$+ \left[F_i + (u^i - v^i)\frac{\partial m}{\partial t} + v \cdot (u - v)\frac{\partial m}{\partial q^i}\right] dq^i \wedge dt +$$
$$+ \left[v(u - v)\frac{\partial m}{\partial v^i} - mv^i\right] dv^i \wedge dt + \left[(u - v) \times \frac{\partial m}{\partial q^j}\right]^i dq^j \wedge dq^k.$$

PROPOSITION 6. *La condition nécessaire et suffisante que la partie géométrique* $\omega = a_{ij}dv^j \wedge dq^i$, de la forme Ω^*, soit sur $T(V_3)$, la forme symplectique cononique $mdv^i \wedge dq^i$, est que $\forall i, j$, $(u^i - v^i)\frac{\partial m}{\partial v} = 0$.

Ω^*, avec $\omega = mdv^i \wedge dq^i$, s'appelle „forme dyn que". La forme géométrique Ω, et la fonction $m = m(q, v, t)$ sont compatibles si *Ω st une forme dynamique, la relation $(u - v)\frac{\partial m}{\partial v^j} = 0$ est app lée „cond i n e compatibilité". On a les cas:

a) Si $u(q, v, t) = v$ il résulte la forme:

$$\Omega^* = m^i dv^i \wedge dq^i + (F_i dq^i - mv^i dv^i) \wedge dt, \tag{6}$$

appelée „forme dynamique réduite" (cas dégénéré) du système.

b) Si $\frac{\partial m}{\partial v^i} = 0$, on obtient:

$$\Omega_s^* = mdv^i \wedge dq^i + (E_i dq^i - mv^i dv^i) \wedge dt + B^i dq^j \wedge dq^k, \tag{7}$$

appelée „forme dynamique semi-réduite", et où

$$E = F + (u - v)\frac{\partial m}{\partial t} + v \cdot (u - v)\frac{\partial m}{\partial q}, \quad B = (u - v) \times \frac{\partial m}{\partial q}. \tag{8}$$

Propriétés: 1° A une forme Ω_r^* on associe le champ de forces

$$F' = E + v \times B = F + (u - v)\left(\frac{\partial m}{\partial q} v + \frac{\partial m}{\partial t}\right);$$

2° Sur les trajectoires on obtient $F' = F + (u - v)\frac{dm}{dt}$;

c) Si l'on vérifie simultanément les conditions: $\frac{\partial m}{\partial v^i} = \frac{\partial m}{\partial q^i} = 0$, on obtient la forme réduite (cas non dégénéré):

$$\Omega_r'^* = m \cdot dv^i \wedge dq^i + (F_i' dq^i - mv^i dv^i) \wedge dt, \tag{9}$$

où $F' = F + (u - v)\frac{dm}{dt}$ (cas de Méchtcherski — T. Levi — Civita).

DÉFINITION. On appelle système dynamique l'ensemble (M, F, u, m), formé par un système géométrique (M, F, u) et une fonction $m = m(q, v, t)$, satisfaisant la condition de compatibilité.

Généralement les formes (6), (7) ou (9) ne sont pas fermées.

a') En appliquant le principe de Maxwell à la forme (6) on obtient le système:

$$\text{rot}\,_q F = 0, \quad \frac{\partial m}{\partial q} = \frac{\partial m}{\partial v} = 0, \quad \frac{\partial F_i}{\partial v^h} + \delta_{ih}\frac{dm}{dt} = 0, \tag{10}$$

d'où

$$F_i = -\alpha(t)v^i + f_i(q, t), \text{ à rot}\,_q f = 0, \ m = m(t), \ m' = \alpha. \tag{11}$$

Si Ω_r^* est fermée, a') est un sous-cas du cas c).

b') Supposons le cas c) avec $\Omega_r'^*$ fermée, on obtient:

$$F = -u\frac{dm}{dt} + f(q, t), \tag{12}$$

$$F' = f(q, t) - v\frac{dm}{dt}.$$

c') Considérons le cas b) et demandons que (7) soit fermée.

$$\text{rot}\,_q E + \frac{\partial B}{\partial t} = 0, \ \text{div}\, B = 0,$$

$$\frac{\partial E_i}{\partial v^h} + v^h\frac{\partial m}{\partial q^i} + \delta_{hi}\frac{dm}{dt} = 0, \tag{13}$$

$$\frac{\partial B^i}{\partial v^h} - \left(\frac{\partial m}{\partial q^j}\delta_{kh} - \frac{\partial m}{\partial q^h}\delta_{kj}\right) = 0.$$

Supposons que la forme (7) n'est pas fermée. Soient la forme géométrique complète (3), la fonction $m = m(q, v, t)$ et la condition de compatibilité :

$$(u - v)^i \frac{\partial m}{\partial v^k} + m[(u - v) \times Q]^k \frac{\partial m}{\partial q^i} = 0. \tag{14}$$

On obtient pour $\Omega_c^* = m^*\Omega_c$, l'expression :

$$\Omega_c^* = m \cdot dv^i \wedge dq^i + (E_i' dq^i + mP'^i dv^i) \wedge dt + B'^i dq^j \wedge dq^k + m^2 Q_i' dv^j \wedge dv^k, \tag{15}$$

où

$$E' = E + (u - v)\frac{\partial m}{\partial t} - [(u - v)P]\frac{\partial m}{\partial q}, \quad B' = B + (u - v) \times \frac{\partial m}{\partial q},$$

$$P' = P - (u - v)P\frac{\partial \ln m}{\partial v} - [(u - v) \times Q]\frac{\partial m}{\partial t}, \quad Q' = Q - [(u - v) \times Q]\frac{\partial \ln m}{\partial v} \tag{16}$$

La force totale $F' = E' + v \times B'$ se compose de F et d'une „force réactive" $\varphi = (u - v)\left[\frac{\partial m}{\partial q} v + \frac{\partial m}{\partial t}\right] - (u - v)(P + v)\frac{\partial m}{\partial q}$.

On distingue les cas particuliers suivants :

a'') Si $u - v = 0$ il résulte $m^*\Omega_c = \Omega_c$.

b'') Si $\frac{\partial m}{\partial q} = \frac{\partial m}{\partial v} = 0$, on a :

$$E' = E + (u - v)\frac{dm}{dt}, \quad B' = B,$$

$$P' = P - [(u - v) \times Q]\frac{dm}{dt}, \quad Q' = Q, \tag{17}$$

$$F' = F + (u - v)\frac{dm}{dt}.$$

c'') Si $\frac{\partial m}{\partial v} = 0$, $(u - v) \parallel Q$, on obtient :

$$E' = E + (u - v)\frac{\partial m}{\partial q} - (u - v)P\frac{\partial m}{\partial q}, \quad P' = P; \tag{18}$$

$$B' = B + (u - v) \times \frac{\partial m}{\partial q}, \quad Q' = Q.$$

Nous nous proposons maintenant de déterminer les conditions que les coefficients de la forme (15) doivent accomplir de manière que cette forme soit fermée. L'annulation des coefficients de la 3-forme $d\Omega_c^*$ nous conduit à

$$
\begin{gathered}
\operatorname{rot}_q E' + \frac{\partial B'}{\partial t} = 0, \ \operatorname{div}_q B' = 0, \ \operatorname{rot}_v (mP') + \frac{\partial (m^2 Q')}{\partial t} = 0, \\
\operatorname{div}_v (m^2 Q') = 0, \ \frac{\partial B'^i}{\partial v^e} + \delta_{ej} \frac{\partial m}{\partial q^h} - \delta_{eh} \frac{mq}{\partial q^j} = 0, \\
\frac{\partial (m^2 Q'_i}{\partial q^e} + \delta_{ej} \frac{\partial m}{\partial v^e} - \delta_{eh} \frac{\partial m}{\partial v^j} = 0, \\
\frac{\partial E'_i}{\partial v^k} - \frac{\partial (mP'^k)}{\partial q^i} + \delta_{ik} \frac{\partial m}{\partial t} = 0.
\end{gathered}
\tag{19}
$$

En remplacant dans (19) les valeurs de E', B', P' et Q', données par (16) on obtient les conditions que les coeficients de la forme (3) doivent satisfaire, à côté de (14) pour qu'une telle forme décrive la dynamique du point à masse variable, donnée par les équations (1), avec la vérification du principe de Maxwell.

Dans les cas particuliers considérés, on a :

a^{iv}) Les coefficients E', B', P' et Q' étant egaux à E, B, P et Q, on obtient pour ceux derniers les mêmes équations (19).

b^{iv}) En tenant compte en (19) des conditions $\frac{\partial m}{\partial q} = \frac{\partial m}{\partial v} = 0$, aussi que des relations (17), on arrive au système :

$$
\begin{gathered}
\operatorname{rot}_q E + \frac{\partial B}{\partial t} + \operatorname{rot}_q u \frac{dm}{dt} = 0, \ \operatorname{div}_q B = 0, \frac{\partial B}{\partial v} = 0, \\
\operatorname{rot}_v P + m \frac{\partial Q}{\partial t} + \left(\operatorname{div}_v u \cdot Q - \frac{\partial u}{\partial v^h} Q_h \right) \frac{\partial m}{\partial t} = 0, \ \operatorname{div}_v Q = 0, \frac{\partial Q}{\partial q} = 0, \\
\frac{\partial E_i}{\partial v^k} - m \frac{\partial P^k}{\partial q^i} + \frac{\partial u^i}{\partial v^k} \frac{dm}{dt} + m \left[\frac{\partial u}{\partial q^i} \times Q \right]^k = 0.
\end{gathered}
$$

c^{iv}) Si on tient compte en (19) des relations (18) il résulte :

$$
\begin{gathered}
\operatorname{rot}_q E + \frac{\partial B}{\partial t} + \operatorname{rot}_q u \frac{\partial m}{\partial t} + \left[\frac{\partial u}{\partial t} - \frac{\partial (u - v) P}{\partial q} \right] \times \frac{\partial m}{\partial q} = 0, \\
\operatorname{div}_q B + \operatorname{rot}_q u \frac{\partial m}{\partial q} = 0, \ \operatorname{rot}_q P + m \frac{\partial Q}{\partial t} + 2Q \frac{\partial m}{\partial t} = 0, \ \operatorname{div}_v Q = 0, \\
\frac{\partial B^i}{\partial v^h} + \frac{\partial u^j}{\partial v^j} \frac{\partial m}{\partial q^k} - \frac{\partial u^k}{\partial v^k} \frac{\partial m}{\partial q^j} = 0, \frac{\partial Q_i}{\partial q^h} + Q_i \frac{\partial \ln m^2}{\partial q^h} = 0, \\
\frac{\partial E_i}{\partial v^k} - m \frac{\partial P^k}{\partial q^i} + \frac{\partial u^i}{\partial v^k} \frac{\partial m}{\partial t} - \left(P^h \frac{\partial u^h}{\partial v^k} + (u - v)^h \frac{\partial P^h}{\partial v^h} \right) \frac{\partial m}{\partial q^i} = 0.
\end{gathered}
$$

3. **Lois de conservation.** Une fois donnée la forme Ω_c^*, qui décrit la dynamique du point, on peut écrire les lois fondamentales de la dynamique sous la forme :

$$q \times mv = \int q \times F \cdot dt + \int q \times \varphi \cdot dt + c,$$

$$m(q - vt) = -\int tF \cdot dt - \int t\varphi dt + c,$$

$$mv = \int F \cdot dt + \int \varphi \cdot dt + c, \tag{20}$$

$$\frac{mv^2}{2} = \int F \cdot dvt + \int \varphi \cdot vdt + c,$$

où

$$\varphi = (u - v)\frac{dm}{dt} - (u - v)(P + v)\frac{\partial m}{\partial q},$$

formules par les quelles on met en évidence la digression vis-à-vis des formules connus au cas de la dynamique du point à masse constante [2], [1].

Conséquences. Le champ F' calculé le long des trajectoires s'obtient par la formule :

$$F' = m\frac{dv}{dt} - (u - v)(P + v)\frac{\partial m}{\partial q}. \tag{21}$$

Cette expression introduite dans les relations (20) et à l'usage de la formule d'intégration par parties, conduit aux lois de conservation :

$$\int q \times v\frac{dm}{dt}dt = -\int q \times \left[(u - v)(P + v)\frac{\partial m}{\partial q}\right]dt + c,$$

$$\int (q - vt)\frac{dm}{dt}dt = +\int t[(u - v)(P + v)]\frac{\partial m}{\partial q}dt + c,$$

$$\int v\frac{dm}{dt}dt = -\int (u - v)(P + v)\frac{\partial m}{\partial q}dt + c, \tag{22}$$

$$\int \frac{v^2}{2}\frac{dm}{dt}dt = -\int (u - v)(P + v)\left(v\frac{\partial m}{\partial q}\right)dt + c.$$

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PHOTOELECTRIC OBSERVATIONS OF V_{382} CYGNI

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ABSTRACT. — 139 mean points were calculated from the photoelectric observations carried on the star of V_{382} CYGNI between 1977–1981, to the end of obtaining its light variability curve. 9 new moments of the light minima were determined. No variation in the orbital period time was detected.

The light variability of the star V_{382} Cygni (BD + 35°4062) was discovered by Morgenroth [3]. It is a massive 0-type, double-line spectroscopic and eclipsing binary.

The period variation has been studied by various authors with widely different conclusions. Landolt [1] considers the period of V_{382} Cygni as being constant, whereas Koch et al. (see Mayer [2]) belive that they have found large changes of the corresponding period. Mayer [2] has found an increase of the orbital period, the first two minima, observed before 1933, beieng considered as an evidence of period change. The mass loss is admited as a main cause of the ,,observed'' increase of the period.

Having in view the above presented discrepancy concerning the period variation of V_{382} Cygni, this star was included in our program for photoelectric observations. So, from 2.09.1977 to 9.09.1981, a number of 1491 observations has been performed. With this end in view the 50-cm reflector of the

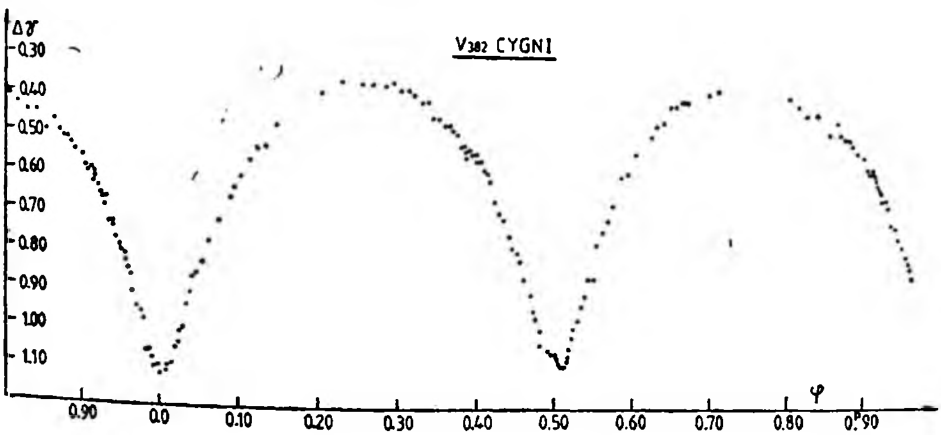

Fig. 1.

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Astronomical observatory in Cluj-Napoca, with the filter Corrning 3384 for V, and an EMI photomultiplier, has been used.

The individual observations were grouped into 139 normal points which are listed in Table 1. The mean light-curve is graphically shown in Figure 1.

Table 1

V_{382} Cygni

φ	Δv	n	φ	Δv	n	φ	Δv	n
$0^P.0029$	1.123	11	$0^P.4108$	0.592	11	$0^P.6707$	0.422	10
.0080	1.113	11	.4134	0.596	10	.7002	0.400	11
.0125	1.103	12	.4176	0.622	10	.7139	0.392	11
.0163	1.094	12	.4233	0.667	10	.8076	0.413	10
.0199	1.054	12	.4286	0.698	10	.8169	0.429	10
.0237	1.043	12	.4366	0.716	10	.8281	0.454	11
.0272	1.013	11	.4437	0.764	10	.8415	0.454	11
.0309	1.002	12	.4482	0.787	10	.8564	0.495	10
.0350	0.940	12	.4522	0.801	10	.8643	0.468	11
.0389	0.910	12	.4562	0.825	10	.8704	0.499	10
.0429	0.871	12	.4614	0.873	10	.8750	0.510	10
.0478	0.859	10	.4692	0.910	10	.8806	0.515	10
.0540	0.829	10	.4734	0.952	10	.8861	0.530	10
.0637	0.780	10	.4767	0.973	10	.8921	0.550	10
.0786	0.724	11	.4820	0.996	11	.8992	0.557	11
.0927	0.667	11	.4822	1.040	10	.9052	0.594	11
.0973	0.643	11	.4909	1.051	12	.9084	0.596	11
.1178	0.566	10	.4943	1.061	12	.9110	0.595	10
.1260	0.535	9	.4969	1.066	11	.9140	0.599	11
.1340	0.528	9	.4993	1.074	11	.9163	0.624	11
.1492	0.475	10	.5019	1.078	12	.9184	0.629	12
.2085	0.395	10	.5046	1.084	12	.9202	0.644	11
.2287	0.356	10	.5072	1.086	12	.9231	0.657	11
.2540	0.370	10	.5097	1.084	12	.9262	0.669	12
.2737	0.373	10	.5122	1.081	11	.9296	0.670	12
.2851	0.377	10	.5146	1.068	11	.9318	0.693	12
.2950	0.372	10	.5178	1.048	11	.9343	0.730	11
.3040	0.386	9	.5228	1.019	10	.9373	0.727	11
.3136	0.393	10	.5270	0.994	11	.9415	0.743	12
.3225	0.401	10	.5314	0.976	12	.9458	0.768	11
.3320	0.420	10	.5355	0.940	11	.9490	0.793	11
.3420	0.424	10	.5393	0.916	12	.9517	0.799	11
.3479	0.461	10	.5451	0.868	12	.9540	0.816	11
.3530	0.457	12	.5523	0.868	11	.9570	0.831	12
.3591	0.482	11	.5572	0.782	12	.9597	0.848	12
.3658	0.483	11	.5635	0.747	11	.9632	0.866	11
.3708	0.487	11	.5689	0.721	12	.9665	0.910	12
.3752	0.499	10	.5768	0.684	12	.9705	0.947	10
.3796	0.527	10	.5870	0.611	10	.9744	0.960	12
.3821	0.532	10	.5955	0.596	10	.9788	0.979	11
.3858	0.557	10	.6067	0.554	10	.9827	1.062	10
.3886	0.540	10	.6249	0.506	11	.9874	1.053	10
.3922	0.540	10	.6338	0.477	10	.9907	1.080	10
.3950	0.550	10	.6395	0.471	11	.9946	1.097	10
.3994	0.549	10	.6491	0.429	11	0.9981	1.105	11
.4029	0.569	10	.6567	0.434	10			
0.4056	0.565	10	0.6644	0.423	10			

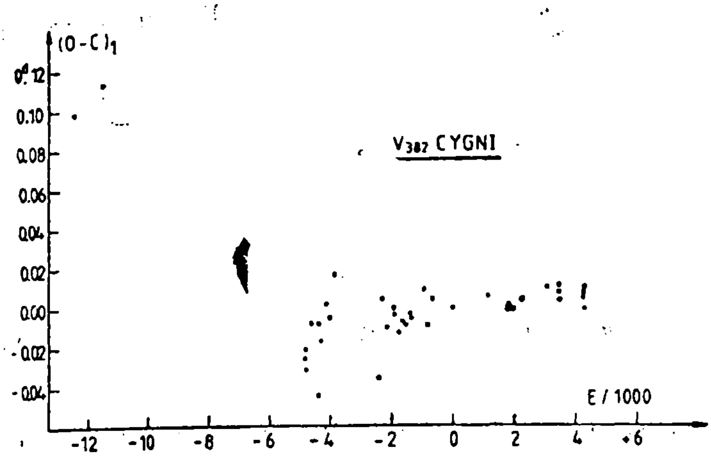

Fig. 2.

The differences Δv are considered as V(V_{382} Cygni)—C(BD + 35°4059); (V = = variable star; C = comparison star). The phases are refering to L a n d o l t's [1] ephemeris

$$\text{Prin. Min.} = \text{JD}2436814.7735 + 1^d.8855121\ \text{E}$$

The individual observations were plotted, and a best time of minimum was calculated by bisecting the branches of the light-curve on either side of the minimum. The appropriate heliocentric correction was applied.

The times of minima resulting from our photoelectric observations are listed in Table 2. The epoch number, E, and values in column $(0-C)_1$ were also calculated according to L a n d o l t ' s ephemeris. These data are plotted in Figure 2 and make up M a y e r ' s [2] diagram.

Table 2

Observed minima of V_{382} Cygni

2440000+	*n*	$(O-C_1)$	$(O-C_2)$	E	
3389.5572	36	$+0^d.0030$	$-0^d.0019$	3487	p
3390.5001	61	.0032	— .0017	3487.5	s
3391.4487	85	.0090	+ .0041	3488	p
3394.2760	72	.0081	+ .0032	3489.5	s
3425.3852	83	.0063	+ .0014	3506	p
3426.3281	48	.0065	+ .0016	3506.5	s
4808.4080	20	.0060	+ .0001	4239,5	s
4837.6358	45	.0083	+ .0024	4255	p
4856.4880	60	0.0054	−0.0006	4265	p

Now, if we do not take into consideration the first two minima, observed before 1933 (see M a y e r [2]), it s easy to see that no large variation of the period of V_{382} Cygni has occured in the last years. In such a case a small correction for the orbital period ($\Delta P = 0^d.0000014$) may be used and the corresponding linear ephemeris becomes

$$\text{Prin. Min.} = \text{JD}2436814.7735 + 1^d.8855135\ E$$

The $(0-C)_2$ values resulting from the new light elements are shown in the fourth column (see Table 2).

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