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SPAŢII D—H-RECURENTE

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ABSTRACT. — D—H—Recurrent Spaces. The results in [3], [4] and [9] are further carried in the present paper. The D-H-recurrences (2,1), (3,4), (4,1), (5,2), (6,1) establisched and the relationships between D-H-recurrences tensors $\underset{s}{H}^{k}_{ijk}$ (3,9), (4,5), (5,7), (5,8), (6,4), (6,8), (6,9) are evidenced, showing that they are analogue to those between the tensors for which the D-H-recurrence has been defined.

Introducere. În prezenta lucrare se fac extinderi a rezultatelor din [3], [4], [9].

Se stabilesc $D-H$-recurenţele şi se pun în evidenţă relaţiile dintre tensorii de $D-H$—recurenţă $\underset{s}{H}^{h}_{ijh}$ arătînd că ele sînt analoge cu cele dintre tensorii pentru care s-a definit $D-H$—recurenţă.

§ 1. Fie L o varietate diferenţiabilă de calsă C^∞ înzestrată cu o metrică riemanniană g de componente g_{ij} într-o hartă locală $(u\,;\,x^i)$. Vom nota cu ∇, conexiunea Levi-Civita corespunzătoare, de coeficienţi $\left\{{i \atop jk}\right\}$ în harata locală $(u,\ x^i)$, prin R^i_{jkh} componentele tensorului de curbură [1], prin $R_{ij} = R^s_{ijs}$ tensorul lui Ricci iar prin $R = g^{ij} R_{ij}$ curbura scalară.

O conexiune D semisimetrică metrică în L_n [2], [14], în harta locală considerată $(u,\ x^i)$, are coeficienţii:

$$\Gamma^i_{jk} = \left\{{i \atop jk}\right\} + \omega_j \delta^i_k - g_{jk}\omega^i \tag{1.1}$$

$$\omega^i = g^{is}\omega_s \tag{1.2}$$

şi

$$T^i_{jk} = \omega_j \delta^i_k - \omega_k \delta^i_j \tag{1.3}$$

$$g_{ij/k} = 0 \tag{1.4}$$

unde T este tensorul de torsiune a lui D iar prin / s-a notat derivarea covariantă în raport cu D.

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Fie $\bar{R}^i_{jkh}$ tensorul de curbură a lui D, $\bar{R}_{ij} = \bar{R}^s_{ijs}$ tensorul lui Ric $\bar{R} = g^{ij}\bar{R}_{ij}$ curbura scalară, $\bar{T}^i_{jkh}$ tensorul D-concircular de curbură [1l $\bar{Z}^i_{jkh}$ tensorul D—coharmonic de curbură [11], $\bar{P}^i_{jkh}$ tensorul D-pı iectiv de curbură [10] şi $\bar{C}^i_{jkh}$ respectiv tensorul de curbură conformă lui D.

Dacă D este o conexiune K. Yano (adică $\bar{R}^i_{jkh} = 0$) atunci g e conform plată [14] şi avem:

$$\omega_{i,j} - \omega_{j,i} = 0 \qquad (1.$$

unde prin virgulă s-a notat derivata covariantă în raport cu ∇.

Din (1,5) rezultă relaţiile echivalente [6] $\omega_{i/j} - \omega_{j/i} = 0$, $\partial_i \omega_j - \partial_j \omega_i = 0$, $T_{i/j} - T_{j/i} = 0$, $T^k_{ij/k} = 0$, (div T=0), $d\omega = 0$, (ω închisă

(1.

Dacă D este mai generală ca o conexiune K. Yano, $\bar{R}^i_{jkh} \neq 0$, d tensorul D-concircular de curbură [11], $\bar{T}^i_{jkh}$ este nul, atunci pent $n \geqslant 3$, g este conform plată [7] şi avem

$$\bar{R}_{/r} - 2\omega\bar{R} = 0, \quad \bar{R}_{/r} = \partial_r \bar{R} \qquad (1.$$

deci [4], [7] rezultă $\bar{R}^s_{ijk}$ recurent respectiv D—recurent [4] cu cove tor de recurenţă 2ω

$$\bar{R}^i_{jkh,r} = 2\omega_r \bar{R}^i_{jkh} = \bar{R}^i_{jkh/r} \qquad (1.$$

Avem deci

PROPOZIŢIA 1.1. *D-conexiunile ce nu sînt conexiuni K. Yano şi a tensorul D-circular nul, sînt D-recurente cu covector* 2ω.

§ 2. *DEFINIŢIA* 1. Vom spune că spaţiul (L_n, D) este *D—H-re curent,* dacă

$$\bar{R}^i_{jkh/r} = \sigma_r \bar{R}^i_{jkh} + \underset{1}{H}{}^i_{jkh}(\bar{R}_{/r} - \sigma_r \bar{R}) \qquad (2.1$$

unde σ_r este un convector iar $\underset{1}{H}{}^i_{jkh}$ un tensor de tip (1 ; 3).

Din (2.1) rezultă

$$\bar{R}_{jk/r} = \sigma_r \bar{R}_{jk} + \underset{1}{H}{}_{jk}(\bar{R}_{/r} - \sigma_r \bar{R}) \qquad (2.2$$

unde $\underset{1}{H}{}_{jk} = \underset{1}{H}{}^i_{jki}$ şi spaţiul se va numi *D—H-Ricci-recurent.*

Observaţia 1. Un spaţiu D—H-recurent este şi D—H-Ricci recurent, reciproca nefiind în general adevărată.

Notînd $\underset{1}{H} = g^{jk}\underset{1}{H}_{jk}$ din (2.2) rezultă

$$\bar{R}_{/r} - \sigma_r\bar{R} = \underset{1}{H}(\bar{R}_{/r} - \sigma_r\bar{R}) \tag{2.3}$$

şi dacă (L_n, g) este $D-H$-recurent propriu $(R_{/r} - \sigma_r R \neq 0)$ atunci $\underset{1}{H} = 1$ şi avem

PROPOZIŢIA 2.1. *Intr-un spaţiu (L_n, D) $D-H$-recurent propriu avem $\underset{1}{H} = g^{ij}\underset{1}{H}_{ij} = 1$.*

Este evident că un spaţiu (L_n, D), $D-H$-recurent în care $\bar{R}_{/r} - \sigma_r\bar{R} = 0$ este D-recurent [4] cu acelaşi covector σ_r.

Să presupunem că (L_n, D) este $D-H$-recurent de covector σ şi D-recurent de covector $\bar{\sigma} \neq \sigma$

$$\bar{R}^i_{jkh/r} = \bar{\sigma}_r\bar{R}^i_{jkh} \tag{2.4}$$

Din (2.1) şi (2.4) rezultă

$$(\bar{\sigma}_r - \sigma_r)\bar{R}^i_{jkh} = \underset{1}{H}^i_{jkh}(\bar{R}_{/r} - \sigma_r\bar{R}) \tag{2.5}$$

şi cum din (2.4) avem $\bar{R}_{/r} = \bar{\sigma}_r\,\bar{R}$, din (2.5) rezultă

$$(\bar{\sigma}_r - \sigma_r)\bar{R}^i_{jkh} = \bar{R}\underset{1}{H}^i_{jkh}(\bar{\sigma}_r - \sigma_r) \tag{2.6}$$

şi în ipoteza în care lucrăm $\bar{\sigma} \neq \sigma$, deducem:

$$\bar{R}^i_{jkh} = \bar{R}\underset{1}{H}^i_{jkh} \tag{2.7}$$

iar din (2.1) şi (2.7) pentru $\bar{\sigma}_r$ rezultă

$$\bar{\sigma}_r = \partial_r \ln \bar{R} \tag{2.8}$$

Avem deci

PROPOZIŢIA 2.2. *Intr-un spaţiu (L_n, D) $D-H$-recurent de vector σ şi D-recurent de vector $\bar{\sigma} \neq \sigma$, tensorul de $D-H$-recurenţă este dat de (2.7), iar $\bar{\sigma}$ verifică (2.8).*

COROLAR 2.1. *Din (2.1) şi (2.7) rezultă pentru $R \neq 0$,*

$$\underset{1}{H}^i_{jkh/r} = 0 \tag{2.9}$$

COROLAR 2.2. *Dacă spaţiul (L_n, D) este D-recurent de vector $\bar{\sigma}_r$ dat de (2.8) el este şi $D-H$ recurent de tensor de H recurenţă dat de (2.7) iar σ_r arbitrar.*

În particular dacă $\omega = 0$, avem $D = \nabla$ şi se obţin rezultatele din [9].

Considerînd tensorul D-concircular de curbură

$$\bar{T}^i_{jkh} = \bar{R}^i_{jkh} - \bar{R}\ \underset{1}{H}^i_{jkh} \tag{2.10}$$

unde

$$H^i_{jkh} = \frac{1}{n(n-1)} (g_{jk}\delta^i_h - g_{jh} \delta^i_k) \quad (2.1$$

dacă spaţiul (L_n, D) are proprietatea $\bar{T}^i_{jkh} = 0$ şi $n \geqslant 3$, atunci confor propoziţiei 1.1 rezultă (1.8) şi dacă (L_n, D) este $D-H$ recurent propr avem $\sigma \neq 2\omega$, iar din (2.7) rezultă

$$\underset{1}{H^i_{jkh}} = H^i_{jkh} \quad (2.1$$

Avem deci:

PROPOZIŢIA 2.3. *Dacă D este o conexiune semi-simetrică mai gener cu o conexiune K. Yano, dar cu tensorul D-concircular de curbură nul spaţiul (L_n, D) este $D-H$-recurent, atunci avem (2.12).*

Observaţia 2. $\bar{R} \neq 0$, deoarece din anularea tensorului D-concircul de curbură ar rezulta $\bar{R}^i_{jkh} = 0$ şi conexiunea ar fi K. Yano. La fel $\bar{R}$ $\neq$ const. deoarece din (1.7) ar rezulta $\omega_r = 0$ şi D ar fi egală cu ∇.

Observaţia 3. Proprietatea 2.3 are loc pentru $D \neq \nabla$. Pentru $D =$ adică $\omega = 0$ din anularea tensorului concircular de curbură pentru $n \geqslant$ din faptul că curbura riemanniană este constantă, rezultă $R =$ const. deci $R^i_{jkh,r} = 0$, de unde:

COROLAR 2.3. *Dacă (L_n, ∇) este $\nabla-H$ recurent, atunci spaţiul nu po fi recurent cu convector de recurenţă $\bar{\sigma} \neq \sigma$.*

Fie pentru D, tensorul lui Einstein

$$E_{ij} = R_{ij} - \frac{R}{n} g_{ij} \quad (2.1$$

Evident, dacă $\bar{T}^i_{jkh} = 0$ avem $\bar{E}_{ij} = 0$. Să presupunem deci $\bar{T}^i_{jkh} \neq 0$ că $\bar{E}_{ij}$ este D-recurent

$$\bar{E}_{ij/r} = \sigma_r \bar{E}_{ij} \quad (2.1$$

Din (2.2), (2.13), (2.14) rezultă

$$(\bar{R}_{/r} - \sigma_r\bar{R})\left(\underset{1}{H_{ij}} - \frac{1}{n} g_{ij}\right) = 0 \quad (2.1$$

de unde avem

$$\underset{1}{H_{ij}} = \frac{1}{n} g_{ij} = H_{ij} \quad (2.1$$

Invers, din (2.2) şi (2.16) rezultă (2.14), deci avem:

PROPOZIŢIA 2.4. *Dacă (L_n, D) este $D-H$-Ricci recurent, atunci condiţ necesară şi suficientă ca tensorul lui Einstein să fie D-recurent este (2.16*

COROLAR 2.4. *Dacă (L_n, D) este $D-H$ recurent atunci (2.16) este condiţia necesară şi suficientă ca tensorul E (2.13) să fie D-recurent cu acelaşi covector de recurenţă.*

§3. Fie Z^i_{jkh} tensorul coharmonic de curbură [5] şi $\bar{Z}^i_{jkh}$ tensorul D-coharmonic de curbură [8], [11]. Avem [4],

$$C^i_{jkh} = Z^i_{jkh} + \frac{n}{n-2} R\, H^i_{jkh} \tag{3.1}$$

$$\bar{C}^i_{jkh} = \bar{Z}^i_{jkh} + \frac{n}{n-2} \bar{R}\, H^i_{jkh} \tag{3.2}$$

unde H^i_{jkh} este dat de (2.11).

Cum avem [12] $C^i_{jkh} = \bar{C}^i_{jkh}$, din (3.1) şi (3.2) rezultă

PROPOZIŢIA 3.1. *Intre tensorul coharmonic de curbură al lui ∇ şi D, există relaţia*

$$\bar{Z}^i_{jkh} - Z^i_{jkh} = \frac{1}{n-2} H^i_{jkh}(R - \bar{R}) \tag{3.3}$$

DEFINIŢIA 2. Vom spune că spaţiul (L_n, D) este *$D-H$-coharmonic recurent* dacă

$$\bar{Z}^i_{jkh/r} = \sigma_r \bar{Z}^i_{jkh} + \underset{2}{H^i_{jkh}}(\bar{R}_{/r} - \sigma_r \bar{R}) \tag{3.4}$$

unde σ_r este un convector şi $\underset{2}{H^i_{jkh}}$ un tensor de tip (1.3) cu proprietatea

$$\underset{2}{H^s_{ijs}} = \underset{2}{H_{ij}} = -\frac{1}{n-2} g_{ij}. \tag{3.5}$$

Din (3.4) rezultă

$$\left(-\frac{1}{n-2} g_{ij} - \underset{2}{H_{ij}}\right)(R_{/r} - \sigma_r \bar{R}) = 0 \tag{3.6}$$

şi dacă H_{ij} ar fi diferit de $-\frac{1}{n-2} g_{ij}$ ar rezulta $\bar{R}_{/r} - \sigma_{/r} \bar{R}_n = 0$ şi deci [4] spaţiul (L_n, D) ar fi D-coharmonic recurent.

Din (3.5) rezultă

$$\underset{2}{H} = g^{ij} \underset{2}{H_{ij}} = -\frac{n}{n-2} \tag{3.7}$$

Să presupunem spaţiul (L_n, D), $D-H$-recurent (2.1). Derivînd covariant în raport cu D tensorul D-coharmonic $\bar{Z}^h_{ijk}$ şi ţinînd seama de (2.1) şi (2.2) avem:

$$\bar{Z}^h_{ijk/r} = \sigma_r \bar{Z}^h_{ijk} + \left[\underset{1}{H}{}^h_{ijk} + \frac{1}{n-2}(\underset{1}{H}_{ik}\delta^h_j - \underset{1}{H}_{ij}\delta^h_k + g_{ik}\underset{1}{H}{}^h_j - g_{ij}H^h_k)\right] \cdot [\bar{R}_{/r} - \sigma_r \bar{R}] \qquad (3.8)$$

unde $H^h_j = g^{hs}H_{sj}$

Din (3.8) rezultă:

PROPOZIŢIA 3.2. *Un spaţiu (L_n, D), $D-H$-recurent este şi $D-H$-coharmonic recurent cu acelaşi covector σ şi cu tensor $\underset{2}{H}{}^i_{jkh}$ de H-recurenţă dat de*

$$\underset{2}{H}{}^h_{ijk} = \underset{1}{H}{}^h_{ijk} + \frac{1}{n-2}(\underset{1}{H}_{ik}\delta^h_j - \underset{1}{H}_{ij}\delta^h_k + g_{ik}\underset{1}{H}{}^h_j - g_{ij}H^h_k) \qquad (3.9)$$

Reciproc, din (3.4) şi (2.2) rezultă:

PROPOZIŢIA 3.3. *Un spaţiu (L_n, D), $D-H$-coharmonic recurent este $D-H$-recurent, dacă şi numai dacă este $D-H$-Ricci-recurent cu acelaşi covector σ_r şi tensor de $D-H$-Ricci-recurenţă $\underset{1}{H}_{ij}$. Tensorul $\underset{1}{H}{}^i_{jkh}$ de $D-H$-recurenţă fiind dat de (3.9).*

Dacă $\omega = 0$, atunci $D = \Delta$ şi $\bar{Z}^i_{jkh} = Z^i_{jkh}$ şi se obţin rezultatele din [9].

Există un $\underset{2}{H}{}^i_{jkh}$ cu proprietatea (3.5) dat de

$$\underset{2}{H}{}^i_{jkh} = -\frac{n}{n-2}H^i_{jkh} \qquad (3.10)$$

§4. Fie $\bar{T}^i_{jkh}$, tensorul D-concircular de curbură (2.10).

DEFINIŢIA 3. Spaţiu. (L_n, D) este $D-H$ *concircular recurent,* dacă

$$\bar{T}^i_{jkh/r} = \sigma_r \bar{T}^i_{jkh} + \underset{3}{H}{}^i_{jkh}(\bar{R}_{/r} - \sigma_r \bar{R}) \qquad (4.1)$$

cu $\underset{3}{H} = 0$, unde $\underset{3}{H} = g^{ij}\underset{3}{H}_{ij}$; $\underset{3}{H}_{ij} = \underset{3}{H}{}^s_{ijs}$.

Din (4.1) rezultă

$$\bar{T}_{ij/r} = \sigma_r \bar{T}_{ij} + \underset{3}{H}_{ij}(\bar{R}_{/r} - \sigma_r \bar{R}) \qquad (4.2)$$

unde $\bar{T}_{ij} = \bar{T}^s_{ijs} = \bar{E}_{ij}$ (4.3) iar din (4.2) avem

$$\underset{3}{H}(\overline{R}_{/r} - \sigma_r \overline{R}) = 0$$

şi dacă $\underset{3}{H}$ ar fi diferit de zero, ar rezulta, [4], spaţiul (L_n, D), D-concircular recurent. Prin urmare condiţia $\underset{3}{H} = 0$ în (4.1) este esenţială.

Dacă presupunem spaţiul $D-H$-recurent, derivînd covariant (2.10) n raport cu D şi ţinînd seama de (2.1) obţinem:

$$\overline{T}^i_{jkh/r} = \sigma_r \overline{T}^i_{jkh} + (\underset{1}{H}^i_{jkh} - H^i_{jkh})(\overline{R}_{/r} - \sigma_r \overline{R}) \tag{4.4}$$

şi reciproc. Avem deci:

PROPOZIŢIA 4.1. *Orice spaţiu (L_n, D), $D-H$-recurent este şi $D-H$ concircular recurent şi reciproc. Intre tensorii de $D-H$-recurenţă şi $D-H$-concircular recurenţă avem relaţia*

$$\underset{3}{H}^i_{ikh} = H^i_{jkh} - \underset{1}{H}^i_{jkh} \tag{4.5}$$

COROLAR 4.1. *Condiţia necesară şi suficientă ca (L_n, D) să fie $D-H$-concircular recurent este ca (L_n, D) să fie $D-H$-recurent.*

COROLAR 4.2. *Condiţia necesară şi suficientă ca spaţiul (L_n, D) concircular recurent să aibă tensorul lui Einstein recurent, este ca $\underset{3}{H}_{ij} = 0$.*

§. 5 Pentru o conexiune semi-simetrică D în [10] se stabilesc transformările proiective de conexiune care au ca invariant tensorul

$$\overline{W}^s_{ijk} = \overline{R}^s_{ijk} - \frac{1}{n-1}(\delta^s_k \overline{R}_{ij} - \delta^s_j \overline{R}_{ik}) \tag{5.1}$$

analog cu tensorul proiectiv de curbură a lui Weyl, iar în [9] se stabilesc condiţiile în care $\overline{W}^s_{ijk}$ este egal cu tensorul proiectiv de curbură a lui Weyl pentru ∇.

DEFINIŢIA 4. Vom spune că spaţiul (L_n, D) este *$D-H$-proiectiv recurent,* dacă

$$\overline{W}^s_{ijk/r} = \sigma_r \overline{W}^s_{ijk} + \underset{4}{H}^s_{ijk}(\overline{R}_{/r} - \sigma_r \overline{R}) \tag{5.2}$$

cu $\underset{4}{H} = 0$, unde $\underset{4}{H} = g^{ij}\underset{4}{H}_{ij}$, $\underset{4}{H}_{ij} = g_{si}\underset{4}{H}^s_j$ iar $\underset{4}{H}^s_k = g^{ij}\underset{4}{H}^s_{ijk}$.

Dacă în (5.1) înmulţim contractat cu g^{ij} avem

$$\overline{W}^s_k = \frac{n}{n-1}\left(\overline{R}^s_k - \frac{\overline{R}}{n}\delta^s_k\right) \tag{5.3}$$

unde

$$\overline{W}^s_k = g^{ij}\overline{W}^s_{ijk} \tag{5.4}$$

$$\overline{W}_{jk} = g_{js}W^s_k = \frac{n}{n-1}\overline{E}_{jk} \tag{5.5}$$

Din (2.13) şi (5.1) rezultă

$$\overline{W}^s_{ijk/r} - \sigma_r \overline{W}^s_{ijk} = \overline{R}^s_{ijk/r} - \sigma_r \overline{R}^s_{ijk} - \tag{5.6}$$

$$- \frac{1}{n-1} [\delta^s_k (\overline{E}_{ij/r} - \sigma_r \overline{E}_{ij}) - \delta^s_j (\overline{E}_{ik/r} - \sigma_r \overline{E}_{ik})] - H^s_{ijk}(\overline{R}_{/r} - \sigma_r \overline{R})$$

iar folosind (4.2), (4.3), (4.4) şi (5.6) rezultă:

PROPOZIŢIA 5.1. *Orice spaţiu D—H-concircular recurent este D—H-proiectiv recurent cu*

$$\underset{4}{H^s_{ijk}} = \underset{3}{H^s_{ijk}} - \frac{1}{n-1} (\delta^s_k \underset{3}{H_{ij}} - \delta^s_j \underset{3}{H_{ik}}) \tag{5.7}$$

Reciproc, dacă în (5.2) înmulţim contractat cu g^{ij}, în baza lui (4.3) şi (5.5) avem:

PROPOZIŢIA 5.2. *Orice spaţiu (L_n, D) D—H-proiectiv recurent este şi D—H-concircular recurent cu*

$$\underset{3}{H^s_{ijk}} = \underset{4}{H^s_{ijk}} + \frac{1}{n} (\delta^s_k \underset{4}{H_{ij}} - \delta^s_j \underset{4}{H_{ik}}) \tag{5.8}$$

COROLAR 5.1. *Spaţiile (L_n, D) sînt în acelaşi timp, D—H-recurente, D—H-concircular recurente şi D—H-proiectiv recurente. Intre tensorii de D—H-recurenţă existînd relaţiile: (4.5), (5.7), (5.8).*

COROLAR 5.2. *Dacă în spaţiul (L_n, D) tensorul lui Einstein este D-recurent, atunci între tensorii de D—H-proiectiv recurenţă şi D—H concircular recurenţă avem*

$$\underset{4}{H^i_{jkh}} = \underset{3}{H^i_{jkh}} \tag{5.9}$$

În particular pentru $\omega = 0$, $D = \Delta$, obţinem rezultatul din [9], iar pentru $\underset{3}{H^i_{jkh}} = 0$, obţinem rezultatele din [4].

§ 6. Fie $\overline{C}^i_{jkh}$ tensorul D-conform de curbură.

DEFINIŢIA 5. Vom spune că spaţiul (L_n, D) este *D—H-conform recurent* $(n > 3)$, dacă

$$\overline{C}^s_{ijk/r} = \sigma_r \overline{C}^s_{ijk} + \underset{5}{H^s_{ijk}}(\overline{R}_{/r} - \sigma_r \overline{R}) \tag{6.1}$$

cu $\underset{5}{H_{ij}} = \underset{5}{H^s_{ijs}} = 0$.

Dacă D este mai generală decît o conexiune K. Yano şi tensorul D-concircular de curbură nu este nul $\overline{T}^i_{jkh} \neq 0$, între $\overline{C}^s_{ijk}$ şi $\overline{T}^s_{ijk}$ există o relaţie [11] analoagă cu cea dintre $\overline{C}^s_{ijk}$ şi T^s_{ijk} [3] şi anume:

$$C^s_{ijk} = \overline{T}^s_{ijk} + \frac{1}{n-2}(\overline{T}_{ik}\delta^s_j - \overline{T}_{ij}\delta^s_k + g_{ik}\overline{T}^s_j - g_{ij}\overline{T}^s_k) = C^s_{ijk} \tag{6.2}$$

unde $\overline{T}^s_j = g^{jk}\overline{T}_{kj}$. De unde

$$\overline{C}^s_{ijk} - \sigma_r\overline{C}^s_{ijk} = \overline{T}^s_{ijk/r} - \sigma_r\overline{T}^s_{ijk} + \frac{1}{n-2}[\delta^s_j(\overline{E}_{ik/r} - \sigma_r\overline{E}_{ik}) -$$

$$- \delta^s_k(\overline{E}_{ij/r} - \sigma_r\overline{E}_{ij}) + g_{ik}(\overline{E}^s_{j/r} - \sigma_r\overline{E}^s_j) - g_{ij}(\overline{E}^s_{k/r} - \sigma_r\overline{E}^s_k)] \tag{6.3}$$

Dacă spaţiul (L_n, D) este $D-H$-concircular recurent, din (4.1), (4.2), (4.3) şi (6.3) rezultă (6.1) cu

$$\underset{5}{H^s_{ijk}} = \underset{3}{H^s_{ijk}} + \frac{1}{n-2}(\delta^s_j \underset{3}{H_{ik}} - \delta^s_k \underset{3}{H_{ij}} + g_{ik}\underset{3}{H^s_j} - g_{ij}\underset{3}{H^s_k}) \tag{6.4}$$

De unde

PROPOZIŢIA 6.1. *Dacă spaţiul (L_n, D) este $D-H$ concircular recurent, atunci este $D-H$-conform recurent cu tensor de $D-H$ recurenţă dat de (6.4).*

COROLAR 6.1. *Orice spaţiu (L_n, D) $D-H$-concircular recurent cu tensorul lui Einstein D-recurent cu acelaşi covector σ, este în acelaşi timp $D-H$ conform recurent şi $D-H$ proiector recurent cu*

$$\underset{5}{H^s_{ijk}} = \underset{4}{H^s_{ijk}} = \underset{3}{H^s_{ijk}}$$

COROLAR 6.2. *Dacă spaţiul (L_n, D) este $D-H$-recurent, atunci din propoziţia 4.1 şi corolarul 6.1 rezultă că spaţiul (L_n, D) este $D-H$-conform recurent cu $\underset{5}{H^s_{ijk}}$ dat de (6.4) şi $\underset{3}{H^s_{ijk}}$ dat de (4.5).*

Din (3.2) rezultă

$$\overline{C}^s_{ijk/r} = \overline{Z}^s_{ijk/r} + \frac{n}{n-2} R_{/r}H^s_{ijk} \tag{6.6}$$

sau

$$\overline{C}^s_{ijk/r} - \sigma_r\overline{C}^s_{ijk} = \overline{Z}^s_{ijk/r} - \sigma_r\overline{Z}^s_{ijk} + \frac{n}{n-2} H^s_{ijk}(\overline{R}_{/r} - \sigma_r\overline{R}) \tag{6.7}$$

De unde

PROPOZIŢIA 6.2. *Condiţia necesară şi suficientă ca un spaţiu (L_n, D) să fie $D-H$-conform recurent ($n > 3$) este ca spaţiul (L_n, D) să fie $D-H$-coharmonic recurent cu*

$$\underset{5}{H^s_{ijk}} = \underset{2}{H^s_{ijk}} + \frac{n}{n-2} H^s_{ijk} \tag{6.8}$$

COROLAR 6.3. *Un spaţiu $D-H$ conform recurent ($n > 3$) este D-recurent, dacă şi numai dacă este $D-H$-Ricci recurent cu acelaşi covector*

COROLAR 6.4. *Condiţia necesară şi suficientă ca spaţiul (L_n, D) fie D-conform recurent este ca (L_n, D) să fie $D-H$ coharmonic recurent $\underset{2}{H}{}^s_{ijk}$ dat de (3.10),* [4].

COROLAR 6.5. *Dacă (L_n, D) este $D-H$-concircular recurent, atunci* (*$D-H$ coharmonic recurent cu*

$$\underset{2}{H}{}^s_{ijk} = \underset{5}{H}{}^s_{ijk} - \frac{n}{n-2} H^s_{ijk} \quad (6$$

unde

$\underset{5}{H}{}^s_{ijk}$ este dat de (6.4).

Observaţia 4. Relaţiile între tensorii de $D-H$-recurenţă (3.9), (4. (5.7), (4.8), (6.4), (6.8) sînt analoage cu relaţiile ce există între tens(pentru care s-a definit $D-H$-recurenţa.

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ON SOME CLASSES OF DIFFERENTIAL SUBORDINATIONS (II)

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ABSTRACT. — The study on differential subordinations of the form $\psi(p(z), zp'(z)) \prec h(z)$, begun in [1] and [2] is further carried in the present paper for the case in which $\psi(p(z), zp'(z)) = \alpha(zp'(z)) + \beta(zp'(z))\gamma(p(z))$ by employing the admissible functions method of [2], obtaining generalizations of the results in [6], some consequences and examples being then presented.

Introduction. Let f and g be analytic in the unit disk U and let $H(U)$ be the space of functions analytic in U. We say that f is subordinate to g ($f \prec g$ or $f(z) \prec g(z)$) if g is univalent in U, $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

If $\psi : C^2 \to C$ is analytic in a domain D, if h is univalent in U and if p is analytic in U with $(p(z), zp'(z)) \in D$ when $z \in U$, then p is said to satisfy the first-order differential subordination

$$\psi(p(z), zp'(z)) \prec h(z), \; z \in U.$$

In [1] the authors determine conditions on ψ and h so that $p(z) \prec h(z)$ in the case

$$\psi(p(z), zp'(z)) = \theta(p(z)) + zp'(z)\Phi(p(z))$$

and they give applications of these results in univalent function theory.

În [3] the author study the differential subordination in the case

$$\psi(p(z), zp'(z)) = \alpha(p(z)) + \beta(p(z))\gamma(zp'(z))$$

and applications of these results are given.

In this paper we shall study the differential subordination when

$$\psi(p(z)), zp'(z)) = \alpha(zp'(z)) + \beta(zp'(z))\gamma(p(z))$$

and we give some particular interesting cases.

Preliminaries. We will need the next two lemmas to prove our theorem.

LEMMA 1. [4] *Let $g \in H(U)$, with $g(0) = 0$, be univalent and starlike in U. If $f \in H(U)$ and $Re[zf'(z)/g(z)] > 0$, $z \in U$, then f is univalent in U.*

We said that $L : U \times [0, +\infty) \to C$ is a subordination (or Loewner) chain if $L(\cdot, t)$ is analytic and univalent in U for all $t \geqslant 0$, $L(z, \cdot)$ is continuosly differentiable on $[0, +\infty)$ for all $z \in U$ and $L(z, s) \prec L(z, t)$ when $0 \leqslant s < t$.

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LEMMA 2. [5, p. 159] *The function* $L(z, t) = a_1(t)z + \ldots$ *with* $a_1(t) \neq 0$ *for all* $t \geqslant 0$ *is a subordination chain if and only if*

$$\mathrm{Re}\left[z\frac{\partial L}{\partial z}\Big/\frac{\partial L}{\partial t}\right] > 0$$

for all $z \in U$ *and* $t \geqslant 0$.

THEOREM A. [2] *Let* $h, q \in H(U)$ *be univalent in* U *and suppose* $q \in H(\bar{U})$. *If* $\psi : C^3 \to C$ *satisfies:*

a) ψ *is analytic in a domain* $D \subset C^3$,

b) $(q(0), 0, 0) \in D$ and $\psi(q(0), 0, 0) \in h(U)$,

c) $\psi(r, s, t) \notin h(U)$ *when* $(r, s, t) \in D$, $r = q(\zeta)$, $s = m\zeta q'(\zeta)$, $\mathrm{Re}(1 + t/s) \geqslant m\ \mathrm{Re}\ (1 + \zeta q''(\zeta)/q'(\zeta))$ *where* $|\zeta| = 1$, $m \geqslant 1$,

then for all $p \in (H(U)$ *so that* $(p(z), zp'(z), z^2p''(z)) \in D$, *when* $z \in U$ *we have:*

$$\psi(p(z), zp'(z), z^2p''(z)) \prec h(z) \quad \textit{implies that} \quad p(z) \prec q(z).$$

MAIN. RESULTS. THEOREM. *Let* q *be convex (univalent) in* U, *and let* α, β *be analytic in* C *and* γ *analytic in a domain* $D \supset q(U)$. *Suppose that*

(i)
$$\mathrm{Re}\ \frac{\beta((1+t)zq'(z))\gamma'(q(z))}{\alpha'((1+t)zq'(z)) + \beta'((1+t)zq'(z))\gamma(q(z))} \geqslant 0$$

for all $z \in U$ *and* $t \geqslant 0$

(ii) $Q(z) = zq'(z)(\alpha'(zq'(z)) + \beta'(zq'(z))\gamma(q(z)))$ *is starlike (univalent) in* U.

If p *is analytic in* U *with* $p(0) = q(0)$, $p(U) \subset D$ *and* $\alpha(zp'(z)) + \beta(zp'(z))\gamma(p(z)) \prec \alpha(zq'(z)) + \beta(zq'(z))\gamma(q(z))$ *then* $p(z) \prec q(z)$.

Proof. Without loss of generality we can assume that p and q satisfy the conditions of the theorem on the closed disk $\bar{U}$; if not, then we can replace $p(z)$ by $p_r(z) = p(rz)$ and $q(z)$ by $q_r(z) = q(rz)$ where $0 < r < 1$. The new functions satisfy the conditions of the theorem on $\bar{U}$ and we would then prove that $p_r(z) \prec q_r(z)$, for all $0 < r < 1$. By letting $r \uparrow 1^-$ we obtain $p(z) \prec q(z)$.

The function

$$L(z, t) = \alpha((1+t)zq')z)) + \beta((1+t)zq'(z))\gamma(q(z))$$

is continuously differentiable on $[0, +\infty)$ for all $z \in U$ and analytic in U for all $t \geqslant 0$. Because $q'(0) \neq 0$, $Q'(0) \neq 0$ from (i), for $z = 0$ we deduce that

$$\frac{\partial L}{\partial z}(0, t) = q'(0)(\alpha'(0) + \beta'(0)\gamma(q(0)))\left(1 + t + \frac{\beta(0)\gamma'(q(0))}{\alpha'(0) + \beta'(0)\gamma(q(0))}\right)$$

and

$$\frac{\partial L}{\partial z}(0, t) \neq 0 \text{ for all } t \geqslant 0.$$

Because q is convex in U, a simple calculation combined with (i) yields

$$\operatorname{Re}\left[z\frac{\partial L}{\partial z}\Big/\frac{\partial L}{\partial t}\right] > 0 \text{ for all } z \in U \text{ and } t \geqslant 0$$

hence by Lemma 2, $L(z, t)$ is a subordination chain.

If we let $h(z) = L(z, 0) = \alpha(zq'(z)) + \beta(zq'(z))\gamma(q(z))$ and using (i) for $t = 0$ we obtain $\operatorname{Re}[zh'(z)/Q(z)] > 0$ for all $z \in U$, hence by Lemma 1, h is univalent in U.

Let $\psi(r, s) = \alpha(s) + \beta(s)\gamma(r)$ analytic in the domain $E = D \times C$; then $(q(0), 0) \in E$, $\psi(q(0), 0) = h(0) \in h(U)$ and because $L(z, t)$ is a subordination chain we have

$$\alpha((1+t)\zeta q'(\zeta)) + \beta((1+t)\zeta q'(\zeta))\gamma(q(\zeta)) \notin h(U)$$

for $t \geqslant 0$ and $|\zeta| = 1$. Using Theorem A we conclude that $p(z) \prec q(z)$.

This theorem give us some particular cases presented in the next corollaries.

If we take $\gamma(w) = 1$, $w = C$ then from Theorem we obtain:

COROLLARY 1. *Let q be convex (univalent) in U, α and β be analytic in C and suppose that*
$Q(z) = zq'(z)(\alpha'(zq'(z)) + \beta'(zq'(z)))$ is starlike (univalent) in U. If p is analytic in U with $p(0) = q(0)$, then
$\alpha(zp'(z)) + \beta(zp'(z)) \prec \alpha(zq'(z)) + \beta(zq'(z))$ implies that $p(z) \prec q(z)$.

If we take $\alpha(w) = w$, $\beta(w) = aw^2$, $w \in C$ then from Corollary 1 we obtain:

Example 1.1. Let q be convex (univalent) in U, $a \in C$ and suppose that $Q(z) = zq'(z)(1 + 2azq'(z))$ is starlike (univalent) in U. If p is analytic in U with $p(0) = q(0)$, then
$zp'(z) + a(zp'(z))^2 \prec zq'(z) + a(zq'(z))^2$ implies that $p(z) \prec q(z)$.

If we take in this example $a = 0$ we obtain the well-known result of T. J. Suffridge [6].

This example give us some interesting particular cases if we replace qq by simple convex functions.

Example 1.1.1. If $a, \lambda \in C$ so that $|a\lambda| \leqslant 1/4$ and p is analytic in U with $p(0) = 0$, then
$zp'(z) + a(zp'(z))^2 \prec \lambda z + a(\lambda z)^2$ implies that $p(z) \prec \lambda z$.

Proof. If we take in Example 1.1., $q(z) = \lambda z$, $\lambda \in C$, $z \in U$ we obtain that

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)} = \operatorname{Re}\frac{1 + 4a\lambda z}{1 + 2a\lambda z} > \frac{1 - 2|2a\lambda z|}{1 - |2a\lambda z|} \geqslant 0$$

when $|2a\lambda z| \leqslant 1/2$, $z \in U$ and this last inequality is equivalent with $|a\lambda| \leqslant 1/4$.

Example 1.1.2. Let a, $\lambda \in C$ so that $|\lambda| = r_0$ where $r_0 \in (0, 1)$ the root of the equation

$$1 - r - \rho r e^r(3 + 3r + 2\rho r^2 e^r) = 0, \quad \rho = 2|a|.$$

If p is analytic in U, $p(0) = 1$ then

$$zp'(z) + a(zp'(z))^2 \prec \lambda z e^{\lambda z} + a(\lambda z e^{\lambda z})^2 \text{ implies that } p(z) \prec e^{\lambda z}.$$

Proof. We can easily prove that $q(z) = e^{\lambda z}$ is convex in U wh $|\lambda| \leqslant 1$. By letting $\lambda z = \zeta = re^{it}$, $0 \leqslant r < 1$ and $c = 2a = \rho e^{i\varphi}$, $\rho = 2$ a simple calculation yields

$$\text{Re}\,\frac{zQ'(z)}{Q(z)} \geqslant \frac{1}{|1+c\zeta e^{\zeta}|^2\zeta}(1 - r - \rho r e^r(3 + 3r + 2\rho r^2 e^r))$$

and
$$|1 + c\zeta e^{\zeta}|^2 \geqslant (1 - \rho r e^{r \cdot \cos t})^2.$$

Let $\varphi : [0, 1] \to R$, $\varphi(r) = 1 - r - \rho r e^r(3 + 3r + 2\rho r^2 e^r)$. Since $\varphi'(r)$ < 0, $\varphi(0) = 1$, $\varphi(1) = -2\rho e(3 + \rho e) < 0$, we conclude that there exi $r_0 \in (0, 1)$ such that $\varphi(r_0) = 0$. Moreover r_0 is the only root of the fu tion φ and for all $r \in [0, r_0)$ we have $\varphi(r) > 0$.

Let $\psi : [0, 1] \to R$, $\psi(r) = 1 - \rho r e^r$. Because $\psi'(r) < 0$, $\psi(0) = \varphi(0) =$ and $\varphi(r) = \psi(r) - r(1 + \rho e^r(2 + 3r + 2\rho r^2 e^r))$ then $\varphi(r) \leqslant \psi(r)$ for $r \in [0, 1]$, we obtain that $\psi(r) > 0$ for all $r \in [0, r_0]$, hence $|1 + c\zeta e^{\zeta}|^2 \geqslant (1 - \rho r e^r)^2 > 0$ and $\text{Re}\,\frac{zQ'(z)}{Q(z)} > 0$ for all $z \in U$ when $|\lambda| \leqslant$

Example. 1.1.3. Let a, $\lambda \in C$ so that $|\lambda| \leqslant \min\{r_0, r_1\}$ whe $r_0 = \min\{|r| : r^2 + 2(1 + a)r + 1 = 0\}$ and

$$r_1 = \min\{r;\ r > 0,\ r^6 + 2(3|a| - 2)r^5 + (8|a|^2 - 12|a| + 1)r^4 -$$
$$- (8|a|^2 - 16|a| + 1)r^2 + 2(3|a| + 2)r - 1 = 0\}.$$

If p is analytic in U with $p(0) = 0$, then

$$zp'(z) + a(zp'(z))^2 \prec \frac{\lambda z}{(1 + \lambda z)^2} + a\left(\frac{\lambda z}{(1 + \lambda z)^2}\right)^2 \text{ implies that } p(z) \prec \frac{\lambda z}{1 + \lambda z}$$

Proof. We can easily prove that $q(z) = \frac{\lambda z}{1 + \lambda z}$ is convex in U wh $|\lambda| \leqslant 1$. By letting $\lambda z = \zeta = re^{it}$, $0 \leqslant r < 1$ and $c = 2a = \rho e^{i\varphi}$, we obt

$$\text{Re}\,\frac{zQ'(z)}{Q(z)} \geqslant \frac{-r^6 - (3\rho - 4)r^5 - (2\rho^2 - 6\rho + 1)r^4 + (2\rho^2 - 8\rho + 1)r^2 - (3\rho + 4)r + 1}{|\zeta^3 + (c + 3)\zeta^2 + (c + 3)\zeta + 1|^2}$$

Let $\varphi : [0, 1] \to R$, $\varphi(r) = -r^6 - (3\rho - 4)r^5 - (2\rho^2 - 6\rho + 1)r^4$ $+ (2\rho^2 - 8\rho + 1)r^2 - (3\rho + 4)r + 1$; because $\varphi(0) = 1$, $\varphi(1) = -8\rho <$ there exists $r' \in (0, 1)$ so that $\varphi(r') = 0$; hence $\varphi(r) > 0$ for all $r \in [0,$ where r_1 is the smallest positive root of the equation $\varphi(r) = 0$. A sim calculation yields

$\zeta^3 + (c + 3)\zeta^2 + (c + 3)\zeta + 1 \neq 0$, for all $z \in U$, when $|\lambda|$ $\leqslant \min\{1, r_0\}$, hence $\text{Re}\,\frac{zQ'(z)}{Q(z)} > 0$ for all $z \in U$ when $|\lambda| \leqslant \min\{r_0, r$

Remarks. From the proof of Example 1.1.3. we observed that this result is not sharp; better upper bounds may be found in the case when $a \in R$.

Case 1. Let $0 < a < 1$ and $\lambda \in C$ with

$$|\lambda| \leqslant \min\{1 + a - \sqrt{a^2 + 2a}\,,\ r_*\} \text{ where}$$

$$r^* = \min\{r : r > 0,\ r^4 - 2(3a + 2)r^3 + 2(4a^2 + 8a + 1)r^2 - 2(3a + 2)r - 1 = 0\}.$$

If p is analytic in U, $p(0) = 0$, then

$$zp'(z) + a(zp'(z))^2 \prec \frac{\lambda z}{(1 + \lambda z)} + a\left(\frac{\lambda z}{(1 + \lambda z)^2}\right)^2 \text{ implies that } p(z) \prec \frac{\lambda z}{1 + \lambda z}.$$

Proof. In the case $0 < a < 1$ we deduce that

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)} \geqslant \frac{r^2 - 1}{|\zeta^3 + (c + 3)\zeta^2 + (c + 3)\zeta + 1^2|}(-r^4 + 2(3a + 2)r^3 - 2(4a^2 + 8a + 1)r^2 + 2(3a + 2)r + 1)$$

where $\zeta = \lambda z = re^{it}$, $0 \leqslant r < 1$ and the right-hand term is defined for all $z \in U$ when $|\lambda| \leqslant r_0 = 1 + a - \sqrt{a^2 + 2a} \in (0, 1)$. If we let $\psi : [0,1] \to R$

$\psi(r) = -r^4 + 2(3a + 2)r^3 - 2(4a^2 + 8a + 1)r^2 + 2(3a + 2)r - 1$ we have

$\psi(0) = -1$, hence $\operatorname{Re}\dfrac{zQ'(z)}{Q(z)} > 0$ for all $z \in U$ if $|\lambda| \leqslant \min\{r_0, r_*\}$.

Case 2. Let $a \geqslant 1$ and $\lambda \in C$ with

$$|\lambda| \leqslant \min\{1 + a - \sqrt{a^2 + 2a}\,;\ r_*\} \text{ where}$$

$$r_* = \min\{r : r > 0,\ r^4 - 2(3a + 2)r^3 + 2(4a^2 + 6a + 3)r^2 - 2(3a + 2)r + 1 = 0\}.$$

If p is analytic in U, $p(0) = 0$, then

$$zp'(z) + a(zp'(z))^2 \prec \frac{\lambda z}{(1 + \lambda z)^2} + a\left(\frac{\lambda z}{(1 + \lambda z)^2}\right)^2 \text{ implies that } p(z) \prec \frac{\lambda z}{1 + \lambda z}.$$

Proof In the case $a \geqslant 1$ we deduce by using the proof of Example 1.1.3., that

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)} \geqslant \frac{r^2 - 1}{|\zeta^3 + (c + 3)\zeta^2 + (c + 3)\zeta + 1|^2}(-r^4 + 2(3a + 2)r^3 - 2(4a^2 + 6a + 3)r^2 - 2(3a + 2)r - 1),$$

where $\zeta = \lambda z = re^{it}$, $0 \leqslant r < 1$ and as in the Case 1 we deduce the above result.

When $a = 1$ we can easily show the following result.

Case 2'. If $|\lambda| \leqslant 3 - 2\sqrt{2}$ and p is analytic in U, $p(0) = 0$, then

$$zp'(z) + (zp'(z))^2 \prec \frac{\lambda z}{(1+\lambda z)^2} + \left(\frac{\lambda z}{(1+\lambda z)^2}\right)^2$$

implies that $p(z) \prec \frac{\lambda z}{1+\lambda z}$.

Case **3**. Let $-2/3 < a < 0$ and $\lambda \in C$ with $|\lambda| \leqslant \min\{1, r_0, r_*\}$ where $r_0 = \min\{|r| : r^2 + 2(1+a)r + 1 = 0\}$ and

$r^* = \min\{|r| : r^4 - 2(3a+2)r^3 + 2(4a^2 + 8a + 1)r^2 - 2(3a+2)r + 1 = 0\}$.

If p is analytic in U, $p(0) = 0$, then

$$zp'(z) + a(zp'(z))^2 \prec \frac{\lambda z}{(1+\lambda z)^2} + a\left(\frac{\lambda z}{(1+\lambda z)^2}\right)^2$$

implies that $p(z) \prec \frac{\lambda z}{1+\lambda z}$.

Proof. If $-2/3 < a < 0$ we can easily show that

$$\mathrm{Re}\frac{zQ'(z)}{Q(z)} \geqslant \frac{r^2-1}{|\zeta^3 + (c+3)\zeta^2 + (c+3)\zeta + 1|^2}(-r^4 + 2(3a+2)r^3 - $$
$$- 2(4a^2 + 8a + 1)r^2 + 2(3a+2)r - 1)$$

where $\zeta = \lambda z = re^{it}$, $0 \leqslant r < 1$. We have $\zeta^3 + (c+3)\zeta^2 + (c+3)\zeta + 1 \neq 0$ for all $z \in U$ when $|\lambda| \leqslant \min\{1, r_0\}$ and the right-hand term is positive when $|\lambda| \leqslant \min\{1, r_0, r_*\}$.

Case **4**. Let $a \leqslant -2/3$ and $\lambda \in C$ with $|\lambda| \leqslant \min\{1, r_0, r_*\}$ where $r_0 = \min\{|r| : r^2 + 2(1+a)r + 1 = 0\}$ and

$r_* = \min\{|r| : r^4 + 2(3a+2)r^3 + 2(4a^2 + 8a + 1)r^2 + 2(3a+2)r + 1 = 0\}$

If p is analytic in U, $p(0) = 0$, then

$$zp'(z) + a(zp'(z))^2 \prec \frac{\lambda z}{(1+\lambda z)^2} + a\left(\frac{\lambda z}{(1+\lambda z)^2}\right)^2 \text{ implies that } p(z) \prec \frac{\lambda z}{1+\lambda z}.$$

Proof. As in the Case 3, we can show that if $a \leqslant -2/3$

$$\mathrm{Re}\frac{zQ'(z)}{Q(z)} \geqslant \frac{r^2-1}{|\zeta^3 + (c+3)\zeta^2 + (c+3)\zeta + 1|^2}(-r^4 - 2(3a+2)r^3 - $$
$$- 2(4a^2 + 8a + 1)r^2 - 2(3a+2)r - 1)$$

where $\zeta = \lambda z = re^{it}$, $0 \leqslant r < 1$ and $c = 2a$ and the right-hand term is positive for all $z \in U$ when $|\lambda| \leqslant \min\{1, r_0, r_*\}$.

When $a = -2/3$ we can easily deduce the following result:
Case 4'. If $|\lambda| \leqslant (4 - \sqrt{7})/3$ and p is analytic in U, $p(0) = 0$ then

$$zp'(z) - \frac{2}{3}(zp'(z))^2 \prec \frac{\lambda z}{(1+\lambda z)^2} - \frac{2}{3}\left(\frac{\lambda z}{(1+\lambda z)^2}\right)^2$$

imlpies that $p(z) \prec \frac{\lambda z}{1+\lambda z}$.

Example 1.1.4. Let $a, \lambda \in C$ with $|\lambda| \leqslant r_0$, for $a = -1/2$ and $|\lambda| \leqslant \min\left\{r_0, \frac{1}{|2a+1|}\right\}$ for $a \neq -1/2$, where

$$r_0 = \min \{r : r > 0, -(8|a|^2 + 6|a| + 1)r^3 + (8|a|^2 - 12|a| + 1)r^2 - - 3(2|a| + 1)r + 1 = 0\}.$$

If p is analytic in U, $p(0) = \log 1 = 0$, then

$$zp'(z) + a(zp'(z))^2 \prec \frac{\lambda z}{1+\lambda z} + a\left(\frac{\lambda z}{1+\lambda z}\right)^2 \text{ implies that } p(z) \prec \log(1+\lambda z).$$

Proof. If $|\lambda| \leqslant 1$ the function $q(z) = \log(1 + \lambda z)$, $\log 1 = 0$ is convex (univalent) in U; if we let $c = 2a = \rho e^{i\varphi}$ and $\lambda z = \zeta = re^{it}$, $0 \leqslant r < 1$, by using Example 1.1. we deduce

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} \geqslant \frac{(-2\rho^2 - 3\rho - 1)r^3 + (2\rho^2 - 6\rho + 1)r^2 - (3\rho + 3)r + 1}{|1+\zeta|^2 |1 + \zeta + c\zeta|^2}.$$

The right-hand term is defined and positive when $|\lambda| \leqslant r_0$ in the case $a = -1/2$ and for $|\lambda| \leqslant \min\left\{r_0, \frac{1}{|2a+1|}\right\}$ in the case $a \neq -1/2$ for all $z \in U$, and using Example 1.1. we obtain the above result.

Example 1.2. Let q be convex (univalent) in U and $a \in C\setminus\{-1\}$ so that $Q(z) = zq'(z)(1 + ae^{zq'(z)})$ is stalike in U. If p is analytic in U, $p(0) = q(0)$, then

$$zp'(z) + ae^{zq'(z)} \prec zq'(z) + ae^{zq'(z)} \text{ implies that } p(z) \prec q(z).$$

Proof. If we take, in Corollary 1, $\alpha(w) = w$ and $\beta(w) = ae^{w}$, $w \in C$, then we obtain the above result.

Example 1.2.1. Let $a \in C\setminus\{-1\}$ and $\lambda \in C$ so that $|\lambda| \leqslant \min\{r_0, r^*\}$ where $r_0 = \min\{|r| : 1 + ae^{r} = 0\}$ and

$$r_* = \min\{r : r > 0, 1 - 2|a|e^{r} - |a|re^{r} - |a|^2re^{2r} + |a|^2e^{-2r} = 0\}.$$

If p is analytic in U, $p(0) = 0$, then

$$zp'(z) + ae^{zp'(z)} \prec \lambda z + ae^{\lambda z} \text{ implies that } p(z) \prec \lambda z.$$

Proof. We use Example 1.2. in the case $q(z) = \lambda z$, $z \in U$. The funct $Q(z) = \lambda z(1 + ae^{\lambda z})$ is starlike in U if

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} = \operatorname{Re} \frac{a^{-1} + e\zeta + \zeta e\zeta}{a^{-1} + e\zeta} > \frac{\rho^2 - 2\rho e^r - \rho re^r - re^{2r} + e^{-2r}}{|a^{-1} + e\zeta|^2} \geqslant 0,$$

where $a^{-1} = \rho e^{i\varphi}$ and $\zeta = \lambda z = re^{it}$, $r \geqslant 0$. We can easily show that inequality is satisfied under the conditions of the example.

If we take, in Corollary 1, $\alpha(w) = w$ and $\beta(w) = aw^n$, $w \in C$ obtain:

Example 1.3. Let q be convex (univalent) in U, $a \in C$, $n \in N^*$ suppose that $Q(z) = zq'(z)(1 + an(zq'(z))^{n-1})$ is starlike in U. If p is an tic in U, $p(0) = q(0)$, then

$$zp'(z) + a(zp'(z))^n \prec zq'(z) + a(zq'(z))^n \text{ implies that } p(z) \prec q(z).$$

Remark. If $n = 1$ or $a = 0$, this example yields the well-known r of T. J. Suffridge [6], and for $n = 2$ we obtain the Example 1.1.

Example 1.3.1. Let $a \in C$ and $\lambda \in C$ with $|\lambda| \leqslant (n^2|a|)^{\frac{1}{1-n}}$, is analytic in U, $p(0) = 0$, then

$$zp'(z) + a(zp'(z))^n \prec \lambda z + a(\lambda z)^n \text{ implies that } p(z) \prec \lambda z.$$

Proof. If we let $\zeta = \lambda z = re^{it}$ and $a = \rho e^{i\varphi}$ we obtain, for $q(z) =$ $z \in U$ that

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} \geqslant \frac{n^3\rho^2r^{2(n-1)} - n(n+1)\rho r^{n-1} + 1}{|1 + na\zeta^{n-1}|^2}$$

and if $|\lambda| \leqslant (n^2|a|)^{\frac{1}{n-1}}$ we can prove that the right-hand term is posi for all $z \in U$.

Remark. For $a = 0$ this result holds for all $\lambda \in C$, and for n we obtain the Example 1.1.1.

Example 1.3.2. Let $a \in C$ and $\lambda \in C$ so that $|\lambda| \leqslant r_0$ where r $\in (0, 1]$ is the root of equation

$$1 - r - n|a|r^{n-1}e^{(n-1)r}((n+1)(r+1) + n^2|a|r^ne^{(n-1)r}) = 0.$$

If p is analytic in U, $p(0) = 1$, then

$$zp'(z) + a(zp'(z))^n \prec \lambda ze^{\lambda z} + a(\lambda ze^{\lambda z})^n \text{ implies that } p(z) \prec e^{\lambda z}.$$

Proof. The function $q(z) = e^{\lambda z}$, $|\lambda| \leqslant 1$ is convex (univalent) in If we let $2a = \rho e^{i\varphi}$ and $\zeta = \lambda z = re^{it}$, $0 \leqslant r < 1$, then

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} \geqslant \frac{\varphi(r)}{|1 + na\zeta^{n-1}e^{(n-1)\zeta}|^2} \text{ where}$$

$$\varphi(r) = 1 - r - \frac{n}{2}\rho r^{n-1}e^{(n-1)r}((n+1) + (n+1)r + \frac{n^2}{2}\rho r^ne^{(n-1)r}).$$

A simple calculation yields

$$|1 + na\zeta^{n-1}e^{(n-1)\zeta}|^2 \geqslant \left(1 - \frac{n}{2}\rho r^{n-1}e^{(n-1)r\cdot\cos}\right)^2 =: \theta(r)$$

and if we let $\psi(r) = 1 - \frac{n}{2}\rho r^{n-1}e^{(n-1)r}$, then $\varphi(r) \leqslant \psi(r)$ for all $0 \leqslant r < 1$. Because $\varphi'(r) < 0$, $0 \leqslant r < 1$, $\varphi(1) \leqslant 0$ and $\varphi(0) = 1 > 0$, there exists $r_0 \in (0, 1]$ so that $\varphi(r_0) = 0$ and for all $r \in [0, r_0)$ we have

$$\psi(r) \geqslant \varphi(r) > 0. \text{ If } r \in [0, r_0) \text{ then } \theta(r) \geqslant \psi^2(r) > 0$$

and by using Example 1.3. we obtain the above result.

Remark. For $n = 2$ we obtain the Example 1.1.1. and for $a = 0$ this result holds for all $\lambda \in C$.

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A FIXED POINT THEOREM FOR DECREASING FUNCTIONS

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It is well-known that an increasing functiont on a complete lattice has at least a fixed point (see [1]). An analogous result for decreasing functions does not hold. Indeed, if B is a complete boolean lattice and $f: B \to B$ where $f(x) = \bar{x}$ is the complement of x for every $x \in B$, f is a decreasing funciton without fixed points. Consequently, we must give other conditions for the lattice or (and) for the function to assure the existence of a fixed point. The aim of this note is to give sufficient conditions for a decreasing function $f: L \to L$ where L is a chain to have a fixed point.

In what follows, L will be a chain and $f: L \to L$ will be a decreasing function. The fixed point set of f will be denoted by F_f. If $a, b \in L$ and $a \leqslant b$, we will denote the set $\{x \in L \mid a < x < b\}$ by (a, b). The chain L is dense if $(a, b) \neq \Phi$ for every $a, b \in L$ with $a < b$. We also consider the two subsets of L : $D = \{x \in L \mid x \leqslant f(x)\}$ and $I = \{x \in L \mid x \geqslant f(x)\}$. We have obviously $F_f = D \cap I$ and $L = D \cup I$.

LEMMA 1. *Let L be a chain and $f: L \to L$ a decreasing function. Then*

(i) *f has at most one fixed point.*

(ii) *$x \leqslant y$ for every $x \in D$, $y \in I$.*

(iii) *$f(D) \subseteq I$ and $f(I) \subseteq D$.*

Proof. (i) Let x, y be fixed points, say $x \leqslant y$. Then $f(y) \leqslant f(x)$, that is $y \leqslant x$, hence $x = y$.

(ii) Suppose $x > y$ for some $x \in D$ and $y \in I$. Then $f(x) \leqslant f(y)$ and $x \leqslant f(x)$, $f(y) \leqslant y$, hence $x \leqslant y$ by transitivity; but this contradicts the hypothesis.

(iii) Obvious.

Remark. If, moreover, f is surjective, then $f(D) = I$ and $f(I) = D$.

Proof. To show, e.g., that $f(D) = I$, we take $x \in I$ and prove that $x \in f(D)$. But $x = f(y)$ for some $y \in L$. If $y \in D$ then $x \in f(D)$. If $y \in I$, then $x \in f(I) \subseteq D$, hence $x \in D \cap I$ is a fixed point, therefore $x = f(x) \in f(D)$.

LEMMA 2. *Let L be a complete chain and $f: L \to L$ a decreasing surjective function. Set $a = \sup D$ and $b = \inf I$. Then*

(i) *$a \leqslant b$ and $(a, b) \neq \Phi$.*

(ii) *$a = f(b) \in D$ and $b = f(a) \in I$.*

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Proof. (i) $a \leqslant b$ follows from Lemma 1, (ii). Now suppose $a < c < b$. If $c \in D$ then $a \neq \sup D$ and if $c \in I$ then $b \neq \inf I$, contradiction.

(ii) From $b \leqslant x$ for every $x \in I$, it follows that $f(x) \leqslant f(b)$ for every $x \in I$, hence $y \leqslant f(b)$ for every $y \in D$ because $f(I) = D$. As $a = \sup D$ it follows that $a \leqslant f(b)$ and similarly $f(a) \leqslant b$. On the other hand $a \leqslant b$ implies $f(b) \leqslant f(a)$, therefore $a \leqslant f(b) \leqslant f(a) \leqslant b$, which shows that $a \in D$ and $b \in I$. Moreover, $f(b) \in D$ and since $a = \sup D$ it follows that $a = f(b)$ and similarly $b = f(a)$.

We can now state the main result of this note:

THEOREM. *Let L be a complete chain and $f: L \to L$ a decreasing surjective function. Set $a = \sup D$ and $b = \inf I$. Then f has a fixed point if and only if $a = b$, in which case the fixed point is $a = b$.*

Proof. If $a = b$ then $a = f(b) = f(a)$ by Lemma 2, (ii). Conversely, if c is a fixed point then $c \in D \cap I$, hence $c \leqslant a \leqslant b \leqslant c$, therefore $a = b = c$.

COROLLARY. *Let L be a complete dense chain and $f: L \to L$ a decreasing surjective function. Then f has an unique fixed point.*

Proof. This follows immediately from Lemma 2 (i) and the Theorem: $a = b$ by the density assumption and Lemma 2 (i), so $a = b$ is the unique fixed point of f by the Theorem.

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A GENERALIZATION OF A COINCIDENCE THEOREM OF HADŽIĆ

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ABSTRACT. — The purpose of this note is to generalize a coincidence theorem of Hadzic in [1] and to show that the remark in [2] about the mentioned theorem is not true.

1. In the sequel we shall use the following notations. For a metric space (X, d), $CB(Y)$ $(Cl(Y))$ stands for the family of all nonempty closed bounded (closed, resp.) of $Y \subset X$, $d(x, Y)$ — the nearest distance from a point x to a set Y, $H(Y, Z)$ — the Hausdorff distance between two sets Y and Z, N — the set of all natural numbers.

In [1] Hadžić has proved the following

THEOREM H. *Let X be a complete metric space, S and T continuous mappings from X into itself, A a closed mapping from X into $CB(SX \cap TX)$ such that $ATx = TAx$, $ASx = SAx$ for every $x \in X$ and*

$H(Ax, Ay) \leqslant q\, d(Sx, Ty)$ for every $x, y \in X$, where $q \in (0, 1)$. Then there exists a sequence $\{x_n\}$ such that

1) *For every $n \in N$, $Sx_{2n+1} \in Ax_{2n}$, $Tx_{2n} \in Ax_{2n-1}$,*

2) *There exists $z = \lim Tx_{2n} = \lim Sx_{2n+1}$,*

3) *$Tz \in Az$, $Sz \in Az$.*

Theorem H can be generalized as follows

THEOREM 1. *Let X be a complete metric space, S, T continuous mappings from X into itself, A, B, closed mappings from X into $Cl(X)$. Suppose that*

(i) $A(X) \subset T(X)$, $B(X) \subset S(X)$, $SA = AS$, $TB = BT$,

(ii) *There is an upper semicontinuous from the right function $q: [0, \infty) \to [0, 1)$ such that*

$$H(Ax, By) \leqslant q(d(Sx, Ty)) \cdot \max\left\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2}[d(Sx, By) + d(Ty, Ax)]\right\}$$

for every $x, y \in X$.

Then there exists a $z \in X$ such that $Sz \in Az$, $Tz \in Bz$.

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PROOF. Take $x_0 \in X$ and put $y_0 = Sx_0$ then fix $r > d(y_0, Ax_0)$ and choose $y_1 \in Ax_0$ so that $d(y_0, y_1) < r \cdot By$ (i), there is an $x_1 \in X$ with $y_1 = Tx_1$. From (ii) we have

$$d(y_1, Bx_1) \leqslant H(Ax_0, Bx_1) \leqslant q(d(y_0, y_1)) \cdot \max\Big\{d(y_0, y_1), d(y_0, Ax_0),$$

$$d(y_1, Bx_1), \frac{1}{2}[d(y_0, Bx_1) + d(y_1, A_0)]\Big\}.$$

In view of $y_1 \in Ax_0, d(y_0, Bx_1) \leqslant d(y_0, y_1) + d(y_1, Bx_1)$ and $q(d(y_0, y_1)) < 1$ from this we get $d(y_1, Bx_1) \leqslant q(d(y_0, y_1))d)y_0, y_1)$ and hence

$$d(y_1, Bx_1) < \min\{d(y_0, y_1), q(d(y_0, y_1))r\} = t.$$

Select $y_2 \in Bx_1$ so that $d(y_1, y_2) < t$. By (i), there is an $x_2 \in X$ with $Sx_2 = y_2$. Analogously, there is an $y_3 \in Ax_2$ with

$$d(y_2, y_3) < \min\{d(y_1, y_2), q(d(y_1, y_2))q(d(y_0, y_1))r\}$$

and $y_3 = Tx_3$.

Generally, we can construct two sequences $\{x_n\}$, $\{y_n\}$ with the following properties

$$y_{2n} = Sx_{2n} \in Bx_{2n-1}, \; y_{2n+1} = Tx_{2n+1} \in Ax_{2n}, \tag{1}$$

$$c_{n+1} < \min\{c_n, q(c_n) \ldots q(c_0)r\}, \text{ where } c_n = d(y_n, y_{n+1}). \tag{2}$$

From (2), $c_n \to c \geqslant 0$. By the upper semicontinuity of q, $\overline{\lim}\, q(c_n) \leqslant q(c)$. Fix k with $q(c) < k < 1$, there is an $n_0 \in N$ such that $q(c_n) \leqslant k$ for $n \geqslant n_0$. Hence, for $n \geqslant n_0$ we have from (2) $c_{n+1} \leqslant k^n R$, where $R = k^{-n_0} q(c_{n_0}) \ldots q(c_0)r$. Since $k < 1$, $\{y_n\}$ is a Cauchy sequence and hence $y_n \to z$. By continuity of S and T, $Ty_{2n} \to Tz$, $Sy_{2n+1} \to Sz$. From (i) and (1) we have $Ty_{2n} \in BTx_{2n-1} = By_{2n-1}$, $Sy_{2n+1} \in ASx_{2n} = Ay_{2n}$. By closedness of A and B we get $Tz \in Bz$, $Sz \in Az$. The proof is complete.

2. In [2] Sanderson claims that „the truth of Theorem H is in doubt as the proof is incomplete". But Theorem 1 shows that Theorem H is true and it seems to me that the proof in [1] is standard and clear enough. Moreover, the counter-example in [2]:

$$X = 1, \ldots, 2^{-n}, \ldots, 0\}, \; S = T = \text{identity}, \; A(0) = 1, \; A(1) = A(2^{-n}) \equiv X$$

is not true. In fact, A is not contractive, for

$$H\left(A(0), A\left(\frac{1}{2}\right)\right) = H(1, X) = 1 > d\left(0, \frac{1}{2}\right).$$

Besides, A is not closed, for $2^{-n} \in A(2^{-n}) \equiv X$, but $0 \notin A(0) = 1$. So this counter-example has no relations with Theorem H.

3. The following result shows that closedness of A and B can be replaced by commutativity of S and T. Namely, we have

THEOREM 2. *Let X be complete, S, T continuous on X, A and B multi-valued mappings from X into $Cl(X)$. Suppose that each of S, T commutes with the there others, $A(X) \cup B(X) \subset ST(X)$ and Condition (ii) in Theorem 1 is satisfied. Then the conclusion of Theorem 1 still holds.*

Proof. Denote $U = ST$, take $x_0 \in X$, put $y_0 = Ux_0$, fix $r > d(y_0, ATx_0)$, choose $y_1 \in ATx_0$ with $d(y_0, y_1) < r$, then select $x_1 \in X$ with $y_1 = Ux_1$. From (ii) we have

$$d(y_1, BSx_1) \leqslant H(ATx_0, BSx_1) \leqslant q(d(y_0, y_1)) \max\Big\{d(y_0, y_1), d(y_1, BSx_1), \frac{1}{2} d(y_0, BSx_1)\Big\} = q(d(y_0, y_1))d(y_0, y_1).$$

Choose $y_2 \in BSx_1$ so that $d(y_1, y_2) < \min\{d(y_0, y_1), q(d(y_0, y_1))r\}$ then select x_2 with $y_2 = Ux_2$.

Repeat this process, we get two sequences $\{x_n\}$, $\{y_n\}$ with

$$y_{2n} = Ux_{2n} \in BSx_{2n-1}, \ y_{2n+1} = Ux_{2n+1} \in ATx_{2n} \tag{3}$$

and for which (2) still holds. So $y_n \to y \in X$. Now by (ii), we have

$$d(Uy, ATy) \leqslant d(Uy, Uy_{2n}) + d(Uy_{2n}, ATy) \leqslant d(Uy, Uy_{2n}) + H(BSy_{2n-1}, ATy) \leqslant d(Uy, Uy_{2n}) + q(d(Uy, Uy_{2n-1})) \max\Big\{\{d(Uy, Uy_{2n-1}), d(Uy, ATy), d(Uy_{2n-1}, Uy_{2n}), \frac{1}{2}[d(Uy, Uy_{2n}) + d(Uy_{2n-1}, ATy)]\Big\}$$

Since $d(Uy, Uy_{2n-1}) \to 0$ and $q(0) < 1$, we have $q(d(Uy, Uy_{2n-1})) \leqslant k < 1$ for n large enough. From this by letting $n \to \infty$ we get $d(Uy, ATy) \leqslant kd(Uy, ATy)$. This shows that $Uy \in ATy$ in view of closedness of ATy. Similarly, we have $Uy \in BSy$. Putting $z = Uy$, from this we get the desired result: $Sz \in Az$, $Tz \in Bz$.

REMARK When $S = T$=the identity, Theorem 2 reduces to Theorem 1 in [3].

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A FIXED POINT THEOREM FOR MULTI-VALUED FUNCTIONS OF CONTRACTION TYPES WITHOUT HYPOTHESIS OF CONTINUITY

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ABSTRACT. — In this paper we present a fixed point theorem for multi-valued functions of contraction type. The class of all multivalued functions which satisfy our condition is more large than those classes considered in [1], [2], [4], and [6].

Definition and notations. In the sequel we shall use the following notations. For a metric space X by $CL(X)$ we denote the class of all non-empty closed subsets of X. By H we denote the Hausdorff distance in $CL(X)$ generated by the metric

$$H(A,\ B) = \max \{\operatorname{Sup}_{b \in B} \inf_{a \in A} d(a,\ b),\ \operatorname{Sup}_{a \in A} \inf_{b \in B} d(a,\ b)\}$$

for all $A,\ B \in CL(X)$

And, as usual, $d(x,\ A) = \inf \{d(x,\ y),\ y \in A\}$

Let $F : X \to CL(X)$ be a multi-valued function.

DEFINITION. A sequence $\{x_n,\ n = 0,\ 1,\ 2,\ \ldots\}$ is called an orbit of F at x iff $x_0 = x,\ x_{n+1} \in Fx_n,\ n = 0,\ 1,\ 2,\ \ldots$

THEOREM. *Let X be a metric space; $F : X \to CL(X)$ be a function satisfying the following conditions:*

i) There is an orbit of F at a point x_0, containing two successive convergent subsequences

$$x_{n_i} \underset{i\to\infty}{\longrightarrow} x_*, \qquad x_{n_i+1} \underset{i\to\infty}{\longrightarrow} x_*$$

ii) There exist real numbers q_1 and q_2:

$$q_2 < 1 \text{ such that}$$

$H(Fx,\ Fy) \leqslant q_1 d(x,\ y) + q_2 \max \{d(x,\ Fx) + d(y,\ Fy),\ d(x,\ Fy) + d(y,\ Fx)\}$ *for all x, y in X.*

Then

$$x_* \in Fx_*$$

Proof. Suppose $x_* \notin Fx_*$. Since Fx_* is nonempty and closed we have $d(x_*,\ Fx_*) = r > 0$. From the condition *i)* of the theorem it follows that for every $\varepsilon > 0$, there is a non-negative integer $i(\varepsilon)$ such that for all

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$i \geqslant i(\varepsilon)$ both x_{n_i} and x_{n_i+1} belong to the opèn ball centered at x_* of radius $\varepsilon : x_{n_i} \in 0 \ (x_*, \varepsilon), \ x_{n_i+1} \in 0 \ (x_*, \varepsilon)$.

And hence for all $i \geqslant i(\varepsilon)$ we have

$$d(x_{n_i}, x_{n_i+1}) \leqslant d(x_{n_i}, x_*) + d(x_*, x_{n_i+1}) \leqslant 2\varepsilon$$

From here we have:

$$d(x_{n_i}, Fx_{n_i}) \geqslant 2 \qquad (1)$$

From the definition of the distance between a point and a set in metric space, it follows:

$$d(x_{n_i}, Fx_*) \leqslant d(x_{n_i}, x_*) + d(x_*, Fx_*)$$

And thus, for all $i \geqslant i(\varepsilon)$ we have

$$d(x_{n_i}, Fx_*) \leqslant \varepsilon + r \qquad (2)$$

From the condition *ii)* of the theorem and using (1) and (2) have for $i \geqslant i(\varepsilon)$

$$H(Fx_{n_i}, Fx_*) \leqslant q_1\varepsilon + q_2 \max\{2\varepsilon + r, \ (\varepsilon + r) + \varepsilon\}$$

Hence

$$H(Fx_{n_i}, Fx_*) \leqslant q_1\varepsilon + q_2(r + 2\varepsilon) \qquad (3)$$

In the other hand

$$d(x_*, Fx_*) \leqslant d(x_*, X_{n_i+1}) + d(x_{n_i+1}, Fx_*)$$

From this for $i \geqslant i(\varepsilon)$ we have

$$d(x_{n_i+1}, Fx_*) \geqslant r - \varepsilon \qquad (4)$$

Since $q_2 < 1$, it is clear that (3) contradicts (4) when ε is chioseen sufficiently small and $i \geqslant i(\varepsilon)$

Thus $x_* \in Fx_*$

Remark 1. In the condition *ii)* of the theorem q_1 is arbitrary and q_2 can be more than $\frac{1}{2}$.

Remark 2. In the proof of the theorem the condition *ii)* need be fulfilled only for all pairs of tipe (x_{n_i}, x_*)

By considering the simple-valed function we have the following.

COROLLARY *Let X be a metric space, $f: X \to X$ be a mapping satisfying the following conditions:*

i) There is an orbit of f at a point x_0 containing two succesive convergent subsequences

$$x_{n} \xrightarrow[i\to\infty]{} x_*, \quad x_{n_i+1} \xrightarrow[i\to\infty]{} x_*$$

ii) There exist real numbers q_1 *and* q_2, $q_2 < 1$ *such that*

$$d(f(x_{n_i}), f(x_*)) \leqslant q_1 d(x_{n_i}, x_*) + q_2 \max \{d(x_{n_i}, f(x_{n_i})) +$$

$$+ d(x_*, f(x_*)), d(x_{n_i}, f(x_*)) + d(x_*, f(x_{n_i}))\}$$ *for all integers* i.

Then x_* *is a fixed point of* f.

The following example shows that the theorem does not hold if q_2 is replaced by 1.

Example. $X = \left\{-\frac{1}{2^n}, n = 0, 1, 2, \ldots\right\} \cup \{0\} \cup \{1\}$,

$$f: X \to X \text{ defined by } f\left(-\frac{1}{2^n}\right) = -\frac{1}{2^{n+1}},$$

$n = 0, 1, 2, \ldots, f(0) = 1 : f(1) = -1$.

The reader can verify the fulfilment of all conditions of the theorem with $q_1 = q_2 = 1$ and f has no fixed point.

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SOME NONNEGATIVE DETERMINANTS IN INNER PRODUCT SPACES

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ABSTRACT. — Cauchy — Schwarz inequality is generalized in the paper, under the form of a n-order dererminant.

A fertile source of inequalities is provided by the notion of the inner product of a vector with itself in a finite dimensional vector space over the field of the real numbers **R**. Let u **and** v be vectors in such an n-dimensional vector space $\mathbf{R}^n$. Thus, u is identified with an n-tuple of real numbers, say, $(a_1, a_2, \ldots, a_n)$ and v is identified with an n-tuple of real numbers, say $(b_1, b_2, \ldots, b_n)$. Denoting the *inner product* of u and v by $\langle u, v \rangle$, we have according to the usual definition:

$$\langle u, v \rangle = a_1b_1 + a_2b_2 + \ldots + a_nb_n \tag{1}$$

Replacing in (1) the vector v by u, we have

$$\langle u, u \rangle = a_1^2 + a_2^2 + \ldots + a_n^2 \tag{2}$$

Since the right side of the equality sign in (2) is a sum of squares of the elements of **R** (the set of all real numbers), we have:

$$\langle u, u \rangle \geqslant 0 \quad \textit{for every vector } u \textit{ in } \mathbf{R}^n \tag{3}$$

Obviously, (3) is an inequality and as shown below, it is the motivating factor behind many inequalities. For instance, let us take instead of u the sum $v + w$ of vectors v and w. But then we have:

$$\langle v + w, v + w \rangle = \langle v, v \rangle + 2\langle v, w \rangle + \langle w, w \rangle \tag{4}$$

which by (2) yields the following inequality:

$$\langle v, v \rangle + 2\langle v, w \rangle + \langle w, w \rangle \geqslant 0 \textit{ for every } v, w \textit{ in } \mathbf{R}^n \tag{5}$$

The inequality (5) itself can be rewritten in various ways, each giving rise to an inequality. Thus, from (5) the following two inequalities follow immediately:

$$\langle v, v \rangle + \langle w, w \rangle \geqslant -2\langle v, w \rangle \textit{ for every } v, w \textit{ in } \mathbf{R}^n \tag{6}$$

and

$$\langle v, v \rangle + \langle v, w \rangle \geqslant -\langle v, w \rangle - \langle w, w \rangle \textit{ for every } v, w \textit{ in } \mathbf{R}^n \tag{7}$$

True, that (5), (6), (7) are inequalities, however, most probably they are neither too interesting nor too useful. For instance, neither seems to be as interesting or as useful as the Cauchy-Schwarz inequality. A reason

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for this perhaps lies in the fact that $v + w$ is a quite trivial linear combination of v and w and in a way one should not expect to obtain an interesting inequality by merely replacing u in (3) by $v + w$.

Let us now consider a less trivial linear combination involving v and w. For instance, let us consider a linear combination involving v and w which is also orthogonal to v. In particular, let us consider the linear combination of v and w given by:

$$\langle v, v\rangle w - \langle v, w\rangle v \tag{8}$$

which is orthogonal to v. Indeed, it is trivial to verify that the inner product of $\langle v, v\rangle w - \langle v, w\rangle v$ with v is 0. Now, let us replace u in (3) by (8). Thus,

$$\langle (\langle v, v\rangle w - \langle v, w\rangle v), (\langle v, v\rangle w - \langle v, w\rangle v) \rangle \geqslant 0 \tag{9}$$

Applying the distributivity law to the above inner product and observing that $r\langle v, w\rangle = \langle w, v\rangle r$ for every v, w in $\mathbf{R}^n$ and every r in $\mathbf{R}$, we obtain, after obvious simplification:

$$\langle v, v\rangle\langle v, v\rangle\langle w, w\rangle - \langle v, v\rangle\langle v, w\rangle\langle v, w\rangle \geqslant 0 \tag{10}$$

If $v \neq 0$ then $\langle v, v\rangle \neq 0$ and therefore upon dividing both sides of the inequality (10) by $\langle v, v\rangle$ we have:

$$\langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle\langle v, w\rangle \geqslant 0 \textit{ for every } v, w \textit{ in } \mathbf{R}^n \tag{11}$$

regardless whether $v = 0$ or $v \neq 0$.

Inequality (11) is quite interesting and quite useful. Indeed, it is the Cauchy-Schwarz inequality. Thus, starting with an interesting linear combination (8) of v and w and using it in an obvious (but very basic) inequality (3), we obtained a trather interesting inequality (11).

We may rewrite inequality (11) in the determinant form as follows:

$$\begin{vmatrix} \langle v, v\rangle & \langle v, w\rangle \\ \langle v, w\rangle & \langle w, w\rangle \end{vmatrix} \geqslant 0 \qquad \textit{for every } v, w \textit{ in } \mathbf{R}^n \tag{12}$$

Thus, the Cauchy-Schwarz inequality lends itself to be expressed as a nonnegative determinant.

Looking at $\langle x, y\rangle$ as an entry in a matrix indicating the entry at the x-row and y-column, we rewrite (12) in the following form:

$$\begin{vmatrix} \langle x, x\rangle & \langle x, y\rangle \\ \langle y, x\rangle & \langle y, y\rangle \end{vmatrix} \geqslant 0 \qquad \textit{for every } x, y \textit{ in } \mathbf{R}^n \tag{13}$$

An immediate generalization of (13) to any finite number of vectors is known in the literature as the Gramian of these vectors. Thus, for vectors x, y, z the inequality corresponding to (13) is

$$\begin{vmatrix} \langle x, x\rangle & \langle x, y\rangle & \langle x, z\rangle \\ \langle y, x\rangle & \langle y, y\rangle & \langle y, z\rangle \\ \langle z, x\rangle & \langle z, y\rangle & \langle z, z\rangle \end{vmatrix} \geqslant 0 \qquad \textit{for every } x, y, z \textit{ in } \mathbf{R}^n \tag{14}$$

Clearly, (14) is another example of nonnegative determinants.

Gramian type nonnegative determinants are known in the literature.

Below, pursuing our approach of considering the inner product with itself of an interesting linear combination of vectors, we obtain a new class of non-negative determinants.

Let us observe that in the case of vectors v and w the nonnegative determinant (12) is obtained as a result of considering the inner product with itself of a nontrivial linear combination of v and w which is orthogonal to v. Motivated by this, for vectors u, v, w let us consider a nontrivial linear combination which is orthogonal to both u and v. Such is for instance the linear combination of u, v, w given by:

$$(\langle u, v\rangle\langle v, w\rangle - \langle u, v\rangle\langle v, v\rangle u + (\langle u, v\rangle\langle u, w\rangle - \langle u, u\rangle\langle v, w\rangle)v + \\ + (\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle\langle u, v\rangle)w \tag{15}$$

It is not difficult to verify that the inner product of the vector given by (15) with itself (which is a nonnegative real number) can be written as the following determinant (which, accordingly, is also nonnegative):

$$\begin{vmatrix} \begin{vmatrix} \langle u, u\rangle & \langle u, v\rangle \\ \langle u, v\rangle & \langle v, v\rangle \end{vmatrix} & \begin{vmatrix} \langle u, u\rangle & \langle u, w\rangle \\ \langle u, v\rangle & \langle v, w\rangle \end{vmatrix} \\ \begin{vmatrix} \langle u, u\rangle & \langle u, w\rangle \\ \langle u, v\rangle & \langle v, w\rangle \end{vmatrix} & \begin{vmatrix} \langle u, u\rangle & \langle u, w\rangle \\ \langle u, w\rangle & \langle w, w\rangle \end{vmatrix} \end{vmatrix} \geqslant 0 \tag{16}$$

Let us observe that the nonnegative determinant given by (16) is a 2 by 2 determinant whose entries, in their turn, are also 2 by 2 determinants. Moreover, the (1, 1) entry in that determinant is the 2 by 2 determinant given by (12), and, the (1, 2) as well as the (2, 1) entry in that determinant is obtained from determinant given by (12) by substituting w for v in every occurence of v in the rightmost column of the table of (12), and, the (2, 2) entry in that determinant is obtained from determinant given by (12) by substituting w for v in every occurence of v in (12).

Appliyng our scheme to four vectors u, v, w, z we obtain the following (quite nontrivial) 2 by 2 nonnegative determinant:

$$\begin{vmatrix} \begin{vmatrix} \begin{vmatrix} \langle u, u\rangle & \langle u, v\rangle \\ \langle u, v\rangle & \langle v, v\rangle \end{vmatrix} & \begin{vmatrix} \langle u, u\rangle & \langle u, w\rangle \\ \langle u, v\rangle & \langle v, w\rangle \end{vmatrix} \\ \begin{vmatrix} \langle u, u\rangle & \langle u, w\rangle \\ \langle u, v\rangle & \langle v, w\rangle \end{vmatrix} & \begin{vmatrix} \langle u, u\rangle & \langle u, w\rangle \\ \langle u, w\rangle & \langle w, w\rangle \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} \langle u, u\rangle & \langle u, v\rangle \\ \langle u, v\rangle & \langle v, v\rangle \end{vmatrix} & \begin{vmatrix} \langle u, u\rangle & \langle u, z\rangle \\ \langle u, v\rangle & \langle v, z\rangle \end{vmatrix} \\ \begin{vmatrix} \langle u, u\rangle & \langle u, w\rangle \\ \langle u, v\rangle & \langle v, w\rangle \end{vmatrix} & \begin{vmatrix} \langle u, u\rangle & \langle u, z\rangle \\ \langle u, w\rangle & \langle w, z\rangle \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} \langle u, u\rangle & \langle u, v\rangle \\ \langle u, v\rangle & \langle v, v\rangle \end{vmatrix} & \begin{vmatrix} \langle u, u\rangle & \langle u, z\rangle \\ \langle u, v\rangle & \langle v, z\rangle \end{vmatrix} \\ \begin{vmatrix} \langle u, u\rangle & \langle u, w\rangle \\ \langle u, v\rangle & \langle v, w\rangle \end{vmatrix} & \begin{vmatrix} \langle u, u\rangle & \langle u, z\rangle \\ \langle u, w\rangle & \langle w, z\rangle \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} \langle u, u\rangle & \langle u, v\rangle \\ \langle u, v\rangle & \langle v, v\rangle \end{vmatrix} & \begin{vmatrix} \langle u, u\rangle & \langle u, z\rangle \\ \langle u, v\rangle & \langle v, z\rangle \end{vmatrix} \\ \begin{vmatrix} \langle u, u\rangle & \langle u, z\rangle \\ \langle u, v\rangle & \langle v, z\rangle \end{vmatrix} & \begin{vmatrix} \langle u, u\rangle & \langle u, z\rangle \\ \langle u, z\rangle & \langle z, z\rangle \end{vmatrix} \end{vmatrix} \end{vmatrix} \geqslant 0$$

Naturally, all the results mentioned above are equally well applicable for the case of the real *inner product spaces*, and, more generally, for the case of the *unitary spaces*.

We summarize the method of construction of our (new class) of 2 by 2 nonnegative determinants as follows.

Let $v_1, v_2, v_3, \ldots$ be elements of a real inner product (or a unitary) space with $\langle v_i, v_j \rangle$ indicating the inner product of v_i and v_j. For every inteber $n \geqslant 2$ we define inductively a 2 by 2 symmetric matrix S_n as follows:

$$S_2 = \begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle \\ \langle v_1, v_2 \rangle & \langle v_2, v_2 \rangle \end{pmatrix} \text{ and } \quad S_{n+1} = \begin{pmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{pmatrix}$$

where $a_{11} = S_n$ and a_{21} is obtained from S_n by substituting v_{n+1} for v_n in every occurence of v_n in the rightmost column of the table of S_n and a_{22} is obtained from S_n by substituting v_{n+1} for v_n in every occurrence of v_n in S_n.

Replacing every matrix S_i which occurs in S_n by its determinant $|S_i|$ we obtain a nonnegative determinant $|S_n|$, i.e., $|S_i| \geqslant 0$ for every integer $n \geqslant 2$.

HYPERVALUABILITY OF A RING

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ABSTRACT. — The aim of this paper in to whow that there exist a semigroup S which although without zero divisors yet is not cancellative, and moreover a Ring exists that is hypervaluated by such a semigroup.

§ 1. **Introduction.** We wish to consider the following question: Is it possible to have a semigroup S that has no zero divisors and non-cancelative, and a ring **R** that can be hypervaluated by this se group?

We mean here our semigroup S to have a zero element 0 and a u element 1, is we have $0 \cdot s = s \cdot 0 = 0 \ \forall s \in S$ and $1 \cdot s = s \cdot 1 = s \ \forall s \in S$. We remark that 1 and 0 are unique.

DEFINITION 1 *We say that a semigroup S is ordered if it is suppl with an order $\leqslant$ such that*

1. *if $a, b, c \in S$ then $a \leqslant b \Rightarrow c \cdot a \leqslant c \cdot b$ and $a \cdot c \leqslant b \cdot c$*
2. $0 \leqslant 1$ (*hence* $0 = 0 \cdot c \leqslant 1 \cdot c = c \ \forall c \in S$)

If the order is total, S is called totally ordered

DEFINITION 2 *An hypervaluation on a ring R is a function (II) fr* **R** *onto a totally ordered semigroup S satisfying the following conditions*

1. $|a| = 0 \Leftrightarrow a = 0 \qquad \forall a \in \mathbf{R}$
2. $|a| = |-a| \ \forall a \in R$
3. $|a + b| \leqslant \operatorname{Max}\{|a|, |b|\} \ \forall a, b \in \mathbf{R}$
4. $|a \cdot b| = |a||b| \ \forall a, b \in \mathbf{R}$

Remarks 1. If the semigroup S does not have any zero divisors th the ring R does not have any either. Indeed suppose $a, b \in \mathbf{R}$ $a \neq 0$ $b \neq$ but with $a \cdot b = 0$. We then have $|a \cdot b| = |0| = 0$. So $|a \cdot b| = |a| \cdot |b| =$ But $a \neq 0$ implies $|a| \neq 0$ and $b \neq 0$ implies $|b| \neq 0$ and yet $|a||b| =$ contradicting our hypothesis that S has no zero divisors.

2. We easily see that a cancellative semigroupe has no zero divisors, how ver the converse is not true in general as we shall see in what follow

We are able to give an affirmativ answer to our question thus pr ving the following theorem:

THEOREM: *There exists a totally ordered a semigroup S, with no ze divisors and yet non cancellative, and a ring* **R** *that can be hypervaluat by this semigroup.*

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The theorem was proved by constructing an example in steps. We construct first a semigroup S_2 with the desired properties (i.e. totally ordered, no zero divisors, and not cancellative) starting out from a given but arbitrary totally ordered semigroup S_1. Then we construct a ring R that is hypervaluated by S_2.

§ 2. **Construction of** S_2. We begin with an arbitrary given totally ordered semigroup $\{S_1, \cdot, >\} = \{0_1, a, b, \ldots\}$ where 0_1 its absorbent (zero) element. Consider now the set $S_2 = S_1 \cup \{0_2\}$ that we get if we adjoint a new element 0_2 to the set S_1 and an operation $*$ defined on S_2 by $a * b = a \cdot b$ if $a, b \in S_1$ and $0_2 * a = a * 0_2 = 0_2 \; \forall \, a \in S_2$. (In particular $0_2 * 0_1 = 0_1 * 0_2 = 0_2$).

PROPOSITION 1 *$\{S_2, *\}$ is a semigroup*. The proof of this is straight forward. Let's show for example the associativity: Let be $a, b, c \in S_2$. If $a, b, c \in S_1$ the associativity results from the associativity in S_1. And if for emample $a = 0_2$ we then have $(0_2 * b) * c = 0_2 * c = 0_2 * (b * c)$.

PROPOSITION 2 *$(S_2 *)$ does not have any zero divisors*

Proof: Indeed it is impossible to have $a, b \in S_2$ $a, b \neq 0_2$ with $a * b = 0_2$, because since $a, b \neq 0_2$ it follows that $a, b \in S_1$ and so their product in S_2 (which coincides with their product in S_1) $a * b = a \cdot b$ also belongs to $S_1 \cdot S_0$ $a \cdot b \in S_1 \Rightarrow a \cdot b \neq 0_2$ because $0_2 \notin S_1$.

PROPOSITION 3 *$(S_2, *)$ is not cancellative*

Proof: Indeed suppose $a, b, \in S_1$ $a, b, \neq 0_1$ $a \neq b$.
We have $0_1 * a = 0_1 \cdot a = 0_1 = 0_1 \cdot b = 0_1 * b$ since 0_1 is the absorbent (zero) element in S_1 (but not in S_2). So in S_2 we can have $0_1 * a = 0_1 * b$ without having $a = b$ (we chose $a \neq b$).

PROPOSITION 4 *There is a compatible total ordering $\succ$ on S_2 that makes $(S_2, *, \succ)$ into a totally ordered semigroup.*

Proof: Define $0_2 \prec a \; \forall \, a \in S_1$ and $a \prec b$ iff $a < b \; \forall \, a, b \in S_1$. The conclusion follows immediately.

§ 3. PROPOSITION 5 *Let I be a two-sided ideal of an integral domain* $\mathbf{R}$. *If $\mathbf{R}/I$ can be hypervaluated by S_1 then $\mathbf{R}$ can be hypervaluated by S_2.* Before proving this proposition let's make two remarks:

1. In what follows, we employ, with no risk of confusion the symbol $\cdot$ to denote the composition in S_1 as well as in S_2.
2. The term „ideal" in a ring not necessarilly commulative signifies a two-sided ideal.

Proof of proposition **5**. Suppose we have the valuation $|\;|$:

$$R/I \xrightarrow{|\;\;|} S_1 = \{0_1, a, b \ldots\}$$

We construct a valuation $||\;\;||$:

$$R \xrightarrow{||\;\;||} S_2 = \{0_2\} \cup S_1 \text{ by posing:}$$

If $a \in \mathbf{R}$ $a = 0$ then $||a|| = 0_2$
If $a \in \mathbf{R}$ $a \neq 0$ then $||a|| = |a + I|$

This implies that if $a \in I$, then $||a|| = 0_1$

We show now that $|| \quad ||$ is a valuation of R onto $S_2 = \{0_2\} \cup S_1$.

1) $\forall a \in \mathbf{R}$ we have $||a|| = 0_2$ if and only if $a = 0$ by definition of

2) *We show that* $||-a|| = ||a|| \ \forall a \in \mathbf{R}$

i) if $a \neq 0$ then $-a \neq 0$ and we have

$||a|| = |a + I| = |-a + I|$ (because $| \quad |$ is an (hyper) valuatic $\mathbf{R}/I$ and $-a + I = -(a + I)$ in $\mathbf{R}/I = ||-a||$

ii) if $a = 0$ then $-a = 0$ and so $||a|| = ||-a|| = 0_2$.

3) *We show that* $||a + b|| \prec Max\{||a||, , ||b||\} \ \forall a, b \in \mathbf{R}$. Indeed

i) if $a = b = 0$ then $a + b = 0$ evident case

ii) if $a = 0$ $b \neq 0$ then $a + b = b$

$$||a|| = 0_2 \ ||b|| \prec 0_2 \text{ and } ||a + b|| = ||b||$$

iii) if $a, b \neq 0$ we can have $a + b \neq 0$ or $a + b = 0$

α) if $a + b = 0$ then $||a + b|| = 0_2 \prec ||a||, ||b||$ so $\prec$Max $\{||a||$

β) if $a + b \neq 0$ then we have

$$||a|| = |a + I| \qquad ||b|| = |b + I|$$

$$||a + b|| = |a + b + I| \text{ (by definition)}$$

$$= |(a + I) + (b + I)| \leqslant \text{Max}\{|a + I|, |b + I|\}$$

(since $| \quad |$ is an hypervaluation for R/I)

$$= \text{Max}\{||a||, ||b||\}.$$

4) *We show that* $||a \cdot b|| = ||a|| \cdot ||b|| \ \forall a, b \in \mathbf{R}$. Indeed, we disti the following cases:

i) if $a = b = 0$ then $ab = 0$ evident

ii) if $a = 0$ $b \neq 0$ then $a \cdot b = 0$
so $||a|| = 0_2$ $||b|| \neq 0_2$ and $||a|| \cdot ||b|| = 0_2$.
Also $||a \cdot b|| = ||0|| = 0_2$ so $||a \cdot b|| = ||a|| \cdot ||b||$

iii) $a \neq 0$, $b \neq 0$ we have $a \cdot b \neq 0$ (since $\mathbf{R}$ is taken to be an in domain)

and so we have $||ab|| = |ab + I|$ (by definition)
and so $||a|| = |a + I|$
$||b|| = |b + I|$

$||a|| \cdot ||b|| = |a + I| \, |b + I| = |ab + I|$ (because)$| \quad |$ is an hypervalu $= ||ab||$ for $\mathbf{R}/I$)

We have verified that the condition 1), 2), 3), 4) that define an (h valuation are satisfied by the function $|| \quad || : R \rightarrow S_2$

So $|| \quad ||$ defines an (hyper) valuation from R onto S_2 and Proposit is thus proved.

§ 4. Coffi's condition for hypervaluability of a ring. DEFINITION 3. *Let A be a ring $a \in A$. We call the set of left annulators of a to be the set $\{x| \ x \in A | x \cdot a = 0\}$ and we denote it by $Ng(a)$. In an analogous way we define the set of the right annulators of a denoted by $Nd(a)$.*
Coffis theorem of valuability of a ring. Let A be a ring with a unit element 1.
A can be hypervaluated by a totally ordered semigroup S if and only if it satisfies the following conditions:

1. For all $a \in A$ $Ng(a) = Nd(s)$ (and we denote this set by $N(a)$)
2. For all $a, b, \in A$ we have $N(a \cdot b) = N(b \cdot a)$
3. The family $N = \{N(a) | a \in A\}$ is totally ordered by inclusion.

In particular, A posesses an hyprvaluation st$|a| \to N(a)$ is a one-to-one correscpondence between S and N.

We remark that Coffi in his construction suppose the semigroup Commutative. The ring A is not supposed necessarily commutative, but with an indentilty element 1. The details can be found in [1]. The idea is the following:

For each $a \in A$ its „value" $|a|$ is $N(a)$.
So, $|| : A \to N = S_1$. Moreover S_1 is totally ordered by the total order defined by:

$$a, \ b, \neq A \quad a \leqslant b \ \text{ iff } \ N(a) \supseteq N(b)$$

Now according to our previous discussion, we can take a ring A of the form $\mathbf{R}/I$ (ie $A = \mathbf{R}/I$ where I is non-zero two-sided ideal of the ring R) and $s \cdot t \ A$ satisfies the Coffi theorem conditions. By Coffi's theorem then we have an hypervaluation $||_1 : A \simeq \mathbf{R}/I \to N = S_1$, which in turn induces (according to our proposition 5 in § 3) an hypervaluation $||_2 : \mathbf{R} \to S_2$ (where S_2 is the semigroup with the desired properties, as it was constructed in § 2) and this provides us with the desired example.

§ 5 **A concrete case** Let's take $\mathbf{R} = Z$ the ring of integers. $I = (16)$ the ideal generated by the integer 16, $A = \mathbf{R}/(16)$ and A satisfies the Coffis theorem conditions as we can easily verify (we observe that $\forall \ b \in A$ $N(b) = \{x \in Z | 16 | b \ x\}$. So A is hypervaluated by a certain semigroup $S_1 = N = \{N(a) | a \in A\}$. By our discussion in § 3 then, $\mathbf{R}$ is hypervaluated by $S_2 = \{0_2\} \cup S_1$ which has the desired properties.

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ON THE CONVERGENCE OF A METHOD OF INTEGRATING CAUCHY'S PROBLEMS

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ABSTRACT. — For integrating Cauchy's problems

$$\dot{x} = f(t, x) \quad x(t_0) = x^0 \tag{1}$$

on each interval t_k, t_{k+1} of the division $t_0 < t_1 < \ldots < t_n$ ($t_i = ih$, $i=0, 1, \ldots, n$), another Cauchy problem $y = g_k(t, y)$, $y(t_k) = y_{k-1}(t_k)$ is formulated, with the solution $y_k(t)$, $k = 0, 1, \ldots, n(y(t_0) = x^0)$. The function x_h, defined by $x_h(t) = y_k(t)$ if $t \in [t_k, t_{k+1}]$, is an approximation of solution (1). The relationships between $f(t, x)$ and $g_k(t, x)$, $k = 0, 1, \ldots, n-1$, are establis hed, which ensure the discrete convergence of the approximative solution x_h twerds x.

Let be the Cauchy's problem

$$\begin{aligned} \dot{x} &= f(t, x) \\ x(t_0) &= x^0 \end{aligned} \tag{1}$$

with the solution $x(t)$ defined on $[t_0, T]$. We supose that $f: [t_0, T] \times \times \mathbf{R}^n \to \mathbf{R}^n$ is a continuous function. To integrate this problem we use the following method. Let be $h = (T - t_0)/n$ and $t_i = t_0 + ih$, $i = 0, 1, \ldots, \ldots, n$. On each interval $[t_k, t_{k+1}]$ it's defined another Cauchy's problem

$$\begin{aligned} \dot{y} &= g_k(t, y) \\ y(t_k) &= y_{k-1}(t_k) \end{aligned} \tag{2}$$

with the solution $y_k(t)$ on $[t_k, t_{k+1}]$. For $k = 0$ we use $y(t_0) = x^0$. We note by x_h the function defined by $x_h(t) = y_k(t)$, if $t \in [t_k, t_{k+1}]$ x_h is called an approximation of the solution of problem (1). This method was used by I x a r u L. [2] and P a v e l G. [4] to integrate linear diferential equation with variable coefficients. M a r i n e s c u C. [3] consider such method to integrate linear systems of differential equations with variable coefficients. A direct proof of convergence is given.

We are interesed to establish a connection between the equations (1) and (2) which assure the convergence of the approximation x_h to x. We say that x_h concerges discretely to x if $\lim\limits_{h \searrow 0} \max\limits_{k=\overline{0,n}} ||x_h(t_k) - x(t_k)|| = 0$.

THEOREM 1. *If* $||f(t, x) - g_k(t, y)|| \leq a_k(t)||x - y|| + c|t - \bar{t}_k|^\gamma$ *on* $[t_k, t_{k+1}]$, *where* $\bar{t}_k \in [t_k, t_{k+1}]$, $a_k(t)$ *is a non-negative continuous function on* (t_k, t_{k+1}) *and* $c, \gamma > 0$, $k = 0, 1, \ldots, n-1$, *then the approximation* x_h *converges discretely to* x, *the solution of the Cauchy's problem.*

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Proof. Let $a : [t_0, t_n] \to \mathbf{R}$ be the function defined by $a(t) = a_k(t)$ if $t \in [t_k, t_{k+1})$. For $t \in [t_k, t_{k+1}]$ we have

$$x(t) = x(t_k) + \int_{t_k}^{t} f(s, x(s))ds,$$

$$y_k(t) = y_k(t_k) + \int_{t_k}^{t} g_k(s, y_k(s))ds$$

and, further

$$x(t) - y_k(t) = x(t_k) - y_k(t_k) + \int_{t_k}^{t} [f(s, x(s)) - g_k(s, y_k(s))]ds$$

Using the hypothesis of the theorem we obtain

$$\|x(t) - y_k(t)\| \leqslant \|x(t_k) - y_k(t_k)\| + \int_{t_k}^{t} a_k(s)\|x(s) - y_k(s)\|ds +$$

$$+ C\int_{t_k}^{t} |s - t_k|^{\gamma} ds \leqslant \|x(t_k) - y_k(t_k)\| + \frac{2Ch^{\gamma+1}}{\gamma + 1} + \int_{t_k}^{t} a_k(s)\|x(s) - y_k(s)\|ds.$$

Applying Gronwall's lemma it results

$$\|x(t) - y_k(t)\| \leqslant \left(\|x(t_k) - y_k(t_k)\| + \frac{2Ch^{\gamma+1}}{\gamma + 1}\right) e^{\int_{t_k} a(s)ds}$$

and particullary for $t = t_{k+1}$

$$\|x(t_{k+1}) - y_k(t_{k+1})\| \leqslant \left(\|x(t_k) - y_k(t_k)\| + \frac{2Ch^{\gamma+1}}{\gamma + 1}\right) e^{\int_{t_k}^{t_{k+1}} a(s)ds} \qquad k = 1, 2, \ldots, (\sqrt{n} - 1)$$

For $k = 0$ one has the inequality

$$\|x(t_1) - y_1(t_1)\| \leqslant \frac{2Ch^{\gamma+1}}{\gamma + 1} e^{\int_{t_0}^{t_1} a(s)ds} .$$

For $k = 1$ we deduce

$$||x(t_2) - y_2(t_2)|| \leqslant \left(||x(t_1) - y_1(t_1)|| + \frac{2Ch^{\gamma+1}}{\gamma + 1}\right) e^{\int_{t_1}^{t_2} a(s)ds} \leqslant$$

$$\leqslant \left(\frac{2Ch^{\gamma+1}}{\gamma + 1} e^{\int_{t_0}^{t} a(s)ds} + \frac{2Ch^{\gamma+1}}{\gamma + 1}\right) e^{\int_{t_1}^{t_2} a(s)ds} =$$

$$= \frac{2Ch^{\gamma+1}}{\gamma + 1}\left(e^{\int_{t_0}^{t_2} a(s)ds} + e^{\int_{t_1}^{t_2} a(s)ds}\right).$$

Inductively, it results

$$||x(t_k) - y_k(t_k)|| \leqslant \frac{2Ch^{\gamma+1}}{\gamma + 1}\left(e^{\int_{t_0}^{t_k} a(s)ds} + e^{\int_{t_1}^{t_k} a(s)ds} + \ldots + e^{\int_{t_{k+1}}^{t_k} a(s)ds}\right) \leqslant$$

$$\leqslant \frac{2kCh^{\gamma+1}}{\gamma + 1} e^{\int_{t_0}^{t_k} a(s)ds} \leqslant \frac{2(t_n - t_0)Ch^{\gamma}}{\gamma + 1} e^{\int_{t_0}^{t_u} a(s)ds} \quad (k = 0, 1, \ldots, n).$$

and hence $\lim\limits_{h \downarrow 0} \max\limits_{k=\overline{0,n}} ||x(t_k) - y_k(t_k)|| = 0.$ *

We apply this theorem to prove the convergence of mentioned meth[od] used in [3] to integrate linear sysrems of differential equations with vari[a]ble coefficients:

$$\dot{x}_i = \sum_{j=1}^{p} a_{ij}(t)x_j + b_i(t) \quad i = 1, 2, \ldots, p. \qquad (3)$$

We attach to the system (3) on each interval $[t_k, t_{k+1}]$ the system wi[th] constant coefficients

$$\dot{y}_i = \sum_{j=1}^{p} a_{ij}(\tilde{t}_k)y_j + b_i(\tilde{t}_k) \quad i = 1, 2, \ldots, p,$$

where $\tilde{t}_k \in [t_k, t_{k+1}]$. We suppose that $a_{ij}(t)$, $i, j = 1, 2, \ldots, p$ and $b_i(t)$ $i = 1, 2, \ldots, p$ are continuous wirh their derivatives on $[t_0, t_n]$. This met[h]od is known as the step method. In this case $f(t, x) = A(t)x + b(t)$ and $g_k(t, y) = A(\tilde{t}_k)y + b(\tilde{t}_k)$ where

$$A(t) = \begin{pmatrix} a_{11}(t) \ldots a_{1p}(t) \\ \cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \\ a_{p1}(t) \ldots a_{pp}(t) \end{pmatrix}, \quad b(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_p(t) \end{pmatrix}$$

Then

$$\|f(t, x) - g_k(t, y)\| = \|A(t)x + b(t) - A(\tilde{t}_k)y - b(\tilde{t}_k)\| \leqslant$$
$$\leqslant \|A(t)x - A(\tilde{t}_k)x + A(\tilde{t}_k)x - A(\tilde{t}_k)y\| + \|b(t) - b(\tilde{t}_k)\| \leqslant$$
$$\leqslant \|A(\tilde{t}_k) - A(t)\| \cdot \|x\| + \|A(\tilde{t}_k)\|\|x - y\| + \|b(t) - b(\tilde{t}_k)\| \leqslant$$
$$\leqslant \|A(\tilde{t}_k)\|\|x - y\| + C|t - \tilde{t}_k| \leqslant (\max_t \|A(t)\|)\|x - y\| + C|t - \tilde{t}_k|,$$

where $C = (p + 1)M$ with $|a'_{ij}(t)|r \leqslant M$, $|b'_i(t)| \leqslant M$ for all $t \in [t_0, t_n]$, $i, j = 1, 2, \dots p$, and $\|x(t)\| \leqslant r$, $t \in [t_0, t_0 + nh]$, $x(t)$ being the solution of the equation $\dot{x} = A(t)x + b(t)$.

The conditions of THEOREM 1 can be weakened. Let k be a compact set which contains the sets $\{x(t) : t \in [t_0, t_0 + nh]\}$ and $\{x_k(t) : t \in [t_0, t_0 + nh]\}$. Then we have

THEOREM 2. *If* $\left\| \int_{t'}^{t''} [f(s, x) - g_k(s, x)]ds \right\| \leqslant \xi(h)$ *for every* $t_k \leqslant t' < t'' \leqslant t_{k+1}$ *and every* $x \in K$, *such that* $\lim_{h \downarrow 0} \frac{\xi(h)}{h} = 0$ *and* $\|g_k(t, x) - g_k(t, y)\| \leqslant L\|x - y\|$, $k = 0, 1, \dots, n - 1$ *then* x_k *converges discretely to* x.

Proof. For every $t \in [t_i, t_{i+1}]$ one has the equalities

$$x(t) - x_k(t) = x(t_i) - x_k(t_i) + \int_{t_i}^{t} [f(s, x(s)) - g_i(s, x_k(s)]ds =$$

$$= x(t_i) - x_k(t_i) + \int_{t_i}^{t} [f(s, x(s) - g_i(s, x(s)]ds + \int_{t_i}^{t} [g_i(s, x(s) - g_i(s, x_k(s))] ds$$

and hence

$$\|x(t) - x_k(t)\| \leqslant \|x(t_i) - x_k(t_i)\| + \xi(h) + L\int_{t_i}^{t} \|x(s) - x_k(s)\|ds.$$

Using Gronwall's lemma we obtain

$$\|x(t) - x_k(t)\| \leqslant [\|x(t_i) - x_k(t_i)\| + \xi(h)]e^{L(t-t_i)} \tag{4}$$

In the same way (for $t = t_{i+1}$) we find

$$x(t_{i+1}) - x_k(t_{i+1}) = x(t_i) - x_k(t_i) + \int_{t_i}^{t_{i+1}} [f(s, x(s)) - g_i(s, x_k(s)(]ds$$

and adding up these equalities for $i = 0, 1, \ldots, k-1$ we obtain

$$x(t_k) - x_h(t_k) = \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} [f(s, x(s)) - g_h(s, x_i(s))]ds =$$

$$= \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} [f(s, x(s)) - g_i(s, x(s))]ds + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} [g_i(s, x(s)) - g_i(s, x_h(s))]ds.$$

Futher we deduce

$$||x(t_k) - x_h(t_k)|| \leqslant \sum_{i=0}^{k-1} \left\| \int_{t_i}^{t_{i+1}} [f(s), x(s)) - g_i(s, x(s))]ds \right\| +$$

$$+ \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} ||g_i(s, x(s)) - g_i(s, x_h(s))||ds \leqslant k\xi(h) + L \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} ||x(s) - x_h(s)||ds$$

Using (4) the last inequality becomes

$$||x(t_k) - x_h(t_k)|| \leqslant k\xi(h) + L \sum_{i=0}^{k-1} [||x(t_i) - x_h(t_i)|| + \xi(h)] \int_{t_i}^{t_{i+1}} e^{L(s-t_i)} ds =$$

$$= k\xi(h) + \sum_{i=0}^{k-1} [||x(t_i) - x_h(t_i)|| + \xi(h)](e^{L(t_{i+1}-t_i)} - 1) =$$

$$= (e^{Lh} - 1) \sum_{i=0}^{k-1} ||x(t_i) - x_h(t_i)|| + e^{Lh}\xi(h)k.$$

Applying the discretely form of Gronwall's lemma we obtain

$$||x(t_k) - x_h(t_k)|| \leqslant e^{Lhk}\xi(h)k.$$

Finally

$$\max_{k=\overline{0,n}} ||x(t_k) - x_h(t_k)|| \leqslant n\xi(h)e^{Lhn} = (t_n - t_0)e^{L(t_n-t_0)} \frac{\xi(h)}{h}$$

and hence $\lim\limits_{h\downarrow 0} \max\limits_{k=\overline{0,n}} ||x(t_k) - x_h(t_k)|| = 0.$ *

Now, we show that the theorem 2 implies theorem 1. Indeed, if $||f(t, x) - g_i(t, y)|| \leqslant a_i(t)||x - y|| + C|t - \tilde{t}_i|^\gamma$ then for every $t_i \leqslant t' < t'' \leqslant t_{i=1}$ one have the inequalities

$$\left\| \int_{t'}^{t''} [f(t, x) - g_i(t, x)]dt \right\| \leqslant \int_{t'}^{t''} ||f(t, x) - g_i(t, x)||dt \leqslant$$

$$\leqslant C \int_{t'}^{t''} |t - \tilde{t}_i|^\gamma dt \leqslant \frac{2C}{\gamma + 1} (t'' - t')^{\gamma+1} \leqslant \frac{2Ch^{\gamma+1}}{\gamma + 1}.$$

Taking $\xi(h) = \dfrac{2Ch^{\gamma+1}}{\gamma + 1}$ we observe that $\lim\limits_{h \downarrow 0} \dfrac{\xi(h)}{h} = 0$.

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ON THE COMMUTATIVITY OF SOME FAMILIES OF CLOSED OPERATIONS IN A HETEROGENEOUS CLONE

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ABSTRACT. — The heterogeneous algebras, introduced by Birkhoff and Lipson [1], play a very important role in computer science and essentialy in the study of abstract types [4,7]. We introduce the concepts of heterogeneous clone and abstract hetrogeneous clone of operations, and a commutativity property between two families of closed heterogenous operations. This commutativity is generaly complex and restrictive, but in the particular forms is very powerful in the specification of abstract types.

1. **Introduction.** Following the notations in [4], let S be a nonvoid the elements of which will be called sorts. Each indexed family of $A = (A_s)_{s \in S}$ will be called S-sorted family of sets and each inede family of mappings $f = (f_s)_{s \in S}$, where $f_s : A_s \to B_s$ is a mapping $(s \in$ will be called S-sorted mapping. An S-sorted operator domain (signat consist of a set Σ equipped with two mappings: $d : \Sigma \to S^*$ and $c : \Sigma$ called domain and respectively codomain; For each $\sigma \in \Sigma$ with $d(\sigma)$ $= w \in S^*$ and $c(\sigma) = s \in S$, we say that σ has functionality $(w, s) \in S^*$ $\times S$. So we can see Σ as a disjoint union

$$\Sigma = \bigcup_{(w,s) \in S^* \times S} \Sigma_{w,s} = \bigcup_{w \in S} \Sigma_w = \bigcup_{s \in S} \Sigma_s$$

where $\Sigma_w = \{\sigma \in \Sigma / d(\sigma) = w \in S^*], \Sigma_s = \{\sigma \in \Sigma / c(\sigma) = s \in S^*\}$ a

$$\Sigma_{w,s} = \Sigma_w \cap \Sigma_s$$

Let $\sigma \in \Sigma_{w,s}$, $w = s_1 \ldots s_n$; If $s \in \{s_1, \ldots, s_n\}$ we will say that closed; Otherwise σ is called open.

A Σ-algebra (or heterogeneous algebra) A consist of an S-sorted fam of sets $(A_s)_{s \in S}$ called carrier sets, and for each $(w, s) \in S^* \times S$ and $\sigma \in \Sigma_{w,s}$, there is a function $\sigma_A : A^w \to A_s$ named operation of typ and sort s, where $A^w = A_{s_1} \times \ldots \times A_{s_n}$. If $w = \varepsilon$ is the unit elem of the free monoid S^*, then σ_A is a nullary operation. If at most one ope tion σ_A is a partial function, A will be called partial Σ-algebra.

A Σ-algebra B is called Σ-subalgebra of A if $B_s \subseteq A_s$ for all $s \in$ and for all $\sigma \in \Sigma$, $\sigma_B = \sigma_{A/B}$, where $\sigma_{A/B}$ is the restriction of σ_A to

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Let A, B two Σ-algebras. An S-sorted function $f: A \to B$ is called Σ-homomorphism if for all $\sigma \in \Sigma_{w,s}$ ($w \in S^*$, $s \in S$) the following diagram commute:

$$\begin{array}{ccc} A^w & \xrightarrow{\sigma_A} & A_s \\ {\scriptstyle f^w}\downarrow & & \downarrow{\scriptstyle f_s} \\ B^w & \xrightarrow{\sigma_B} & B_s \end{array} \qquad (1.1)$$

Let $X = (X_s)_{s \in S}$ an S-sorted family of variables and $W_\Sigma(X)$ the word Σ-algebra freely generated by X([1], [2], [5], [6]). The properties of Σ-algebras can be expressed by formulas builts over equations of the form $(t, t')_s$, $s \in S$, t, $t' \in [W_\Sigma(X)]_s$, using the firstorder predicate calculus. In the most general case we can consider sentences in the prenex normal form

$$Q_1 s_1 x_1 \ldots Q_n s_n x_n \bigwedge_{1 \leqslant i \leqslant k} \left(\left(\bigvee_{1 \leqslant j \leqslant 1} p_{ij} \neq q_{ij} \right) \vee \left(\bigvee_{1+1 \leqslant j \leqslant m} r_{ij} = t_{ij} \right) \right) \qquad (1.2)$$

where $Q_i \in \{\forall, \exists\}$.

The specification of an abstract type consist of a triple $SP = (S, \Sigma, E)$ where Σ is an S-sorted signature and E a set of sentences in the form (1, 2) called axioms. If the axiom in E are simple equations, the type will be called equational and then the category Alg_{SP} of all Σ-algebras satisfiyng E form a variety in the sense of [2].

For the formal description of an abstract type let take the following example:

type DATA *is* (BOOL) +
sort data
opns E_{data} : $\to$ *data*
COND_{data} : (*bool*, *data*, *data*) $\to$ *data*
EQ_{data} :(*data*, *data*) $\to$ *bool*
OK_{data} :(*data*)$\to$ *bool*
axioms $\forall$ *data* D, D_1, D_2
(1) $\mathrm{COND}_{data}(T, D, D_1) = D$
(2) $\mathrm{COND}_{data}\ (F, D, D_1) = D_1$
(3) $EQ_{data}\ (D, D) = T$
(4) $EQ_{data}\ (D, D_1) = EQ_{data}\ (D_1, D)$
(5) $EQ_{data}\ (D, D_1) = T \& EQ_{data}\ (D_1, D_2) = T \Rightarrow EQ_{data}\ (D, D_2) = T$
(6) $OK_{data}\ (D) = \mathrm{COND}_{data}\ (EQ_{data}\ (D, E_{data}\ ), F,\ T)$
eot

where BOOL is the usual type of the truth values (see [4]).
Now $S = \{bool, data\}$, $\Sigma = \{T, F, \&, \vee, \Rightarrow, \mathrm{EQ}_{bool}, \mathrm{COND}_{bool}, \mathrm{E}_{data}$ $\mathrm{COND}_{data}, \mathrm{EQ}_{data}, \mathrm{OK}_{data}\}$, E contains the axioms of BOOL and, the axioms (1) ... (6) above.

2. Clones of Heterogeneous Operations. Let A be a novoid S-sorted family of sets and denote by $H(A)$ the set of all finitary heterogeneous operations on A. Then $H(A)$ can be viewd as a disjoint union:

$$H(A) = \bigcup_{w,s\, S^* \times S} H_{w,s}(A), \text{ where } H_{w,s}(A) = \{\sigma \in H(A)/d(\sigma) = w, c(\sigma) = s\}.$$

Let $w = s_1 \ldots s_n \in S^*$, $\tau_i \in H_{u,s_i}(A)$, $(i = 1, \ldots, n)$, $u \in S^*$ and $\sigma \in H_{w,s}(A)$. Then there is a unique operation θ on $H(A)$ defined by:

$$\theta(\sigma, \tau_1, \ldots, \tau_n)(a) = \sigma(\tau_1(a), \ldots, \tau_n(a)) \text{ for all } a \in A^u \quad (2.1)$$

and let denote $\theta(\sigma, \tau_1, \ldots, \tau_n) = \sigma[\tau_1, \ldots, \tau_n]$, calling it „composition" of $\tau_1, \ldots, \tau_n$ with σ.
For any $w \in S^+$, where $S^+ = S^* - \{\varepsilon\}$, with $w = s_1 \ldots s_n$, there are n operations on A denoted $1^{w,s_i}$ and defined by:

$$1^{w,s_i}(a) = a_i \text{ for all } a \in A^w, \quad i = 1, \ldots, n. \quad (2.2)$$

Let call $1^{w,s_i}$ unit operations or projections on the i^{-th} coordinate. We can now regard $H(A)$ as a partial heterogeneous algebra with sorts $S^* \times S$, nullary operations the units and the other operations defined in (2.1) with $d(\theta) = (w, s)(u, s_1) \ldots (u, s_n)$ and $d(\theta) = (u, s)$.

Definition A set H of heterogeneous operations on an S-sorted family A of sets, containing the unit operations defined in (2.2) and closed under the compositions (2.1) is called heterogeneous clone of operations on A. This notion was introduced by P. Hall (1958) and studied for the homogeneous case ([2], [6]).
Generaly, giving an S-sorted family A of sets and the clone $H(A)$ of all operations on A, we will call H_1 a clone on A if H_1 is a subclone of $H(A)$, i.e. a subalgebra in the sense mentioned above.

Let H_1, H_2 be heterogeneous clones on A; An $S^* \times S$-sorted mapping $f: H_1 \to H_2$ with peoperties:

(i) $d(f(\sigma)) = d(\sigma)$ and $c(f(\sigma)) = c(\sigma)$ for all $\sigma \in H_1$

(ii) $f(1^{w,s_i} = 1^{w,s_i}$, $w \in S^*$, $i = 1, \ldots, n$ and

(iii) $f(\sigma[\tau_1, \ldots, \tau_n]) = f(\sigma)[f(\tau_1), \ldots, f(\tau_n)]$, for all composable operations $\sigma, \tau_1, \ldots, \tau_n \in H_1$,

will be called homomorphism of heterogeneous clones on A.
The set of all heterogeneous clones on an S-sorted family A of sets with theirs homomorphisms, form a category.

Let Σ an S-sorted signature and A a Σ-algebra. The actions of the operations in Σ determines a heterogeneous clone on A, denoted Σ_A^* and called heterogeneous clone of action of Σ on A.

DEFINITION. An abstract heterogenous clone is a partial heterogenous algebra H defined as follows:

1° there are two mappings $d: H \to S^*$ and $c: H \to S$ which associates to each $\sigma \in H$ the domain $d(\sigma)$ and the target $c(\sigma)$ and.

2° with each $w \in S^+$, $w = s_1 \ldots s_n$, H contains the unit operators $1^{w,s_i}: w \to s_i$, $i = 1, \ldots, n$ and

3° for $\sigma, \tau_1, \ldots, \tau_n$ in H with $d(\tau_1) = \ldots = d(\tau_n)$ and $d(\sigma) = c(\tau_1) \ldots c(\tau_n) \in S^*$, there is an operation on H denoted by $\sigma[\tau_1, \ldots, \tau_n] : d(\tau_1) \to c(\sigma)$ with properties:

(i) $(\sigma[\tau_1, \ldots, \tau_n])[\eta_1, \ldots, \eta_m] = \sigma[\tau_1[\eta_1, \ldots, \eta_m], \ldots, \tau_n[\eta_1, \ldots, \eta_m]]$ where $d(\tau_1) = c(\eta_1) \ldots c(\eta_m) \in S^*$

(ii) $1^{w,s_i}[\tau_1, \ldots, \tau_n] = \tau_i$, $i = 1, \ldots, n$

As a consequence of this definition we have the

DEFINITION *Every heterogeneous clone of operations is abstract.*

3. **(i, j) — Commutativity of Two Families of Operations.** Let Σ an S-sorted signature, A a Σ-algebra and $\tau = (\tau_1, \ldots, \tau_n)$ $\sigma = (\sigma_1, \ldots, \sigma_m)$ two families of operations in Σ_A^*, with:

$$\begin{aligned}
\sigma_1 &: s_{11} \ldots s_{1j} \ldots s_{1n} \to s_{1j} \\
&\vdots \\
\sigma_i &: s_{i1} \ldots s_{ij} \ldots s_{in} \to s_{ij} \\
&\vdots \\
\sigma_m &: s_{m1} \ldots s_{mj} \ldots s_{mn} \to s_{mj}, \quad \text{and}
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
\tau_1 &: s'_{11} \ldots s'_{i1} \ldots s'_{mj} \to s'_{i1} \\
&\vdots \\
\tau_j &: s'_{1j} \ldots s'_{ij} \ldots s'_{mj} \to s'_{ij} \\
&\vdots \\
\tau_n &: s'_{1n} \ldots s'_{in} \ldots s'_{mn} \to s'_{in}
\end{aligned} \tag{3.2}$$

We call $\tau = (\tau_1, \ldots, \tau_n)$ in (3.2) compsable with σ_i in (3.1) if $d(\sigma_i) = c(\tau_1) \ldots c(\tau\ ) \in S^*$.

DEFINITION. Giving two families of closed operations in Σ_A^*, $\sigma = (\sigma_1, \ldots, \sigma_m)$ and $\tau = (\tau_1, \ldots, \tau_n)$, we will say that σ and τ commute (i, j) if:

(i) σ is composable with τ_j

(ii) τ is composable with σ_i, and

(iii) for all $a_{ij} \in A_{s_{ij}}$, $(i = 1, \ldots, m;\ j = 1, \ldots n)$ we have the identity:

$$\begin{aligned}
&\sigma_i(\tau_1(a_{11}, \ldots, a_{m1}), \ldots, \tau_j(a_{1j}, \ldots, a_{mj}), \ldots, \tau_n(a_{1n}, \ldots, a_{mn})) = \\
&= \tau_j(\sigma_1(a_{11}, \ldots, a_{1n}), \ldots, \sigma_i(a_{in}, \ldots, a_{in}), \ldots, \sigma_m(a_{m1}, \ldots, a_{mn}))
\end{aligned} \tag{3.3}$$

Let now $w'_j = s_{1j} \ldots s_{mj}$ for $j = 1, \ldots, n$ and take $\tau_j = 1^{w'_j, s_{ij}}$; We the have:

$$\begin{aligned}
&\sigma_i(1^{w'_i, s_{i1}}(a_{11}, \ldots, a_{m1}), \ldots, 1^{w'_n, s_{in}}(a_{1n}, \ldots, a_{mn})) = \sigma_i(a_{i1}, \ldots, a_{in}) = \\
&= 1^{w'_j, s_{ij}}(\sigma_1(a_{11}, \ldots, a_{1n}), \ldots, \sigma_i(a_{i1}, \ldots, a_{in}), \ldots, \sigma_m(a_{m1} \ldots, a_{mn}))
\end{aligned} \tag{3.4}$$

and therefore we can state:

PROPOSITION. *Every family of closed o perations of the form* (3.1) *co mmute* (i, j) *with a cooresponding family of units.*

If we denote $w_i = s_{i1} \ldots s_{in}$ for $i = 1, \ldots, m$, and take

$$\sigma_1 = 1^{w_1, s_{1j}}, \ldots, \sigma_{i-1} = 1^{w_{i-1}, s_{i-1,j}}, \ \sigma_{i+1} = 1^{w_{i+1}, s_{i+1,j}}, \ldots, \sigma_m =$$

$$= 1^{w_m, s_{mj}} \text{ and } \tau_1 = 1^{w'_1, s_{i1}}, \ldots, \tau_{j-1} = 1^{w'_{j-1}, s_{i,j-1}}, \ \tau_{j+1} =$$

$$= 1^{w'_{j+1}, s_{i,j+1}}, \ldots, \tau_n = 1^{w'_i, s_{in}}, \text{ then (3.3) becomes:}$$

$$\sigma(a_{i1}, \ldots, a_{ij-1}, \tau(a_{1j}, \ldots, a_{mj}), a_{ij+1}, \ldots, a_{in}) =$$

$$= \tau(a_{1j}, \ldots, a_{i-1j}, \sigma(a_{i1}, \ldots, a_{in}), a_{i+1j}, \ldots, a_{mj}) \quad (3.5)$$

where $\sigma = \sigma_i$ and $\tau = \tau_j$. When this is the case, we call (3.5) (i, j)-commutativity of σ and τ.

The (i, j)-comutativity of two families of closed operations defined above, generalises the commutativity of two operations in the homogeneous case, ([2], III, 3), and is powerfull in the specifications of the abstract types ([3], [4]).

4. Examples

4.1. Let M a monoid acting on a set A. This is a heterogeneous algebra (see [1]) with $\Sigma = \{1_M, *, \circ\}$, where $1_M : M \to M, * : M \times M \to M$ and $0 : M \times A \to A$ and axioms:

$$1_M \circ a = a \text{ for all } a \in A, \text{ and}$$

$$(m * n) \circ a = m \circ (n \circ a) \text{ for all } m, n \in M, \ a \in A$$

Let take first the families of operations $(*, \circ)$ and $(*, \circ)$. The for $(i, j) = (2, 2)$ the relation (3, 3) becomes:

$$(m * n) \circ (p \circ a) = (m * p) \circ (n \circ a) \text{ for all } m, n, p \in M, a \in A \quad (4.1.1)$$

If $p = 1_M$ then using the first axiom we have:

$$(m * n) \circ a = m \circ (n \circ a) \quad (4.1.2)$$

which is the second axiom stated above. On the other hand, taking $m = 1_M$ in (4.1.1.) and using the first axiom we obtain:

$$n \circ (p \circ a) = p \circ (n \circ a) \quad (4.1.3)$$

Now if M is the monoid of all functions $A \to A$, $*$ is the composition and $\circ$ is the value function, then (4.1.3) establishes the center of M.

Secondly, taking the (1,1) commutativity of the families $(*, *)$, $(*, *)$ we have:

$$(m * n) * (p * q) = (m * p) * (n * q) \quad (4.1.4)$$

which contains simultaneousely the associativity and commutativity of $*$ in M.

4.2. As a second example, let take the type CIRCULAR–LIST (DATA) (see [3], Fig. 4.2.2.). The last axiom assert:

$$\text{JOIN (C, INSERT (C1, D))} = \text{INSERT (JOIN (C, C1), D)} \qquad (4.2.1)$$

for all circular_list C, C1 ; data D
where INSERT : (circular_list, data) → circular_list and

JOIN : (circular_list, circular_list) → circular_list

The (4.2.1) aseert the (2.1) commutativity of JOIN and INSERT in the sense of (3.4).
Finaly, the axiom 17 in the same specification is:

$$\text{RIGHT (INSERT (INSERT (C, D), D1))} = \text{INSERT (RIGHT (INSERT (C, D1)D)} \qquad (4.2.2)$$

where RIGHT : circular_list → circular_list.
We have RIGHT ∘ INSERT : (circular_list, data) → circular_list and then (4.2.2) express the (1,1)-commutativity of RIGHT ∘ INSERT and INSERT.

Naturaly, for more complicated types the commutativity relations are more complicated.

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SUR CERTAINES FORMULES DE QUADRATURE OPTIMALES

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ABSTRACT. — On Certain Optimal Quadrature Formulas. The quadrature formulae of (1)-type form are studied, with an exactness degree of (3), for which the rest is minimum with the functionscloss W^{r+1} M ; x_0, x_m. It is proved that such formulae are only extent in the case when m = 2p + 1, and such formulas are effectively construed when p = 3 and p = 4 (formulae (9) and (11)), also estimations of their rest (formulae (10) and (12), respectively) being given.

Soit $W^{r+1}[M; x_0, x_m]$ l'ensemble des fonctions définies sur l'intervalle $[x_0, x_m]$, qui satisfont aux conditions: $f \in C^r[x_0, x_m]$, $f^{(r+1)}$ segmentaire continue et $|f^{(r+1)}(x)| \leqslant M$, $x \in [x_0, x_m]$.

On considère la formule de quadrature

$$\int_{x_0}^{x_m} f(x)\mathrm{d}x = A_0[f(x_0) + f(x_m)] + A_1[f(x_1) + f(x_{m-1})] + \ldots + A_p[f(x_p) + f(x_{p-1})] + h[f(x_{p+1}) + \ldots + f(x_{m-p-1})] + R_{m+1}[f], \quad (1)$$

où $f \in W^{r+1}[M; x_0, x_m]$, $A_0, A_1, \ldots, A_p$ sont les coefficients, $0 \leqslant p \leqslant \left[\frac{m-1}{2}\right]$ et $x_i = x_0 + ih$, $i = 0, 1, \ldots, m$ les noeuds de la formule.

Dans ce travail, nous proposons de determiner les coefficients A_i, $i = 0, 1, \ldots, p$ de manière à ce que le degré d'exactitude soit égale a 3 et le reste $R_{m+1}(f)$ soit minime, quand $f \in W^4[M; x_0, x_m]$.

Ce problème a été aussi considéré par D u r a n d [9] dans le cas $r = 1$, $p = 1$; G. C o u l m y [3] dans le cas $r = 1$, $p = 3$; L a c r o i x [9] pour $r = 3$, $p = 2$. Dans ces articles le problème du reste n'a pas été posé.

D. V. I o n e s c u [5], [6], [7], [8] a déterminé les restes de formules (1) dans le cas $r = 1, 3, 5$, $p \leqslant 4$, en supposant que $f \in C^r[x_0, x_m]$

G h. C o m a n [1], [2] a détérminé les formules de quadrature (1) optimales pour la classe $W^2[M; x_0, x_m]$ dans l'hypothèse que le degré d'exactitude de la formule est $r = 1$.

En appliquant al mèthode de „la fonction φ" donnée par le prof. D. V. I o n e s c u [4], nous prenons sur l'intervalle $[x_0, x_m]$ les noeuds $x_0, x_1, \ldots, x_m$; $x = x_0 + ih$; $i = 0, 1, \ldots, m$. Nous attachons aux intervalles

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$[x_0, x_1], [x_1, x_2), \ldots, [x_{m-1}, x_m]$ les fonctions $\varphi_1, \varphi_2, \ldots, \varphi_m$ solutions des équations différentielles :

$$\varphi_i^{(IV)}(x) - 1, \ i = 1, 2, \ldots, m, \tag{2}$$

avec les conditions aux limites

$$\begin{aligned}
&\varphi_1(x_0) = \varphi_1'(x_0) = \varphi_1''(x_0) = 0 ; \ -\varphi_1'''(x_0) = \varphi_m'''(x_m) = A_0 \\
&\varphi_m(x_m) = \varphi_m'(x_m) = \varphi_m''(x_m) = 0 \\
&\varphi_k^{(s)}(x_k) = \varphi_{k+1}^{(s)}(x_k) ; \ s = 0, 1, 2 ; \ k = 1, 2, \ldots, m-1 \\
&\varphi_k'''(x_k) - \varphi_{k+1}'''(x_k) = \varphi_{m-k}'''(x_{m-k}) - \varphi_{m-k+1}'''(x_{m-k}) = A_k \\
&\qquad k = 1, 2, \ldots, p \\
&\varphi_k'''(x_k) - \varphi_{k+1}'''(x_k) = h ; \ k = p+1, p+2, \ldots, m-p-1.
\end{aligned} \tag{3}$$

On obtient la formule

$$\int_{x_0}^{x_m} f(x)\mathrm{d}x = -\varphi_1'''(x_0)f(x_0) + \sum_{k=1}^{m-1} [\varphi_k'''(x_k) - \varphi_{k+1}'''(x_k)]\, f(x_k) +$$

$$+ \varphi_m'''(x_m) f(x_m) + \int_{x_0}^{x_m} \varphi(x) f^{(IV)}(x) dx,$$

avec le reste

$$R_{m+1}[f] = \int_{x_0}^{x_m} \varphi(x) f^{(IV)}(x)\, dx. \tag{4}$$

Les fonctions

$$\varphi_k(x) = \frac{(x - x_0)^4}{4!} - \sum_{i=0}^{p} A_i \frac{(x - x_i)_+^3}{3!} - h \sum_{i=p+1} \frac{(x - x_i)_+^3}{3!}, \tag{5}$$

où

$$u_+ = \begin{cases} u \text{ si } u > 0 \\ 0 \text{ si } u \leqslant 0 \end{cases}.$$

vérifient les équations différentielles (2) et les conditions aux limites (3) relativement aux points $x_0, x_1, \ldots, x_{m-1}$.
Il reste à déterminer les constantes A_k, de manière à satisfaire aussi les conditions au point x_m.

On obtient

$$\begin{aligned}
&\sum_{k=0}^{p} A_p = \frac{2p+1}{2} h \\
&\sum_{k=1}^{p} k^2 A_k - m \sum_{k=1}^{p} k A_k = \frac{h}{6} p(p+1)(2p+1) - \frac{mh}{6} (6p(p+1)+1).
\end{aligned} \tag{6}$$

On observe que pour la détermination complète de la solution du prol (2)+(3), $p-2$ conditions sont encore nécessaires

D'après (4) nous obtenons l'évaluation

$$|R_{m+1}(f)= \leqslant MJ,$$

où

$$J=\int_{x_0}^{x_m}|\varphi(x)|dx=\sum_{k=1}^{m} I_k \text{ et } I_k=\int_{x_{k-1}}^{x_k}|\varphi_k(x)|dx.$$

De cette manière le problème posé se reduit à la détermination des (cients de la formule (1) tels que les intégrales

$$I_k=\int_{x_{k-1}}^{x_k}|\varphi_k(x)|dx\,;\ k=4,5,\ldots,p+1,$$

soient minimes

LEMME *Le polynome de Tchébychev de seconde espèce*

$$h_1^r Q_r\left(\frac{x-a}{h_1}\right);\ Q_r(x)=\frac{\sin(r+1)\arccos x}{2^r\sqrt{1-x^2}};\ -1\leqslant x\leqslant 1,$$

est l'unique polynome pour lequal l'intégrale $\int_{a-h_1}^{a+h_1}|P_r(x)|dx$

atteint son minimum. Ici $P_r(x)$ est un polynome arbitraire de degré r, lequel le coefficient de la puissance la plus élevée est égal à l'unité. Pour la démonstration de ce lemme, voy [10].

De cette manière le problème posé se reduit à tels que les poly φ_k, $k=4,5,\ldots,p+1$ coincident avec le polynome de Tchéb $h_1^4\cdot Q_4\left(\frac{x-a}{h_1}\right)$, sur l'intervalle $[a-h_1,\ a+h_1]$, où

$$a=\frac{x_{k-1}+x_k}{2},\ h_1=\frac{x_k-x_{k-1}}{2}.$$

On obtient le système d'équations

$$\sum_{i=0}^{k-1} A_i=\frac{2k-1}{2}h$$

$$\sum_{i=1}^{k-1} iA_i=\frac{32k(k-1)+7}{64}h$$

$$\sum_{i=1}^{k-1} i^2A_i=\frac{2k-1}{2}\cdot\frac{32k(k-1)+5}{96}h$$

$$\sum_{i=1}^{k-1} i^3A_i=\frac{4(2k-1)^2(16k(k-1)+1)+1}{1024}h.$$

En tenant compte des conditions (6) + (7) il resulte que

$$m = 2p + 1. \tag{8}$$

La solution du système d'équations (7), avec la condition (8) est :

1° pour $p = 3$ $(m = 7)$

$$A_0 = \frac{1995}{6144} h, \ A_1 = \frac{8255}{6144} h, \ A_2 = \frac{4481}{6144} h, \ A_3 = \frac{6773}{6144} h.$$

La formule de quadrature correspondente este

$$\int_{x_0}^{x_7} f(x)dx = \frac{h}{6144} [1995(f(x_0) + f(x_7)) + 8255(f(x_1) + f(x_6)) +$$

$$4481(f(x_2) + f(x_5)) + 6773(f(x_3) + f(x_4))] + \int_{x_0}^{x_7} \varphi(x) f^{(IV)}(x)dx, \tag{9}$$

où

$$|R_7(f)| \leqslant 0{,}0899494 \ \mathrm{h}^5 M. \tag{10}$$

2°. Dans le cas $p = 4$, le système d'èquations (7) a la solution

$$A_0 = A_4 - \frac{4469}{6144}, \ A_1 = -4A_4 + \frac{33951}{6144} h, \ A_2 = 6A_4 - \frac{33663}{6144} h,$$

$$A_3 = -4A_4 + \frac{31829}{6144} h, \ A_4 \text{ arbitraire}$$

ainsi la formule de quadrature optimale est :

$$\int_{x_0}^{x_9} f(x)dx = \left(A_4 - \frac{4469}{6144} h\right)(f(x_0) + f(x_9)) + \left(-4A_4 + \frac{33951}{6144} h\right) \cdot$$

$$\cdot (f(x_1) + f(x_8)) + \left(6A_4 - \frac{33663}{6144} h\right)(f(x_2) + f(x_7)) +$$

$$+ \left(-4A_4 + \frac{31829}{6144} h\right)(f(x_3)) + f(x_6) + A_4(f(x_4) + f(x_5)) + R_9[f].$$

En choissisant $A_4 = \frac{4469}{6144} h$, on obtient une formule de quadrature du type ouvert

$$\int_{x_0}^{x_9} f(x)dx = \frac{h}{6144}\,[16075(f(x_1) + f(x_8)) - 6849(f(x_2) + f(x_7)) +$$

$$+ 13953(f(x_3) + f(x_6)) + 4469(f(x_4) + f(x_5)) + \int_{x_0}^{x_9} \varphi(x) f^{(IV)}(x)dx,$$

avec le reste

$$|R(f_9)| \leqslant 0{,}7032546\; Mh^5.$$

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FUNDAMENTAL THEOREM OF ALGEBRA FOR GENERALIZED POLYNOMIAL MONOSPLINES

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ABSTRACT. — This paper presents a Fundamental Theorem of Algebra for Generalized Monosplines as introduced by Braess and Dyn [5]. Such monosplines generated by the polynomial spline kernel are of primary interest, but similar results are obtained for totally positive generalized monosplines where the corresponding kernel satisfies the cone condition of Burchard [6].

I. **Introduction.** An extended totally positive (ETP) kernel $K(x, y)$ is a function $K:[a, b] \times [c, d] \to \mathbf{R}$ such that for any set of points $a \leqslant x_1, \leqslant \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant b$ and $c \leqslant y_1 \leqslant y_2 \leqslant \ldots \leqslant y_n \leqslant d$, the corresponding determinant $\det \{K(x_i, y_j)\}_{i,j=1}^{n} = K\begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ y_1 & y_2 & \ldots & y_n \end{pmatrix} > 0$. We call K a totally positive (TP) kernel if this determinant is nonnegative. Where the points coincide, we replace the function by increasing partial derivatives of the function and require sufficient smoothness of the kernel.

Let $K(x, y)$ be an ETP kernel on

$[a, b] \times [c, d]$ and define $Z_m^+ = \{\nu = (\nu_0, \ldots, \nu_{m+1}) : \nu_i \geqslant 0$

for $i = 0, 1, \ldots, m + 1\}$. For $\nu \in Z_n^+$, $\omega \in Z_m^+$ let $\sum_{i=0}^{m} \omega_i = N = \sum_{i=1}^{n} \nu_i$.

Define $\Delta^n[a, b] = \{x = (x_1, \ldots, x_n) : a = x_0 < x_1 < \ldots < x_n < x_{n+1} = b\}$,

let $1 = (0, 1, \ldots, 1, 0) \in Z_m^+$ and let $K_j(x, t) = \dfrac{\partial^j}{\partial t^j} K(x, t)$.

Define the sign function $\underset{t,\omega+1}{\sigma}(t)$ to be

$$\underset{t,\omega+1}{\sigma}(t) = (-1)^{\sum_{j=1}^{i}(\omega_j+1)} \quad \text{for } t_i \leqslant t < t_{i+1},\ i = 0, 1, \ldots, m.$$

Here $t_0 = c$ and $t_{m+1} = d$. The sign is normalized by

$\sigma_{t,\omega+1}(t) = +1$ for $c < t < t_1$ in accordance with Braess and Dyn [5]. We define the generalized monospline $M(x)$ by

$$M(x) = \int_c^d K(x, t)\underset{t,\omega+1}{\sigma}(t)\, dq(t) - \sum_{i=0}^{m} \sum_{j=0}^{\omega_i - 1} a_{ij} K_j(x, t_i), \tag{$*$}$$

where $d\rho$ is a nonnegative, nonatomic measure.

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A generalized monospline can also be defined in the case where the generating kernel $K(x, y)$ on $(*)$ is only totally positive. If the kernal satisfies certain cone conditions, a fundamental theorem of algebra can be obtained (see Section II). The most studied kernel of this type is the polynomial spline kernel

$$K_n(x, t) = (x - t)_+^{n-1} \text{ where } x_+^r = \begin{cases} x^r & x \geqslant 0 \\ 0 & x < 0. \end{cases}$$

In this case we define the „Generalized Polynomial Monospline" M_n for $n - \omega_i \geqslant 1$

$$\text{by } M_n(x) = \int_c^d (x - t)_+^{n-1} \sigma_{t,\omega+1}(t)\, d\rho(t) - \sum_{i=0}^{m} \sum_{j=1}^{\omega_i} a_{ij}(x - t_i)_+^{n-j}. \quad (1)$$

Then $M_n(x)$ is a polynominal of degree n on each of the intervals (t_i, t_{i+1}), $i = 0, \ldots m - 1$, and $M \in C^{n-\omega_i-1}$ in a neighborhood of t_i

It is necessary to study the zeros of such monosplines and as a result obtain a bound on the coefficients of generalized polynomial monosplines with a full set of zeros.

Throughout the following we will count multiplicities in the manner of Micchelli [13].

The following theorem which arises from the theory of generalized signs, will be of use in the following:

Theorem A [9]. *If the number of sign changes of a monospline of the form* $(*)$ *is given by* Z, *then* $Z \leqslant N$. *Moreover, if* $Z = N$, *then if the generalized sign vector is of the form* $(S_1, S_2, \ldots, S_{N+1})$, *then*

$$S_j = sgn\ M(x_j),\ j = 1,2, \ldots, N + 1.$$

Here $a < x_1 < x_2 < \ldots < x_{N+1} < b$ define the sign changes of $M(x)$.

II. The Zero Structure of Generalized Polynomial. Monosplines.

LEMMA 1: *Let* $M_n(x)$ *be as defined in* (1). *Then* M_n *has at most*

$$\sum_{i=0}^{m} \omega_i + m \textit{ zeros, counting multiplicities.}$$

Proof. We first consider the case $n = 1$. Then $\sigma_{t,\omega+1}(t) = (-1)^{i+1} = +1$ and so this reduces to lemma 2.2 of Karlin and Schumacher [11] which states that M_1 has at most $2m + 1$ zeros, noting that $\omega_i = 1$ for all i.

As a monospline of the above type is of class $C^{n-2}(-\infty, \infty)$, for $n \geqslant 2$ we may use the theory of generalized signs. Therefore, using Theorem A the result is shown.

LEMMA 2: *Let M_n be a monospline of the form* (1) *which vanishes at* $x_1 < x_2 < \ldots < x_N$ *where* $N = \sum_{i=0}^{m} (\omega_i + 1) - 1$. *If* $\omega_i < n$ *then* $x_{k(i)} < t_i < x_{n+k(i-1)+1}$ *where* $k(i) = \sum_{j=1}^{i} (\omega_j + 1)$. *In the case that* $\omega_i = n$ *then* $t_i = x_{k(i)}$

Proof. Suppose $\omega_i < n$ and $t_i \leqslant x_{k(i)}$. Define M_+ to be the monospline which agress with M_+ to the right of t_i and has no knots to the left. Then M_+ has at least $\omega_0 + \sum_{j=i+1}^{m} (\omega_i + 1) + 1$ zeros since M_n is continuous at t_i, but M_+ has only $m - i$ knots. By lemma 1, M_+ can haveat-most $\omega_0 + \sum_{j=i+1}^{m} (\omega_j + 1)$ zeros, so the first inequality must hold. The reamining assertions follow in a similar manner.

PROPOSITION 1: *Given any* $K > 0$ *there exists a* $\lambda > 0$ *such that whenever* $M(x)$ *is of the form* (1)

and M has $\sum_{i=0}^{m} \omega_i + m$ *distinct zeros in* $(-K, K)$ *then* $|a_{ij}| \leqslant \lambda$. for $i = 0, \ldots, m$, $j = 0, \omega_i - 1$.

Proof. The proof follows that of Micchelli [13, pg. 426]. It proceeds by simultaneous induction on n and m. The case $m = 0$, $n \geqslant 1$ is obvious. If $n = 1$ and $m \geqslant 1$ then $\omega_i = 1$, $i = 1, \ldots, r$ and this case is handled by Karlin and Schumacher [11].

Now suppose the proposition is true for all generalized monosplines of the form (1) with degree n and $m - 1$ knots. Let M be a monospline of form (1) of degree n with m knots. Consider first the case $\omega_i < n$, $i = 1, \ldots, m$.

Define $D_+ M(x) = \lim_{h \to 0^+} \frac{M(x+h) - M(x)}{h}$. Then $D_+ M$ is of the form (1) and by Rolle's theorem and lemma 1, $D_+ M$ has $\sum_{i=0}^{m} \omega_i + m - 1$ distinct zeros. Therefore the induction hypothesis implies that all coefficients of $D_+ M$ are bounded. Hence the same is true for M except possibly for the constant term λ_0. Since M has certainly one zero and all of its knots are in $(-K, K)$, we see that λ_0 is also bounded.

In the case of $m = n$ for some i, we can appeal to lemma 2 to conclude that the two monosplines M_+ and M_-, as defined above, both have a maximum number of zeros in $(-K, K)$. Applying the induction hypothesis to M_+ and M_- we again conclude that M has bounded coefficients.

We now include the possibility of multiple zeros, using a limiting procedure similar to that of Karlin and Schumacher [11].

PROPOSITION 2: *Given any $K > 0$ there exists a $\lambda > 0$ such that whenever M is of form* (1) *with* $\sum_{i=0}^{m} \omega_i + m$ *zeros up to order n in $(-K, K)$ then* $|a_{ij}| \leqslant \lambda$ *for* $i = 0, \ldots, m$ *and* $j = 0, \ldots, \omega_i - 1$.

Proof. Let ν_i be the multiplicity of the zero x_i, where $\sum_{i=1}^{n} \nu_i = N = \sum_{i=0}^{m} \omega_i + m$. We then „spread apart" the multiple zero x_i by defining $S_{m_i+j}(l) = x_i + j\varepsilon/2^l$ for $j = 0, 1, \ldots, \nu_i - 1$ and $m_i = \sum_{j=1}^{i-1} \nu_j + 1$, where ε is a sufficiently small positive number to insure that $-\infty < S_1 < S_2 < \ldots < S_N < \infty$.

By proposition 1, given any $K > 0$ there exists a $\lambda > 0$ such that whenever M is of the form (1) with zeros $S_i(l)$ in $(-K, K)$ then the corresponding coefficients a^l_{ij} satisfy $|a^l_{ij}| \leqslant \lambda$ for $i = 0, 1, \ldots, m$, $j = 0, 1, \ldots, \omega_i - 1$. Noting that λ is independent of l, there must be a subsequence of coefficients converging as $l \to \infty$, where, in the limit, $|a_{ij}| \leqslant \lambda$ for $i = 0, 1, \ldots, m$, $j = 0, 1, \ldots, \omega_i - 1$. By Rolle's theorem the resulting monospline $M(x)$ has zeros at the x_i with the desired multiplicities ν_i.

III. Generalized Gaussian Quadrature Formulas with Multiple Nodes for Weak Chebysev Systems. In this section we discuss multiple node Gaussian quadrature formulas for weak Chebysev systems where the integral contains a sign function as in the previous section. This will later be used to obtain a fundamental theorem of algebra for totally positive kernels.

An N-dimensional space of functions is called a weak Chebysev space if $u \in U$ implies that u has at most $N - 1$ sign changes.

Let $\{u_i\}_{i=1}^{N}$ be a basis for U, where the domain of U is $[-\delta, 1 + \delta]$ for some $\delta > 0$. Given a set of positive integers $\{\omega_i\}_{i=1}^{m}$ and two non-negative integers ω_0 and ω_{m+1}, we have the following two relationships:

$$\text{(a)}\ N = \sum_{i=0}^{m+1} \omega_i + m$$

and (b) U is a subspace of $C^k[-\delta, 1 + \delta]$, where

$$k \geqslant \max\{\max_{1 \leqslant i \leqslant m} \omega_i, \max_{i=0,\, m+1} (\omega_i - 1)\}.$$

Notice that if $\omega_0 \leqslant 1$ and $\omega_{m+1} \leqslant 1$, we can set $\delta = 0$.

Define the convexity cone $K(U)$ by

$$K(U) = \left\{ f \varepsilon C^k[-\delta, 1 + \delta] : 0 < t_1 < \ldots < t_{N+1} \Rightarrow U\begin{pmatrix} 1, & \ldots, & N f \\ t, & \ldots, & t_{N+1} \end{pmatrix} > 0 \right\}.$$

We then have the following assumption on the cone:

For each set $0 < t_1 < t_2 < \ldots < t_m < 1$,

$$U[t_1, \ldots, t_m] = \{(f(0), \ldots, \overset{(\omega_0-1)}{f}(0), f(t_1), \ldots, \overset{(\omega_1)}{f}(t_1), f(t_2), \ldots, \overset{(\omega_2)}{f}(t_2), \ldots, \overset{(\omega_m)}{f}(t_m),$$

$$f(1), \ldots, \overset{(\omega_{m+1})-1)}{f}(1)) : f \in K(U)\}$$ contains a basis for $\mathbf{R}^N$.

Consider now a measure $d\alpha$ which has the property: For each subspace U_f generated by the functions $\{u_1, \ldots, u_N, f\}$ where $f \in K(U)$, $d\alpha$ is a positive measure. By this we mean that for every nontrivial nonnegative $u \in U_f$, $\int_0^1 u d\alpha > 0$. Let $\sigma_{t,\omega+1}(t)$ be defined on $[-\delta, 1+\delta]$ as in Section I.

A quadrature formula of the form

$Q(u) = \sum_{i=0}^{m+1} \sum_{j=0}^{\omega_i - 1} a_{ij} u^j(t_i)$ where $0 = t_0 < t_1 < \ldots < t_m < t_{m=1} = 1$, such that

$$Q(u) = \int_0^1 u(t) \sigma_{t,\omega+1}(t)\, d\alpha(t) \text{ for all } u \in U$$

will lead us to a fundamental theorem of algebra as desired.

Consider, therefore, the Gaussian transform of $u_i(x)$, defined by

$$u_i(x; \varepsilon) = \frac{1}{|\varepsilon|\sqrt{2\pi i}} \int_{-\delta}^{1+\delta} \exp\left(-\frac{1}{2}\left(\frac{y-x}{\varepsilon}\right)^2\right) u_i(y) dy$$

for each $\varepsilon \neq 0$ and $i = 1, \ldots, N$. For each $\varepsilon \neq 0$, it is well known that $\{u(x; \varepsilon) : i = 1, 2, \ldots, N\}$ forms an N-dimensional extended Chebyshev system. A result of Dyn [8]) tells us that for each $\varepsilon \neq 0$ there is a unique quadrature formula of the type

$$Q_\varepsilon(f) = \sum_{i=0}^{m+1} \sum_{j=0}^{\omega_i - 1} a_{ij}(\varepsilon) \overset{(j)}{f}(t_i)$$

$$\text{so that } Q_\varepsilon(u_i(\cdot\,; \varepsilon)) = \int_0^1 u_i(x; \varepsilon) \sigma_{t((\varepsilon),\omega+1}(x) d\alpha(x)$$

$$\text{for } i = 1, 2, \ldots, N,$$

$$\text{where } 0 = t_0(\varepsilon) < t_1(\varepsilon) < \ldots < t_m(\varepsilon) < t_{m+1}(\varepsilon) = 1.$$

By going to an appropriate subsequence we can assume that as $\varepsilon \downarrow 0$,

$$t_i(\varepsilon) \to t_i, \text{ where } 0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_m \leqslant t_{m+1} = 1.$$

Actually, for these limit points it is true that:

LEMMA 3: *The limit points satisfy* $0 = t_0 < t_1 < \dots < t_m < t_{m+1} =$

Proof. Assume, for example, that $0 = t_0 < t_1 = t_2 < t_3 < \dots < t_m <$
A sequence of functions $\{u_\varepsilon\}$ will be constructed where u_ε in the sp of $\{u_i(\cdot\,;\,\varepsilon)\}_{i=1}^{N}$ is such that $Q_\varepsilon(u) = 0$ for each $\varepsilon > 0$. Further, as $\varepsilon \downarrow$

$$u \to u_\varepsilon \text{ uniformly where } \int u(t)\sigma_{t,\omega+1}(t)\, d\alpha \neq 0,$$

a contradiction.

To accomplish this, select a u which satisfies

$$u_\varepsilon^{(j)}(0) = 0 \quad j = 0, \dots, \omega_0 - 1$$

$$u_\varepsilon(\varepsilon) = 0,\ u_\varepsilon'(\varepsilon) > 0,\ ||u_\varepsilon|| = \max_{x\in[0,1]} |u_\varepsilon(x)| = 1$$

$$u_\varepsilon^{(j)}(t_i(\varepsilon)) = 0 \quad j = 0, \dots, \omega_i - 1 \quad i = 1, 2$$

$$u_\varepsilon^{(j)}(t_i(\varepsilon)) = 0 \quad j = 0, \dots, \omega_i \quad i = 3, 4, \dots, m$$

$$u_\varepsilon^{(j)}(1) = 0 \quad j = 0, \dots, \omega_{m+1} - 1.$$

Recall that $\sigma_{t(\varepsilon),\omega+1}(t)$ is normalized so that $\sigma_{t(\varepsilon),\omega+1}(t) = +1$ for $0 < t < t_1(\varepsilon)$.

Notice that u_ε has $\sum_{i=0}^{2} \omega_i + \sum_{i=3}^{m} (\omega + 1) + \omega_{m+1} + 1 = N - 1$ zeros, allowing the certainty that u has no further sign changes.

By going to a subsequence it can be assumed that $u_\varepsilon \to u \in U$ u formly, where $||u|| = 1$ and $u(t)\sigma_{t,\omega+1}(t) \geqslant 0$. Clearly $Q_\varepsilon(u_\varepsilon) = 0$ for each $\varepsilon >$ but $\int u\sigma d\alpha > 0$, which is the desired contradiction.

LEMMA 4: *For these limit knots,* $0 < t_1 < t_2 < \dots < t_m < 1$, the det minant $D([t_1, \dots, t_m])$ of

$$\begin{pmatrix} u_1(t_0)u_1'(t_0) \dots u_1^{(\omega_0-1)}(t_0)u_1(t_1) \dots u_1^{(\omega_1)}(t_1), \\ \vdots \\ u_N(t_0)u_N'(t_0) \dots u_N^{(\omega_0-1)}(t_0)u_N(t_1) \dots u_N^{(\omega_1)}(t_1) \\ u_1(t_2) \dots u_1(t_m) \dots u_1^{(\omega_m)}(t_m)u_1(t_{m+1}) \dots u^{(\omega_{m+1}-1)}(t_m) \\ \vdots \\ u_N(t_2) \dots u_N(t_m) \dots u_N^{(\omega_m)}(t_m)u_N(t_{m+1} \dots u_N^{(\omega_{m+1}-1)}(t_{m+1}) \end{pmatrix}$$

is positive, where $t_0 = 0$ *and* $t_{m+1} = 1$.

Proof. Assume that the conclusion is not valid. Then there is a set $\{c_i, c_{ij}, d_i\}$ of elements not all zero such that

$$F(u_l) = \sum_{i=0}^{\omega_0-1} c_i u_l^{(i)}(0) + \sum_{i=1}^{m}\sum_{j=0}^{\omega_i} c_{ij}\, u_l^{(j)}(t_i) + \sum_{i=0}^{\omega_{m+1}-1} d_i u_l^{(i)}(t_{m+1}) = 0 \qquad (2)$$

$$\text{for } l = 1, 2, \ldots, N.$$

Since we have assumed that $U[t_1, \ldots, t_m]$ contains a basis for $\mathbf{R}^N$, there is an $f_0 \in K(U)$ such that $F(f_0) \neq 0$. Define $f_0(x; \varepsilon)$ to be the Gaussian transform of f_0 and define

$$\hat{U}(\varepsilon) \equiv \left\{u(x;\ \varepsilon) = \sum_{i=1}^{N} \alpha_i u_i(x;\ \varepsilon) : \sum_{i=1}^{N} \alpha_i^2 \leqslant 1\right\},$$

$$\hat{U}(f_0, \varepsilon) \equiv \left\{v(x;\ \varepsilon) = \alpha_0 f_0(x;\ \varepsilon) + \sum_{i=1}^{N} \alpha_i u_i(x;\ \varepsilon) : \sum_{i=0}^{N} \alpha_i = 1\right\}$$

and

$$F_\varepsilon(g) = \sum_{i=0}^{\omega_i-1} c_i g^{(i)}(0) + \sum_{i=1}^{m}\sum_{j=0}^{\omega_i} c_{ij} g(t_i(\varepsilon)) + \sum_{i=0}^{\omega_{m+1}-1} d_i g^{(i)}(1).$$

As we have assumed that $F(u) = 0$ for $u \in U$, letting $t_i(\varepsilon) \to t_i$ we can secure for each $\eta > 0$ and $\varepsilon(\eta) > 0$ with the property $\varepsilon < \varepsilon(\eta)$ and $u(\cdot\,;\ \varepsilon) \in \hat{U}(\varepsilon) \Rightarrow |F_\varepsilon(u(\cdot\,;\ \varepsilon))| < \eta$. (3)

Also, as $F(f_0) \neq 0$, for small $\varepsilon > 0$, $F_\varepsilon(f_0(\cdot\,;\ \varepsilon))$ is bounded away from zero. Thus, for small $\varepsilon > 0$ we can find a C_ε which is uniformly bounded so that

$$\int_1^0 f_0(x;\ \varepsilon)\sigma_{t(\varepsilon),\omega+1}(x)\,d\alpha(x) = Q_\varepsilon(f_0(\cdot\,;\ \varepsilon)] + C_\varepsilon F_\varepsilon[f_0(\cdot\,;\ \varepsilon)]. \qquad (4)$$

On the other hand, (3) and the properties of Q_ε and F imply

$$\int_0^1 u_\varepsilon \sigma_{t(\varepsilon),\omega+1}\, r\,dx = Q_\varepsilon(u_\varepsilon) = Q_\varepsilon(u_\varepsilon) + C_\varepsilon F_\varepsilon(u_\varepsilon) + o(1) \qquad (5)$$

for all $u_\varepsilon \in \hat{U}(\varepsilon)$ as $\varepsilon \downarrow 0$.

Now for each $\varepsilon > 0$, choose a v_ε in $\hat{U}(f_0, \varepsilon)$ which satisfies

$$v_\varepsilon^{(j)}(0) = 0 \qquad j = 0, \ldots, \omega_0 - 1$$

$$v_\varepsilon^{(j)}(t_i(\varepsilon)) = 0 \qquad j = 0, 1, \ldots, \omega_i,\ i = 1, \ldots, m$$

$$v_\varepsilon^{(j)}(1) = 0 \qquad j = 0, 1, \ldots, \omega_{m+1} - 1$$

$$v_\varepsilon(x) > 0 \qquad \text{for } x \in (0, t(\varepsilon)).$$

Hence, as v_ε is a function in the span of $\{u_i(\cdot\,;\,\varepsilon)\}_{i=1}^N$ and $f_0(\cdot\,;\,\varepsilon)$, we can use equations (4) and (5) and the fact that $Q_\varepsilon(v_\varepsilon) = F_\varepsilon(v_\varepsilon) = 0$ by the construction of v_ε to conclude that

$$\int_0^1 v_\varepsilon\, \sigma_{t(\varepsilon),\omega+1}\, d\alpha = 0(1) \qquad (6)$$

On the other hand, by going to a subsequence we can find a function v in $\hat{U}(f_0,\, 0)$ which is the uniform limit of $\{v_\varepsilon\}$. Moreover, v has sign $+1$ on $(0,\, t_1)$ and sign $(-1)^{\sum_{j=1}^{i} \omega_i}$ for $x \in (t_i,\, t_{i+1})$.

Thus

$$\int_0^1 v(x) \underset{t,\omega+1}{\sigma(x)}\, d\alpha(x) = \lim_{\varepsilon\to 0} \int_0^{t_1(\varepsilon)} |v_\varepsilon(x)|\, (+1)\, (+1)\, d\alpha(x) +$$

$$+ \lim_{\varepsilon\to 0} \int_{t_1(\varepsilon)}^{t_2(\varepsilon)} |v_\varepsilon(x)|(-1)^{\omega_1+1}(-1)^{\omega_1+1}\, d\alpha(x) + \dots$$

$$+ \lim_{\varepsilon\to 0} \int_{t_m(\varepsilon)}^{1} |v_\varepsilon(x)|\, (-1)^{\sum_{i=1}^{m} \omega_i + m} (-1)^{\sum_{i=1}^{m} \omega_i + m}\, d\alpha(x)$$

which is strictly positive, contradicting (6).

We now prove a general existence theorem for generalized Gaussian quadrature formulas with respect to weak Chebyshev systems.

THEOREM 1. *There exists a generalized Gaussian quadrature formula of the form*

$$Q(u) = \sum_{i=0}^{m+1} \sum_{j=0}^{\omega_i - 1} a_{i_j}\, u^{(j)}(t_i) \text{ such that} \qquad (7)$$

$$Q(u) = \int_0^1 u(t) \underset{t,\omega+1}{\sigma(t)}\, d\alpha(t) \text{ for all } u \in U,$$

where $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$.

Proof. By the result of Dyn, for each $\varepsilon \to 0$ there is a unique quadrature formula Q_ε with the property

$$Q_\varepsilon\,[u_i(\cdot\,;\,\varepsilon)] = \int_0^1 u_i(t\,;\,\varepsilon) \underset{t,\omega+1}{\sigma(t)}\, d\alpha(t) \qquad i = 1, \dots, N.$$

Letting $\varepsilon \downarrow 0$, the coefficients associated with Q_ε must be bounded uniformly, for otherwise one could construct a set of coefficients $\{c_i, c_{i_j}, d_i\}$ not all zero such that the corresponding F [see (2)] $Fu = 0$ for all $u \in U$. To construct such a set of coefficients, assume that exists a coefficient $a_i \in \{c_i, c_{i_j}, d_i\}$ which is unbounded. Upon dividing by this coefficient, in the limit one obtains a non-zero F for which $Fu = 0$ for all $u \in U$.

As such a relationship contradicts lemma 4, the coefficients of $\{Q_\varepsilon\}$ are bounded as $\varepsilon \to 0$. Therefore, by compactness, there exists a limit for the coefficients of $Q = \lim_{\varepsilon \to 0} Q_\varepsilon$. Hence we have the existence of the desired quadrature formula.

LEMMA 5. *If* $Q(u) = \sum_{i=0}^{m+1} \sum_{j=0}^{\omega_i - 1} a_{i_j} u^{(j)}(t_i)$ *is such that*

$$Q(u) = \int_0^1 u(t) \sigma_{t, \omega+1}(t) \, d\alpha(t) \text{ for all } u \in U,$$

where $0 \leqslant t_1 \leqslant \ldots \leqslant t_m \leqslant 1,\ t_0 = 0,\ t_{m+1} = 1,$

then (a) $0 = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = 1$ *and*

(b) $$\operatorname{sgn} a_{i, \omega_i - 1} = (-1)^{\sum_{j=1}^{i} (\omega_j + 1)}$$

Proof. Assume that (a) is not correct. Then a contraction can be reached as in lemma 3.

To prove (b), for each $1 \leqslant k \leqslant m$ and $\varepsilon > 0$ one can find a function $u(\cdot\,;\varepsilon)$ in the span of $\{(u_i(\cdot\,;\varepsilon)\}_{i=1}^{N}$ which satisfies

$$\|u(\cdot\,;\varepsilon)\| = \max_{x \in [0,1]} |u(x;\varepsilon)| = 1$$

$$u^{(j)}(0;\varepsilon) = 0 \qquad j = 0, \ldots, \omega_0 - 1$$

$$u(\varepsilon;\varepsilon) = 0$$

$$u^{(j)}(t_i;\varepsilon) = 0 \qquad j = 0, 1, \ldots, \omega_i, \quad i = 1, \ldots, k-1, \quad k+1, \ldots m$$

$$u^{(j)}(t_k;\varepsilon) = 0 \qquad j = 0, \ldots, \omega_k - 2$$

$$(-1)^{\sum_{i=1}^{k} (\omega_i + 1)} u^{(\omega_i + 1)}(t_k;\varepsilon) > 0$$

$$u^{(j)}(1;\varepsilon) = 0 \qquad j = 0, 1, \ldots, \omega_{m+1}$$

By letting $\varepsilon \downarrow 0$ we can secure a uniform limit function on $[0, 1]$, $u_k \in U$. As t_i is a zero of multiplicity $\omega_i + 1$ for $i \neq k$, $i = 1, \ldots, m$, the sign of the integrand $u_k(t) \underset{t,\omega+1}{\sigma}(t)$ remains constant over $[0, 1]$. As $u^{(\omega_k-1)}(t_k; \varepsilon) > 0$, $u_k \underset{t,\omega+1}{\sigma}(t)$ is a nonnegative product. Consider, then, the result of the fact that

$$Q(u) = \int_0^1 u(t) \underset{t,\omega+1}{\sigma}(t)\, d\alpha(t) \text{ for all } u \in U;$$

$$0 < \int_0^1 u_k(t) \underset{(t,\omega+1)}{\sigma(t)}\, d\alpha(t) = Q(u_k) = a_{k,\omega_k-1}\, u_k^{(\omega_k-1)}(t_k)$$

implies that

$$\operatorname{sgn} a_{k,\omega_k-1} = \operatorname{sgn} u_k^{(\omega_k-1)}(t_l) = (-1)^{\sum_{i=1}^{k}(\omega_i+1)}.$$

THEOREM 2. *There is a unique quadrature formula of the form* (7) *such that*

$$Q(u) = \int u \underset{t,\omega+1}{\sigma}(t)\, d\alpha \qquad (8)$$

for all $u \in U$. [Note that by Lemma 3 all the t_i are distinct and lie in (0, 1).]

Proof. By Theorem 1 there exists a formula Q^* of the form (7) which satisfies (8). Let

$$A^* = [a_{00}^*, \ldots, a_{0,\omega_0-1}^*, a_{10}^*, \ldots, a_{1,\omega_1-1}^*, a_{20}^*, \ldots, a_{m,\omega_m-1}^*, a_{m+1,0}^*, \ldots, a_{m+1,\omega_{m+1}-1}^*, t_1^*, \ldots, t_m^*] \subset \mathbf{R}^N$$

be the set of values which define Q^*, and for $|\varepsilon| > 0$, let $u_i(\cdot\,; \varepsilon)$ be the Gaussian transform of $u_i(\cdot\,; 0) \equiv u_i$. Consider the nonlinear system of N equations

$$Q[u_i(\cdot\,; \varepsilon), A] = \int_0^1 u_i(t; \varepsilon) \underset{t,\omega+1}{\sigma}(t)\, d\alpha(t) \qquad (9)$$

with the vector of N unknowns:

$$A = [a_{00}, \ldots, a_{0,\omega_0-1}, a_{01}, \ldots, a_{1,\omega_1-1}, a_{20}, \ldots, a_{m,\omega_m-1}, a_{m+1,0}, \ldots, a_{i,+1,\omega_{m+1}-1}, t_1, \ldots, t_m]$$

associated with any quadrature of the form (7). For $\varepsilon = 0$, clearly $Q^* = Q(\cdot\,; A^*)$ satisfies (9). We indicate his dependence by letting $A^* = A^*(\varepsilon)$ for $\varepsilon = 0$. We would like to aaply the implicit function theorem to (9) with the parameter ε near $\varepsilon = 0$. At $A = A^*$ and $\varepsilon = 0$, the

Jacobian determinant at $A = A^*$ and $\varepsilon = 0$ is $\pm \prod_{i=1}^{m} a^*_{i,\omega_i-1} D[t_1^*, t_2^*, \ldots, t_m^*]$ where $D(t_1, \ldots, t_m)$ is as defined in lemma 4. By lemma 5,

$$\operatorname{sgn} a_{i,\omega_i-1} = (-1)^{\sum_{j=1}^{i}(\omega_j+1)} \qquad i = 1, \ldots, m,$$

hence $a_{i,\omega_i-1} \neq 0$ for $i = 1, \ldots, m$. Further, an argument fashioned after Lemma 4 shows that $D(t_1^*, \ldots, t_m^*) > 0$. By the implicit function theorem, for small $|\varepsilon| > 0$ one can find a solution $A^*(\varepsilon)$ of the nonlinear system (9) close to A^*.

Assume now that there is another solution to (9) at $\varepsilon = 0$, say $\hat{A} = \hat{A}(0)$. Then by the same reasoning one could find a solution $\hat{A}(\varepsilon)$ of (9) close to $\hat{A}$ for small $|\varepsilon| > 0$. For even smaller $|\varepsilon| > 0$, $\hat{A}(\varepsilon) \neq \neq A^*(\varepsilon)$. Thus far such ε, (9) has at least two solutions, contradicting the result of Dyn.

IV. **Fundamental Theorem of Algebra for Generalized Polynomial Monosplines.** Consider the polynomial spline kernel $\Phi_p(x, t) = \frac{(x-t)_+^{p-1}}{(p-1)!}$, where $p \geq 3$. We are given a nonnegative integer $\omega_0 \leq p$, positive integers $\{\omega_i\}_{i=1}^{m}$ and positive integers $\{\nu_i\}_{i=1}^{n}$ which satisfy the relation ships

(a) $$N = \sum_{i=0}^{m} \omega_i + m = \sum_{i=1}^{n} \nu_i,$$

(b) If $M_1 = \max_{1 \leq i \leq m} \omega_i$ and $M_2 = \max_{1 \leq i \leq m} \nu_i$,

then $$M_1 + M_2 \leq p - 1.$$

THEOREM 3. *For each set of n distinct numbers, $0 < x_1 < x_2 < \ldots \ldots < x_n < 1$, there is a unique generalized monospline of the form*

$$M(x) = \int_0^1 \Phi_p(x, t) \underset{t,\,\omega+1}{\sigma(t)}\, d\alpha(t) - \sum_{j=0}^{\omega_0-1} a_j \Phi_p^{(j)}(x, 0) - \sum_{i=1}^{m} \sum_{j=0}^{\omega_i-1} a_{i_j} \Phi_p^{(j)}(x, t_i)$$

such that M_p has a zero of order ν_i at x_i, $i = 1, \ldots, n$.

Here

$$\Phi_p^{(j)}(x, t) = \frac{\partial^j}{\partial t_j} \Phi_p(x, t) \quad \text{and} \quad 0 \leq t_1 \leq \ldots \leq t_m \leq 1.$$

Indeed, for this monospline, $0 < t_1 < t_2 < \ldots < t_m < 1$ and

$$\operatorname{sgn} a_{i,\omega_i-1} = (-1)^{\sum_{j=1}^{i}(\omega_j+1)}, \quad i = 1, \ldots, m.$$

Proof:

We set $S(k) = \sum_{j=0}^{k-1} \nu_j$, $k = 1, \ldots, n$ where $\nu_0 = 0$, and define

$$u_{S(k)+l}(t) = \frac{\partial^{l-1}}{\partial x_k - 1} \Phi_p(x_k, t) \qquad \begin{matrix} l = 1, \ldots, \nu_k \\ k = 1, \ldots, n. \end{matrix}$$

So $M^{(l-1)}(x_k)$ translates into

$$\int_0^1 u_{S(k)+l}(t) \underset{t,\,\omega+1}{\sigma}(t)\, d\alpha(t) = \sum_{j=0}^{\omega_0-1} a_j\, u^{(j)}_{S(k)+l}(0) + \sum_{i=1}^{m} \sum_{j=0}^{\omega_i-1} a_{i_j}\, u^{(j)}_{S(k)+l}(t_i) \qquad \begin{matrix} l = 1, \ldots, \nu_k \\ k = 1, \ldots, n \end{matrix}$$

From the fundamental theorem of determinants for polynomial splines [12], one can infer that

(a) $\left\{u_i(t)\right\}_{i=1}^{N}$ form a weak Chebyshev system.

(b) For each $x \in [0, 1]$ one of the functions $f(t) = \pm\, \Phi_p(x, t)$ is in the convex cone $K(U)$ of $\left\{u_i(t)\right\}_{i=1}^{N}$.

(c) For each sequence $0 < t_1 < \ldots < t_m < 1$, the set of N functions

$$\Phi_p(x, 0), \ldots, \Phi_p^{(\omega_0-1)}(x, 0), \Phi_p(x, t_1), \ldots, \Phi_p^{(\omega_1)}(x, t),$$

$$\Phi_p(x, t_2), \ldots, \Phi_p^{(\omega_{m-1})}(x, t_{m-1}), \Phi_p(x, t_m), \ldots, \Phi_p^{(\omega_m)}(x, t_m)$$

is independent.

The fact that $\pm\Phi_p(x, t)$ is in the convexity cone of $\{u_i(t)\}_{i=1}$ combined with the independence in (c) tells us that for each $0 < t_1 < \ldots < t_m < 1$, $U[t_1, \ldots, t_m]$ contains a basis for $\mathbf{R}^N$.

Therefore, the result follows directly from theorem 2.

One can also obtain similar results by including the right hand end point $t_{m+1} = 1$. We seek an expression of the type

$$Q(u) = \sum_{i=1}^{m} \sum_{j=0}^{\omega_i-1} a_{i_j}\, u^{(j)}(t_i) + \sum_{i=0}^{\omega_{m+1}-1} b_i\, u^{(i)}(1), \tag{10}$$

where $0 \leqslant t_1 \leqslant \ldots \leqslant t_m \leqslant 1$, such that

$$Q(u) = \int_0^1 u(t) \underset{t,\omega+1}{\sigma(t)}\, d\alpha(t) \tag{11}$$

for all u in the N-dimensional subspace generated by $\left\{u_i\right\}_{i=1}^{N}$.

The $\{\omega_i\}$, $\{\nu_i\}$ and ω_{m+1} satisfy the same restraints as in the first application.

Proceeding exactly as before, we can show:

THEOREM 4. *There exists a unique Q of the form* (10) *which satisfies* (11). *Furthermore, for such a Q,* $0 < t_1 < \ldots < t_m < 1$ *and* a_{i,ω_i-1} *has sign* $(-1)^{\sum_{j=1}^{i}(\omega_j+1)}$ *for* $i = 1, \ldots, m$.

Remark:

For the cases $p = 1$ and $p = 2$ in the defining polynomial spline kernel

$$\Phi_p(x, t) = \frac{(x-t)_+^{p-1}}{(p-1)!}$$

similar results can be obtained. In the case that $p = 1$, the relationships (a) and (b) preceeding Theorem 3 reduce to the restriction that

(a)' $\quad N = 2m + 1$ where $\omega_i = 1$ for $i = 0, \ldots, m$

and

(b)' $\quad M_1 = M_2 = 1. \qquad \nu_i = 1$ for $i = 1, \ldots, n$

In this setting $\underset{t,\omega+1}{\sigma}(t) = +1$ for all t and therefore the generalized monospline reduces to the Tchebycheffian (T^-) monospline of degree 1 with m knots considered by Karlin and Schumacher [11]. The fundamental theorem of algebra for the case $p = 1$ is found in Theorem 1.1 of that paper.

Consider the case $p = 2$, where

$$M(x) = \int_{c=t_0}^{d} (x-t)_+^1 \underset{t,\omega+1}{\sigma}(t)\, dt - \sum_{i=0}^{m}\sum_{j=1}^{\omega_i} a_{ij}(x-t)_+^{2-j}. \tag{12}$$

We wish to show the existence and uniqueness of such a generalized monospline where $\{x_i\}_{i=1}^{N}$ are given and we require that $M(x_i) = 0$ for $i = 1, 2, \ldots, N$. We assume first that $a < x_1 < x_2 < \ldots < x_n < b$.

Consider first the monospline $M_1(x)$ which is the restriction of M to (t_0, t_1). By lemma 4 Section III, $x_{2i} < t_i < x_{2i+1}$ for $i = 1, 2, \ldots, m$. By definition, therefore

$$M_1(x) = \frac{(x-c)^2}{2} - a_{01}(x-c) - a_{02}.$$

In the case $\omega_0 = 1$, $M_1(x) = (x-c)\left(\frac{(x-c)}{2} - a_{01}\right)$ and so $M_1(x_1) = M_1(x_2) = 0$ implies that we must have $x_1 = c$ and $a_{01} = \frac{x_2 - c}{2}$.

If $\omega_0 = 2$, then a_{01} and a_{02} are given by unique solutions to a linear system induced by the zero structure. The determinant is nonzero as a result of the fact that $c < x_1 < x_2$. The right hand side is nonzero for the same reason, giving a unique set of defining coefficients for $M(x)$.

In considering $M(x)$ in the interval $t_1 < x < t_2$, as $x_4 < t_2 < x_5$ by lemma 2 of Section I

$$M(x) = M_1(x) + [(-1)^{\omega_1} + 1]\left(\frac{(x - t_1)^2}{2}\right) - a_{11}(x - t_1) - a_{12}.$$

We then use the fact that $M(x_3) = M(x_4) = 0$ to determine the unknowns a_{11}, a_{12} and t_1.

If $\omega_1 = 1$ then we are in the classical case and Theorem 1 of Micchelli [13] gives the desired result.

If $\omega_1 = 2$ then by lemma 4 of Section III, $t_1 = x_3$. Thus we can solve for a_{11} and a_{12} using the equations $M(x_3) = 0 = M_1(x_3) - a_{12}$ and $M(x_4) = 0 = M_1(x_4) + (x_4 - x_3)^2 - a_{11}(x_4 - x_3) - a_{12}$. Therefore $a_{12} = M_1(x_3)$ and $a_{11} = M_1(x_3, x_4] + (x_4 - x_3)$
where $M_1[x_3, x_4]$ denotes the divided difference of M_1 with respect to x_3 and x_4 (see ref. [7], page 195). Note that $M(x)$ is not identically zero for if it were $M_1(x)$ would be identically equal to $p_2(x) = -(x - x_3)^2 + a_{11}(x - x_3) + a_{12}$. Upon examining the roots of $p_2(x)$ one finds that it has one real root to the right of x_3. Thus M has a third root and must be identically zero, a contradiction.

The above process may be repeated to recursively determine the set $\left\{a_{k1}, a_{k2}, t_k\right\}_{k=1}^{m}$ and so follows the existence and uniquences of a generalized polynomial monospline when $p = 2$ and we have simple $\left\{x_i\right\}_{i=1}^{N}$.

To allow the zeros to have multiplicity two when $p = 2$ we employ a limiting argument similar to that of Karlin and Schumacher ([II], page 267). In this case

$$N = \sum_{i=0}^{m} \omega_i + m = \sum_{i=1}^{r} n_i \text{ where if } \omega = \max_{0 \leq i \leq m} \omega_i \text{ and}$$

$n = \max_{1 \leq i \leq r} n_i$, then $\omega + n \leq 2$, and we have prescribed zeros of $M(x)$ at x_i of order n_i, $i = 1, \ldots, r$. Here $M(x)$ is of the form (1).

For each $l \geq 1$ consider a set of points $\left\{y_i(l)\right\}_{i=1}^{N}$ formed from $\left\{x_i\right\}_{i=1}^{r}$ by „spreading apart'' the multiple zeros. Specifically, if $x_{m-1} < x_m = x_{m+1} < x_{m+2}$ for some $2 \leq m \leq r - 1$, then define $y_{m+1}(l) = x_m + \frac{\varepsilon}{2l}$ where ε is a sufficiently small positive number to insure that $y_i(l) < y_{i+1}(l)$ for $i = 1, \ldots, N - 1$. For each l there exists a generalized monospline of the form (12), call it M_l, with zeros $\left\{y_i(l)\right\}_{i=1}^{N}$. Suppose that M_l has the representation

$$M_l(x) = \int_c^d (x - t)_+^{n-1} \underset{l,\omega+1}{\sigma}(t)\, dt - \sum_{i=0}^{m} \sum_{j=1}^{\omega_i} a_{ij}^l (x - t_i^l)_+^{n-j}.$$

The sequence of coefficients $\left\{a_{ij}^{l}\right\}_{i=0}^{m}{}_{j=1}^{\omega_i}$ and the sequence of knots $\left\{t_i^l\right\}_{i=0}^{m}$ depend continuously on the variable x and hence on l. By Proposition 1 of Section II the coefficients are uniformly bounded as $l \to \infty$. The knots $\left\{t_i\right\}_{i=0}^{m}$ are trivially bounded as noted by lemma 2, Section II.

Thus there exists a subsequence $\{l_k\}$ such that all coefficients and knots converge. By continuity and Rolle's theorem, the limit generalized monospline has the desired zeros $\left\{x_i\right\}_{i=1}^{r}$.

We can now state an extension to theorem 3:

THEOREM 5. *The results of theorem* **3** *remain valid if* $p = 1$ *or* $p = 2$.

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ON SOME CLASSES OF BI-UNIVALENT FUNCTIONS

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ABSTRACT. — Let the functions $f(z) = z + a_2 z^2 + \dots$ and its inverse f^{-1} be analytic and univalent in the unit disc. The authors obtain upper bounds for $|a_2|$ and $|a_3|$ under various additional hypotheses — namely, that f and f^{-1} are both (i) strongly starlike of order α, (ii) starlike of order β, (iii) convex of order β.

1. **Introduction.** In this note we discuss several classes of functic

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.}$$

that are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$. T class of all such functions we denote by S. We denote by σ the class all functions of the form (1.1) that are analytic and bi-univalent in t unit disc, that is $f \in S$ and f^{-1} has a univalent analytic continuation $\{|w| < 1\}$. We also introduce the following classes:

(i) The class $S^*_\sigma[\alpha]$ of strongly bi-starlike functions of order α, $0 < \alpha \leqslant$

(ii) The class $S^*_\sigma(\beta)$ of bi-starlike functions of order β, $0 \leqslant \beta < 1$.

(iii) The class $C_\sigma(\beta)$ of bi-convex functions of order β, $0 \leqslant \beta < 1$.

For the above classes we give bounds for $|a_2|$, $|a_3|$; also for the cla $C_\sigma(0)$ we give the bound for $|a_n|$ and the extremal function.

The class σ was first investigated by L e w i n [1]; the showed th $|a_2| < 1.51$. Later B r a n n a n [2, Problem 6.82] conjectured that $|a_2| \leqslant \sqrt{2}$. The class $S^*_\sigma[\alpha]$ and the class $C_\sigma(0) \equiv C_\sigma$ were first introduced in [

2. **The class** $S^*_\sigma[\alpha]$

A function $f(z)$ of the form (1.1) belongs to the class $S^*_\sigma[\alpha]$, $0 < \alpha <$ if it satisfies the following set of conditions:

$$f \in \sigma, \tag{2.}$$

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2}, \quad |z| < 1, \tag{2.}$$

$$\left|\arg \frac{wg'(w)}{g(w)}\right| < \frac{\alpha\pi}{2}, \quad |w| < 1, \tag{2.}$$

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where

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \dots, \tag{2.4}$$

is the extension of f^{-1} to the whole of $|w| < 1$.

THEOREM 2.1. *Let*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

belong to $S_\sigma^*[\alpha]$. *Then*

$$|a_2| \leqslant \frac{2\alpha}{\sqrt{1+\alpha}} \quad \text{and} \quad |a_3| \leqslant 2\alpha.$$

Proof. We are going to follow the notation used in [4]; namely, we denote by P_α, $0 < \alpha \leqslant 1$, the class of functions

$$P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

that are analytic in the unit disc U and subordinate to the function $\left[\frac{1+z}{1-z}\right]^\alpha$. Now, $P(z) \in P_\alpha$ if an only if $P(z) = [h(z)]^\alpha$, where $h(z) \in P_1$; and P_1 is the class of functions of positive real part in U.

Conditions (2.2) and (2.3) can be written as

$$\frac{zf'(z)}{f(z)} = [Q(z)]^\alpha \tag{2.5}$$

and

$$\frac{wg'(w)}{g(w)} = [P(w)]^\alpha, \tag{2.6}$$

respectively, where $Q(z)$, $P(w)$ belong to P_1 and have the forms

$$Q(z) = 1 + c_1 z + c_2 z + \dots$$

and

$$P(z) = 1 + p_1 w + p_2 w^2 + \dots$$

If $f(z) \in S_\sigma^*(\alpha)$, then by (2.5)

$$\frac{zf'(z)}{f(z)} = [Q(z)]^\alpha = [1 + c_1 z + c_2 z^2 + \dots]^\alpha.$$

From this, it follows that

$$a_2 = \alpha c_1$$

$$2a_3 = a_2^2 + \alpha c_2 + \frac{\alpha(\alpha-1)}{2} c_1^2.$$

Also by (2.6)

$$\frac{wg'(w)}{g(w)} = [p(w)]^{\alpha} = [1 + p_1 w + p_2 w^2 + \ldots]^{\alpha}.$$

This gives

$$a_2 = -\alpha p_1$$

$$3a_2^2 = 2a_3 + \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2.$$

Combining the set of equations for a_2, a_3 we obtain

$$a_2^2 = \frac{\alpha^2(c_2 + p_2)}{\alpha + 1}.$$ $|p_n| \leq 2$

By a well known theorem due to Carathéodory [5, page 41], $|p_n| \leqslant 2$, $|c_n| \leqslant 2$. Hence

$$|a_2| \leqslant \frac{2\alpha}{\sqrt{1 + \alpha}}.$$

For a_3 we have

$$4a_3 = \alpha(p_2 + 3c_2) + 2\alpha(\alpha - 1)c_1^2. \quad (2.7) \qquad (2.7)$$

If $\alpha = 1$, then $|a_3| \leqslant 2$. So we consider the case $0 < \alpha < 1$. By (2.7)

By (2.7) $$4 \operatorname{Re} a_3 = \alpha \operatorname{Re} \{p_2 + 3c_2 - 2(1 - \alpha)c_1^2\}. \quad (2.8) \qquad (2.8)$$

For the functions $Q(z)$, $P(w)$, Herglotz's representation formula [5, page 40] states that

$$Q(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, d\mu_1(t)$$

and

$$P(w) = \int_0^{2\pi} \frac{1 + we^{-it}}{1 - we^{-it}}\, d\mu_2(t),$$

where $\mu_i(t)$ are increasing on $[0, 2\pi]$ and $\mu_i(2\pi) - \mu_i(0) = 1$, $i = 1, 2$.
We also have

$$c_n = 2\int_0^{2\pi} e^{-int}\, d\mu_1(t), \quad n = 1, 2, \ldots,$$

and

$$p_n = 2\int_0^{2\pi} e^{-int}\, d\mu_2(t), \quad n = 1, 2, \ldots .$$

Now (2.8) becomes

$$4 \operatorname{Re} a_3 = 2\alpha \int_0^{2\pi} \cos 2t \, d\mu_2(t) + 6\alpha \int_0^{2\pi} \cos 2t \, d\mu_1(t) -$$

$$- 8\alpha(1-\alpha)\left\{\left[\int_0^{2\pi} \cos t \, d\mu_1(t)\right]^2 - \left[\int_0^{2\pi} \sin t \, d\mu_1(t)\right]^2\right\},$$

$$\leqslant 2\alpha \int_0^{2\pi} \cos 2t \, d\mu_2(t) + 6\alpha \int_0^{2\pi} \cos 2t \, d\mu_1(t) + 8\alpha(1-\alpha)\left[\int_0^{2\pi} \sin t \, d\mu_1(t)\right]^2 =$$

$$= 2\alpha\left\{1 - 2\int_0^{2\pi} \sin^2 t \, d\mu_2(t) + 3 - 6\int_0^{2\pi} \sin^2 t \, d\mu_1(t) +\right.$$

$$\left. + 4(1-\alpha)\left[\int_0^{2\pi} \sin t \, d\mu_1(t)\right]^2\right\}.$$

By Jensen's inequality [6, page 61], we have that

$$\left[\int_0^{2\pi} |\sin t| \, d\mu(t)\right]^2 \leqslant \int_0^{2\pi} \sin^2 t \, d\mu(t).$$

Hence

$$4 \operatorname{Re} a_3 \leqslant 2\alpha\left\{4 - 2\int_0^{2\pi} \sin^2 t \, d\mu_2(t) - 2(1+2\alpha)\int_0^{2\pi} \sin^2 t \, d\mu_1(t)\right\}.$$

Therefore $\operatorname{Re} a_3 \leqslant 2\alpha$, which implies that

$$|a_3| \leqslant 2\alpha.$$

The effect of the bi-univalency condition can be easily seen by looking at the coefficients of the corresponding class $S^*[\alpha]$ introduced in [4]; this is the class of functions f of the form (1.1) univalent in $|z| < 1$ and satisfying the condition (2.2). There the sharp coefficient bounds are

$$|a_2| \leqq 2\alpha,$$

and

$$\text{if } 0 < \alpha < \frac{1}{3}, \text{ then } |a_3| \leqq \alpha,$$

$$\text{if } \frac{1}{3} < \alpha \leqq 1, \text{ then } |a_3| \leqq 3\alpha^2,$$

and

$$\text{if } \alpha = \frac{1}{3}, \text{ then } |a_3| \leqslant \frac{1}{3} \cdot$$

In each case the stated coefficient bound is sharp.

It would be of interest to know what the sharp bounds on the coefficients a_2, a_3 are in the class $S_\sigma^*[\alpha]$.

3. The class $S_\sigma^*[\beta]$

We define the class $S_\sigma^*(\beta)$, $0 \leqslant \beta < 1$, to be the class of functions of the form (1.1) satisfying the following conditions:

$$f \in \sigma,$$

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, \ |z| < 1, \tag{3.1}$$

and

$$\text{Re}\left\{\frac{wg'(w)}{g(w)}\right\} > \beta, \ |w| < 1, \tag{3.2}$$

where $g(w)$ is the same function as in (2.4). We call $S_\sigma^*(\beta)$ the class of bi-starlike functions of order β.

THEOREM 3.1. *Let*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

belong to $S_\sigma^*(\beta)$, $0 \leqslant \beta < 1$. *Then*

$$|a_2| \leqslant \sqrt{2(1-\beta)} \ \textit{and} \ |a_3| \leqslant 2(1-\beta).$$

Proof: Let $P(\beta)$ be the class of functions $V(z)$ analytic in $|z| < 1$ with $V(0) = 1$, $\text{Re} > (z) > \beta$ in $|z| < 1$.

In fact $P(0)$ is just the class of functions

$$P(z) = 1 + p_1 z + p_2 z^2 + \ldots$$

for which $\text{Re}\, P(z) > 0$.

Note that $V(z) \in P(\beta)$ if and only if

$$P(z) = \frac{1}{1-\beta}(V(z) - \beta) \text{ belongs to } P(0).$$

Hence, it follows that there exists a unique $P(z) \in P(0)$ such that

$$V(z) = \beta + (1-\beta)p(z), \tag{3.3}$$

for $V(z)$ in $P(\beta)$.

Now conditions (3.1) and (3.2) are equivalent to

$$\frac{zf'(z)}{f(z)} = \beta + (1 - \beta)Q(z) \tag{3.4}$$

and

$$\frac{wg'(w)}{g(w)} = \beta + (1 - \beta)P(w), \tag{3.5}$$

respectively, where $Q(z)$, $P(w)$ belong to $P(0)$ and have the forms

$$Q(z) = 1 + c_1 z + c_2 z^2 + \dots$$

and

$$P(z) = 1 + p_1 w + p_2 w^2 + \dots .$$

Now, it follows from (3.4) that

$$a_2 = (1 - \beta)c_1 \tag{3.6}$$

and

$$2a_3 = a_2(1 - \beta)c_1 + (1 - \beta)c_2. \tag{3.7}$$

Also from (3.5) it follows that

$$a_2 = -(1 - \beta)p_1 \tag{3.8}$$

and

$$4a_2^2 = 2a_3 - a_2(1 - \beta)p_1 + (1 - \beta)p_2. \tag{3.9}$$

The four equations give

$$2a_2^2 = (1 - \beta)(c_2 + p_2).$$

Using the bounds for $|c_2|$ and $|p_2|$, we obtain

$$|a_2| \leqslant \sqrt{2(1 - \beta)}$$

and

$$|a_3| \leqslant 2(1 - \beta).$$

In comparison, let $S^*(\beta)$, $0 < \beta \leqslant 1$, denote the class of functions starlike of order β in $|z| < 1$; this is the class of functions f of the form (1.1) univalent in $|z| < 1$ and satisfying the condition (3.1). It was shown in [7] that the sharp coefficient bounds for a_2, a_3 are

$$|a_2| \leqslant 2(1 - \beta),$$

$$|a_3| \leqslant (1 - \beta)(3 - 2\beta).$$

It would be of interest to know what are the sharp bounds on the coefficients a_2, a_3 in the class $S_\sigma^*(\beta)$.

4. The class $C_\sigma(\beta)$

A function $f(z)$ of the form (1.1) belongs to the class $C_\sigma(\beta)$ it satisfies the following set of conditions:

$$f \in \sigma, \tag{4.1}$$

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > \beta, \quad |z| > 1, \tag{4.2}$$

$$\operatorname{Re}\left\{\frac{wg''(w)}{g'(w)} + 1\right\} > \beta, \quad |w| > 1, \tag{4.3}$$

where $g(w)$ is the function defined in (2.4).

THERORЕM 4.1. *Let*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

belong to $C_\sigma(\beta)$. *Then*

$$|a_2| \leqslant \sqrt{1-\beta} \text{ and } |a_3| \leqslant 1 - \beta.$$

Moreover, for the class $C_\sigma(0)$, *the extremal function is given by* $f(z) = \dfrac{z}{1-z}$ *and its rotations.*

Proof. Using the same notation as in Theorem 3.1, conditions (4.2), (4.3) give

$$\frac{zf''(z)}{f'(z)} + 1 = \beta + (1-\beta)Q(z) \tag{4.4}$$

and

$$\frac{wg''(w)}{g'(w)} + 1 = \beta + (1-\beta)P(w), \tag{4.5}$$

where $Q(z)$, $P(w) \in P(0)$.

Equation (4.4) gives us that

$$2a_2 = (1-\beta)c_1 \tag{4.6}$$

and

$$6a_3 = (1-\beta)c_2 + 2a_2(1-\beta)c_1. \tag{4.7}$$

from these two equations we obtain

$$6a_3 = 4a_2^2 + (1-\beta)c_2. \tag{4.8}$$

Now, by (4.5) we obtain that

$$2a_2 = -(1-\beta)p_1 \tag{4.9}$$

and

$$12a_2^2 = 6a_3 + (1-\beta)p_2 - 2a_2(1-\beta)p_1. \tag{4.10}$$

The two equations give

$$8a_2^2 = 6a_3 + (1 - \beta)p_2. \tag{4.11}$$

Combining (4.8) and (4.11) and using the bounds for $|p_2|$ and $|c_2|$, we obtain that

$$|a_2| \leqslant \sqrt{1 - \beta}$$

and

$$|a_3| \leqslant 1 - \beta.$$

In the case $\beta = 0$, we have $C_\sigma(0) \subset C$, where C is the class of all λ normalised functions convex in the unit disc. This implies that

$$|a_n| \leqslant 1, \quad n = 2, 3, \ldots,$$

which is sharp as seen from the function

$$f(z) = \frac{z}{1 - z}, \tag{4.12}$$

which is in $C_\sigma(0)$.

The question arises whether the class $C_\sigma(0)$ and the class C are the same. The function

$$f(z) = \frac{1}{2\alpha}\left[\left(\frac{1 + z}{1 - z}\right)^\alpha - 1\right], \qquad \frac{1}{2} < \alpha < 1,$$

belongs to C; since it is not bi-univalent, it is not in $C_\sigma(0)$ — consequently $C_\sigma(0)$ is a proper subclass of C.

We emphasize that it is *not* true that: A function $f(z)$ is bi-convex in U if and only if $zf'(z)$ is bi-starlike in U. This is clear from the function in (4.12) which is bi-convex; however for that function $zf'(z)$ is the Koebe function which is not bi-starlike (since it is not bi-univalent).

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RECENZII

Measure Theory and Its Applications. Proceedings, Sherbrooke, Canada 1982, Lectures Notes in Mathematics, vol. 1033, Springer-Verlag, Berlin Heidelberh New York, 1983, 317 pp.

These are the Proceedings, edited by J—M. Belley, J. Dubois and P. Morales, of the Workshop on Measure Theory and its Applications, held at the Université de Sherbrooke from June 7 to 18, 1982. The Workshop was attented by 87 mathematicians from 12 countries presenting new and significant results in Ergodic Theory, Choquet Representation Theory, Vector Measures and Topology. The book contains 29 contributions of the participants. Let us remark the survey papers by G. Choquet, ,,Représéntation integrale", p. 114—143 and by J. Diestel and J.J. Uhl, Jr., ,,Progress in Vector Measures" — 1977—1983, p. 144—192. There are also other valuable papers written by leading specialists in measure theory as M. Ackoglu, J. Batt, N. Dinculeanu, A. Bellow, J.K. Brooks, G.A. Edgar, P. Greim, J. Oxtoby, F. Topse. By presenting the State of the Art, new results and putting problems which open new ways of investigations, this book is a valuable contributions to measure thory and related fields.

S. COBZAŞ

Complex Analysis — Methods, Trends and Applications, Edited by E. Lanckau and W. Tutschke, Akademie Verlag, Berlin 1983, 398 pp.

This is the first book in a series initiated by the organizers of the conference on ,,Complex Analysis and Its Applications to Partial Differential Equations", regularly held at the Halle-Wittenberg-Martin Luther University. The aim of the series is to present surveys giving a comprehensive explanation of complex analysis.

The book is divided into two parts: I. *Complex Analysis and Its, Relations to Other Spheres in Mathematics*, and; II. *Complex Methods in Partial Differential Equations and Other Applications of Complex Analysis*, and contains twenty-two papers written by eminent specialists in the field as W. Tu chke, E. Lanchau, B. Bojarski, J. Lawr nowicz, S. Prössdorff V.S. Vnogradov, L. Wolfersdorff et al. The papers present vario aspects of the holomorphy in the whole ar of mathematics and its applications, empha sising the new concepts of generalized analy tic functions of I.N. Vekua, pseudo-analyti functions of L. Bers, (p. q)-analytic functio of G.N. Položiĭ, having deep and fruiti applications to PDE. The book is a valu ble contribution to the modern comp function theory and its applications, we recommend it warmly to all people rested in this field.

S. CO

M. M. Rao, **Probability Theory and Applications,** Academic Press, New Yo 1984, 495 pp.

The book is designed as a graduate co on probability theory and its applicatio All the proofs are given in detail and so key results are given multiple proofs.

The author avoids excessive generaliz ons (for instance Banach space valued rand variables have not been included), the p requisites being a knowledge of Lebe integral. The necessary results from analysis are reviewed in Chapter I and so of them, usually not covered in stand courses, are given with proofs. The book very well and carefully written. The au explains the special character of the sub the notions are gradually introduced the need of an abstract theory is very motivated on apparently simple real-w problems. The book also contains very in resting historical and philosophical comme on the evolution of ideas and concepts probability theory. Many classical proble are discussed in detail and others are p sented in the problems at the end of ea chapter. Some of these problems are ro tine but there are also some more diffic problems, usually provided with hints.

The result is a fine book on probabil theory and we recommend it warmly to people interested in learning, applying teaching probability theory.

S. COBZ

Hugo Steinhaus, **Selected Papers,** PWN Warszawa, 1985, 899 pp.

The book contains 84 from 255 papers of the eminent Polish mathematician Hugo Steinhaus (1887—1972), one of the founders (together with S. Banach, J. Schauder *et al.*) of functional analysis. The articles were chosen to cover the wide area of interests of H. Steinhaus, the fields he made fundamental achievements being: trigonometric and orthogonal series, functional analysis, probability theory, game theory, topology, applications of mathematics, popularization. The papers are arranged chronologically, in order to help the reader in following the developement of scientific ideas of H. Steinhaus. The book also contains an article on the life and work of H. Steinhaus written by E. Marczewski, a list of scientific publications of H. Steinhaus and some of his polemics, pamphlets and programmatic talks.

S. COBZAŞ

Conference on Applied Mathematics, Ljubljana, September 2—5, 1986, Edited by Z. Bohte, University of Ljubljana, Ljubljana, 1986

Prezenta carte cuprinde 27 de lucrări prezentate la a „V-a Conferinţă de matematici aplicate" ţinută la Ljubljana în 2—5 septembrie 1986. La această conferinţă au participat 126 matematicieni din universităţi şi centre de cercetare din Jugoslavia. Cele 27 de lucrări, menţionate mai sus, tratează probleme actuale din următoarele domenii: Analiză numerică' Informatică, Ecuaţii diferenţiale ordinare şi Ecuaţii cu derivate parţiale. Recomandăm această carte tuturor cercetătorilor antrenaţi în aceste domenii.

I.A. RUS

Discrete Geometry and Convexity, Editors Jacob E. Goodman, Erwin Lutwak, Joseph Malkevitch, Richard Pollack, *Annals of the New York Academy of Sciences*, Vol. 440, The New York Academy of Sciences, New York, 1985. (XII+392 pages).

The amin of the volume is to collect under one cover some representative current work in the areas of geometry which could be subsummed under the heading Discrete Geometry and Convexity. These areas include a rather wide spectrum of problems including purely combinatorial questions involving the geometry of finite sets of points on one extreme and integral geometry at the other. The contained 35 papers, signed by outstanding specialists of the field are distributed as follows:
1. *Discrete Problems* (8); 2. *Quantitative Convexity* (7);
3. *Qualitative Convexity* (5); 4. *Polyhedral Geometry* (5);
5. *Tilling, Packing, Covering and Weaving* (5); 6. *Computational Aspects* (5). Most of the papers have both expository and research paper characteristics. The reader can find in them an extended literature and an important amount of open problems as well. The volume ends with *Index of Contributors, Author Index* and *Subject Index*.

A. B. NÉMETH

D. P. Parent, **Exercices in Number Theory,** Springer — Verlag, New York, Berlin, Heidelberg, Tokyo, *Problem Books in Mathematics*, 1984, pp.

This problem book ia a very good and attractive introduction to number theory. The book contains ten chapters in the following order: *Prime Numbers; Arithmetic Functions; Selberg's Sine; Additive Theory; Rational Series; Algebraic Theory; Distribution Modulo 1; Transcendal Numbers; Congruences Mod p; Modular Forms; Quadratic Forms; Continued Fractions; p-Adic Analysis.*

Each chapter is divided in three sections: introduction and basic results, problems, solutions. The solutions are complete and contain many remarks and bibliographycal comments.

The book is useful for all interested in number theory and related fields.

D. ANDRICA

A. Langenbach, **Vorlesung zur höheren Analysis. Hochschulbücher für Mathematik.** Band 84, VEB Deutscher Verlag der Wissenschaften Berlin 1984, 280 pages.

The book presents some fundamental methods of linear and nonlinear functional analysis, useful for those students and specialists, (mathematicians, physicists etc.), who use analytic methods in their research domain as the theory of differential and partial differential equations, maximum and minimum problems, optimization and control theory, approximation and numerical methods etc.

To read the book one needs relatively few previous knowledge, a very clear way of presentation is chosen, too general results are not discussed. The book is written with very much pedagogical sense, so it is available to the students of mathematics, physics and engineering of lower years.

The titles of the chapters and appendices are: *Metric and Normed Linerar Spaces, Topological Spaces, Functionals and Minimum problems, Hilbert Spaces, Construction Methods for Minimum Problems and Equations, Application of Prolongation and Computation Methods, Classification of Partial Differential Equations, Theory of Elliptic Equations, Linear Parabolic and Hyperbolic Equations, Theory of Evolution Equations, The Stone-Weierstrass Theorem, Measure-theoretical Basis of Integration of Continuous Functions*

P. SZILÁGYI

ÎNTREPRINDEREA POLIGRAFICĂ CLUJ, Comanda nr. 502/1986

Revista științifică a Universității din Cluj-Napoca, **STUDIA UNIVERSITATIS BABEȘ-BOLYAI,** apare începînd cu anul 1986 în următoarele condiții:

matematică — trimestrial

fizică — semestrial

chimie — semestrial

geologie-geografie — semestrial pentru geologie și anual pentru geografie

biologie — semestrial

filozofie — semestrial

științe economice — semestrial

științe juridice — semestrial

istorie — semestrial

filologie — semestrial

STUDIA UNIVERSITATIS BABEȘ-BOLYAI, the scientific journal of the University of Cluj-Napoca, starting with 1986 in issued as follows:

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biology: biannually

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history: binnually

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