# STUDIA
## UNIVERSITATIS BABEȘ-BOLYAI

## MATHEMATICA

1

1987

BCU Cluj-Napoca
PMATE 2014 00369

CLUJ-NAPOCA

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**Anul XXXII** **1987**

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### MATHEMATICA

**1**

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# THE RITZ VARIATIONAL METHOD FOR THE HEAT TRANSFER IN A CIRCULAR CYLINDER

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*Received: October 7, 1986*

ABSTRACT. — The paper is a variational study based on the lemma Lax — Milgram for the mixed boundary value problem of axisymmetrical heat conductivity in a solid circular cylinder which has interior heat sources. The Ritz global method with trial functions of the form (17): $\varphi_k(x, y) = x^h \cos k\pi y$ is applied for the approximate solving of the problem; $(x, y)$ are dimensionless polar coordinates ($x = X/R$, $y = r/R$, $R$ = cylinder radius).

1. **Formulation of the mixed boundary value problem.** Let us consider the solid cylindrical body with respect to the cylindrical coordinates

$$(C) = \{(X, r, \theta) \mid 0 \leqslant X \leqslant L, 0 \leqslant r \leqslant R, 0 \leqslant \theta \leqslant 2\pi\}$$

with the radius $R$ and the length $L$ ($OX$ is the symmetry axis of the cylinder). In $(C)$ there are internal heat sources with flux per unit volume $q_v$.

The differential equation governing stationary heat flow in the solid cylinder is

$$\nabla \cdot (\lambda \nabla T) + q_v = 0 \text{ in } (C)$$

where $T$ is the temperature and $\lambda$ is the conductivity cofficient in the cylinder $(C)$. The unitary thermic flux $q_{wi}$ through the solid surface $\Gamma_{wi}$ of the cylinder, in the case of conduction and for convection fluid-surface $\Gamma_{wi}$, is

$$q_{wi} = -\lambda \frac{\partial T}{\partial n}\bigg|_{wi};\ q_{wi} = \alpha_i \,|\, T_{wi} - T_{ei} \,|$$

$$\left([q_{wi}] = \frac{W}{m^2},\ [\lambda] = \frac{W}{m \cdot K} \qquad [\alpha_i] = \frac{W}{m^2 \cdot K}\right)$$

where $\alpha_i$ is the convective heat transfer coefficient on the surface $\Gamma_{wi}$, $T_{ei}$ is the temperature of a fluid that flows over the cylinder and $T_{wi}$ is the temperature of the surface $\Gamma_{wi}$.

The axial symmetry of heat propagation in $(C)$ is admitted. We use cyndrical coordinates and we introduce dimensionless values, by considering

$$X = Rx,\ r = Ry,\ L = Rh,\ T = T_e(1 + u);$$

$$\sigma_i = \frac{R\alpha_i}{\lambda},\ g_i = \sigma_i\left(\frac{T_{ei}}{T_e} - 1\right),\ f = \frac{R^2}{\lambda T_e} q_v,\ \alpha_i > 0,\ \lambda > 0$$

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In order to determine the dimensionless temperature $u(x, y)$ on $(C)$ it is enough to consider the rectangle $\Omega = (0, h) \times (0,1)$ limited by the contour $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ on which the two-dimensional boundary value problem is formulated $(\partial/\partial\theta = 0)$:

$$Au \equiv -y\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\partial u}{\partial y} = yf(x, y), \quad (x \cdot y) \in \Omega \tag{1}$$

$$u(0, y) = g(y) \quad \text{on} \quad \Gamma_0(0 < y < 1\,;\; x = 0) \tag{1a}$$

$$\frac{\partial u}{\partial x} + \sigma_1(y)u = g_1(y) \quad \text{on} \quad \Gamma_1(0 < y < 1\,;\; x = h) \tag{1b}$$

$$\frac{\partial u}{\partial y} + \sigma_2(x)u = g_2(x) \quad \text{on} \quad \Gamma_2(0 < x < h\,;\; y = 1) \tag{1c}$$

$$\frac{\partial u}{\partial y}(x, 0) = 0 \quad \text{on} \quad \Gamma_3(0 < x < h\,;\; y = 0) \tag{1d}$$

$$(0 < \sigma_i^{(1)} \leqslant \sigma_i \leqslant \sigma_i^{(2)})$$

**2. Variational formulation and generalized solution for the differential boundary value problem (1)–(1d).** We add the function sets

$$H^1(\Omega) = \left\{u \,|\, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in L_2(\Omega)\right\}$$

$$H_0^1(\Omega\,;\,\Gamma_0) = \{u \in H^1(\Omega) \,|\, u = 0 \;\text{ on }\; \Gamma_0\}$$

$$H_g^1(\Omega) = \{u \in H^1(\Omega) \,|u(0, y) = g(y) \;\text{ on }\; \Gamma_0(0 < y < 1)\}$$

to the problem (1)–(1d).

Be $f \in L_2(\Omega)$, $u \in H_g^1(\Omega)$ [with $u \in H_g^1(\Omega) \cap C(\bar{\Omega})$, $g \in H^{1/2}(\Gamma_0)$]. We put $u = w + u_1$ where $w \in H_0^1(\Omega\,;\,\Gamma_0)$ and $u_1 \in H_g^1(\Omega)$ is a given element. We use the Lax-Milgram lemma. On this purpose the triplet $(H, a, \Phi_1)$ is introduced, where $H$ is a Hilbert space, in the form

$$H = H_0^1(\Omega\,;\,\Gamma_0), \quad ||\cdot||_H = ||\cdot||_1 \;\text{(Sobolev norm)} \tag{3}$$

$$a(w, v) = \iint_\Omega y\nabla w \cdot \nabla v \, dx\, dy + \int_{\Gamma_1} \sigma_1 wvy\, dy + \int_{\Gamma_2} \sigma_2 wv\, dx\,; \tag{4}$$

$$\Phi_1(v) = \iint_\Omega fvy\, dxdy + \int_{\Gamma_1} g_1 vy dy + \int_{\Gamma_2} g_2 v\, dx - a(u_1, v)\,; \tag{5}$$

$$w,\, v \in H_0^1(\Omega\,;\,\Gamma_0)$$

and we put

$$\Phi(v) = \Phi_1(v) + a(u_1, v)$$

*The properties of the form $a(w, v)$ with $w, v \in H_0^1(\Omega; \Gamma_0)$:*

a) $a(w, v)$ is a bilinear symmetrical form.

b) $a(w, v)$ is a coercive form:

$$a(w, w) = \iint_\Omega y\,|\nabla w|^2\,dx\,dy + \int_{\Gamma_1} \sigma_1 w^2 y\,dy + \int_{\Gamma_2} \sigma_2 w^2\,dx \geqslant$$

$$\geqslant \frac{1}{2} \iint_\Omega |\nabla w|^2\;dxdz + \frac{1}{2} \int_{\Gamma_1} \sigma_1 w^2\,dz \geqslant \bar{\alpha}\,|\,|w|\,|_1^2 \tag{5'}$$

where $\bar{\alpha} = \rho \min\left(\frac{1}{2}, \sigma_1^{(1)}\right)$ if the second Friederichs inequality is applied (with the constant $\rho > 0$).

c) $a(w, v)$ is a bounded (symmetrical) form:

$$a(w, w) \leqslant \iint_\Omega (\,|\nabla w\,|^2 + w^2)\,dx\,dy + \int_{\Gamma_1} \sigma_1 w^2 dy + \int_{\Gamma_2} \sigma_2 w^2 dx \leqslant$$

$$\leqslant ||\,w\,|\,|_1^2 + \sigma_1^{(2)}\mu_1\,||\,u\,||_1^2 + \sigma_2^{(2)}\mu_2\,||\,w\,||_1^2 = \bar{\beta}\,||\,w\,||_1^2 \tag{5''}$$

if the trace inequality is used, according to which a constant $\mu > 0$ exists so that

$$\int_\Gamma w^2\,ds \leqslant \mu\,|\,|w\,|\,|_1^2,\; w \in H^1(\Omega)$$

The form $\Phi_1(v)$ is linear and bounded.

— Under these conditions, according to the Lax-Milgram lemma there exists a unique function $\tilde{w} \in H_0^1(\Omega; \Gamma_0)$ such that

$$a(\tilde{w}, v) = \Phi_1(v),\; \forall v \in H_0^1(\Omega; \Gamma_0) \tag{6}$$

$$J_1(\tilde{w}) = \min J_1(w),\; w \in H_0^1(\Omega; \Gamma_0) \tag{7}$$

where $J_1$ is the energy functional

$$J_1(w) = \frac{1}{2}\,a(w, w) - \Phi_1(w),\; w \in H_0^1(\Omega; \Gamma_0) \tag{7'}$$

*Remark* 1. The test for (7) can be made. We consider the function $\varepsilon \rightarrow$ $\rightarrow J_1(w + \varepsilon v)$ with $\varepsilon \in \mathbf{R}^1$ and $v$ a function arbitrarily given in the linear space of the perturbation functions $H_p = H_0^1(\Omega; \Gamma_0)$. Whe have

$$J_1(w + \varepsilon v) = \frac{1}{2}\,a(w + \varepsilon v,\, w + \varepsilon v) - \Phi_1(w + \varepsilon v) =$$

$$= J_1(w) + \varepsilon\,[a(w, v) - \Phi_1(v)] + \frac{1}{2}\,\varepsilon^2 a(v, v)$$

Therefore $J_1(w)$ is a quadratic functional and if $a(\tilde{w}, v) - \Phi_1(v) = 0$, $\forall v \in H_0^1$ $(\Omega; \Gamma_0)$ we have $J_1(\tilde{w} + \varepsilon v) - J_1(\tilde{w}) = \varepsilon^2 a(v, v)/2 > 0$.
We return to the function $u$ by means of the

PROPOSITION 1. *The equation*

$$a(u, v) = \Phi(v), \quad \forall v \in H_0^1(\Omega; \Gamma_0) \tag{8}$$

*has the unique solution* $u = \tilde{u}(= \tilde{w} + u_1)$ *in* $H_g^1(\Omega)$ *which is also the unique global minimizer of energy functional*

$$J(u) = \frac{1}{2}\, a(u, u) - \Phi(u), \; u \in H_g^1(\Omega) \tag{9}$$

*on the subspace* $H_g^1(\Omega)$, i.e.

$$J(\tilde{u}) = \min_{u \in H_g^1(\Omega)} J(u)$$

*Proof.* The equation (6) with $\Phi_1(w) = \Phi(w) - a(w, u_1)$ is written in the form

$$a(\tilde{w} + u_1, v) = \Phi(v), \quad \tilde{w} + u_1 \in H_g^1(\Omega)$$

Therefore, as (6) has the unique solution $\tilde{w}$, the equation (8) has the unique solution $u = \tilde{w} + u_1$.

On the other hand, the functional $J_1(w)$ is written as

$$J_1(w) = J_1(u - u_1) = \frac{1}{2}\, a(u - u_1, u - u_1) - \Phi(u - u_1) + a(u - u_1, u_1) =$$
$$= J(u) - J(u_1)$$

and since $J(u_1)$ is a constant value from a variational point of view, the unique minimizer $\tilde{w}$ of $J_1(w)$ is the unique minimizer $\tilde{u}$ of $J(u)$.

PROPOSITION 2. *The equation* (8) *ia a weak variational formulation and its unique solution is the generalized solution of the differential problem* (1) — (1d).

*Proof.* Let us begin by noticing that the form $a(u, v)$, $u \in H_g^1(\Omega)$ cannot be integrated by parts (Green) unless u $\in C^1(\Omega)$. Using the expressions of the forms $a(u, v)$ and $\Phi(v)$ we have

$$a(u, v) - \Phi(v) = \iint_{\Omega} (\nabla u \cdot \nabla v - fv) y\, dx\, dy -$$
$$- \int_{\Gamma_1} (g_1 - \sigma_1 u) v y\, dy - \int_{\Gamma_2} (g_2 - \sigma_2 u) v y\, dx$$

Taking into account the identity

$$\nabla u \cdot \nabla(yv) = y \nabla u \cdot \nabla v + v \frac{\partial u}{\partial y}$$

and formally applying the Green identity

$$\iint_{\Omega} y\nabla u\cdot\nabla v dx\,dy=\iint_{\Omega}\left[\nabla u\cdot\nabla(yv)-v\frac{\partial u}{\partial y}\right]dx\,dy=$$

$$=-\iint_{\Omega} yv\Delta u dx\,dy+\int_{\Gamma} v\frac{\partial u}{\partial n}ds-\iint_{\Omega}\frac{\partial u}{\partial y}v\,dx\,dy$$

we obtain the formula $(\forall v\in H_0^1(\Omega;\Gamma_0))$

$$a(u,v)-\Phi(v)=-\left[\iint_{\Omega}\left(y\Delta u+\frac{\partial u}{\partial y}+yf\right)v\,dx\,dy+\right.$$

$$\left.+\int_{\Gamma_0}\frac{\partial u}{\partial x}vy\,dy+\int_{\Gamma_1}\left(g_1-\sigma_1 u-\frac{\partial u}{\partial x}\right)vy\,dy+\int_{\Gamma_2}\left(g_2-\sigma_2 u-\frac{\partial u}{\partial y}\right)v\,dx\right] \tag{10}$$

From (10) we infer that the unique solution $u$ of the equation (8) having the property $u=g(y)$ on $\Gamma_0$ is the solution of the problem (1)—(1d).

Consequently, the finding of the generalized solution $\tilde{u}$ solves the differential boundary value problem (1)—(1d). In order to compute the generalized solution $\tilde{u}$ we use:

3. **The Ritz variational method (global approximation).** This method is applied for the problem (6)—(7) on the Hilbert subspace $H_0^1(\Omega;\Gamma_0)$ which implies a homogeneous essential boundary condition. We define the subspace $N$-dimensional $H_N$:

$$H_N=\operatorname{span}\{\varphi_1,\dots,\varphi_N\}=\{w=\textstyle\sum_1^N\lambda_i\varphi_i\,|\,\lambda_i\in\mathbf{R}^1,\ \varphi_i\in H_0^1(\Omega;\Gamma_0)\}$$

where $\{\varphi_i\}_{i=1}^N$ forms a given basis, chosen from the complete system $\{\varphi_n\}_1^\infty\subset H_0^1(\Omega;\Gamma_0)$. The Lax-Milgram lemma is also valid on $H_N$ since $H_N$ is a Hilbert spece, too. The conditions of the lemma will stand if $H_0^1(\Omega;\Gamma_0)$ is replaced by $H_N$. Then, the functional, (7'), $J_1(w)$, $w\in H_N$ has the unique minimizer $w_N\in H_N$ and

$$w_N=\sum_{k=1}^N c_k\varphi_k(x,y),\ c_k\in\mathbf{R}^1 \tag{11}$$

If $c=(c_1,\dots,c_N)^T$, $\lambda=(\lambda_1,\dots,\lambda_N)^T$, $K=[a(\varphi_j,\ \varphi_k)]_{N\times N}$, $b=(\Phi\varphi_1),\dots,\Phi_1(\varphi_N))^T$ we have $J_1(w_N)=(1/2)c^TKc-c^Tb\equiv F_1(c)$. Therefore, $F_1(\lambda)$ is a quadratic functional on $\mathbf{R}^N$ whose Hessian $H_s(=K)$ is a constant and definite positive matrix $[\lambda^TK\lambda=a(w,w)\geqslant\bar{\alpha}\,||w||_1^2\geqslant 0;\ \lambda^TK\lambda=0\Leftrightarrow\lambda=0]$ and then $F_1(\lambda)$ has a minimizer $\lambda=c$, that is $J_1(w_N)=F_1(c)=\min\{F_1(\lambda)\,|\,\lambda\in\mathbf{R}^N\}$.

Consequently, we obtain the Ritz matrix equation (Cramer system)

$$\nabla F_1(c)=Kc-b=0 \tag{12}$$

in order to determine the vector $c$, from (11).

We choose the approximate solution $u_N(=w_N+u_1)$ for the generalized solution $\tilde{u}(=\tilde{w}+u_1)$ of the problem (1) – (1d) taking $u_1 = \sum_{N+1}^{N+P} c_k\varphi_k$ where $N$ and $P$ are integers $\geqslant 1$:

$$u_N(x,y)=\sum_{k=1}^{N+P} c_k\varphi_k(x,y)\,;\ [\varphi_k\in H_0^1(\Omega\,;\,\Gamma_0),\ k=\overline{1,N}] \tag{13}$$

in which the coefficients $c_k$, $k=\overline{1,N}$ are determined by means of the Ritz system

$$\sum_{k=1}^{N+P} a(\varphi_j,\varphi_k)c_k=\Phi(\varphi_j),\ j=\overline{1,N} \tag{14}$$

where the constant $c_k$ and the functions $\varphi_k$, $k=\overline{N+1,N+P}$, are chosen so that

$$\sum_{k=N+1}^{N+P} c_k\varphi_k(y)=g(y),\ y\in\Gamma_0 \tag{15}$$

*The error.* As it is known, the Ritz method gives an optimal estimation of the error with respect to the energetic norm $\|\cdot\|_A$ and a cvasioptimal estimation with respect to the norm in $H$, respectively

$$\|\tilde{w}-w_N\|_A=\min_{v\in H_N}\|\tilde{w}-v\|_A\,;\quad \|\tilde{w}-w_N\|_H\leqslant C\min_{v\in H}\|\tilde{w}-v\|_H,$$

$$C=\sqrt{\frac{\bar{\beta}}{\bar{\alpha}}}$$

from where, with $\tilde{w}=\tilde{u}-u_1$ and $w_N=u_N-u_1$, we infer that

$$\|\tilde{u}-u_N\|_H\leqslant C\min_{v\in\overline{H}_N}\|\tilde{u}-v\|_H,\ C=\sqrt{\frac{\bar{\beta}}{\bar{\alpha}}} \tag{16}$$

$$(\overline{H}_N=\{v\in H_g^1(\Omega)\,|\,v=w+u_1,\ w\in H_N\})$$

where $\bar{\alpha}$ is the coerciveness constant, (5′), and $\bar{\beta}$ is the boundness constant, (5″), for the bilinear functional. From this we also infer the convergence of the Ritz method: if the system $\{\varphi_n\}_{n=1}^{\infty}$ in complete ($H_N$ — a complete sequence of subspaces) then, for $\tilde{u}$ there exists $v\in\overline{H}_N$ so that foreach $\varepsilon(\tilde{u},N)$ we should have $\|\tilde{u}-v\|_H\leqslant\varepsilon(\tilde{u},N)\to 0$ for $N\to\infty$; therefore $\|\tilde{u}-u_N\|_H\to 0$, $N\to\infty$.

**4. Choice of approximation solution.** In the Ritz system we choose

$$\varphi_k(x,y)=x^k\cos k\pi y,\quad k=\overline{1,N} \tag{17}$$

$$(\text{and } P=1,\ c_{N+1}=1,\ \varphi_{N+1}(x,y)=g(y),\ 0<y<1)$$

The system of functions $\{\varphi_k\}_{k=1}^{\infty}$ is complete. Indeed, if the function $\psi \in C[\Omega]$ orthogonal over $\Omega$ with all elements $\varphi_{jk} = x^j \cos k\pi y$ is introduced, we have (for all $j$, $k$)

$$(\psi, \varphi_{jk}) = 0 = \left(\int_0^1 \psi(x, y) \cos k\pi y \, dy, \; x^j\right) \Rightarrow \int_0^1 \psi \cos k\pi y dy = 0$$

$$\text{or } (\psi, \cos k\pi y) = 0 \Rightarrow \psi(x, y) = 0$$

if we take into account that $\{x^j\}$ and $\{\cos \; k\pi y\}$ are complete systems.

For approximate solution $u_N$, for $u$, has the form

$$u_N(x, y) = \sum_{k=1}^{N} c_k x^k \cos k\pi y + g(y) \tag{18}$$

where $c_k$, $k = \overline{1, N}$, are determined by means of the Ritz linear system (14) :

$$\sum_{k=1}^{N} a(\varphi_j, \varphi_k) c_k = \Phi(\varphi_j) - a(\varphi_j, g), \quad j = \overline{1, N} \tag{19}$$

in which $a(\varphi_j, \varphi_k)$, $\Phi(\varphi_j)$ and $a(\varphi_i, g)$ are calculated with the formulas (4) — (5).

5. **A particular problem.** Let us solve the given problem, by choosing the following functions

$$f(x, y) = q_0 e^{-x/\alpha}(1 - \beta y^2)\,; \quad g(y) = \cos \pi y$$

$$\sigma_i = \text{const.}, \; g_i = \text{const.}, \; (i = \overline{1\,;\,2})$$

in this case we obtain

$$a(\varphi_j, \varphi_m) = h^{j+m}\left(\sigma_1 + \frac{jm}{j+m-1} \, \frac{1}{h}\right) A_c(j, m) + $$
$$+ \frac{h^{j+m+1}}{j+m+1} \left[jm\pi^2 A_s(j,m) + (-1)^{j+m} \sigma_2\right], \; j = \overline{1, N}$$

where

$$A_c(j, m) = \begin{cases} \dfrac{1}{4} & , \; j = m \\ -\dfrac{2}{\pi^2} \, \dfrac{j^2 + m^2}{(j^2 - m^2)^2} & , \; j + m = \text{odd} \\ 0 & , \; j + m = \text{even} \end{cases}$$

$$A_s(j, m) = \begin{cases} \dfrac{1}{4} & , \; j = m \\ -\dfrac{4}{\pi^2} \, \dfrac{jm}{(j^2 - m^2)^2} & , \; j + m = \text{odd} \\ 0 & , \; j + m = \text{even} \end{cases}$$

$$\Phi(\varphi_j) = q_0 \, B(j) \left[B_1(j) - \beta B_3(j)\right] + g_1 h^j B_1(j) + (-1)^j \frac{h^{j+1}}{j+1} \, g_2, \; j = \overline{1, N}$$

where

$$B(j) = \alpha[jB(j-1) - h^j e^{-h/\alpha}],\ j = \overline{1, N};\quad B(0) = \alpha(1 - e^{-h/\alpha})$$

$$B_1(j) = \begin{cases} 0 & , j = \text{even} \\ -\dfrac{2}{j^2\pi^2} & , j = \text{odd} \end{cases};\quad B_3(j) = \begin{cases} \dfrac{3}{j^2\pi^2} & , j \doteq \text{even} \\ -\dfrac{3}{j^2\pi^2}\left(1 - \dfrac{4}{j^2\pi^2}\right), & j = \text{odd} \end{cases}$$

and

$$a(\varphi_j, g) = \frac{h^{j+1}}{j+1}\left[\pi^2 j A_s(j, 1) + \frac{\sigma_1}{h}\cdot(j+1)A_e(j, 1) + (-1)^{j+1}\sigma_2\right],\ j = \overline{1, N}$$

— The third order approximation ($N = 3$). We have

$$a(\varphi_1, \varphi_1) = \frac{h^2}{4}\left(\sigma_1 + \frac{1}{h}\right) + \frac{h^3}{3}\left(\sigma_2 + \frac{\pi^2}{4}\right)$$

$$a(\varphi_1, \varphi_2) = -\frac{10h^3}{9\pi^2}\left(\sigma_1 + \frac{1}{h}\right) - \frac{h^4}{4}\left(\sigma_2 + \frac{16}{9}\right)$$

$$a(\varphi_1, \varphi_3) = \frac{h^5}{5}\sigma_2$$

$$a(\varphi_2, \varphi_2) = \frac{h^4}{4}\left(\sigma_1 + \frac{4}{3h}\right) + \frac{h^5}{5}(\sigma_2 + \pi^2)$$

$$a(\varphi_2, \varphi_3) = -\frac{26h^5}{25\pi^2}\left(\sigma_1 + \frac{3}{2h}\right) - \frac{h^6}{6}\left(\sigma_2 + \frac{144}{25}\right)$$

$$a(\varphi_3, \varphi_3) = \frac{h^6}{4}\left(\sigma_1 + \frac{9}{5h}\right) + \frac{h^7}{7}\left(\sigma_2 + \frac{9}{4}\pi^2\right)$$

$$\Phi(\varphi_1) = \frac{q_0}{\pi^2}B(1)\left[-2 + 3\beta\left(1 - \frac{4}{\pi^2}\right)\right] - \frac{2h g_1}{\pi^2} - \frac{h^2 g_2}{2};$$

$$B(1) = \alpha[\alpha - (\alpha + h)e^{-h/\alpha}];$$

$$\Phi(\varphi_2) = -\frac{3\beta q_0}{4\pi^2}B(2) + \frac{h^3}{3}g_2;$$

$$B(2) = \alpha[2B(1) - h^2 e^{-h/\alpha}];$$

$$\Phi(\varphi_3) = \frac{q_0}{9\pi^2}B(3)\left[-2 + 3\beta\left(1 - \frac{4}{9\pi^2}\right)\right] - \frac{2h^3 g_1}{9\pi^2} - \frac{h^4 g_2}{4};$$

$$B(3) = \alpha[3B(2) - h^3 e^{-h/\alpha}]$$

$$a(\varphi_1, g) = \frac{h^2}{2}\left(\frac{\pi^2}{4} + \frac{\sigma_1}{2h} + \sigma_2\right);\quad a(\varphi_2, g) = -\frac{h^3}{3}\left(\sigma_2 + \frac{30\sigma_1}{9h\pi^2} + \frac{16}{9}\right);$$

$$a(\varphi_3, g) = \frac{h^4}{4}\sigma_2$$

The temperature distribution in the cylinder $(C)$, on the sides $\Gamma_1$ and $\Gamma_2$ is given by the formulas:

$$T(X, r) = T_e\left[1 + \cos\frac{\pi r}{R} + \sum_{k=1}^{3} c_k \left(\frac{X}{R}\right)^k \cos k\frac{\pi r}{R}\right];$$

$$T_1(r) = T_e\left[1 + \cos\frac{\pi r}{R} + \sum_{k=1}^{3} c_k \left(\frac{L}{R}\right)^k \cos k\frac{\pi r}{R}\right]$$

$$T_2(X) = T_e\sum_{k=1}^{3} (-1)^k c_k\left(\frac{X}{R}\right)^k$$

where the coefficients $c_1$, $c_2$ and $c_3$ are obtained by solving the system

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ \text{symmetry} & & a_{33} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{Bmatrix} - \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$(a_{ij} \equiv a(\varphi_i, \varphi_j),\ \Phi_i \equiv \Phi(\varphi_i),\ a_i = a(\varphi_i, g))$$

*Numerical example.* We consider a cylindrical metalic beam whose basis $\Gamma_{u0}$ is fixed onto a solid wall and that is attacked perpendicularly by a fluid flow (water). We choose

$$L = 1\ m,\ R = \frac{1}{5}\ \text{m},\ \alpha = 2,\ \beta = \frac{1}{2},\ q_0 = 3$$

$$\lambda = 58\frac{W}{m \cdot K},\ \alpha_e(= \alpha_1 = \alpha_2) = 4640\frac{W}{m^2 \cdot K},\ T_{e1} = T_{e2} = T_e = 50\,°\text{C}$$

where $\alpha_e$ is the convective heat transfer coefficient (cylinder-water). We get

$$h = 5,\ \sigma_1 = \sigma_2 = 16,\ g_1 = g_2 = 0$$

and the Ritz system

$$\begin{bmatrix} 870{,}725 & -3005{,}75 & 10^4 \\ & 18710{,}17 & -62034{,}16 \\ \text{symmetry} & & 490319{,}31 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} -251{,}81 \\ 789{,}25 \\ -2500{,}45 \end{Bmatrix}$$

The solutions of this system are (the Gauss method)

$$c_1 = -0{,}3223638;\ c_2 = -0{,}0081212;\ c_3 = 0{,}0004473$$

The distribution of the temperature is given by Table 1.

*Table* 1

**The values of the temperature $T(X, r)$**

| $r$[m] \ $X$[m] | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 0 | 100 | 83.50 | 66.34 | 48.67 | 35.65 | 12.4 |
| 1/20 | 85.36 | 73.94 | 62.42 | 50.68 | 38.64 | 26.16 |
| 1/10 | 50.00 | 50.41 | 51.62 | 53.65 | 56.48 | 60.18 |
| 3/20 | 14.64 | 26.06 | 37.59 | 49.32 | 61.37 | 73.85 |
| 1/5 | 0 | 15.70 | 30.42 | 44.04 | 56.40 | 67.35 |

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STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXII, 1, 1987

# DESPRE IRAŢIONALITATEA UNOR SERII FACTORIALE

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*Intrat în redacţie la 15 februarie 1986*

**ABSTRACT.** — **On Some Irrational Factorial Series.** In this paper one finds some results on special infinite series, named factorial and weakly-factorial series, which contain many known and particular cases of irrationality. Our propositions are based upon some ideas of W. Schwarz [5] and utilize the approximation theorems of K. Roth [2] and Th. Schneider [4], and a theorem [3] which follows from this.

În articolul de faţă vom găsi cîteva teoreme despre iraţionalitatea şi transcendenţa unor serii factoriale şi slab factoriale. Lucrarea noastră are ca punct de plecare rezultatele lui W. Schwarz [5], şi P. Bundschuh [1], care printre altele au demonstrat că pentru $k, t, b$, numere naturale, $k \geq 2$, $t \geq 2$, $o < b < t^{1-1/k}$, seria $\sum_{n=1}^{\infty} \frac{b^{k^n}}{t^{k^n} - b^{k^n}}$ este iraţională iar dacă $k = 2$ sau inpar şi $|b| < t^{1-1/k}$ sau $k \geq 4$ par şi $|b| < t^{1-1/(k-1)}$, atunci $\sum_{n=1}^{\infty} \frac{b^{k^n}}{t^{k^n} + b^{k^n}}$ este iraţională ($b \in \mathbf{Z}$).

1. **Definiţii.** a) Spunem că şirul $(x_n)$ de numere naturale este „şir factorial", dacă proprietatea" $x_n | x_m$ ($x_n$ divide $x_m$) pentru orice $n \leq m$" are loc pentru orice $m \in \mathbf{N}^*$.

b) Şirul $(x_n)$ este numit „slab-factorial", dacă proprietatea „$x_n | x_m$ pentru orice $n \leq m$" are loc pentru o infinitate de numere naturale $m \geq 1$.

c) Dacă $(b_n)$ este şir factorial şi seria $\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ este convergentă, numim această serie „factorială". Analog în cazul cînd $(b_n)$ este slab-factorial, vorbim de „serie slab-factorială".

2. **Observaţii.** a) Orice serie factorială este şi slab factorială. Reciproca însă nu mai este adevărată. Fie de exemplu $x_n = 2^{y_n}$, unde $y_n = 1$, pentru $n = 1$; $n$ dacă $n$ este par şi $n - 2$, pentru $n$ impar. Şirul $(y_n)$ este „slab-crescător" fără să fie crescător. Într-adevăr, fie $m$ număr par. Atunci pentru orice $n \leq m$ avem $y_n \leq y_m$. Pentru numere $m$ impare se poate verifica uşor că proprietatea nu mai are loc. De exemplu, pentru $m = 5$ avem $y_5 = 3$, $y_4 = 4$.

b) Şirul $(x_n)$ este exact atunci factorial, dacă $x_n | x_{n+1}$, $n = 1, 2, \ldots$ Într-adevăr, dacă $x_n | x_{m+1}$ pentru $n \leq m + 1$ are loc pentru orice $m$, alegînd $n = m$ rezultă $x_m | x_{m+1}$. Invers, dacă $x_1 | x_2$, $x_2 | x_3$, $\ldots$, $x_m | x_{m+1}$,

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... din proprietatea de tranzitivitate a relației de divizibilitate rezultă imediat proprietatea din definiția a)

c) Exemple de șiruri factoriale (termen general) $x_n = n!$, $x_n = a^n$ $(a \in N^*)$, $x_n = a^{k^n} - b^{k^n}$ $(k \in N^*,\ a > b,\ a, b \in N^*)$ etc.

3. **Rezultatele principale.** TEOREMA 1. *Dacă* $\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ *este factorială și avem* $\lim_{N\to\infty} b_N \cdot \sum_{n>N} \frac{a_n}{b_n} = 0$, *atunci seria este irațională.*

TEOREMA 2. *Dacă seria* $\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ *este slab-factorială și avem* $\lim_{N\to\infty} b_N \times \sum_{n>N} \frac{a_n}{b_n} = 0$ *atunci seria este irațională.*

TEOREMA 3. *Dacă* $\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ *serie factorială cu* $\lim_{n\to\infty} b_n = \infty$ *și pentru care avem* $b_N^{\lambda} \cdot \sum_{n>N} \frac{a_n}{b_n} < 1$, *pentru o infinitate de numere naturale* $N$, *unde* $\lambda > 2$ *este un număr real fixat dat. Atunci seria este transcendentă.*

TEOREMA 4. *Dacă* $\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ *este serie factorială, astfel încît să avem*

i) $\overline{\lim_{N\to\infty}}\, b_N^2 \cdot \sum_{n>N} \frac{a_n}{b_n} < \infty$

ii) *cel mai mare factor prim al lui* $b_n$ *nu tinde către infinit împreună cu* $n$

iii) *seria este irațională.*

*In aceste condiții seria este transcendentă.*

4. **Consecințe.** 1) Fie $\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ serie factorială și să presupunem că există $0 < \alpha < 1$ pentru care avem

(i) $$\lim_{n\to\infty} b_n \cdot \alpha^{k^{n+1}} = 0 \qquad (k \in N^*,\ \text{fixat})$$

(ii) $$\frac{a_n}{b_n} \leq 2 \cdot \alpha^{k^n} \qquad (n \in N^*)$$

Atunci seria este irațională.

2) Fie seria $\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ slab-factorială pentru care avem

(i) $\lim_{n\to\infty} b_n = \infty$

(ii) $a_{n+k} \leq \frac{b_{n+k}}{b_n^{2k}}$

Atunci seria este iraţională.

3) Fie seria $\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ serie factorială şi să presupunem că

(i) există $0 < \alpha < 1$ cu $\varliminf_{n \to \infty} b_n^\lambda \cdot \alpha^{k^{n+1}} = 0$ ($\lambda > 2$, $k \in N^*$ fixate)

(ii) $\lim_{n \to \infty} b_n = \infty$

Atunci seria este transcendentă.

4) Fie $\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ serie factorială pentru care sînt satisfăcute (i), (ii) din 1) şi încă în plus

(iii) $\varlimsup_{n \to \infty} b_n^2 \cdot \alpha^{k^{n+1}} < \infty$

(iv) cel mai mare factor prim al lui $b_n$ nu tinde la infinit împreună cu $n$. Atunci seria este transcendentă.

5. **Demonstraţii.** TEOREMA 1. Fie $\sum_{n \geq 1} \frac{a_n}{b_n} = \frac{a}{b}$ cu $a, b \in N^*$ şi să considerăm sumele $S_1 = b \cdot b_N \cdot \sum_{n \geq 1} \frac{a_n}{b_n}$, $S_2 = b \cdot b_N \cdot \sum_{n \leq 1} \frac{a_n}{b_n}$. Este evident că sumele considerate sînt numere întregi, prima din cauza presupunerii, a doua din cauza factorialităţii şirului $(b_n)$. ($N$ deocamdată este arbitrar.) Atunci şi $S = S_1 - S_2$ este număr întreg, cu $|S| \geq 1$. Vom arăta că totuşi pentru $N$ convenabil vom avea $|S| < 1$. Într-adevăr, $S = b \cdot b_N \cdot \sum_{n > N} \frac{a_n}{b_n} < 1$, unde alegem $N$ cu această condiţie, (ce se poate din cauza presupunerii din enunţul teoremei) Deci seria este iraţională.

TEOREMA 2. Se demonstrează analog cu teorema 1, numai că aici alegem $N$ suficient de mare cu proprietatea $b_n | b_N$, $n \leq N$.

TEOREMA 3. Aplicăm teorema lui Roth, într-o formă găsită de Schneider ([2], [4]).

„Fie $\alpha$ un număr algebric cu gradul mai mare decît 1. Fie $q_\gamma, p_\gamma$ ($\gamma \in N^*$) numere întregi cu proprietăţile $0 < q_\gamma \leq q_{\gamma+1}$, $q_\gamma = q'_\gamma \cdot q''_\gamma$ unde $q''_\gamma$ este puterea unui număr întreg $g$.

Dacă $\eta = \varlimsup_{\gamma \to \infty} \frac{\ln q'_\gamma}{\ln q_\gamma}$ şi $\lambda > \eta + 1$, atunci

inegalitatea $\left| \alpha - \frac{p_\gamma}{q_\gamma} \right| < \frac{1}{q_\gamma^\lambda}$ are un număr finit de soluţii $\frac{p_\gamma}{q_\gamma}$"

Dacă alegem $g = 1$, $\eta = 1$ şi $(b_n)$ şir factorial, cu notaţia $\alpha = \sum_{n \geq 1} \frac{a_n}{b_n}$ vom avea $\left| \alpha - \sum_{n \leq N} \frac{a_n}{b_n} \right| = \left| \sum_{n < N} \frac{a_n}{b_n} \right| < \frac{1}{b_N^\lambda}$ pentru o infinitate de numere $N$, prin presupunere.

Însă $\lambda > 2$ şi $0 < b_b \leqslant b_{n+1}$, avem evident din teorema lui Roth că $\alpha$ este sau număr raţional sau transcendent. Însă nu poate fi raţional căci

$$b_N \cdot \sum_{n>N} \frac{a_n}{b_n} < \frac{1}{b_N^{\lambda-1}} \quad \text{şi} \quad \lim_{N\to\infty} b_N \cdot \sum_{n>N} \frac{a_n}{b_n} = 0 \quad (\lambda - 1 > 1; \quad b_N \to \infty)$$

şi Teorema 1 se aplică.

TEOREMA 4. Demonstrăm mai întîi propoziţia (mai vezi [3]) „Dacă $\alpha$ este un număr algebric, iraţional, astfel încît pentru o infinitate de numere raţionale $(x_k/y_k)$ să avem $\left|\alpha - \frac{x_k}{y_k}\right| < \frac{A}{y_k^2}$ ($A$ o constantă) atunci cel mai mare factor prim al lui $y_k$ tinde la infinit împreună cu $k$"

Pentru *demonstraţia* acestei propoziţii să presupunem că $y_k = p_1^{m_{1,k}} \cdot p_2^{m_{2,k}} \ldots p_r^{m_{r,k}}$, unde $p_1, \ldots, p_r$ sînt $r$ numere prime ($r$ nu depinde de $k$), $m_{i,k} \geqslant 0$ sînt nişte numere întregi. Deoarece avem un număr finit de indici $r$, cel puţin un număr prim — să fie el $p_1$ — pentru o infinitate de indici $k$ apare la cea mai mare putere. Acest subşir să-l notăm tot cu $m_{i,k}$ şi eventual renumerotăm indicii. Obţinem $y_k = p_1^{m_{1,k}} \ldots p_r^{m_{r,k}}$ cu $p_1^{m_{1,k}} > p_i^{m_{i,k}}$ $(i = \overline{2,k})$ deci $t_k = \frac{\ln p_i^{m_{i,k}}}{\ln p_1^{m_{1,k}}} < 1$ $(i = \overline{1,r})$. Şirul $(t_k)$ fiind mărginit, admite un subşir convergent, aşadar

$$\frac{\ln p_i^{m_{i,k_j}}}{\ln p_1^{m_{1,k_j}}} \to \eta_i \quad (0 \leqslant \eta_i \leqslant 1)$$

Cu scopul de a aplica teorema Roth-Schneider fie $m = p_1$, $v_{k_j} = p_1^{m_{1,k_j}}$. Avem atunci

$$\frac{\ln v'_{k_j}}{\ln v_{k_j}} = \frac{\sum_{i=2}^{r} \ln p_i^{m_{i,k_j}}}{\sum_{i=1}^{r} \ln p_i^{m_{i,k_j}}} \to \frac{\sum_{i=2}^{r} \eta_i}{\sum_{i=2}^{r} \eta_i + 1} = \eta < 1$$

Fie $0 < \varepsilon < 1 - \eta$. Avem atunci $\left|\alpha - \frac{x_{k_j}}{y_{k_j}}\right| < \frac{A}{y_{k_j}^2} < \frac{1}{y_{k_j}^{2-\varepsilon}}$ este satisfăcută de o infinitate de ori. Alegînd $\lambda = 2 - \varepsilon$, $\lambda > \eta + 1$, obţinem o contradicţie, deci cel mai mare factor prim al şirului $(y_k)$ trebuie să tindă către infinit.

Trecem acum la demonstraţia teoremei.

Fie $\alpha = \sum_{n=1}^{\infty} \frac{a_n}{b_n}$, $\frac{x_N}{y_N} = \frac{a_1}{b_1} + \ldots + \frac{a_N}{b_N}$

Aici putem alege $y_N = b_N$, şirul $(b_n)$ fiind factorial. Atunci condiţiile (i), (ii), (iii) satisfac condiţiile de aplicabilitate a propoziţiei de mai sus, deci $\alpha$ nu poate fi număr algebric.

**6. Demonstrațiile consecințelor.**

1) Avem $b_N \cdot \sum_{\lambda>N} \frac{a_n}{b_n} \leqslant 2 \cdot b_N \cdot \sum_{n>N} \alpha^{k^n} < 2b_N \cdot \sum_{m>k^n+1} \alpha^m \leqslant 4 \cdot b_N \cdot \alpha^{k^N+1}$

şi pe baza lui (i), $\underline{\lim}\, b_N \cdot \sum_{n>N} \frac{a_n}{b_n} = 0$ şi teorema 1 se aplică.

2) $b_N \cdot \sum_{n>N} \frac{a_n}{b_n} \overset{(ii)}{\leqslant} b_N \cdot \left(\frac{1}{b_N^2} + \frac{1}{b_N^4} + \ldots\right) = \frac{b_N}{b_N^2 - 1} \overset{(i)}{\longrightarrow} 0$

Teorema 2 se verifică.

3) $\sum_{n>N} \frac{a_n}{b_n} \leqslant 4 \cdot \alpha^{k^N+1}$ (vezi 1)) şi $4 \cdot \alpha^{k^N+1} < \frac{1}{b_N^\lambda}$ pentru o infinitate de indici $N$, din cauza condiției (i).

Teorema 3 se poate folosi.

4) Aplicăm Teorema 4 şi Consecința 1.

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STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXII, 1, 1987

# ON THE APPROXIMATE SOLUTION OF OPERATOR EQUATIONS IN HILBERT SPACES

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Received: October 15 1986

**REZUMAT. — Asupra rezolvării aproximative a ecuaţiilor operatoriale în spaţii Hilbert.** În prezenta lucrare se dă o metodă de rezolvare iterativă de tip coardă a ecuaţiilor operatoriale neliniare P(x) = 0 în spaţii Hilbert. Avantajul acestei metode faţă de cele cunoscute de acest tip din spaţii Banach este acela că nu trebuie inversată diferenţa divizală a operatorului P. Metoda este dată de formulele (2).

**0.** For the approximate solution of the operator equations in Banach spaces the method of the chord is known [5]. But the applicability of this method supposes the existence and the continuity of the inverse of certain operators (divided difference). It is known that a linear operator does not have always an inverse, and when it has one the evaluation of its norm is difficult. In the present paper we give an iterative method for approximate solution of non-linear operator equations in Hilbert spaces, wich neither supposes the existence of the invers of the divided difference of the operator, nor the evaluation of this inverse.

**1.** Consider the non-linear equation

$$P(x) = 0. \tag{1}$$

where $P$ is a continuous operator defined on the real Hilbert space with values in $H$, $P: H \to H$. We note by $\langle v_1, v_2 \rangle$ the scalar product of the vectors $v_1, v_2 \in H$. It is known that the operator $P$ has divided differences [4]. Let $[u, v; P]$ a symmetrical divided difference of the operator $P$ for the points $u, v \in \in H$, $u \neq v$. In this case the linear functional defined in $H$ by $x \to \langle [u, v; P](x), P(u) + P(v) \rangle$ is a symmetrical divided difference of the functional $\| P \|^2 : H \to \mathbf{R}$ [2].

The iterative process for solving equation (1) given in the present paper is defined by the sequence $(x_n)$, given by

$$\begin{aligned} x_1 &= x_0 - \frac{\| P(x_0) \|^2}{\langle [x_0, x_{-1}; P] P(x_0), P(x_0) + P(x_{-1}) \rangle} P(x_0), \\ x_{n+1} &= x_n - \frac{\| P(x_n) \|^2}{\langle [x_n, x_{n-1}; P] P(x_n), P(x_n) + P(x_{n-1}) \rangle} P(x_n), \quad (n \geqslant 0) \end{aligned} \tag{2}$$

where $x_0, x_{-1} \in H$ are two initial approximations of equation (1).

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THEOREM. *Suppose that there exist two points* $x_0$, $x_{-1} \in H$ *and the positive constants* $\eta_0$, $B$, $D$ *and* $K$ *such that the following conditions hold:*

1° $\| x_1 - x_0 \| = \dfrac{\| P(x_0) \|^3}{\langle [x_0, x_{-1}; P] P(x_0), P(x_0) + P(x_{-1}) \rangle} = \eta_0$,

$\| x_0 - x_{-1} \| \leqslant \eta_0$;

2° $\| P(x) \| \leqslant D$ for every $x$ of the sphere $S[x_0, r]$, where

$$r = \frac{\eta_0}{1 - 2(BD)^2 K}$$

that is $P$ is bounded in $S[x_0, r]$;

3° $| \langle [u, v; P] P(u), P(u) + P(v) \rangle | \geqslant B^{-1} \| P(u) \|$ for every $u, v \in S[x_0, r]$;

4° $\| \langle [u, v; P], P(u) + P(v) \rangle - \langle [v, w; P], P(v) + P(w) \rangle \| \leqslant K \| u - v \|$ for every $u, v, w \in S[x_0, r]$;

5° $BD\sqrt{K} < \dfrac{1}{2}$.

*Under these conditions the equalities* (2) *define by recurrence a sequence* $(x_n)$ *having the following properties:*

i) *The sequence* $(x_n)$ *is convergent and*

$$\lim_{n \to \infty} x_n = x^* \in S[x_0, r];$$

ii) $x^*$ *is a solution of equation* (1);

iii) *For the error estimate we have the inequality:*

$$\| x^* - x_n \| \leqslant [2(BD)^2 K]^n \frac{\eta_0}{1 - 2(BD)^2 K}.$$

*Proof.* If $P(x_0) \neq 0$, then from condition 3° it results that $\langle [x_0, x_{-1}; P] P(x_0), P(x_0) + P(x_{-1}) \rangle \neq 0$, therefore $x_1$ can be constructed by (2) and $x_1$, $x_{-1}$ are in the sphere $S[x_0, r]$.

Taking the value of the linear operator $\langle [x_0, x_{-1}; P] \cdot, P(x_0) + P(x_{-1}) \rangle$ for the point $(x_1 - x_0)$ and using the first equality from (2), we obtain

$$\langle [x_0, x_1; P] (x_1 - x_0), P(x_0) + P(x_{-1}) \rangle = - \| P(x_0) \|^2$$

whence we have the identity

$$\| P(x_1) \|^2 = \| P(x_1) \|^2 - \| P(x_0) \|^2 - \langle [x_0, x_{-1}; P] (x_1 - x_0), P(x_0) + P(x_{-1}) \rangle. \tag{3}$$

Using the definition and the form of the divided difference in the case of the functional $||P||^2$, from (3) we have

$$||P(x_1)||^2 = \langle [x_1, x_0; P](x_1 - x_0),\ P(x_1) + P(x_0)\rangle -$$
$$- \langle [x_0, x_{-1}; P](x_1 - x_0),\ P(x_0) + P(x_{-1})\rangle = \{\langle [x_1, x_0; P]\cdot,$$
$$P(x_1) + P(x_0)\rangle - \langle [x_0, x_{-1}; P]\cdot,\ P(x_0) + P(x_{-1})\rangle\}(x_1 - x_0)$$

whence the conditions 1°, 4° and the inequality: $\eta_0 \leqslant BD^2$ give

$$||P(x_1)||^2 \leqslant K||x_1 - x_{-1}|| \cdot ||x_1 - x_0|| \leqslant 2BD^2K||x_1 - x_0|| \leqslant$$
$$\leqslant 2BD^2K\eta_0. \tag{4}$$

If $P(x_1) \neq 0$ by condition 3°, using equality (2) we construct $x_2$ and we have

$$||x_2 - x_1|| \leqslant B||P(x_1)||^2 \leqslant 2(BD)^2K||x_1 - x_0|| \leqslant 2(BD)^2K\eta_0$$

From $||x_0 - x_2|| \leqslant ||x_0 - x_1|| + ||x_1 - x_2|| \leqslant \eta_0\ [1 + 2(BD)^2K]$ it results that $x_2 \in S[x_0, r]$.

Taking the value of the linear operator $\langle [x_1, x_0; P]\cdot, P(x_1) + P(x_0)\rangle$ for $(x_2 - x_1)$ and using the equality (2), we obtain

$$\langle [x_1, x_0; P](x_2 - x_1),\ P(x_1) + P(x_0)\rangle = -||P(x_1)||^2,$$

whence it results

$$||P(x_2)||^2 = ||P(x_2)||^2 - ||P(x_1||^2 -$$
$$-\langle [x_1, x_0; P](x_2 - x_1),\ P(x_1) + P(x_0)\rangle$$

or

$$||P(x_2)||^2 = \{\langle [x_2, x_1; P]\cdot,\ P(x_2) + P(x_1)\rangle -$$
$$- \langle [x_1\cdot, x_0; P]\cdot,\ P(x_1) + P(x_2)\rangle\}(x_2 - x_1)$$

thus

$$||P(x_2)||^2 \leqslant K||x_2 - x_0|| \cdot ||x_2 - x_1|| \leqslant 2BD^2K||x_2 - x_1||$$

where the following relation has been used:

$$||x_2 - x_0|| = \left\| \frac{||P(x_1)||^2}{\langle [x_1, x_0; P]\, P(x_1), P(x_1) + P(x_0)\rangle} P(x_1) + \right.$$
$$\left. + \frac{||P(x_0)||^2}{\langle [x_0, x_{-1}; P]\, P(x_0), P(x_0) + P(x_{-1})\rangle} P(x_0) \right\| \leqslant 2BD.$$

If $P(x_2) \neq 0$, we construct $x_3$ and we have

$$||x_3 - x_2|| \leqslant 2(BD)^2K||x_2 - x_1|| \leqslant [2(BD)^2K]^2\eta_0$$

From

$$||x_0 - x_3|| \leqslant \eta_0\{1 + 2(BD)^2K + [2(BD)^2K]^2\}\eta_0$$

we get $x_3 \in S[x_0, r]$.

If $P(x_{n-1}) \neq 0$, using equality (2) we construct $x_n$. Suppose that

$$||x_n - x_{n-1}|| \leqslant [2(BD)^2K]^{n-1}\eta_0$$

In this case $x_n \in S[x_0, r]$. *If* $P(x_n) \neq 0$ we construct $x_{n+1}$ and by induction, we shall prove that the following inequalities hold:

$$||P(x_n)||^2 \leqslant 2BD^2K||x_n - x_{n-1}|| \tag{5}$$

and

$$||x_{n+1} - x_n|| \leqslant [2(BD)^2K]^n\eta_0, \tag{6}$$

whence we obtain $x_{n+1} \in S[x_0, r]$.

From the inequality (6) we have

$$||x_{n+p} - x_n|| \leqslant [2(BD)^2K]^n \frac{\eta_0}{1 - 2(BD)^2K} \tag{7}$$

for every $p \in \mathbf{N}$. Therefore by (7) $||x_{n+p} - x_n|| \to 0$, $n \to \infty$ for every $p \in \mathbf{N}$. Because the space $H$ is a Banach space it results the existence of the limit of the sequence $(x_n)$ and obviously $x^* \in S[x_0, r]$.

From the inequality (5) for $n \to \infty$ we obtain that $x^* = \lim_{n\to\infty} x_n$ is a solution of the equation (1).

*Remark* 1. The condition 3° of Theorem 1 can be replaced by

$$|\langle [u, v; P]x, x + y\rangle| \geqslant B^{-1}||x||$$

for every $u, v, x, y \in S[x_0, r]$.

*Remark* 2. The condition 4° of Theorem 1 can be replaced by the following: the second order divided difference of the functional $||P||^2$ is bounded, that is there exists a constant $K$ such that

$$||\langle [x', x'', x'''; P], P(x') + P(x'')\rangle + \langle [x', x''; P], [x'', x'''; P]\rangle|| \leqslant K$$

for every $x', x'', x''' \in S[x_0, r]$.

**2.** Now we consider the linear equation

$$L(x) = a \qquad (x, a \in H), \tag{8}$$

where $L$ is a bounded linear operator defined on $H$ with values in $H$ and $a$ a fixed element of $H$.

Using the sequence definde by (2) we obtain a method for the approximate solution of the linear operator equation (8).

The following theorem is a particular case of Theorem 1.

THEOREM 2. *Suppose that the bounded linear operator L satisfies the following conditions:*

1° *There exists a positive constant B such that*

$$|\langle L(x), L(x) + L(y)\rangle| \geqslant B^{-1}\|x\|$$

*or every x, y of H;*

2° $$\sqrt{2}B(\|L\| + \|a\|)\,\|L\| = h < 1.$$

*Then the sequence of approximate solutions* $(x_n)$ *defined by*

$$x_{n+1} = x_n - \frac{\|L(x_n) - a\|^2}{\langle L(L(x_n) - a),\, L(x_n) + L(x_{n-1}) - 2a\rangle}\,[L(x_n) - a](n \geqslant 0) \tag{9}$$

*where* $x_0$, $x_{-1}$ *are two initial approximations of the equation* (8) *have the following properties:*

i) *The sequence* $(x_n)$ *converges for arbitrary initial approach* $x_0, x_{-1}$;

ii) *The limit* $\lim\limits_{n\to\infty} x_n = x^*$ *is a solution of the equation* (8);

iii) *For the error estimate we have the following inequality:*

$$\|x^* - x_n\| \leqslant \|x_0 - x_{-1}\|\frac{h^{n-1}}{1-h},\ n \geqslant 1.$$

*Proof.* We note that, if $x_n(n \geqslant 0)$ is not a solution of equation (8), then from 2° it results: $\langle L(L(x_n) - a), L(x_n) + L(x_{n-1}) - 2a\rangle \neq 0$, therefore $x_{n+1}$ can be constructed.

Taking the value of the linear operator $\langle L\cdot, L(x_n) + L(x_{n-1}) - 2a\rangle$ for the point $(x_{n+1} - x_n)$ and using the relation (9) we obtain

$$\langle L(x_{n+1} - x_n), L(x_n) + L(x_{n-1}) - 2a\rangle = -\|L(x_n) - a\|^2,$$

whence we derive the obvious equality

$$\|L(x_{n+1}) - a\|^2 = \|L(x_{n+1}) - a\|^2 - \|L(x_n) - a\|^2 - \\ -\langle L(x_{n+1} - x_n), L(x_n) + L(x_{n-1}) - 2a\rangle$$

Using the definition and the form of the divided difference of an operator of the form: $\|P\|^2$ we have

$$\|L(x_{n+1}) - a\|^2 = \langle L(x_{n+1} - x_n), L(x_{n+1}) + L(x_n) - 2a\rangle - \\ -\langle L(x_{n+1} - x_n), L(x_n) + L(x_{n-1}) - 2a\rangle = \\ = \langle L(x_{n+1} - x_n), L(x_{n+1} - x_{n-1})\rangle.$$

Therefore

$$\|L(x_{n+1}) - a\|^2 \leqslant \|L\|^2 \cdot \|x_{n+1} - x_n\| \cdot \|x_{n+1} - x_{n-1}\|. \tag{10}$$

whence using condition 1° of Theorem 2 and the relation (9), we obtain

$$||x_{n+2} - x_{n+1}|| \leqslant B||L||^2||x_{n+1} - x_n|| \cdot ||x_{n+1} - x_{n-1}||. \quad (11)$$

Now we estimate $||x_{n+1} - x_{n-1}||$. From (9), using 1° of Theorem 2, it results

$$||x_{n+1} - x_{n-1}|| = ||x_n - \frac{||L(x_n) - a||^2}{\langle L(L(x_n) - a), L(x_n) + L(x_{n-1}) - 2a\rangle}[L(x_n) - a] - $$

$$- x_{n-1}|| \leqslant 2B(||L|| + ||a||)^2,$$

whence from (11) we have

$$||x_{n+2} - x_{n+1}|| \leqslant 2B^2||L||^2(||L|| + ||a||)^2||x_{n+1} - x_n|| =$$

$$= h||x_{n+1} - x\ ||. \quad (12)$$

The convergence of the sequence $(x_n)$ to a certain element $x^*$ of $H$ results from (12). It follows from (10) that $x^*$ is the solution of the equation (8).

Taking limits as $p \to \infty$ in

$$||x_{n+p} - x_n|| \leqslant \frac{h^{n+1}}{1-h}||x_0 - x_{-1}||,$$

we obtain the inequality iii) of Theorem 2.

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STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXII, 1, 1987

# NUMERICAL METHODS IN FUZZY HIERARCHICAL PATTERN RECOGNITION

## II. Divisive hierarchical clustering

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*Received: October 18, 1986*

**ABSTRACT.** — A polarization relation between fuzzy partitions is considered. The polarization degree of a two-atom fuzzy partition is defined and three examples are considered. Using a polarization degree a condition that clusters described by a fuzzy partition be "real" is given. The concept of fuzzy hierarchy is defined. A divisive procedure to obtain a binary fuzzy hierarchy is given. The entropy of a fuzzy partition is used to define a stability coefficient of a fuzzy hierarchy.

1. **Polarization of a fuzzy partition.** By the polarization of a fuzzy partition $P \in F(C)$ we understand the tendency of the membership degrees of its atoms to concentrate near by the extreme values 0 and $C(x)$. A polarization relation between fuzzy partitions is introduced by the next

DEFINITION 1.1. Let $P$, $Q$ be from $F_2(C)$, $P = \{A_1, A_2\}, Q = \{B_1, B_2\}$. The fuzzy partition $P$ is less polarized than $Q$, $P \leqslant_p Q$, if and only if

$$|A_1(x) - A_2(x)| \leqslant |B_1(x) - B_2(x)|, \qquad \forall x \in X.$$

PROPOSITION 1.1. *The relation $\leqslant_p$ is a preorder on $F_2(C)$.*

*Proof.* It results from the definition of the $\leqslant_p$ relation, taking into account that $A_1(x) + A_2(x) = B_1(x) + B_2(x)$, $\forall x \in X$.

The relation $\leqslant_p$ is generalized to $n$-atoms fuzzy partitions by the next

DEFINITION 1.2. Let $P$, $Q$ be from $F_n(C)$, $P = \{A_1, \ldots, A_n\}$, $Q = \{B_1, \ldots, B_n\}$. $P$ is less polarized than $Q$, $P \leqslant_p Q$, if and only if

$$\max_{i,j} |A_i(x) - A_j(x)| \leqslant \max_{i,j} |B_i(x) - B_j(x)|, \qquad \forall x \in X.$$

PROPOSITION 1.2. *The relation $\leqslant_p$ is a preorder on $F_n(C)$.*

Let $P = \{C_1, C_2\}$ be from $F_2(C)$. If $C_1$ and $C_2$ are well-separated fuzzy classes, then the memberships $C_1(x)$, $C_2(x)$ tend to accumulate towards 0 and $C(x)$, for every sample $x$. The degree of polarization of a fuzzy partition may thus be considered as a measure of its quality.

DEFINITION 1.3. The polarization degree of a two-atom fuzzy partition is a function $R : F_2(C) \to \mathbf{R}$, which satisfies the axioms:

(i) $a \leqslant R(P) \leqslant 1$, $\forall P \in F_2(C)$, where $a \in [0, 1)$.

(ii) $R(P) = 1$ iff $P = \{C_1, C_2\}$, where $C_1(x) = C(x)$ or $C_2(x) = C(x)$, $\forall x \in X$.

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(iii) $R(P) = a$ iff $P = \{C_1, C_2\}$ with $C_1(x) = C_2(x)$, $\forall x \in X$.

(iv) $P \leqslant_p Q \Rightarrow R(P) \leqslant R(Q)$.

We may define a polarization degree in a variety of ways. Let us consider first

$$R_1(P) = \frac{\sum_x \max(C_1(x), C_2(x))}{\sum_x C(x)},$$

where $P = \{C_1, C_2\} \in F_2(C)$.

PROPOSITION 1.3. *$R_1$ is a polarization degree with* $a = \frac{1}{2}$.

*Proof.* (i) From $C_1(x) + C_2(x) = C(x)$ we have $C(x)/2 \leqslant \max(C_1(x), C_2(x)) \leqslant C(x)$. The summation over all $x \leqslant X$ leads to the result.

(ii)
$$R_1(P) = 1 \text{ iff } \sum_x \max(C_1(x), C_2(x)) = \sum_x C(x).$$

From this equality, taking into account that $\max(C_1(x), C_2(x)) \leqslant C(x)$ and $C_1(x)$, $C_2(x) \geqslant 0$, for every $x$, we obtain $\max(C_1(x), C_2(x)) = C(x)$. This is possible only if $C_1(x) = C(x)$, $C_2(x) = 0$ or $C_2(x) = C(x)$, $C_1(x) = 0$, for every $x$.

(iii)
$$R_1(P) = \frac{1}{2} \text{ iff} \sum_x \max(C_1(x), C_2(x)) = \frac{1}{2} \sum_x C(x).$$

From this equality, since $\max(C_1(x), C_2(x)) \geqslant C(x)/2$ and $C_1(x)$, $C_2(x) \geqslant 0$, we obtain $\max(C_1(x), C_2(x)) = C(x)/2$,

$$\text{i.e. } C_1(x) = C_2(x) = C(x)/2 \text{ for every } x.$$

(iv) If $P = \{A_1, A_2\}$, $Q = \{B_1, B_2\}$, $P, Q \in F_2(C)$, and $P \leqslant_p Q$, then

$$|A_1(x) - A_2(x)| \leqslant |B_1(x) - B_2(x)|, \text{ for every } x.$$

From this inequality, taking into account that $A_1(x) + A_2(x) = B_1(x) + B_2(x)$, it results that $\max(A_1(x), A_2(x)) \leqslant \max(B_1(x), B_2(x))$, for every $x$, and thus $R_1(P) \leqslant R_1(Q)$.

*Remark.* If $P \in F_2(X)$ then $R_1(P) = 1$ iff $P$ is a hard partition, i.e. $P \in F_{2h}(X)$.

Another way to define a polarization degree is

$$R_2(P) = \frac{\sum_x C_{1, \frac{C(x)}{2}}(x) + \sum_x C_{2, \frac{C(x)}{2}}(x)}{\sum_x C(x)},$$

where $C_{i,\alpha}$ is the $\alpha$-cut of $C_i$, i.e.

$$C_{i,\alpha}(x) = \begin{cases} C_i(x), \text{ if } C_i(x) > \alpha \\ 0, \text{ otherwise.} \end{cases} \quad \forall x \in X.$$

PROPOSITION 1.4. *$R_2$ is a polarization degree with $a = 0$.*

According as the atoms of successive partitions are splitted (see Section 2) the membership degrees preserve their initial polarization near by 0 and 1, if the corresponding clusters are „real". Hence, for „real" clusters one does not observe an increase of the uniformity by successive decompositions. „Real" clusters have therefore the memberships degrees far-off the value 1/2. Aslight modification of $R_2$ leads to a new measure of polarization defined by

$$R_3(P) = \frac{\sum_x C_{1,\frac{1}{2}}(x) + \sum_x C_{2,\frac{1}{2}}(x)}{\sum_x C(x)}.$$

The previous reasons permit us to consider $R_3$ to be more appropriate than $R_1$ and $R_2$ as a measure of polarization. The conclusion is supported by the experimental results, too.

We may now state the next

PROPOSITION 1.5. *With respect to $R_3$ the following affirmations are true:*

(1) $0 \leqslant R_3(P) \leqslant 1, \ \forall P \in F_2(C)$.

(2) $R_3(P) = 1$ iff $C_1(x) = C(x) > \frac{1}{2}$ *or* $C_2(x) = C(x) > \frac{1}{2}, \ \forall x \in X$.

(3) $C_1(x) = C_2(x), \ \forall x \in X \Rightarrow R_3(P) = 0$.

(4) $P \leqslant_p Q \Rightarrow R_3(P) \leqslant R_3(Q)$.

*Remark*. The properties (2) and (3) are generalizations of (ii) and (iii) from Definition 1.3. $R_3$ is not thus a polarization degree. We may call it an *index of polarization*.

Experimental results indicate a decreasing tendency of the value of $R_3$ with the decomposition level. This fact also recommend $R_3$ to be more appropriate for practical purposes than $R_1$ and $R_2$.

In what follows we consider that the partition $P = \{C_1, C_2\}$ describes „real" clusters if for either of its atoms there exists a pattern vector $x$ with the membership degree greather than $\frac{1}{2}$ and $R_3(P) \geqslant t$, where $t \in (0, 1)$ is an appropriate threshold.

2. **Fuzzy hierarchy.** Let $H$ be a class of fuzzy on $X$. We denote by $S(A)$ the set of succesors of $A$ in $H$, i.e.

$$S(A) = \{A_i \in H \mid A_i \subseteq A, \ A_i \neq A \text{ and } \forall B \in H, \ B \neq X, \ B \neq A, \\ A_i \subseteq B \Rightarrow B = A_i\}.$$

DEFINITION 2.1. A class $H$ of fuzzy sets on $X$ ($H \subset L(X)$) is called a fuzzy hierarchy if and only if the following axioms are satisfied

$(H_1)$ $A \cap B = \emptyset$ or $A \subseteq B$, $A \neq B$ or $B \subseteq A$, $A \neq B$ for every $A, B$ from $H$.

$(H_2)$ $\bigcup_{A_i \in S(A)} A_i = A$, for every $A \in H$, $S(A) \neq \emptyset$.

A fuzzy hierarchy is binary iff card $S(A)$ is 0 or 2. $H$ is called total iff $X$ is from $H$.

*Remarks.* i) If $A \in H$ and card $S(A) = 0$, then $A$ is minimal in $H$. ii) A total binary fuzzy hierarchy corresponds to a binary tree whose nodes are the elements of $H$. $X$ is the root node and the minimal sets in $H$ correspond to the terminal nodes of the tree.

We may now to give a divisive procedure to obtain a binary fuzzy hierarchy.

Let $P^{(l)}$) be the fuzzy partition of $X$ corresponding to the decomposition level $l$ and $C$a non-empty atom of $P^{(l)}$. Using the Generalized Fuzzy ISODATA algorithm (see Part I), a fuzzy partition $\{C_1, C_2\}$ of $C$ is computed. If $C_1$ and $C_2$ represent "real" clusters, then they become atoms of $P^{(l+1)}$ — the fuzzy partition at the level $l + 1$. Otherwise $C$ remains unchanged and it is allocated to $P^{(l+1)}$. The procedure repeats for every $C \in P^{(l)}$, $C \neq \emptyset$.

The initial partition is $P^{(0)} = \{X, \emptyset\}$. The decomposition process ends at the level $k$ if none "real" cluster is obtained at this level i.e. $P^{(k+1)} = P^{(k)}$. The partitions $P^{(0)}, P^{(1)}, \ldots, P^{(k)}$ are all fuzzy partitions of $X$

The algorithm corresponding to this procedure is the next:

FUZZY DIVISIVE HIERARCHICAL (FDH) CLUSTERING ALGORITHM

$S_1$. Set $l := 0$, $N := 0$, $P^{(l)} = \{X, \emptyset\}$. ($l$ is the decompositions level).

S2. For every $C \in P^{(l)}$, $C$ not marked and $C \neq \emptyset$, perform S4. Allocate every marked $C$ to $P^{(l+1)}$.

S3. If $P^{(l+1)} = P^{(l)}$ Stop, else set $l := l + 1$ and go to $S_2$.

S4. Compute the fuzzy partition $\{C_1, C_2\}$ of $C$ and its representation $\{L_1, L_2\}$, using the GFI algorithm. If $C_1, C_2$ are "real" clusters, then allocate $C_1$ and $C_2$ to $P^{(l+1)}$.

Othervise mark $C$ with $+1$ and set $N := N + 1$.

*Remark.* $N$, the number of marked classes, equals the optimal number of clusters in $X$.

PROPOOITION 2.2. *The sequence of fuzzy partitions obtained by the* FDH *algorithm is a chain*

$$P^{(0)} \prec P^{(1)} \prec \ldots \prec P^{(l)}.$$

This chain induces a total binary fuzzy hierarchy $H$, given by

$$H = \{A \in P^{(i)} \mid i \in \{0, 1, \ldots, l\}\}.$$

**3. Entropy and stability in a fuzzy hierarchy.** In this section the concept of entropy [7] of a fuzzy partition is used to give a measure of stability of a fuzzy hierarchy.

Let $F$ be a $\sigma$-algebra of fuzzy sets on $X$ and $m$ a fuzzy measure over $F$ [5]. *A fuzzy measure space* is a triple $(X, F, m)$. A fuzzy partition $P$ of $X$ is called *admissible* if $A_i \in F$, $\forall A_i \in P$. The family of all finite admissible fuzzy partitions of $X$ is denoted by $Z = Z(F)$. In this section we consider only partitions from $Z$.

DEFITION 3.1. Let $P = \{A_1, \ldots, A_n\}$ be an admissible fuzzy partition. The *entropy* of $P$ is defined by

$$H(P) = -\sum_i m(A_i) \log m\,(A_j),$$

where $m$ is a fuzzy measure with $m(X) = 1$.

*Remark.* $H(P) \geqslant 0$ for every $P$. $H(P^{(0)}) = 0$, where $P^{(0)} = \{X, \emptyset\}$.

DEFINITION 3.2. The entropy $H(P|D)$ of a fuzzy partition $P = \{A_1, \ldots, A_n\}$, $P \in Z$, conditioned by a fuzzy set $D$, $m(D) > 0$, is defined by

$$H(P|D) = -\sum_i \frac{m(A_i \cdot D)}{m(D)} \log \frac{m(A_i \cdot D)}{m(D)},$$

where $A_i \cdot D$ is the product $(A_i \cdot D)\,(x) = A_i(x) \cdot D(x)$, $\forall x \in X$.

*Remark.* $H(P|X) = H(P)$, $\forall P \in Z$.

DEFINITION 3.3. Let $P, Q$ be from $Z$, $P = \{A_1, \ldots, A_n\}$, $Q = \{B_1, \ldots, B_m\}$. The entropy $H(P|Q)$ of $P$ conditioned by $Q$ is defined by

$$H(P|Q) = \sum_j m(B_j)\, H(P|B_j) = -\sum_{i,j} m(A_i B_j) \log \frac{m(A_i \cdot B_j)}{m(B_j)},$$

where $m(B_j) > 0$, $\forall j$.

*Remarks.* (i) $H(P|Q) \geqslant 0$; $H(P|P^{(0)}) = H(P)$.
(ii) If $P$ and $Q$ are independent, i.e.

$$m(A_i \cdot B_j) = m(A_i) m\,(B_j), \ \forall i, j \text{ then } H(P|Q) = H(P).$$

PROPOSITION 3.1. *If $P, Q, R$ are admissible fuzzy partitions, then the next affirmations are true*:

(i) $P \prec Q \Rightarrow H(P|R) \leqslant H(Q|R)$, $\forall R$.
(ii) $P \prec Q \Rightarrow H(P) \leqslant H(Q)$.
(iii) $Q \prec R \Rightarrow H(P|Q) \geqslant H(P|R)$, $\forall P$.
(iv) $H(P) \geqslant H(P|R)$, $\forall R$.

Since in the classification processes the information carried by a fuzzy class $C$ concerns the membership to $C$, we consider the particular fuzzy measure $m(C) = \sum_x C(x)$.

The entropy $H(P)$ is interpreted as the information carried by $P$ about the cluster structure of $X$. The trivial partition $P^{(0)} = \{X, \emptyset\}$ means no structure in $X$. $P^{(0)}$ does not carry any onformation and this is in agreement with $H(P^{(0)}) = 0$. $H(P|Q)$ is interpreted as the extra information carried by $P$ given the partition $Q$.

Let $P^{(l)}$ be the fuzzy partition of $X$ obtained at the level $l$ of the fuzzy hierarchy generated by the FDH algorithm. From $P^{(l)} \prec P^{(l+1)}$ it follows that $H(P^{(l)}) \leqslant H(P^{(l+1)})$. The entropy is thus an increasing function of the level $l$.

For our hierarchical classification procedure $H(P^{(l)} | P^{(l-1)})$ is more suitable than $H(P^{(l)})$ as a measure of information associated with the level $l$.

We may consider that a hierarchy becomes stable up to level $l$ if the information $H(P^{(l)} | P^{(l-1)})$ carried by this level is negligible.

DEFINITION 3.4. The stability coefficient of a fuzzy hierarchy is a function $S : \mathbf{N} \to \mathbf{R}$, given by

$$S(l) = \frac{H(P^{(l)}) - H(P^{(l)} | P^{(l-1)})}{H(P^{(l)})}, \quad l \geqslant 1.$$

*Remarks.* i) $S(l)$ is the stability coefficient at the level $l$.

(ii) $0 \leqslant S(l) \leqslant 1, \ \forall l \geqslant 1. \ S(1) = 0.$

(iii) $S$ is not a monotone function.

The hierarchy may be considered stable up the level $l$ iff $S(l)$ exceeds a prescribed threshold $S^*$.

Using $S(l)$ we may complete the termination condition in the FHD algorithm. The step $S2$ may be replaced by $S2'$, where

$S2'$. Compute $S(l)$. If $S(l) \geqslant S^*$, where $S^*$ is a suitable threshold, STOP. Else for every $C \in P^{(l)}$, $C$ not marked and $C \neq \emptyset$, perform $S4$. Allocate every marked $C$ to $P^{(l+1)}$.

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# ABELIAN GROUPS WITH CONTINUOUS LATTICE OF SUBGROUPS

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*Received: October 24, 1986*

ABSTRACT. — If $A$ is an abelian group, the lattice of all the subgroups of $A$ is lower continuous iff $A$ is a torsion group with finitely cogenerated $p$-components.

A complete lattice $L$ is called lower continuous if $a \vee (\bigwedge_{c\in C} c) = \bigwedge_{c\in C} (a \vee c)$ holds for every element $a \in L$ and every chain $C$ in $L$. If $A$ is an abelian group we denote by $L(A)$ the lattice of all the subgroups of $A$. This is a modular, compactly generated and hence upper continuous lattice. Our result is the following

THEOREM *$L(A)$ is lower continuous iff $A$ is a torsion group with finitely cogenerated $p$-components (i.e. direct sums of cocyclic $p$-groups).*

This result does not exceed very much the following general one

1. PROPOSITION. *Every complete artinian lattice is lower continuous. Indeed, if a poset $P$ satisfies the descending chain condition, every chain $C$ from $P$ has a least element and the above condition is obvious.*

2. COROLLARY. *If $A$ is a finitely cogenerated abelian group then $L(A)$ is lower continuous.*

This is immediate using a well-known characterization [1,25.1]: $L(A)$ is artinian iff $A$ is finitely cogenerated.

Our theorem will follow from a few lemmas.

3. LEMMA. *If $B$ is a subgroup of $A$ and $L(A)$ is lower continuous then $L(B)$ is lower continuous too.*

4. LEMMA *$L(\mathbf{Z})$ is not lower continuous.*

*Proof.* Let $p$ and $q$ be different primes, $C = \{p^n\mathbf{Z}\}_{n\in N}$ the descending chain of subgroups and $B = q\mathbf{Z}$. Then $B + p^\omega \mathbf{Z} = B \neq \mathbf{Z} = \bigcap_{n\in N} (B+p^n\mathbf{Z})$ because $p^\omega\mathbf{Z} = 0$ and $q\mathbf{Z} + p^n\mathbf{Z} = \mathbf{Z}$.

5. LEMMA. *If $A$ is an infinite elementary group, $L(A)$ is not lower continuous.*

*Proof.* Using well-known reduction theorems we can suppose $A$ to be a countable direct sum of $\mathbf{Z}(p)$ for a prime $p$. Let $\{e_n = (0, \ldots, 0, 1, 0, \ldots)/ n \in N^*\}$ the canonic basis of $A$ (as linear space over $(\mathbf{Z}p)$), $B$ the subgroup generated by $\{e_n/n \geqslant 2\}$ and $C_n$ the subgroups generated by $\{v_k/k \geqslant n\}$

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where $v_k = \sum_{s=1}^{k} e_s$ ($n \in N^*$). The following relations hold: $A = C_1 \supset C_2 \supset \ldots \supset \ldots \supset C_n \supset \ldots$, $\bigcap_{n \in N^*} C_n = 0$, $e_1 \notin B$ and $e_1 = (p-1) \sum_{s=2}^{n} e_s + v_n \in B + C_n$ for each $n \in N^*$. Hence $A$ is as stated. This example was suggested by Zoltan Finta.

*Proof of the theorem.* Let $A$ be an abelian group and $L(A)$ a lower continuous lattice. From 3 and 4, $A$ is a torsion group. If $p$ is a prime let $A_p$ be the $p$-component of $A$ and $A[p]$ its socle. Using 5, $A[p]$ is finite (otherwise it would contain a countable (elementary) subsocle) and hence $A_p$ is finitely cogenerated [1, 25.1].

Conversely, let $A$ be a torsion group such that $A_p$ is finitely cogenerated for every prime $p$, $\{C_i\}_{i \in I}$ a chain of subgroups of $A$ and $B$ a subgroup of $A$. The inclusion $\bigcap_{i \in I} (B + C_i) \subseteq B + (\bigcap_{i \in I} C_i)$ is readily verified going down to $p$-components, using 2 and the obvious equality $(\bigcap_{i \in I} (X_i)_p) = (\bigcap_{i \in I} X_i)_p$.

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STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXII, 1, 1987

# TOLERANCES IN LOGIC

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*Received: October 25, 1986*

**REZUMAT.** — **Toleranțe în logică.** Avînd o metrică discretă $d: P_n \to Q$ pe mulțimea $P_n$ a funcțiilor propoziționale n-are, se definesc relațiile de toleranță $\tau_i$,

$$x\tau_i y \Leftrightarrow d(x, y) \leqslant \frac{i}{2^n}$$

Se demonstrează anumite proprietăți ale semigrupului $(T, \circ)$ al toleranțelor $\tau_i$.

The tolerance spaces have more and more application in mathematics. The propositional logic is treated more like boolean algebra and less as tolerance space, as we do in this paper.

**1.** Let $P$ be the set of (bivalent) propositions, $P_n = \{x\colon P^n \to P\}$ the set of $n$-ary propositional functions and $V = \{0, 1\}$. For each operation or relation from $P$, it corresponds operation or relation in $P_n$. Using the (bivalent) valuation $v\colon P \to V$, we define a mapping from $P_n$ into $V^{V^n}$ ($n$-ary boolean functions), which ensures the commutativity of the enclosed diagram:

that is, $v \circ x = v_x \circ v^n$.
$$\begin{array}{ccc} P^n & \xrightarrow{\;x\;} & P \\ {\scriptstyle v^n}\downarrow & & \downarrow{\scriptstyle v} \\ V^n & \xrightarrow{\;v_x\;} & V \end{array}$$

We mention that $P^n = \underbrace{PX\ldots XP}_{n \text{ times}}$, analogously for $V^n$.

Here, $v^n$ is defined through: $v^n(p_1, \ldots, p_n) = (v(p_1), \ldots, v(p_n))$.

Let's note $A_x = \overset{-1}{v_x}\langle 1\rangle = \{\alpha \in V^n \mid v_x(\alpha) = 1\}$ and $F_x = \overset{-1}{v_x}\langle 0\rangle$. In [2] there is defined the distance in $P_n$ by:

$$(d).\quad (d(x,y) = \frac{1}{2^n}\ |(A_x \cap F_y) \cup (A_y \cap F_x)|.$$

This corresponds to the similar notion defined in [3], where $|M|$ represents the cardinal number of the set $M$. Observe that $0 \leqslant d(x,y) \leqslant 1$, for each $x,y \in P_n$. Based on [2] (or [3]) one shows that $(P_n, d)$ is a quasimetric space, namely:

$$\text{if } x = y \text{ then } d(x,y) = 0$$
$$d(x,y) = d(y,x) \text{ and}$$
$$d(x,y) \in d(x,z) + d(y,z).$$

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LEMMA 1. $d(x,y) = d(x, x \wedge y) + d(x \wedge y, y)$.

As $A_{x \wedge y} = A_x \cap A_y$, the Lemma results immediately from the definition of the distance $d$.

LEMMA 2. *If* $d(x,y) = \frac{h+1}{2^n}$ *then there exists* $z \in P_m$ *so that* $d(x,z) = \frac{h}{2^n}$ *and* $d(z,y) = \frac{1}{2^n}$.

*Proof.* The case $h = 0$ is obvious.

a). Let $x,y$ be two comparable elements in $P_n$, that is, for example, $A_x \subseteq A_y$ and so $F_y \subseteq F_x$. Therefore $\frac{h+1}{2^n} = d(x,y) = \frac{1}{2^n} \mid (A_x \cap F_y) \cup \cup (A_y \cap F_x) \mid = \frac{1}{2^n} \mid A_y \cap F_x \mid$. Taking $\alpha \in A_y \cap F_x$ we define $z \in P_n$ so that $A_z = A_y - \{\alpha\}$. It follows that $F_z = F_y \cup \{\alpha\}$ and moreover

$$\text{(i).} \quad A_x \cap A_y \subseteq A_z \subseteq A_y.$$

Herewith, $d(x,z) = \frac{1}{2^n} \mid (A_y \cap F_x) - \{\alpha\} \mid = \frac{h+1}{2^n} - \frac{1}{2^n} = \frac{h}{2^n}$ and $d(z,y) = = \frac{1}{2^n} \mid A_y \cap \{\alpha\} \mid = \frac{1}{2^n}$.

b). If $x,y$ are not comparable, we apply first Lemma 1: $\frac{h+1}{2^n} = d(x,y) = d(x, x \wedge y) + d(x \wedge y, y)$, where $d(x, x \wedge y) = \frac{h'}{2^n}$, $d(x \wedge y, y) = \frac{h''+1}{2^n}$ with $h' + h'' = h$. As $x \wedge y$, $y$ are comparable, from a) we obtain the existence of an element $z \in P_n$ so that $d(x \wedge y, z) = \frac{h''}{2^n}$ and $d(z, y) = \frac{1}{2^n}$. From (i) it follows that $A_x \cap A_y = A_x \cap A_z$, that is, $d(x, x \wedge z) = d(x, x \wedge y)$ and $d(x \wedge y, z) = d(x \wedge z, z)$ and so, with Lemma 1, $d(x,z) = d(x, x \wedge z) + + d(x \wedge z, z) = d(x, x \wedge y) + d(x \wedge y, z) = \frac{h'}{2^n} + \frac{h''}{2^n} = \frac{h}{2^n}$.

COROLLARY. For *every* $x, y \in P_n$ *with* $d(x,y) > \frac{1}{2^n}$ *there existe* $z$ *in* $P_n$ *with the property*:

$$d(x,y) = d(x,z) + d(z,y).$$

The preceding equality may be formulated as: „$z$ is metrically between $x$ and $y$" (see [1]). So the above Corollary gives an interpolation property.

2. In [4] it is given the following result:

*If* $(A,\mu)$ *is a quasimetric space, then* $\tau_{\mu(\varepsilon)} \subseteq A \times A$, *defined by*:

$$x \tau_{\mu(\varepsilon)} y \Rightarrow \mu(x,y) \leqslant \varepsilon, \quad \varepsilon \in \mathrm{R}_+$$

*is a tolerance relation and* $(A, \tau_{\mu(\varepsilon)})$ *is a connected tolerance space.*

As $d(P_n^2) = \{d(x,y) \mid x,y \in P_n\} = \left\{\frac{k}{2^n} \mid k = 0, 1, \ldots 2^n\right\}$, in the case of the quasimetric space $(P_n, d)$, there are $2^n + 1$ tolerances, $\tau_k = \tau_{d\left(\frac{k}{2^n}\right)}$ defined as above. Moreover, $i \leqslant j \Rightarrow \tau_i \subseteq \tau_j$ and so we obtain the following sequence of tolerances in $P_n$:

$$\tau_0 \subseteq \tau_1 \subseteq \ldots \subseteq \tau_{2^n}$$

where $\tau_0$ is the logical equivalence and $\tau_{2^n}$ is the universal relation on $P_n$.

We note $T_n = \{\tau_k \mid k = 1, \ldots 2^n\}$.

THEOREM $\tau_i \circ \tau_1 = \tau_{i+1}$, $i = 1, \ldots, 2^{n-1}$

*Proof.* $x(\tau_i \circ \tau_1)y \Rightarrow \exists z : x\, \tau_1 z$ and $z\, \tau_i y \Rightarrow \exists z : d(x,z) \leqslant \frac{1}{2^n}$ and $d(z,y) \leqslant \leqslant \frac{i}{2^n} \Rightarrow \exists z : d(x,y) \leqslant d(x,z) + d(z,y) \leqslant \frac{i+1}{2^n} \Rightarrow x\tau_{i+1}y$.

Conversely, suppose that $x\tau_{i+1}y$, namely $d(x,y) \leqslant \frac{i+1}{2^n}$. Then there exist the following two possibilities:

a). $d(x,y) < \frac{i+1}{2^n} \Rightarrow d(x,y) \leqslant \frac{i}{2^n} \Rightarrow x\tau_i y$. As $\tau_i \subseteq \tau_i \circ \tau_1$, we have $x\tau_i y \Rightarrow x(\tau_i \circ \tau_1)y$.

b). $d(x,y) = \frac{i+1}{2^n}$; with Lemma 2 it results the existence of $z$ in $P_n$ so that $d(x,z) = \frac{1}{2^n}$ and $d(z,y) = \frac{i}{2^n}$, that is, $x\tau_1 z$ and $z\tau_i y$, which means that $x(\tau_i \circ \tau_1)y$.

*Remarks.* 1) $\tau_i \circ \tau_h = \tau_{\overline{i+h}}$

$$\text{where } \overline{i+h} = \begin{cases} i+h & \text{if } i+h \leqslant 2^n \\ 2^n & \text{if } i+h > 2^n. \end{cases}$$

2) $(T_n, \circ)$ is generated by $\tau_1$.

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STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXII, 1, 1987

# IO'S INFLUENCE UPON THE JOVISTATIONARY SATELLITE MOTION

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Received: November 3, 1986

**REZUMAT. — Influenţa lui Io asupra mişcării satelitului jovistaţionar.** Utilizînd modelul matematic al problemei restrînse circulare a celor trei corpuri, se studiază perturbaţiile în raza vectoare a unui satelit staţionar al lui Jupiter pe seama atracţiei gravitaţionale a satelitului natural Io al planetei. Curba de variaţie a razei vectoare, obţinută pe baza integrării numerice a ecuaţiilor mişcării, este aproximată printr-o expresie analitică.

**1. Notations.** We shall use the following notations:

$T_J$ = Jupiter's axial rotation period;

$\mu_J$ = product between the gravitational constant and Jupiter's mass;

$a$ = semi-major axis of Io's orbit;

$e$ = eccentricity of Io's orbit;

$i$ = inclination of Io's orbit (onto Jupiter's equatorial plane);

$T_{sid}$ = Io's sidereal period;

$n$ = orbital angular velocity of Io;

$k$ = ratio between Io's mass and Jupiter's mass;

$G$ = common mass centre of the system Jupiter—Io;

$a_1, a_2$ = distances between $G$ and the mass centres of Jupiter and Io, respectively;

$r, r'$ = radii vectors of the jovistationary satellite with respect to the mass centres of Jupiter and Io, respectively;

$T$ = orbital period of the jovistationary satellite;

$T_s$ = jovistationary satellite period with respect to the rotating frame;

$\omega$ = angular velocity of the jovistationary satellite with respect to the rotating frame.

**2. The jovistationary satellite.** In a previous paper [1] we have defined — analogously to the case of a geostationary satellite — the jovistationary satellite, through the conditions:

(i) the orbital period is equal to Jupiter's rotation period ($T = T_J = 590^m$), the satellite orbiting in the same direction as the planet rotates around its axis;

(ii) the orbital plane coincides with the jovian equator plane;

(iii) the orbit is circular, with a radius $r = 159\,110.746$ km (value deduced from Kepler's third law for $T = 590^m$).

A presumably artificial jovian satellite for which these conditions are fulfilled will remain fixed above a point lying on the equator of Jupiter.

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In the paper [1] we have studied the influence of the solar attraction upon the motion of such a jovistationary satellite. Now we will take separately into account, as a disturbing factor, the gravitational attraction of the natural jovian satellite Io. We shall study the influence of this one upon the jovicentric radius vector (r) of the satellite.

3. **Motion equations.** Taking into account the orbital features of Io ($a = 421\,600$ km, $e = 0.004$, $i = 0° \cdot 04$, $T_{sid} = 1^d \cdot 76914$), we can use the mathematical model of the circular restricted three-body problem (see e.g. [2], [3], [5]). We choose a uniformly rotating reference frame $Gxyz$, having the origin G in the common mass centre of the system Jupiter — Io, the $Gx$ — axis directed towards the mass centre of Io, and the $Gxy$ — plane coinciding with Io's circumjovian orbit plane. In this frame, the differential equations which describe the motion of the infinitesimal body (the jovistationary satellite) have the form (cf. [1], [4], [6], [7]):

$$\begin{aligned} \ddot{x} - 2n\dot{y} - n^2x &= -\mu_J(x + a_1)/r^3 - \mu_J k(x - a_2)/r'^3, \\ \ddot{y} + 2n\dot{x} - n^2y &= -\mu_J y/r^3 - \mu_J ky/r'^3, \\ \ddot{z} &= -\mu_J z/r^3 - \mu_J kz/r'^3, \end{aligned} \tag{1}$$

where:

$$n = 24.6635 \times 10^{-4} \text{ rad/min};$$
$$\mu_J = 126\,897\,115{\cdot}5 \text{ km}^3/\text{s}^2;$$
$$k = 4.68019 \times 10^{-5};$$
$$a_1 = 19.73 \text{ km};$$
$$a_2 = 421\,580.27 \text{ km},$$

while the radii vectors $r$ and $r'$ are givne by the formulae:

$$\begin{aligned} r^2 &= (x + a_1)^2 + y^2 + z^2, \\ r'^2 &= (x - a_2)^2 + y^2 + z^2. \end{aligned} \tag{2}$$

4. **Integration of the motion equations.** In order to reduce the order of the differential system (1), we have introduced the new variables:

$$y_1 = \dot{x},\ y_2 = \dot{y},\ y_3 = \dot{z};\ y_4 = x,\ y_5 = y,\ y_6 = z. \tag{3}$$

So, we have obtained a first order differential system with the unknowns $y_i$, $i = \overline{1,6}$. This one has to be integrated with the following initial conditions (for the jovicentric radius vector and the velocity):

$$\begin{aligned} \vec{r}_0 &= \vec{r}_0(x_0, y_0 = 0,\ z_0), \\ \vec{v}_0 &= \vec{v}_0(\dot{x}_0 = 0,\ \dot{y}_0,\ \dot{z}_0 = 0). \end{aligned} \tag{4}$$

For the initial coordinates, given by the obvious formulae:

$$\begin{aligned} x_0 &= r\cos i - a_1, \\ y_0 &= 0, \\ z_0 &= r\sin i, \end{aligned} \tag{5}$$

we have obtained:

$$\vec{r}_0 = \vec{r}_0(x_0 = 159\,090.976,\ y_0 = 0,\ z_0 = 111.080), \tag{6}$$

with $x_0, y_0, z_0$ expressed in km.

For the component of the initial velocity, a special condition established by S e h n a l [4] (but for geostationary satellites) must be taken into account, namely: after a period $T_s$ of the satellite with respect to the rotating frame (or a synodic period of the satellite with respect to Io), the jovistationary orbit covered by the satellite must remain as near-circular as possible.

Considering the values of $T$ and $T_{sid}$, and the well-known relation:

$$1/T_s = 1/T - 1/T_{sid}, \tag{7}$$

one obtains immediately $T_s = 767^m \cdot 823 \cong 768^m$. Then, we have determined the theoretical value of the linear velocity on a circular orbit with a radius of 159 110.746 km and a period of 768 minutes, obtaining:

$$v_0 = |\vec{v}_0| = \dot{y}_0 = 1302.0218 \text{ km/min}. \tag{8}$$

The system (1), written with respect to the variables (3), has been integrated numerically, with the initial conditions (6) and (8), by using the Runge-Kutta algorithm (Gill's variant). For this purpose, a computer program written in BASIC [6], [7] was used. The integration was performed for one period $T_s = 768^m$, with an integration step of 2 minutes.

In order to fulfil the condition given by S e h n a l [4], the integration was performed several times, with different values for $v_0$ around the theoretical value (8). The above mentioned condition (annulment of the variation of the jovicentric radius vector at the moment $t = T_s$) is satisfied for:

$$v_0 = \dot{y}_0 = 1302.0309 \text{ km/min}. \tag{9}$$

5. **Results.** Integrating the motion equations with the initial conditions (6) and (9) over a period $T_s$ (with the same integration step of 2 minutes), we have obtained the variation $\Delta r$ of the jovicentric radius vector of the satellite due to the disturbing attraction of Io during a time interval equal to $T_s$. This variation is plotted versus time in Figure 1.

Examining this figure, one notices that the curve is symmetrical with respect to $t = T_s/2$. It shows the maximum variations at $t \cong 160^m$, $t \cong 608^m$ ($\Delta r = +1.656$ km) and $t \cong 384^m$ ($\Delta r = -1.521$ km). This curve keeps its aspect (with the period $T_s$) for however long time intervals.

One remarks that, since the variation amplitude is small, the orbit constantly remains near-circular. One also notices that at $t = T$ the variation reaches $\Delta r \cong +1.6$ km (during the first revolution).

It seemed to us interesting to determine the annulment of $\Delta r$ after a whole number of periods $T$. This event occurs after 157.33 days ($384\ T = 295\ T_s$) for $T_s = 768^m$, or after 573.2 days ($1399\ T = 1075\ T_s$) for $T_s = 767^m \cdot 823$.

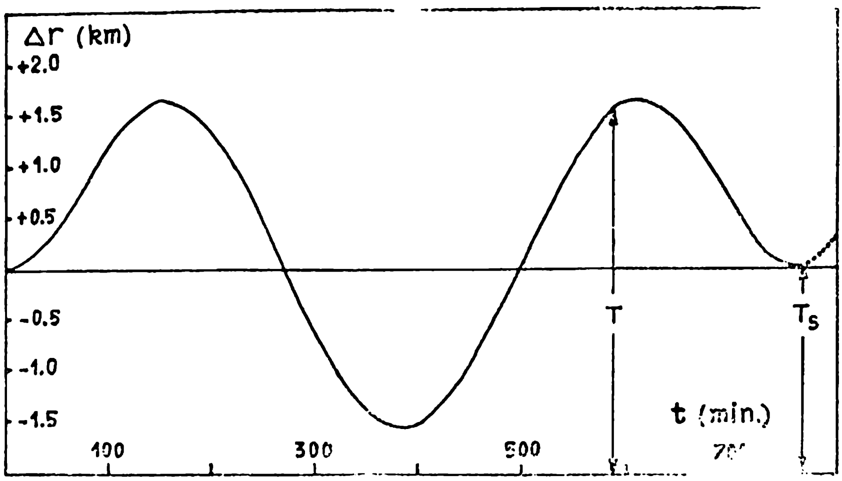


Fig. 1

Coming back to the amplitude of the jovicentric radius vector variation, and comparing the present results to those obtained in [1], we see that the solar attraction produces more significant effects in $r$ than Io's attraction. Indeed, the difference $r_{max} - r_{min}$ reaches some 70 km if the disturbing factor is the Sun's attraction, and only about 3 km if one takes into account the separate influence of Io's attraction.

6. **Analytical approach.** We said that the curve $\Delta r(t)$ plotted in Figure 1 has a periodical character, with a period $T_s = 768^m$. In order to approximate this curve by an analytical expression, we resorted to the harmonic analysis, trying to establish an empirical formula of the kind:

$$f(x) = \sum_{j=0}^{n} (A_j \cos (jx) + B_j \sin (jx)). \tag{10}$$

For this purpose, dividing the interval [0°, 360°] into 12 subintervals of 30° each (which is equivalent to the division of the period $T_s$ into intervals of $64^m$ each), we have applied the calculation scheme for the case when one uses 12 ordinates [8]. So, we have obtained the following expression for the variation of the deviation $\Delta r$ with the time:

$$\Delta r(t) = \sum_{j=0}^{6} (A_j \cos (j\omega t) + B_j \sin (j\omega t)), \tag{11}$$

where the value of $B_0$ is not needed and the convention that $B_6 \equiv 0$ is made. In (11), $\omega = 0.46875°/\text{min}$, $t$ is expressed in minutes, while the coefficients $A_j$, $B_j$ (expressed in km) have the values:

$$\begin{aligned} &A_0 = +0.3815,\ A_1 = +0.9020,\ A_2 = -1.1129,\ A_3 = -0.1338, \\ &A_4 = -0.0262,\ A_5 = -0.0062,\ A_6 = -0.0013,\ B_1 = +0.0008, \\ &B_2 = +0.0007,\ B_3 = +0.0002,\ B_4 = -0.0002,\ B_5 = -0.0001; \end{aligned} \tag{12}$$

obviously, the deviation $\Delta r$ results to be expressed in km.

Considering the values (12), one sees that (11) can be written in the more simple form:

$$\Delta r(t) = \sum_{j=0}^{6} A_j \cos (j\omega t), \tag{13}$$

neglecting the terms which contain sin $(j\omega t)$; in this case $\Delta r$ is determined with an error which does not exceed $\pm 0.0005$ km. Moreover, if the sum in (13) is considered only up to $j = 5$, the error in determining $\Delta r$ does not exceed $\pm 0.001$ km.

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STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXII, 1, 1987

# OPTIMAL ALGORITHMS WITH RESPECT TO THE COMPLEXITY

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*Received: November 11, 1986*

**REZUMAT. — Algoritmi optimali în raport cu complexitatea.** În lucrare se studiază comparativ, în raport cu complexitatea, mai mulţi algoritmi de rezolvare numerică a sistemelor algebrice liniare precum şi de calcul a valorii polinomului de interpolare Lagrange pe un punct dat. Concluziile obţinute sînt prezentate prin propoziţiile 1—4.

**0.** Let $X$ be a linear space over the real or complex field $K$, $(Y, ||\cdot||)$ a normed linear space over the $K$, $X_0$ a subset of $X$ and $S : X_0 \to Y$ a given operator.

One considers the $(X_0, S)$ — problem: for a given $\varepsilon > 0$ to find an element $y \in Y$ such that $||s - y|| \leqslant \varepsilon$ where $s = S(x)$ and $y = y(x)$, i.e. an $\varepsilon$ — approximation of s for all $x \in X_0$ [3].

For $X_1 \subseteq X$ such that $X_0 \subseteq X_1$ one denotes by $\mathfrak{I} : X_1 \to Z$, where $Z$ is a given linear space, the information operator of the problem $(X_0, S)$, i.e. $\mathfrak{I}(x)$ is the information of the problem element $x$.

Let also $\alpha : \mathfrak{I}(X_0) \to Y$ be an algorithm to compute the approximation $\alpha(\mathfrak{I}(x))$ of the solution $S(x)$ and $\mathcal{A}(S, \mathfrak{I})$ the set of all algorithms $\alpha$ for the problem $(X_0, S)$ with the information $\mathfrak{I}$.

If $\mathfrak{R}$ is a set of operations, the information operator $\mathfrak{I}$ respectively an algorithm $\alpha \in \mathcal{A}(S, \mathfrak{I})$ are named $\mathfrak{R}$ — admissible if $\mathfrak{I}(x)$ and $\alpha(\mathfrak{I}(x))$ can be computed, for any $x \in X_0$, with a finite numbers of operations from $\mathfrak{R}$.

Next, one suppose that $\mathfrak{I}$ and $\alpha$ are $\mathfrak{R}$ — admissibles.

Let $r_1, \ldots, r_m \in \mathfrak{R}$ be the necessary operations to compute $\mathfrak{I}(x)$, $x \in X_0$. The value

$$\mathrm{CPE}\,(\mathfrak{I}(x)) = \sum_{i=1}^{m} p_i\, \mathrm{CP}(r_i)$$

where $p_i$ is the performing number of the operation $r_i$ and CP $(r_i)$ is the complexity of the same operation, is named the complexity of the information $\mathfrak{I}(x)$ and the value

$$\mathrm{CPE}\,(\mathfrak{I}) = \sup_{x \in X_0} \mathrm{CPE}\,(\mathfrak{I}(x))$$

is the information complexity.

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Also, for $\rho_1, \dots, \rho_n \in \mathfrak{R}$ the necessary operations to compute $\alpha(\mathfrak{I}(x))$, one denotes by

$$\text{CPC}(\alpha(\mathfrak{I}(x))) = \sum_{i=1}^{n} q_i \,\text{CP}(\rho_i) \quad \text{and} \quad \text{CPC}(\alpha) = \sup_{x \in X_0} \text{CPC}(\alpha(\mathfrak{I}(x)))$$

with $q_i$ the performing number of the operation $\rho_i$, the combinatorial complexity of the algorithm $\alpha$.

The value

$$\text{CP}(\alpha) = \text{CPE}(\mathfrak{I}) + \text{CPC}(\alpha)$$

is the complexity of the algorithm $\alpha$.

In order to get an $\varepsilon$ — approximation of the solution element $s = S(x)$, $x \in X_0$ with the information $\mathfrak{I}$, the error $e(S, \mathfrak{I}, \alpha)$ of the algorithm $\alpha$;

$$e(S, \mathfrak{I}, \alpha) = \sup_{x \in X_0} ||S(x) - \alpha(\mathfrak{I}(x))||$$

must satisfy the condition

$$e(S, \mathfrak{I}, \alpha) \leqslant \varepsilon$$

i.e. the information $\mathfrak{I}$ and the algorithm $\alpha$ must be $\varepsilon$ — admissible.

Let $\mathfrak{A}(S, \mathfrak{I}, \varepsilon)$ be the set of algorithms $\alpha \in \mathfrak{A}(S, \mathfrak{I})$ with $e(S, \mathfrak{I}, \alpha) \leqslant \varepsilon$.

The value

$$\text{CP}(S, \mathfrak{I}, \varepsilon) = \begin{cases} \inf\limits_{\alpha \in \mathfrak{A}(S, \mathfrak{I}, \varepsilon)} \text{CP}(\alpha), & \mathfrak{A}(S, \mathfrak{I}, \varepsilon) \neq \emptyset \\ +\infty, & \mathfrak{A}(S, \mathfrak{I}, \varepsilon) = \emptyset \end{cases}$$

is named the $\varepsilon$ — complexity of the problem $S$ with the information $\mathfrak{I}$.

An algorithm $\alpha^* \in \mathfrak{A}(S, \mathfrak{I}, \varepsilon)$ for which $\text{CP}(\alpha^*) = \text{CP}(S, \mathfrak{I}, \varepsilon)$ is named an optimal algorithm with regard to the complexity.

Thus, the complexity or the $\varepsilon$ — complexity can be used as a criterion to evaluate the „goodness" of an algorithm.

The goal of this paper is to study, using the complexity, some numerical algorithms for the solution of linear algebraic systems and for the computation of interpolating polynomial of a given function.

**1. Linear algebraic systems.** One considers the system

$$As = b, \quad s^T \in \mathbb{R}^n \tag{1}$$

where $A$ is a real $n \times n$ matrix, $b$ is a real column vector and $s^T$ is the transpose of s. Next, one supposes that $A$ is nonsingular.

In this case $X$ is the set of the matrices $[A|b]$, $X_0$ is the set of elements $[A|b]$ with $A$ nonsingular, $Y$ is the set of column vectors $r$ with $r^T \in \mathbb{R}^n$ and the solution operater is $S: X_0 \to Y$; $[A|b] \mapsto s$, $s = A^{-1}b$.

For a linear system the given information consists of the matrix $A$ and the vector $b$. Hence $Z = X_0$ and the information operator $\mathfrak{I}$ is the identity operater, $\mathfrak{I} = I$. This means that $\mathfrak{I}$ is $\varepsilon$ — admissible for any $\varepsilon > 0$ and CPE $(\mathfrak{I}) = 0$.

The operations set is $\mathfrak{R} = \{+, -, *, /\}$. Thus, any direct algorithm (performing exact arithmetic operations on real numbers) is $\varepsilon$ — admissible.

It follows that for a direct algorithm $\alpha$,

$$\text{CP}\,(S, \mathfrak{I}, \varepsilon) = \text{CPC}\,(\alpha).$$

Next, we consider six direct algorithms and two iterative algorithms for the solution of the system (1).

*Cramer's algorithm* $(\alpha_C)$. We have

$$s_i = \frac{D_i}{D}\ ,\ i = \overline{1, n}$$

where $D$ and $D_i$, $i = \overline{1, n}$ are the corresponding determinants from the Cramer's formulas.

Knowing that the computational complexity of a determinant of order $n$ is [2],

$$\text{CP}\,(D) = (n\,! - 1)\,\text{CP}\,(+) + n\,!\,r\,\text{CP}\,(*)$$

with

$$r = 1 + \frac{1}{2\,!} + \ldots . + \frac{1}{(n-1)\,!}$$

one obtains

$$\text{CP}\,(\alpha_C) = (n+1)(n\,! - 1)\,\text{CP}\,(+) + r(n+1)\,!\,\text{CP}\,(*) + n\,\text{CP}(/). \quad (2)$$

Taking into acount that on the Felix system

$\text{CP}\,(+) = \text{CP}\,(-) = 8{,}10\ \mu$ (microseconds)

$\text{CP}\,(*) = 24{,}40\ \mu$

$\text{CP}\,(/) = 27{,}30\ \mu$

and we abbreviate by $\text{CP}_F$ the computational complexity on the Felix system, we have

$$\text{CP}_F(\alpha_C) = [8{,}10\,(n+1)(n\,! - 1) + 24{,}40\,r(n+1)\,! + 27{,}30\,n]\ \mu \quad (3)$$

*Gaussian elimination algorithm* $(\alpha_G)$. It is well known that the computations in the Gaussian elimination method are described by the relations

$$a_{ij}^{(1)} := a_{ij},\ b_i^{(1)} := b_i,\ i, j = \overline{1, n}$$

$$a_{ij}^{(p+1)} = a_{ij}^{(p)} - a_{pj}^{(p)}(a_{ip}^{(p)}/a_{pp}^{(p)})$$

$$b_i^{(p+1)} = b_i^{(p)} - b_p^{(p)}(a_{ip}^{(p)}/a_{pp}^{(p)})$$

for $p = \overline{1, n-1}$ and $i, j = \overline{p+1, n}$. If all $a_{pp}^{(p)} \neq 0$, $p = \overline{1, n}$, this process yields an upper triangular matrix of coefficients in the system of $n$ equations

$$\sum_{j=p}^{n} a_{pj}^{(p)} s_j = b_p^{(p)},\ p = \overline{1, n}$$

This triangular system is then solved by the back substitution process:

$$s_n = b_n^{(n)} / a_{nn}^{(n)}$$

$$s_p = \frac{1}{a_{pp}^{(p)}} \left[ b_p^{(p)} - \sum_{j=p+1}^{n} a_{pj}^{(p)} s_j \right],\quad p = \overline{n-1, 1}.$$

It follows that the complexity of the algorithm $\alpha_G$ is:

$$\mathrm{CP}\,(\alpha_G) = \frac{n(n-1)(2n+5)}{6}\,[\mathrm{CP}(+) + \mathrm{CP}\,(*)] + \frac{n(n+1)}{2}\,\mathrm{CP}(/), \tag{4}$$

respectively

$$\mathrm{CP}_F(\alpha_G) = \frac{n(32{,}50\,n^2 + 89{,}70\,n - 40{,}30)}{3}\,\mu. \tag{5}$$

*Total elimination algorithm* ($\alpha_T$). By the transformations

$$a_{pj}^{(p)} := a_{pj}^{(p-1)} / a_{pp}^{(p-1)},\ j := \overline{p+1, n}$$

$$b_p^{(p)} := b_p^{(p-1)} / a_{pp}^{(p-1)}$$

$$a_{ij}^{(p)} := a_{ij}^{(p-1)} - a_{pj}^{(p)} \cdot a_{ip}^{(p-1)},\ j = \overline{p+1, n},\ i = \overline{1, n},\ i \neq p$$

$$b_i^{(p)} := b_i^{(p-1)} - b_p^{(p)} \cdot a_{ip}^{(p-1)},\ i = \overline{1, n},\ i \neq p$$

for $p = \overline{1, n-1}$ and

$$b_i^{(n)} := b_i^{(n-1)} - b_n^{(n)} \cdot a_{in}^{(n-1)},\ i = \overline{1, n-1},$$

where

$$a_{ij}^{(0)} := a_{ij},\ b_i^{(0)} := b_i,\ i, j = \overline{1, n},$$

the system (1) becomes

$$s_i = b_i^{(n)},\ i = \overline{1, n}.$$

It follows that

$$\mathrm{CP}\,(\alpha_T) = \frac{n(n^2-1)}{2}\,[\mathrm{CP}(+) + \mathrm{CP}(*)] + \frac{n(n+1)}{2}\,\mathrm{CP}(/) \tag{6}$$

respectively

$$\mathrm{CP}_F(\alpha_T) = n\,(16{,}25\,n^2 + 13{,}65\,n - 2{,}60)\,\mu. \tag{7}$$

*Cholesky's algorithm* ($\alpha_{Ch}$). The matrix $A$ is written in the form

$$A = P \cdot Q$$

where

$$P = \begin{bmatrix} p_{11} & 0 & \dots & 0 \\ p_{21} & p_{22} & \dots & 0 \\ & \cdot & \cdot & \cdot & \cdot \\ p_{n1} & p_{n2} & \dots & p_{nn} \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & q_{12} & \dots & q_{1n} \\ 0 & 1 & \dots & q_{2n} \\ & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

with

$$p_{i1} = a_{i1} \quad , i = \overline{1, n}$$

$$q_{1j} = a_{1j}/p_{11}, \ j = \overline{2, n}$$

$$p_{ij} = a_{ij} - \sum_{k=1}^{j-1} p_{ik} q_{kj}, \ j = \overline{2, i}$$

$$q_{ij} = \frac{1}{p_{ii}} \left( a_{ij} - \sum_{k=1}^{i-1} p_{ik} q_{kj} \right), \ j = \overline{i + 1, n}$$

for $i = \overline{2, n}$. This way, the system (1) is reduced to the following triangular systems:

$$Py = b$$

$$Qs = y$$

that can be solved by back substitution method.

One obtains:

$$\text{CP}(\alpha_{Ch}) = \frac{n(n-1)(2n+5)}{6} [\text{CP}(+) + \text{CP}(*)] + \frac{n(n+1)}{2} \text{CP}(/) \tag{8}$$

and

$$\text{CP}_F(\alpha_{Ch}) = \frac{1}{3} n(32{,}50 \, n^2 + 89{,}70 \, n - 40{,}30) \, \mu. \tag{9}$$

*Onicescu's algorithm* ($\alpha_0$). Using a linear transformation

$$\alpha_{kl} = a_{1l} \lambda_{k1} + \dots + a_{nl} \lambda_{kn}, \ k, l = \overline{1, n}$$

$$\beta_k = b_1 \lambda_{k1} + \dots + b_n \lambda_{kn}, \ k = \overline{1, n} \tag{10}$$

from the initial system (1) the equivalent system can be obtained

$$\alpha s = \beta \tag{11}$$

with the property that

$$\begin{cases} \alpha_{1,p+1}s_{p+1} + \dots + \alpha_{1n}s_n = s_{p+1}(\alpha_{1,p+1} + \dots + \alpha_{1n}) \\ \dots \\ \alpha_{p+1,p+1}s_{p+1} + \dots + \alpha_{p+1,n}s_n = s_{p+1}(\alpha_{p+1,p+1} + \dots + \alpha_{p+1,n}). \end{cases}$$

Thus, the first $(p+1)$ equations of the system (11), become:

$$\begin{cases} \alpha_{11}s_1 + \dots + \alpha_{1p}s_p + s_{p+1}(\alpha_{1,p+1} + \dots + \alpha_{1n}) = \beta_1 \\ \dots \\ \alpha_{p+1,1}s_1 + \dots + \alpha_{p+1,p}s_p + s_{p+1}(\alpha_{p+1,p+1} + \dots + \alpha_{p+1,n}) = \beta_{p+1}. \end{cases} \quad (12)$$

Solving this system one obtains the first $(p+1)$ components, $\dot{s}_i$, $i = \overline{1, p+1}$, of the solution. From the last $(n-p-1)$ equations of the initial system (1):

$$\begin{cases} a_{p+2,p+2}s_{p+2} + \dots + a_{p+2,n}s_n = b_{p+2} - (a_{p+2,1}\dot{s}_1 + \dots + a_{p+2,p+1}\dot{s}_{p+1}) \\ \dots \\ a_{n,p+2}s_{p+2} + \dots + a_{n,n}s_n = b_n - (a_{n1}\dot{s}_1 + \dots + a_{n,p+1}\dot{s}_{p+1}) \end{cases} \quad (13)$$

one obtains the next components, $\dot{s}_k$, $k = \overline{p+2, n}$, of the solution.

This way, the problem to solve a $n \times n$ system is reduced to the problem to solve two systems of order $(p+1) \times (p+1)$ respectively $(n-p-1) \times (n-p-1)$.

Of course, we also have to determine the transformation (10), i.e. the parameters $k_{kl}$, $k, l = \overline{1, n}$.

*Remark 1*. The number $p$ can be chosen arbitrarily.

If, for the beginning, $p := n-3$ then (12) is a $(n-2) \times (n-2)$ system and (13) is a $2 \times 2$ system.

Now, if we apply the same method to the system (12) for $p := n-5$ one obtains a $(n-4) \times (n-4)$ system respectively a $2 \times 2$ system. So, we can obtain $n/2$, $2 \times 2$ systems if $n$ is even respectively $(n-3)/2$, $2 \times 2$ systems and one $3 \times 3$ system if $n$ is an odd number. Solving these systems, beginning with the last one we get the solution.

The algorithm is:

1. The matrix $A$ is written in the form

$$A = \left[ B \,\middle|\, \frac{C}{D} \right]$$

where

$$B = \begin{bmatrix} a_{11} & \dots & a_{1,p+1} \\ & \dots & \\ a_{n1} & \dots & a_{n,p+1} \end{bmatrix}, \quad C = \begin{bmatrix} a_{1,p+2} & \dots & a_{1n} \\ & \dots & \\ a_{p+1,p+2} & \dots & a_{p+1,n} \end{bmatrix}, \quad D = \begin{bmatrix} a_{p+2,p+2} & \dots & a_{p+2,n} \\ & \dots & \\ a_{n,p+2} & \dots & a_{nn} \end{bmatrix}$$

2. One determines the matrix $\lambda$:

$$\lambda = \begin{bmatrix} \lambda_{1,p+2} & \dots & \lambda_{1n} \\ \lambda_{p+1,p+2} & \dots & \lambda_{p+1,n} \end{bmatrix}$$

solving the following $(p + 1)$ systems:

$$D^T\lambda_i = M_i, \ i = \overline{1, p + 1} \tag{14}$$

where

$$\lambda_i = \begin{bmatrix} \lambda_{i,p+2} \\ \dots \\ \lambda_{in} \end{bmatrix}, \quad M_i = \begin{bmatrix} -a_{i,p+2} \\ \dots \\ -a_{in} \end{bmatrix}, \quad i = \overline{1, p + 1}$$

*Remark 2.* Taking $p = n - 3$, (14) are $2 \times 2$ systems.

3. One determines the matrices $\alpha$ and $\beta$:

$$\alpha = (I_{p+1} \,|\lambda)(P\,|F)$$
$$\beta = (I_{p+1} \,|\lambda)b$$

where $I_k$ is the unity matrix of order $k$ and

$$P = \begin{bmatrix} a_{11} & \dots & a_{1p} \\ \cdot & \cdot & \cdot \\ a_{n1} & \dots & a_{np} \end{bmatrix}, \quad F = \begin{bmatrix} \sum_{j=p+1}^{n} a_{1j} \\ \dots \\ \sum_{j=p+1}^{n} a_{nj} \end{bmatrix}$$

4. One solves the system

$$\alpha s = \beta$$

to get $s_1^*, \dots, s_{p+1}^*$, then the system

$$DT = G$$

where

$$T = \begin{bmatrix} s_{p+2} \\ \cdots \\ s_n \end{bmatrix}, \quad G = \begin{bmatrix} b_{p+2} - \sum_{j=1}^{p+1} a_{p+2,j}\, s_j^* \\ \cdots\cdots \\ b_n - \sum_{j=1}^{p+1} a_{nj} s_j^* \end{bmatrix}$$

in order to get $s_{p+2}^*, \dots, s_n^*$.

If we take $p = n - 3, n - 5, \dots, 3, 1$ for $n$ even, the complexity of the corresponding algorithm denoted by $\alpha_0^2$ is

$$\mathrm{CP}(\alpha_0^2) = \frac{4n^3 + 15n^2 - 4n - 48}{12}\mathrm{CP}(+) + \frac{n(4n^2 + 9n - 16)}{12}\mathrm{CP}(*) + \frac{3n^2}{4}\mathrm{CP}(/), \tag{15}$$

while the same algorithm for $n$ odd, has the complexity

$$CP(\alpha_0^2) = \frac{4n^3 + 15n^2 - 4n - 99}{12} CP(+) + \frac{4n^3 + 89n^2 - 16n - 9}{12} CP(*) + + \frac{3(n^2 - 1)}{4} CP(/) \tag{16}$$

We also have

$$CP_F(\alpha_0^2) = \left(\frac{32,50}{3} n^3 + 48,90n^2 - \frac{105,7}{3} n\right) \mu \tag{17}$$

for $n$ even respectively

$$CP_F(\alpha_0^2) = \left(\frac{32,50}{3} n^3 + 48,90n^2 - \frac{105,7}{3} n - 105,60\right) \mu \tag{18}$$

for $n$ an odd number.

Now, if one denotes by $\alpha_0^1$ the Onicescu's algorithm $\alpha_0$ for $p = n - 2$, $n - 3, \ldots, 1$, we have

$$CP(\alpha_0^1) = \frac{n^3 + 3n^2 - n - 9}{3} CP(+) + \frac{n(n-1)(2n+5)}{6} CP(*) + \frac{n(n+1)}{2} CP(/) \tag{19}$$

respectively

$$CP_F(\alpha_0^1) = \left(\frac{32,50}{3} n^3 + 33,95n^2 + 13,55n - 24,30\right)\mu. \tag{20}$$

Taking into attention the complexity of the above algorithms it follows:

PROPOSITION 1. *For any $n \in \mathbf{N}$, $n \geqslant 4$ we have*

$$CP(\alpha_G) = CP(\alpha_{Ch}) < CP(\alpha_0^1) < CP(\alpha_0^2) < CP(\alpha_C) \tag{21}$$

$$CP(\alpha_T^-) < CP(\alpha_C)$$

*and for $n \geqslant 5$* $CP(\alpha_0^2) < CP(\alpha_T^-)$.

Indeed, from (4) and (6) we have $CP(\alpha_G) = CP(\alpha_{Ch})$. Also, from (19) it can be seen that the algorithm $\alpha_0^1$ contains the same number of multiplications and divisions as the algorithms $\alpha_G$ o, $\alpha_{Ch}$, but more additions. Hence $CP(\alpha_0^1) > CP(\alpha_G)$. The relations (15) – (16) and (19) imply that $CP(\alpha_0^1) < CP(\alpha_0^2)$ if $n > 3$ for both $n$ even or odd. The inequalities $CP(\alpha_0^2) <$ $< CP(\alpha_C)$ and $CP(\alpha_T) < CP(\alpha_C)$ follow immediately. The last inequality follows by the observation that $CP(*) > 2CP(+)$ $(CP_F(*) = 3CP_F(+))$ and $CP(/) > CP(*)$ but the difference $CP(/) - CP(*)$ is very small. For $n = 5$ we have $CP(\alpha_T^-) - CP(\alpha_0^2) = -3CP(+) + 7CP(*) - 3CP(/)$ and it increases with $n$.

The values of the complexity $CP_F$ for some $n \in \mathbf{N}$ are given in the following table:

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|
| $CP_F(\alpha_G)$ | 179 | 521 | 1118 | 2034 | 3335 | 5086 | 7352 | 10198 | 13689 | 98358 |
| $CP_F(\alpha_0^1)$ | 225 | 614 | 1266 | 2246 | 3619 | 5449 | 7803 | 10745 | 14339 | 100493 |
| $CP_F(\alpha_0^2)$ | 211 | 521 | 1334 | 2294 | 3889 | 5759 | 8394 | 11435 | 15371 | 105522 |
| $CP_F(\alpha_T)$ | 179 | 553 | 1248 | 2359 | 3485 | 6224 | 9172 | 12928 | 17589 | 135408 |

| $n$ | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
|---|---|---|---|---|---|---|---|---|
| $CP_F(\alpha_G)$ | 319007 | 740636 | 1428250 | 2446830 | 3861400 | 5736950 | 8138480 | 11131000 |
| $CP_F(\alpha_0^1)$ | 323437 | 748171 | 1439690 | 2463010 | 3883110 | 5765010 | 8173690 | 11174200 |
| $CP_F(\alpha_0^2)$ | 335453 | 770164 | 1474660 | 2513930 | 3952980 | 5856810 | 8290420 | 11318800 |
| $CP_F(\alpha_T)$ | 450957 | 1061740 | 2065240 | 3558980 | 5640450 | 8407150 | 11956600 | 16386200 |

It can be seen that the inequalities from the Proposition 1 are preserved.

Next, one considers the Jacobi and Gauss–Siedel iterative algorithms. First, the system (1) is written in the form (one supposes that $a_{ii} \neq 0$, $i = \overline{1, n}$):

$$s_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{n} \alpha_{ij} s_j\right), \quad i = 1, n$$

where $\alpha_{ij} = a_{ij}$ for $i \neq j$ and $\alpha_{ii} = 0$.

One denotes by $\alpha_J$ the Jacobi algorithm defined by the recurrence formulas

$$s_i^{(k)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{n} \alpha_{ij} s_j^{(k-1)}\right), \ i = \overline{1, n}; \quad k = 1, \ldots, n_J$$

and by $\alpha_{GS}$ the Gauss–Siedel algorithm defined by

$$s_i^{(k)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=i}^{n} \alpha_{ij} s_j^{(k-1)} - \sum_{j=1}^{i-1} \alpha_{ij} s_j^{(k)}\right), \ i = \overline{1, n}; \ k = \overline{1, n_{GS}}$$

where $s^{(0)}$ is the start value and $n_J$, respectively $n_{GS}$ are the necessary number of iteration to get an ε-approximation of the solution by the Jacobi and Gauss–Siedel methods.

It follows that the ε-complexity is

$$CP(\alpha_J; \varepsilon) = n[(n-1)CP(+) + (n-1)CP(*) + CP(/)]n_J \tag{22}$$

$$CP_F(\alpha_J; \varepsilon) = (32{,}50n^2 - 5{,}20n)n_J \tag{23}$$

respectively

$$CP(\alpha_{GS}; \varepsilon) = n[(n-1)CP(+) + (n-1)CP(*) + CP(/)]n_{GS} \tag{24}$$

$$CP_F(\alpha_{GS}; \varepsilon) = (32{,}50n^2 - 5{,}20n)n_{GS}. \tag{25}$$

Taking into account that the Gauss–Siedel algorithm is faster than the Jacobi algorithm, we have $n_{GS} < n_J$.

It follows:

PROPOSITION 2. *For a given $\varepsilon > 0$ and $n \in \mathbf{N}$, $n \geq 2$ we have*

$$CP(\alpha_{GS}; \varepsilon) \leq CP(\alpha_J; \varepsilon)$$

Hence, the Gauss–Siedel algorithm is better from the complexity point of view.

To compare a direct algorithm with an iterative one you must know the iterations number to get an ε-approximation. But this number is not *apriori* known.

Using the complexity we can get the following conclusion:

PROPOSITION 3. *If* $(2/3)[CP(+)+CP(*)] < CP(/) < CP(+)+CP(*)$ *then*

$$CP(\alpha_G)/CP(\alpha_{GS}) > [(n+2)/3],$$

*where* $[\cdot]$ *is the entier function.*

Indeed, we have

$$\frac{CP(\alpha_G)}{CP(\alpha_{GS})} = \frac{1}{6}\,\frac{(n-1)(2n+5)\,[CP(+)+CP(*)]+3(n+1)CP(/)}{(n-1)\,[CP(+)+CP(*)]+CP(/)}$$

From the hypothesis inequalities, it follows that

$$\frac{CP(\alpha_G)}{CP(\alpha_{GS})} > \frac{1}{6}\,\frac{2n^2+5n-3}{n} > \frac{n+2}{3}.$$

*Remark 3.* If the iteration number necessary to get an ε-approximation by Gauss–Siedel method is less than $[(n+2)/3]$, then the algorithm $\alpha_{GS}$ is better, from the complexity point of view, than $\alpha_G$.

2. **Lagrange interpolation.** Let $S$ be the Lagrange interpolation problem with the information $\mathfrak{I}(f) = (f_1, \ldots, f_n)$, where $f_i = f(x_i)$, $x_i \in [a, b]$, $i = \overline{1, n}$ and $x_i \neq x_j$ for $i \neq j$.

It is well known that the Lagrange interpolation polynomial has the form

$$(L_{n-1}f)(x) = \sum_{i=1}^{n} l_i(x) f(x_i) \qquad (26)$$

with

$$l_i(x) = \frac{(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}$$

respectively the Newton form

$$(N_{n-1}f)(x) = f(x_1) + \sum_{i=1}^{n}(x-x_1)\ldots(x-x_{i-1})[x_1, \ldots, x_i; f] \qquad (27)$$

where $[x_1, \ldots, x_i; f]$ is the divided difference of $f$.

One denotes by $\alpha_L$, $\alpha_L^A$ and $\alpha_L^N$ the algorithms that compute the value $(L_{n-1}f)(x)$ using the formula (26), the Aitken's method respectively the Newton's formula (27), i.e.:

*Algorithm* $\alpha_L$:

1. Input Data ;$n \in N$; $x, x_i, f_i \in R$, $i = \overline{1, n}$

2. Compute:

$a_i := x - x_i,\ i = \overline{1, n}$

$u := a_1 * \ldots * a_n$

$y_{ki} := x_k - x_i,\ i = \overline{1, k-1};\ k = \overline{2, n}$

$s_k := f_k / (a_k * y_{k1} * \ldots * y_{k,k-1} * (-y_{k,k+1}) * \ldots * (-y_{kn})),\ k = \overline{1, n}$

$VL := u * (s_1 + \ldots + s_n)$

2. Output Data: $VL$.

*Algorithm* $\alpha_L^A$:

1. Input Data: $n \in \mathbb{N},\ x,\ x_i, f_i \in \mathbb{R},\ i = \overline{1, n}$
2. Compute:

$a_i := x_i - x,\ i = \overline{1, n}$

$f_{i1} := f_i,\ i = \overline{1, n}$

$y_{ij} := x_i - x_j,\ j = \overline{1, i-1};\ i = \overline{2, n}$

$f_{i,j+1} := (a_i * f_{jj} - a_j * f_{ij}) / y_{ij},\ j := \overline{1, i-1};\ i = \overline{2, n}$

$VL := f_{nn}$

3. Output Data: $VL$

*Algorithm* $\alpha_L^N$:

1. Input Data: $n \in \mathbb{N};\ x,\ x_i, f_i \in \mathbb{R},\ i = \overline{1, n}$
2. Compute:

$a_i := x - x_i,\ i = \overline{1, n-1}$

$D_{i1} := f_i,\ i = \overline{1, n};\ VL := D_{11};\ s := 1$

$y_{ij} := x_i - x_j,\ j = \overline{1, i-1};\ i = \overline{2, n}$

$D_{i,j+1} := (D_{ij} - D_{i-1,j}) / y_{i,i-j-1};\ j = \overline{1, i-1};\ i = \overline{2, n}$

$(s := s * a_{i-1},\ VL := VL + s * D_{ii}),\ i = \overline{2, n}$

3. Output Data: $VL$

It can be seen that for all these algorithms the information complexity is the same:

$$\mathrm{CPE}(\alpha_L) = \mathrm{CPE}(\alpha_L^A) = \mathrm{CPE}(\alpha_L^N) = n\mathrm{CP}(f),$$

where $\mathrm{CP}(f)$ is the complexity of the computation of $f(x)$, for a given $x \in \mathbb{R}$.

For the combinatorial complexity one obtains:

$$CPC(\alpha_L) = \frac{n^2 + 3n - 2}{2} CP(+) + n^2 CP(*) + nCP(/)$$

$$CPC(\alpha_L^A) = n^2 CP(+) + n(n - 1)CP(*) + \frac{n(n-1)}{2} CP(/)$$

$$CPC(\alpha_L^N) = (n - 1)(n + 2)CP(+) + 2(n - 1)CP(*) + \frac{n(n-1)}{2} CP(/)$$

It follows that:

$$CP(\alpha_L) = nCP(f) + \frac{n^2 + 3n - 2}{2} CP(+) + n^2 CP(*) + nCP(/)$$

$$CP(\alpha_L^A) = nCP(f) + n^2 CP(+) + n(n - 1)CP(*) + \frac{n(n-1)}{2} CP(/)$$

$$CP(\alpha_L^N) = nCP(f) + (n - 1)(n + 2)CP(+) + 2(n - 1)CP(*) + \frac{n(n-1)}{2} CP(/)$$

respectively

$$CP_F(\alpha_L) = nCP(f) + (28{,}45n^2 + 39{,}45n - 8{,}10)\mu$$

$$CP_F(\alpha_L^A) = nCP(f) + (46{,}15n^2 - 21{,}75n)\mu$$

$$CP_F(\alpha_L^N) = nCP(f) + (21{,}75n^2 + 43{,}15n - 65)\mu$$

PROPOZITION 4. We have:

$CP(\alpha_L^N) < CP(\alpha_L^A)$ for any $n \geq 2$

$CP(\alpha_L^N) < CP(\alpha_L)$ for any $n \geq 2$

$CP(\alpha_L) < CP(\alpha_L^A)$ for any $n \geq 5$.

These inequalities follow from a direct comparison of the corresponding complexity.

*Remark* 5. The best algorithm, on regard with the compexity is the Newton algorithm $\alpha_L^N$.

The same conclusions can be obtained if we consider the complexity $CP_F$.

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STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXII, 1, 1987

# MAXIMUM PRINCIPLE FOR SOME SYSTEMS OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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*Received: December 30, 1986*

**REZUMAT. — Principiul maximului pentru sisteme de ecuaţii diferenţiale cu argument modificat.** Rolul principiilor de maxim în teoria ecuaţiilor diferenţiale ordinare este bine cunoscut. În prezenta lucrare se stabilesc principii de maxim pentru sisteme de ecuaţii diferenţiale cu argument modificat. Folosind principiile de maxim obţinute se stabilesc teoreme de unicitate pentru anumite probleme la limite relative la ecuaţii diferenţiale cu argument modificat.

1. **Introduction.** Let us consider the following second order differential operator with deviating arguments

$$L(y)(x) := y''(x) + p(x)y'(x) + q(x)y(x) + \sum_{i=1}^{m} q_i(x)y(g_i(x)),$$

for all $x \in [a, b]$; where $p, q, q_i, g_i \in C[a, b], i = \overline{1, m}$. Let $a_1, b_1 \in \mathbf{R}$ be such that $a_1 \leqslant a, b \leqslant b_1, a_1 \leqslant g_i(x) \leqslant b_1$, for all $x \in [a, b]$. For example

$$a_1 = \min\left(a, \min_{1 \leqslant i \leqslant m} \left\{\begin{matrix}\min g_i(x) \\ x \in [a, b]\end{matrix}\right\}\right),$$

$$b_1 = \max\left(b, \max_{1 \leqslant i \leqslant m} \left\{\begin{matrix}\max g_i(x) \\ x \in [a, b]\end{matrix}\right\}\right),$$

We investigate the following differential inequalities with deviating arguments:

$$L(y) \geqslant 0 \quad (1); \quad L(y) > 0 \quad (2); \quad L(y) \leqslant 0 \quad (3);$$
$$L(y) < 0 \quad (4); \quad L(y) = 0 \quad (5)$$

By definition, a solution of an above differential inequality is a function $y \in C[a_1, b_1] \cap C^2[a, b]$ which satisfy that inequality.

The maximum principle for the solutions of the ordinary differential equations or of the ordinary differential inequalities has been studied by many authors, for example, Protter and Weinberger [7], Rus [8].

The object of this paper is to establish maximum principles for some types of differential inequalities with deviating arguments. To the author's knowledge, these maximum principles are new. For other types of ma-

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ximum principle for differential equations with deviating argument see Bellen and Zennaro [1] and Zennaro [10].

**2. Second order differential inequalities with deviating arguments.** We begin with

DEFINITION 1. A function $y \in C[a_1, b_1] \cap C^2[a, b]$ is said to satisfy the maximum principle if

$(\max_{x \in [a_1, b_1]} y(x) = M > 0$ and $y(x_0) = M)$ imply

$(x_0 \in [a_1, a] \cup [b, b_1])$

DEFINITION 2. A function $y \in C[a_1, b_1] \cap C^2[a, b]$ is said to satisfy the minimum principle if

$(\min_{x \in [a_1, b_1]} y(x) = m < 0$ and $y(x_0) = m)$ imply

$(x_0 \in [a_1, a] \cup [b, b_1])$

We have

THEOREM 1. *Let $y$ be a solution of* (1). *If $q_i(x) \geqslant 0$ and $q(x) + \sum_{i=1}^{m} q_i(x) < 0$ for all $x \in ]a, b[$* (6)
*then $y$ satisfies the maximum principle.*

*Proof.* Let $\max_{x \in [a_1, b_1]} y(x) = M > 0$, and $y(x_0) = M$. We suppose that $x_0 \in ]a, b[$. We shall show this leads to a contradiction. We obtain

$$L(y)(x)_0 = y''(x_0) + q(x_0)y(x_0) + \sum_{i=1}^{m} q_i(x_0)y(g_i(x_0)) \leqslant$$

$$\leqslant (q(x_0) + \sum_{i=1}^{m} q_i(x_0))M < 0.$$

THEOREM 2. *Let $y$ be a solution of* (3). *If $L$ satisfy* (6), *then $y$ satisfies the minimum principle.*

*Proof.* If $y$ is a solution of (3), then $-y$ is a solution of (1). The theorem follows from the Theorem 1.

THEOREM 3. *Let $y$ be a solution of* (5). *If $L$ satisfy* (6), *then $y$ satisfies the maximum principle and the minimum principle.*

*Proof.* $y$ is a solution of (1) and of (3). The theorem follows from the Theorem 1 and Theorem 2.

By a similar arguments we have

THEOREM 4. *Let $y$ be a solution of* (2). *If $q_i(x) \geqslant 0$ and $q(x) + \sum_{i=1}^{m} q_i(x) \leqslant 0$ for all $x \in ]a, b[$* (7) *then $y$ satisfies the maximum principle.*

THEOREM 5. *Let $y$ be a solution of* (4). *If $L$ satisfy* (7), *then $y$ satisfies the minimum principle.*

*Remark 1.* Let $y \in \mathbf{C}^2(\mathbf{R})$ be a solution of the following differential equation with deviating arguments

$$L(y)(x) := y''(x) + p(x)y'(x) + q(x)y(x) + \sum_{i=1}^{m} q_i(x)y(g_i(x)) = 0$$

for all $x \in \mathbf{R}$, where $p, q, g_i \in \mathbf{C}(\mathbf{R})$, and $q_i(x) \geqslant 0$, $q(x) + \sum_{i=1}^{m} q_i(x) < 0$,

$$\min (0, \underline{\lim}_{x\to+\infty} y(x), \underline{\lim}_{x\to-\infty} y(x)) \leqslant y(x) \leqslant \max (0, \overline{\lim}_{x\to+\infty} y(x), \overline{\lim}_{x\to-\infty} y(x))$$

*Remark 2.* Let $y \in \mathbf{C}[a_1, b_1] \cap \mathbf{C}^2[a, b]$ be a solution of the following differential equation with deviating arguments

$$y''(x) + p(x, y(h(x, y(x))))y'(x) + q(x)y(x) + \sum_{i=1}^{m} q_i(x)y(g_i(x, y(x))) = 0$$
$$, x \in [a, b],$$

where $a_1 \leqslant g_i(x, r) \leqslant b_1$ for all $x \in [a, b]$ and $r \in \mathbf{R}$. If $q_i(x) \geqslant 0$ and $q(x) + \sum_{i=1}^{m} q_i(x) < 0$ for all $x \in [a, b]$, then $y$ satisfies the maximum principle and the minimum principle.

*Remark 3.* Let $y \in \mathbf{C}[a_1, b_1] \cup \mathbf{C}^2[a, b]$ be a solution of the following inequality

$$y''(x) + f(x, y'(x), y(x), y(g_1(x)), \ldots, y(g_m(x))) \geqslant 0, \quad x \in [a, b],$$

where

$$a_1 \leqslant g_i(x) \leqslant b_1, \; g_i \in \mathbf{C}[a, b] \text{ and } f \in \mathbf{C}([a, b) \times \mathbf{R}^{m+2}).$$

If

(i) $t, s \in \mathbf{R}^{m+1}$, $t \leqslant s$ implies $f(x, 0, t) \leqslant f(x, 0, s)$, $x \in [a, b]$,

(ii) $f(x, 0, r, \ldots, r) < 0$, for all $r \in \mathbf{R}$, $r < 0$, then $y$ satisfies the maximum principle.

*Example 1.* (see [3]). Let $y \in \mathbf{C}[-2, 2] \cap \mathbf{C}^2[0, 2]$ be a solution of the following equation

$$y''(x) - y(x) + \frac{1}{3} y(2 \sin y(x)) = 0, \; x \in [0, 2].$$

Then $y$ satisfies the maximum principle.

*Example 2.* Let $y \in \mathbf{C}[a - r, b + r] \cap \mathbf{C}^2[a, b]$ be a solution of the following equation with retarded and advanced arguments

$$y''(x) + p(x)y'(x) + q(x)y(x) + q_1(x)y(x - r) + q_2(x)\, y(x + r) = 0,$$
$$x \in [a, b],$$

where

$p, q, q_1 \in C[a, b]$, $r > 0$. If $q_1(x) \geq 0$, $q_2(x) \geq 0$ and $q(x) + q_1(x) + q_2(x) < 0$ for $x \in ]a, b[$, then $y$ satisfies the maximum principle and the minimum principle.

*Example 3.* Let $y \in C[-\lambda a, \lambda a] \cap C^2[-a, a]$ be a solution of the following equation

$$y''(x) + p(x)y'(x) + q(x)y(x) + q_1(x)y(\lambda x) = 0, \; x \in [-a, a],$$

where $p, q, q_1 \in C[-a, a]$, $a > 0$, $\lambda > 1$. If $q_1(x) \geq 0$, and $q(x) + q_1(x) < 0$ for all $x \in ]-a, a[$, then $y$ satisfy the maximum principle and the minimum principle.

**3. Second order system of differential equations with deviating arguments.** We begin with

DEFINITION 3. *If* $y, z \in C([a, b], \mathbf{R}^n)$, $y = (y_1, \ldots, y_n)$, $z = (z_1, \ldots, z_n)$ then $y \leq z$ if and only if $y_k \leq z_k$, $k = \overline{1, n}$. If $y \in C([a, b], \mathbf{R}^n)$ and $M \in \mathbf{R}$, then $y \leq M$ if and only if $y_k \leq M$, $k = \overline{1, n}$.

DEFINITION 4. A function $y \in C([a_1, b_1], \mathbf{R}^n) \cap C^2([a, b], \mathbf{R}^n)(a_1 \leq a, b \leq b_1)$, satisfies the maximum principle if there exists a component $y_k$ of $y$ with the following properties

(i) $\max\limits_{x \in [a_1, b_1]} y_k(x) = M > 0,$

(ii) $y \leq M,$

(iii) $\{x \in [a_1, b_1] \mid y_k(x) = M\} \subset [a_1, a] \cup [b, b_1].$

DEFINITION 5. A function $y \in C([a_1, b_1], \mathbf{R}^n) \cap C^2([a, b], \mathbf{R}^n)$ satisfies the minimum principle if there exists a component $y_k$ of $y$ with the following properties

(i) $\min\limits_{x \in [a_1, b_1]} y_k(x) = m < 0$

(ii) $y \geq m$

(iii) $\{x \in [a_1, b_1] \mid y_k(x) = m\} \subset [a_1, a] \cup [b, b_1].$

Consider the second order system of differential equations with deviating arguments

$$L_k(y)(x) := y_k''(x) + p_k(x)y_k'(x) + q_k(x)y_k(x) + \\ + \sum_{i=1}^{m} \sum_{j=1}^{n} r_{k,j,i}(x)y_j(g_{k,j,i}(x)) = 0 \tag{8}$$

$$x \in [a, b]$$

$$k = \overline{1, n}$$

where $p_k$, $q_k$, $r_{k,j,i} \in \mathbf{C}[a, b]$, $k = \overline{1, n}$, $i = \overline{1, m}$, $j = \overline{1, n}$ and $a_1 \leqslant g_{k,j,i}(x) \leqslant b_1$, $a_1 \leqslant a$, $b \leqslant b_1$.

We have

THEOREM 7. *Let* $y \in \mathbf{C}([a_1, b_1], \mathbf{R}^n) \cap \mathbf{C}^2([a, b], \mathbf{R}^n)$, $y$ is not $\leqslant 0$ and $y$ is not $\geqslant 0$, *be a solution of* (8). *If* $r_{k,j,i}(x) \geqslant 0$, $x \in ]a, b[$, $k = 1, n$, $i = 1, \overline{m}, j = 1, \overline{n}$, *and* $q_k(x) + \sum_{i=1}^{m} \sum_{j=1}^{n} r_{k,j,i}(x) < 0$ $x \in ]a, b[$ *and* $k = \overline{1, n}$, *then* $y$ *satisfies the maximum principle and the minimum principle.*

*Proof.* We begin with the maximum property for $y$. First we prove that

$$(\max_{x \in [a_1, b_1]} y_k(x) = M > 0,\ y \leqslant M,\ y_k(x_0) = M) \text{ implies } (x_0 \in [a_1, a] \cup [b, b_1]) \tag{9}$$

If $x_0 \in ]a, b[$ then

$$L_k(y)(x_0) = y_k''(x_0) + q_k(x_0)y_k(x_0) + \sum_{i=1}^{m} \sum_{j=1}^{n} r_{k,j,i}(x_0)y_j(g_{k,j,i}(x_0)) \leqslant$$

$$\leqslant (q_k(x_0) + \sum_{i=1}^{m} \sum_{j=1}^{n} r_{k,j,i}(x_0))\, M < 0$$

Now, let $y$ be a solution of (8) such that $y \leqslant 0$. This means that there exist $i \in \{1, \ldots, n\}$ and $x \in [a_1, b_1]$ such that $y_i(x) > 0$. We denote $\max_{x \in [a_1, b_1]} y_i(x) = M_i$. Let $k \in \{1, \ldots, n\}$ be such that $\max(M_1, \ldots, M_n) = M_k$. It is clear the $M_k > 0$ and $y \leqslant M_k$. For the component $y_k$ we have (9). From (9), it follows the maximum property for $y$.

If $y$ is a solution of (8) then $-y$ is a solution of (8). The minimum principle follows from the maximum principle.

*Remark 4.* Let $y$ be a solution of the system (8). In the conditions of the Theorem 7 for $q_k$, $r_{k,j,i}$, we have $\min(0, \min_{x \in \Omega} y_1(x), \ldots, \min_{x \in \Omega} y_n(x)) \leqslant y(x) \leqslant \max(0, \max_{x \in \Omega} y_1(x), \ldots, \max_{x \in \Omega} y_1(x))$ where $\Omega = [a_1, a] \cup [b, b_1]$.

*Remark 5.* Let $y$ be a solution of the following system of differential inequalities

$$L_k(y) \geqslant 0,\ k = \overline{1, n}.$$

If $p_k$, $q_k$, $r_{k,j,i}$ are as in the Theorem 7, then $y$ satisfies the maximum principle.

*Remark 6.* Let $y$ be a solution of the following system of differential inequalities

$$L_k(y) \leqslant 0,\ k = \overline{1, n}.$$

If $p_k$, $q_k$, $r_{k,j,i}$ are as in the Theorem 7, then $y$ satisfies the minimum principle.

**4. Boundary value problems.** We consider the following boundary value problem

$$y''(x) + p(x)y'(x) + q(x)y(x) + \sum_{i=1}^{n} q_i(x)y(g_i(x)) = f(x), \quad x \in [a, b] \tag{10}$$

$$\begin{aligned} y(x) &= \varphi(x), \ x \in [a_1, a], \\ y(x) &= \Psi(x), \ x \in [b, b_1], \end{aligned} \tag{11}$$

where $p, q, q_i, f, g_i \in C[a, b]$, $\varphi \in C[a_1, a]$, $\Psi \in C(b, b_1]$.

We have

THEOREM 8. *If $q$, $q_i$ are as in the Theorem 3, then the problem* (10)+ +(11) *has at most one solution.*

*Proof.* The problem (10)+(11) has at most one solution if and only if

$$\left\{ \begin{array}{c} L(y) = y''(x) + p(x)y'(x) + q(x)y(x) + \sum_{i=1}^{m} q_i(x)y(g_i(x)) = 0 \\ y|_{[a_1, a]} = 0, \ y|_{[b, b_1]} = 0 \end{array} \right\} \Rightarrow (y = 0) \tag{12}$$

Now we prove (12). If $y \leqslant 0$, then $y$ has a positive maximum $M$ and there exists $x_0 \in [a, a_1] \cup [b, b_1]$ such that $y(x_0) = M$. But $y|_{[a, a_1]} = 0$ and $y|_{[b, b_1]} = 0$. Thus we have $y \leqslant 0$. By a similar argument we prove that $y \geqslant 0$. Hence $y = 0$.

THEOREM 9. *If $q_k$, $r_{k,j,i}$ are as in the Theorem 7 then the problem*

$$L_k(y) = f, \ k = \overline{1, n} \tag{13}$$

$$y|_{[a_1, a]} = \varphi, \ y|_{[b, b_1]} = \Psi \tag{14}$$

has at most one solution.

*Proof.* The problem (13)+(14) has at most one solution if and only if

$$\left\{ \begin{array}{c} L_k(y) = 0, \ k = \overline{1, n}, \\ y|_{[a_1, a]} = 0, \ y|_{[b, b_1]} = 0 \end{array} \right\} \Rightarrow (y = 0) \tag{15}$$

Now we prove (15). If $y \leqslant 0$, then by the Theorem 7 there exists a component $y_k$ of $y$ such that $\max_{x \in [a_1, b_1]} y_k = M > 0$, and $\max_{[a_1, a] \cup [b_1, b]} y_k = M$. But $y(x) = 0$ for all $x \in [a_1, a] \cup [b, b_1]$. Thus we have $y \leqslant 0$. By a similar argument we prove that $y \geqslant 0$. Hence $y=0$.

*Example 4.* In the conditions of the Example 2 we have that the probleme

$$y''(x) + p(x)y'(x) + q(x)y(x) + q_1(x)y(x-r) + q_2(x)y(x+r) = f(x)$$
$$y(x) = \varphi(x), \ x \in [a-r, a]$$
$$y(x) = \varphi(x), \ x \in [b, b+r]$$

has at most one solution.

*Example 5.* In the condition of the Example 3 the probleme

$$y''(x) + p(x)y'(x) + q(x)y(x) + q_1(x)y(\lambda x) = f(x), \ x \in [-a, a]$$
$$y|_{[-\lambda a, -a]} = \varphi. \ y|_{[a, \lambda a]} = \Psi$$

has at most one solution.

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STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXII, 1, 1987

# RELATIONS AMONG SOME CLASSES OF REAL FUNCTIONS ON A COMPACT INTERVAL

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*Received: January 14, 1987*

„Ordo et connexio idearum idem est ac ordo et connexio rerum"
(Spinoza: Ethica, pars II, prop. VII)

**REZUMAT. — Relaţii între unele clase de funcţii reale pe un interval compact.** Se propune o clasificare a 15 mulţimi de funcţii reale, definite pe un acelaşi interval compact, care se întîlnesc frecvent în analiza matematică. Relaţiile între aceste mulţimi de funcţii se reprezintă în plan atît ca un graf orientat de discuri (diagrama Hasse), cît şi sub forma unei configuraţii exhaustive şi mai sugestive de dreptunghiuri (diagrama Venn). Se construiesc 44 de exemple de funcţii prin care se arată că fiecare porţiune a diagramei Venn reprezintă o clasă nevidă de funcţii.

This paper proposes a classification of fifteen sets of real-valued functions defined on the same compact interval, which often occur in real analysis. The connections among these function sets are represented both by an oriented graph of disks (Hasse diagram), and by an exhaustive and more suggestive configuration of rectangles (Venn diagram). Forty-four appropriate examples are constructed to show that each portion of the Venn diagram represents a nonempty class of functions.

1. **The Function Classes.** The ultimate goal of any branch of mathematics is to describe and classify the objects it studies. A desideratum of this program is to determine when two objects are essentially the same, and to establish what are the interrelations between two or more distinct objects. The objects considered in this paper are fifteen sets of real-valued functions of a real variable which frequently appear in real analysis. We list below these sets and label them with suitable symbols:

**B** = the bounded functions,
**C** = the continuous functions,
**Da** = the Darboux functions,
**D** = the differentiable functions,
**Bd** = the differentiable functions with a bounded derivative,
**L** = the Lipschitz functions,
**M** = the monotonic functions,
**Sp** = the functions admitting a strict primitive,
**R** = the Riemann-integrable functions,
**Rg** = the regulated functions,
**P** = the functions admitting a primitive,

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**Bv** = the functions of bounded variation,
**Ac** = the absolutely continuous functions,
**I** = the injective functions,
**Sm** = the strictly monotonic functions.

Recall that a real-valued function $f$ defined on an interval $J$ of $\mathbb{R}$ is said to be *Darboux function* if whenever $a$ and $b$ are in $J$, $a < b$, and $y$ is any number between $f(a)$ and $f(b)$ there is a point $c$ in $[a, b]$ such that $f(c) = y$. A function $f: J \to \mathbb{R}$ is said to admit a *primitive* (or a *strict primitive*) if there exist a continuous function $F: J \to \mathbb{R}$ and a countable subset $A$ of $J$ such that $F$ be differentiable on $J \setminus A$ (on $J$) and $F'(x) = f(x)$ for all $x$ in $J \setminus A$ (for all $x$ in $J$, respectively). We say that $f: J \to \mathbb{R}$ is a *regulated function* if it has finite one-sided limits at every point of $J$.

Our attention will be focused in the following on the important case of real-valued functions defined on the same nondegenerate compact interval in $\mathbb{R}$. S. K. Berberian [1] presented a Hasse diagram to point out the relationships among the function classes **C**, **Bv**, **Rg**, **R**, **P** and $\mathbf{L}^1$ (= the Lebesgue-integrable functions). A Venn diagram for the first nine and the first eleven function sets in the above list has been proposed in [23] and [24], respectively. Recently, H. Diamond and G. Gellès [9] produced a Venn diagram for some classes of subsets of the real line. In this paper we exhibit both a Hasse diagram and a Venn diagram to disply all inclusion relations existing among the above fifteen sets and their complements. Forty-four examples are constructed to prove that each portion of the Venn diagram represents a nonempty class of functions.

2. **Diagrams of Function Classes.** Now we establish the hierarchy of function classes listed in Section 1. From now on we shall deal solely with real-valued functions defined on the same compact interval $[a, b]$ with $a < b$. In this case the following inclusions are well-known: $\mathbf{M} \subset \mathbf{B}$, $\mathbf{L} \subset \mathbf{B}$, $\mathbf{C} \subset \mathbf{B}$, $\mathbf{C} \subset \mathbf{Da}$, $\mathbf{C} \subset \mathbf{Sp}$, $\mathbf{D} \subset \mathbf{C}$, $\mathbf{Sp} \subset \mathbf{Da}$, $\mathbf{M} \subset \mathbf{R}$, $\mathbf{C} \subset \mathbf{R}$, $\mathbf{C} \cup \mathbf{M} \subset \mathbf{Rg}$, $\mathbf{Bv} \subset \mathbf{Rg}$, $\mathbf{L} \subset \mathbf{Ac}$, $\mathbf{Rg} \subset \mathbf{R}$ ([8], pp. 90–92), $\mathbf{Rg} \subset \mathbf{P}$ ([2], pp. 60–62), and so on. In order to systematize these inclusions we represent each function class by a plane disk labelled with the associate symbol. If a function class **X** is included in another function class **Y**, the corresponding disks are joined either by a line segment oriented from the disk representing **X** to the disk representing **Y** or by more such segments when other classes interpose between **X** and **Y**. The resulting oriented graph is the Hasse diagram in Figure 1.

To emphasize all interrelations existing among our classes we identify each function class by a rectangle (except the classes **P**, **I** and **Sm** rendered by unions of two adjacent rectangles each), whose lower side lies on the same line $AA'$, and upper vertices are adequately marked. Each rectangle is labelled by the symbol of the corresponding class, so that relations among the function classes translate into relations among the corresponding rectangles. The supplementary relations

2.1. $\mathbf{L} \cap \mathbf{D} = \mathbf{Bd}$, $\mathbf{Rg} \cap \mathbf{Da} = \mathbf{C}$, $\mathbf{I} \cap \mathbf{M} = \mathbf{Sm}$,

2.2. $\mathbf{I} \cap \mathbf{Da} = \mathbf{C} \cap \mathbf{Sm}$ (see [17]),

2.3. $\mathbf{Bv} \cap \mathbf{D} \subset \mathbf{Ac}$ (see [25]), together with a careful analysis of the Hasse diagram lead to the conclusion that from the total of $2^{15} = 32{,}768$ intersections, obtained with the above rectangles and their complements, only 44 may be nonempty. This is also confirmed by two computer programs written in FORTRAN on the FELIX C 256 at the University „Babeş-Bolyai" in Cluj-Napoca.

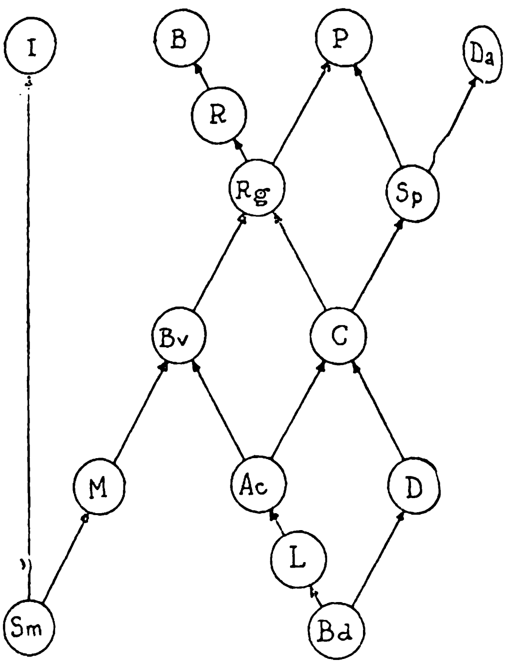



Figure 1

The intersection $\mathbf{X} \cap \mathbf{Y}$ of two classes $\mathbf{X}$ and $\mathbf{Y}$ will be denoted by $\mathbf{XY}$ (or $\mathbf{X} \cdot \mathbf{Y}$) and, given a class $\mathbf{X}$, we write $\bar{\mathbf{X}}$ for the complement $\mathbf{E} \setminus \mathbf{X}$ of $\mathbf{X}$, where $\mathbf{E}$ is the set of all real-valued functions defined on $[a, b]$. For instance, $\mathbf{I} \cap \mathbf{Bv} \cap \cap (\mathbf{E} \setminus \mathbf{Da}) \cap (\mathbf{E} \setminus \mathbf{M})$ will be denoted by $\mathbf{IBv}\overline{\mathbf{Da}}\overline{\mathbf{M}}$. Accordingly, the remaining 44 intersections are symbolized in this way and placed in the Venn diagram as shown in Figure 2. The contour of the unions of adjacent rectangles representing the classes **P** and **I** is drawn by a heavy line and a dotted line, respectively.

**3. Preliminary Examples.** Our discussion in Section 4 will make extensive use of the examples below.

3.1. Given a real number $k$, we define the function $s_k$ on $[0, 1]$ by $s_k(x) = \sin \frac{1}{x}$ for $x \neq 0$, and $s_k(0) = k$. We have:

3.1.a. $s_k \in \mathbf{RP}\overline{\mathbf{Rg}}$ for any $k$;

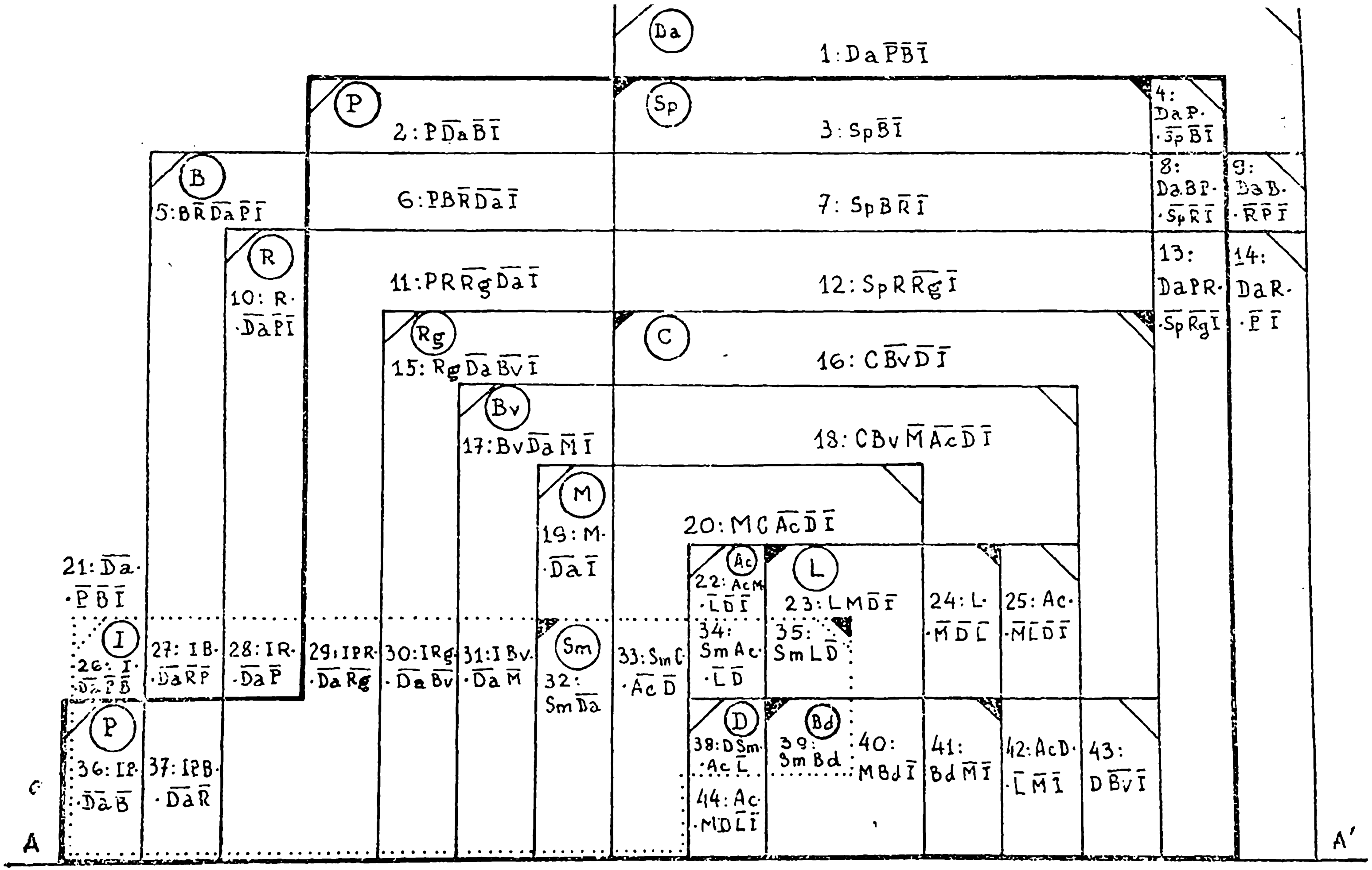
3.1.b. $s_k \in \mathbf{Da}$ if and only if $|k| \leqslant 1$ ([8], p. 108; see also [23]);

3.1.c. $s_k \in \mathbf{Sp}$ if and only if $k = 0$ (see [23]).

3.2. Let $C_0$ be the Cantor ternary set of zero measure ([5], Ammerkung 11; see also [14], pp. 85–86).

3.2.a. Define the function $f$ on $[0, 1]$ by $f(x) = 1$ for $x \in C_0$, and $f(x) = 0$ for $x \notin C_0$. Clearly, $f \in \mathbf{R}$. We assert that $f \in \bar{\mathbf{P}}$. If not, there exist a continuous function $F$ on $[0, 1]$ and a countable subset $A$ of $[0, 1]$ such that $F'(x) = fx)$ for all $x$ in $[0, 1] \setminus A$. Then the Fundamental Theorem of Calculus in the form in [16], p. 299, yields

$$F(x) = F(0) + \int_0^x f(t)\, dt = F(0) \text{ for all } x \text{ in } [0, 1],$$



Da
1: Da P B I
P
2: P Da B I
Sp
3: Sp B I
4: Da P· ·Sp B I
B
5: B R Da P I
6: P B R Da I
7: Sp B R I
8: Da B P· ·Sp R I
9: Da B· ·R P I
R
10: R· ·Da P I
11: P R Rg Da I
12: Sp R Rg I
13: Da P R· ·Sp Rg I
14: Da R· ·P I
Rg
15: Rg Da Bv I
C
16: C Bv D I
Bv
17: Bv Da M I
18: C Bv M Ac D I
M
19: M· ·Da I
20: M C Ac D I
21: Da· ·P B I
Ac
22: Ac M· ·L D I
L
23: L M D I
24: L· ·M D L
25: Ac· ·M L D I
I
26: I· ·Da P B
27: I B· ·Da R P
28: I R· ·Da P
29: I P R· ·Da Rg
30: I Rg· ·Da Bv
31: I Bv· ·Da M
Sm
32: Sm Da
33: Sm C· ·Ac D
34: Sm Ac· ·L D
35: Sm L D
P
36: I P· ·Da B
37: I P B· ·Da R
D
38: D Sm· ·Ac L
Bd
39: Sm Bd
40: M Bd I
41: Bd M I
42: Ac D· ·L M I
43: D Bv I
44: Ac· ·M D L I
A
A′


**Fig. 2**

so that $f(x) = F'(x) = 0$ for all $x$ in $[0, 1]\setminus A$, hence $C_0 \subset A$, a contradiction.

3.2.b. The Cantor function $\psi$ on $[0, 1]$ ([6]) satisfies $\psi \in \overline{\mathbf{CMAc}}$ (see [14], pp. 96–98; the relation $\psi \subset \overline{\mathbf{Ac}}$ was proved by G. Vitali [29]).

3.3. Let $C_p$ be the Cantor ternary set of measure $p \in [0, 1[$ on the interval $[0, 1]$, and let $\{]a_n, b_n[ : n \in \mathbb{N}\}$ be the infinite countable family of its complementary open intervals (see [14], pp. 88–89). Consider the Volterra function $V_p$ on $[0, 1]$ defined by $V_p(x) = 0$ for $x \in C_p$, and

$$V_p(x) = (x - a_n)^2(x - b_n)^2 \cdot \sin [(b_n - a_n)(x - a_n)(x - b_n)]^{-1}$$

for $x \notin C_p$, where $n$ is the single natural number with $a_n < x < b_n$ ([30]). We have $V_p \in \mathbf{Bd}$, $V_p'(x) = 0$ for all $x$ in $C_p$, and $V_p'$ is discontinuous at every point of $C_p$ (see [14], pp. 107–108; [3], pp. 45–46, and [7]). Therefore, by the Lebesgue characterization of Riemann integrability [19], p. 29, we obtain

3.3.a. $V_p' \in \mathbf{BSp\bar{R}}$ when $0 < p < 1$.

The function $f_p$, defined on a nondegenerate subinterval $J$ of $[0, 1]$ by $f_p(x) = V_p'(x)$ for $x \notin C_p$, and $f_p(x) = \frac{1}{2}$ for $x \in C_p$, satisfies:

3.3.b. $f_p \in \mathbf{R}$ on $J$ if and only if $p = 0$;

3.3.c. $f_p \in \mathbf{Da}$ on $J$ for any $p$ in $[0, 1[$.

To prove 3.3.c, let $a$ and $b$ be two points in $J$, $a < b$, and let $y$ be any number between $f_p(a)$ and $f_p(b)$. First, suppose that $a \in C_p$ and $b \notin C_p$, and admit that $f_p(a) < y < f_p(b)$. Then there exists a unique $n \in \mathbb{N}$ with $a \leqslant a_n < b < b_n$ and, since $\liminf_{x \to a_n} V_p'(x) = -1$, we can choose a point $d$ in $]a_n, b[$ such that $V_p'(d) \leqslant \frac{1}{2}$. Taking into account that $V_p'$ belongs to $\mathbf{Da}$ on $]d, b[$, from $V_p'(d) \leqslant \frac{1}{2} = f_p(a) < y < f_p(b) = V_p'(b)$ it follows that there is a point $c$ in $]d, b[ \subset ]a_n, b_n[$ such that $f_p(c) = V_p'(c) = y$. When $f_p(b) < y < f_p(a)$ a similar argument applies with $\limsup_{x \to a_n} V_p'(x) = 1$ instead of $\liminf_{x \to a_n} V_p'(x) = -1$.

The case $a \notin C_p$ and $b \in C_p$ can be treated in a like manner, and the analysis of the case $a \notin C_p$ and $b \notin C_p$ uses the same argument.

3.4. Consider the sequences

$x_n = 1 - 2^{-n}$, $y_n = x_n + 4 \cdot 8^{-n-1}$ and $z_n = \frac{1}{2}(x_n + y_n)$ for $n \geqslant 0$, and the functions $g$ and $G$ defined on $[0, 1]$ by $g(1) = 0$ and

$$g(x) = \begin{cases} 2^n(z_n - x_n)^{-1}(x - x_n) & \text{for } x_n \leqslant x \leqslant z_n, \\ 2^n(z_n - y_n)^{-1}(x - y_n) & \text{for } z_n < x \leqslant y_n, \\ 0 & \text{for } y_n < x < x_{n+1}, \end{cases}$$

and $G(x) = \int_0^x g(t)\, dt$,

where $n \geqslant 0$ is the unique integer with $x_n \leqslant x < x_{n+1}$ when $x$ is in $[0, 1[$. We have $g(z_n) = 2^n \to \infty$ as $n \to \infty$, and

$$G(1) = \sum_{n=0}^{\infty} \int_{x_n}^{x_{n+1}} g(x)dx = \frac{1}{2} \cdot \sum_{n=0}^{\infty} 4 \cdot 8^{-n-1} \cdot 2^n = \frac{1}{3}.$$

The function $G$ is differentiable on $[0, 1]$ and $G' = g$ since, if $x_n \leqslant x < x_{n+1}$, then

$$0 \leqslant \frac{G(1) - G(x)}{1 - x} \leqslant 2^{n+1} \cdot \sum_{k=n}^{\infty} 4^{-k-1} = \frac{4}{3} \cdot 2^{-n-1} \to 0 \text{ as } x \to 1.$$

Consequently, by 2.3, we have $G \in$ **MDAcB̄d**. The function $G$ appeared in [22], pp. 91—92. Other examples of functions in **MDAcB̄d** can be found in [28] and [26].

4. **Main Examples.** In this section we shall construct some appropriate examples of functions for proving that each of the 44 intersections shown in Figure 2 is really nonempty. All these functions are real-valued and defined on the compact unit interval [0, 1]. The order number of each example below is followed in Figure 2 by the intersection containing it.

4.1. The restriction $f$ to $[0, 1]$ of an unbounded and nonmeasurable solution $g$ of the Cauchy functional equation

$$g(x + y) = g(x) + g(y) \text{ for all } x \text{ and } y \text{ in } \mathbb{R},$$

belongs to **DaP̄B̄Ī** (see [15]). For, if $f$ had a primitive $F$ in the sense of Section 1, one would have $F'(x) = f(x)$ for all but countably many values of $x$ in $[0, 1]$. Then a well-known theorem of Luzin ([21], § 14, Theorem 1; see also [3], pp. 113—114) implies the measurability of $f$, a contradiction.

4.2. $f \in$ **PD̄aB̄Ī**, where $f(x) = \left(x^2 \cdot \cos \frac{1}{x^2}\right)'$ for $x \in ]0, 1[$, and $f(0) = f(1) = 0$.

4.3. $G' \in$ **SpB̄Ī**, where $G(x) = x^2 \cdot \cos \frac{1}{x^2}$ for $x \neq 0$, and $G(0) = 0$

4.4. $f = s_1 + G' \in$ **DaPS̄pB̄Ī**, where $s_1$ is the function defined in 3.1 and $G'$ is the derivative of $G$ in 4.3. The relation which needs a proof is $f \in$ **Da**. Let $a$ and $b$ be two points in $[0, 1]$, $a < b$, and let $y$ be any number between $f(a)$ and $f(b)$. We may assume that $a = 0$ and $f(0) < f(b)$, so that $1 < y < f(b)$. Since $f$ is unbounded from below on the open interval $]0, b[$, there is a positive number $d < b$ with $f(d) < y$. Now, the continuous function $f$ on $]d, b[$ is a Darboux function on the last interval, hence one can find a point $c$ in $]d, b[$ such that $f(c) = y$.

4.5. $\chi + f \in$ **BR̄D̄aP̄Ī**, where $\chi$ is the Dirichlet function ([10], p. 169) and $f$ is the function in 3.2.a. (Remark that the classes **B**, **R** and **P** are closed under addition of functions.)

4.6. The function $\chi$ in 4.5 belongs to **PBR̄D̄aĪ**.

4.7. $V'_{1/2}$ belongs to **SpBR̄Ī** as it follows from 3.3.a.

4.8. Let $c \in [0, 1]$ be a point of continuity of the function $f = V'_{1/2}$ defined in 3.3, and denote by $g$ the function given by $g(x) = \sin\frac{1}{x-c}$ for $x$ in $[0, 1] \setminus \{c\}$, and $g(c) = 1$. Since $f$ and $g$ are Darboux functions without common points of discontinuity, we have $f + g \in$ **Da** (see [3], p. 15) so that, by 2.2 and 3.3.a, $f + g \in$ **DaBPS̄pR̄Ī**.

4.9. The function $f$, defined on $[0, 1]$ by $f(x) = V'_0(x)$ for $x$ in $\left[0, \frac{1}{3}\right] \setminus C_0$, $f(x) = V'_{2/3}(x)$ for $x$ in $\left]\frac{1}{3}, 1\right] \setminus C_{2/3}$, and $f(x) = \frac{1}{2}$ otherwise, belongs to **P̄**. If not, there exist a continuous function $F$ on $[0, 1]$ and a countable subset $A$ of $[0, 1]$ such that $F'(x) = f(x)$ for all $x$ in $[0, 1] \setminus A$. Then

$$F(x) = F(0) + \int_0^x f(t)dt \text{ for all } x \text{ in } [0, 1],$$

hence

$$F(x) = F(0) + \int_0^x V'_0(t)dt = F(0) + V_0(x) - V_0(0)$$

$$\text{for all } x \text{ in } \left[0, \frac{1}{3}\right].$$

Now, choosing an $x_0$ in $\left(C_0 \cap \left[0, \frac{1}{3}\right]\right) \setminus A$, we arrive at the contradiction $\frac{1}{2} = f(x_0) = F'(x_0) = V'_0(x_0) = 0$.

We shall prove that $f \in$ **Da**. Let $a$ and $b$ be two points in $[0, 1]$, $a < b$, and let $y$ be any number between $f(a)$ and $f(b)$. In virtue of 3.3.c, we may admit that $a < \frac{1}{3} < b$. Suppose $f(a) < y < f(b)$. When $y \leqslant \frac{1}{2}$, from $f(a) < y \leqslant \frac{1}{2} = f\left(\frac{1}{3}\right)$ and 3.3.c we conclude that there exists a point $c$ in $\left]a, \frac{1}{3}\right] \subset ]a, b[$ such that $f(c) = y$. Similarly, when $y > \frac{1}{2}$ there exists a point $c$ in $\left]\frac{1}{3}, b\right[ \subset ]a, b[$ such that $f(c) = y$.

Consequently, by 3.3.b, we must have $f \in$ **DaBR̄P̄Ī**.

4.10. The function $f$ in 3.2.a belongs to **RD̄aP̄Ī**.

4.11. $s_1 \in$ **PRR̄gD̄aĪ** as it follows from 3.1.a and 3.1.b.

4.12. $s_0 \in$ **SpRR̄gI** as it follows from 3.1.a and 3.1.c.

4.13. $s_2 \in$ **DaPRS̄pR̄gĪ** as 3.1 shows.

4.14. The function $f_0$ in 3.3 with $J = [0, 1]$ belongs to $\mathbf{Da R \overline{P} \overline{I}}$ as the argument in the first part of 4.9 shows.

4.15. We have $f \in \mathbf{Rg \overline{Da} \overline{Bv} \overline{I}}$, where $f$ is the function defined by $f(x) = \frac{1}{n}$ if $x = \frac{m}{n}$ is a positive rational number with $m$ and $n$ mutually prime, and $f(x) = 0$ otherwise (see [19], p. 30).

4.16. We have $f \in \mathbf{C Bv \overline{D} \overline{I}}$, where $f(x) = x \cdot \cos \frac{1}{x}$ for $x \neq 0$, and $f(0) = 0$. K. A. Bush [4] constructed a nonconstant continuous function $g$ on $[0, 1]$ which is „very noninjective" in the sense that each neighborhood of any point $x$ in $[0, 1]$ contains a point $y \neq x$ with $g(x) = g(y)$.

4.17. $f \in \mathbf{Bv \overline{Da} \overline{M} \overline{I}}$, where $f(x) = x$ for $0 \leqslant x \leqslant \frac{1}{2}$, and $f(x) = x - \frac{1}{2}$ for $\frac{1}{2} < x \leqslant 1$.

4.18. We have $\psi - i \in \mathbf{C Bv \overline{M} \overline{Ac} \overline{D} \overline{I}}$, where $\psi$ is the function in 3.2.b and $i(x) = x$ for $x \in [0, 1]$. J. Foran [12] constructed a function $f \in \mathbf{CBv}$ which is „very nonmonotonic" in the sense that $f$ is not monotonic on any measurable subset of positive measure of the interval $[0, 1]$.

4.19. $f \in \mathbf{M \overline{Da} \overline{I}}$, where $f(x) = 0$ for $0 \leqslant x \leqslant \frac{1}{2}$, and $f(x) = 1$ for $\frac{1}{2} < x \leqslant 1$.

4.20. The Cantor function $\psi$ in 3.2.b belongs to $\mathbf{MC \overline{Ac} \overline{D} \overline{I}}$. F. Ferro [11] exhibited a function $f \in \mathbf{MC\overline{Ac}}$ with $f'(x) = 0$ a.e. on $[0, 1]$, which is Hölderian with each exponent $p$ in $[0, 1[$, i. e., there is a number $H_p \geqslant 0$ such that $|f(x) - f(y)| \leqslant H_p|x - y|^p$ for all $x$ and $y$ in $[0, 1]$. On the other hand, Z. J. Lu and C. F. Wang [20] constructed a continuous function which is not Hölderian for any exponent in $[0, 1[$.

4.21. $f \in \mathbf{\overline{Da} \overline{P} \overline{B} \overline{I}}$, where $f(x) = \frac{1}{x}$ for $x$ in $\left]0, \frac{1}{2}\right[$, $f(x) = -2x + 3$ for $x$ in $\left[\frac{1}{2}, 1\right[ \setminus C_0$, $f(x) = 2x - 1$ for $x$ in $\left]\frac{1}{2}, 1\right] \cap C_0$, and $f(0) = 2$. Here $C_0$ is the set in 3.2.

4.22. $f \in \mathbf{Ac M \overline{L} \overline{D} \overline{I}}$, where $f(x) = \sqrt{x}$ for $x$ in $\left[0, \frac{1}{2}\right]$, and $f(x) = \frac{1}{2}\sqrt{2}$ for $x$ in $\left]\frac{1}{2}, 1\right]$.

4.23. $f \in \mathbf{L M \overline{D} \overline{I}}$, where $f(x) = 0$ for $x$ in $\left[0, \frac{1}{2}\right]$, and $f(x) = x - \frac{1}{2}$ for $x$ in $\left]\frac{1}{2}, 1\right]$. K. M. Garg [13] proved that the set $\mathbf{Ac}$ coincides with the closure of $\mathbf{L}$ in the Banach space $\mathbf{Bv}$, endowed with the topology of the norm $||f|| = |f(0)| + \bigvee_0^1 (f)$, $f \in \mathbf{Bv}$.

4.24. $f \in \mathbf{L\overline{M}\overline{D}\overline{I}}$, where $f(x) = \left|x - \frac{1}{2}\right|$.

4.25. $f \in \mathbf{Ac\overline{M}\overline{L}\overline{D}\overline{I}}$, where $f(x) = \sqrt{x - x^2}$.

4.26. $g \in \mathbf{I\overline{D}a\overline{P}\overline{B}}$, where $g(x) = f(x)$ for $x \neq 0$, and $g(0) = 0$, and $f$ is the function in 4.21.

4.27. We have $f \in \mathbf{IB\overline{D}a\overline{R}\overline{P}}$, where $f(x) = x$ for $x$ in $\left(\left[0, \frac{1}{2}\right[ \setminus C_0\right) \cup \cup \left(\left[\frac{1}{2}, 1\right] \setminus \mathbf{Q}\right)$, and $f(x) = -x$ otherwise. Here $C_0$ is the set in 3.2 and $\mathbf{Q}$ is the set of rational numbers.

4.28. We have $f \in \mathbf{IR\overline{D}a\overline{P}}$, where $f(x) = x$ for $x$ in $C_0$, and $f(x) = -x$ otherwise.

4.29. $f \in \mathbf{IP\overline{R}\overline{D}a\overline{R}g}$, where $f(x) = x + 1$ for $x \in \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$, and $f(x) = x$ otherwise.

4.30. $f \in \mathbf{IRg\overline{D}a\overline{B}v}$, where $f(0) = 2$, $f(x) = x$ for $x \in \left\{\frac{1}{4n-1} : n \in \mathbb{N}\right\}$, $f(x) = -x$ for $x \in \left\{\frac{1}{4n-3} : n \in \mathbb{N}\right\}$, and $f(x) = x \cdot \operatorname{sign}\left(\cos \frac{\pi}{2x}\right)$ otherwise.

4.31. $f \in \mathbf{IBv\overline{D}a\overline{M}}$, where $f(x) = x$ for $x$ in $\left[0, \frac{1}{2}\right]$, and $f(x) = 2 - x$ otherwise.

4.32. $f \in \mathbf{Sm\overline{D}\overline{a}}$, where $f(x) = x$ for $x$ in $\left[0, \frac{1}{2}\right]$, and $f(x) = x + 1$ otherwise.

4.33. We have $\psi + i \in \mathbf{SmC\overline{A}c\overline{D}}$, where $\psi$ and $i$ are the functions in 4.18. A more dramatic example of a function $f$ in $\mathbf{SmC\overline{A}c\overline{D}}$ has been constructed by R. Salem [27] (see also [16], pp. 278–282): the derivative $f'$ exists a. e. and $f'(x) = 0$ a. e. on $[0, 1]$.

4.34. $f \in \mathbf{SmAcL\overline{D}}$, where $f(x) = \sqrt{x}$.

4.35. $f \in \mathbf{SmL\overline{D}}$, where $f(x) = \frac{x}{2}$ for $x$ in $\left[0, \frac{1}{2}\right]$, and $f(x) = x - \frac{1}{4}$ otherwise.

4.36. $f \in \mathbf{IP\overline{D}a\overline{B}}$, where $f(0) = 0$, $f(x) = x$ for $x$ irrational, and $f(x) = \frac{1}{x}$ otherwise.

4.37. $f \in \mathbf{IPB\overline{D}a\overline{R}}$, where $f(x) = x$ if $x$ is rational, and $f(x) = 1 - x$ if $x$ is irrational.

4.38. $G + i \in \mathbf{DSmAc\overline{L}}$, where $G$ and $i$ are the functions in 3.4 and 4.18, respectively.

4.39. The function $i$ in 4.18 belongs to $\mathbf{SmBd}$.

4.40. The function $f$, defined by $f(x) = 0$ for $x$ in $\left[0, \frac{1}{2}\right]$, and $f(x) = \left(x - \frac{1}{2}\right)^2$ otherwise, belongs to $\mathbf{MBd\overline{I}}$.

4.41. $f \in \mathbf{Bd\overline{M}\overline{I}}$, where $f(x) = \left(x - \frac{1}{2}\right)^2$. Nowhere monotonic functions in **Bd** are presented in [3], pp. 32–34, and [18].

4.42. The function $f$, defined by $f(x) = x^{3/2} \cdot \cos \frac{2\pi}{x}$ for $x \neq 0$, and $f(0) = 0$, belongs to **Bv** as it immediately follows from the convergence of the series $\sum_{n=1}^{\infty} n^{-3/2}$. Hence, by 2·3, we obtain $f \in \mathbf{AcD\overline{L}\overline{M}\overline{I}}$.

4.43. The function $f$, given by $f(x) = x^2 \cdot \cos \frac{\pi}{x^2}$ for $x \neq 0$, and $f(0)=0$, belongs to $\mathbf{D\overline{Bv}\overline{I}}$. Indeed, for $a_k = 2^{1/2}(4k+1)^{-1/2}$ and $b_k = (2k)^{-1/2}$, $k \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| = \sum_{k=1}^{n} \frac{1}{2k} \to \infty \text{ as } n \to \infty.$$

4.44. The function $G$ in 3.4 belongs to $\mathbf{AcMD\overline{L}\overline{I}}$.

**Acknowledgment.** I would like to express my sincere thanks to Professors D. Duca and C. Tarţia for their computer assistance.

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STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXII, 1, 1987

# SOME INEQUALITIES IN PREHILBERTIAN SPACES

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*Received: January 22, 1987*

**REZUMAT.** — **Cîteva inegalităţi în spaţii prehilbertiene.** Obiectul acestei lucrări este de a pune în evidenţă cîteva rafinări ale inegalităţilor Schwarz şi triunghiului în spaţii prehilbertiene, precum şi extinderii în cazul acestor spaţii a inegalităţilor Beurling şi Clarkson ([1], [2]).

The main purposes of this paper are to establish some refinements of Schwarz and triangle inequalities in prehilbertian spaces over the real or complex number field and to extend in this spaces the well-known inequalities of Beurling and Clarkson [1], [2].

1. Let $(H, \langle , \rangle)$ be a prehilbertian spaces over the field $K(K = \mathbf{C}$ or $\mathbf{R})$, and $||\cdot|| = \sqrt{\langle \cdot, \cdot \rangle}$. The following theorem holds.

THEOREM 1. *Let be $x, y, a, b \in H$ such that*

$$||a||^2 \leqslant 2 \operatorname{Re}\langle x, a \rangle, \quad ||b||^2 \leqslant 2 \operatorname{Re}\langle y, b \rangle. \tag{1}$$

*Then we have*

$$||x|| \, ||y|| \geqslant (2 \operatorname{Re}\langle x, a \rangle - ||a||^2)^{1/2} (2 \operatorname{Re}\langle y, b \rangle - ||b||^2)^{1/2} + \\ + |\langle x, y \rangle - \langle x, b \rangle - \langle a, y \rangle + \langle a, b \rangle|.$$

*Proof.* We note that, for every $m, n, p, q \in \mathbf{R}$ the following inequality

$$(m^2 - n^2)(p^2 - q^2) \leqslant (mp - nq)^2 \tag{3}$$

is true.

Since

$$|\langle x, y \rangle - \langle x, b \rangle - \langle a, y \rangle + \langle a, b \rangle|^2 = |\langle x - a, y - b \rangle|^2 \leqslant \\ \leqslant ||x - a||^2 ||y - b||^2 = (||x||^2 + ||a||^2 - 2 \operatorname{Re}\langle x, a \rangle) \cdot (||y||^2 + \\ + ||b||^2 - 2 \operatorname{Re}\langle y, b \rangle),$$

we obtain by (3)

$$|\langle x, y \rangle - \langle x, b \rangle - \langle a, y \rangle + \langle a, b \rangle|^2 \leqslant \\ \leqslant (||x|| \, ||y|| - (2 \operatorname{Re}\langle x, a \rangle - ||a||^2)^{1/2} \cdot (2 \operatorname{Re}\langle y, b \rangle - ||b||^2)^{1/2})^2.$$

On the other hand

$$0 \leqslant (2 \operatorname{Re}\langle x, a \rangle - ||a||^2)^{1/2} \leqslant ||x||, \\ 0 \leqslant (2 \operatorname{Re}\langle y, b \rangle - ||b||^2)^{1/2} \leqslant ||y||$$

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what implies

$$||x||\ ||y|| \geq (2\,\mathrm{Re}\langle x, a\rangle - ||a||^2)^{1/2}\,(2\,\mathrm{Re}\langle y, b\rangle - ||b||^2)^{1/2}$$

and the theorem is proved.

COROLLARY 1.1. *Let be* $x, y, e \in H$ *such that* $||e|| = 1$. *Then the following inequality holds*

$$||x||\ ||y|| \geq |\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle| + |\langle x, e\rangle\langle e, y\rangle| \geq |\langle x, y\rangle|. \tag{4}$$

*Proof*. Putting $a = \langle x, e\rangle\, e$, $b = \langle y, e\rangle\, e$, then the condition (1) is satisfied. Moreover:

$$|\langle x, y\rangle - \langle x, \langle y, e\rangle e\rangle - \langle\langle x, e\rangle e, y\rangle + \langle\langle x, e\rangle e, \langle y, e\rangle e\rangle| =$$
$$= |\langle x, y\rangle - \langle x, e\rangle\overline{\langle y, e\rangle} - \langle x, e\rangle\langle e, y\rangle + \langle x, e\rangle\overline{\langle y, e\rangle}| =$$
$$= |\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle|.$$

Applying the theorem T.1., we obtain (4).

COROLLARY 1.2. *Let be* $x, y \in H$ *such that* $||x||^2, ||y||^2 \leq 2$. *Then the following inequality*

$$||x||\ ||y|| \geq |\langle x, y\rangle|^2 (2 - ||x||^2)^{1/2}\,(2 - ||y||^2)^{1/2} +$$
$$+ |\langle x, y\rangle|\cdot|1 - ||x||^2 - ||y||^2 + |\langle x, y\rangle|^2| \tag{5}$$

is true.

*Proof*. Putting $a = \langle x, y\rangle y$, $b = \langle y, x\rangle x$, then the condition (1) is satisfied. Moreover:

$$(2\,\mathrm{Re}\langle x, a\rangle - ||a||^2)^{1/2}\,(2\,\mathrm{Re}\langle y, b\rangle - ||b||^2)^{1/2} =$$
$$= |\langle x, y\rangle|^2 (2 - ||x||^2)^{1/2}\,(2 - ||y||^2)^{1/2}$$

and

$$|\langle x, y\rangle - \langle x, b\rangle - \langle a, y\rangle + \langle a, b\rangle| = |\langle x, y\rangle|\,|1 - ||x||^2 - ||y||^2 + |\langle x, y\rangle|^2|.$$

Applying the theorem T.1., we obtain (5).

THEOREM 2. *Let be* $x, y, e \in H$ *such that* $||e|| = 1$. *Then we have*

$$(||x||^2 - |\langle x, e\rangle|^2)(||y||^2 - |\langle y, e\rangle|^2) \geq |\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle|^2. \tag{6}$$

*Proof*. Let consider the mapping

$$p: H \times H \to K,\ p(x, y) = \langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle.$$

It is easy to see that the mapping $p$ satisfies the relations

(i) $p(x, x) \geq 0$, $x \in H$;

(ii) $p(\alpha x + \beta y, z) = \alpha p(x, z) + \beta p(y, z)$, $\alpha, \beta \in K$, $x, y, z \in H$;

(iii) $p(y, x) = \overline{p(x, y)}$, $x, y \in H$;

Then the Schwarz inequality holds

$$|p(x, y)|^2 \leq p(x, x)p(y, y),\quad x, y \in H,$$

i.e. (6).

*Remark* 1. By the relation (3) we have

$$(||x||\,||y|| - |\langle x, e\rangle\langle e, y\rangle|)^2 \geqslant (||x||^2 - |\langle x, e\rangle|^2)(||y||^2 - |\langle y, e\rangle|^2) \geqslant$$
$$\geqslant |\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle|^2.$$

Since $||x||\,||y|| \geqslant |\langle x, e\rangle\langle e, y\rangle|$, we obtain

$$||x||\,||y|| - |\langle x, e\rangle\langle e, y\rangle| \geqslant |\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle|,$$

from where results the corollary C.1.1. with an other proof.

COROLLARY 2.1. *Let be* $x, y, e \in H$ *such that* $||e|| = 1$. *Then the following inequality holds*

$$(||x+y||^2 - |\langle x+y, e\rangle|^2)^{1/2} \leqslant (||x||^2 - |\langle x, e\rangle|^2)^{1/2} +$$
$$+ (||y||^2 - |\langle y, e\rangle|^2)^{1/2} \quad (7)$$

*Proof*. It results by the triangle inequality for the seminorm $s: H \to \mathbf{R}^+$ given by $s(x) = p(x, x)^{1/2}$.

*Remark* 2. Putting $x = a$, $y = \lambda b$, $e = \frac{b}{||b||}$, where $a, b \in H$, $b \neq 0$ and $\lambda \in K$, by the relation (7) we obtain

$$0 \leqslant ||a + \lambda b||^2||b||^2 - |\langle a + \lambda b, b\rangle|^2 \leqslant ||a||^2||b||^2 - |\langle a, b\rangle|^2 \quad (8)$$

and

$$0 \leqslant ||b||^2 - |\langle a \pm b, b\rangle|^2 \leqslant ||a||^2||b||^2 - |\langle a, b\rangle|^2 \quad (9)$$

COROLLARY 2.2. *Let be* $x, y, z \in H - \{0\}$. *Then the following inequality holds*

$$\left|\frac{\langle x, y\rangle}{||x||\,||y||}\right|^2 + \left|\frac{\langle y, x\rangle}{||y||\,||x||}\right| + \left|\frac{\langle z, x\rangle}{||z||\,||x||}\right| \leqslant 1 + 2\frac{|\langle x, y\rangle\langle y, z\rangle\langle z, x\rangle|}{||x||^2\,||y||^2\,||z||^2} \quad (10)$$

The proof results by theorem T.2.

OBSERVATION. If $(H, \langle, \rangle)$ is a real prehilbertian space and $\alpha, \beta, \gamma$ given by

$$||a||\,||b||\cos\alpha = \langle a, b\rangle, \quad ||b||\,||c||\cos\beta = \langle b, c\rangle,$$
$$||c||\,||a||\cos\gamma = \langle c, a\rangle$$

then the following inequalities holds

$$|\cos\gamma| \leqslant |\cos\alpha\cos\beta| + |\sin\alpha\sin\beta| \quad (11)$$
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma \leqslant 1 + 2|\cos\alpha\cos\beta\cos\gamma| \quad (12)$$

Indeed, putting in theorem T.2. $x = a$, $y = c$, $e = \frac{b}{||b||}$, we obtain

$$(||a||^2||b||^2 - |\langle a, b\rangle|^2)(||b||^2||c||^2 - |\langle b, c\rangle|^2) \geqslant$$
$$\geqslant |\langle a, c\rangle||b||^2 - \langle a, b\rangle\langle b, c\rangle|^2$$

It results

$$\sin^2\alpha \sin^2\beta \geqslant |\cos\gamma - \cos\alpha\cos\beta|^2 \geqslant ||\cos\gamma| - |\cos\alpha\cos\beta||^2$$

from where results (11).

The inequality (12) results by (10).

2. Further, we shall give a generalisation of corollary C.1.1., for the orthonormal families in the prehilbertian space $(H, \langle , \rangle)$.

THEOREM 3. *Let* $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ *be a orthonormal family in* $H$ *and* $x, y$ *two elements of* $H$. *Then the following inequality holds*

$$\|x\|\,\|y\| \geqslant \left|\langle x, y\rangle - \int_{\alpha\in\mathcal{A}} \langle x, e_\alpha\rangle\langle e_\alpha, y\rangle\right| + \int_{\alpha\in\mathcal{A}} |\langle x, e_\alpha\rangle\langle e_\alpha, y\rangle| \geqslant \geqslant |\langle x, y\rangle|. \tag{1}$$

*Proof*. Let be $I \subset \mathcal{A}$, with card $I < \aleph_0$.

By the Schwarz inequality we have

$$\left|\langle x - \sum_{\alpha\in I} \langle x, e_\alpha\rangle e_\alpha,\ y - \sum_{\alpha\in I} \langle y, e_\alpha\rangle e_\alpha\rangle\right|^2 \leqslant$$

$$\leqslant \left\|x - \sum_{\alpha\in I} \langle x, e_\alpha\rangle e_\alpha\right\|^2 \left\|y - \sum_{\alpha\in I} \langle y, e_\alpha\rangle e_\alpha\right\|^2.$$

It is easy to see that

$$\left|\langle x, y\rangle - \sum_{\alpha\in I} \langle x, e_\alpha\rangle\langle e_\alpha, y\rangle\right|^2 \leqslant \left(\|x\|^2 - \sum_{\alpha\in I} |\langle x, e_\alpha\rangle|^2\right)\left(\|y\|^2 - \sum_{\alpha\in I} |\langle y, e_\alpha\rangle|^2\right). \tag{2}$$

Applying the Aczél inequality ([3] p. 57) we obtain:

$$\left(\|x\|^2 - \sum_{\alpha\in I} |\langle x, e_\alpha\rangle|^2\right)\left(\|y\|^2 - \sum_{\alpha\in I} |\langle y, e_\alpha\rangle|^2\right) \leqslant \leqslant \left(\|x\|\,\|y\| - \sum_{\alpha\in I} |\langle x, e_\alpha\rangle\langle e_\alpha, y\rangle|\right)^2. \tag{3}$$

Applying the Bessel and Cauchy—Buniakovski—Schwarz inequalities, one gets

$$\|x\|^2\|y\|^2 \geqslant \sum_{\alpha\in I} |\langle x, e_\alpha\rangle|^2 \sum_{\alpha\in I} |\langle y, e_\alpha\rangle|^2 \geqslant \left(\sum_{\alpha\in I} |\langle y, e_\alpha\rangle\langle e_\alpha, x\rangle|\right)^2$$

what implies

$$\left|\langle x, y\rangle - \sum_{\alpha\in I} \langle x, e_\alpha\rangle\langle e_\alpha, y\rangle\right| \leqslant \|x\|\,\|y\| - \sum_{\alpha\in I} |\langle x, e_\alpha\rangle\langle e_\alpha, y\rangle|.$$

Then we have

$$\left|\langle x, y\rangle - \sum_{\alpha\in I} \langle x, e_\alpha\rangle\langle e_\alpha, y\rangle\right| + \sum_{\alpha\in I} |\langle x, e_\alpha\rangle\langle e_\alpha, y\rangle| \leqslant \|x\|\,\|y\| \tag{4}$$

for every $I \subset \mathcal{A}$ with the property card $I < \aleph_0$.

Consequently, the family $\{\langle x, e_\alpha\rangle\langle e_\alpha, y\rangle\}_{\alpha\in I}$ is absolute sumable over the number field $K$, and

$$\left|\langle x, y\rangle - \int_{\alpha\in\mathfrak{A}} \langle x, e_\alpha\rangle\langle e_\alpha, y\rangle\right| + \int_{\alpha\in\mathfrak{A}} |\langle, e_\alpha\rangle\langle e_\alpha, y\rangle| \leqslant ||x||\ ||y||.$$

This proves the theorem.

*Remark* 1. If $\{e_\alpha\}_{\alpha\in\mathfrak{A}} = \{e\}$, we obtain the corollary C.1.1.

COROLLARY 3.1. *Let* $\{e_\alpha\}_{\alpha\in\mathfrak{A}}$ *be an orthonormal family in* $H$ *and* $x$ $y \in H$ *such that* $x \perp y$. *Then*

$$||x||\ ||y|| \geqslant 2\int_{\alpha\in\mathfrak{A}} |\langle x, e_\alpha\rangle\langle e_\alpha, y\rangle| \tag{5}$$

*Particularly, if* $\{e_\alpha\}_{\alpha\in\mathfrak{A}} = \{e\}$, *then we have*

$$||x||\ ||y|| \geqslant 2|\langle x, e\rangle\langle e, y\rangle|. \tag{6}$$

COROLLARY 3.2. *Let* $\{e_\alpha\}_{\alpha\in\mathfrak{A}}$ *be an orthonormal family in* $H$ *and* $x$ $y \in H$. *Then the following inequalities hold:*

$$(||x|| + ||y||)^2 - ||x+y||^2 \geqslant 2\left(\left|\langle x, y\rangle - \int_{\alpha\in\mathfrak{A}} \langle x, e_\alpha\rangle\langle e_\alpha, y\rangle\right| + \right.$$
$$\left. + \int_{\alpha\in\mathfrak{A}} |\langle x, e_\alpha\rangle\langle e_\alpha, y\rangle| - |\langle x, y\rangle|\right) \geqslant 0. \tag{7}$$

*Particularly, if* $\{e_\alpha\}_{\alpha\in\mathfrak{A}} = \{e\}$, *then we have*

$$(||x|| + ||y||)^2 - ||x+y||^2 \geqslant 2(|\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle| + \tag{8}$$
$$+ |\langle x, e\rangle\langle e, y\rangle| - |\langle x, y\rangle| \geqslant 0.$$

*Proof*. Let be $x, y \in H$, then

$$(||x|| + ||y||)^2 - ||x+y||^2 = 2(||x||\ ||y|| - \mathrm{Re}\langle x, y\rangle| \geqslant$$
$$\geqslant 2(||x||\ ||y|| - |\langle x, y\rangle|) \geqslant 0.$$

But

$$||x||\ ||y|| - |\langle x, y\rangle| \geqslant \left|\langle x, y\rangle - \int_{\alpha\in\mathfrak{A}} \langle x, e_\alpha\rangle\langle e_\alpha, y\rangle\right| +$$
$$+ \int_{\alpha\in\mathfrak{A}} |\langle x, e_\alpha\rangle\langle e_\alpha, y\rangle| - |\langle x, y\rangle| \geqslant 0$$

and the corollary is proved.

3. Let $\mu$ be a positive Radon measure and $f, g \in L^p(\mu)$. Then we have the Beurling inequality:

$$||f+g||^p + ||f-g||^p \geqslant (||f|| + ||g||)^p + \big|\,||f|| - ||g||\,\big|^p \qquad (1)$$

if $1 < p < 2$, and the Clarkson inequality:

$$||f+g||^p + ||f-g||^p \geqslant 2(||f||^p + ||g^p||) \qquad (2)$$

if $p \geqslant 2$ ([1], [2], [3] p. 82).

We shall prove the following theorem:

THEOREM 4. *Let be $x, y \in H$, then*

$$||x+y||^p + ||x-y||^p \geqslant (||x|| + ||y||)^p + \big|\,||x|| - ||y||\,\big|^p \qquad (3)$$

*if $1 < p < 2$ and*

$$||x+y||^p + ||x-y||^p \geqslant 2(||x||^p + ||y||^p) \qquad (4)$$

*if $p \geqslant 2$.*

*Proof.* Let be $x, y \in H$. Then we have

$$||x+y||^p + ||x-y||^p = (||x+y||^2)^{p/2} + (||x-y||^2)^{p/2} =$$
$$= (||x||^2 + ||y||^2 + 2\mathrm{Re}\langle x, y\rangle)^{p/2} + (||x||^2 + ||y||^2 - 2\mathrm{Re}\langle x, y\rangle)^{p/2}.$$

Let be now the function $d$: $[0, 2\pi] \to \mathbf{R}$ given by

$$d(\varphi) = (||x||^2 + ||y||^2 + 2||x||\,||y||\cos\varphi)^{p/2} + (||x||^2 + ||y||^2 - 2||x||\,||y||\cos\varphi)^{p/2}.$$

Then

$$d'(\varphi) = p||x||\,||y||\sin\varphi[(||x||^2 + ||y||^2 - 2||x||\,||y||\cos\varphi)^{p/2-1} - (||x||^2 + ||y||^2 + 2||x||\,||y||\cos\varphi)^{p/2-1}].$$

If $1 < p < 2$, then $\inf\limits_{\varphi \in [0, 2\pi]} d(\varphi) = d(0) = (||x|| + ||y||)^p + |\,||x|| - ||y||\,|^p$.

If $p \geqslant 2$, then $\inf\limits_{\varphi \in [0, 2\pi]} d(\varphi) = d\left(\frac{\pi}{2}\right) = 2(||x||^2 + ||y||^2)^{p/2}$.

But $(||x||^2 + ||y||^2)^{p/2} \geqslant ||x||^p + ||y||^p$ and the theorem is proved.

COROLLARY. 4.1. *Let be $a, b \in H$. Then*

$$2^p(||a||^p + ||b||^p) \geqslant (||a+b|| + ||a-b||)^p + \big|\,||a+b|| - ||a-b||\,\big|^p \qquad (5)$$

*if $1 < p < 2$ and*

$$2^{p-1}(||a||^p + ||b||^p) \geqslant ||a+b||^p + ||a-b||^p \qquad (6)$$

*if $p \geqslant 2$.*

*Proof.* Putting in theorem T.4. $x = a + b$, $y = a - b$ the corollary is therefore proved.

*Remark.* For every $x, y \in H$ and $p \geqslant 2$ we have:

$$2^{p-1}(||x||^p + ||y||^p) \geqslant ||x+y||^p + ||x-y||^p \geqslant 2(||x||^p + ||y||^p). \quad (7)$$

If $p = 2$, we have the parallelogram identity

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2). \quad (8)$$

4. Finally, we shall prove the following theorem:

THEOREM 5. *Let $x, y \in H$. Then for every $\lambda \geqslant 2$ the following inequality holds*

$$(||x|| + ||y||)^\lambda \left(\left\|\frac{x}{||x||} - \frac{y}{||y||}\right\|^\lambda + \left\|\frac{x}{||x||} + \frac{y}{||y||}\right\|^\lambda\right) \leqslant$$
$$\leqslant 2^{2\lambda-1}(||x||^\lambda + ||y||^\lambda), \quad x, y \neq 0. \quad (1)$$

Firstly, we shall prove

LEMMA 6. *Let be $x, y \in H \setminus \{0\}$. Then we have*

$$||x - y|| \geqslant \frac{||x|| + ||y||}{2}\left\|\frac{x}{||x||} - \frac{y}{||y||}\right\| \quad (2)$$

*Proof.* It is easy to see that

$$(||x||^2 + ||y||^2 + 2||x||\,||y||)\left(1 - \frac{1}{||x||\,||y||}\operatorname{Re}\langle x, y\rangle\right) =$$
$$= ||x||^2 + ||y||^2 + 2||x||\,||y|| - \operatorname{Re}\langle x, y\rangle\left(\frac{||x||}{||y||} + \frac{||y||}{||x||}\right) - 2\operatorname{Re}\langle x, y\rangle.$$

Since $\frac{||x||}{||y||} + \frac{||y||}{||x||} \geqslant 2$, $||x||^2 + ||y||^2 \geqslant 2||x||\,||y||$, we obtain

$$||x||^2 + ||y||^2 + 2||x||\,||y|| - \operatorname{Re}\langle x, y\rangle\left(\frac{||x||}{||y||} + \frac{||y||}{||x||}\right) - 2\operatorname{Re}\langle x, y\rangle \leqslant$$
$\leqslant 2||x||^2 + 2||y||^2 - 4\operatorname{Re}\langle x, y\rangle$, what implies

$$(||x|| + ||y||)^2\left\|\frac{x}{||x||} - \frac{y}{||y||}\right\|^2 \leqslant 4||x - y||^2$$

from where results (2).

The theorem's proof. Applying lemma L.6. we have

$$||x - y||^\lambda \geqslant \frac{(||x|| + ||y||)^\lambda}{2^\lambda}\left\|\frac{x}{||x||} - \frac{y}{||y||}\right\|^\lambda$$

and

$$||x + y||^\lambda \geqslant \frac{(||x|| + ||y||)^\lambda}{2^\lambda}\left\|\frac{x}{||x||} + \frac{y}{||y||}\right\|^\lambda$$

from where results

$$||x-y||^{\lambda}+||x+y||^{\lambda} \geqslant \frac{(||x||+||y||)^{\lambda}}{2^{\lambda}}\left(\left\|\frac{x}{||x||}-\frac{y}{||y||}\right\|^{\lambda}+\right.$$

$$\left.+\left\|\frac{x}{||x||}+\frac{y}{||y||}\right\|^{\lambda}\right).$$

Applying the inequality (6) of corollary C.4.1., we obtain (1). The theorem is proved.

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# RECENZII

Wolfgang Hackbusch, **Multi-Grid Methods and Applications,** Springer-Verlag, Berlin, Heidelberg, New-York, Tokyo, XIV + 377 pages.

This monograph deals with the basic concepts of multi-grid methods. It is divided into five parts: liniar multi-grid algorithms convergence analysis, special multi-grid applications, additional techniques and application to integral equations. We recommend this book to all intersted in computational Mathematics.

IOAN A. RUS

H. Kästner, **Architektur und Organisation digitaler Rechenanlagen,** B. G. Teubner, Stuttgart, 1978, 224 pp.

This book is a clear and systematic presentation of the important problems of computer architecture. It is recommended as a student textbook; also it may be useful to all interested in the mentioned subject.

M. FRENŢIU

Claude George, **Exercises in Integration, Problem Books in Mathematics,** Springer Verlag, Berlin Heidelberg New York 1984 (550 pp.)

The book is an excellent collection of 230 exercises in integration theory and its applications. The majority of the exercises are at the master's level but some of them are advanced, treating more specialized and deep questions in measure theory. The emphasis is on Lebesgue integral and its aplication, especially to trigonometric series, a theory which lies at the origin of Lebesgue integration theory. The book contains many results which usually are not included in standard courses of integration theory, being scattered about in appendices in various works or as technical lemmas. The material is arranged in twelve chapters: 0. Outline of the course, 1. Measurable sets, 2. $\sigma$ — algebras and positive measures, 3. The fundamentals theorems, 4. Asymptotic evaluation of integrals, 5. Fubini's theorem, 6. The L spaces, 7. The space $L^2$, 8. 'Convolution products and Fourier transforms, 9. Function with bounded variation. Absolutely continuous functions. Differentiation and integration, 10. Summation processes. Trigonometric polynomials, 1.. Trigonometric series.

As can be seen from the sketch presented in the first chapter, the Cours d'integration published by the author at the Ecole de Mines, Nancy 1977, seems to be a very good companion of the book and we suggest its publication (eventually in a revised form) in a popular series such as Springer's Universitext.

S. COBZAŞ

A. Langenbach, **Vorlesungen zur höheren Analysis,** VEB Berlin 1984, 280 pp.

The aim of the book is to present, in an accessible and relatively short way, the functional analytic tools used in solving partial differential equations — fixed point theorems, monotonicity and compactness methods, Sobolev spaces etc. A good idea on the content of the book is given by the headings of its chapters: 1. Metric and normed spaces, 2. Topological spaces, 3. Functionals and minimum problems, 4. Hilbert spaces, 6. Constructive methods for solving minimum problems and equations, 7. The application of extension and completion methods, 8. Classification of partial differential equations, 9. Theory of elliptic equations, 10. Hipperbolic and parabolic linear equations, 11. Evolution equations, Appendix 1. The theorem of Stone-Weierstass, Appendix 2. Integration of continuous functions.

The book is clearly written, very well organized and we recommend it to all interested in applications of functional anaysis to partial differential equations.

S. COBZAŞ

E. Walter, **Biomathematik für Mediziner, Teubner Studienbücher, Mathematik,** Stuttgart 1980, 206 pp.

This is a text of a course taught at the University of Freiburg and is concerned with topics as descriptive statistics, probability theory, statistics, medical informatics. The book also contains many examples of applications of mathematics, statistics and informatics, in clinical practice.

The book will be useful to all interested in applying mathematical methods in medicine and biology.

S. COBZAŞ

**Transportsystemanalyse (Autorenkollektiv)**, Transpress, VEB Berlin 1986, 192 pp.

The book is dealing with models (especially matrix models), analysis, optimization, a simulation and prognosis in transportation and traffic problems. The book is carrefully written, contains many concrete examples and will be useful to all working in transport problems.

S. COBZAŞ

W l a d y s l a w M. T u r s k i, **Informatics, A Propaedeutic View**, PWN — Polish Scientific Publishers — Warszawa and North-Holland — Amsterdam ● New York ● Oxford, 1985, 342 p.

The book is intended as a text for a first course in informatics. It provides a common basis for a variety of aspects and constituentes of informatics. The book is well written. It neither explicitly depends on any prior specific knowledge, nor contains direct advice for further study. The book is recommanded as well for students in informatics as for students in other sciences.

The chapters of the book are the following: 1. Informatics, information and coding; 2. Informatic hardware for recording, storing; and displaying information; 3. Binary arithmetic and computing; 4. Algorithm — A description of information processing; 5. Programming in machine language; 6. Formal languages; 7. Language and process of programming; 8. Principles of automatic translation; 9. Computing systems; 10. Applications of informatics.

C. TARŢIA

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