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SOME CLASSES OF IRRATIONAL NUMBERS

SÁNDOR JÓZSEF

1. Let $1 < n_1 < n_2 < \dots$ be a sequence of positive integers with

$$\lim_{k\to\infty} \frac{n_{k+1}}{n_1 n_2 \dots n_k} = \infty.$$

Then we know (see Erdős [1]) that

$$\sum_{k=1}^{\infty} \frac{1}{n_k}$$

is irrational.

I extend this result in the following way:

PROPOSITION 1. *Let (n_k) and (m_k) be two sequences of positive integers which have the properties*

$$\overline{\lim_{k\to\infty}} \frac{n_k}{n_1 n_2 \dots n_{k-1}} \cdot \frac{1}{m_k} = \infty \tag{1.1}$$

and

$$\underline{\lim_{k\to\infty}} \frac{n_k}{n_{k-1}} \cdot \frac{m_{k-1}}{m_k} > 1. \tag{1.2}$$

If

$$\xi = \sum_{k=1}^{\infty} \frac{m_k}{n_k} < \infty, \tag{1.3}$$

then ξ is an irrational number.

Proof. According to (1.2), for each real number $A > 1$ we can find a positive integer k_0 such that

$$\frac{n_k}{n_{k-1}} \cdot \frac{m_{k-1}}{m_k} > A \tag{1.4}$$

for all $k \geqslant k_0$.

Assume, on the contrary, that ξ is rational, i.e. there exist positive integers p, q such that

$$\sum_{i=1}^{\infty} \frac{m_i}{n_i} = \frac{p}{q} \tag{1.5}$$

Multiplying both sides of (1.5) with $n_1 n_2 \dots n_{k-1} q$, $(k \in \mathbb{N})$ we obtain

$$\sum_{i=1}^{k-1} \frac{n_1 n_2 \dots n_{k-1}}{n_i} \cdot q m_i + \sum_{i=k}^{\infty} \frac{n_1 n_2 \dots n_{k-1}}{n_i} \cdot q m_i = n_1 n_2 \dots n_{k-1} p \tag{1.6}$$

for $k = 1, 2, 3, \ldots$. (1.6) implies at once that

$$S = S(k) = \sum_{i=k}^{\infty} \frac{n_1 n_2 \ldots n_{k-1}}{n_i} \cdot q m_i \tag{1.7}$$

must be integer, as the difference of two integer numbers. In what follows we shall construct a corresponding k for which S is not integer, an evident contradiction.

By using (1.1) we can find a $k > k_0$ with

$$\frac{n_1 n_2 \ldots n_{k-1}}{n_k} \cdot m_k < \frac{1}{M} \tag{1.8}$$

(we shall choose a suitable M).

We can now write

$$S = \frac{n_1 n_2 \ldots n_{k-1}}{n_k} \cdot m_k \cdot q + \frac{n_1 n_2 \ldots n_{k-1}}{n_{k+1}} \cdot m_{k+1} q + \ldots =$$

$$= \left(\frac{n_1 n_2 \ldots n_{k-1}}{n_k} \cdot m_k\right) q + \left(\frac{n_1 n_2 \ldots n_{k-1}}{n_k} \cdot m_k\right) \frac{m_{k+1} n_k}{m_k \cdot n_{k+1}} \cdot q + \ldots <$$

$$< \frac{q}{m} \cdot \left(1 + \frac{m_{k+1}}{m_k} \cdot \frac{n_k}{n_{k+1}} + \frac{m_{k+2}}{m_k} \cdot \frac{n_k}{n_{k+2}} + \ldots + \frac{m_{k+p}}{m_k} \cdot \frac{n_k}{n_{k+p}} + \ldots\right) \tag{1.9}$$

Identity

$$\frac{m_{k+p}}{m_k} \cdot \frac{n_k}{n_{k+p}} = \left(\frac{m_{k+1}}{m_k} \cdot \frac{n_k}{n_{k+1}}\right) \left(\frac{m_{k+2}}{m_{k+1}} \cdot \frac{n_{k+1}}{n_{k+2}}\right) \ldots \left(\frac{m_{k+p}\, n_{k+p-1}}{m_{k+p-1}\, n_{k+p}}\right)$$

with (1.4) implies

$$\frac{m_{k+p}}{m_k} \cdot \frac{n_k}{n_{k+p}} < \frac{1}{A^p}, \ (\forall)\ k \geqslant k_0,\ p \in \mathbf{N}, \tag{1.10}$$

and so from (1.9), (1.10) we get

$$S < \frac{q}{M} \cdot \left(1 + \frac{1}{A} + \frac{1}{A^2} + \ldots\right) = \frac{q}{M} \cdot \frac{A}{A-1} \tag{1.11}$$

Taking $M > q \cdot \dfrac{A}{A-1}$, then evidently we have $S < 1$. On the other hand it is obvious that $S > 0$.

Corollary 1. *If conditions (1.1) and (1.2) of Proposition 1 are satisfied, and the sequence (n_k/m_k) is strictly icreasing, with terms greater than 1, then (1.3) is irrational.*

COROLLARY 2. (Erdős) *If* (n_k) *satisfies the conditions*

1) $1 < n_1 < n_2 < \ldots$;

2). $\overline{\lim\limits_{k\to\infty}} \dfrac{n_k}{n_1 n_2 \ldots n_{k-1}} = \infty$;

3) $\underline{\lim\limits_{k\to\infty}} \dfrac{n_k}{n_{k-1}} > 1$,

then

$$\sum_{k=1}^{\infty} \frac{1}{n_k}$$

is irrational.

COROLLARY 3. *If* (n_k), (m_k) *are sequences of positive integers with*

1) $1 < n_1 < n_2 < \ldots$;

2) $\lim\limits_{k\to\infty} \cdot \dfrac{n_k}{n_1 n_2 \ldots n_{k-1}} \cdot \dfrac{1}{m_k} = \infty$;

then (1.3) is irrational.

Application. For any sequence (m_k) of positive integers satisfying

$$\lim_{k\to\infty} \frac{(k-1)!}{m_k} = \infty,$$

$$\sum_{k=1}^{\infty} m_k / \left(\frac{2^{2^k} + 1}{k} \right)$$

is irrational.

2. A theorem of Cantor [2] asserts that if (n_k) is a sequence of positive integers with $n_{k+1} > n_k^2$ for all large k, then

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{n_k}\right) \text{ is irrational.}$$

We have the following similar proposition.

PROPOSITION 2. *Let* (m_k) *be a sequence of primes, with*

$$\lim_{k\to\infty} m_k = \infty. \tag{2.1}$$

and let (n_k) *be a sequence of positive integers which verify the inequalities*

$$n_{k+h} \geqslant m_{k+h} n_k^{2^h}, \; h = 1, 2, \ldots; k = 1, 2, \ldots \tag{2.2}$$

Then the infinite product

$$\prod_{k=1}^{\infty} \left(1 + \frac{m_k}{n_k}\right) \tag{2.3}$$

is irrational.

Proof. The infinite product evidently is convergent.
Let us assume, on the contrary, that

$$\alpha = \prod_{k=1}^{\infty} \left(1 + \frac{m_k}{n_k}\right)$$

is rational and put

$$\alpha_h = \prod_{k=h}^{\infty} \left(1 + \frac{m_k}{n_k}\right).$$

We have $\alpha_h \in \mathbf{Q}$, $h = 1, 2, \ldots$, say $\alpha_h = \frac{a_h}{b_h}$ with

$$a_h,\ b_h \in \mathbf{N},\ (a_h, b_h) = 1 \tag{2.4}$$

By using the identity

$$\frac{a_h}{b_h} = \left(1 + \frac{m_h}{n_h}\right) \cdot \frac{a_{h+1}}{b_{h+1}},$$

we get

$$\frac{a_{h+1}}{b_{h+1}} = \frac{n_h \cdot a_h}{(n_h + m_h) b_h}$$

which implies the existence of $d_h \in \mathbf{N}$, $h = 1, 2, \ldots$, such that

$$n_h \cdot a_h = d_h \cdot a_{h+1};\ (n_h + m_h) \cdot b_h = d_h \cdot b_{h+1} \tag{2.5}$$

ubtracting the equalities in (2.5) we obtain

$$n_h \cdot (a_h - b_h) - m_h \cdot b_h = d_h(a_{h+1} - b_{h+1}) \geqslant a_{h+1} - b_{h+1} \tag{2.6}$$

Now, we have

$$\prod_{k=h}^{h+n-1} \left(1 + \frac{m_k}{n_k}\right) = \prod_{k=0}^{n-1} \left(1 + \frac{m_{h+k}}{n_{h+k}}\right) \leqslant \prod_{k=0}^{n-1} \left(1 + \frac{1}{n_k^{2^k}}\right) = \left(1 - \frac{1}{n_k}\right)^{2^n} \Big/ \left(1 - \frac{1}{n_k}\right)$$

which is smaller than

$$1/\left(1 - \frac{1}{n_k}\right) = n_k/(n_k - 1).$$

If $n \to \infty$ we get

$$\alpha_h \leqslant n_h/(n_h - 1). \tag{2.7}$$

On the other hand

$$\alpha_h = \left(1 + \frac{m_h}{n_h}\right) \cdot \alpha_{h+1} > 1 + \frac{m_h}{n_h} \geqslant 1 + \frac{1}{n_h},$$

thus $\alpha_h > 1 + \frac{1}{n_h}$, hence $\frac{\alpha_h}{\alpha_{h-1}} < n_h + 1$.

We observe that in fact,

$$n_h = \frac{\alpha_h}{\alpha_{h-1}} \text{ and, since}$$

$$n_h \leqslant \frac{\alpha_h}{\alpha_{h-1}} = \frac{a_h}{a_h - b_h}, \text{ by } m_h > 1$$

from (2.6) we have $a_{h+1} - b_{h+1} \leqslant a_h - b_h$. But the sequence $(a_h - b_h)$ is of positive integers, thus there exist an index $h = n$ so that for each $h = n, n+1, \ldots,$ we have

$$a_h - b_h = a \text{ (} a \text{ natural number)}, \tag{2.8}$$

which implies that (2.6) may be written also in the form $n_h \cdot a - m_h \cdot b_h = d_h \cdot a$ and this means that $a \mid m_h \cdot b_h$. Condition (2.1) and that m_h is a prime for each $h = 1, 2, \ldots,$ gives us $a \mid b_h$ for h sufficiently large. By (2.1) $a = 1$ is the single possibility, i.e. $a_h - b_h = 1$, $h \geqslant N$ $(N \in \mathbf{N})$ and so $\frac{\alpha_h}{\alpha_h - 1} = a_h$, $n_h = a_h$. By (2.5) we get $n_h^2 \geqslant n_{h+1}$. On the other hand (2.2) yields $n_{h+1} \geqslant m_{h+1} \cdot n_h^2 \geqslant n_h^2$ with $h \geqslant N$ large $(m_h \to \infty)$, a contradiction.

Applications

1) If p_k denotes the k — th prime and

$$n_{h+k} \geqslant p_{h+k}\, n_h^{2^k},$$

then

$$\prod_{k=1}^{\infty} \left(1 + \frac{p_k}{n_k}\right)$$

is irrational.

2) Denote by $p_k^{(a,b)}$ the k — th prime in the arithmetic progression $a \cdot k + b$ with $(a, b) = 1$.
If

$$n_{h+k} \geqslant p_{h+k}^{(a,b)} \cdot n_h^{2^k},$$

then

$$\prod_{k=1}^{\infty} \left(1 + \frac{p_k^{(a,b)}}{n_k}\right)$$

is irrational.

3) In 1947 Mills [3] proved that there is a $\theta > 1$ for which $[\theta^{3^n}]$ is a prime for each $n = 1, 2, \ldots$. If

$$n_{h+k} \geqslant [\theta^{3^{h+k}}] \cdot n_h^{2^k},$$

then

$$\prod_{k=1}^{\infty}\left(1+\frac{[\theta^{s^k}]}{n_k}\right)$$

is irrational.

3. In this section we shall give another theorem, which utilises a reciprocal of a lemma of Hinčin [4].

LEMMA 1. *Let* $\eta : \mathbf{N} \to \mathbf{R}^+$ *be a function with condition* $\lim_{n\to\infty} n \cdot \eta(n) = 0$ *and let* α *be an arbitrarily chosen real number. If there are infinitely many distinct rational numbers* p/q *such that*

$$|\alpha - p/q| < \eta(q), \tag{3.0}$$

then α *is irrational.*

Proof. (3.0) may be written in the form

$$-q_k \cdot \eta(q_k) + q_k \cdot \alpha < p_k < q_k \cdot \eta(q_k) + q_k\alpha, \quad k = 1, 2, \ldots.$$

In other words for each q_k we have a finite number of p' s, hence $q_k \to \infty$ if $k \to \infty$.

Let us assume, on the contrary, that $\alpha = \frac{a}{b} \in \mathbf{Q}$, and choose k with $\frac{a}{b} \neq \frac{p_k}{q_k}$.

Then

$$\left|\frac{a}{b} - \frac{p_k}{q_k}\right| = \frac{|aq_k - bp_k|}{b \cdot q_k} \geqslant \frac{1}{b \cdot q_k} > \eta(q_k)$$

if we take $k \geqslant k_0$, with k_0 — the first natural index from which we have $q_k\eta(q_k) < \frac{1}{b}$.

PROPOSITION 3. *Let* (a_n), (b_n) *be sequences of positive integers,* $v_n = a_n/b_n$, $n = 1, 2, \ldots$ *and*

$$\theta = \sum_{n=1}^{\infty} v_n < \infty.$$

Consider the function $\eta : \mathbf{N} \to \mathbf{R}^+$ *as follows:*

$$\lim_{n\to\infty} n\eta(n) = 0.$$

If

$$v_{k+p} \leqslant v_{k+1}^p \quad (k, p = 1, 2, \ldots) \tag{3.1}$$

and

$$v_{k+1} < \frac{\eta(b_1b_2\ldots b_k)}{1+\eta(b_1b_2\ldots b_k)}, \quad (k = 1, 2, \ldots), \tag{3.2}$$

then θ *is irrational.*

Proof. Denoting by $\frac{p_k}{q_k} = \frac{a_1}{b_1} + \ldots + \frac{a_k}{b_k}$, $k = 1, 2, \ldots,$

we have

$$\theta = \frac{a_1}{b_1} + \ldots + \frac{a_k}{b_k} + \ldots \text{ and by (3.1), (3.2)}$$

$$|\theta - p_k/q_k| = \frac{a_{k+1}}{b_{k+1}} + \frac{a_{k+2}}{b_{k+2}} + \ldots < \frac{a_{k+1}}{b_{k+1}} + \left(\frac{a_{k+1}}{b_{k+1}}\right)^2 + \ldots =$$

$= \frac{a_{k+1}}{b_{k+1} - a_{k+1}} < \eta(b_1 b_2 \ldots b_k)$. Taking $q_k = b_1 b_2 \ldots b_k$, we have proved (3.0) for infinitely many k's.

COROLLARY. *If* $b_k \geqslant 2$, $k = 1, 2, \ldots,$ *and*

(1) $v_{k+p} \leqslant v_{k+1}^k$

(2) $v_{k+1} < \frac{1}{1 + (b_1 b_2 \ldots b_k)^\lambda}$, $k = 1, 2, \ldots; \lambda < 1$

than

$$\sum_{k=1}^{\infty} \frac{a_k}{b_k}$$

is irrational.

Application:

$$\sum_{k=1}^{\infty} \frac{1}{2^{3^k} \cdot 3^{4^k}}$$

is irrational.

4. Finally, we obtain by simple arguments a generalization of a result of Estermann [5].
First we state a lemma.

LEMMA 2. *Let* a, b *be integers, with* $b \neq 0$ *and* $f_n : \mathbf{R}^+ \to \mathbf{R}$, $f_n(t) = \frac{(a - bt^k)^n}{n!}$ *with* $k \in \mathbf{N}$. *Then we can find some integers* A_n^i $(i = \overline{1, k})$ *such that*

$$f_n^{(k)}(t) = A_n^1 \cdot f_{n-1}(t) + A_n^2 f_{n-2}(t) + \ldots + A_n^k f_{n-k}(t) \tag{4.1}$$

Proof. By

$$f_n'(t) = -bk \cdot t^{k-1} \cdot f_{n-1} \tag{4.2}$$

and Leibniz's derivation rule,

$$f_n^{(k)}(t) = -b \cdot k \cdot \sum_{p=0}^{k-1} C_{k-1}^p (k-1)(k-2) \ldots (k-p) t^{k-1-p} f_{n-1}^{(k-1-p)}(t) \tag{4.3}$$

We prove by induction (on m) that $t^m \cdot f_{n-1}^{(m)}$ may be written as the combination of $f_{n-1}, f_{n-2}, \ldots, f_{n-m-1}$ with integer coefficients. Indeed,

$$t^k \cdot f_{n-2} = a \cdot f_{n-2} - (n-1)f_{n-1} \text{ and (4.2) yields}$$

$$t \cdot f_{n-1}' = -k[af_{n-2} - (n-1)f_{n-1}] \tag{4.4}$$

Then if for some integers $A_n, \ldots, L_n$ we have

$$t^m f_{n-1}^{(m)} = A_n f_{n-1} + \ldots + L_n f_{n-m-1},\ m > 1,$$

then by derivation and by (4.4) we get

$$t^{m+1} f_{n-1}^{(m+1)} = U_n \cdot f_{n-1} + \ldots + Z_n f_{n-m-1}.$$

Now, the lemma follows at once from (4.3).

PROPOSITION 4. *Let $f_n(t)$ be given as in Lemma 2 and let $l : \mathbf{R} \to \mathbf{R}$ be a solution to the problem*

$$l_{(0)}^{(k)} = \pm 1,\ l^{(k-1)}(0) = 0 \tag{4.5}$$

(with „+" for even k and "−" for odd k).
Let

$$z \in \mathbf{R}^+,\ z^k = \frac{a}{b} \tag{4.6}$$

(a, b positive integers), and

$$I_n = \int_0^z f_n(t)\, l(t)\, dt,\ n = 0, 1, 2, \ldots \tag{4.7}$$

with $I_0 \neq 0$.

Then for each $k \geqslant 2$ at least one of the numbers $I_1/I_0, \ldots, I_{k-1}/I_0$ is irrational.

Proof. Let g be defined in the following manner

$$g(t) = \sum_{p=0}^{k} (-1)^p \cdot f_n^{(p)}(t)\, l^{(k-1-p)}(t) \tag{4.8}$$

We can easily verify that

$$g'(t) = f_n^{(k)}(t) \cdot l(t) + (-1)^{k-1} \cdot f_n(t) \cdot l^{(k)}(t),$$

which on the basis of (4.5) yields

$$g'(t) = l(t) \cdot (f_n^{(k)}(t) \pm f_n(t)). \tag{4.9}$$

By integration and Lemma 2

$$g(0) = 0 \text{ and } g(z) = 0 \text{ gives us}$$

$$0 = \pm I_n + A_n^1 \cdot I_{n-1} + \ldots + A_n^k \cdot I_{n-k}, \text{ where } I_0 \neq 0. \tag{4.10}$$

Assume now, on the contrary, that each I_1/I_0 $(i = \overline{1, k-1})$ is rational, i.e. $I_1/I_0 = \alpha_i/\beta_i \in \mathbf{Q}(i = \overline{1, k-1})$.

Then $\beta_i \cdot I_i/I_0$ are integers. Put

$$J_n = \beta_1\beta_2 \ldots \beta_{k-1} \cdot I_n/I_0. \tag{4.11}$$

Evidently $J_0, J_1, \ldots, J_{k-1}$ are integers and by (4.10) we get

$$0 = \pm J_n + A_n^1 \cdot J_{n-1} + \ldots + A_n^k \cdot J_{n-k} \tag{4.12}$$

By induction, (J_k) is a sequance of integers. But $I_0 \neq 0$ shows that not all I_k are zero, therefore

$$|J_n| + |J_{n-1}| + \ldots + |J_{n-k-1}| \geqslant 1, \ (\forall) n \in \mathbf{N} \tag{4.13}$$

On the other hand

$$|I_n| = \left| \int_0^z \frac{(a - bt^k)^n}{n!} l(t)dt \right| \leqslant \int_0^z \frac{|a|^n}{n!} |l(t)| dt = \frac{|a|^n}{n!} \cdot \Phi(z),$$

where $\Phi(z) \neq 0$.

Then $\lim_{n\to\infty} I_n = 0$, and by (4.11) $\lim_{n\to\infty} J_n = 0$. This contradicts (4.13).

COROLLARY 1. (Estermann). *For* $\operatorname{sh} z \neq 0$, *one of* z^2 *and* $z \cdot \operatorname{cth} z$ *is irrational*

(Take $k = 2$, $l(t) = \operatorname{ch} t$. Then $I_0 = \operatorname{sh} z$,

$I_1 = 2b \cdot (z \cdot \operatorname{ch} z - \operatorname{sh} z)$. If $z^2 = \frac{a}{b} \in \mathbf{Q}$, therefore I_1/I_0 is irrational).

2. For $\sin z \neq 0$, one of z^2 and $z \cdot \operatorname{ctg} z$ is irrational. (Take $k = 2$, $l(t) = \cos t$.)

Remark. By putting $z = \pi/2$ we get that π^2 is irrational.

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UNELE CLASE DE NUMERE IRAŢIONALE

(Rezumat)

În lucrare, plecînd de la un rezultat al lui P. Erdős [1], se demonstrează că în anumite condiţii

$$\sum_{k=1}^{\infty} \frac{m_k}{n_k}$$

este un număr iraţional. În ultima parte a lucrării se generalizează un rezultat al lui Esterman [5].

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THE FIXED-POINT SET FOR INJECTIVE MAPPINGS

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The aim of this note is to give a characterization for the fixed-point set of an injective mapping in terms of the maximal total-variant subsets.

1. **Notations and definitions.** Let X be a set, $Y \subseteq X$ a subset of X and $f: X \to X$ a mapping. Let $F_f = \{x \in X \mid f(x) = x\}$ the set of the fixed-points of f, $f(Y) = \{f(y) \mid y \in Y\}$, $f^{-1}(Y) = \{x \in X \mid f(x) \in Y\}$, $\overline{Y} = X \setminus Y$ and $|Y|$ the cardinality of Y.

A subset Y of X is called total f-variant if $Y \cap f(Y) = \Phi$. If $f = 1_X$ it is evident that exists no total f-varaint subsets. But this case is trivial and it will be assumed for the remainder of this paper that $f \neq 1_X$. Our assumption assures that $\mathcal{A} = \{Y \subset X \mid Y \cap f(Y) = \Phi\} \neq \Phi$. Our results need the following

LEMMA *If $f: X \to X$ is a mapping then $\mathcal{A}$ has maximal elements (with respect to set inclusion).*

Proof: see Abian [1].

2. **Main results.** THEOREM 1. *Let $f: X \to X$ be a mapping and $Y \subset X$ a maximal total f-variant subset of X. Then*

$$\overline{Y} \cap \overline{f(Y)} \cap \overline{f^{-1}(Y)} \subseteq F_f \tag{1}$$

Proof. Let $x \in \overline{Y} \cap \overline{f(Y)} \cap \overline{f^{-1}(Y)}$. Because $x \in \overline{Y}$, the maximality of Y implies that $f(Y \cup \{x\}) \cap (Y \cup \{x\}) \neq \Phi$ and then

$$(f(Y) \cap Y) \cup (Y \cap \{f(x)\}) \cup (f(Y) \cap \{x\}) \cup (\{x\} \cap \{f(x)\}) \neq \Phi.$$

But $f(Y) \cap Y = \Phi$, $x \in \overline{f(Y)}$, $x \in \overline{f^{-1}(Y)}$, so $\{x\} \cap \{f(x)\} \neq \Phi$ and $x \in F_f$.

THEOREM 2. *Let $f: X \to X$ be an injective mapping and $Y \subset X$ a maximal total f-variant subset of X. Then*

$$F_f = \overline{Y} \cap \overline{f(Y)} \cap \overline{f^{-1}(Y)} \tag{2}$$

Proof. By (1), we must prove only that $F_f \subseteq \overline{Y} \cap \overline{f(Y)} \cap \overline{f^{-1}(Y)}$. Let $x \in F_f$, i.e. $x = f(x)$. It is clear that $x \in \overline{Y}$.

If $x \in f(Y)$ then $f(x) = x \in f(Y)$ and $x \in Y$ by injectivity of f, wich contradicts the fact that $x \in \overline{Y}$. Thus, $x \in \overline{f(Y)}$.

If $x \in f^{-1}(Y)$ then $f(x) = x \in Y$, contradiction. Thus $x \in \overline{f^{-1}(Y)}$, as required.

Remarks. 1) Unfortunatelly the four subsets in (2) are not mutually disjoint. Indeed, if f is the permutation given by the table $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{pmatrix}$, then $F_f = \{5\}$, $Y = \{1, 3\}$, $f(Y) = f^{-1}(Y) = \{2, 4\}$.

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* Şc. gen. nr. 3, Deva.

2) The injectivity hypothesis in Theorem 2 is necessary, Indeed, if f is given by the table $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 2 & 1 \end{pmatrix}$ we have $F_f = \{1\}$, $Y = \{2, 5\}$, $f(Y) = \{1, 3\}$ and $F_f = \{1\} \subseteq \overline{f(Y)} = \{2, 4, 5\}$.

3. **Aplications.** As an immediate consequence of Theorem 1 we prove the following factorization theorem:

THEOREM 3. *Let $f: X \to X$ be a mapping and $Y \subset X$ a total f-variant maximal subset of X. Then:*

$$X = F_f \cup Y \cup f(Y) \cup f^{-1}(Y) \tag{3}$$

Proof. Immediate by (1) and by one of de Morgan's laws.

THEOREM 4. *Let X be a continuum (i.e. conex and compact) in the topological Hausdorff space $\tilde{X}$ and let $f: X \to X$ an injective continuous mapping such that exists a closed total f-variant maximal subset of X. Then $F_f = \Phi$.*

Proof. Let us point out at the beginning some consequences of the hypothesis:

a) By our initial assumption, $F_f \neq X$ and $Y \neq \Phi$ where Y is the closed total f-variant maximal subset of X.

b) X Hausdorff and f continuous implies the well-known fact that F_f is closed.

c) Because X is compact, f is continuous and Y is closed it results by (2) that F_f is open.

d) By the conexity of X, the only simultaneously open and closed subsets are Φ and X.

Then, by b), c) and d) we have $F_f = \Phi$ or $F_f = X$. The last possibility contradicts a).

In what follows, if G is a group and $f \in \mathrm{Aut}\, G$, f will be called fixed-point-free (f.p.f.) automorphism if $F_f = \{1\}$ where 1 is the identity element of G.

THEOREM 5. *Let G be a finite group and f a f.p.f. automophism of G. If there exists a maximal total-f-variant subset Y of X such that $f^{-1}(Y) = f(Y)$ then G is (solvable) of odd order.*

Proof. By (3), $G = F_f \cup Y \cup f(Y) \cup f^{-1}(Y) = \{1\} \cup Y \cup f(Y)$. The subsets in the union being muttually disjoint and f being one-toone, $|G| = 1 + |Y| + |f(Y)| = 1 + 2|Y|$.

THEOREM 6. *Let G be a finite group and f a f.p.f. automorphism of G such that there exists a maximal total f-variant subset Y of X with $f^{-1}(Y) \cup f(Y) = \Phi$. Then G is solvable.*

Proof. Note that the minimal counterexamples for the assertion „A finite group wich has a f.p.f. automorphism is solvable" are all simple (nonabelian) groups. On the other hand, (see G o r e n s t e i n [2]) the only finite simple groups whose orders are not divisible by 3 are the Sjuzuki groups $Sz\,(2^{2n+1})$, which admit no f.p.f. automorphisms. So, it is sufficient to consider a simple group G in Theorem 6 and to show that its order is not divisible by 3. But cf. (3) we have $G = \{1\} \cup Y \cup f(Y) \cup f^{-1}(Y)$ and then $|G| = 1 + |Y| + |f(Y)| + |f^{-1}(Y)| = 1 + 3|Y|$ because all the subsets are disjoint.

Remark: the case $0 < |f(Y) \cup f^{-1}(Y)| < |Y|$ is very difficult because the last argument fails.

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MULŢIMEA PUNCTELOR FIXE PENTRU APLICAŢII INJECTIVE

(Rezumat)

În această notă se dă o teoremă de caracterizare a mulţimii punctelor fixe ale unei aplicaţii injective în termenii submulţimilor total f-variante maximale şi se stabilesc aplicaţii în topologie şi teoria grupurilor finite.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXIX, 1984

ON SOME ITERATED INVERSE VECTOR OPERATORS

CONSTANTIN TUDOSIE*

1. **Introduction.** In my doctorate thesis, as a mathematical instrument of research I made use of an operator Algebra and Analysis, created on the basis of two fundamental operators I introduced. They also appear in my paper [6], being defined, in the Euclidean space E_3, by the following expresions

$$O_s = \cdot\, x\bar{s}, \quad O_{rs} = \bar{r}x(\cdot\, x\bar{s}), \tag{1}$$

where s and r, s respectively are „indices of the operators".

The operator O_s is of „the first kind", and the operator O_{rs} is of „the second kind". Between the operators (1) there exists the relation [6]

$$O_{rs} = -O_r O_s. \tag{2}$$

By complete induction, as in [6], the following formulae of reducing the order of the first and second kind iterated operators are deduced

$$O_s^k = (-1)^{\frac{k}{2}} s^{k-2} O_{ss}, \quad (k = 2,\ 4,\ 6,\ \ldots), \tag{3}$$

$$O_s^k = (-1)^{\frac{k+3}{2}} s^{k-1} O_s, \ (k = 1,\ 3,\ 5,\ \ldots), \tag{4}$$

$$O_{rs}^k = (\bar{r}\bar{s})^{k-1} O_{rs}, \qquad (k = 1,\ 2,\ 3,\ \ldots), \tag{5}$$

where O_s^k, O_{rs}^k are „the k-order iterants" of the operators O_s, O_{rs}.

In [3] the theorems of existence of the first and second kind inverse operators were established. They have the expressions

$$O_s^{-1} = -s^{-2} O_s, \quad O_{rs}^{-1} = (\bar{r}\bar{s})^{-2} O_{rs}, \tag{6}$$

the first inverse operator existing for the sub-set of the vectors $\bar{\omega}$, with the property $\bar{\omega}\bar{s} = 0$, whereas the second inverse operator existing for the same sub-set $\bar{\omega}$, with the property $\bar{\omega}\bar{r} = 0$.

By observing (6), it results that the sub-set of the vectors $\bar{\omega}$ admits the inverse operators O_q^{-1} and O_{pr}^{-1} if the properties $\bar{\omega}\bar{q} = 0$, $\bar{\omega}\bar{p} = 0$ take place.

In this paper there are established the expressions of first and second kind iterated inverse vector operators, and the relations of dependence between the iterated and non-iterated direct and inverse operators.

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2. **Iterated inverse operators.** The „k-order iterants" of the inverse operators O_s^{-1} and O_{rs}^{-1} are defined by the recurrence relations

$$O_s^{-k} = O_s^{-1}(O_s^{1-k}),\ O_{rs}^{-k} = O_{rs}^{-1}(O_{rs}^{1-k}), \quad (k = 1, 2, 3, \ldots). \tag{7}$$

By observing (6), the relations (7) become, for $k = 2$,

$$O_s^{-2} = s^{-4}O_s^2,\ O_{rs}^{-2} = (\bar{r}\,\bar{s})^{-4}O_{rs}^2. \tag{8}$$

By complete induction, the expressions of the k-order iterants of the two inverse operators are obtained:

$$O_s^{-k} = (-1)^k s^{-2k} O_s^k, \quad (k = 1, 2, 3, \ldots), \tag{9}$$

$$O_{rs}^{-k} = (\bar{r}\,\bar{s})^{-2k} O_{rs}^k, \quad (k = 1, 2, 3, \ldots). \tag{10}$$

For $\bar{r} = \lambda\bar{s}$, (λ parameter), (10) becomes

$$O_{rs}^{-k} = \lambda^{2k} r^{-4k} O_{rs}^k, \quad (k = 1, 2, 3, \ldots), \tag{11}$$

or

$$O_{rs}^{-k} = \lambda^{-2k} s^{-4k} O_{rs}^k, \quad (k = 1, 2, 3, \ldots). \tag{12}$$

For $k = 1$, the expressions (11) and (12) take the form

$$O_{rs}^{-1} = \lambda^2 r^{-4} O_{rs},\ O_{rs}^{-1} = \lambda^{-2} s^{-4} O_{rs}, \tag{13}$$

and hence

$$O_{rs} = \lambda^{-2} r^4 O_{rs}^{-1},\ O_{rs} = \lambda^2 s^4 O_{rs}^{-1}. \tag{14}$$

By using the evident relations, for $\bar{r} = \lambda\bar{s}$,

$$O_{rs}^{-k} = \lambda^{-k} O_{ss}^{-k},\ O_{rs}^k = \lambda^k O_{ss}^k,\ (k = 1, 2, 3, \ldots), \tag{15}$$

from the second expression (13), it results, for $k = 1$,

$$O_{ss}^{-1} = s^{-4} O_{ss},\ O_{ss} = s^4 O_{ss}^{-1}, \tag{16}$$

what is also obtained from the second relation (6), for $\bar{r} = \bar{s}$.

Taking into account (2), for $\bar{r} = \bar{s}$, the first transformation (16) becomes

$$O_{ss}^{-1} = -s^{-4} O_s^2, \tag{17}$$

and observing the first relation (6), one obtains

$$O_{ss}^{-1} = -O_s^{-1}(O_s^{-1}) = -O_s^{-2}, \tag{18}$$

or

$$O_s^{-1} = s^{-2} O,\quad O = s^2 O_{ss}^{-1}, \tag{19}$$

where O is "the operator of identity".

The k-order iterants of the operators (16) are

$$O_{ss}^{-k} = s^{-4k} O_{ss}^{k}, \quad O_{ss}^{k} = s^{4k} O_{ss}^{-k}, \quad (k = 1,\ 2,\ 3,\ \ldots). \tag{20}$$

By using (2), for $\bar{r} = \bar{s}$, and (18), the formulae (20) become

$$O_{ss}^{-k} = (-1)^{k} s^{-4k} O_{s}^{2k}, \quad O_{ss}^{k} = (-1)^{k} s^{4k} O_{s}^{-2k}, \quad (k = 1,\ 2,\ 3,\ \ldots). \tag{21}$$

By observing (5), for $\bar{r} = \bar{s}$, the first formula (20) is written

$$O_{ss}^{-k} = s^{-2(k+1)} O_{ss}, \quad (k = 1,\ 2,\ 3,\ \ldots), \tag{22}$$

whence

$$O_{ss} = s^{2(k+1)} O_{ss}^{-k}, \ (k = 1,\ 2,\ 3,\ \ldots), \tag{23}$$

By a k-times iteration, the relations (19) are transformed into

$$O_{ss}^{-k} = s^{-2k} O, \ O = s^{2k} O_{ss}^{-k}, \quad (k = 1,\ 2,\ 3,\ \ldots), \tag{24}$$

from (20) and (24) resulting

$$O_{ss}^{k} = s^{2k} O. \tag{25}$$

If we take into account the expressions of „the 2-product operators" [6]

$$O_{s}^{2} = O \begin{bmatrix} 1, & 1 \\ s, & s \end{bmatrix}, \quad O_{s}^{-2} = O \begin{bmatrix} -1, & -1 \\ s, & s \end{bmatrix},$$

the transformations (21) may also be written under the form

$$\begin{aligned} O_{ss}^{-k} &= (-1)^{k} s^{-4k} O^{k} \begin{bmatrix} 1, & 1 \\ s, & s \end{bmatrix}, \\ O_{ss}^{k} &= (-1)^{k} s^{4k} O^{k} \begin{bmatrix} -1, & -1 \\ s, & s \end{bmatrix}, \end{aligned} \quad (k = 1,\ 2,\ 3,\ \ldots), \tag{26}$$

in which

$$O_{s}^{2k} = \underbrace{O_{s}^{2} O_{s}^{2} \ldots O_{s}^{2}}_{k\text{-times}} = O^{k} \begin{vmatrix} 1, & 1 \\ s, & s \end{vmatrix},$$

$$O_{s}^{-2k} = \underbrace{O_{s}^{-2} O_{s}^{-2} \ldots O_{s}^{-2}}_{k\text{-times}} = O^{k} \begin{vmatrix} -1, & -1 \\ s, & s \end{vmatrix}.$$

From (24) and (25) two presentations of the operator of identity result

$$O = s^{2k} O_{ss}^{-k} = s^{-2k} O_{ss}^{k}, \quad (k = 1,\ 2,\ 3,\ \ldots). \tag{27}$$

By observing the formulae established in [6],

$$O_s^k = (-1)^{\frac{k}{2}} s^{k-2} O_{ss}, \quad (k = 2, 4, 6, \ldots), \tag{28}$$

$$O_s^k = (-1)^{\frac{k+3}{2}} s^{k-1} O_s, \quad (k = 1, 3, 5, \ldots), \tag{29}$$

the operator (9) becomes

$$O_s^{-k} = (-1)^{\frac{3k}{2}} s^{-(k+2)} O_{ss}, \quad (k = 2, 4, 6, \ldots), \tag{30}$$

$$O_s^{-k} = (-1)^{\frac{3}{2}(k+1)} s^{-(k+1)} O_s, \quad (k = 1, 3, 5, \ldots). \tag{31}$$

By applying the operator (22) to the operator (30), one obtains

$$O_{ss}^{-k} O_s^{-k} = O_s^{-k} O_{ss}^{-k} = (-1)^{\frac{3k}{2}} s^{-(3k+4)} O_{ss}^2, \tag{32}$$
$$(k = 2, 4, 6, \ldots).$$

The relation (32) shows that the operators O_s^{-k} and O_{ss}^{-k} are "commutable", to the even values of the index k.

By substituting (29) in (9) one obtains

$$O_s^{-k} = (-1)^{\frac{3}{2}(k+1)} s^{-(k+1)} O_s, \quad (k = 1, 3, 5, \ldots). \tag{33}$$

By applying the operator (22), under the form

$$O_{ss}^{-k} = - s^{-2(k+1)} O_s^2, \quad (k = 1, 2, 3, \ldots), \tag{34}$$

to the operator (33), it results

$$O_{ss}^{-k} O_s^{-k} = O_s^{-k} O_{ss}^{-k} = (-1)^{\frac{3k+5}{2}} s^{-3(k+1)} O_s^3, \tag{35}$$
$$(k = 1, 3, 5, \ldots).$$

The relation (35) shows that the operators O_s^{-k} and O_{ss}^{-k} and "commutable" also to the uneven values of the index k.

By applying the direct operator O_{rs}^k to the vector $\bar{\omega}$, conditioned by the existence of the inverse operator, $\bar{\omega}\bar{r} = 0$, we have

$$O_{rs}^k \bar{\omega} = (\bar{r}\,\bar{s})^k \bar{\omega}, \quad (k = 1, 2, 3, \ldots),$$

resulting the operator

$$O_{rs}^k = (\bar{r}\,\bar{s})^k O, \quad (k = 1, 2, 3, \ldots), \tag{36}$$

By substituting (36) in (10), one obtains

$$O_{rs}^{-k} = (\bar{r}\ \bar{s})^{-k}\, O, \quad (k = 1, 2, 3, \ldots), \tag{37}$$

and for $\bar{r} = \bar{s}$, it results the operator

$$O_{ss}^{-k} = s^{-2k} O, \quad (k = 1, 2, 3, \ldots), \tag{38}$$

what is also obtained from (27).

3. **Applications.** α). Let the vector $\bar{\omega}$ be applied to the direct operator (29), for $k = 5$. We have

$$O_s^5 \bar{\omega} = s^4 O_s \bar{\omega}. \tag{39}$$

By applying the inverse operator (33) to the expression (39), observing the condition $\bar{\omega}\bar{s} = 0$, and having $O_s^0 = 0$, one obtains

$$O\bar{\omega} = -s^{-2} O_s^2 \bar{\omega} = s^{-2} O_{ss} \bar{\omega} = \bar{\omega}.$$

β). By observing Poincaré's [1] terminology, let the "consequent Iterant of the vector $\bar{\omega}$" be determined, in relation to the operator (5), for $k = 2$, $\bar{\omega}\bar{r} = 0$ It follows

$$O_{rs}^2 \bar{\omega} = (\bar{r}\ \bar{s}) O_{rs} \bar{\omega} = (\bar{r}\ \bar{s})^2 \bar{\omega}. \tag{40}$$

By applying the operator (10) to the expression (40), we obtain the initial vector. By noting $O_{rs}^0 = O$, we have

$$O\bar{\omega} = (\bar{r}\ \bar{s})^{-2} O_{rs}^2 \bar{\omega},$$

and by using (5), we obtain

$$O\bar{\omega} = (\bar{r}\ \bar{s})^{-1} O_{rs} \bar{\omega} = \bar{\omega}.$$

The same result may be obtained by the successive application of the operators (36) and (37) to the vector $\bar{\omega}$.

γ). Let the consequent iterant of the vector $\bar{\omega}$, relative to the operator (5), be determined, for $k = 2$, $r = \lambda s$, $\omega r = 0$, We have

$$O_{rs}^2 \bar{\omega} = \lambda^2 s^4 \bar{\omega}, \tag{41}$$

and by applying the operator (11) to the iterant (41), one obtains

$$O\bar{\omega} = \lambda^{-2} s^{-4} O_{rs}^2 \bar{\omega} = \bar{\omega}.$$

By applying the first operator (15) to the iterant (41), for $k = 2$, we have

$$O\bar{\omega} = s^4 O_{ss}^{-2} \bar{\omega},$$

and by substituting the first operator (24), for $k = 2$, it results $O\bar{\omega} = \bar{\omega}$.
The same $\bar{\omega}$ is obtained by applying the operator (12) to the iterant (41)

$$O\bar{\omega} = s^{-4} O_{ss}^2 \bar{\omega}.$$

By observing (25), for $k = 2$, the preceding expression becomes $O\bar{\omega} = \bar{\omega}$.

Application γ) shows that different operator procedures permit to pass from the iterant (41) to the vector $\bar{\omega}$.

δ). Let the iterant (39) be applied to the operator (33), for $k = 3$. One obtains

$$O_s^2\bar{\omega} = -O_{ss}\bar{\omega} = -s^2\bar{\omega}. \tag{42}$$

Iterant (42) shows that the application of an iterated inverse operator of an iteration index $(-n)$ to a direct operator of iteration index k, $(k > n)$, reduces the iteration index of the direct operator to the value $k - n$.

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ASUPRA UNOR OPERATORI VECTORIALI INVERŞI ITERAŢI

(Rezumat)

În prezenta lucrare se continuă cercetările, din [6], [3] şi [5], asupra unor operatori vectoriali, pe care i-am introdus în Algebra şi Analiza operatorială. Lucrarea are ca obiect studiul unor operatori vectoriali inverşi iteraţi, şi relaţiile lor cu operatorii direcţi.

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THE PROPERTY W^2 FOR THE MULTIPLICATIVE GROUP OF THE QUATERNIONS FIELD

JAN AMBROSIEWICZ*

We say that a group G has the property W^n if there exists such $n \in N$ that for each w, $K_w^{2^n}$ is a subgroup of group G, where $K_w = \{g \in G : |g| = w\}$, w runs all orders of elements of group G.

When $n = 1$, we say that a group G has property W. In [1] it was shown that the multiplicative group G_1 of quaternions with norm 1 does not have property W. In this work we shall prove that the group G_1 has property W^2 while the factor group $G1/\{1, -1\}$ has the property W. We shall also investigate a possession of the property W in the group $SO(3)$ and $PSL(2, F)$.

THEOREM 1. *The multiplicative group G_1 of quaternions with norm 1 has the property W^2.*

In the proof of Theorem 1, we will use the following lemmas:

LEMMA 1. *If $n \geqslant 7$ and $n \neq 12$, then between numbers $\frac{1}{8}n$ and $\frac{3}{8}n$ there is at least one number $k \in N$ $(k \neq 1)$ such that $(k, n) = 1$.*

Indeed, if

1° $n = 8s$, $s \geqslant 1$, then $k = 2s + 1$,
2° $n = 8s + 1$, $s \geqslant 1$, then $k = 2s$,
3° $n = 8s + 2$, $s \geqslant 1$, then $k = 2s + 1$,
4° $n = 8s + 3$, $s \geqslant 1$, then $k = 2s + 1$,
5° $n = 8s + 4$, $s \geqslant 1$, then $k = 2s - 1$,
6° $n = 8s + 5$, $s \geqslant 1$, then $k = 2s + 2$,
7° $n = 8s + 6$, $s \geqslant 1$, then $k = 2s + 1$,
8° $n = 8s + 7$, $s \geqslant 0$, then $k = 2s + 2$.

It is easy to check that so defined k fulfils the conditions of Lemma 1.

LEMMA 2. *In the multiplicative group G_1 of quaternions with norm 1, $K_n^4 = G_1$ for $n \geqslant 7$ and $n \neq 8, 12$.*

Proof. The quaternion $q_1 = p_1 + x_1 i + x_2 j + x_3 k$ with $p_1 = \cos\frac{2\pi}{n}$ belongs to a set K_n. By Lemma 1 there is such number $k \neq 1$ that K_n contains also quaternion q_2 with a real part $p_2 = \cos\frac{2k\pi}{n} \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, because $\frac{\pi}{4} \leqslant \frac{2k\pi}{n} \leqslant \frac{3\pi}{n}$. If q_2, q_3 are quaternions with real parts $p_3 = p_2 \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ then real parts of quaternions $q_2 q_3$ cover the interval $\langle 2p_2^2 - 1, 1\rangle$ which contains

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negative numbers. Therefore we can show in K_nK_n quaternions with real parts p and $-p$ such that $p \in \langle 2p_2^2 - 1,1\rangle \cap \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

The set K_nK_n is a normal set so quaternions with real parts p and $-p$ create in K_nK_n the subset $\bar{K}_m$ such that $\bar{K}_m \subset K_m$ for certain m. By Theorem 2 (see [1]) $\bar{K}_m\bar{K}_m = G_1$ then also $(K_nK_n)^2 = G_1$ for $n \geqslant 7$, $n \neq 8, 12$.

In order to answer the question whether the group G_1 has the property W^2, we must investigate also cases for $n = 2, 3, 4, 5, 6, 8, 12$. Of course, $K_2K_2 = \{1, -1\} \leqslant G_1$. By Corollary 2(see [1]), we have $K_4K_4 = K_8K_8\, G_1$.

Lemma 3. *In the group G_1 of quaternions with norm 1, $K_n^4 = G_1$ for $n = 3, 5, 6, 12$.*

Indeed the quaternion $q_1 = p + x_1i + x_2j + x_3k$ with $p = \cos\frac{2\pi}{n}$ belongs to the set $K_n(n = 3, 5, 6, 12)$. Since

$$q_1^{2^r} = \cos\frac{2^{r+1}\pi}{n} + y_1i + y_2j + y_3k,$$

then real parts of elements of set K_n^2 $(n = 3, 5, 6)$ cover the interval $\left\langle \cos\frac{4\pi}{n}, 1\right\rangle$ (see Lemma in [1]). If these intervals for all investigated n, contain negative numbers, then sets K_n^2 $(n = 3, 5, 6)$ contain quaternions with real parts p, $-p$, where $p \in \left\langle\cos\frac{4\pi}{n}, 1\right\rangle$. Real parts of quaternions of set K_{12}^2 cover intervals $\left\langle -1, -\frac{1}{2}\right\rangle$ and $\left\langle\frac{1}{2}, 1\right\rangle$ what follows from Lemma (see [1]) and from fact that numbers 5, 11 are relative prime with 12. Therefore also the set K_{12}^2 contains certain set of quaternions with real parts p and $-p$ $\left(p \in \left\langle\frac{1}{2}, 1\right\rangle, -p \in \left\langle -1, -\frac{1}{2}\right\rangle\right)$.

A further part of proof is the same as in the proof of Lemma 1.

Therefore for all $n \neq 2$ we have $K_n^2 = G_1$ or $K_n^4 = G_1$ while for $n = 2$, $K_nK_n = \{1, -1\}$ which means that Theorem 1 is true.

By Corollary 3 (see [1]), Theorem 1 and by the fact that in the multiplicative group G of quaternions field, $K_\infty K_\infty = G$, where K_∞ denotes the set of elements of order ∞ with norm 1, we have corollary:

Corollary 1. *The multiplicative group G of quaternions field has the property W^2.*

Theorem 2. *The factor group $\bar{G}_1 = G_{1/\{1,-1\}}$ has the property W.*

In the proof we use the following lemma:

Lemma 4. *If $n \geqslant 5$ and $n \neq 6$, then between numbers $\frac{n}{4}$ and $\frac{n}{2}$ there exists at least one number k such that $(k, n) = 1$.*

Indeed, if

1° $n = 4s$, $s \geqslant 2$, then $k = 2s - 1$,
2° $n = 4s + 1$, $s \geqslant 1$, then $k = 2s$,

3° $n = 4s + 2$, $s \geqslant 2$, then $k = 2s - 1$,
4° $n = 4s + 3$, $s \geqslant 1$, then $k = 2s + 1$.

It is easy to check that so defined k fulfils conditions of Lemma 4.

Proof. of Theorem 2. Since the unit of group $\bar{G}$ is $e = \{1, -1\}$, then quaternions with real part $p_1 = \cos\frac{2\pi}{n}$ or $p_2 = \cos\frac{\pi}{n}$ belong to set $\bar{K}_n$ of all elements of order n from group $\bar{G}_1$. By Lemma 4 follows existence of such number $k \neq 1$ that $\bar{K}_n$ $(n \geqslant 5, n \neq 6)$ contains a quaternion q_2 with real part $p_2 = \cos\frac{k\pi}{n} \in \left\langle 0, \frac{1}{\sqrt{2}} \right\rangle$ because $\frac{\pi}{4} \leqslant \frac{k\pi}{n} \leqslant \frac{\pi}{2}$. Since $(n, k) = 1$ implies $(n - k, n) = 1$ then the set $\bar{K}$ $(n \geqslant 5, n \neq 6)$ contains also the quaternion with real part $-p_2$. By Theoren 2 (see [1]), $K_nK_n = G_1$ and therefore $\bar{K}_n\bar{K}_n = \bar{G}_1$ for each $n \geqslant 5$, $n \neq 6$.

Since $K_4K_4 = G_1$ (see [1], Corollary 2) thus $\bar{K}_4\bar{K}_4 = \bar{G}_1$. We must to investigate yet sets $\bar{K}_2\bar{K}_2$, $\bar{K}_3\bar{K}_3$ and $\bar{K}_6\bar{K}_6$. The set $\bar{K}_2$ contains quaternions with real parts 0, $\bar{K}_3$ — quarternions with real part $\frac{1}{2}$ or $-\frac{1}{2}$. So by Lemma (see [1]) the real parts of elements of sets $\bar{K}_2\bar{K}_2$, $\bar{K}_3\bar{K}_3$ cover all interval $\langle -1, 1\rangle$. Theerfore $\bar{K}_2\bar{K}_2 = \bar{K}_3\bar{K}_3 = G_1$, thus $\bar{K}_2\bar{K}_2 = \bar{K}_3\bar{K}_3 = \bar{G}_1$.

The set $\bar{K}_6 = K_6' \cup K_6''$ where K_6' contains quaternions with real part $\frac{1}{2}$ while K_6'' contains quaternions with real parts $\pm\frac{\sqrt{3}}{2}$. By Lemma (see [1]), real parts of quaternions from the set $K_6'K_6'$ cover the interval $\left\langle -\frac{1}{2}, 1 \right\rangle$ while real parts of quaternions from the set $K_6''K_6''$ cover the interval $\left\langle -1, -\frac{1}{2} \right\rangle$. Therefore $\bar{K}_6\bar{K}_6 = G_1$, thus also $\bar{K}_6\bar{K}_6 = \bar{G}_1$.

It is easy to prove the following propositions:

PROPOSITION 1. *Real parts of elements of the set of quaternions CC, where C is a class of conjugate elements determined by quaternion with real part* $\cos\varphi$, *cover all interval* $\langle \cos 2\varphi, 1\rangle$.

By the induction, we have proposition:

PROPOSITION 2. *If C is a class of conjugate elements determined by quaternion with real part* $\cos\varphi$, *then real parts of quaternions from the set* C^{2^r} cover the interval $\langle \cos 2^r\varphi, 1\rangle$.

THEOREM 3. *In the multiplicative group G_1 of quaternions with norm* 1, *for each class C of conjugate elements* ($C \neq \{1\}, \{-1\}$), there exists number h such that $C^{2^h} = G_1$.

Proof. Let the class of conjugate elements be determined by quaternion q with real part $\cos\frac{2\pi}{n}$. According to the Proposition 2, real parts of elements from the set C^{2^r} cover the interval $\left\langle \cos\frac{2^{r+1}\pi}{n}, 1 \right\rangle$. Therefore there exists number

r_0 such that $\cos \frac{2^{r_0+1}\pi}{n} < 0$, which means that $C^{2^{r_0}}$ contains quaternion q with real part 0, so also $C^{2^{r_0}}$ contains all class C_0 of elements which are conjugate with q.

By Theorem 2 (see [1]), $C_0 C_0 = G_1$. Since $C_0 \subset C^{2^{r_0}}$, then

$$C^{2^{r_0}} C^{2^{r_0}} = C^{2^{r_0+1}} = G_1.$$

Corollary 2. *The group $\bar{G}_1$ does not have normal subgroups diffrent from $\{1, -1\}$.*

Indeed, let A be a normal subgroup of group G_1 and let $a \in A$, $a \neq 1, -1$. The element a determines certain class C of conjugate elements. From normality A, we have that for each $n \in N$, $C^n \subset A$ and in particular, $C^{2^h} = G_1 \leqslant A$.

By the same way as in Theorem 3, we have theorem:

Theorem 4. *In the group $\bar{G}_1 = G_1/\{1, -1\}$ for each class $\bar{C}$ of conjugate elements there exists $h \in N$ such that*

$$\bar{C}^{2^h} = \bar{G}_1.$$

Corollary 3. *The group $\bar{G}_1$ is a simple group.*

Let us accept the following notations:
$U(2)$ — the group of matrices

$$\begin{vmatrix} x & y \\ -\bar{y} & \bar{x} \end{vmatrix}$$

over the field of complex numbers $\mathbf{C}$, $SU(2)$ — the group of matrices

$$\begin{bmatrix} x & y \\ -\bar{y} & \bar{x} \end{bmatrix}, \quad x\bar{x} + y\bar{y} = 1, \quad x, y \in \mathbf{C},$$

$SO(3)$ — the group of turns of Euclid space E_3.

Using the following well known facts:
(i) $U(2) \simeq G$ (G — multiplicative group of quaternions),
(ii) $SU(2) \simeq G_1$ (G_1 — multiplicative group of quaternions with norm 1),
(iii) $SO(3) \simeq SU(2)/\{E, -E\} \simeq G_1/\{1, -1\}$, and using suitable theorems for quaternions field: Theorem 1, 2, 3, Corollary 1 and Theorem 1 (see [1]), we have a theorem:

Theorem 5. *The group $U(2)$ does not have property W, but the group $U(2)$ has property W^2.*

Theorem 6. *The group $SU(2)$ does not have property W, but has property W^2.*

Theorem 7. *The group $SO(3)$ has property W.*

THEOREM 8. *In the group $SO(3)$ for each C_m there exists $h \in N$ such that $C^{2^h} = SO(3)$.*

By $SL(2, F)$ we denote the group of matrices

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} a_{11}a_{22} - a_{12}a_{21} = 1, \ a_{ij} \in F.$$

We have $PSL(2, F) \simeq SL(2, F)/\{E, -E\}$.

Let $A_a = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$, $A_b = \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix}$, $A_a, A_b \in K_m \subset SL(2, F)$.

We will investigate the set $K \subseteq K_m K_m$ such that

$$K = \{T_1 A_a T_1^{-1} T_2 A_b T_2^{-1}, \ T_i \in SL(2, F)\}. \tag{1}$$

Since the set is a normal set then it is enough to limit our investigation to matrices of form

$$Y = A_a X A_b X^{-1}, \text{ where } \ X = T_1^{-1} T_2. \tag{2}$$

Let $X = \begin{bmatrix} x & y \\ z & u \end{bmatrix}$, $xu - yz = 1$. After transformations, we have

$$Y = \begin{bmatrix} abxu - ab^{-1}yz, & (-ab + ab^{-1})\, xy \\ (a^{-1}b - a^{-1}b^{-1})\, uz, & -a^{-1}bzy + a^{-1}\, b^{-1}\, ux \end{bmatrix}. \tag{3}$$

According to the notations used above, we have the following theorem:

THEOREM 9. *If*

(i) $m \neq 2$, $b^2 \neq 1$, *then* $K = K_m K_m = SL(2, F)$,

(ii) $m = 2$, $\mathrm{char} F \neq 2$, *then* $K_m K_m = K \neq SL(2, F)$.

Proof. (i). It is enough to show that matrices (3) cover all the group $SL(2, F)$. From Jordan Theorem we know that the matrix Y has the form

$$\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}, \ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Let us investigate the case $Y = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$. From a comparison of traces of matrices Y and $A_a X A_b X^{-1}$, we have

$$ab + a^{-1}b^{-1} + yza^{-1}b^{-1}(a^2 - 1)(b^2 - 1) = t + t^{-1}.$$

If $(a^2 - 1)(b^2 - 1) \neq 0$, then t can assume each value and the set K contain all matrices of form

$$T \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} T^{-1}, \ T \in SL(2, F).$$

If $(a^2-1)(b^2-1)=0$ then $Y=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ or $Y=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$.

Matrices $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ or $\begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix}$ belong to the set (3) iff following system equations

$$\begin{cases} a^{-1}uz(b-b^{-1})=0\\ xub-yzb^{-1}=\pm a^{-1}\\ a(b^{-1}-b)xy=1\\ xub^{-1}-yzb=\pm a\end{cases} \tag{4}$$

has a solution.

The system equations (4) has a solution over any field F if $b^2\neq 1$, $a+$ $+a^{-1}=b+b^{-1}$ and so it finishes the proof (i).

Proof (ii). If char $F\neq 2$, then in the group $SL(2, F)$,

$$K_2=\left\{T_1\begin{bmatrix}a & 0\\ 0 & a^{-1}\end{bmatrix}T_1^{-1},\ T_2\begin{bmatrix}b & 0\\ 0 & b^{-1}\end{bmatrix}T_2^{-1},\ T_i\in SL(2, F),\ a^2\neq 1,\ b^2\neq 1\right\}.$$

Therefore elements of the set K_2K_2 can be only of form (3). However the set of matrices of form (3) does not contain matrices $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$, $\begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix}$, because the system (3) with assumptions (ii) does not have a solution, then the set K_2K_2 cannot cover all of group $SL(2, F)$.

COROLLARY 4. *The group $SL(2, \mathbf{C})$ has the property W.*

Indeed, by Theorem 9 for $m\neq 2$, $K_mK_m=SL(2, \mathbf{C})$ while $K_2K_2=\{E, -E\}$.

COROLLARY 5. *The group $PSL(2, \mathbf{C})$ has the property W.*

Indeed, if $b^2\neq 1$, then $K=PSL(2, \mathbf{C})-\{E, -E\}$, if $b^2=1$, we receive the matrices E, $-E$. Therefore matrices (3) cover all the group $PSL(2, \mathbf{C})$. Observe that if $a=b$, then the set (3) is a product C_mC_m of classes of conjugate elements of order m. The system (4) is over field C solvable if $b^2\neq 1$. Therefore for the group $PSL(2, \mathbf{C})$ we have a corollary:

COROLLARY 6. *In the group $PSL(2, \mathbf{C})$ for each class $C_m(m\neq 1)$ of conjugate elements, $C_mC_m=PSL(2, \mathbf{C})$.*

We also have a corollary:

COROLLARY 7. *The group $PSL(2, \mathbf{C})$ is a simple group.*

Let us investigate the group $PSL(2, F)$, which char$F\neq 2$.

THEOREME 10. *If char $F\neq 2$ and a field F does not have such an element b that $b^2=-1$, then in the group $PSL(2, F)$ the set K_2K_2 does not cover of the group $PSL(2, F)$.*

Indeed, if char $F \neq 2$, then the set K_2 in the group $PSL(2, F)$ has the form:

$$K_2 = \left\{ T \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} T^{-1},\ T \in PSL(2, F),\ a^2 = 1 \right\}.$$

Therefore elements of the set K_2K_2 can be only of form (3). From (ii) of Theorem 9, $K_2K_2 \neq PSL(2, F)$.

COROLLARY 8. *If* char$F \neq 2$, $|F| \neq 3$ *and if the field* F *does not have such elements* b *that* $b^2 = -1$, *then the group* $PSL(2, F)$ *has no property* W.

Indeed, by Theorem 10, $K_2K_2 \neq PSL(2, F)$. Since K_2K_2 is a normal set and the group $PSL(2, F)$ with $|F| \neq 3$ is a simple group, then $K_2K_2 \leqslant PSL(2, F)$

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PROPRIETATEA W^2 PENTRU GRUPUL MULTIPLICATIV AL CÎMPULUI DE CUATERNIONI

(Rezumat)

În prezenta notă se demonstrează că grupul multiplicativ G_1 al cuaternionilor cu norma 1 are proprietatea W^2 şi că grupul factor $G_1/\{1, -1\}$ are proprietatea W.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXIX, 1984

SOME SEQUENTIAL PROPERTIES OF THE WEAK* DUAL OF A BANACH SPACE

MIGUEL A. CANELA[1]

1. **Introduction.** Let (E, τ) be a locally convex space, and E^* its topological dual. We denote by $\mu(E, E^*)$ the Mackey topology, and by $\sigma(E, E^*)$ the weak topology on E. τ^+ stands for the finest locally convex topology on E having the same convergent sequences as τ, and E^+ for the topological dual of (E, τ^+), i.e. the space of sequentially continuous linear functionals on (E, τ). The properties of τ^+ have been studied by Webb [14]. The spaces which satisfy the formula $\tau = \tau^+$ are called *C-sequential* by Wilansky [15], and *almost sequential* by Néméthi [11]. A weakening of this property is the identity $E^* = E^+$, which defines a class of spaces, called by Wilansky [15] *Mazur Spaces*.

C. S. Néméthi [11, 12] has examined the topology τ^+ for the case in which (E, τ) is the weak* dual of a Banach space X, showing that this topology is not hornological when X is infinite dimensional, and not even a Mackey topology in the separable case. In this note, we calculate explicity the topology $\sigma(X^*, X)^+$ showing it to coincide with the topology $\mu(X, X^*)^0$ of the convergence on compact subsets of X, when X^* is weak* angelic. We also show situations in which this result is not valid, $\sigma(X^*, X)$ not even being Mazur. Thus, we exhibit en example of a Banach space X whose closed dual unit ball is weak* sequentially compact, but $\sigma(X^*, X)$ is not Mazur. We say that a topological space is *angelic* when for every relatively countably compact subset A, the following holds:

i) A is relatively compact.

ii) Every closure point of A is the limit of a sequence contained in A.

The basic facts about angelic spaces can be found in [8]. The locally convex spaces whose weak* dual is angelic have been studied in [1].

We say that a Banach space X satisfies the *property D* when, if (Ω, Σ) is a measurable space and $f: \Omega \to X$ is totally scalarly measurable (i.e. the set of the $x^* \in X^*$ such that $x^* \circ f$ is measurable is weak* dense in X^*), f is scalarly measurable ($x^* \circ f$ is measurable for every $x^* \in X^*$). Property D has been introduced by A. Gulisashvili [9] in connection with the Pettis integral in interpolation spaces.

2. THEOREM. *Let X be a Banach space, and suppose that $(X^*, \sigma(X^*, X))$ is angelic. Then $\sigma(X^*, X)^+ = \mu(X, X^*)^0$.*

Proof. By the Banach-Dieudonné theorem, $\mu(X, X^*)^0$ is the finest locally convex topology which coincides with $\sigma(X^*, X)$ on the closed unit ball U of X^*. On the other hand, $\sigma(X^*, X)^+$ is the finest locally convex topology which coincides with $\sigma(X^*, X)$ on every weak* metrizable compact subset of X^*(see [15]). Thus, $\mu(X, X^*)^0$ is always coarser than $\sigma(X^*, X)^+$.

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A $\sigma(X^*, X)^+$ closed subset of U must be weak* sequentially closed, and, by the angelicity, weak* closed. Hence, $\sigma(X^*, X)$ and $\sigma(X^*, X)^+$ coincide on U, and $\sigma(X^*, X)^+$ is coarser than $\mu(X, X^*)^0$.

This result is valid for a large class of spaces, the weakly compactly generated Banach spaces, in particular for the reflexive and separable Banach spaces. In fact, it has been proved for the separable case by Webb [14, Proposition 4.12], following a different way. Actually, it is valid for $c_0(\Gamma)$, Γ arbitrary, for $L_1(\Omega, \Sigma, \mu)$, when μ is σ-finite, and for $C(K)$ when K is an Eberlein compactum. [3] is a good reference for the weakly compactly generated spaces.

3. **First example.** We are going to see now that the angelicity is not necessary for the equality $\sigma(X^*, X)^+ = \mu(X, X^*)^0$, though this formula seems to define a sequential condition for the weak* topology on the dual ball. Let ω_1 denote the first uncountable ordinal, and $[0, \omega_1]$ the set of ordinals $\leqslant \omega_1$. G.A. Edgar has proved that the space $X = \ell_1([0, \omega_1])$ is such that $\sigma(X^*, X)$ is Mazur (keeping in mind that $\aleph_1$ is not a *real-measurable cardinal*, use theorem 5.10 of [6]). $\sigma(X^*, X)$ being Mazur, the relation:

$$\mu(X, X^*)^0 \leqslant \sigma(X^*, X)^+ \leqslant \mu(X^*, X)$$

holds. But every weakly compact subset of $\ell_1([0, \omega_1])$ is norm compact, and therefore:

$$\mu(X, X^*)^0 = \mu(X^*, X).$$

Finally, we see that the closed unit ball of $X^* = \ell_\infty([0, \omega_1)]$ is not weak* angelic. It is not difficult to check that this ball is homeomorphic to the space $[-1, 1]^{[0,\omega_1]}$ and it is well known that $[-1, 1]^I$ contains a sequentially closed subset which is not closed, for any uncountable I (e.g. [8, 1.4]).

4. **Second example.** Consider now the order topology on $[0, \omega_1]$. Endowed with this topology, $[0, \omega_1]$ is a scattered compact space. Thus, the dual of the Banach space $X = C([0, \omega_1])$ is isometric to $\ell_1([0, \omega_1])$ [13, 19.7.7]. $X^* = \ell_1([0, \omega_1])$ has the Radon-Nikodym property [4, III.3.8.], and a result of Hagler and Johnson [10, Corollary 2] implies that the closed dual unit ball U is weak* sequentially compact. We will see now that this sequential compactness is not a sufficient condition for $(X^*, \sigma(X^*, X))$ to be Mazur nor for the property D.

1. G. A. Edgar [5, 6.2] has proved that $C([0, \omega_1])$ is not a *real-compact space* for its weak topology. According to an older result of Corson [2], the realcompactness of $(X, \sigma(X, X^*))$ is equivalent to the following condition: every $\alpha \in X^{**}$ which is weak* — continuous on all separable subsets of $(X^*, \sigma(X^*, X))$ is an element of X. If $(X^*, \sigma(X^*, X))$ is Mazur, this condition is obviously fulfilled. So this is not the case for $X = C([0, \omega_1])$.

At this point, a remark must be made. B. Faires has given in [7] a proof of the following fact: If X is a locally convex space which is complete and such that every closed equicontinuous subset of X^* is weak* sequentially compact, then $(X^*, \sigma(X^*, X))$ is Mazur. The proof uses the Grothendieck compactness theorem uncorrectly, because some type of completeness is always needed in applications of this theorem. The proof is reproduced in [15]. Our example shows that this result is not true. The author acknowledges. Cs. Némethi to share his skepticism on this point.

2. A. Gulisashvili [9] has proved that the angelicity of $(X^*, \sigma(X^*, X))$ implies the property D mentioned at the Introduction and has raised the problem of the reverse implication. We make here an approximation to this problem, showing that the weak* sequential compactness of the dual ball is not a sufficient condition for the property D.

We are going to exhibit now a $C([0, \omega_1])$ — valued function which is totally scalarly measurable but not scalarly measurable. For this we consider:

$$\Omega = [0, \omega_1] \times \{0, 1\},$$

and $f: \Omega \rightarrow C[(0, \omega_1])$ defined by:

$f(\alpha, 0) = \chi_{[0, \alpha]}$ (characteristic function of the clopen subset

$$[0, \alpha] \subset [0, \omega_1])$$

$f(\alpha, 1) = 1 = \chi_{[0, \omega_1]}$

for $\alpha \in [0, \omega_1]$. We consider now the σ-algebra Σ of the subsets of Ω which are countable or have countable complement. The space $[0, \omega_1]$ can be embedded in $C([0, \omega_1])^* = e_1([0, \omega_1])$ in the usual way, and the open segment $[0, \omega_1)$ is weak* total. For $\alpha \in [0, \omega_1]$, we have:

$$\alpha \circ f(\beta, 0) = 0 \qquad \text{if} \quad \beta < \alpha$$

$$\alpha \circ f(\beta, 0) = 1 \qquad \text{if} \quad \alpha \leqslant \beta \leqslant \omega_1$$

$$\alpha \circ f(\beta, 1) = 1 \qquad \text{for all } \beta.$$

Thus, $(\alpha \circ f)^{-1}(0)$ is countable for $0 \leqslant \alpha < \omega_1$, and so f is totally scalarly measurable. But $(\omega_1 \circ f^{-1})(0) = [0, \omega_1[x\{0\}]$ is not countable nor has countable complement, and so f is not scarlarly measurable.

5. **Problem.** If $\sigma(X^*, X)$ is Mazur, does the identity $\sigma(X^*, X)^+ = \mu(X, X^*)^0$ hold?

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UNELE PROPRIETĂŢI SECVENŢIALE ALE TOPOLOGIEI SLABE A DUALULUI UNUI SPAŢIU BANACH

(Rezumat)

În lucrare se arată că modificarea aproape secvenţială a topologiei *a* slabe a dualului unui spaţiu Banach coincide cu topologia convergenţei uniforme pe compacte, dacă bila unitate indusă a dualului este un spaţiu angelic în raport cu topologia slabă. Aceasta are loc în particular pentru spaţiile Banach separabile, caz în care se reobţine un rezultat al lui Webb [9]. Pe baza unui exemplu se arată că angelicitatea bilei unitate din spaţiul dual nu este esenţială pentru validitatea teoremei din lucrare. În încheierea lucrării se prezintă două probleme deschise.

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UNE THÉORIE DE COHOMOLOGIE SUR LA CATÉGORIE DES VARIÉTÉS FEUILLETÉES

GHEORGHE PITIŞ*

§1. **Préliminaires.** Soit V^{n+m} une variété feuilletée de codimension n. En partant de la décomposition de la différentielle extérieure sur une telle variété, dans [5] sont introduits les groupes de d_{01} — cohomologie de V^{n+m}, à coefficients réels, $H^{pq}(V;R)$.

Soit $\mathcal{CF}$ la catégorie des variétés feuilletées paracompactes. Dans [3] nous avons démontré que:

THÉORÈME 1.1. *Pour tout $p \geqslant 0$ il existe un foncteur contravariant $\mathcal{C}^p : \mathcal{CF} \to Ab$ grade qui associe á chaque variété feuilletée paracompacte V, le groupe gradué de cohomologie*

$$H^p(V;R) = \bigoplus_{q \geqslant 0} H^{pq}(V;R)$$

et á un morphisme feuilleté $f: V \to W$, l'homomorphisme

$$H^p(\tilde{f}^p) : H^p(W;R) \to H^p(V;R)$$

Le but de cette note est de démontrer que la catégorie des variétés feuilletées paracompactes est admissible pour une théorie de cohomologie et de construire une telle théorie sur la catégorie considérée. Nous obtenons aussi un exemple de théorie de cohomologie généralisée sur la même catégorie.

Les notations sont celles de [3] et [4].

§2. **Cohomologie feuilletée.** Soit $p\mathcal{CF}$ la catégorie des couples (V, V'), où V est une variété feuilletée paracompacte et V' une sous-variété feuilletée fermée de V. Dans $p\mathcal{CF}$ les morphismes de source (V, V') et de but (W, W') sont les morphismes feuilletés $f: V \to W$ tels que $f(V') \subseteq W'$.

PROPOSITION 2.1. *La catégorie $p\mathcal{CF}$ est admissible pour une théorie de cohomologie.*

Pour démontrer cette proposition, remarquons que si $(V, V') \in \mathrm{Ob}\, p\mathcal{CF}$ alors $(V, V') \times R \in \mathrm{Ob}\, p\mathcal{CF}$ et pour $t \in (-\infty, 0] \cup [1, +\infty)$, l'application $k_t : (V, V') \to (V, V') \times R$, définie par $k_t(x) = (x, t)$, est feuilletée.

Les morphismes dans la catégorie $p\mathcal{CF}$ étant feuilletés, nous pouvons donner la

DÉFINITION 2.1. Les morphismes $f, g : (V, V') \to (W, W')$, dans la catégorie $p\mathcal{CF}$, sont F — homotopes s'il existe un morphisme feuilleté $h : (V, V') \times R \to (W, W')$ tel que $h \cdot k_t = f$ pour $t \leqslant 0$ et $h \cdot k_t = g$ pour $t \geqslant 1$.

Compte tenu de la définition précédente, remplaçons l'axiome de l'homotopie par le suivant

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AXIOME. Si $f, g: (V, V') \to (W, W')$ sont deux morphismes F — homotopes dans $p\mathcal{C}\mathcal{F}$ alors

$$H^p(\tilde{f}^p) = H^p(\tilde{g}^p), \quad p \geqslant 0$$

Nous obtenons ainsi la notion de théorie de cohomologie feuilletée sur la catégorie $p\mathcal{C}\mathcal{F}$ ou sur une de ses sous-catégories admissibles.

Désignons par $\mathcal{A}^{pq}(V, V')$ le sous-espace de $\mathcal{A}^{pq}(V)$, dont les éléments sont les champs de formes différentielles de bidegré (p, q), nuls sur V'. Il en résulte que $\omega \in \mathcal{A}^{pq}(V, V')$ si et seulement si $\omega \in \mathcal{A}^{pq}(V)$ et $\tilde{j}^{pq}\, \omega = 0$, $j: V' \to V$ étant le morphisme d'inclusion. De plus

$$\tilde{j}^{p,q+1}\, d_{01}^{pq}\, \omega = d_{01}^{pq}\, \tilde{j}^{pq}\, \omega = 0$$

donc $d_{01}^{pq}\, \omega \in \mathcal{A}^{p,q+1}(V, V')$, d'où il résulte la suite semi-exacte

$$\mathcal{A}^p(V, V') \qquad \ldots \to \mathcal{A}^{pq}(V, V') \xrightarrow{d_{01}^{pq}} \mathcal{A}^{p,q+1}(V, V') \to \ldots$$

Le groupe de cohomologie de dimension q de la suite $\mathcal{A}^p(V, V')$ sera noté par $H^{pq}(V, V'; R)$.

Si $f: (V, V') \to (W, W')$ est un morphisme dans la catégorie $p\mathcal{C}\mathcal{F}$ alors pour $\omega \in \mathcal{A}^{pq}(W, W')$ on a

$$\tilde{j}^{pq}\tilde{f}^{pq}\, \omega = \tilde{f}^{pq}\, \tilde{j}^{pq}\, \omega = 0$$

donc $\tilde{f}^{pq}\, \omega \in \mathcal{A}^{pq}(V, V')$, d'où il résulte que f induit un morphisme de suites semi-exactes

$$\tilde{f}^p = \{\tilde{f}^{pq} : \mathcal{A}^{pq}(W, W') \to \mathcal{A}^{pq}(V, V')\}_{q \geqslant 0}$$

Nous avons construit ainsi un foncteur contravariant $\mathcal{A}^p : p\mathcal{C}\mathcal{F} \to \mathcal{S}e^s$, qui associe au couple admissible (V, V'), la suite semi-exacte $\mathcal{A}^p(V, V')$ et à un morphisme admissible f, le morphisme $\tilde{f}^p$. En composant ce foncteur par le foncteur contravariant de cohomologie H, nous obtenons le

THÉORÈME 2.1. *Pour tout $p \geqslant 0$ il existe un foncteur contravariant H^p: $: p\mathcal{C}\mathcal{F} \to$ Ab grade, qui associe au couple (V, V') le groupe gradué de cohomologie*

$$H^p(V, V'; R) = \bigoplus_{0 \geqslant q} H^{pq}(V, V'; R)$$

et au morphisme admissible $f: (V, V') \to (W, W')$ le morphisme

$$H^p(\tilde{f}^p) : H^p(W, W'; R) \to H^p(V, V'; R)$$

LEMME 2.1. *Soit $(V, V') \in$ Ob $p\mathcal{C}\mathcal{F}$ et $j: V' \to V$ le morphisme d'inclusion. Pour tout $\omega \in \mathcal{A}^{pq}(V')$ il existe $\bar{\omega} \in \mathcal{A}^{pq}(V)$ tel que $\tilde{j}^{pq}\bar{\omega} = \omega$.*
La démonstration du lemme 2.1 est la même que celle de la proposition 9.6, § 9, chap. IX, [2].

Du lemme 2.1 il résulte que pour $\omega \in \mathcal{A}^{pq}(V')$, d_{01} — fermée, il existe $\overline{\omega} \in \mathcal{A}^{pq}(V)$ telle que $\tilde{j}^{pq}\overline{\omega} = \omega$ et alors

$$\tilde{j}^{p,\,q+1}\, d_{01}^{pq}\overline{\omega} = 0$$

donc $d_{01}^{pq}\overline{\omega} \in \mathcal{A}^{p,\,q+1}(V, V')$ et c'est un cocycle. Nous pouvons donc définir l'homomorphisme

$$\delta^{pq}(V,\ V') : H^{pq}(V'\,;\ \ R) \to H^{p,\,q+1}(V,\ V'\,;\ R)$$

de la manière suivante: $\delta^{pq}(V,\ V') = 0$ pour $q < 0$ et

$$\delta^{pq}(V,\ V')[\omega] = [d_{01}^{pq}\overline{\omega}],\ \ \tilde{j}^{pq}\overline{\omega} = \omega$$

On démontre aisément que la classe de cohomologie du cocycle $d_{01}^{pq}\overline{\omega}$ dans $H^{p,\,q+1}(V,\ V'\,;\ R)$ dépend seulement de la classe de cohomologie du ω dans $H^{pq}(V'\,;\ R)$, donc δ^{pq} est bien défini.

LEMME 2.2. *Soient k_0, $k_1 : (V,\ V') \to (V,\ V') \times R$ les applications définies par $k_0(x) = (x,\ 0)$, resp. $k_1(x) = (x,\ 1)$. Les morphismes de suites semi-exactes*

$$\tilde{k}_0^p,\ \tilde{k}_1^p : \mathcal{A}^p(V \times R,\ V' \times R) \to \mathcal{A}^p(V,\ V')$$

sont homotopiquement équivalentes.

Démonstration. Nous allons construire une famille d'homomorphismes

$$\tilde{h}^{pq} : \mathcal{A}^{pq}\ (V \times R,\ V' \times R) \to \mathcal{A}^{p,\,q-1}(V,\ V')$$

tels que la relation suivante soit vérifiée

$$d_{01}^{p,\,q-1}\,\tilde{h}^{pq} + \tilde{h}^{p,\,q+1}\,d_{01}^{pq} = \tilde{k}_1^{pq} - \tilde{k}_0^{pq} \tag{1}$$

Pour $q \leqslant 0$ posons $\tilde{h}^{pq} = 0$ et pour $q > 0$ considérons d'abord le cas $V_0 = R^{n+m}$ et V_0' une sous-variété feuilletée fermée de V_0. Soit $\omega \in \mathcal{A}^{pq}(R^{n+m} \times R,\ V_0' \times R)$, R étant considéré comme une variété feuilletée de codimension 1. Si

$$\omega = a\ \ dx^{a_1} \wedge \ldots \wedge dx^{a_p} \wedge \theta^{u_1} \wedge \ldots \wedge \theta^{u_q} \tag{2}$$

alors $\tilde{h}_0^{pq}\,\omega = 0$ et pour

$$\omega = b\ \ dx^{a_1} \wedge \ldots \wedge dx^{a_p} \wedge dt \wedge \theta^{u_1} \wedge \ldots \wedge \theta^{u_{q-1}} \tag{3}$$

définissons l'homomorphisme $\tilde{h}_0^{pq}$ par

$$\tilde{h}_0^{pq}\,\omega = (-1)^p\Big(\int_0^1 b\,dt\Big)\,dx^{a_1} \wedge \ldots \wedge dx^{a_p} \wedge \theta^{u_1} \wedge \ldots \wedge \theta^{u_{q-1}} \tag{4}$$

Pour les formes différentielles du type (2) nous avons

$$\tilde{h}_0^{p,q+1} d_{01}^{pq}\omega = \tilde{h}_0^{p,q+1}\left[(-1)^p \frac{\partial u}{\partial t}\, dx^{a_1}\wedge \ldots \wedge dx^{a_p}\wedge dt \wedge \theta^{u_1}\wedge \ldots \theta^{u_q} + \text{formes ne contenant pas la différentielle } dt\right] = \left(\int_0^1 \frac{\partial a}{\partial t}\, dt\right) dx^{a_1}\wedge \ldots \wedge dx^{a_p}\wedge \theta^{u_1}\wedge \ldots \theta^{u_q}$$

D'autre part on a

$$(\tilde{k}_1^{pq} - \tilde{k}_0^{pq})\,\omega = [a(x,\ 1) - a(x,\ 0)]\ dx^{a_1}\wedge \ldots \wedge dx^{a_p}\wedge \theta^{u_1} \ldots \wedge \theta^{u_q}$$

donc (1) est vérifiée pour les formes (2).
Si ω est donnée par (3) alors

$$d_{01}^{p,q-1}\tilde{h}_0^{pq}\omega = (-1)^{p+q-1}\left(\int_0^1 \frac{\partial b}{\partial x^{u_q}}\, dt\right) dx^{a_1}\wedge \ldots \wedge dx^{a_p}\wedge dt \wedge \theta^{u_1}\wedge \ldots \wedge \theta^{u_q}$$

$$\tilde{h}_0^{p,q+1} d_{01}^{pq}\omega = (-1)^{p+q}\left(\int_0^1 \frac{\partial b}{\partial x^{u_q}}\, dt\right) dx^{a_1}\wedge \ldots \wedge dx^{a_p}\wedge dt \wedge \theta^{u_1}\wedge \ldots \wedge \theta^{u_q}$$

Mais $(\tilde{k}_1^{pq} - k_0^{pq})\omega = 0$, donc (1) est vérifiée pour toute forme différentielle du type (3).

Pour le cas général soit $\mathcal{A} = \{(U_\alpha,\ h_\alpha)\}_\alpha$ un atlas localement fini de V. Nous pouvons supposer que U_α est homéomorphe à R^{n+m} par h_α. Alors $\{U_\alpha \times R,\ \psi_\alpha\}_\alpha$, $\psi_\alpha(x,\ t) = (h_\alpha(x),\ t)$, est un altas sur $V \times R$ et si $\{a_\alpha\}_\alpha$ est une partition de l'unité subordonnée au recouvrement $\{U_\alpha\}_\alpha$ alors $\{b_\alpha : (x,\ t) \to a_\alpha(x)\}_\alpha$ est une partition de l'unité, subordonnée au recouvrement $\{U_\alpha \times R\}_\alpha$. Posons

$$\tilde{h}^{pq}\omega = \sum_\alpha \tilde{h}_\alpha^{p,q-1}\tilde{h}_0^{pq}(\tilde{\psi}_\alpha^{-1})^{pq} b_{\alpha*}(\omega)$$

$\tilde{h}^{pq}$ ainsi défini vérifie l'égalité (1).

Théorème 2.1. *Pour $p \geq 0$ le couple $H^{*p} = (H^p,\ \delta^{*p})$, $\delta^{*p} = \{\delta^{pq}\}_{q\geq 0}$ définit une théorie de cohomologie généralisée feuilletée à coefficients réels sur la catégorie $p\mathcal{C}\mathcal{F}$. H^{*0} définit une théorie de cohomologie feuilletée sur $p\mathcal{C}\mathcal{F}$.*

Démonstration 1. Axiome de la commutativité. On vérifie aisément que le diagramme suivant est commutatif pour tout $q \in Z$

$$\begin{array}{ccc} H^{pq}(W';\ R) & \xrightarrow{\delta^{pq}(w,w')} & H^{p,q+1}(W,\ W';\ R) \\ \tilde{j}^{*pq}\downarrow & & \downarrow \tilde{j}^{*p,q+1} \\ H^{pq}(V';\ R) & \xrightarrow{\delta^{pq}(V,V')} & H^{p,q+1}(V,\ V';\ R) \end{array}$$

2. Nous allons montrer que la suite

$$\ldots \to H^{p,q-1}(V';\ R) \xrightarrow{\delta^{p,q-1}} H^{pq}(V,\ V';\ R) \xrightarrow{\tilde{j}^{*pq}} H^{pq}(V;\ R) \to$$
$$\xrightarrow{\tilde{j}^{*pq}} H^{pq}(V';\ R) \xrightarrow{\delta^{pq}} H^{p,q+1}(V,\ V';\ R) \to \ldots$$

est exacte ($\bar{j} : (V,\ \emptyset) \to (V,\ V')$ est le morphisme d'inclusion).

2.a. Pour $[\omega] \in \operatorname{Im} \delta^{p,q-1}$ il existe $[\omega'] \in H^{p,q-1}(V'; R)$ tel que

$$\delta^{p,q-1}[\omega'] = [\omega]$$

D'après le lemme 2.1 il existe $\overline{\omega}' \in \mathcal{A}^{p,q-1}(V)$ tel que $\tilde{j}^{pq}\overline{\omega}' = \omega'$ et alors $[d_{01}^{p,q-1}\omega'] = [\omega] \in H^{pq}(V, V'; R)$, donc

$$\omega = d_{01}^{p,q-1}\overline{\omega}' + d_{01}^{p,q-1}\theta, \quad \theta \in \mathcal{A}^{p,q-1}(V, V')$$

$$\tilde{j}^{pq}\omega = d_{01}^{p,q-1}\tilde{j}^{p,q-1}\overline{\omega}' + d_{01}^{p,q-1}\tilde{j}^{p,q-1}\theta = d_{01}^{p,q-1}\omega'$$

Il en résulte $\operatorname{Im} \delta^{p,q-1} \subseteq \operatorname{Ker} \tilde{j}^{*pq}$.

Si $\tilde{j}^{*pq}[\omega] = 0$ alors il existe $\theta \in \mathcal{A}^{p,q-1}(V)$ tel que $\tilde{j}^{pq}\omega = d_{01}^{p,q-1}\theta$. Mais $\tilde{j}^{p,q-1}\theta \in \mathcal{A}^{p,q-1}(V')$ et

$$d_{01}^{p,q-1}\tilde{j}^{p,q-1}\theta = \tilde{j}^{pq} d_{01}^{p,q-1}\theta = \tilde{j}^{pq}\tilde{j}^{pq}\omega = 0$$

donc $[\tilde{j}^{p,q-1}\theta] \in H^{p,q-1}(V'; R)$ et parce que

$$\delta^{p,q-1}[\tilde{j}^{p,q-1}\theta] = [\omega]$$

il résulte $\operatorname{Ker} \tilde{j}^{*pq} \subseteq \operatorname{Im} \delta^{p,q-1}$.

2.b. Il est évident que $\operatorname{Im} \tilde{j}^{*pq} \subseteq \operatorname{Ker} \tilde{j}^{*pq}$.

Si $[\omega] \in \operatorname{Ker} \tilde{j}^{*pq}$ alors il existe $\theta \in \mathcal{A}^{p,q-1}(V')$ tel que $\tilde{j}^{pq}\omega = d_{01}^{p,q-1}\theta$ et pour $\sigma \in \mathcal{A}^{p,q-1}(V)$, $\tilde{j}^{pq}\sigma = \theta$ on a

$$\tilde{j}^{pq}(\omega - d_{01}^{p,q-1}\sigma) = 0$$

Mais $d_{01}^{pq}(\omega - d_{01}^{p,q-1}\sigma) = 0$, donc $[\omega - d_{01}^{p,q-1}\sigma] \in H^{pq}(V, V'; R)$ et alors

$$\tilde{j}^{*pq}[\omega - d_{01}^{p,q-1}\sigma] = [\omega]$$

ce qui démontre que $\operatorname{Ker} \tilde{j}^{*pq} \subseteq \operatorname{Im} \tilde{j}^{*pq}$.

2.c. Si $[\omega'] = \tilde{j}^{*pq}[\omega]$ alors $\omega' = \tilde{j}^{pq}\omega$ et

$$\delta^{pq}[\omega'] = [d_{01}^{pq}\omega] = 0$$

donc $\operatorname{Im} \tilde{j}^{*pq} \subseteq \operatorname{Ker} \delta^{pq}$.

Soit $[\omega] \in \operatorname{Ker} \delta^{pq}$ et $\overline{\omega} \in \mathcal{A}^{pq}(V)$ tel que $\tilde{j}^{pq}\overline{\omega} = \omega$. On a $[d_{01}^{pq}\overline{\omega}] = 0$ et considérons $\theta \in \mathcal{A}^{pq}(V, V')$ tel que $d_{01}^{pq}\overline{\omega} = d_{01}^{pq}\theta$. Alors $[\overline{\omega} - \theta] \in H^{pq}(V; R)$ et de plus

$$\tilde{j}^{*pq}[\overline{\omega} - \theta] = [\tilde{j}^{pq}\overline{\omega}] = [\omega]$$

donc $\operatorname{Ker} \delta^{pq} \subseteq \operatorname{Im} \tilde{j}^{*pq}$.

3. L'axiome de l'homotopie résulte du lemme 2.2.

4. Axiome de l'excision. Soit $(V, V') \in \mathrm{Ob}\, p\,\mathcal{CF}$, $U \subset V$ un ouvert tel que $\bar{U} \subseteq \mathring{V}'$ et supposons que $(V \setminus U, V' \setminus U) \in \mathrm{Ob}\, p\mathcal{CF}$ et que l'inclusion $e : (V \setminus U, V' \setminus U) \to (V, V')$ est un morphisme feuilleté. Si $\bar{\omega} \in \mathcal{A}^{pq}(V \setminus U, V' \setminus U)$ alors définissons $\omega \in \mathcal{A}^{pq}(V, V')$ par

$$\omega_{|V \setminus U} = \bar{\omega}, \quad \omega_{|\bar{U}} = 0$$

Il en résulte $\tilde{e}^{pq}\,\omega = \bar{\omega}$, donc $\tilde{e}^{pq}$ est surjectif.

Soit $\omega \in \mathcal{A}^{pq}(V, V')$ tel que $\tilde{e}^{pq}\,\omega = 0$. De la définition de l'homomorphisme $\tilde{e}^{pq}$ il résulte que $\omega = 0$ sur $V \setminus U$ et parce que $(V \setminus U) \cup V' = V$ on en déduit que $\tilde{e}^{pq}$ est injectif. Donc e induit un isomorphisme des suites semi-exactes $\mathcal{A}^{p}(V, V')$ et $\mathcal{A}^{p}(V \setminus U, V' \setminus U)$.

5. Pour $p = 0$, l'axiome de la dimension résulte de la proposition 3, 1, [3], et alors H^{*0} définit une théorie de cohomologie feuilletée sur la catégorie $p\mathcal{CF}$.

Remarque. Les conséquences des axiomes d'Eilenberg — Steenrod restent valables pour la théorie de cohomologie construite, avec les modifications imposées par la notion de F — homotopie.

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O TEORIE DE COOMOLOGIE PE CATEGORIA VARIETĂŢILOR FOILETATE

(Rezumat)

În lucrare se consideră categoria varietăţilor foliate paracompacte $\mathcal{CF}$ şi se construieşte pentru aceasta o teorie de coomologie plecînd de la anumite rezultate ale lui I. Vaisman [5], [6] privind descompunerea diferenţialei exterioare şi grupurile de d_{01}—coomologie ale unei varietăţi V^{n+m} foliate de codimensiune n. În paragraful 2 se dau mai multe detalii asupra coomologiei foliate. În Teorema 2.1 se demonstrează posibilitatea construirii teoriei de coomologie folietată cu coeficienţi reali pe categoria $p\mathcal{CF}$ a perechilor (V, V') unde V e varietate foliată paracompactă, iar V' o subvarietate foliată închisă a lui V.

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NOTE ON THE INFINITE SYSTEM OF DIFFERENTIAL EQUATIONS

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Introduction. In this note we consider the infinite initial value problem

(PC) $$x_n' = g_n(t, x_n) + f_n(t, x_1, x_2, \ldots, x_{i_n}), \; x_n(0) = x_n^0$$

$(n = 1, 2, \ldots)$ with given x_n^0 in $\mathbf{R}$ and real functions f_n, g_n defined on $I \times \mathbf{R}^{i_n}$ and $I \times \mathbf{R}$, respectively. Here $I = [0, a]$, R^q is the q-dimensional Euclidean space and $C(I)$ denote the Banach space of continuous functions from I to $\mathbf{R}(=\mathbf{R}^1)$ with the usual supremum norm $||\cdot||_0$.

A function $x = (x_1, x_2, \ldots)$ is said to be a *solution* of

(PC) if $x_n \in C^1(I)$, $x_n(0) = x_n^0$ and

$$x_n'(t) = g_n(t, x_n(t)) + f_n(t, x_1(t), x_2(t), \ldots, x_{i_n}(t)) \text{ in } I, \text{ for each } n \geqslant 1.$$

Our purpose is to find assumptions of f_n and g_n which guarantee the existence of solution of the problem (PC) on I, using the fixed point theorem of Schauder type established in Sec. 2.

2. **Fixed point theorem.** Let X be a Fréchet space (see e.g. [4]) with a nonempty convex closed subset K. Denote by $\mathcal{P} = \{p_n : n = 1, 2, \ldots\}$ a saturated family of seminorms which generates the topology of X. Suppose we are given: T — a mapping of K into itself such that $T[K]$ is a closed set, and Q — a continuous mapping from K into a compact subset of X, and F — a mapping from $K \times K$ to X with $F[K \times K] \subset T[K]$. Assume, moreover, that for each p_n in $\mathcal{P}$ there is a constant k_n, $0 \leqslant k_n < 1$, and a constant $c_n > 0$ such that

$$p_n(F(x_1, y) - F(x_2, y)) \leqslant k_n \cdot p_n(Tx_1 - Tx_2),$$

and

$$p_n(F(x, y_1) - F(x, y_2)) \leqslant c_n \cdot p_n(Qy_1 - Qy_2)$$

for all x_1, x_2, y and x, y_1, y_2 in K.

Under these hypotheses the equation $F(x, Tx) = Tx$ has a solution in K.

The above result is a slight generalization of fixed point theorem given in [2] and [3]. Next, for the convenience of the reader we sketch a proof of this result:

First, assume that Ω is a closed set in X. Let $h_i (i \geqslant 1)$ be a sequence of Ω into itself such that (1) there exists $\lim_{i\to\infty} h_i(x)$ for every x in Ω, and (2) $p(h_i(u) - h_i(v)) \leqslant k_p \cdot p(u - v)$ for all u, v in Ω, $p \in \mathcal{P}$ and with $0 \leqslant k_p < 1$ (here k_p is a constant depending of a seminorm p). Further, let us put

$$h_0(x) = \lim_{i\to\infty} h_i(x) \text{ in } \Omega.$$

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Since Ω is a complete space, so by Cain and Nashed theorem ([1], Th. 2.2) we obtain that each $h_j (j = 0, 1, \dots)$ has a unique fixed point x_j in Ω. Moreover, if $y_0^{(i)} = x_0$ and $y_n^{(i)} = h_i(y_{n-1}^{(i)})$, then $p(x_i - y_n^{(i)}) \leqslant (1 - k_p)^{-1} \cdot k_p^n \cdot p(y_1^{(i)} - x_0)$ for $p \in \mathfrak{P}$, $i \geqslant 1$. and $n \geqslant 1$. Hence $\lim_{i \to \infty} x_i = x_0$.

Now, if we employ the above remarks then, in the same way as in the proof of theorem 2.2 from [2], our assertion follows easily.

3. Result. *Let* i_n $(n = 1, 2, \dots)$ *be positive integers with* $\sup_{n \geqslant 1} i_n = +\infty$. *Let* f_n, g_n $(n = (n = 1, 2, \dots)$ *be a real continuous bounded functions defined on* $I \times \mathbf{R}^{i_n}$ *and* $I \times \mathbf{R}$, *respectively. Suppose that* $|f_n(t, u_1, u_2, \dots, u_{i_n})| \leqslant A_n$ *on* $I \times \mathbf{R}^{i_n}$, $|g_n(t, u)| \leqslant B_n$ *on* $I \times \mathbf{R}$, *and* $|g_n(t, u_1) - g_n(t, u_2)| \leqslant L_n |u_1 - u_2|$ *for every* t *in* I *and* u_1, u_2 *in* $\mathbf{R}$. *Then the problem* (PC) *has at least one solution defined on the interval* I.

4. Proof. Let $X = C(I) \times C(I) \times \dots$. In the vector space X define a sequence (p_n) of seminorms as $p_n(x) = ||x_n||_0$ for $x = (x_1, x_2, \dots)$. It is known that the space X equipped with topology generated by a family $\mathfrak{P} = \{p_n : n = 1, 2, \dots\}$ is a Fréchet space.

Without loss of generality we may suppose that $x_n^0 = 0$ for $n \geqslant 1$. Let n be a fixed index. Let $r_n > L_n$, and let

$$U_n(t) = \exp(r_n t), \quad V_n(t) = \exp(-r_n t)$$

for t in I. Moreover, H_n is defined as

$$H_n(x_1, x_2, \dots, x_{i_n})(t) = \int_0^t f_n(s, U_1(s)x_1(s), U_2(s)x_2(s), \dots, U_{i_n}(s)x_{i_n}(s))ds$$

for $x_1, x_2, \dots$ in $C(I)$.

Now, let us put:

$$K = \{(x_1, x_2, \dots) \in X : ||x_n||_0 \leqslant a(A_n + B_n) \text{ for } n \geqslant 1\},$$

and

$$(Tx)(t) = (V_1(t)x_1(t), V_2(t)x_2(t), \dots),$$

$$(Qx)(t) = (H_1(x_1, x_2, \dots, x_{i_1})(t), H_2(x_1, x_2, \dots, x_{i_2})(t), \dots),$$

$$F(x, y)(t) = ((H_1(y_1, y_2, \dots, y_{i_1})(t) + \int_0^t g_1(s, x_1(s))ds)V_1(t),$$

$$(H_2(y_1, y_2, \dots, y_{i_2})(t) + \int_0^t g_2(s, x_2(s))ds)V_2(t),$$

$$\dots$$

for $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ in X.

Obviously, $F[K \times K] \subset T[K] \subset K$, K is convex set, and $p(F(x, y) - F(x, z)) \leqslant p(Qy - Qz)$ for each p in $\mathfrak{P}$ and x, y, z in K. The convergence in X is equivalent to the coordinate-wise convergence, so K and $T[K]$ are closed subsets of X. Further, Q is continuous on K, and by Ascoli-Arzela Theorem the set $Q[K]$ is conditionally compact.

For $n \geqslant 1$, and $(x_1, x_2, \ldots)$, $(y_1, y_2, \ldots)$ in K, and t in I, we have

$$\int_0^t |g_n(s, x_n(s)) - g_n(s, y_n(s))| ds \leqslant$$

$$\leqslant L_n ||V_n(x_n - y_n)||_0 \cdot \int_0^t U_n(s) ds \leqslant r_n^{-1} L_n ||V_n(x_n - y\ )||_0 U_n(t)$$

and it follows that

$$\sup_{t \in I} V_n(t) \left| \int_0^t [g_n(s, x_n(s)) - g_n(s, y_n(s))] ds \right| \leqslant r_n^{-1} \cdot L_n \cdot \sup_{t \in I} V_n(t) \,|\, x_n(t) - y_n(t) \,|$$

This means that $p\,(F(x, z) - F(y, z)) \leqslant r_n^{-1} \cdot L_n \cdot p_n(Tx - Ty)$ for each p_n in $\mathfrak{P}$ and x, y, z in K.

Consequently, according to our Schauder type theorem, there exists a least one point $(x_1, x_2, \ldots, x_n, \ldots)$ in K such that

$$V_n(t) \left[\int_0^t g_n(s, x\ (s)) ds + H_n (V_1 x_1, V_2 x_2, \ldots, V_{t_n} x_{t_n})(t) \right] = V_n(t) \cdot x_n(t)$$

for t in I. Thus

$$x_n(t) = \int_0^t g_n(s, x_n(s)) ds + \int_0^t f(s, x_1(s), x_2(s), \ldots, x_{t_n}(s)) ds$$

$(n = 1, 2, \ldots)$ on I. This completes the proof.

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O NOTĂ DESPRE UN SISTEM INFINIT DE ECUAŢII DIFERENŢIALE

(Rezumat)

În lucrare se foloseşte o teoremă de punct fix a lui Schauder pentru a stabili existenţa unei soluţii (globale) a problemei cu valori iniţiale

$$x_n = g_n(t, x_n) + f_n(t, x_1, x_2, \dots), \quad x_n(0) = x_n^0$$

pentru $n = 1, 2, \dots$. Aici $x_n^0 \in R$ şi f_n, g_n sînt funcţii continue astfel că f_n depinde numai de primele n componente şi g_n satisfac condiţiei lui Lipschitz referitor la variabila a doua.

GENERALIZED CONVEX SEQUENCES

GH. TOADER

The notion of convexity was generalized in many ways. Some of these generalizations are based on the geometric interpretation of convexity and resort to an alteration of the finite differences. In this case it was impossible to transpose them for high order convexities. In this paper, we propose another generalization of convexity based on the notion of finite differences. For the moment we study the convexity of sequences of elements of an abelian group.

Let $(X, +)$ be an abelian group and $(x_m)_{m=1}^{\infty}$ a sequence of elements of X. With usual notations, we define the finite differences by the relations:

$$\Delta^0 x_m = x_m, \quad \Delta^{n+1} x_m = \Delta^n x_{m+1} - \Delta^n x_m \quad \text{for } n \geq 0. \tag{1}$$

One proves by induction, as in the classical case, the validity of the following relation:

$$\Delta^n x_m = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} x_{m+i} \tag{2}$$

where the second member must be interpreted in the natural way by means of the operation of the group. In fact, finite differences defined for sequences of elements of a commutative field was considered previously by M. D. Torres in [10], but obviously a group structure is enough for our purpose.

Let P be an arbitrary proper subset of X.

DEFINITION 1. The sequence $(x_m)_{m=1}^{\infty}$ is said to be $P - n -$ convex if $\Delta^n x_m \in P$ for any m.

Before passing to the study of the notion just introduced, let us give some examples.

Example 1. For the group $(\mathbf{R}, +)$ with $P = \mathbf{R}_+$ we obtain the usual $n -$ convexity (see [5] for more references).

Example 2. In the same group, for $P = \{0\}$ we obtain $n -$ polynomial sequences (met esspecially in the case of functions). Particularly, for $n = 2$ one get the arithmetical progressions.

Example 3. Ju. N. Subbotin has considered in [8] the set of the sequences with the property: $|\Delta^n x_m| \leq 1$ for any m. This may be obtained by choosing $P = [-1, 1]$.

Example 4. The case $(\mathbf{R} - \{0\}, .)$ with $P = [1, \infty)$ corresponds to logarithmic $n -$ convexity. In fact, one obtains a generalization of this because the sequences so defined need not to be positive.

Example 5. In the same group, but with $P = \{1\}$ we obtain sequences which we can call logarithmic $n -$ polynomial. Particularly, for $n = 2$ one get geometrical progressions.

Example 6. In the group $(Q - \{0\}, .)$ with $P = \mathbf{N}$ we obtain sequences which we name $n -$ divisible.

Remark 1. Although the way which we have chosen to arrive at the definition 1 is suitable, the following method may be regarded more natural. Let $(P, +)$ be a semigroup and $(x_m)_{m=1}^{\infty}$ be a sequence of elements of P. We say that x_m has finite difference of first order if there is $d \in P$ such that $x_{m+1} = x_m + d$. In this case we denote d by $\Delta^1 x_m$. Similarly may exist the differences of higher order. A sequence is named $n -$ convex if all his elements have differences of order n. This method is suggested by example 6.

Remark 2. Analogously, we way define the convexity of a function with values in a group (in which we have fixed certain subset P).

Although the definition 1 seems to be too general, we may transpose for it all the results which we obtained in [9] concerning the representation of $n -$ convex sequences. We begin with the following useful result which is easy to prove by induction:

LEMMA 1. *If the sequences $(x_m)_{m=1}^{\infty}$ and $(y_m)_{m=1}^{\infty}$ are related by:*

$$x_m = \sum_{i=1}^{m} y_i \text{ for } m \geqslant 1 \tag{3}$$

then:

$$\Delta^n x_m = \Delta^{n-1} y_{m+1} \text{ for } n \geqslant 1. \tag{4}$$

As a direct consequence, we have:

LEMMA 2. *The sequence $(x_m)_{m=1}^{\infty}$ is $P - n -$ convex if and only if there is a sequence $(y_m)_{m=1}^{\infty}$ such that holds (3) and $(y)_{m=2}^{\infty}$ be $P - \overline{n-1} -$ convex.*

To formulate the following result (which may be obtained by successive application of lemma 2) we need the following:

DEFINITION 2. The sequence $(y_m)_{m=1}^{\infty}$ is a $n - P$ sequence if $y_m \in P$ for $m > n$.

LEMMA 3. *There are the natural numbers $p_{m,i}^n$ (for any n, m and $i \leqslant m$) such that a sequence $(x_m)_{m=1}^{\infty}$ is $P - n -$ convex if and only if it may be represented by:*

$$x_m = \sum_{i=1}^{m} p_{m,i}^n y_i, \text{ for any } m \tag{5}$$

with a $n - P$ sequence $(y_m)_{m=1}^{\infty}$.

In [9] we have determined the numbers $p_{m,i}^n$ for the usual case of $n -$ convex sequences (example 1). We shall see that they are generally valid. We prove first:

LEMMA 4. *For an arbitrary sequence* $(y_m)_{m=1}^{\infty}$, *define the sequence* $(x_m)_{m=1}^{\infty}$ *by:*

$$x_m = \begin{cases} \sum_{i=1}^{m} \binom{m-1}{i-1} y_i \ \textit{for} \ m < n \\ \sum_{i=1}^{n-1} \binom{n-1}{i-1} y_i + \sum_{i=n}^{m} \binom{n+m-i-1}{n-1} y_i, \ \textit{for} \ m \geqslant n. \end{cases} \tag{6}$$

Then:

$$\Delta^n x_m = y_{n+m}, \ \textit{for any} \ m \geqslant 1 \tag{7}$$

and

$$\Delta^k x_1 = y_{k+1}, \ \textit{for} \ k < n. \tag{7'}$$

Proof. We shall prove only the relations (7) for $m < n$. The other cases may be proved analogously. From (2) and (6) we have:

$$\Delta^n x_m = \sum_{j=0}^{n-m-1} (-1)^{n-j} \binom{n}{j} \sum_{i=1}^{m+j} \binom{m+j-1}{i-1} y_i +$$

$$+ \sum_{j=n-m}^{n} (-1)^{n-j} \binom{n}{j} \sum_{i=1}^{n-1} \binom{m+j-1}{i-1} y_i + \sum_{i=n}^{m+j} \binom{m+n+j-i-1}{n-1} y_i$$

or, changing the order of addition:

$$\Delta^n x_m = \sum_{i=1}^{m-1} y_i \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \binom{m+j-1}{i-1} +$$

$$+ \sum_{i=m}^{n-1} y_i \sum_{j=i-m}^{n} (-1)^{n-j} \binom{n}{j} \binom{m+j-1}{i-1} +$$

$$+ \sum_{i=n}^{n+m} y_i \sum_{j=i-m}^{n} (-1)^{n-j} \binom{n}{j} \binom{m+n+j-i-1}{n-1},$$

the first sum missing for $m = 1$. Because, for any m and n we have:

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \binom{n+j}{k} = 0, \text{ if } k < m \text{ and } k \leqslant n \tag{8}$$

(as is proved, for example, in [7] p. 48), the first sum is zero. Making in the other two sums the changement of variable: $j = i - m + k$, by (8), we get (7).

So we get the coefficients $p_{m,j}^{n}$ from lemma 3, that is:

THEOREM 1. *A sequence* $(x_m)_{m=1}^{\infty}$ *is* $P - n -$ *convex if and only if there is a* $n - P$ *sequence* $(y_m)_{m=1}^{\infty}$ *such that* (6) *holds.*

Remark 3. In the usual case of $n -$ convex sequences, in [9] we found the representation (6) by induction from lemmas 2 and 3. Taking in account the lemma 4, we can obtain a similar representation by solving the system of equations (7) and (7') (see [1]). Prof. A. Lupaş pointed out to me „the fundamental formula of transformation of divided differences'' given by T. Popoviciu in [6], from which (6) may be also deduced if we make the notations (7) and (7'). In [2] and [3] may be also found some formulas related to Popoviciu's formula.

Remark 4. As is usually done in defining the logarithmic — convexity (see [4] for αm — convexity also), we may assume that the transformed sequence, by some fixed function, is convex. That is, given the set M, the group $(X, +)$, the set $P \subset X$, and the function $f: M \to X$, we may define the $f - P - n$ — convexity of a sequence $(x_m)_{m=1}^{\infty}$ from M, taking $\Delta^0 x_m = f(x_m)$. For example, for $f: \mathbf{R} - \{0\} \to \mathbf{R}$ defined by $f(x) = 1/x$ and the addition on $\mathbf{R}$, we obtain "harmonic progressions" for $P = \{0\}$ and a related convexity for $P = [0, \infty)$. If f is injective, we may obtain also the representation of such sequences using $f^{-1}: f(M) \to M$.

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ŞIRURI CONVEXE GENERALIZATE

(Rezumat)

Fie $(X, +)$ un grup abelian şi P o submulţime proprie a lui X. Spunem că un şir $(x_m)_{m=1}^{\infty}$ de elemente ale lui X este P-n-convex dacă $\Delta^n x_m \in P$ pentru orice m. Particularizînd grupul şi mulţimea P se obţin diverse noţiuni de convexitate (v. exemplele 1—6). În lucrare se obţine o teoremă de reprezentare a acestor şiruri.

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DOUBLE CONDENSATION OF SINGULARITIES FOR SYMMETRIC MAPPINGS

PETRU JEBELEAN

1. **Introduction.** First, we recall some notions and results needed in the sequel.

A subset S of a topological space T is called G_δ if S can be written as a countable intersection of open subsets of T. A countable union of nowhere dense sets in T is said to be of *I-st Baire category* (or a *meager set*). An uncountable dense G_δ subset of T is said to be *superdense* in T. If no open nonempty subset of T is meager in T then T is said to be a *Baire space*. As it is well known, every complete semimetric space is a Baire space. A topological vector space ($T.V.S.$ in short) X is Baire space if and only if X is not meager in X. All the vector spaces will be considered over the field K, where K stands for C—the field of complex numbers or R—the field of real numbers.

A subset M of a $T.V.S.$ X is said to be *bounded* if for every o-neighbourhood V in X there exists $\lambda > 0$ such that $M \subset \lambda \cdot V$. If X an arbitrary set, (Y, ρ) a semimetric space, $y_0 \in Y$ and $\mathcal{A}$ is a family of mappings from X to Y then the set

$$S_{\mathcal{A}}(y_0) \mathrel{\dot=} \{x \in X : \sup \{\rho(y_0, A(x)) : A \in \mathcal{A}\} = \infty\}$$

is said to be *set of singularities of the family* $\mathcal{A}$ *with respect to* y_0.

In the proof of main result (Theorem 2.3) we use the following lemma whose proof may be found in [4], p. 103:

1.1. LEMMA *If T is a nonempty complete metric space with no isolated points, then the intersection of any countable family of open and dense subsets of T is superdense in T.*

Rudin [4] pp. 101—103, emphasizes the phenomenon of double condensation of singularities for Fourier series of continuous functions on [0, 1] both in the space C[0, 1] and in the interval [0, 1].

S. Cobzaş and I. Muntean [1] proved a principle of double condensation of singularities for families of continuous sub-homogeneous mappings between normed spaces, depending upon a parameter ranging over a metric space. They derive the Rudin's result as well as other divergence results for some approximation methods as Lagrange interpolation polynomials, biorthogonal systems, quadrature formulae.

The exact statement of the principle of double condensation of singularities proved in [1] is:

1.2. THEOREM. *Let X be a nonzero Banach space, Y a normed space and T a nonempty separable complete metric space with no isolated points. Let A be a family of mappings $A : X \times T \to Y$ satisfying the following conditions:*

a) $A(.,t): X \to Y$ *is continuous,* $||A(x+y, t)|| \leqslant ||A(x; t)|| + ||A(y, t)||$, *and* $||A(\lambda x, t)|| \leqslant ||A(x, t)||$ *for all* $A \in \mathcal{A}$, $t \in T$, $x, y \in X$, *and* $\lambda \in K$ *with* $|\lambda| \leqslant 1$;

b) $A(x.,): T \to Y$ *is continuous for each* $A \in \mathcal{A}$ *and* $x \in X$;

c) *there exists a dense subset* T_0 *of* T *such that*

$$\sup \{||A(x, t)||: x \in X, ||x|| \leqslant 1, A \in \mathcal{A}\} = \infty$$

for all $t \in T_0$.

Then there exists a superdense subset X_0 *of* X *such that for each* $x \in X_0$ *the set* $\{t \in T: \sup\{||A(x, t)||: A \in \mathcal{A}\} = \infty\}$ *is superdense in* T.

The aim of this paper is to prove a more general principle of double condensation of singularities for families of continuous and symmetric mappings defined on a metrizable $T.V.S.$ and with values in a semimetric space (Theorem 2.3). The proof of this theorem is based upon an extension given in Theorem 2.1 of a result in Edwards [3], Theorem 7.5.1, on the condensation of singularities for countable families of lower semicontinuous functions. The last section of the paper contains some applications. First it is shown by an example in $C[0, \infty[$ that our Theorem 2.3 applies in situations wheere Theorem 2.1 does not work. Then it is shown that the set of functions in the space $C[a, b]$ having unbounded variation is superdense in this space.

2. Double condensation of singularities for symmetric mappings. The following theorem is an extension of a result in [3], Theorem 7.5.1.

2.1. THEOREM. *Let* X *be a Baire T.V.S., let* B *be a bounded subset of* X *and let* $f_n: X \to [0, \infty]$, $n \in N$, *be a family of mappings verifying the following conditions:*

a) *each* f_n *is lower semicontinuous;*

b) $f_n(x+y) \leqslant f_n(x) + f_n(y)$ *whenever* $f_n(x)$ *and* $f_n(y)$ *are finite and* $f_n(x) = f_n(-x)$, *for all* $x, y \in X$ *and* $n \in N$;

c) $\sup \{f_n(x): x \in B\} = \infty$ *for each* $n \in N$.

Then $S = \{x \in X: f_n(x) = \infty, n \in N\}$ *is a dense* G_δ *subst of* X.

Proof. Let $S_{n,m} = \{x \in X: f_n(x) > m\}$ for $n, m \in N$. Then

$$S = \bigcap \{S_{n,m}: n, m \in N\}.$$

Because f_n is lower semicontinuous, the sets $S_{n,m}$ are open in X for all $n, m \in N$. Let us show that $S_{n,m}$ are also dense in X for all $n, m \in N$. Suppose on the contrary that there exists $n, m \in N$ such that $S_{n,m}$ is not dense in X. Let $x_0 \in X \setminus S_{n,m}$ and let V be a balanced o-neighbourhood in X such that $(x_0 + V) \cap S_{n,m} = \emptyset$. As the set B is bounded, there exists $\lambda > 0$ such that $B \subset \lambda V$. Let $p \in N$, $p \geqslant \lambda$. The set V being balanced we have $B \subset \lambda \cdot V \subset p \cdot V$, so that $(x_0 + p^{-1} \cdot B) \cap S_{n,m} = \emptyset$. Consequently $f_n(x_0 + p^{-1} \cdot x) \leqslant m$, for all $x \in B$. But then

$$f_n(p^{-1} \cdot x) = f_n(x_0 + p^{-1} \cdot x - x_0) \leqslant f_n(x_0 + p^{-1} \cdot x) + f_n(-x_0) =$$
$$= f_n(x_0 + p^{-1} \cdot x) + f_n(x_0) \leqslant 2\, m$$

and

$$f_n(x) = f_n(p \cdot p^{-1} \cdot x) \leqslant p \cdot f_n(p^{-1} \cdot x) \leqslant 2pm$$

for all $x \in B$, contradicting the hypothesis $\sup \{f_n(x) : x \in B\} = \infty$. Hence the set $S_{n,m}$ is dense in X for every $m, n \in N$. It follows that S is a dense G_δ subset of X.

2.2. COROLLARY. *Suppose the hypothesis and notations of Theorem 2.1 are preserved, except the space X which is supposed to be a nonzero complete metrizable T.V.S. Then the set S is superdense in X.*

The Corollary 2.2 results from the proof of Theorem 2.1 and the Lemma 1.1.

Now we are ready to prove the main result of this paper.

2.3. THEOREM. *Let X be a nonzero complete metrizable T.V.S., (Y, ρ) a semimetric space, $y_0 \in Y$ and T a nonempty separable complete metric space without isolated points. Let $\mathcal{A}$ be a family of mappings $A : X \times T \to Y$ satisfying the following conditions:*

a) *$A(.,t) : X \to Y$ is continuous, $\rho(y_0, A(x + y, t)) \leqslant \rho(y_0, A(x, t)) + \rho(y_0, A(y, t))$ and $\rho(y_0, A(x, t)) = \rho(y_0, A(-x, t))$ for all $A \in \mathcal{A}$, $t \in T$ and $x, y \in X$;*

b) *$A(x,.) : T \to Y$ is continuous for each $A \in \mathcal{A}$ and $x \in X$;*

c) *there exists a dense subset T_0 of T and a bounded subset B of X such that:*

$$\sup \{\rho(y_0, A(x, t)) : A \in \mathcal{A},\ x \in B\} = \infty$$

for each $t \in T_0$.

Then there exists a superdense subset X_0 of X such that the set $\{t \in T : \sup \{\rho(y_0, A(x, t)) : A \in \mathcal{A}\} = \infty\}$ is superdense in T for each x in X_0.

Proof. Because T_0 is dense in the separable metric space T, there exists a countable subset $T_0' = \{t : n \in N\}$ of T_0, which is dense in T. For each $n \in N$ define $f_n : X \to [0, \infty]$ by

$$f_n(x) = \sup\{\rho(y_0, A(x, t_n)) : A \in \mathcal{A}\},\ x \in X.$$

By the continuity of $A(.,t_n)$ and of the semimetric ρ it follows that f_n is lower semicontinuous on X. By a), $f_n(x + y) \leqslant f_n(x) + f_n(y)$ and $f_n(x) = f_n(-x)$, for all $x, y \in X$. By c)

$$\sup\{f_n(x) : x \in B\} \geqslant \sup\{\rho(y_0, A(x, t_n)) : x \in B,\ A \in \mathcal{A}\} = \infty$$

for all $n \in N$.

Therefore by Corollary 2.2 the set $X_0 = \{x \in X : f_n(x) = \infty,\ n \in N\}$ is superdense in X.

By the continuity of $A(x,.)$ and of the semimetric ρ, it follows that the sets $T_{m,A}(x) = \{t \in T : \rho(y_0, A(x, t)) > m\}$ are open in T, for $m \in N$, $A \in \mathcal{A}$ and so will be the union $T_m(x) = \bigcup \{T_{m,A}(x) : A \in \mathcal{A}\}$. Let us prove that $T_m(x) = T$ for $m \in N$ and $x \in X_0$. To do this it suffices to show that $T_0' \subset T_m(x)$. If this inclusion were not true, one can find $n \in N$ such that $t_n \notin T_m(x)$ so that $f_n(x) \leqslant m$ contradicting the definition of X_0. Taking into account Lemma 1.1 the set $\{t \in T : \sup\{\rho(y_0, A(x, t)) : A \in \mathcal{A}\} = \infty\} = \bigcap \bigcap \{T_m(x) : m \in N\}$ is superdense in T for each $x \in X_0$.

2.4. THEOREM. *Let X be a nonzero complete metrizable T.V.S., (Y, ρ) a semimetric space, $y_0 \in Y$ and $\mathcal{A}$ a family of continuous mappings $A : X \to Y$ satisfying the following conditions:*

a) $\rho(y_0, A(x+y)) \leqslant \rho(y_0, A(x)) + \rho(y_0, A(y))$ *and* $\rho(y_0, A(x)) = \rho(y_0, A(-x))$ *for all* $A \in \mathcal{A}$ *and* $x, y \in X$;

b) *there exists a bounded set B of X such that*

$$\sup\{\rho(y_0, A(x)) : x \in B,\ A \in \mathcal{A}\} = \infty.$$

Then the set of singularities of the family $\mathcal{A}$ with respect to y_0 is superdense in X.

Proof. For each $n \in N$ and $x \in X$ we put $f_n(x) = f(x) = \sup\{\rho(y_0, A(x)) : A \in \mathcal{A}\}$. By Corollary 2.2 it follows that the set $S_{\mathcal{A}}(y_0) = \{x \in X : f(x) = \infty\}$ is superdense in X.

2.5. *Remarks.* a) Theorems 5.2 and 5.4 in [1] are consequences of our Theorems 2.3 respectively 2.4.

b) Taking in Theorem 2.4, X a Banach space, Y a normed space and a family of continuous linear mappings from X to Y, one obtains the classical Banach-Steinhaus principle of condensation of singularities.

c) Theorem 2.4 can also be compared with the uniform boundedness principle for F-spaces as proved in [2], p. 53.

3. **Some applications.** The following examples show that Theorem 2.3 is indeed more general than Theorem 1.2.

3.1. *Example.* Consider the locally convex space $C[0, \infty[$, of all continuous functions on $[0, \infty[$ with values in K endowed with the topology generated by the family of seminorms $\{p_n : n \in N\}$ where

$$p_n(x) = \max\ \{|x(t)| : t \in [0, n]\},\quad x \in C[0, \infty[,\ n \in N.$$

$C[0, \infty[$ is a non-normable complete locally convex metric space with respect to the metric

$$\rho(x, y) = \sum_{n=1}^{\infty} 2^{-n} p_n(x-y)/(1 + p\ (x-y)),\quad x, y \in C[0, \infty[.$$

Let $B = \{x \in C[0, \infty[: |x(t)| \leqslant t \text{ for all } t \geqslant 0\}$, $T = [1, 2]$, $Y = R$, $y_0 = 0$ and $A_m : C[0, \infty[\times T \to R$ defined by

$$A_m(x, t) = x(t^m),\ \text{for}\ x \in C[0, \infty[,\ m \in N,\ t \in T.$$

Let $T_0 =]1, 2]$. Then $\sup\ \{|A_m(x, t)| : x \in B,\ m \in N\} = \infty$ for each $t \in T_0$, since the function $x_0 \in C[0, \infty[$, defined by $x_0(t) = t$, $t \in [0, \infty[$ is in B. By applying Theorem 2.3 it follows that there exists a superdense subset X_0 of $C[0, \infty[$ such that for every $x \in X_0$, the set

$$\{t \in [1, 2] : \sup\{|x(t^m)| : m \in N\} = \infty\}$$

is superdense in $[1, 2]$.

Let $\mathcal{D}$ be the set of all divisions of a compact interval [a, b] with $a < b$; if $d \in \mathcal{D}$, $d = (a = t_0 < t_1 < \ldots < t_n = b)$ and $y : [a, b] \to K$, we put

$$V(y, d) = \sum_{i=1}^{n} |y(t_i) - y(t_{i-1})|$$

and

$$\overset{b}{\underset{a}{V}}(y) = \sup\{V(y, d) : d \in \mathcal{D}\}.$$

If $\overset{b}{\underset{a}{V}}(y) < \infty$ the function y is said to be *with bounded variation* on $[a, b]$ and if $\overset{b}{\underset{a}{V}}(y) = \infty$ then y is said to be *with unbounded variation* on $[a, b]$.

Let $C[a, b]$ be the Banach space of all continuous functions on $[a, b]$ with values is K endowed with the usual sup-norm.

3.2. THEOREM. *The set* $\left\{x \in C[a, b] : \overset{b}{\underset{a}{V}}(x) = \infty\right\}$ *is superdense in* $C[a, b]$.

Proof. For $d \in \mathcal{D}$, let $A_d : C[a, b] \to R$ defined by

$$A_d(x) = V(x, d), \ x \in C[a, b].$$

Observe that A_d is subadditive and $A_d(-x) = A_d(x)$ for all $x \in C[a, b]$ and $d \in \mathcal{D}$. If $d = (a = t_0 < t_1 < \ldots < t_n = b)$ then $|A_d(x - y)| \leqslant 2n\|x - y\|$, which shows that A_d is also continuous for every $d \in \mathcal{D}$.

Taking in Theorem 2.4 $X = C[a, b]$, $Y = R$, $y_0 = 0$, $\mathcal{A} = \{A_d : d \in \mathcal{D}\}$ and $B = \{x_0\}$ where $x_0 : [a, b] \to R$ is defined by

$$x_0(t) = \begin{cases} 0 & \text{for } t = a \\ \dfrac{t-a}{b-a} \sin^2 \dfrac{b-a}{t-a} & \text{for } t \in\,]a, b], \end{cases}$$

it follows that the set $S_{\mathcal{A}}(0) = \left\{x \in C[a, b] : \overset{b}{\underset{a}{V}}(x) = \infty\right\}$ is superdense in $C[a, b]$.

3.3. *Remark.* Theorem 3.2 can be derived also from Theorem 5.4 in [1].

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CONDENSAREA DUBLĂ A SINGULARITĂŢILOR PENTRU APLICAŢII SIMETRICE

(Rezumat)

Lucrarea prezintă un principiu general al condensării duble a singularităţilor pentru familii de aplicaţii continue şi simetrice definite pe un spaţiu vectorial topologic, metrizabil, indexate într-un spaţiu topologic separabil, cu valori într-un spaţiu semimetric. Obţinem o generalizare a rezultatului similar din [1].

Se arată că mulţimea funcţiilor cu variaţie nemărginită din $C[a, b]$ este superdensă în acest spaţiu.

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DESCRIEREA METODEI ELEMENTULUI FINIT CU FUNCŢII SPLINE PE O PROBLEMĂ BILOCALĂ SIMPLĂ

DOINA BRĂDEANU

1. Formularea problemei. Fie ecuaţia diferenţială ordinară liniară

$$-\frac{d^2u}{dx^2}+ax^k u=0 \tag{1}$$

unde u este funcţie de x iar a şi k sînt constante reale ($k \geq 0$). Se pune problema găsirii funcţiei $u(x)$ care verifică ecuaţia (1) în interiorul intervalului $I=(0,\ 2) \in R^1$ şi satisface condiţiile la extremităţi: $u(0)=0$ şi $u(2)=e^2$, [2]. Cu transformarea de funcţie

$$u(x)=z(x)+\frac{e^2-1}{2}x+1 \tag{2}$$

se ajunge la următoarea

Problemă bilocală cu condiţii omogene: să se determine funcţia $u(x)$ astfel ca

$$-\frac{d^2z(x)}{dx^2}+ax^k z(x)=-ax^k\left(1+\frac{e^2-1}{2}x\right),\quad x\in I=(0,2) \tag{3}$$

$$z(0)=0,\ z(2)=0$$

În cele ce urmează se consideră A ca un operator pe z astfel încît Az este partea stîngă din (3).

PROPOZIŢIA 1. *Operatorul liniar A este autoadjunct, pozitiv definit şi pozitiv mărginit inferior (strict pozitiv) cu constanta* $1/2$ *pe spaţiul liniar* $Z=\{z\in C^2[0,2]\,|\,z(0)=0,\ z(2)=0\}$, *dacă* $a\geq 0$.

Demonstraţie. Pentru demonstraţie se introduce un spaţiu fundamental cu produs scalar (spaţiul $L_2[0,2]$ cu produsul scalar şi norma definite în mod obişnuit). După cum este cunoscut, $C^2[0,2]$ este dens în spaţiul $L_2[0,2]$. Dacă $z,\ v\in Z$ şi calculăm produsul scalar L_2

$$(Az,v)=\int_0^2(z'v'+ax^kzv)dx=\int_0^2 z(-v''+ax^kv)dx=(z,A^*v), \tag{4}$$

deducem din $A^*v=-v''+ax^kv$, A^* fiind adjunctul lui A, că $A^*=A$ pentru toţi $z,\ v\in Z$. Prin urmare, operatorul A este autoadjunct pe Z.
Dacă $z=v$, din (4) obţinem

$$(Az,z)=\int_0^2 z'^2\,dx+a\int_0^2 x^kz^2\,dx\geq 0,\ \text{dacă}\ a\geq 0 \tag{5}$$

Egalitatea are loc dacă şi numai dacă $z(x) \equiv 0$. Într-adevăr, $z'(x) = 0$ implică $z(x) = c$ (const) ; dar $z(0) = 0$ aşa încît $c = 0$. Inegalitatea (5) dovedeşte că operatorul A este pozitiv definit.

Să arătăm, acum, că A este un operator strict pozitiv (pozitiv mărginit inferior), adică există o constantă $\alpha > 0$ astfel ca

$$(Az, z) \geqslant \alpha^2 (z, z). \tag{6}$$

Pentru demonstraţie să observăm că putem scrie

$$z(x) = \int_0^x z'(s)\, ds, \text{ cu } z(0) = 0$$

Ridicînd la patrat şi aplicînd inegalitatea lui Cauchy-Schwartz, obţinem

$$z^2(x) \leqslant \int_0^x ds \int_0^x z'^2\, ds = x \int_0^x z'^2\, ds$$

De aici, prin majorare pe [0,2] şi integrare, primim

$$\int_0^2 z^2(x)\, ds \leqslant 4 \int_0^2 z'^2\, ds,$$

de unde, dacă $a \geqslant 0$, deducem că are loc inegalitatea

$$\frac{1}{4} \int_0^2 z^2(x)\, ds \leqslant \int_0^2 z'^2(x)\, dx + a \int_0^2 z^2(x) x^k\, dx$$

care, după (4) se poate scrie în forma $(Az, z) \geqslant (z, z)/4$. Prin urmare, luînd $\alpha = 1/2$ este demonstrată inegalitatea (6).

Observaţia 1. Calculele precedente arată, de asemenea, că funcţionala biliniară $a(z, v) = (Az, v)$ dată de (4) este un produs scalar (energetic) notat prin $(z, v)_A$; avem

$$(z, v)_A = (Az, v) = \int_0^2 (z'v' + a x^k z v)\, dx$$

Observaţia 2. Inegalitatea (6) se poate demonstra şi direct, cu ajutorul inegalităţii lui Friedrichs

$$\| u \|_2 \leqslant (b - a) \| u \|_A$$

unde $\| \cdot \|_A$ este norma energetică a operatorului A iar $\| \cdot \|_2$ este norma în spaţiul $L_2([a, b]$. În cazul problemei formulate vom avea

$$\sqrt{(z, z)} \leqslant 2\sqrt{(Az, z)} \text{ şi } (Az, z) \geqslant \frac{1}{4} (z, z)$$

Propoziţia 1 asigură, din teorema de existenţă şi unicitate a soluţiei ecuaţiei operatoriale $Az = f$, ca să aibă loc

PROPOZIŢIA 2. *În spaţiul Z problema bilocală (3) are o singură soluţie; fie $z_0 \in Z$ această soluţie*

De asemenea, din teorema fundamentală a lui Ritz, pentru problema (3) are loc următoarea propoziţie

PROPOZIŢIA 3. *Funcţionala patratică a energiei $F: Z \to R^1$ definită prin*

$$F(z) = (Az,\ z) - 2(f,\ z) =$$

$$= \int_0^2 (z'^2 + ax^k z^2)\, dx + 2a \int_0^2 x^k \left(1 + \frac{e^2-1}{2} x\right) dx \tag{7}$$

are un minim absolut pentru $z = z_0$, adică

$$F(z_0) = \inf\{F(z) \,|\, z \in Z\}$$

şi reciproc: dacă $z_0 \in Z$ realizează un minim pentru funcţionala energiei $F(z)$ atunci z_0 este soluţie a problemei diferenţiale (3).

După cum este cunoscut, se spune în acest caz că funcţionala energiei (7) asociază o formulare variaţională de minim echivalentă (principiu variaţional de minim) la problema diferenţială (3).

Pentru rezolvarea problemei variaţionale aplicăm metoda aproximativă a elementului finit de tip Rayleigh-Ritz, într-un spaţiu finit dimensional (de dimensiune N), cu funcţii de formă (interpolare, baza) date de funcţiile spline.

2. **Procedeul lui Rayleigh-Ritz. Aproximarea soluţiei prin funcţii spline cubice.** Se consideră problema (3) în cazul $a = 1$; $k = 0$ pentru care avem

$$A(z) \equiv -\frac{d^2z}{dx^2} + z(x) = -\left(1 + \frac{e^2-1}{2} x\right),\ x \in I = (0,2) \tag{8}$$

$$z(0) = 0,\ z(2) = 0$$

În scopul determinării soluţiei aproximative alegem în locul spaţiului Z un subspaţiu finit dimensional Z_N (de dimensiune N), caracterizat printr-un ordin de netezime egal cu acela al spaţiului Z. Pentru N fixat se alege în Z_N o bază de funcţii $\{\Phi_i\}$, $i = \overline{1, N}$, funcţii care îndeplinesc următoarele condiţii: funcţiile Φ_i reprezintă un şir complet de funcţii liniar independente (baza spaţiului), sînt definite pe porţiuni (elemente finite, subintervale) şi aparţin spaţiului $C^2[0,2]$, au suport compact şi verifică condiţiile la limită omogene (se anulează în $x = 0$ şi $x = 2$). Un astfel de subspaţiu al lui Z îl oferă mulţimea

$$Z_N = \{z_N \in \tilde{S}_3(\pi) \,|\, z_N(0) = z_N(2) = 0\}$$

unde $\tilde{S}_3(\pi)$ este spaţiul liniar al funcţiilor spline cubice pe diviziunea

$$\pi: 0 = x_0 < x_1 < x_2 < x_3 < x_4 = 2$$

cu pasul $h = 1/2$, de dimensiune $N = 5$. Baza acestui spaţiu este formată din funcţiile spline cubice $\Phi_i = \widetilde{B}_i$, $i = \overline{0,4}$, care îndeplinesc condiţiile de mai sus. Prin urmare, subspaţiul Z_N al lui Z are forma

$$Z_N = \text{span}\,\{\widetilde{B}_0, \widetilde{B}_1, \widetilde{B}_2, \widetilde{B}_3, \widetilde{B}_4\}.$$

Pentru construirea bazei $\{\widetilde{B}_i\}$, $i = \overline{0,4}$, se foloseşte cunoscuta bază $\{B_i\}$ $i = -\overline{1,5}$ a spaţiului de funcţii spline cubice $S_3(\pi) = \text{span}\{B_{-1}, B_0, B_1, B_2, \ldots, \ldots, B_5\}$, [3], după cum urmează

$$\widetilde{B}_0 = B_0 - 4B_{-1},\ \widetilde{B}_1 = B_0 - 4B_1,\ \widetilde{B}_2 = B_2,\ \widetilde{B}_3 = B_4 - 4B_3,\ \widetilde{B}_4 = B_4 - 4B_5$$

Se obţin formulele:

$$\widetilde{B}_0(x) = 8\begin{cases} x(-9x + 3 + 7x^2), & x \in [0,1/2] \\ (1 - x), & x \in [1/2,1] \\ 0, & x \geqslant 1 \end{cases}$$

$$\widetilde{B}_1(x) = 8\begin{cases} -3x(1 + 3x - 5x^2), & x \in [0,1/2] \\ -\frac{1}{2} - 3(1 - x) - 6(1 - x)^2 + 13(1 - x)^3 & x \in [1/2,1] \\ -4\left(\frac{3}{2} - x\right)^3 & x \in [1,3/2] \\ 0 & x \geqslant 3/2 \end{cases}$$

$$\widetilde{B}_2(x) = 8\begin{cases} x^3, & x \in [0,1/2] \\ \frac{1}{8} + \frac{3}{4}\left(x - \frac{1}{2}\right) + \frac{3}{2}\left(x - \frac{1}{2}\right)^2 - 3\left(x - \frac{1}{2}\right)^3, & x \in [1/2,1] \\ \frac{1}{8} + \frac{3}{4}\left(\frac{3}{2} - x\right) + \frac{3}{2}\left(\frac{3}{2} - x\right)^2 - 3\left(\frac{3}{2} - x\right)^3, & x \in [1,3/2] \\ (2 - x)^3, & x \in [3/2,2] \end{cases}$$

$$\widetilde{B}_3(x) = 8\begin{cases} 0, & x \leqslant 1/2 \\ -4\left(x - \frac{1}{2}\right)^3, & x \in [1/2,1] \\ -\frac{1}{2} - 3(x - 1) - 6(x-1)^2 + 13(x-1)^3, & x \in [1,3/2] \\ 3(1 - x)(5x^2 - 17x + 13), & x \in [3/2,2] \end{cases}$$

$$\widetilde{B}_4(x) = 8\begin{cases} 0, & x \leqslant 1 \\ (x - 1)^3, & x \in [1,3/2] \\ \frac{1}{8} + \frac{3}{4}\left(x - \frac{3}{2}\right) + \frac{3}{2}\left(x - \frac{3}{2}\right)^2 - 7\left(x - \frac{3}{2}\right)^3, & x \in [3/2,2] \end{cases}$$

Graficele funcţiilor $\widetilde{B}_i$ sînt reprezentate în fig. 1

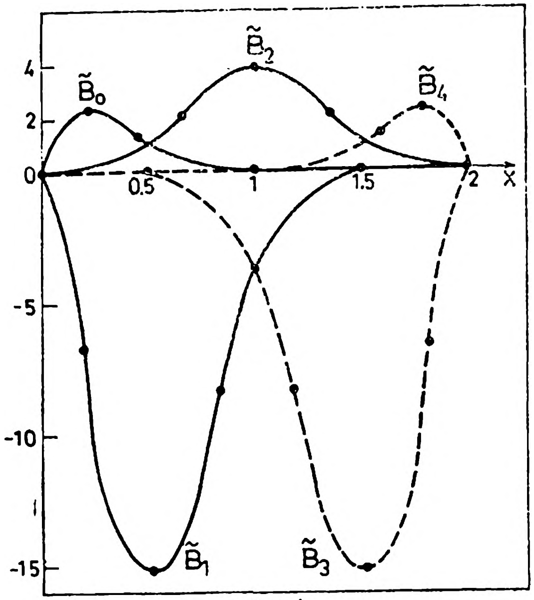

Fig. 1

În scopul rezolvării efective a problemei diferențiale (8) căutăm o aproximație Rayleigh-Ritz pentru problema variațională echivalentă pusă asupra funcționalei $F(z)$. Se încearcă o soluție aproximativă de forma funcției spline cubice

$$z_N(x) = \sum_{k=0}^{4} c_k \tilde{B}_k(x), \quad x \in [0,2] \tag{9}$$

unde c_k sînt coeficienți necunoscuți constanți. Aceşti coeficienți pot fi determinați prin rezolvarea sistemului Rayleigh-Ritz:

$$\sum_{k=0}^{4} (A\tilde{B}_i, \ \tilde{B}_k)c_k = (f, \ \tilde{B}_i), \ i = \overline{0,4} \tag{10}$$

în care notăm $(' = d/dx)$

$$b_i = (f, \ \tilde{B}_i) = \int_0^2 f\tilde{B}_i dx, \ i = \overline{0,4}; \tag{11}$$

$$a_{ik} = (A\tilde{B}_i, \ \tilde{B}_k) = \int_0^2 (-\tilde{B}_i'' + \tilde{B}_i)\tilde{B}_k dx = \int_0^2 (\tilde{B}_i'\tilde{B}_k' + B_i B_k)dx, \ i=\overline{0,4}; \ k=\overline{0,4}; \tag{12}$$

$$a_{ik} = \begin{cases} a_{ki} \\ 0 \text{ dacă } |i-k| = 4 \end{cases}; \ i, \ k = \overline{0,4}$$

Sistemul algebric (10) se scrie în forma matricială

$$[\widetilde{A}]\{c\} = \{b\} \tag{13}$$

$$([\widetilde{A}] = [a_{ik}],\ \{c\} = (c_0 \ldots c_4)^T,\ \{b\} = (b_0 \ldots b_4)^T)$$

Pentru a determina valorile coeficienţilor a_{ik} se vor calcula în prealabil derivatele $\widetilde{B}_i'(x)$ ale funcţiilor spline, scrise mai sus, $\widetilde{B}_i(x)$ după regulile de derivare obişnuite. Dacă se pune

$$\alpha_{ik} = \widetilde{B}_i'\widetilde{B}_k' + \widetilde{B}_i\widetilde{B}_k$$

şi se ţine seama de simetrie avem de calculat numai coeficienţii

$$a_{i(i+m)} = \sum_{k=0}^{3-m} \int_{x_{i-k+1}}^{x_{i-k+2}} \alpha_{i(i+m)}(x)dx\,;\ m = \overline{0,3}\,;\ i = \overline{0,4-m} \tag{14}$$

În aceste formule se va face convenţia ca acei termeni pentru care indicii limitelor de integrare sînt mai mici ca zero sau mai mari ca patru să nu intervină în calcul. Mai mult, volumul de calcule se poate reduce atît datorită simetriei operatorului A cît şi simetriei funcţiilor $\widetilde{B}_i$ şi $\widetilde{B}_i'$, $i = 0{,}4$ (fig. 1). Aceasta induce o simetrie a matricei de rigiditate $[\widetilde{A}]$ faţă de ambele diagonale. De aceea, în formulele (14) avem

$$a_{ii} = a_{(4-i)(4-i)},\ i = \overline{0,2}\,;\ a_{i(i+1)} = a_{(3-i)(4-i)},\ i = \overline{0,1}\,;$$

$$a_{i(i+2)} = a_{(2-i)(4-i)},\ i = \overline{0,1}\,;\ a_{i(i+3)} = a_{(1-i)(4-i)},\ i = 0\,;$$

În consecinţă, rămîne să se calculeze efectiv cu formulele (14) numai 8 elemente ale matricei $[\widetilde{A}]$, celelalte elemente se obţin prin simetrie faţă de cele două diagonale:

$$[\widetilde{A}] = \begin{vmatrix} a_{00} & a_{01} & a_{02} & a_{03} & 0 \\ . & a_{11} & a_{12} & a_{13} & . \\ . & & a_{22} & & . \\ . & & . & & . \\ 0 & . & . & . & . \end{vmatrix}$$

Termenii liberi b_i se calculează cu formulele (11) care se pot scrie în forma

$$b_i = -\int_0^2 \widetilde{B}_i(x)\,dx - a_0\int_0^2 x\widetilde{B}_i(x)\,dx\,;\ a_0 = \frac{e^2-1}{2} = 3{,}194528 \tag{15}$$

Valorile numerice b_i calculate cu (15) sînt scrise în sistemul (16).

Sistemul algebric al lui Rayleigh-Ritz (13), devine

$$\begin{bmatrix} 3484 & -4513 & -811 & 167 & 0 \\ . & 66562 & 351 & -15314 & . \\ . & . & 3964 & . & . \\ . & . & . & . & . \\ 0 & . & . & . & . \end{bmatrix} \begin{Bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} -1,958358 \\ 28,208810 \\ -12,583584 \\ 55,681751 \\ -6,430698 \end{Bmatrix} \qquad (16)$$

Acest sistem s-a rezolvat cu metoda eliminării a lui Gauss. S-au obţinut următoarele soluţii

$$c_0=-0,049061;\ c_1=0,042327;\ c_2=-0,264463;\ c_3=0,062205;\ c_4=-0,112217 \qquad (16')$$

Aceste valori împreună cu relaţia (9) dau soluţia aproximativă a problemei (8): se pot obţine valorile aproximative ale soluţiei problemei (8) în fiecare punct x din intervalul $[0,2]$.

3. **Evaluarea erorii.** Referitor la eroarea pe care o introduce $z(x)$, în [3] este dată următoarea evaluare

$$\| z - z_N \|_\infty \leqslant K h^3 \qquad (17)$$

unde $z(x)$ este soluţia exactă, $z_N(x)$ este aproximaţia lui Ritz de tipul funcţiilor spline cubice, h este pasul diviziunii uniforme pe $[0,2]$ (lungimea elementului finit rectiliniu) iar K este un număr pozitiv independent de N. Teoria matematică a procedeului lui Ritz, [1], [2], care este valabilă şi în cazul discretizării prin elemente finite unidimensionale, ca şi estimarea (17) asigură convergenţa metodei în problema considerată aici. În lucrarea [2] s-a aplicat pentru problema (1) o soluţie de aproximaţie prin polinoame Lagrange liniare pe porţiuni (o aproximaţie de clasă C^0). S-a întocmit tabelul 1 în care sînt date valorile nodale pentru soluţia exactă $u(x)$, soluţia de tip Lagrange u_L şi soluţia prin funcţii spline cubice (de clasă C^2) $u_s(x)$. Se obţine un înalt grad de exactitate în cazul folosirii funcţiilor spline cubice (tabelul 1).

Tabel 1

x	0,5	1,0	1,5
u(x)	1,648721	2,718281	4,481689
$u_s(x)$	1,648835	2,718546	4,482029
$u_L(x)$	1,634821	2,696116	4,460750

(Intrat în redacţie la 12 octombrie 1981)

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THE DESCRIPTION OF THE FINITE ELEMENT METHOD WITH SPLINE FUNCTIONS FOR A SIMPLE BILOCAL PROBLEM

(Summary)

In this paper, the Rayleigh-Ritz variational method on one-dimensional finite elements was outlined, within the general context of mathematical approximation, for a simple bilocal differential problem (1). This problem is also considered and is studied by piecewise linear polynomials in [2]. Here, another approximate solution, is proposed (9), given by means of the spline functions. This trial solution (9), is then determined by computing the $B_i(x)$ splines and by calculating the coefficients c_k (16'), using the Ritz procedure. The trial solution $u_s(x)$, from (2), is compared to the exact solution $u(x)$, which can be found analytically, and to the linear solution $u_L(x)$ (Lagrange). The spline solution $u_s(x)$ is better than the linear solution (Table 1).

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ON SOME CLASSES OF REGULAR FUNCTIONS

PETRU T. MOCANU and GRIGORE ŞT. SĂLĂGEAN

1. Introduction. Let m and n be two integers such that $m \geqslant 1$, $m + n \geqslant 1$, and let α be a real number, $\alpha < 1$. We denote by $T_{m,n}(\alpha)$ the class of functions

$$f(z) = z^m + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \dots, \tag{1}$$

which are regular in the unit disc $U = \{z\,;\,|z| < 1\}$ and satisfy

$$\operatorname{Re}\frac{[D_{m,n}f(z)]'}{m\,z^{m-1}} > \alpha, \qquad z \in U, \tag{2}$$

where

$$D_{m,n}f(z) = \frac{z^m}{(1-z)^{m+n}} * f(z). \tag{3}$$

Here $(*)$ stands for the Hadamard product (convolution) of power series, i.e. if $r(z) = \sum_{j=0}^{\infty} r_j z^j$ and $s(z) = \sum_{j=0}^{\infty} s_j z^j$, then $(r * s)(z) = \sum_{j=0}^{\infty} r_j s_j z^j$.

For $\alpha \in [0, 1)$ the classes $T_{m,n}(\alpha)$ were introduced by R. M. Goel and N. S. Sohi [1], who proved that $T_{m,n+1}(\alpha) \subset T_{m,n}(\alpha)$ and deduced that all functions in $T_{m,n}(\alpha)$ are m-valent.

In this paper we extend some results obtained in [1]. Our results, which are expressed in terms of subordination, are sharp and they yield best improvements of the main theorems stated in [1].

2. Preliminaries. Let r and s be regular functions in U. We say that r is *subordinate* to s, written $r \prec s$, or $r(z) \prec s(z)$, if s is univalent, $r(0) = s(0)$ and $r(U) \subset s(U)$.

We will make use of the following result, the more general form of which may be found in [3].

THEOREM A. *Let γ be a complex number with* $\operatorname{Re}\gamma \geqslant 0$ *and let h be a regular function in U such that $h(0) = 1$, $h'(0) \neq 0$ and*

$$\operatorname{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\min\left\{\operatorname{Re}\gamma,\ \frac{1}{2}\,\frac{|\gamma+1| - |\gamma-1|}{|\gamma+1| + |\gamma-1|}\right\}, \qquad z \in U. \tag{4}$$

If $p(z) = 1 + p_1 z + \dots$ is regular in U and

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z), \tag{5}$$

then

$$p(z) \prec q(z), \tag{6}$$

where

$$q(z) = \frac{\gamma}{z^\gamma}\int_0^z h(t)\, t^{\gamma-1}\, dt \prec h(z). \tag{7}$$

The function q is convex and the subordination (6) *is sharp.*

We note that condition (4) implies

$$\mathrm{Re}\left[1 + \frac{z h''(z)}{h'(z)}\right] > -\frac{1}{2},$$

which shows that h is univalent (close-to-convex).

Theorem A is a generalization of a result due to D. J. Hallenbeck and S. Ruscheweyh [2].

The hypergeometric function, which we shall use in this paper, will be denoted by $F(a, b, c; z)$.

3. **Main results.** THEOREM 1. *Let m and n be two integers, $m \geqslant 1$, $m + n \geqslant 1$ and let h be a regular function in U such that $h(0) = 1$, $h'(0) \neq 0$ and*

$$\mathrm{Re}\left[1 + \frac{z h''(z)}{h'(z)}\right] > -\frac{1}{2(m+n)}, \quad z \in U. \tag{8}$$

If the regular function f is of the form (1) *and satisfies*

$$\frac{[D_{m,n+1} f(z)]'}{m z^{m-1}} \prec h(z), \tag{9}$$

then

$$\frac{[D_{m,n} f(z)]'}{m z^{m-1}} \prec q(z), \tag{10}$$

where

$$q(z) = \frac{m+n}{z^{m+n}}\int_0^z h(t)\, t^{m+n-1}\, dt \prec h(z). \tag{11}$$

The function q is convex and the subordination (10) *is sharp.*

Proof. The function

$$p(z) = \frac{[D_{m,n} f(z)]'}{m z^{m-1}} \tag{12}$$

is regular in U and $p(0) = 1$. From (3) we obtain

$$z[D_{m,n} f(z)]' = (m + n) D_{m,n+1} f(z) - n D_{m,n} f(z) \tag{13}$$

and from (12) and (13) we get

$$\frac{[D_{m,n+1} f(z)]'}{m z^{m-1}} = p(z) + \frac{1}{m+n} z p'(z).$$

Hence the subordination (9) can be rewritten as

$$p(z) + \frac{1}{m+n} zp'(z) \prec h(z) \tag{14}$$

and Theorem 1 easily follows from Theorem A by letting $\gamma = m + n$.

COROLLARY 1.1. *If m and n are integers, $m \geqslant 1$, $m + n \geqslant 1$ and α is a real number, $\alpha < 1$, then*

$$T_{m,n+1}(\alpha) \subset T_{m,n}(\delta(\alpha)), \tag{15}$$

where

$$\delta(\alpha) = 1 - 2(1-\alpha)(m+n)\sum_{k=0}^{\infty} \frac{(-1)^k}{m+n+k+1} = \tag{16}$$

$$= 1 - 2(1-\alpha)\frac{m+n}{m+n+1} F(1, m+n+1, m+n+2; -1)$$

Moreover $\delta(\alpha) \geqslant \alpha$ and the value of $\delta(\alpha)$ is best possible.

Proof. Let $h(z) = h_\alpha(z) = \frac{1+(1-2\alpha)z}{1-z}$ and $f \in T_{m,n+1}(\alpha)$, i.e.

$$\frac{[D_{m,n+1} f(z)]'}{m\, z^{m-1}} \prec h_\alpha(z). \tag{17}$$

For $h = h_\alpha$, from (7) we get $q = q_\alpha$, where

$$q_\alpha(z) = \frac{\gamma}{z^\gamma}\int_0^z \frac{1+(1-2\alpha)t}{1-t} t^{\gamma-1}\, dt = 1 + \frac{2(1-\alpha)\gamma z}{\gamma+1} F(1, \gamma+1, \gamma+2; z). \tag{18}$$

Since, by Theorem A, $q_\alpha(z)$ is convex and for $\gamma = m + n$ it has real coefficients, we deduce

$$\inf_{z \in U} q_\alpha(z) = q_\alpha(-1) = 1 - 2(1-\alpha)\frac{m+n}{m+n+1} F(1, m+n+1, \ m+n+2; -1). \tag{19}$$

By Theorem 1 the subordination (17) implies

$$\frac{[D_{m,n} f(z)]'}{m\, z^{m-1}} \prec q_\alpha(z) \prec h_\alpha(z) \tag{20}$$

and from (19) and (20) we deduce the corollary.

Remark. Since $\delta(\alpha) \geqslant \alpha$, from (15) we obtain

$$T_{m,n+1}(\alpha) \subset T_{m,n}(\alpha).$$

This last result was proved in [1, Theorem 1], for $\alpha \in [0, 1)$.

THEOREM 2. *Let m and n be two integers, $m \geqslant 1$, $m + n \geqslant 1$, and let c be a complex number such that* $\mathrm{Re}(c + m) > 0$. *Let h be a regular function on U such that* $h(0) = 1$, $h'(0) \neq 0$ *and*

$$\mathrm{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\min\left\{\mathrm{Re}\,(c+m), \frac{1}{2}\,\frac{|c+m+1| - |c+m-1|}{|c+m+1| + |c+m-1|}\right\}, \tag{21}$$

If the regular function f is of the form (1) *and satisfies*

$$\frac{[D_{m,n}f(z)]'}{m\,z^{m-1}} \prec h(z), \tag{22}$$

then

$$\frac{[D_{m,n}\,g(z)]'}{m\,z^{m-1}} \prec q(z), \tag{23}$$

where

$$g(z) = \frac{c+m}{z}\int_0^z f(t)\,t^{c-1}\,dt \tag{24}$$

and

$$q(z) = \frac{c+m}{z^{c+m}}\int_0^z h(t)\,t^{c+m-1}\,dt. \tag{25}$$

The subordination (23) *is sharp and* $q(z) \prec h(z)$.

Proof. From (24) we obtain

$$cg(z) + zg'(z) = (c+m)f(z).$$

Hence

$$cD_{m,n}g(z) + D_{m,n}[zg'(z)] = (c+m)D_{m,n}f(z),$$

which can be rewritten as

$$cD_{m,n}\,g(z) + z[D_{m,n}\,g(z)]' = (c+m)D_{m,n}f(z). \tag{26}$$

If we let

$$p(z) = \frac{[D_{m,n}\,g(z)]'}{m\,z^{m-1}},$$

from (26) we get

$$p(z) + \frac{1}{m+c}\,z\,p'(z) = \frac{[D_{m,n}f(z)]'}{m\,z^{m-1}}$$

and (22) becomes

$$p(z) + \frac{1}{m+c}\,z\,p'(z) \prec h(z).$$

Now Theorem 2 easily follows from Theorem A by letting $\gamma = m + c$

COROLLARY 2.1. *Let m and n be two integers,* $m \geqslant 1$, $m + n \geqslant 1$, α *be a real number,* $\alpha < 1$ *and let c be a complex number such that* $\operatorname{Re}(c+m) > 0$

If $f \in T_{m,n}(\alpha)$ *then* $g \in T_{m,n}(\delta(\alpha))$, *where g is given by* (24) *and*

$$\delta(\alpha) = \inf_{z \in U} \operatorname{Re} \left\{ \frac{c+m}{z^{c+m}} \int_0^z \frac{1 + (1-2\alpha)t}{1-t} t^{c+m-1}\, dt \right\}. \tag{28}$$

Moreover $\delta(\alpha) \geqslant \alpha$ *and the value of* $\delta(\alpha)$ *is best possible.*

Remarks. 1. If c is real and $c + m > 0$, then $\delta(\alpha)$ defined by (28) is given by

$$\delta(\alpha) = 1 - 2(1 - \alpha)\frac{c+m}{c+m+1} F(1, c+m+1, c+m+2; -1).$$

2. Since $\delta(\alpha) \geqslant \alpha$, from Corollary 2.1. we deduce

$$f \in T_{m,n}(\alpha) \Rightarrow g \in T_{m,n}(\alpha),$$

where g is given by (24). For c real, $c + m > 0$ and $\alpha \in [0, 1)$ this last result was proved in [1, Theorem 2].

3. Corollary 2.1. shows that for all real α (even negative) for which $\delta(\alpha) \geqslant 0$, the integral operator (24) maps each function f in $T_{m,n}(\alpha)$ onto a function g in $T_{m,n}(0)$, which implies that g is m-valent.

For example, if $c = m = 1$ and $n = 0$ we obtain

$$\operatorname{Re} f'(z) > \alpha_0 \Rightarrow \operatorname{Re} g'(z) > 0, \tag{29}$$

where

$$\alpha_0 = \frac{4 \ln 2 - 3}{4 \ln 2 - 2} = -0.29435 \ldots .$$

We also have

$$\operatorname{Re} f'(z) > 0 \Rightarrow \operatorname{Re} g'(z) > \alpha_1, \tag{30}$$

where

$$\alpha_1 = 3 - 4 \ln 2 = 0.22741 \ldots .$$

These two implications improve the results of R. M. Goel and N. S. Sohi [1], who proved (29) and (30) with α_0 and α_1 repleced by $-1/4$ and $1/5$ respectively.

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ASUPRA UNOR CLASE DE FUNCŢII OLOMORFE

(Rezumat)

Aplicînd un rezultat general din [3] se îmbunătăţesc unele rezultate din [1] privind clasele $T_{m,n}(\alpha)$ definite de (1), (2) şi (3).

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXIX, 1984

E — CONEXIUNI SEMI-SIMETRICE

P. ENGHIŞ

În prezentul articol ne propunem reformularea, precizarea şi completarea rezultatelor lui S. Golab [3] privind spaţiile cu conexiune semi-simetrică.

A_n fiind un spaţiu cu conexiune afină, notăm cu Γ^i_{jk} componentele conexiunii afine într-un sistem de coordonate, cu $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$ componentele tensorului de torsiune a conexiunii Γ şi cu $T_k = T^i_{ik}$ componentele vectorului de torsiune (vectorul lui Vrânceanu).

Spaţiul A_n se numşte semi-simetric (Schouten) dacă există un cîmp vectorial covariant S_i astfel ca

$$T^i_{jk} = S_j \delta^i_k - S_k \delta^i_j \tag{1}$$

unde δ^i_k sînt simbolurile lui Kronecker.

În (1) dacă se aplică o contracţie în i şi j se obţine pentru $n \neq 1$

$$T_k = (1 - n)S_k \tag{2}$$

Din (2) rezultă:

PROPOZIŢIA 1. *Nu există spaţii semi-simetrice cu vectorul lui Vrânceanu nul.*

Dacă ţinem seama de (2) în (1) avem:

$$(1 - n)T^i_{jk} = T_j \delta^i_k - T_k \delta^i_j \tag{3}$$

Avem deci:

PROPOZIŢIA 2. *Intr-un spaţiu A_n cu conexiune semi-simetrică are loc relaţia* (3).

Din (3) putem deduce o nouă definiţie pentru spaţiile A_n cu conexiune semi-simetrică:

DEFINIŢIE. *Un spaţiu A_n se numeşte semi-simetric dacă între tensorul de torsiune şi vectorul de torsiune are loc relaţia* (3).

Dacă în (3) înmulţim contractat cu T_i obţinem

$$T^i_{jk} T_i = 0 \tag{4}$$

Avem deci:

PROPOZIŢIA 3. *Intr-un spaţiu A_n semi-simetric are loc relaţia* (4).

Un spaţiu A_n cu conexiune semi-simetrică se numeşte semisimetric special (S. Golab) dacă cîmpul vectorial S_i este gradient. Din (2) rezultă:

PROPOZIŢIA 4. *Intr-un spaţiu A_n semi-simetric special vectorul de torsiune este gradient şi reciproc, dacă într-un spaţiu A_n semi-simetric vectorul de torsiune este gradient, spaţiul este semi-simetric special.*

Într-o lucrare anterioară [2] am întrodus spaţiile A_n a căror conexiune Γ verifică relaţia

$$T_{i,j} - T_{j,i} = 0 \tag{5}$$

unde prin virgulă s-a notat derivata covariantă în raport cu conexiunea Γ, spaţii notate cu A_n şi numite de S. Golab [3] conexiuni Enghiş. Vom numi în cele ce urmează aceste conexiuni, E — conexiuni.

Scriind dezvoltat relaţiile (5) avem

$$T_{i,j} - T_{j,i} = \frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i} - T_s T^s_{ij} = 0 \tag{6}$$

şi dacă conexiunea este semi-simetrică rezultă

$$\frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i} = 0 \tag{7}$$

iar din propoziţia 4 rezultă că spaţiul este semi-simetric special. Am regăsit astfel rezultatul lui S. Golab [3] potrivit căruia, dacă o conexiune semi-simetrică este o E-conexiune ea este semisimetrică specială.

Să observăm acum, reciproc, că dacă conexiunea Γ este semisimetrică specială, din propoziţia 4 şi relaţiile (4), (6), (7) rezultă că ea este o E-conexiune. Avem deci:

PROPOZIŢIA 5. *Orice conexiune semi-simetrică specială este o E-conexiune.*

Observaţia 1. Faptul că o conexiune semi-simetrică specială verifică relaţia (5) a fost pus în evidenţă şi de P. Stavre [4] într-o altă problemă.

Din propoziţiile 4 şi 5 rezultă că E-conexiunile semi-simetrice sînt caracterizate de faptul că vectorul lui Vrînceanu este gradient.

Considerînd relaţia de definiţie a E-conexiunilor [2]

$$D\sigma = \frac{1}{2} T_i T^i_{hk} [dx^h \, dx^k] \tag{8}$$

unde $\sigma = T_k dx^k$, şi ţinînd seama de (4) rezultă:

PROPOZIŢIA 6. *Într-o E-conexiune semi-simetrică forma σ este închisă.*

Să notăm

$$\Gamma^i_{jkh} = \frac{\partial \Gamma^i_{jh}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^h} + \Gamma^i_{sk} \Gamma^s_{jh} - \Gamma^i_{sh} \Gamma^s_{jk} \tag{9}$$

componentele tensorului de curbură a conexiunii Γ. Se ştie [1] [5] că în acest caz avem

$$\Gamma^i_{jkh} = S^i_{jkh} + \frac{1}{2} \Omega^i_{jkh} \tag{10}$$

unde S^i_{jkh} sînt componentele tensorului de curbură a conexiunii simetrice $S^i_{jk} = \frac{1}{2} [\Gamma^i_{jk} + \Gamma^i_{kj}]$ asociată conexiunii Γ, iar Ω^i_{jkh} este dat de

$$\Omega^i_{jkh} = \frac{\partial T^i_{jh}}{\partial x^k} - \frac{\partial T^i_{jk}}{\partial x^h} - \frac{1}{2} T^i_{sh} T^s_{jk} + \frac{1}{2} T^i_{sk} T^s_{jh} + S^p_{jh} T^i_{pk} + S^i_{pk} T^p_{jh} - S^p_{jk} T^i_{ph} - S^i_{ph} T^p_{jk} \tag{11}$$

Contractînd în (11) în raport cu i şi j avem:

$$\Omega^i_{ikh} = \Omega_{kh} = \frac{\partial T_h}{\partial x^k} - \frac{\partial T_k}{\partial x^h} \tag{12}$$

Dacă conexiunea Γ este acum o E-conexiune semi-simetrică, din (12) rezultă:

$$\Omega_{kh} = 0 \tag{13}$$

relaţie ce reprezintă o condiţie necesară şi suficientă ca o conexiune semi-simetrică să fie o E-conexiune. Avem deci:

PROPOZIŢIA 7. *O condiţie necesară şi suficientă ca o conexiune semi-simetrică să fie o E-conexiune este dată de (13).*

Putem acum obţine mai simplu teoremele 5 şi 6 din lucrarea lui S. G o l a b [3], derivînd covariant (3). Avem:

$$(1 - n)T^i_{jk,r} = T_{j,r}\,\delta^i_k - T_{k,r}\,\delta^i_j$$

şi contractînd în i şi r rezultă:

$$(1 - n)T^i_{jk,i} = T_{j,k} - T_{k,j} \tag{14}$$

Deci:

PROPOZIŢIA 8. *Într-o E-conexiune semi-simetrică divergenţa torsiunii este nulă.*

Observaţia 2. Proprietatea că într-o conexiune semi-simetrică specială divergenţa torsiunii este nulă, a fost pusă în evidenţă pentru prima dată de P. S t a v r e [4].

Să observăm că din (1), (3), (14) rezultă şi reciproc, dacă într-o conexiune semi-simetrică divergenţa torsiunii este nulă, conexiunea este o E-conexiune. Avem deci o altă condiţie necesară şi suficientă ca o conexiune semi-simetrică să fie o E-conexiune, exprimată de:

PROPOZIŢIA 9. *O condiţie necesară şi suficientă ca o conexiune semi-simetrică să fie o E-conexiune este ca divergenţa torsiunii să fie nulă.*

Pentru a obţine o altă condiţie necesară şi suficientă ca o conexiune semi-simetrică să fie o E-conexiune, considerăm bine cunoscuta [5] relaţie dintre contractaţii tensorului de curbură.

$$\Gamma_{jh} - \Gamma_{hj} + R_{hj} = T^i_{jh,i} - T_{h,j} + T_{j,h} + T_s\,T^s_{jh} \tag{15}$$

unde

$$\Gamma_{jh} = \Gamma^i_{jih} \text{ iar } R_{hj} = \Gamma^i_{ihj}$$

În (15) dacă ţinem seama de (3), (4), (14) avem:

$$\Gamma_{jh} - \Gamma_{hj} + R_{hj} = \frac{2 - n}{1 - n}(T_{j,h} - T_{h,j}) \tag{16}$$

Avem deci:

PROPOZIŢIA 10. *Intr-un spaţiu A_n cu conexiune semi-simetrică are loc relaţia (16)*

Din (16) rezultă acum uşor că dacă conexiunea este o E-conexiune, $n \neq z$ avem:

$$\Gamma_{jh} - \Gamma_{hj} + R_{hj} = 0 \tag{17}$$

şi, reciproc, dacă (17) are loc, conexiunea semi-simetrică este o E-conexiune. Deci:

PROPOZIŢIA 11. *O condiţie necesară şi suficientă ca o conexiune semi-simetrică să fie o E-conexiune este ca (17) să aibă loc.*

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E — CONNECTIONS SEMI-SYMÉTRIQUES

(Résumé)

Dans le présent travail on apporte quelques précisions, additions et reformulations des résultats de S. Golab [3] concernant les connections semi-symétriques. On y montre qu'il n'existe pas d'espaces semi-symétriques à vecteur Vrănceanu nul. De la relation (3), on donne une nouvelle définition aux connections semi-symétriques. On montre que toute connection semi-symétrique spéciale est une E — connection (proposition 5) et on en donne par les propositions 6, 8, 10 les propriétés. Dans les propositions 7, 9, 11 on donne les conditions nécessaires et suffisantes pour qu'une connection semi-symétrique soit une E — connection par les relations (13), (14), (17).

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXIX, 1984

CONVEXITY OF SOME PARTICULAR FUNCTIONS

PETRU T. MOCANU

Let U be the unit disc in the complex plane. A regular function f is said to be *convex* in U if it is univalent and $f(U)$ is a convex domain. It is well-known that this geometrical condition is equivalent to $f'(0) \neq 0$ and

$$\operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, \text{ for } z \in U.$$

Consider the particular function

$$f(z) = \frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} z^{2k}, \quad z \in U, \tag{1}$$

where B_{2k} are the Bernoulli numbers. From a result due to R. Libera [1] it is easy to show that $1/f$ is convex in U. We shall prove that f and $\log f$ are convex in U, by using the following lemma, which is a slight modification of a result due to D. R. Wilken and J. Feng [4].

LEMMA 1. *Let μ be a positive measure on $[0, 1]$ and let $g(z, t)$ be a complex-valued function defined on $U \times [0, 1]$, such that $g(z, \cdot)$ is integrable on $[0, 1]$ for each $z \in U$. Let $a(t)$ be an integrable function, $a(t) > 0$, on $[0, 1]$ and suppose that*

$$\operatorname{Re} \frac{1}{g(z, t)} \geqslant \frac{1}{a(t)}, \text{ for } z \in U, \ t \in [0, 1]. \tag{2}$$

If

$$g(z) = \int_0^1 g(z, t) d\mu(t)$$

then

$$\operatorname{Re} \frac{1}{g(z)} \geqslant \frac{1}{\int_0^1 a(t) d\mu(t)}, \quad z \in U. \tag{3}$$

Proof. Condition (2) is equivalent to

$$\left| g(z, t) - \frac{a(t)}{2} \right| \leqslant \frac{a(t)}{2}, \quad z \in U, \ t \in [0, 1].$$

Hence

$$\left| g(z) - \frac{1}{2} \int_0^1 a(t) d\mu(t) \right| = \left| \int_0^1 \left[g(z, t) - \frac{a(t)}{2} \right] d\mu(t) \right| \leqslant \frac{1}{2} \int_0^1 a(t) d\mu(t),$$

which is equivalent to (3).

We also need the following result.

LEMMA 2. *If f is the function defined by* (1) *then*

$$\operatorname{Re}[2f(z) + z] < \frac{e+1}{e-1}, \text{ for } z \in U. \tag{4}$$

Proof. Since $2f(z) + z$ is an even regular function in U, to prove (4) it is sufficient to check the following inequality

$$A(x, y) \equiv a \operatorname{ch} x - x \operatorname{sh} x - (a \cos y + y \sin y) \geqslant 0, \tag{5}$$

where $x^2 + y^2 = 1$, $x \in [0, 1]$, $y \in [0, 1]$ and $a = \frac{e+1}{e-1} > 2$. Since for $a > 2$ the function $k(y) = a \cos y + y \sin y$ is decreasing on $[0, 1]$, we deduce $k(y) \leqslant k(0) = a$. Hence

$$A(x, y) \geqslant a \operatorname{ch} x - x \operatorname{sh} x - a \equiv B(x).$$

Since $B(x) = C(x) \operatorname{sh} x$, where $C(x) = a \operatorname{th} \frac{x}{2} - x$ is concave on $[0, 1]$ and $C(0) = C(1) = 0$, we deduce $B(x) \geqslant 0$, for all $x \in [0, 1]$, which yields (5).

THEOREM. *If f is the function defined by* (1) *then f and* $\log f$ *are convex in U.*

Proof. 1) If f is given by (1), we have

$$1 + \frac{zf''(z)}{f'(z)} = P(-z),$$

where

$$P(z) = 1 + \frac{z^2}{e^z - 1 - z} - \frac{2z}{e^z - 1} = 1 + \frac{z(e^z - 1)}{e^z - 1 - z} - [2f(z) + z]. \tag{6}$$

Let

$$g(z) = \frac{e^z - 1 - z}{z(e^z - 1)} = \int_0^1 g(z, t) d\mu(t),$$

where

$$g(z, t) = \frac{e^{tz} - 1}{t(e^z - 1)} \quad \text{and} \quad d\mu(t) = t\, dt.$$

We have

$$\frac{1}{g(z, t)} = \frac{t}{e^{tz} - 1} \int_0^z e^w\, dw = \frac{1}{h(z)} \int_0^z e^{(1-t)w} h'(w)\, dw,$$

where $h(z) = (e^{tz} - 1)/t$, for a fixed $t \in (0, 1]$. Since h is convex, with $h(0) = 0$ (hence starlike), from the well-known mean-value theorem of Sakaguchi-Merkes-Wright, [2], [3], we deduce

$$\operatorname{Re} \frac{1}{g(z, t)} \geqslant \frac{1}{e^{1-t}}, \; z \in U, \; t \in [0, 1].$$

Applying Lemma 1, with $a(t) = e^{1-t}$, we obtain

$$\operatorname{Re} \frac{1}{g(z)} \geqslant \frac{1}{\int_0^1 e^{1-t}\, t\, dt} = \frac{1}{e-2}$$

which yields

$$\operatorname{Re} \frac{z(e^z-1)}{e^z-1-z} > \frac{1}{e-2}, \quad z \in U. \tag{7}$$

Using Lemma 2 and (7), from (6) we deduce

$$\operatorname{Re} P(z) > 1 + \frac{1}{e-2} - \frac{e+1}{e-1} = \frac{3-e}{(e-1)(e-2)} > 0, \quad z \in U,$$

which shows that f is convex in U.

2) If we let $F(z) = \log f(-z) = \log \frac{ze^z}{e^z-1}$, then

$$1 + \frac{zF''(z)}{F'(z)} = \frac{z(e^z-1)}{e^z-1-z} - \frac{ze^z}{e^z-1} = \frac{z(e^z-1)}{e^z-1-z} - \frac{z}{2} - \left(\frac{z}{e^z-1} + \frac{z}{2}\right) =$$

$$= \frac{1}{2}\left(\frac{z(e^z-1+z)}{e^z-1-z} - [2f(z)+z]\right). \tag{8}$$

Let

$$G(z) = \frac{e^z-1-z}{z(e^z-1+z)} = \int_0^1 G(z,t)\, d\mu(t),$$

where

$$G(z,t) = \frac{e^{tz}+1}{t(e^{tz}-1+z)} \quad \text{and} \quad d\mu(t) = t\, dt.$$

We have

$$\frac{1}{G(z,t)} = \frac{1}{e^{tz}-1}\int_0^z (e^w+1)t\, dw = \frac{1}{h(z)}\int_0^z [e^{(1-t)w} + e^{-tw}]h'(w)dw,$$

where $h(z) = (e^{tz}-1)/t$, for a fixed $t \in [0, 1]$. Using again the mean-value theorem of Sakaguchi-Merkes-Wright, we deduce

$$\operatorname{Re} \frac{1}{G(z,t)} \geqslant e^{t-1} + e^{-t}, \quad z \in U, \ t \in [0, 1].$$

Applying Lemma 1, with $a(t) = 1/(e^{t-1} + e^{-t})$, we obtain

$$\operatorname{Re} \frac{1}{G(z)} = \operatorname{Re} \frac{z(e^z - 1 + z)}{e^z - 1 - z} \geqslant \frac{1}{I},$$

where

$$I = \int_0^1 \frac{t\,dt}{e^{t-1} + e^{-t}}.$$

A simple calculation shows that

$$I = \frac{1}{2}\int_0^1 \frac{dt}{e^{t-1} + e^{-t}} = \frac{\sqrt{e}}{2}\left[\operatorname{arctg}\sqrt{e} - \operatorname{arctg}\frac{1}{\sqrt{e}}\right] < \frac{\sqrt{e}}{12}\pi < \frac{\sqrt{3}}{4} < \frac{e-1}{e+1}.$$

Hence

$$\operatorname{Re} \frac{z(e^z - 1 + z)}{e^z - 1 - z} > \frac{e+1}{e-1},\ z \in U. \tag{9}$$

Using Lemma 2 and (9), from (8) we deduce

$$\operatorname{Re} \frac{zF''(z)}{F'(z)} + 1 > 0,\ z \in U,$$

which shows that $\log f$ is convex in U.

COROLLARY. *If t is real and B_{2k} are the Bernoulli numbers, then*

$$\frac{1}{e-1} \leqslant 1 - \frac{\cos t}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \cos 2kt \leqslant \frac{e}{e-1}.$$

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CONVEXITATEA UNOR FUNCŢII PARTICULARE

(Rezumat)

Se demonstrează că dacă funcţia f este definită de (1) atunci f şi $\log f$ sînt funcţii convexe în discul unitate U.

RECENZII

B e r n a d G e l b a u m, **Problems in Analysis**, Spinger-Verlag, New York, Heidelberg, Berlin, 1982, 228 p.

This book contains over five hundred problems and solutions in modern mathematical analysis including real analysis, measure theory, topology and topological vector spaces. It is structured into fifteen sections, the problems within each section being ordered according to their difficulty. We give here the titles of these sections in order to show the variety of problems offered by the book: Set algebra; Topology; Limits; Continuous functions; Functions from R^n to R^m; Measure and topology; General measure theory; Mesures in R^n; Lebesgue measure in R^n; Lebesgue measurable functions; $L^1(X, \mu)$; $L^2(X, \mu)$ or Hilbert space; $L^p(X, \mu)$, $1 \leq p \leq \infty$; Topological vector spaces; Miscellaneous problems.

Each section is provided at the beginning with some brief explanations of the notations and notions which occur in the folloving problems.

The solutions are given in the second part of the book. Some of them are complete and may enrich the experience of the readers in solving problems, others are rather sketchy leaving the completion to the reader. Some solutions are illustrated by graphs which make them more suggestive.

Although the notations and notions are those widely used in literature, at the end of the book there is a glossary and an index which allow an easy undestanding of the problems and the solutions to those who are familiar with analysis.

The bibliography lists a number of books which constitute the book's background material.

M. BALÁZS jr.

D a n i e l G o r e n s t e i n, **Finite simple groups**, An Introduction to Their Classification, Plenum Press, New York and London, 1982, 333 pag.

This is a fascinating book describing one of the greatest mathematical discoveries, namely the determination of all regular bodies by Euler and the determination of all simple Lie algebras over the complex field by Killing and Cartan. The determination of all finite simple groups is the work of a great number of scientists, more than a hundred, among whom the author of the reviewed book is one of the protagonists.

The book has four chapters. In the first one there is made a description of the way from R. Brauer's character theory to local analyses. The four phases of the classification, its consequences and the future of the finite group theory are sketched.

In the second chapter the simple nonabelian groups are briefly described namely the groups of Lie type, the Mathieu groups and the other sporadic groups.

The third chapter shows us how to make the identification of a group under investigation by means of some known simple groups.

The last chapter illustrates the main methods and results that underlie local groups theoratic analysis.

This book presents in a brief form the work of more than a hundred mathematicians, its author being one of the most successful in establishing simple groups. We must be grateful to the author of this scientific work cho revealed us in an accessible form and in few pages a very extensive work, which would have needed more than 16000 pages to be presented in detail.

G. PIC

W. R e i s i g, **Petrinetze. Eine Einführung.** Springer-Verlag, Berlin-Heidelberg-NewYork, 1982, 158 S. 111 Abb.

Petrinetze sind auf die Dissertation von C.A. Petri im Jahre 1962 zurückzuführen. Sie finden in letzter Zeit ein immer breiteres Interresse in den Reihen der Informatiker, da sie sich besonders für den Entwurf und die Analyse nicht-sequentieller (paralleler) Prozesse und Systeme eignen.

Das Buch setzt sich zum Ziel eine geschlossene und motivierte Einführung in die Theorie der Petrinetze zu vermitteln und den Leser in die Lage zu versetzen sich in der Speziallliteratur, deren Umfang und Vielfalt überwältigend sind, zurecht zu finden.

Es beginnt mit einer Einleitung, in welcher verschiedene Beispiele und einige grundlegende Definitionen gebracht werden. Der eigentliche Stoff des Buches ist in drei Teile gegliedert: Bedingungs/Ereignis-Systeme, Stellen/Transitionen-Netze und Netze mit Individuen als

Marken, entsprechend den möglichen Interpretationsmustern.

Im ersten Teil werden die kausale Abhängigkeit und Unabhängigkeit zwischen Ereignissen, die Metrik der Synchronieabstände und die Formulierung von Systemeigenschaften in der Sprache der Logik, als Fakten vorgestellt.

Die Netze im zweiten Teil entsprechen einer höheren Interpretations ebene und eignen sich besonders zur Formulierung des Blockierungsproblems. Als Untersuchungsmethoden werden Überdeckungsgraphen, sowie der Invariantenkalkül erklärt. Für spezielle Netzklassen werden Lebendigkeits- und Sicherheits- Kriterien gebracht.

Im dritten Teil werden Prädikat/Ereignis-Netze sowie Relationen-Netze vorgestellt und ein verallgemeinerter Invariantenkalkül für den Beweis bestimmter Systemeigenschaften entwickelt.

Das sehr sorgfältig geschriebene Buch bringt auf wenigen Seiten die meisten Konzepte der Petrinetztheorie, ihre exakten Definitionen und die mathematischen Hilfsmittel die benötigt werden. Es kann allen, die Petrinetze anwenden wollen, als unentbehrlich empfohlen werden.

FRIEDRICH LANDA

Nicolaie Lungu, **Pulsaţii stelare. Teorie matematică** (Pulsations stellaires. Théorie mathématique), Ed. ştiinţifică şi enciclopedică, Bucureşti, 1982, 180 pag.

Sans la prétention d'être exhaustif, le livre traite un des plus importants aspects de la recherche astrophysique actuelle — l'étude des pulsations stellaires — en faisant appel à la méthode moderne du modelage mathématique. Le phénomène de pulsation est étudié tant séparément qu'en connexion avec d'autres phénomènes qui influent sur lui, envisageant d'abord une théorie simplifiée, celle des pulsations linéaires, puis le cas le plus général, la théorie des pulsations non-linéaires.

Après une classification des étoiles pulsantes et après l'énumération des paramètres utilisés pour la comparaison de la théorie aux observations, la première partie du livre (consacrée aux pulsations linéaires) présente quelques modèles globaux où l'on entreprend un modelage mathématique et mécanique des pulsations, sans se s oucier de la production d'énergie ou du maintien du phénomène. On remarque entre ceux-ci les modèles à rotation proposés par V. Ureche et par l'auteur. On étudie ensuite les modèles à enveloppe et on présente amplement un modèle théorique de pulsations linéaires à rotation. Les équations qui y interviennent sont eésolues dans le cadre de quelques hypothèses simplificatrices et les résultats théoriques obtenus sont comparés aux données d'observation dans le cas des pulsantes RR Lyrae.

La seconde partie du livre est axée sur l'étude des pulsations non-linéaires. On présente dans ce cadre le modèle à enveloppe profonde introduit par Christy et on étudie ensuite les effets de la rotation sur ce genre de pulsations. Les équations du modèle théorique ainsi obtenu sont intégrées numériquement par la méthode des réseaux plans, les résultats permettant — par comparaison aux observations — quelques conclusions sur les étoiles RR Lyrae.

Afin de faciliter la compréhension de l'exposé, le livre contient trois annexes concernant respectivement la théorie des champs, la théorie de la stabilité et la méthode des réseaux plans.

Bien que destiné en premier lieu aux spécialistes, le livre (le premier publié chez nous qui traite en profondeur un seul aspect des recherches d'astrophysique) constitue — par les méthodes employées et par la rigueur de l'exposé — un excellent matériel qui peut être utile à des cercles bien plus larges de lecteurs, tout particulièrement aux mathématiciens et aux physiciens.

VASILE MIOC

Application and Theory of Petri Nets (in Englisch), Edited by Claude Girault and Wolfgang Reisig, Informatik-Fachberichte, Springer-Verlag, Berlin, Heidelberg, New York, 1982, Band 52, 337 S.

Dieser Band veranschaulicht den Fortschritt auf dem Gebiet der Theorie und der Anwendungen von Petrinetzen seit der fortgeschrittenen Vorlesung über Allgemeine Netztheorie für Prozesse und Systeme, welche in Hamburg, 8.—19.10.1979, gehalten wurde und wo das was auf diesem Gebiet in den 20 Jahren seit seinem Bestehen erreicht worden ist, im Detail vorgestellt wurde. Der Band enthält 34 Beiträge der ersten Arbeitstagung der verschiedenen Forschungsgruppen auf diesem Gebiet, die in Strassburg, 23.—26.09.1980, abgehalten wurde, sowie 10 Beiträge der zweiten Arbeitstagung, in Bad Honnef, 28.—30.09.1981.

Die Beiträge in Strassburg wurden in den folgenden sechs Sektionen vorgestellt:

1) Anwendungen von Netzen auf Realzeitsysteme

Die Nützlichkeit und die Fähigkeiten verschiedener Extensionen des Petrinetz-Konzeptes sowie Probleme bezüglich deren Anwendung für Rechnersysteme werden behandelt.

2) Programmiersprachen und Software- Technik

Man hofft die Software-Entwicklung mit Hilfe von Netzen in ein technisches Fach umzuwandeln, wobei die Netztheorie zur Beschreibung des wachsenden Verstehens des Systemverhaltens durch den Analysten oder als semantisches Mittel in Systementwurfssprachen benutzt werden kann.

3) Informationsfluss und Parallelität

Es wird versucht den Informationsfluss in einem Rechnersystem genauer zu definieren. Ausserdem werden gewichtete Synchronieabstände, Erreichbarkeit in einem Ereignisnetz and eine Methode zum systematischen Aufbau von Netzen vorgestellt.

4) Netzmorphismen und höhere Netzinterpretation

Es werden Morphismen, die gewisse Bedingungen, wie z. B. Beibehalten der Lebendigkeit und der Synchronieabstände erfüllen gesucht. Ausserdem werden über den Morphismus hinausführende Konzepte vorgestellt.

5) Mathematische Analysis und Netzsprachen

Für die Analyse von Plätze/Transitionen-Netzen werden abgewandelte Methoden der Prüfung sequentieller Programme, graph-theoretische Methoden, Methoden aus der Theorie der Formalen Sprachen und Methoden der Formalen Logik vorgestellt.

6) Zuverlässigkeit und Versuche zur Wiederherstellung

Ausgehend von den Fragen: Wie verwaltet man ein System?, Wie erhält man ein Netz welches ein System beschreibt?, wird eine kompakte Beschreibung für ein zuverlässiges System gesucht. Es werden einige Ideen und Hinweise gegeben.

Auf der Arbeitstagung in Bad Honnef wurden Beiträge aus allen Bereichen der Theorie und Anwendungen der Petrinetze gebracht. In den Band wurden aber nur diejenigen aufgenommen welche zum Gebiet der Kommunikations — protokolle gehören. Dieses, weil einerseits vom praktischen Gesichtspunt aus Kommunikationsprotokolle immer mehr an Bedeutung gewinnen, andererseits, weil die Netztheorie für die Beschreibung, Abschätzung und Prüfung von Kommunikationsprotokollen geeigneter als andere Methoden ist.

Der Band gibt einen Überblick der neuesten Forschungsergebnisse, der Entwicklung der Konzepte, Methoden und der Problematik. R Er ist für alle Petrinetz-Interressierten ein nützliches und brauchbares Werk.

FRIEDRICH LANDA

CRONICĂ

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Preprint 1—1982, P. Brădeanu, I. Pop, I. Stan, T. Petrilă, M. Drăganu, D. Brădeanu, C. Gheorghiu, Şt. Maksay, L. Mânea, *Seminar of Numerical Approximation Methods in Hydrodynamics and Heat Transfer.*

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Preprint 4—1983, V. Ureche, Á. Pál, E. I. Popova, L. R. Jungelson, M.I. Kumsiaşvili, C. Cristescu, I. Todoran, Z. Kraiceva, L. Patkós, N. Lungu, T. Oproiu, N. Ionescu-Pallas, L. Sofonea, D. Mihăilescu, V. Pop, D. Chiş, G. Oprescu, M. D. Şuran, R. Dinescu, A. Dumitrescu, A. Imbroane, *Seminar of Stellar Structure and Stellar Evolution.*

Preprint 5—1983, E. Popoviciu, L. Lupşa, M. Ivan, I. Raşa, *Seminar on Best Approximation and Mathematical Programming (Generalized Convexity).*

II. **Participări la manifestări ştiinţifice organizate în afara facultăţii**

1. *Seminar de spaţii Finsler, Braşov, 9—12 februarie 1982*

Din partea Facultăţii de matematică au prezentat comunicări:

M. Ţarină, P. Enghiş, *E-conexiuni Finsler.*

M. Ţarină, *Spaţii Finsler şi algebre Lie.*

2. *Colocviul naţional de mecanică, Bucureşti, 26—27 martie 1982:*

S-au prezentat din partea facultăţii comunicările:

Á. Pál, *Activitatea profesorului Caius Iacob la Universitatea din Cluj.*

V. Ureche, N. Lungu (Inst. Politehnic Cluj-Napoca), T. Oproiu (Centrul de Astronomie şi Ştiinţe Spaţiale — Colectiv Cluj-Napoca), *Proprietăţile hidrostatice şi geometrice ale continuumului spaţiu-timp la stelele politropice relativiste.*

T. Oproiu (CASS), I. Pop, *Asupra mişcării unui satelit artificial într-un mediu rezistent.*

V. Mioc (CASS — Colectiv Cluj-Napoca), *Perturbaţii în perioada nodală a sateliţilor artificiali cauzate de a cincea armonică zonală a geopotenţialului.*

V. Mioc, E. Radu (CASS — Col. Cluj-Napoca), *Asupra mişcării satelitului artificial al Pămîntului în cîmpul gravitaţional necentral.*

T. Petrilă, *Soluţii tip vîrtej pentru mişcări plane ale fluidelor ideale compresibile şi rotaţionale.*

I. Pop, *Convecţia liberă pe o placă verticală într-un cîmp gravitaţional neuniform.*

P. Mocanu, *Condiţii de univalenţă pentru clase de funcţii definite prin formule de structură.*

E. Popoviciu, *Proprietăţi de alură care intervin în aplicaţiile matematicii.*

M. Ţarină, *Conexiuni liniare şi aplicaţii în mecanica fluidelor.*

3. *A V-a Consfătuire a personalului din unitățile de informatică din M.E.I., Timișoara, 26—31 iulie 1982*

Din partea Centrului de calcul electronic al Universității au prezentat comunicări:

Gr. Moldovan, *Operatori convolutivi pozitivi în teoria aproximației.*

I. Chiorean, B. Pârv, R. Pop Deleanu, *CROSS — ASAMBLOR și LINK — EDITOR pentru sistemul MICRO — ARGUS.*

Gr. Moldovan, Gh. Mureșan, T. Toadere, *Optimizarea extracției sării prin dizolvare în sonde.*

4. *A VII-a Conferință de teoria probabilităților, Brașov, 29 august — 4 septembrie 1982*

Au participat din partea facultății cu comunicări:

I. Marușciac, *On the polygonal convex functions and applications.*

W. W. Breckner, *Equicontinuous families of generalized convex mappings.*

5. *Colocviul național de mecanica fluidelor, Galați, 22—23 octombrie 1982*

Au prezentat comunicări:

I. Stan, *Curgere superficială cu gradient de tensiune.*

T. Petrilă, *Procedee de abordare a studiului influenței pereților nelimitați asupra mișcărilor fluidelor ideale.*

6. *Zilele academice clujene, Cluj-Napoca, 22—26 noiembrie 1982:*

Comunicarea prezentată:

E. Popoviciu, *Implicații interdisciplinare ale noilor cercetări matematice.*

7. *Primul Seminar național de cercetări eoliene, Brașov, 26—27 noiembrie 1982*

A participat cu comunicare:

T. Petrilă, *Model matematic al turbinelor eoliene cu n pale*

8. *Sesiunea științifică „18 ani de realizări în domeniul matematicii românești, 1965—1983", București, 25 februarie 1983*

Din partea facultății s-a prezentat comunicarea:

Á. Pál, *Invățămîntul și cercetarea științifică la Facultatea de matematică din Cluj-Napoca.*

9. *Sesiunea Secției de Științe Matematice a Academiei R.S.R., București, martie 1983*

Comunicarea prezentată din partea facultății:

T. Petrilă, *Model matematic al turbinelor eoliene cu ax vertical.*

10. *Simpozionul național „Gheorghe Țițeica", Rm. Vîlcea, 8—9 aprilie 1983*

Comunicarea prezentată:

M. Țarină, *Teoreme de geometrie proiectivă. Aspecte elementare.*

11. *Ședința de comunicări a Seminarului „Th. Angheluță" a Catedrei de matematică din cadrul Institutului Politehnic, Cluj-Napoca, 10—12 iunie 1983*

Din partea Facultății de matematică s-au prezentat comunicările:

D. V. Ionescu, *Observații asupra ecuației diferențiale a lui Halphen.*

I. A. Rus, *Probleme actuale în analiza neliniară.*

Gh. Pic, *Contribuția lui Th. Angheluță în algebră.*

P. Mocanu, *Asupra unor operatori integrali care conservă stelaritatea.*

E. Popoviciu, *Asupra unor noțiuni de alură.*

L. Lupșa, *Proprietăți de optim ale funcțiilor tare convexe.*

D. Duca, *Duale de ordin superior în programarea matematică în domeniul complex.*

I. Marușciac, *Proprietăți diferențiale ale funcțiilor poligonal convexe cu aplicații în optimizare.*

I. Păvăloiu, *Metode iterative de tip interpolator cu ordin de convergență optimal.*

D. D. Stancu, *Contribuțiile lui Th. Angheluță la teoria aproximării funcțiilor.*

P. Enghiș, *Subspații recurente într-un spațiu euclidian.*

M. Țarină, *Variația curbelor autoparalele și ecuația lui Jacobi.*

I. Pop, *Asupra unor probleme de mișcare în medii poroase.*

F. Radó, *Asupra ecuației lui Cauchy.*

Gr. Moldovan, *O proprietate algebrică a operatorilor convolutivi pozitivi, pentru funcții de mai multe variabile.*

D. Dumitrescu, *Partiții nuanțate în recunoașterea formelor.*

A. Vasiu, *Spații eliptice de tip Hjelmslev—Barbilian.*

I. Kolumbán, *Despre modernizarea predării matematicii.*

12. *A XX-a sesiune de comunicări științifice a Institutului de învățămînt superior Oradea, 10—11 iunie 1983*

Au prezentat comunicări:

I. A. Rus, *Convexitate, compactitate, existență.*

M. Țarină, *Conexiuni invariante pe grupuri Lie.*

P. Enghiș, *Spații K dotate cu o D — E conexiune.*

13. *A XIV-a Conferință națională de geometrie și topologie, Piatra Neamț, 16—19 iunie 1983*

Matematicienii clujeni au fost prezenți prin:

F. Radó, *Teoreme de tip Beckman Quarles în plane Minkowski peste un cîmp.*

M. Țarină, *Conexiuni pe grupuri Lie și cîmpuri Jacobi.*

V. Groze, A. Vasiu, *Asupra unor clase de plane preeuclidiene.*

A. Vasiu, *Coordinatizarea unei clase de B — structuri.*

P. Enghiș, *Conexiuni sfert-simetrice T-recurente.*

F. Radó, *Legătura dintre intuiție și deducție în predarea matematicii.*

14. *Cea de-a XVII-a Consfătuire a Grupei de lucru permanente „Fizica cosmică" a academiilor țărilor socialiste participante la programul de colaborare „INTERCOSMOS", București-Măgurele, 7—10 iunie 1983.*

La Secția VI „Folosirea observațiilor asupra sateliților artificiali ai Pămîntului în scopuri astronomice, geodezice și geofizice", din partea Facultății de matematică s-a prezentat comunicarea:

Á. Pál, *Realizări în domeniul studiului atmosferei înalte și al cîmpului gravitațional terestru în anii 1981—1983.*

15. *Sesiunea de comunicări „Direcții moderne în astronomie și astrofizică"*, cu ocazia aniversării a 75 de ani de la înființarea Observatorului Astronomic București, *București—Măgurele, 9—11 noiembrie 1983.*

De la Facultatea de matematică au fost prezentate comunicările:

Á. Pál, *Asupra învățămîntului de astronomie din țara noastră.*

Á. Pál, *Model Roche pentru o stea dublă bazat pe schema problemei restrînse eliptice a trei corpuri.*

L. Burs (Deva), Á. Pál, *Algoritm și program FORTRAN pentru calculul densității atmosferei înalte din datele de frînare a sateliților artificiali.*

V. Ureche, *Stele relativiste*

V. Ureche, A. Imbroane (Într. Carbochim), *Stele relativiste omogene în rotație lentă.*

V. Ureche, N. Lungu (Inst. Politehnic Cluj-Napoca),

T. Oproiu (CASS), B. Pârv, *Diagrame de imersiune la unele modele stelare relativiste.*

V. Pop, *Variații ale perioadelor de pulsație la stelele RR Lyrae.*

I. Todoran (CASS), *Poziționarea oglinzilor plane pentru recepționarea energiei solare.*

I. Todoran (CASS), *Considerații asupra mișcării apsidale la steaua DI Herculis.*

V. Mioc (CASS), E. Radu (CASS), *Asupra perturbațiilor orbitelor sateliților artificiali produse de cea de-a șasea armonică zonală a geopotențialului.*

V. Mioc (CASS), *Evoluția mișcării de rotație a satelitului 1969—94 B.*

T. Oproiu (CASS), M. Cîrșmaru (CASS), *Asupra variației perioadei nodale a satelitului SAMOS 2.*

D. Chiș (CASS), *Variația perioadelor stelelor pulsante de tip RR Lyrae în urma neconservării masei sistemului binar.*

16. *A VI-a Consfătuire a personalului din unitățile de informatică, Suceava, 29 iulie — 4 august 1983.*

Gr. Moldovan, *Sistem informatic pentru conducerea activităților de bază din institutele de învățămînt superior.*

S. Damian, B. Pârv, P. Pop, *Subsistem pentru simularea fundamentării deciziilor într-o unitate agricolă.*

B. Pârv, A. Chisăliță (Inst. Politehnic Cluj-Napoca), *Algoritm și subprograme pentru rezolvarea eficientă a sistemelor de ecuații liniare de dimensiuni mari ($N > 1000$ ecuații)*

B. Pârv, R. Pop, Deleanu, *Cross-Assamblor pentru sistemul Microargus*

D. Chiorean, I. Chiorean, *Link—Editor pentru sistemul Microargus.*

I. Parpucea, S. Damian, M. Topliceanu, *Posibilități de testare automată, simulată prin software, a plachetelor cu circuite electronice cu ajutorul microcalculatorului M—18.*

17. *A II-a Conferință Națională de Cibernetică, 5—8 octombrie 1983, București*

Gr. Moldovan, *O problemă de distribuire a bazelor de date.*

18. *Al IV-lea Colocviu de Informatică INFO—Iași, 27—29 oct. 1983*

Gh. Coman, *Asupra complexității unor algoritmi numerici.*

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L. Țâmbulea, *Determinarea numărului de RC-mulțimi maximale.*

F. Boian, *Determinarea funcțiilor first-follow-1 și eff-1 folosind metode booleene.*

D. Chiorean, B. Pârv, I. Chiorean, *Cross-asamblor pentru sistemul Microargus.*

19. *Al II-lea Simpozion Național „Metode interdisciplinare ale fizicii", 14—15 octombrie 1983, Cluj-Napoca*

[illegible] Modele matematice în fizică; [illegible] structurală.

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20. Sesiunea de comunicări a Academiei R.S. România, Secția de Științe Matematice, București, [illegible] 1983

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21. Seminarul Național de Mecanica Fluidelor și aplicațiile ei tehnice, Pitești, [illegible] 1983.

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I. Pop, *Convecția liberă la 4°C pe o placă verticală.*

T. Petrilă, *Un nou procedeu de abordare a ecuațiilor fluidelor compresibile ideale.*

22. *Zilele academice clujene*, Cluj-Napoca, 21—25 noiembrie 1983

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