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LINEAR OPERATORS THAT TRANSFORM A NORMAL CONE IN COMPLETELY REGULAR CONES

A. B. NÉMETH

The Fredholm resolvents of a wide class of operators, which are sublinear with respect to the ordering induced by the wedge W in the normed space Y, have the property that transform W into completely regular cones [6]. These resolvents approximate indefinitely the identity map in the topology of uniform convergence on norm bounded sets. This advantage is associated with the drawback that composing them with convex mappings with values in Y, the resulting operators fail to be convex with respect to the ordering induced by the transformed cone.

The linear operator A on Y has the property that composed with any W-convex operator yields a mapping which is $A(W)$-convex. The complete regularity of $A(W)$ remains of a crucial interest for applications. But it appears that when W isn't regular, the linear operator with this property cannot approximate indefinitely the identity map (Corollary 3). However, some important operators (see the example in 12) have good properties from this point of view. Hence we devote the present note to investigation of the linear operators A with the cone range $A(W)$ being a completely regular cone.

If the linear and positive operator A maps the closed normal cone C with nonempty interior, contained in the Banach space (B-space), Y, into a completely regular cone, then any abstract Hammerstein operator AF, where F is C-convex and continuous, is subdifferentiable at any interior point of the domain of F (see Proposition 19).

1. Operators with completely regular cone ranges. Let Y be a normed space over the reals and let C be a *cone* in Y, i.e., a subset having the properties $C + C \subset C$, $tC \subset C$ for any positive real number, t, and $C \cap (-C) = \{0\}$. The cone C *induces* a reflexive, transitive and antisymmetric order relation $\leqslant$ on Y if we put $u \leqslant v$ whenever $u - u \in C$. This order relation relates to the linear structure of Y by the properties: $u \leqslant v$ implies $u + w \leqslant v + w$ for any w in Y and $tu \leqslant tv$ for any positive real number t. Since in the sequel we have to do with different cones, we shall call C-ordering the ordering induced by C. Similarly, we shall use terms as C-order bound, C-monotone etc.

The cone C is said to be *normal* if there exists a positive number b such that $\|u\| \leqslant b\|v\|$, whenever $0 \leqslant u \leqslant v$.

The cone C is called *completely regular (regular)* if any C-monotone norm bounded (C-order bounded) sequence in Y is fundamental. Any regular cone which is complete is normal, and any completely regular cone is normal and hence regular ([3], theorems 1.6 and 1.7).

If A is a linear operator on Y, then the *cone range* $A(C)$ of A is obviously a cone. If $A(C) \subset C$, then A is called *positive*. For a wide class of cones the positivity of a linear operator implies its continuity. We shall in the present note ignore this aspect and shall explicitly require in all what follows the continuity of the considered linear and positive operators.

The linear operator A is said to be *of completely regular (regular) type*, if its cone range $A(C)$ is a completely regular (regular) cone. Since any completely regular cone is also regular, any linear operator of completely regular type is also of regular type.

We shall frequently use in the sequel Lemma 4 in [6] and hence we shall state it here in a slightly modified from.

1. LEMMA. *The cone C in Y is completely regular (regular) if it contains no sequence (y_i) having the property that $\|y_i\| \geqslant d$ for any i and some positive d and for which the set $\left\{\sum_{i=1}^{n} y_i : n \in \mathbf{N}\right\}$ is norm bounded (C-order bounded).*

2. PROPOSITION. *If C isn't a regular cone of the normed space Y, then no regular type linear and continuous operator (and hence no completely regular type linear and continuous operator) can have continuous left hand side inverse.*

Proof. The linear operator A has continuous left hand side inverse if and only if there exists a positive b such that

$$\|Ay\| \geqslant b\|y\| \tag{1}$$

for any y in Y (see e.g. V. 4.4. in [4]).

If C isn't regular, it contains by Lemma 1 a sequence (y_i) with the property that $\|y_i\| \geqslant d$ for any i and some positive d, for which the set $\left\{\sum_{i=1}^{n} y_i : n \in N\right\}$ is order bounded. Let y be a C-upper bound for this set. Then Ay will be an $A(C)$-upper bound for the set $\left\{\sum_{i=1}^{n} Ay_i : n \in \mathbf{N}\right\}$. According (1) and the property of (y_i) it holds

$$\|Ay_i\| \geqslant b\|y_i\| \geqslant bd > 0$$

for any i. Applying once again Lemma 1 we conclude that the cone $A(C)$ cannot be regular. Q.E.D.

We shall denote with $\mathfrak{L}(Y)$ the vector space of all linear and bounded operators acting in Y, endowed with the norm topology.

3. CAROLLARY. *Let C be a cone in the B-space Y that isn't regular. Then the open unit sphere in $\mathfrak{L}(Y)$ with the centre at the identity map I ean contain no operator of regular (and henece no operator of completely regular) type.*

Proof. Any operator in the above open sphere has continuous inverse by a theorem of Banach (see e.g. V. 4.5 in [2]). Q.E.D.

4. *Remark.* In [6] it was shown that the identity map can be indefinitely approximated in the topology of the uniform convergence on norm bounded sets by the Fredholm resolvents of some sublinear operators. These resolvents transform the cone C whose closure isn't a subspace in some completely regular subcones of its. Restricted to the linear and continuous operators the considered topology is quite the norm topology. Intuitively the above cited result

means that a cone contains subcones „arbitrarily close to it" which are completely regular. Although, by Corollary 3, the transformation of a cone that isn't regular into a such subcone cannot be realised by a linear operator.

5. Proposition. *The property of a cone in a normed space to be completely regular is preserved by any linear and bounded operator with continuous left hand side inverse.*

Proof. Let C be a completely regular cone and assume that $A(C)$ isn't completely regular for some linear and bounded A with continuous left hand side inverse. We have for any y the relation (1) for some positive b. Invoking Lemma 1, there exists a sequence (y_i) in C with $\|Ay_i\| \geqslant d$ for some positive d and any i, such that the set $\left\{\sum_{i=1}^{n} Ay_i : n \in \mathbf{N}\right\}$ is norm bounded. We have for any i the relation $\|y_i\| \geqslant d/\|A\| > 0$, while from (1), $\left\|\sum_{i=1}^{n} Ay_i\right\| \geqslant b\left\|\sum_{i=1}^{n} y_i\right\|$. Accordingly the set $\left\{\sum_{i=1}^{n} y_i : n \in \mathbf{N}\right\}$ is norm bounded and we have get via Lemma 1 a contradiction with the hypothesis that C is a completely regular cone. Q.E.D.

6. *Remark.* Obviously, any linear operator of finite range preserves the complete regularity of a cone. The operators constructed in 12 and 14 furnish other examples having this property. However, there exist linear and compact operators that transform some completely regular cones onto cones without this property (see the example in 17).

Let $\mathfrak{B}$ and $\mathfrak{C}$ be subsets in $\mathfrak{L}(Y)$. We shall say that $\mathfrak{C}$ *is modular over* $\mathfrak{B}$ if for any $n \in \mathbf{N}$, any B_i in $\mathfrak{B}$ and any C_i in $\mathfrak{C}$, the operator $\sum_{i=1}^{n} B_i C_i$ is in C. If $\mathfrak{B}$ contains the identity map, then it suffices to restrict n in the above definition to be $\geqslant 2$. It is straightforward to show that if $\mathfrak{B}$ contains all the positive multiples of the identity map (respectively, its multiples with scalars in $[0, 1]$), then if $\mathfrak{C}$ is modular over $\mathfrak{B}$, it is a convex cone (respectively, a convex set). If $\mathfrak{C}$ is modular over itself we shall say that it is *automodular*.

We shall need in our next reasonings the

7. Lemma. *The sum of a finite number of completely regular subcones of a normal cone is a completely regular cone.*

Proof. Let C_1 and C_2 be completely regular subcones of the normal cone C and assume that $C_1 + C_2$ isn't completely regular. By Lemma 1 there exist the sequence (y_i^j) in C_j, $j = 1, 2$ and a positive number d so to have $\|y_i^1 + y_i^2\| \geqslant d$ for any i, while the set

$$\left\{\sum_{i=1}^{n} (y_i^1 + y_i^2) : n \in \mathbf{N}\right\} \tag{2}$$

is norm bounded. Passing to a subsequence we can assume without loss of the generality that

$$\|y_{i_k}^1\| \geqslant d/2, \quad k \in \mathbf{N}. \tag{3}$$

On the other hand if $\leqslant$ denotes the C-ordering in Y, we have

$$0 \leqslant \sum_{k=1}^{m} y_{i_k} \leqslant \sum_{k=1}^{m} (y_{i_k}^1 + y_{i_k}^2) \leqslant \sum_{i=1}^{i_m} (y_i^1 + y_i^2),$$

wherefrom, using the normality of the cone C and the norm boundendess of (2) it results that the set

$$\left\{\sum_{k=1}^{m} y_{i_k}^1 : k \in \mathbf{N}\right\}$$

is norm bounded. But this, together with (3) contradicts the complete regularity of C_1. Q.E.D.

8. PROPOSITION. *Let C be a normal cone in the normed space Y. Let B denote the subset of C-positive operators in $\mathfrak{L}(Y)$ that transform any completely regular subcone in C in completely regular cone. Then B is an automodular convex cone in $\mathfrak{L}(Y)$.*

Proof. We have obviously $B_1B_2 \in \mathfrak{B}$ whenever B_1 and B_2 are in $\mathfrak{B}$. Further, by the inclusion

$$(B_1 + B_2)(C) \subset B_1(C) + B_2(C),$$

it follows that the cone in the left is completely regular being the subcone of the cone in the right, which is completely regular by Lemma 7. That is, $B_1 + B_2$ is in $\mathfrak{B}$. From Proposition 5 we have that I and any positive multiple of its are in $\mathfrak{B}$ and hence we are done. Q.E.D.

9. PROPOSITION. *Let C be a normal cone in the normed space Y. Let $\mathfrak{C}$ denote the set in $\mathfrak{L}(Y)$ of the operators that transform C in completely regular subcones of its. Then $\mathfrak{C}$ is an automodular convex cone in $\mathfrak{L}(Y)$, which is modular over $\mathfrak{B}$, where $\mathfrak{B}$ is the set in $\mathfrak{L}(Y)$ of C-positive operators transforming the completely regular subcones of C in completely regular ones.*

Proof. Since $\mathfrak{C}$ is contained in $\mathfrak{B}$ it suffices to prove that it is modular over $\mathfrak{B}$. For any B' in $\mathfrak{B}$ and any C' in $\mathfrak{C}$ the composed operator $B'C'$ is obviously in $\mathfrak{C}$. Because I is contained in $\mathfrak{B}$ we have only to prove that $B_1C_1 + B_2C_2$ is in $\mathfrak{C}$ whenever B_1 and B_2 are in $\mathfrak{B}$ and C_1 and C_2 are in $\mathfrak{C}$. But this follows directly from Lemma 7. Q.E.D.

10. *Remark.* From 14 it follows that the cone $\mathfrak{C}$ in general is not closed in the norm topology of the space $\mathfrak{L}(Y)$.

Let A and B be in $\mathfrak{L}(Y)$. We shall put $A \leqslant B$ if $B - A$ is a C-positive operator.

11. PROPOSITION. *Let C be a normal cone in Y. Let $\mathfrak{C}$ denote the set of C-positive operators in $\mathfrak{L}(Y)$ which transform C in completel)y regular cones. If for some A and B in $\mathfrak{L}(Y)$ there exist the positive scalars α and β such that*

$$\alpha B \leqslant A \leqslant \beta B, \tag{4}$$

then A is in $\mathfrak{C}$ if and only if B is in $\mathfrak{C}$.

Proof. The relation (4) defines in fact an equivalence relation. Hence it is sufficient to show that $B \in \mathfrak{C}$ implies $A \in \mathfrak{C}$. According the normality of C, for any sequence (z_i) in C the norms $\|Bz_i\|$ and $\|Az_i\|$, $i \in \mathbf{N}$ are in the same time bounded from above and respectively, lower bounded by a positive number. Using now Lemma 1 in the way we have done it in the preceeding proofs, we get the required implication. Q.E.D.

2. Examples. The operators of finite range transform a closed cone in completely regular ones. The image by a compact operator of a closed cone is a compactly generated cone and hence the compact operators can be suspected to improve esentially the properties of a cone. Are they of completely regular or of regular type? Unfurtunately they don't. The aim of this paragraph is to show that the property of an operator to be of completely regular as well as of regular type is far to be characterizable with a property like compactness. There are linear and continuous operators of rather general form which are of completely regular type, while some compact operators don't have this property. In the same time we complete the results in the preceeding paragraph.

Let $C[0, 1]$ denote the space of continuous real valued functions defined on $[0, 1]$ endowed with the uniform norm and ordered by the cone C of non-negative functions. This cone is closed and normal.

12. *The linear operators in $C[0, 1]$ with the representing kernels bounded rom above and from below by positive multiples of a measure function, which represents a linear and positive functional, are of completely regular type.*

Let A be a linear and positive operator in $C[0, 1]$ and assume that the representing kernel K of its (see e.g. VI. 9.46 in [1]) satisfies the following conditions:

(i) There exist a normalized function g of bounded variation on $[0, 1]$ and a positive real β, such that

$$0 \leqslant K(s, dt) \leqslant \beta g(dt)$$

for any s and t in $[0, 1]$;

(ii) There exist an s_0 in $[0, 1]$ and a positive scalar α such that

$$\alpha g(dt) \leqslant K(s_0, dt)$$

for any t in $[0, 1]$.

For any y in the cone C it holds by (i)

$$\|Ay\| = \sup_s \int_0^1 y(t)\, K(s, dt) \leqslant \beta \int_0^1 y(t)\, g(dt).$$

If $\|Ay\| \geqslant d$, then this relation together with (ii) yields

$$\alpha d/\beta \leqslant \int_0^1 y(t) K(s_0, dt) = (Ay)(s_0). \tag{5}$$

If (z_i) is a sequence in $A(C)$ with the property that $\|z_i\| \geqslant d$ for some positive d and any i, then we have by (5) the inequality

$$z_i(s_0) \geqslant \alpha d/\beta$$

and hence

$$\left\|\sum_{i=1}^{n} z_i\right\| \geqslant n\alpha d/\beta,$$

that is, the set $\left\{\sum_{i=1}^{n} z_i : n \in \mathbf{N}\right\}$ cannot be norm bounded. Thus by Lemma 1, $A(C)$ is a completely regular cone.

We have in particular, that *if the representing kernel K of the positive and compact operator A satisfies the condition*

$$K(s_0, t) \geqslant \alpha > 0$$

for any t in $[0, 1]$ *and some s_0 in this interval, then A is of completely regular type.*

Indeed, we have then

$$K(s_0, t)dt \geqslant \alpha dt,$$

and

$$K(s, t)dt \leqslant \beta dt$$

for any s and t since K is continuous and hence bounded.

13. *Example of a positive integral operator with continuous kernel acting in* $C[0, 1]$ *that isn't of completely regular type.*

Consider the increasing sequence (a_n) of distinct real numbers in $(0, 1/2)$. Let we construct the functions k_n by putting for any $n \in \mathbf{N}$

$$k_n(s, t) = \max\{0, (a_n - a_{n+1})^2 - (t - a_n - a_{n+1})^2 - (s - a_n - a_{n+1})^2\},$$
$$(s, t) \in [0, 1] \times [0, 1]$$

They have the properties

(i) k_n vanishes outside the square $[2a_n, 2a_{n+1}] \times [2a_n, 2a_{n+1}]$;

(ii) $0 \leqslant k_n(s, t) \leqslant \max\{0, (a_n - a_{n+1})^2 - (t - a_n - a_{n+1})^2\} = k_n(a_n + a_{n+1}, t)$;

(iii) $\max k_n(s, t) = (a_n - a_{n+1})^2$.

According (ii) and (iii) the function

$$K(s, t) = \sum_{n=1}^{\infty} k_n(s, t)$$

is non-negative and continuous on $[0, 1] \times [0, 1]$.

We shall show that the integral operator A defined by the relation

$$(Ay)(s) = \int_0^1 K(s, t)y(t)\, dt$$

isn't of completely regulat type with respect to the cone C of the non-negative functions in $C[0, 1]$. To this end, we consider the sequence (y_n) in C defined by

$$y_n(t) = \max\{0, c_n((a_n - a_{n+1})^2 - (t - a_n - a_{n+1})^2)\}, \quad n \in \mathbf{N},$$

where

$$c_n = \left(\int_{2a_n}^{2a_{n+1}} ((a_n - a_{n+1})^2 - (t - a_n - a_{n+1})^2)^2 \, dt \right)^{-1}. \tag{6}$$

Then we have the properties

(a) The function $z_n(s) = \int_0^1 k_n(s, t)\, y_n(t) \, dt$ vanish outside the interval $[2a_n, 2a_{n+1}]$;

(b) $\|z_n\| = 1$;

(c) $\int_0^1 k_m(s, t)\, y_n(t)\, dt = 0$ for any s, whenever $m \neq n$.

From (a), (c) and the definitions it follows that

$$z_n(s) = (Ay_n)(s).$$

The properties (a) and (b) imply

$$\left\| \sum_{i=1}^{n} z_i \right\| = 1$$

for any n in $\mathbf{N}$, wherefrom via Lemma 1 we conclude that $A(C)$ isn't a completely regular cone.

14. *The integral operator A constructed in 13 can be indefinitely approximated in the norm topology by positive integral operators of completely regular type.*

We refer for the notations to the preceeding point. Consider the function

$$K_m(s, t) = \sum_{n=1}^{m} k_n(s, t)$$

and let the operator A_m be defined by the relation

$$(A_m y)(s) = \int_0^1 K_m(s, t)\, y(t)\, dt.$$

From the properties (i) and (iii) of k_n we have for any y in C

$$0 \leqslant (Ay)(s) - (A_m y)(s) = \int_0^1 (K(s, t) - K_m(s, t)) y(t)\, dt =$$

$$= \int_0^1 \left(\sum_{n=m+1}^{\infty} k_n(s, t) \right) y(t) dt \leqslant \max_{n \geqslant m+1} (a_n - a_{n+1})^2 \int_0^1 y(t) dt \leqslant \|y\| \max (a_n - a_{n+1})^2.$$

and hence

$$\|A - A_m\| \leqslant \max_{n \geqslant m+1} (a_n - a_{n+1})^2$$

wherefrom A_m converges in the norm to A when $m \in \infty$.

We have to check that A_m is for any m of completely regular type. We observe first that for any y in C the function $A_m y$ attains its local maxima at the points $a_1 + a_2, \ldots, a_m + a_{m+1}$. Indeed, suppose that s is in the interval $[2a_j, 2a_{j+1}]$ $(j = 1, \ldots, m)$. Then by the property (i) of k_j,

$$(A_m y)(s) = \int_0^1 K_m(s, t) y(t) dt = \int_0^1 k_j(s, t) y(t) \, dt, \quad (s \in [2a_j, 2a_{j+1}]).$$

Now, by the property (ii) of k_j,

$$\int_0^1 k_j(s, t) y(t) \, dt \leqslant \int_0^1 k_j(a_j + a_{j+1}, t) y(t) \, dt = (A \; y) (a_j + a_{j+1}),$$

that is, for s in $[2a_j, 2a_{j+1}]$,

$$(A_m y)(s) \leqslant (A_m y)(a_j + a_{j+1}). \tag{7}$$

Consider now an arbitrary sequence (z_i) in $A_m(C)$ with the property that $\|z_i\| \geqslant d$ for some positive d and for any i. We have

$$z_i = A_m y_i, \; i \in \mathbf{N},$$

for some y_i in C. According to the property (i) of k_j it follows that $z_i(s) = 0$ for s in $[0, 1] \setminus [2a_1, 2a_{m+1}]$. By the relation (7) we have that the maximum of z must be attained on some point $a_j + a_{j+1}, \; j = 1, \ldots, m$. That is, since $\|z_i\| \geqslant d$, there exists at least a $j (1 \leqslant j \leqslant m)$ so to have

$$z_i(a_j + a_{j+1}) \geqslant d.$$

Because j can have a finite number of values, it follows that there exists an index h $(1 \leqslant h \leqslant m)$ and a subserquence (uz_{i_l}) of (z_i) such that

$$z_{i_l}(a_h + a_{h+1}) \geqslant d$$

for any l in $\mathbf{N}$. This means that

$$\sum_{l=1}^{r} z_{i_l}(a_h + a_{h+1}) \geqslant rd,$$

and hence the set

$$\left\{ \sum_{j=1}^{n} z_j : n \in \mathbf{N} \right\}$$

cannot be norm bounded. From Lemma 1 we have then that $A_{\blacksquare}(C)$ is a completely regular cone.

15. *The operator A constructed in* 13 *is of regular type.* We have to show in accordance with Lemma 1, that if (z_i) is a sequence in $A(C)$ with the property that there exists a positive d such that $\|z_i\| \geqslant d$ for any i, then the set

$$\left\{\sum_{i=1}^{n} z_i : n \in \mathbf{N}\right\} \tag{8}$$

cannot be $A(C)$-order bounded (by any element in $A(C)$).

Let a be the limit of the sequence (a_i). Then $K(2a, t) = 0$ for any t in $[0, 1]$. Hence $z(2a) = 0$ for any z in $A(C)$. Assume that z is an element in $A(C)$ which is a C-order bound for the set (8). This means that

$$\sum_{i=1}^{n} z_i(s) \leqslant z(s), \qquad s \in [0, 1],\ n \in \mathbf{N}.$$

Since z is continuous, $z(a_j + a_{j+1}) \to z(2a) = 0$. Assume that $h \in \mathbf{N}$ has the property that $z(a_j + a_{j+1}) < d/2$ for any $j \geqslant h$. Since $z_i(s) \leqslant z(s)$ for any i and since $\|z_i\| \geqslant d$, it follows that the maximum of any element z_i must be attained at a point $s < a_h + a_{h+1}$. According the reasonings in the point 14, an s with this property must be one of the points $a_j + a_{j+1}$ for $j \leqslant h$. Hence we get a contradiction as in the above point with the norm boundendess of the set (8) which follows from the C-order boundendess of it. Now, if the set (8) would be $A(C)$-order bounded by some element in $A(C)$, then it would be also C-order bounded by the same element. But this contradicts, as we have seen above, the hypothesis that $\|z_i\| \geqslant d$ for any i. Thus $A(C)$ must be regular.

16. *Example of a positive integral operator in* $C[0, 1]$ *with continuous kernel, which isn't of regular type.*

We shall use the constructions in the example 13, restricting the terms fo the sequence (a_i) to satisfy $1/4 < a_i < 1/2$, $i \in \mathbf{N}$. Let be $a_0 = 1/4$ and put

$$k_0(s, t) = \max\{0, (a_0 - a_1)^2 - (t - a_0 - a_1)^2\}$$

Consider also the function

$$y_0(t) = \max\{0, c_0((a_0 - a_1)^2 - (t - a_0 - a_1)^2)\},$$

where c_0 is given by (6) with $n = 0$.

The function

$$K^1(s, t) = \sum_{n=0}^{\infty} k_n(s, t)$$

is continuous and non-negative and have the property that

$$\int_0^1 K^1(s, t)\, y_0(t)\, dt = 1$$

for any s in $[0, 1]$.

The elements

$$z_n(s) = \int_0^1 K^1(s, t)\, y_n(t)\, dt, \quad n \in \mathbf{N}$$

are of the norm 1 and have the property that $\left\|\sum_{n=1}^{m} z_n\right\| = 1$ for any m in $\mathbf{N}$. Let us denote by $e(s)$ the function identically 1 on $[0, 1]$, and let consider the difference

$$u_m(s) = e(s) - \sum_{n=1}^{m} z_n(s), \quad m \in \mathbf{N}.$$

This is for any m a non-negative function of norm $\leqslant 1$.

Consider the sequence (b_i), where $b_i = a_i - 1/4$, $i \in \mathbf{N}$, and put

$$h_n(s, t) = u_n(s) \max \{b_n - b_{n+1})^2 - (t - b_n - b_{n+1})^2\}, \quad n \in \mathbf{N}.$$

h_n is a non-negative continuous function vanishing outside the strip $[0, 1] \times \times [2b_n, 2b_{n+1}]$, satisfying the inequality $h_n(s, t) \leqslant (b_n - b_{n+1})^2$. Hence

$$K^2(s, t) = \sum_{n=1}^{\infty} h_n(s, t)$$

is a continuous non-negative function. Let

$$v_n(t) = \max \{0, c_n((b_n - b_{n+1})^2 - (t - b_n - b_{n+1})^2)\}$$

with c_n given by (6). We shall show that the compact operator A defined by

$$(Ay)(s) = \int_0^1 K(s, t)\, y(t)\, dt,$$

where $K = K^1 + K^2$ isn't of regular type.

We observe first that e and the sequence (z_n) are in $A(C)$. Further, we have

$$\int_0^1 K(s, t)\, v_n(t)\, dt = \int_{2b_n}^{2b_{n+1}} h_n(s, t)\, v_n(t)\, dt =$$

$$= c_n u_n(s) \int_{2b_n}^{2b_{n+1}} ((b_n - b_{n+1})^2 - (t - b_n - b_{n+1})^2)^2\, dt = u_n(s)$$

by the definition of the sequence (b_n) and of the numbers c_n, $n \in \mathbf{N}$.

The obtained relation shows that u_n is in $A(C)$ and that the set

$$\left\{\sum_{n=1}^{m} z_n : m \in \mathbf{N}\right\}$$

is $A(C)$-order bounded by the element e of $A(C)$. But $\|z_n\| = 1$ for any n in $\mathbf{N}$, and invoking Lemma 1 again we conclude that A isn't of regular type.

17. *Example of a linear, positive and compact operator in c that transforms a completely regular cone in a cone that isn't completely regular.*

Denote by c the space of convergent sequences of real numbers, endowed with the usual norm. Let C be the cone of the sequences in c with non-negative terms. The subcome C_1 in C of the nondecreasing sequences is completely regular. Indeed, if y is in C_1, $y = (y^i)$, $y^i \in \mathbf{R}$, then $\|y\| = \lim y^i$. Accordingly, for y_1 and y_2 in C_1 we have $\|y_1 + y_2\| = \|y_1\| + \|y_2\|$ and hence there cannot exist any sequence (y_i) of elements in C_1 such that $\|y_i\| \geq d$ for some positive d and any i, for which $\left\{\sum_{i=1}^{n} y_i : n \in \mathbf{N}\right\}$ is a norm bounded set. That is, C_1 is completely regular by Lemma 1.

Let us consider the infinite matrix of real numbers denoted by A,

$$A = (a_{ij})_{i,j=1,2,\ldots}, \quad a_{ij} = 2^{-i}\,\delta_i^j$$

with δ_i^j standing for the Kronecker symbol. If we define Ay for some y in c as to be the multiplication of A by the (column) vector y, then A can be interpreted as a linear operator in c. It is straightforward to see that A is compact.

Define the sequence (y_i) of the elements in the completely regular cone C_1 by putting

$$y_n = (\underbrace{0, \ldots, 0}_{n-1 \text{ times}}, 2^n, 2^n, \ldots)$$

Then

$$Ay_n = z_n = (z_n^1, z_n^2, \ldots, z_n^m, \ldots),$$

where $z_n^m = 2^{-m} y_n^m$, that is,

$$z_n = (\underbrace{0, \ldots, 0}_{n-1 \text{ times}}, 2^0, 2^{-1}, 2^{-2}, \ldots,).$$

We have $\|Ay_n\| = \|z_n\| = 1$ and

$$\|Ay_1 + Ay_2 + \ldots + Ay_n\| < 2$$

for any n in $\mathbf{N}$. That is, $A(C_1)$ isn't completely regular by Lemma 1.

3. The subdifferentiability of some Hammerstein type operators. A totally ordered subset of the ordered vector space Y is said to be a *chain*. The space Y is said to be *chain complete* if any chain that is bounded from below (from above) has an infimum (a supremum) in Y. If Y is a regular space ordered by a closed cone, then the limit of any monotonically decreasing (increasing) sequence is also the infimum (supremum) of this sequence (see II.3.2 in [8]). Hence isn't difficult to show (see for example the reasoning in the proof of Proposition 2 in [7]), that a space with this property is chain complete. Thus for it we have the conditions used in [5] in order to prove the existence of the subgradients for convex mappings. Ths operator F from the vector space X to the ordered vector space $(Y, \leqslant)$ is said to be *convex* if

$$F(tx_1 + (1-t)x_2) \leqslant tF(x_1) + (1-t)F(x_2)$$

for any x_1 and x_2 in X and any t in $[0, 1]$. The linear operator A from X to Y is said to be a subgradient of F at x if

$$F(x+u) - F(x) \geqslant Au$$

for any u in X.

Suppose that X and Y are B-spaces and that Y is ordered by a closed, normal cone with nonempty interior. Then if F is a continuous convex operator from X to Y, then from the existence of a subgradient of F, it follows its continuity. There exist examples (see e.g. [4]) showing that even for rather nice convex operators there are points in the domain of them at which no subgradient exists. We shall use in this paragraph the results we have estabilished in order to give some sufficient conditions for the existence of subgradients. First of all we prove the following preparatory result:

18. Lemma. *The closure of any completely regular cone is completely regular too.*

Proof. Assume that C is completely regular and $\bar{C}$ isn't. Then there exist a $d > 0$ and a sequence (y_i) in $\bar{C}$ with the property that $\|y_i\| \geqslant d$ for any i, so to $\left\{\sum_{i=1}^{n} y_i : n \in \mathbf{N}\right\}$ be a norm bounded set (Lemma 1). Suppose that $\left\|\sum_{i=1}^{n} y_i\right\| \leqslant \alpha$ for any n. Let z_i be elements in C which satisfy the conditions $\|z_i\| \geqslant d/2$ and $\|z_i - y_i\| < 2^{-i}$ for any i. Then

$$\left| \left\|\sum_{i=1}^{n} y_i\right\| - \left\|\sum_{i=1}^{n} z_i\right\| \right| \leqslant \sum_{i=1}^{n} \|y_i - z_i\| < 1$$

and hence

$$\left\|\sum_{i=1}^{n} z_i\right\| \leqslant \left| \left\|\sum_{i=1}^{n} z_i\right\| - \left\|\sum_{i=1}^{n} y_i\right\| \right| + \left\|\sum_{i=1}^{n} y_i\right\| < 1 + \alpha$$

for any n. That is, the set $\left\{\sum_{i=1}^{n} z_i : n \in \mathbf{N}\right\}$ is norm bounded. Thus we have get a contradiction via Lemma 1 with the hypothesis that C is a completely regular cone. Q.E.D.

19. Proposition. *Let Y be a B-space ordered by a closed normal cone C with nonempty interior and let F be a continuous convex mapping from the B-space X to Y. If A is a positive operator in Y of completely regular type, then the abstract Hammerstein operator AF has continuous subgradients at any point of X.*

Proof. From Lemma 18, $\overline{A(C)}$ will be a closed completely regular cone. The operator AF will be convex with respect to the $\overline{A(C)}$-ordering in Y and hence it will have $\overline{A(C)}$-subgradients in any point of X. Since $\overline{A(C)} \subset C$, these subgradients will be C-subgradients too. Hence they will be continuous operators by our comments at the beginning of this paragraph. Q.E.D.

20. COROLLARY. *Let the space $C[0, 1]$ be ordered by the cone of non-negative functions and let F be a continuous convex operator acting in it. Consider the Hammerstein operator defined by*

$$G(x)(s) = \int_0^1 F(x(t))\, K(s, dt),$$

where the kernel K satisfies the conditions in 12. Then G has continuous subgradients in each point of $C[0, 1]$.

Proof. The positive cone in $C[0, 1]$ is closed, normal and has nonempty interior. The linear operator defined by

$$(Ay)(s) = \int_0^1 y(t) K(s, dt)$$

is of completely regular type by 12. Now, $G = AF$ and hence we are in the conditions of Proposition 19. Q.E.D.

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OPERATORI LINIARI CE TRANSFORMĂ UN CON NORMAL ÎN CONURI COMPLET REGULARE

(Rezumat)

În lucrare sînt studiaţi operatorii liniari şi continui care transformă un con normal în conuri complet regulare. Se dau condiţii suficiente pentru ca un operator liniar şi continuu din spaţiul funcţiilor continue definite pe un interval compact de pe axa reală, să aibă această proprietate. Se construiesc operatori liniari şi compacţi definiţi în acest spaţiu, care nu transformă conul funcţiilor pozitive într-un con regular.

STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXVIII, 1983

THE HARDY CLASS OF CERTAIN FUNCTIONS WITH INTEGRAL REPRESENTATIONS

OTTO FEKETE

1. **Introduction.** Let S denote the class of functions $f(z) = z + a_2z^2 + \dots$ which are regular and univalent in the unit disc U and let S^c, S^*, K and $B(\alpha, \beta, \mathfrak{L}, S^*)$ denote the usual subclasses of S consisting of functions which are, respectively, convex, starlike, close-to-convex and Bazilevič of type (α, β) in U. Let L be an integral operator. Several authors [1], [3], [4], [5], [6] have investigated the invariability of subclasses of univalent functions with respect to some operators L and have shown that $L(S^c) \subset S^c$,

$$L(S^*) \subset S^*,\ L(K) \subset K,\ L(B(\alpha, \beta, \mathfrak{L}, S^*)) \subset B(\alpha, \beta, \mathfrak{L}, S^*).$$

In this note we determine the Hardy class to which $L(f)$ belongs for some integral operators L and for $f \in S^c$, S^*, K or $B(\alpha, \beta, \mathfrak{L}, S^*)$.

2 **Preliminary results.** We shall need the following results.

LEMMA 1. *If f is in one of the classes S^c, S^* or K and F is defined by*

$$F(z) = L(f(z)) = \frac{2}{z}\int_0^z f(t)dt \tag{1}$$

then the function F is likewise in S^c, S^* or K.

LEMMA 2. *If f is in one of the classes S^c, S^* or K and F is defined by*

$$F(z) = L_c(f(z)) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt,\quad 0 \leqq c \leqq 1 \tag{2}$$

then the function F is likewise in S^c, S^*, or K.

LEMMA 3. *If f is in one of the classes S^* or $B(\alpha, 0, \mathfrak{L}, S^*)$ and F is defined by*

$$F(z) = L_c^\alpha(f(z)) = \left[\frac{\alpha+c}{z^c}\int_0^z t^{c-1}f^\alpha(t)dt\right]^{\frac{1}{\alpha}},\ \alpha > 0,\ c \in C,\ \mathrm{Re}\, c \geqq 0 \tag{3}$$

then the function F is likewise in S^* or $B(\alpha, 0, \mathfrak{L}, S^*)$.

LEMMA 4. *If $f' \in H^p$, $0 < p < 1$ then $f \in H^{p/(1-p)}$.*

LEMMA 5. *If $f \in S^*$ and $f(z) \neq k_\tau^0(z) = z/(1+e^{i\tau}z)^2$ then there exists $\varepsilon = \varepsilon(f) > 0$ such that $f \in H^{1/2+\varepsilon}$.*

LEMMA 6. *If $f(z) = \left(\alpha\int_0^z P(t)g^\alpha(t)t^{-1}dt\right)^{\frac{1}{\alpha}} \in B(\alpha, 0, \mathfrak{L}, S^*)$ with $g \neq k_\tau^0$ then there exists $\varepsilon = \varepsilon(f) > 0$ such that $f \in H^{1/2+\varepsilon}$.*

Lemma 1 is in [1], Lemma 2 is in [4], Theorem 4.1., Lemma 3 was proved in [6] for α and c positive integers and in [5] in the general case, Lemma 4 and 5 are well known and Lemma 6 is in [2].

3. Results for operators of the Libera type. Let $f(z) = z + a_2 z^2 + \dots$ be regular in U and $F(z) = \frac{2}{z}\int_0^z f(t)\, dt$. We have

$$F'(z) = -\frac{1}{z} F(z) + \frac{2}{z} f(z)$$

For $0 < p < 1$, $z = re^{i\theta}$, $0 < r < 1$ we define

$$I(r) = \int_0^{2\pi} |F'(re^{i\theta})|^p \, d\theta \tag{4}$$

We obtain

$$I(r) = \int_0^{2\pi} \left| -\frac{F(re^{i\theta})}{re^{i\theta}} + \frac{2f(re^{i\theta})}{re^{i\theta}} \right|^p d\theta \leq$$
$$\leq \frac{1}{r^p}\int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta + \left(\frac{2}{r}\right)^p \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \tag{5}$$

THEOREM 1. *If $f \in S^*$, $f \neq k_\tau^0$ and L is the operator defined by* (1) *then there exists $\varepsilon = \varepsilon(f) > 0$ such that $L(f) \in H^{1+\varepsilon}$.*

Proof. Since $f \in S^*$, Lemma 1 implies that $F = L(f) \in S^*$. By Lemma 5 and (5), $\lim\limits_{r\to 1} I(r)$ exists if $p < 1/2$, i.e. $F' \in H^p$ for all $p < 1/2$. By Lemma 4 we have $F \in H^p$ for all $p < 1$ and this implies that $F \neq k_\tau^0$. Since $f \neq k_\tau^0$, we have from (5), $\lim\limits_{r\to 1} I(r) < \infty$ if $p = 1/2 + \varepsilon$, $\varepsilon = \varepsilon(f) > 0$, i.e. $F' \in H^{1/2+\varepsilon}$. Applying again Lemma 4 we obtain our result.

The condition $f \neq k_\tau^0$ in the theorem is essential. This may be seen by taking $f(z) = z/(1+z)^2$. Then $Lf(z) = \frac{2}{z}\int_0^z \frac{t}{(1+t)^2}\, dt = \frac{2}{z}\left[\ln(1+z) + \frac{z}{1+z}\right]$. We show that $L(f) \notin H^1$. Let

$$J(r) = \int_0^{2\pi} |L(f(re^{i\theta}))|\, d\theta = \int_0^{2\pi} \left| \frac{2}{re^{i\theta}} \left[\ln(1+re^{i\theta}) + \frac{re^{i\theta}}{1+re^{i\theta}}\right]\right| d\theta \geq$$
$$\geq \frac{2}{r}\int_0^{2\pi} \left|\frac{re^{i\theta}}{1+re^{i\theta}}\right| d\theta - \frac{2}{r}\int_0^{2\pi} |\ln(1+re^{i\theta})|\, d\theta = \frac{2}{r} J_1(r) - \frac{2}{r} J_2(r)$$

Since $\lim\limits_{r\to 1} J_1(r) = \infty$, the proof will be complete if we establish that $\lim\limits_{r\to 1} J_2 < \infty$. We have

$$J_2(r) = \int_0^{2\pi} |\ln(1+re^{i\theta})|\, d\theta = \int_0^{2\pi} \left[(\ln|1+re^{i\theta}|)^2 + \left(\operatorname{arctg}\frac{r\sin\theta}{1+r\cos\theta}\right)^2\right]^{\frac{1}{2}} d\theta \leqq$$

$$\leqq \int_0^{2\pi} \ln|1+re^{i\theta}|\, d\theta + \int_0^{2\pi} \operatorname{arctg}\frac{r\sin\theta}{1+r\cos\theta}\, d\theta$$

A simple calculation shows that $\int_0^{2\pi} \operatorname{arctg}\frac{r\sin\theta}{1+r\cos\theta}\, d\theta = 0$. On integrating $\int_0^{2\pi} \ln|1+re^{i\theta}|\, d\theta$ as an integral depending on a parameter we obtain $J_2(r)=0$.

Remark 1. In fact we can prove the following more general form of Theorem 1.

THEOREM 1′. *If $f \in S^*$, $f \neq k_\tau^0$ and L_c is the operator defined by* (2) *then there exists $\varepsilon = \varepsilon(f) > 0$ such that $L_c(f) \in H^{1+\varepsilon}$.*

THEOREM 2. *If $f \in S^c$, $f \neq k_\tau^1$ $\left(k_\tau^1(z) = \frac{z}{1+e^{i\tau}z}\right)$ and L is the operator defined by* (1) *then $L(f) \in H^\infty$.*

Proof. Since $f \in S^c$, Lemma 1 implies that $F = L(f) \in S^c$ and for $f \in S^c$ it is known that $f \in H^p$, for $p < 1$. By (5) we have $\lim\limits_{r\to 1} I(r)$ exists if $p < 1$ and by Lemma 4 we have $F \in H^p$ for all $p < \infty$. Let now assume $p > 1$. The inequality (5) can be replaced by

$$I(r) = \int_0^{2\pi} \left| -\frac{F(re^{i\theta})}{re^{i\theta}} + \frac{2f(re^{i\theta})}{re^{i\theta}} \right|^p d\theta \leqq$$

$$\leqq \left\{\frac{1}{r}\left[\int_0^{2\pi} |F(re^{i\theta})|^p d\theta\right]^{\frac{1}{p}} + \frac{2}{r}\left[\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta\right]^{\frac{1}{p}}\right\}^p \tag{6}$$

Since $f \neq k_\tau^1$, $f \in S^c$, there exists $\varepsilon = \varepsilon(f) > 0$ such that $f \in H^{1+\varepsilon}$. Since $F' \in H^1$ implies $F \in H^\infty$, we conclude that $F \in H^\infty$.

The condition $f \neq k_\tau^1$ is essential because for $f(z) = z/(1-z)$ we have $L(f(z)) = -2 - \frac{2}{z}\ln(1-z) \notin H^\infty$.

Remark 2. In fact, we can prove the following more general form of Theorem 2.

THEOREM 2.′ *If* $f \in S^c$, $f \neq k_\tau^1$ *and* L_c *is the operator defined by* (2) *then* $L_c(f) \in H^\infty$.

The following theorem may be similarly proved.

THEOREM 3. *If* $f \in K$, $f(z) \neq \int_0^z \frac{1}{(1+e^{i\tau}t)^2} P(t)dt$, $P \in \mathfrak{L}$ and L *is the operator defined by* (1), *then there exists* $\varepsilon = \varepsilon(f) > 0$ *such that* $L(f) \in H^{1+\varepsilon}$.

Remark 3. In fact, we can prove the following more general form of Theorem 3.

THEOREM 3′. *If* $f \in K$, $f(z) \neq \int_0^z \frac{1}{(1+e^{i\tau}t)^2} P(t)\,dt$, $P \in \mathfrak{L}$ *and* L_c *is the operator defined by* (2), *then there exists* $\varepsilon = \varepsilon(f) > 0$ *such that* $L_c(f) \in H^{1+\varepsilon}$.

Remark 4. For Theorem 1′ and Theorem 3 we have not shown that the results are sharp.

4. Results for operators of type L_c^α. Let $f(z) = z + a_2z^2 + \ldots$ be regular in U and $F(z) = L_c^\alpha(f(z))$, L_c^α defined in (3). We have

$$\alpha z F^{\alpha-1}(z)F'(z) = (\alpha + c)f^\alpha(z) - cF^\alpha(z) \tag{7}$$

If we let $\Phi(z) = (F(z)/z)^\alpha$ then Φ is regular in U and satisfies

$$\Phi'(z) = \frac{\alpha z F'(z)F^{\alpha-1}(z) - \alpha F^\alpha(z)}{z^{\alpha+1}} \tag{8}$$

From (7) and (8) we obtain

$$\Phi'(z) = \frac{\alpha + c}{z^{\alpha+1}}(f^\alpha(z) - F^\alpha(z)) \tag{9}$$

For $0 < p < 1$, $z = re^{i\theta}$, $0 < r < 1$ we obtain

$$I(r) \equiv \int_0^{2\pi} |\Phi'(re^{i\theta})|^p d\theta = \int_0^{2\pi} \left| \frac{\alpha + c}{(re^{i\theta})^{\alpha+1}} (f^\alpha(re^{i\theta}) - F^\alpha(re^{i\theta})) \right|^p d\theta \leq$$

$$\leq \frac{|\alpha + c|^p}{r^{(\alpha+1)p}} \left(\int_0^{2\pi} |f(re^{i\theta})|^{\alpha p} d\theta + \int_0^{2\pi} |F(re^{i\theta})|^{\alpha p} d\theta \right) \tag{10}$$

THEOREM 4. *If* $f \in S^*$, $f \neq k_\tau^0$ *and* L_c^α *is the operator defined by* (3) *then*
(*i*) *if* $0 < \alpha \leq \frac{1}{2}$, $L_c^\alpha(f) \in H^\infty$.

(*ii*) *if* $\alpha > \frac{1}{2}$ *there exists* $\varepsilon = \varepsilon(f) > 0$ *such that* $L_c^\alpha(f) \in H^{\frac{\alpha}{2\alpha-1}+\varepsilon}$.

Proof. (i) Let $0 < \alpha \leqq \frac{1}{2}$, $f \in S^*$, $p < 1$. Since $\alpha p \leqq \frac{1}{2} p$, using Lemma 3 and (10) we see that $\lim_{r \to 1^-} I(r)$ exists if $p < 1$, i.e. $\Phi' \in H^p$ for all $p < 1$. By Lemma 4 we have $\Phi \in H^p$ for all $p < \infty$ and since $F(z) = z(\Phi(z))^{\frac{1}{\alpha}}$, we have

$$F \in H^p \text{ for all } p < \infty. \tag{11}$$

Let us assume now $f \in S^*$, $f \neq k_c^0$. By Lemma 5 there exists $\varepsilon = \varepsilon(f) > 0$ such that $f \in H^{\frac{1}{2}+\varepsilon}$. For $p > 1$ the inequality (10) becomes

$$I(r) \equiv \int_0^{2\pi} |\Phi'(re^{i\theta})|^p \, d\theta \leqq \frac{|\alpha + c|^p}{r^{(\alpha+1)p}} \left\{ \left[\int_0^{2\pi} |f(re^{i\theta})|^{p\alpha} \, d\theta \right]^{\frac{1}{p}} + \left[\int_0^{2\pi} |F(re^{i\theta})|^{p\alpha} \right]^{\frac{1}{p}} \right\}^p \tag{12}$$

Since $\alpha \leqq \frac{1}{2}$, $\alpha p < \frac{1}{2} + \varepsilon$ for all $p < 1 + \varepsilon$. By (11), (12) and Lemma 5 we therefore conclude that $\lim_{r \to 1^-} I(r) < \infty$ for $p = 1 + \varepsilon$, i.e. $\Phi' \in H^{1+\varepsilon}$. This implies $\Phi \in H^\infty$ and since $F(z) = z(\Phi(z))^{\frac{1}{\alpha}}$, $F \in H^\infty$.

(ii) If $\alpha > \frac{1}{2}$, $f \in S^*$, Lemma 3 implies that $F = L_c^\alpha(f) \in S^*$. By (10) and Lemma 5, $\lim_{r \to 1^-} I(r)$ exists if $\alpha p < \frac{1}{2}$, i.e. $\Phi' \in H^p$ for all $p < \frac{1}{2\alpha}$. By Lemma 4 we have $\Phi \in H^p$ for all $p < \frac{1}{2\alpha - 1}$. Since $F(z) = z(\Phi(z))^{\frac{1}{\alpha}}$, we have

$$\int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta = r^p \int_0^{2\pi} |\Phi(re^{i\theta})|^{\frac{p}{\alpha}} \, d\theta \tag{13}$$

and consequently, this last integral will be bounded provided that $\frac{p}{\alpha} < \frac{1}{2\alpha - 1}$, i.e. $F \in H^p$ for all $p < \frac{\alpha}{2\alpha - 1}$. For $\varepsilon_1 < \frac{\alpha}{2\alpha - 1} - \frac{1}{2}$, $F \in H^{\frac{1}{2}+\varepsilon_1}$ and by Lemma 5 there exists $\varepsilon_2 = \varepsilon_2(f) > 0$ such that $f \in H^{\frac{1}{2}+\varepsilon_2}$. Taking $\varepsilon' = \min\{\varepsilon_1, \varepsilon_2\}$, we have by applying (10), $\lim_{r \to 1^-} I(r) < \infty$ for $\alpha p = \frac{1}{2} + \varepsilon'$ and hence by Lemma 4 and (13) $F \in H^{\frac{\alpha}{2\alpha - 1}+\varepsilon}$.

Remark 5. Setting $\alpha = 1$ and $c = 1$ we obtain Theorem 1. Setting $\alpha = 1$ we obtain Theorem 1'.

Remark 6. A simple calculation shows that for $f(z) = z/(1-z)^2$ and $\alpha \in \mathbf{N}$, $L_c^\alpha(f) \notin H^{\frac{\alpha}{2\alpha-1}}$. In the general case we have not shown that the result is sharp.

The following theorem may be similarly proved.

THEOREM 5. *If* $f(z) = \left(\alpha \int_0^z P(t) g^\alpha(t) t^{-1} dt\right)^{\frac{1}{\alpha}} \in B(\alpha, 0, \mathfrak{L}, S^*)$, $g \neq k_\tau^0$ and L_c^α *is the operator defined by* (3) *then*

(i) *if* $0 < \alpha \leq \frac{1}{2}$, $L_c^\alpha(f) \in H^\infty$

(ii) *if* $\alpha > \frac{1}{2}$ *there exists* $\varepsilon = \varepsilon(f) > 0$ *such that* $L_c^\alpha(f) \in H^{\frac{\alpha}{2\alpha-1}+\varepsilon}$.

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CLASA HARDY A UNOR FUNCȚII CU REPREZENTĂRI INTEGRALE

(Rezumat)

Pornind de la invarianța unor clase de funcții univalente la operatori integrali L de forma (1), (2) sau (3), se determină clasa Hardy pentru $L(f)$ în cazul în care $f \in S^c$, S^*, K, sau $B(\alpha, 0, \mathfrak{L}, S^*)$.

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FORMULE OPTIMALE DE CUADRATURĂ DE TIP INTERVAL

PETRU P. BLAGA

Formulele de cuadratură sînt folosite pentru calculul valorii aproximative a unei integrale definite, cînd se cunosc valori ale funcţiei de integrat pe anumite puncte şi ale derivatelor acesteia. În practică, deseori, sînt cunoscute valori medii ale funcţiei de integrat, pe anumite subintervale ale intervalului de integrare, pe care le numim intervale de evaluare.

Bazat pe această observaţie, în lucrările [2, 3, 4] sînt studiate astfel de formule de cuadratură numite de tip interval.

În cele ce urmează sînt considerate formule de cuadratură de tip interval, care sînt alese dintr-o clasă de astfel de formule. Această clasă de formule de cuadratură de tip interval se obţine considerînd coeficienţii formulei şi distanţele dintre două intervale de evaluare ca parametri. Determinarea acestor parametri se face în aşa fel încît formula să aibă gradul de exactitate unu, iar restul formulei să fie minim, într-un sens precizat mai jos.

Pentru precizarea celor spuse, fie funcţia de integrat din spaţiul de funcţii

$$H_1[0, 1] = \{f \in C[0, 1],\ f' \text{ segmentar continuă}\}$$

şi intervalul de integrare [0, 1] cu intervalele de evaluare $[x_k, y_k]$, $k = \overline{0, m-1}$, definite prin şirul de inegalităţi:

$$0 = y_{-1} \leqslant x_0 < y_0 < x_1 < y_1 < \ldots < x_{m-1} < y_{m-1} \leqslant x_m = 1.$$

Se notează lungimile intervalelor de evaluare şi respectiv distanţele dintre aceste intervale de evaluare prin:

$$l_k = y_k - x_k,\ k = \overline{0, m-1}; \qquad d_k = x_k - y_{k-1},\ k = \overline{0, m}.$$

Se consideră formula de caudratură de tip interval

$$\int_0^1 f(x)dx = \sum_{k=0}^{m-1} \frac{A_k}{l_k} \int_{x_k}^{y_k} f(x)\,dx + R[f], \tag{1}$$

cu proprietatea $R[1] = 0$.

Restul are următoarea reprezentare integrală

$$R[f] = \int_0^1 K(t) f'(t)\,dt,$$

unde

$$K(t) = R[(x-t)_+^0] = (1-t) - \sum_{k=0}^{m-1} \frac{A_k}{l_k} [(y_k - t)_+ + (x_k - t)_+].$$

Din reprezentarea integrală a restului se obține:

$$|R[f]| \leq \sup_{t\in[0,1]} |f'(t)| \int_0^1 |K(t)|dt.$$

Vom spune că formula de cuadratură de tip interval (1) cu $A_0^*, A_1^*, \ldots, A_{m-1}^* \in \mathbf{R}$, respectiv $d_0^*, d_1^*, \ldots, d_m^* \in \mathbf{R}_+$ este o formulă optimală de cuadratură de tip interval, dacă

$$I^* = I(A_0^*, \ldots, A_{m-1}^*; d_0^*, \ldots, d_m^*) = \min_{\substack{A_0,\ldots,A_{m-1}\\ d_0,\ldots,d_m}} I(A_0, \ldots, A_{m-1}; d_0, \ldots, d_m),$$

unde

$$I(A_0, \ldots, A_{m-1}; d_0, \ldots, d_m) = \int_0^1 |K(t)|dt.$$

Pentru determinarea formulei optimale trebuie rezolvate două probleme.

I. Prima dată se determină distanțele dintre intervalele de evaluare, $d_0^*, \ldots, d_m^*$, astfel încît

$$\bar{I}(A_0, \ldots, A_{m-1}) = I(A_0, \ldots, A_{m-1}; d_0^*, \ldots, d_m^*) = \min_{d_0,\ldots,d_m} I(A_0, \ldots, A_{m-1}; d_0, \ldots, d_m).$$

II. Se determină apoi ponderile $A_0^*, \ldots, A_{m-1}^*$ astfel ca:

$$I^* = \bar{I}(A_0^*, \ldots, A_{m-1}^*) = \min_{A_0,\ldots,A_{m-1}} \bar{I}(A_0, \ldots, A_{m-1}).$$

Rezolvarea problemei I. Ținînd seama de expresia nucleului K, se poate scrie

$$I(A_0, \ldots, A_{m-1}; d_0, \ldots, d_m) = \int_0^{x_0} |K(t)|dt +$$

$$+ \sum_{j=0}^{m-1} \int_{x_j}^{y_j} |K(t)|\, dt + \sum_{j=0}^{m-2} \int_{y_j}^{x_{j+1}} |K(t)|\, dt + \int_{y_{m-1}}^{1} |K(t)|\, dt,$$

unde

$$K(t) = \begin{cases} -t, \text{ dacă } t \in [0, x_0], \\ \dfrac{A_k - l_k}{l_k} t + \sum\limits_{j=0}^{k-1} A_j - \dfrac{A_k x_k}{l_k}, \text{ dacă } t \in [x_k, y_k],\ k = \overline{0, m-1}, \\ \sum\limits_{j=0}^{k} A_j - t, \text{ dacă } t \in [y_k, x_{k+1}],\ k = \overline{0, m-2}, \\ 1 - t, \text{ dacă } t \in [y_{m-1}, 1]. \end{cases}$$

Considerînd îndeplinită condiţia $A_k - l_k > 0$, $k = \overline{0, m-1}$, se obţine:

$$I(A_0, \ldots, A_{m-1}; d_0, \ldots, d_m) =$$

$$= \frac{x_0^2}{2} + \frac{1}{2}\sum_{j=0}^{m-1}\frac{l_j}{A_j - l_j}\left[\left(x_j - \sum_{i=0}^{j-1}A_i\right)^2 + \left(\sum_{i=0}^{j}A_i - y_j\right)^2\right] +$$

$$+ \frac{1}{2}\sum_{j=0}^{m-2}\left[\left(\sum_{i=0}^{j}A_i - y_j\right)^2 + \left(x_{j+1} - \sum_{i=0}^{j}A_i\right)^2\right] + \frac{(1 - y_{m-1})^2}{2}$$

Ţinînd seama de faptul că

$$\sum_{i=0}^{m} d_i + \sum_{i=0}^{m-1} l_i = 1$$

se obţine valoarea minimă a lui I, pentru $A_0, \ldots, A_{m-1}$, fixaţi, ca fiind cea realizată pentru valorile $d_0^*, \ldots, d_m^*$ determinate prin rezolvarea sistemului algebric

$$x_k + y_k - 2\sum_{j=0}^{k-1} A_j - A_k = 0, \; k = \overline{0, m-1},$$

$$\sum_{j=0}^{m} d_j + \sum_{j=0}^{m-1} l_j = 1.$$

Soluţia acestui sistem, cu necunoscutele $d_0, \ldots, d_m$ este

$$d_k^* = \frac{A_{k-1} - l_{k-1}}{2} + \frac{A_k - l_k}{2}, \text{ pentru } k = \overline{0, m},$$

unde

$$A_{-1}, l_{-1}, A_m, l_m = 0.$$

În acest fel se obţine

$$x_k^* = \sum_{j=0}^{k} A_j - \frac{A_k + l_k}{2};$$

$$y_k^* = \sum_{j=0}^{k} A_j + \frac{A_k - l_k}{2}, \text{ pentru } k = \overline{0, m-1},$$

iar

$$\bar{I}(A_0, \ldots, A_{m-1}) = \frac{1}{4}\sum_{j=0}^{m-1} A_j(A_j - l_j).$$

Rezolvarea problemei II. Determinarea minimului lui $\bar{I}$ se face ţinîndu-se seama de relaţia

$$\sum_{j=0}^{m-1} A_j = 1.$$

În acest fel se obține că valoarea minimă a lui $\bar{I}$ este atinsă pentru

$$A_k^* = \frac{1}{m} + \frac{l_0 + \ldots + (m-1)l_k + \ldots + l_{m-1}}{2m}, \text{ pentru } k = \overline{0, m-1},$$

iar

$$I^* = \frac{1}{16}\left[\frac{1}{m}\left(2 - \sum_{j=0}^{m-1} l_j\right)^2 - \sum_{j=0}^{m-1} l_j^2\right].$$

În final se obține:

$$d_0^* = \frac{1}{2m} - \frac{l_0}{4} - \frac{1}{4m}\sum_{j=0}^{m-1} l_j,$$

$$d_k^* = \frac{1}{m} - \frac{l_k + l_{k-1}}{4} - \frac{1}{2m}\sum_{j=0}^{m-1} l_j, \text{ pentru } k = \overline{1, m-1},$$

$$d_m^* = \frac{1}{2m} - \frac{l_{m-1}}{4} - \frac{1}{4m}\sum_{j=0}^{m-1} l_j.$$

Se observă că $d_j^* > 0$, $j = \overline{0, m}$, în ipoteza în care s-a lucrat, adică $A_j > d_j$, $j = \overline{0, m-1}$.

Cazuri particulare.

1) Dacă în formula optimală obținută se face $l_k \to 0$, $k = \overline{0, m-1}$, se obține optimală de cuadratură cu evaluări punctuale

$$\int_0^1 f(x)dx = \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{2k+1}{2m}\right) + R[f],$$

pentru care $I^* = \frac{1}{4m}$.

2) Dacă $l_k = l$, pentru $k = \overline{0, m-1}$, se obține formula optimală de cuadratură de tip interval

$$\int_0^1 f(x)\,dx = \frac{1}{ml}\sum_{j=0}^{m-1} \int_{\frac{2j+1}{2m}-\frac{l}{2}}^{\frac{2j+1}{2m}+\frac{l}{2}} f(x)dx + R[f],$$

iar

$$I^* = \frac{1-ml}{4m}.$$

În cazul în care $l = \frac{1}{2m}$ se obține formula optimală de cuadratură de tip interval

$$\int_0^1 f(x)dx = 2\sum_{j=0}^{m-1} \int_{\frac{2j-1}{4m}}^{\frac{2j+1}{4m}} f(x)dx + R[f],$$

cu

$$I^* = \frac{1}{8m}.$$

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OPTIMAL QUADRATURE FORMULAS OF INTERVAL TYPE

(Summary)

Let us consider the quadrature formula of interval type

$$\int_0^1 f(x)\,dx = \sum_{k=0}^{m-1} \frac{A_k}{l_k} \int_{x_k}^{y_k} f(x)\,dx + R[f]$$

with

$$0 = y_{-1} \leqslant x_0 < y_0 < x_1 < y_1 < \ldots x_{m-1} < y_{m-1} \leqslant x_m = 1,$$

$$l_k = y_k - x_k, \quad k = \overline{0, m-1}, \; d_k = x_k - y_{k-1}, \; k = \overline{0, m}$$

of the exactness degree one.

One defines optimal quadrature formulas of interval type, i.e. the formula for which Peano's kernel function is minimized in L_1-norm, with regard to parameters A_k, $k = \overline{0, m-1}$ and d_k, $k = \overline{0, m}$.

In this paper the optimal quadrature formula of the given form is obtained. Also, two remarkable cases are given.

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ON THE ORBITAL ECCENTRICITY AND INCLINATION OF AN ECLIPSING BINARY SYSTEM

IOAN TODORAN

1. **Introduction.** There are lots of problems where an accurate determination of the orbital eccentricity and inclination is required. Among such situations we can mention here, for instance, the problems concerning the geometrical characteristics and physical features of the close binary systems, the nature of *X*-ray binaries where the compact object is linked to a massive star etc. Taking into account the recent accuracy of the photoelectric minima determination and the possibility to dispose of successive *X*-ray minima, it would often be very important to dispose of a suitable method for orbital eccentricity and inclination determination without the use of a whole light curve. That is why we shall review, in the present note, the problem of apsidal motion and the possibility we have to use it in order to determine the two orbital parameters e and i, of course where its effect is observable.

2. **Equations of the problem.** In two previous papers (Todoran [3], [4]) for the establishment of the ephemeris formulae for the primary and secondary minima, we have followed the method exposed by Martynov (see Zverev et al [6]), but taking into account all expansions into series correctly to the third power of e inclusively. In such a way, the ephemeris for the two kinds of minima is given by:

$$(T_E)_I = T_0 + \mathfrak{P}E - \frac{Pe}{2\pi}\left\{\cot^2 i + 2 - e^2\left[\frac{3}{4}\cot^2 i - \frac{1}{4}(2 + \operatorname{cosec}^2 i)\cot^2 i \operatorname{cosec}^2 i\right]\right\}\cos\omega + \frac{Pe^2}{4\pi}\left[\cot^2 i \operatorname{cosec}^2 i + 2\cot^2 i + \frac{3}{2}\right]\sin 2\omega + \frac{Pe^3}{8\pi}\left[(2 + \operatorname{cosec}^2 i)\cot^2 i \operatorname{cosec}^2 i + 3\cot^2 i + \frac{4}{3}\right]\cos 3\omega, \quad (1)$$

and:

$$(T_E)_{II} = T_0 + \frac{P}{2} + \mathfrak{P}E + \frac{Pe}{2\pi}\left\{\cot^2 i + 2 - e^2\left[\frac{3}{4}\cot^2 i - \frac{1}{4}(2 + \operatorname{cosec}^2 i)\cot^2 i \operatorname{cosec}^2 i\right]\right\}\cos\omega + \frac{Pe^2}{4\pi}\left[\cot^2 i \operatorname{cosec}^2 i + 2\cot^2 i + \frac{3}{2}\right]\sin 2\omega - \frac{Pe^3}{8\pi}\left[(2 + \operatorname{cosec}^2 i)\cot^2 i \operatorname{cosec}^2 i + 3\cot^2 i + \frac{4}{3}\right]\cos 3\omega, \quad (2)$$

respectively.

Here we have used the abbreviation:

$$\mathcal{P} = (1 - \omega_1/2\pi)P \tag{3}$$

and the following usual notations:

P = orbital period;
T_0 = initial epoch;
E = whole number of periods;
i = orbital inclination;
e = orbital eccentricity;
ω_0 = longitude of periastron at $E = 0$;
ω_1 = variation of the longitude of periastron.

Now, if we consider that the apsidal motion is going on with constant velocity, the following equation may be written:

$$\omega = \omega_0 + \omega_1 E, \tag{4}$$

with:

$$\omega_1 = \frac{2\pi}{U} P, \tag{5}$$

where U denotes the corresponding apsidal period.

Further, let us put for abbreviation:

$$\begin{aligned} \cot^2 i = y; \quad \operatorname{cosec}^2 i = 1 + \cot^2 i = 1 + y; \\ (O - C)_{\mathrm{I}} = (T_x)_{\mathrm{I}} - T_0 + \mathcal{P}E; \\ (O - C)_{\mathrm{II}} = (T_x)_{\mathrm{II}} - T_0 - \frac{P}{2} - \mathcal{P}E, \end{aligned} \tag{6}$$

and the number E is taken as 0 (zero) when the longitude of periastron is $\omega = 90°$ or $270°$ and the line of apsides is parallel to the line of sight. In such a moment we have:

$$\omega = \omega_0 + \omega_1 \times (E = 0) = 90° \text{ or } 270°$$

and consequently:

$$\cos\omega = \cos\left(90° + \frac{2\pi}{U}E\right) = -\sin\frac{2\pi}{U}E;$$

$$\cos 3\omega = \sin 3\frac{2\pi}{U}E;$$

$$\sin 2\omega = \sin\left(180° + 2\frac{2\pi}{U}E\right) = -\sin 2\frac{2\pi}{U}E;$$

$$U = \frac{2\pi}{\omega_1}P = \text{apsidal period.}$$

Now, using the above mentioned notations, from (1) and (2) we have:

$$\left[\frac{2\pi}{P}(O-C)_{\rm I}\right]_E = A\sin\frac{2\pi}{U}E - B\sin 2\frac{2\pi}{U}E + C\sin 3\frac{2\pi}{U}E, \qquad (7)$$

$$\left[\frac{2\pi}{P}(O-C)_{\rm II}\right]_E = -A\sin\frac{2\pi}{U}E - B\sin 2\frac{2\pi}{U}E - C\sin 3\frac{2\pi}{U}E, \qquad (8)$$

where:

$$A = e\left\{y + 2 + e^2\left(y^2 + \frac{1}{4}y^3\right)\right\};$$

$$B = \frac{e^2}{2}\left(3y + y^2 + \frac{3}{2}\right); \qquad (9)$$

$$C = \frac{e^3}{4}\left(6y + 4y^2 + y^3 + \frac{4}{3}\right).$$

If we dispose of a sufficiently great number of observed minima, we can perform an $O - C$ diagram, the corresponding differences being considered as an effect of the apsidal motion. In this case we can determine the abscissae of those points where we have $O - C = 0$ (see Fig. 1). Consequently we can write:

$$\left.\begin{array}{l}\omega'' = \omega_0 + \omega_1 E'' \\ \omega' = \omega_0 + \omega_1 E'\end{array}\right\} \Rightarrow \omega'' - \omega' = \omega_1(E'' - E') = \pi$$

and:

$$\omega_1 = \frac{\pi}{E'' - E'}. \qquad (10)$$

From (10) and (5) we can write:

$$U = 2(E'' - E')P. \qquad (11)$$

In this way we can determine a preliminary value of the corresponding apsidal period $\tilde{U}$.

If we substitute $\tilde{U}$ in Eqs. (7) and (8), for each observed minimum we can write an equation of condition. A least squares solution of the corresponding equations gives us the preliminary values $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$, and from (9) we can write:

$$\tilde{A} = \tilde{e}(\tilde{y} + 2), \qquad \tilde{B} = \tilde{e}^2\left(3\tilde{y} + \tilde{y}^2 + \frac{3}{2}\right), \qquad (12)$$

where, in a first approximation, we have omitted the terms in e^3. In such conditions we have:

$$\tilde{e} = \tilde{A}(\tilde{y} + 2)^{-1}; \qquad \tilde{B} = \tilde{A}^2\left(3\tilde{y} + \tilde{y}^2 + \frac{3}{2}\right)[2(\tilde{y}^2 + 4\tilde{y} + 4)]^{-1}$$

or:

$$\tilde{y}^2 + (8\tilde{B} - 3\tilde{A}^2)(2\tilde{B} - \tilde{A}^2)^{-1}\tilde{y} + (16\tilde{B} - 3\tilde{A}^2)[2(2\tilde{B} - \tilde{A}^2)]^{-1} = 0 \qquad (13)$$

So, in a first approximation, we have:

$$\cot^2 i = \tilde{y} \quad \text{and} \quad \tilde{e} = \tilde{A}(\tilde{y} + 2)^{-1}$$

with the assumption that $i \leqslant 90°$.

Now, we have to improve our results. For this purpose, we shall put $\tilde{y}$, $\tilde{e}$ and $\tilde{U}$ in Eqs. (1) and (2), and by considering Eqs. (4) and (6) we shall obtain the „computed" $O - C$ differences. In such a way we have:

$$(O - C)' = \frac{2\pi}{P} [(O - C)_{obs} - (O - C)_{comp}], \tag{14}$$

where we have put:

$$\begin{aligned} (O - C)_{obs} &= [(T_E)_I]_{obs} - T_0 - \mathfrak{Q}E ; \\ (O - C)_{comp} &= [(T_E)_{II}]_{comp} - T_0 - \mathfrak{Q}E. \end{aligned} \tag{14'}$$

If we accept:

$$U = \tilde{U} + dU ; \quad e = \tilde{e} + de ; \quad y = \tilde{y} + dy \tag{15}$$

in order to determine the most probable values U, e and i, a set of successive approximations must be used. With this end in view, from Eqs. (1) and (2) we have:

$$(O - C)'_I = \mathcal{A}_I\, de + \mathcal{B}_I\, dy + \mathcal{C}_I\, dU, \tag{16}$$

$$(O - C)'_{II} = \mathcal{A}_{II}\, de + \mathcal{B}_{II}\, dy + \mathcal{C}_{II}\, dU, \tag{17}$$

where:

$$\begin{aligned} \mathcal{A}_I &= \mathcal{A}_I(\tilde{U}, \tilde{i}, \tilde{e}) \\ \mathcal{B}_I &= \mathcal{B}_I(\tilde{U}, \tilde{i}, \tilde{e}) \\ \mathcal{C}_I &= \mathcal{C}_I(\tilde{U}, \tilde{i}, \tilde{e}), \end{aligned} \tag{18}$$

$$\begin{aligned} \mathcal{A}_{II} &= \mathcal{A}_{II}(\tilde{U}, \tilde{i}, \tilde{e}) \\ \mathcal{B}_{II} &= \mathcal{B}_{II}(\tilde{U}, \tilde{i}, \tilde{e}) \\ \mathcal{C}_{II} &= \mathcal{C}_{II}(\tilde{U}, \tilde{i}, \tilde{e}). \end{aligned} \tag{19}$$

For each observed minimum we have an equation of condition of the form (16) or (17) for the primary and secondary minima respectively. Then, by the least squares method, we obtain the corresponding corrections de, dU and dy and from (15) we obtain the improved values, e.g. in the second approximation and so on.

Remark: If we dispose of the primary and secondary minima observed almost successively, the differences between (8) and (7) is:

$$\frac{2\pi}{P} [(O - C)_{II} - (O - C)_I] = -2A \sin \frac{2\pi}{U} E - 2C \sin 3 \frac{2\pi}{U} E$$

or:

$$\frac{\pi}{P}\left[(T_E)_{II} - (T_E)_I - \frac{P}{2}\right] = -A \sin\frac{2\pi}{U}E - C \sin 3\frac{2\pi}{U}E. \tag{20}$$

In a first approximation C may be neglected and for $\tilde{U}$ we are able to determine the corresponding value $\tilde{A}$.

Now, if we suppose, in addition, $\tilde{i} \cong 90°$, $\tilde{y} = 0$, then from (9) we have:

$$\tilde{e} = \frac{\tilde{A}}{2} \tag{21}$$

and with these approximate values: $\tilde{e}, \tilde{y}\ (= 0)$ and $\tilde{U}$, we can resume all the problem; in a second approximation, from (17) and (16), we have:

$$(O - C)'_{II} - (O - C)'_I = (\mathcal{A}_{II} - \mathcal{A}_I)de + (\mathcal{B}_{II} - \mathcal{B}_I)dy + (\mathcal{C}_{II} - \mathcal{C}_I)dU,$$

or:

$$(O - C)'' = \alpha de + \beta dy + \gamma dU. \tag{22}$$

3. Application to Y CYGNI. Having in view the fact that the apsidal motion for *Y Cygni* was well studied by different authors, we shall use the corresponding observational data in order to illustrate the numerical application of the above presented considerations.

From Fig. 1 (see Todoran [5]) we have redetermined:

$$E_{max} = (4315 \pm 20)P; \qquad E_{min} = (1438 \pm 14)P.$$

Consequently,

$$\tilde{U} = 2(\tilde{E}_{max} - \tilde{E}_{min})P = 5754\ P$$

and

$$\tilde{\omega}_1 = (2\pi/\tilde{U})P = \frac{360°}{5754} = 0°{\cdot}06256.$$

Now, from (20) it follows

$$\tilde{A} = -\frac{\pi}{P}\left\{\sum_{i=1}^{n}[(T_{E_i})_{II} - (T_{E_i})_I] - \frac{n}{2}P\right\}\left(\sum_{i=1}^{n}\sin\frac{2\pi}{U}E_i\right)^{-1},$$

but in order to avoid a possible undetermination, we have to write apart the corresponding sums of the positive and negative terms. Thus for *Y Cygni* it follows:

$$\tilde{A}_+ = \frac{\pi}{P}\left\{\sum_{i=1}^{14}[(T_{E_i})_{II} - (T_{E_i})_I] - 7P\right\}\left(\sum_{i=1}^{14}\sin\frac{2\pi}{\tilde{U}}E_i\right)^{-1};$$

$$\tilde{A}_- = \frac{\pi}{P}\left\{\sum_{i=15}^{28}[(T_{E_i})_{II} - (T_{E_i})_I] - 7P\right\}\left(\sum_{i=15}^{28}\sin\frac{2\pi}{\tilde{U}}E_i\right)^{-1},$$

or, if we use the numerical data from Table I (see Todoran [5]) we have:

$$\tilde{A}_{+} = -\frac{1.048481\,(18.5687 - 20.97433)}{8.49026} = 0.29677\,;$$

$$\tilde{A}_{-} = -\frac{1.048481\,(24.2376 - 20.97433)}{-11.83570} = 0.2891$$

and

$$\tilde{e} = \frac{A}{2} = \frac{1}{4}(\tilde{A}_{+} + \tilde{A}_{-}) = \frac{1}{4}(0.29677 + 0.28910)\,;$$

$$\tilde{e} = 0.1464.$$

So, in a first approximation, we have:

$$\tilde{U} = 5754\ P,\ \tilde{e} = 0.1464 \text{ and } \tilde{i} = 90^\circ$$

and in the second approximation:

$$U = 5758\ P\,;\ e = 0.1450\,;\ i = 87^\circ{\cdot}6 \pm 2^\circ{\cdot}5$$
$$\pm 4 \qquad\qquad \pm 16$$

which are comparable with the corresponding values obtained from a very good light curve:

$$U = 5801\ P\,;\quad e = 0.1458 \text{ (see O'Connell, 1977)}$$
$$\pm 1 \qquad\qquad \pm 2$$

and

$$i = 88^\circ{\cdot}2 \pm 1^\circ{\cdot}8 \text{ (see Kopal and Shapley, 1956).}$$

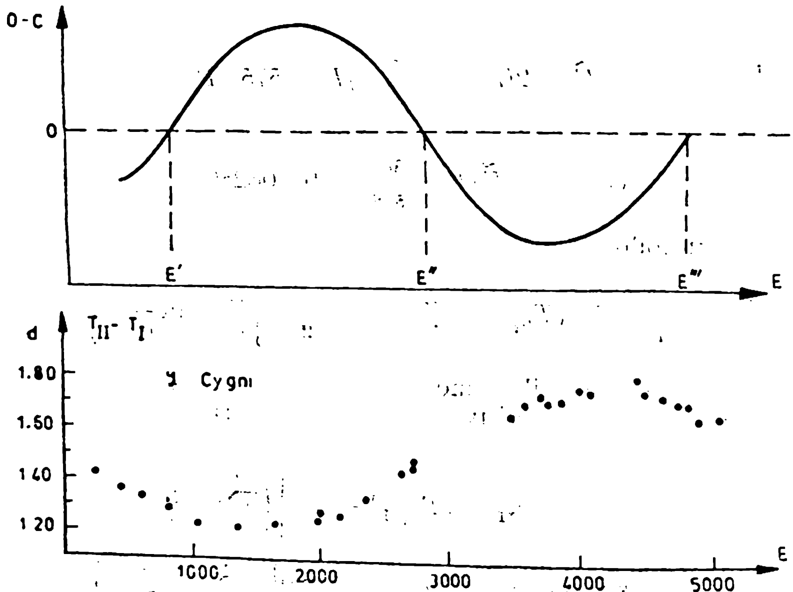

Fig. 1.

4. **Concluding remarks.** By using the above presented method, we have a new possibility to determine the orbital eccentricity and inclination, at least with the same accuracy as they would be determined from the analysis of an excellent light curve. This new method makes use of primary and secondary minima observed within a long time interval, but it does not need a whole observed light curve. Therefore, this new proceeding gives us a supplementary possibility for orbital parameters (e, i and U) determination in such peculiar cases where a whole light curve analysis is not possible, or/and the ength of the orbital period is not known with a very great accuracy, or/and when some physical changes in the orbital period are observed, etc.

In order to perform a numerical application, we have used a sequence of observed minima of *Y Cygni* (see Todoran [5]), whose eccentricity and inclination were also determined by using other different methods. The numerical comparison shows that the above method may be used successfully.

The above presented method gives also important results in the case of the X-ray binaries where the compact object is linked to a massive star. In this last case, the corresponding orbital period can be deduced from the time interval between two successive X-ray minima.

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ASUPRA ÎNCLINĂRII ŞI EXCENTRICITĂŢII ORBITALE ALE UNEI STELE DUBLE CU ECLIPSE

(Rezumat)

În prezenta lucrare este propusă o nouă metodă pentru determinarea înclinării şi excentricităţii orbitale ale unei stele duble cu eclipse. În acest scop se analizează efectele mişcării apsidale asupra momentelor eclipselor corespunzătoare, eficienţa metodei fiind verificată în cazul stelei duble fotometrice *Y Cygni*.

ON STEFFENSEN'S METHOD IN FRÉCHET SPACES

M. BALÁZS and G. GOLDNER

Let X be a real Fréchet space with a quasinorm induced by an invariant distance d(i.e. $\|x\| = d(x, O)$) (see e.g. [4, p. 14]) and $P: X \to X$ a continuous operator. We shall note by $[x', x''; P]$ and $[x', x'', x'''; P]$ the symmetrical divided difference of the first, respectively the second order of P in the considered points [2].

We consider the equation

$$P(x) \equiv x - \Phi(x) = 0. \tag{1}$$

If we have the sequence (x_n) in X, then the sequence (u_n) is defined by $u_n = \Phi(x_n)$, and $\Gamma_n = [x_n, u_n; P]^{-1}$, if this inverse operator exists. There exists the results concering the existence of the solution of the equation (1) in Banach spaces [3], as the limit of the sequence given by

$$x_{n+1} = x_n - \Gamma_n P(x_n),\ n = 0, 1, 2, \ldots \tag{2}$$

The purpose of this paper is to generalize these sufficient conditions for the case of the Fréchet spaces.

THEOREM. *We suppose that there exist the numbers B_0, η_0, K and the point x_0 of X so that:*

(i) $\|P(x_0)\| = \|x_0 - \Phi(x_0)\| = \|x_0 - u_0\| \leq \eta_0$;

(ii) *there exists* $\Gamma_0 = [x_0, u_0; P]^{-1}$, *and* $\|\Gamma_0\| \leq B_0$;

(iii) $\sup\{\|[x', x'', x'''; P]\| : x', x'', x''' \in S(x_0, r)\} \leq K$,

where $S(x_0, r) = \{x \in X : \|x - x_0\| \leq r\}$, $\quad r = \eta_0(2B_0 + 1)$;

(iv) $h_0 = B_0 K \left(\frac{4}{3} + 2B_0\right)\eta_0 < \frac{1}{3}$.

In these conditions there exists the sequence (x_n) defined by (2) having the following properties:

(j) *there exists* $\lim\limits_{n\to\infty} x_n = x^*$, $x^* \in \overline{S(x_0, r)}$, and $P(x^*) = 0$;

(jj) *the rapidity of the convergence of the sequence (x_n) to the solution x^* of the (1) is given by*

$$\|x_n - x^*\| \leq 4B_0\eta_0\left(\frac{2}{3}\right)^n\left(\frac{3}{4}\right)^{2^n} \cdot (3h_0)^{2^n-2}.$$

Remark. We use in this paper the quasinorm of the linear mappings L having Lipschitz property (i.e. there exists $M > 0$ so that for every x of X, $\|L(x)\| \leq M\|x\|$) and $\|L\| = \inf\{M > 0 : \|L(x)\| \leq M\|x\| \text{ for every } x \in X\}$.

Prof. By (ii) and (2) we can construct the points x_1 and u_1, and using (1), (2), (i) and (ii), we have

$$\begin{aligned}
&\|x_0 - u_0\| = \|x_0 - \Phi(x_0)\| = \|P(x_0)\| \leqslant \eta_0, \\
&\|x_0 - x_1\| = \|\Gamma_0 P(x_0)\| \leqslant B_0\eta_0, \qquad (3) \\
&\|x_1 - u_0\| = \|x_0 - [x_0, u_0; P]^{-1}P(x_0) - u_0\| = \|P(x_0) - \\
&\quad - [x_0, u_0; P]^{-1}P(x_0)\| \leqslant \eta_0(1 + B_0).
\end{aligned}$$

By Newton's interpolation formula

$$P(z) = P(x) + [x, y; P](z - x) + [z, x, y; P](z - y)(z - x), \tag{4}$$

choosing $z = x_1$, $x = x_0$, $y = u_0$, the equality (2) gives

$$P(x_1) = [x_1, x_0, u_0; P](x_1 - u_0)(x_1 - x_0),$$

using (3), (iii), and (iv) in quasinorm we have

$$\|P(x_1)\| \leqslant K\|x_1 - x_0\| \cdot \|x_1 - u_0\| \leqslant KB_0\eta_0^2(1 + B_0) < h_0\eta_0 = \eta_1 < \frac{1}{3}\eta_0. \tag{5}$$

We also have

$$\begin{aligned}
\|x_0 - u_1\| &= \|x_0 - x_1 + x_1 - u_1\| \leqslant \|x_0 - x_1\| + \|P(x_1)\| \leqslant \qquad (6) \\
&\leqslant B_0\eta_0 + \frac{1}{3}\eta_0 = \eta_0\left(B_0 + \frac{1}{3}\right).
\end{aligned}$$

We shall show that there exist the numbers B_1, η_1, h_1 so that the points x_1, u_1 satisfy the analogous conditions with (i) — (iv). By the identity

$$\begin{aligned}
[x, y; P] - [z, v; P] &= [x, y; P] - [x, z; P] + [x, z; P] - [z, v; P] = \\
&= [x, y, z; P](y - z) + [x, z, v; P](x - v), \qquad (7)
\end{aligned}$$

if $x = x_0$, $y = u_0$, $z = x_1$, $v = u_1$, using (3) and (6), and the fact that x_1, u_1, $u_0 \in S(x_0, r)$, it results

$$\begin{aligned}
\|\Gamma_0\{[x_0, u_0; P] - [x_1, u_1; P]\}\| &\leqslant KB_0(\|x_1 - u_0\| + \|x_0 - u_1\|) \leqslant \\
&\leqslant B_0K\left(\frac{4}{3} + 2B_0\right)\eta_0 = h_0 \leqslant \frac{1}{3}. \qquad (8)
\end{aligned}$$

If I is the identical mapping of X, then we have

$$\{\Gamma_0[x_1, u_1; P]\}^{-1} = \{I - \Gamma_0\{[x_0, u_0; P] - [x_1, u_1; P]\}\}^{-1}$$

and by (8) it results the existence of the mapping $\{\Gamma_0[x_1, u_1; P]\}^{-1}$ and

$$\|\{\Gamma_0[x_1, u_1; P]\}^{-1}\| \leqslant \frac{1}{1 - h_0},$$

and using the equality

$$\{\Gamma_0[x_1, u_1; P]\}^{-1}\Gamma_0 = \Gamma_1$$

we obtain

$$\|\Gamma_1\| \leqslant \frac{B_0}{1-h_0} = B_1 \leqslant \frac{3}{2} B_0 \qquad (B_0 < B_1).$$

Hence, the hypothesis (ii) is satisfied for x_1, u_1. The hypothesis (i) for x_1 is verified in (5). We have

$$h_1 = B_1 K\left(\frac{4}{3} + 2B_1\right)\eta_1 = \frac{B_0}{1-h_0} K\left(\frac{4}{3} + \frac{2B_0}{1-h_0}\right) h_0 \eta_0 <$$

$$< h_0 \frac{B_0 K\left(\frac{4}{3} + 2B_0\right)\eta_0}{(1-h_0)^2} = \frac{h_0^2}{(1-h_0)^2} \leqslant \frac{1}{4} < \frac{1}{3},$$

and so the constants B_1, η_1, K veritfy the condition (iv).

By induction we can prove the following relations:

a) $x_n \in S(x_0, r)$;

b) $\|P(x_n)\| \leqslant h_{n-1}\eta_{n-1} = \eta_n \leqslant \frac{\eta_0}{3^n}$;

c) $u_n \in S(x_0, r)$;

d) There exists

$\Gamma_n = [x_n, u_n; P]^{-1}$ and $\|\Gamma_n\| \leqslant \frac{B_{n-1}}{1-h_{n-1}} = B_n \leqslant \left(\frac{3}{2}\right)^n \cdot B_0$;

e) $h_n = B_n K\left(\frac{4}{3} + 2B_n\right)\eta_n \leqslant \frac{h_{n-1}^2}{(1-h_{n-1})^2} \leqslant \left(\frac{3}{2}\right)^2 h_{n-1}^2 < \frac{1}{3}$.

By e) and b) we have

$$h_n \leqslant \left(\frac{3}{2}\right)^{2(2^n-1)} \cdot h_0^{2^n}, \qquad n = 1, 2, \ldots \tag{9}$$

$$\eta_n \leqslant 3\eta_0 \left(\frac{4}{9}\right)^{n+1} \left(\frac{3}{4}\right)^{2^n} (3h_0)^{2^n-1}. \tag{10}$$

Using the inequality $\|x_{n+1} - x_n\| \leqslant B_n \eta_n$, and the relations (10) and d) we obtain

$$\|x_{n+p} - x_n\| \leqslant \frac{3}{4} B_0 \eta_0 \left(\frac{2}{3}\right)^n \left(\frac{3}{4}\right)^{2^n} (3h_0)^{2^n-1} \cdot \sum_{i=1}^{p} \left(\frac{2}{3}\right)^i \left(\frac{9h_0}{4}\right)^{2^n(2^{i-1}-1)} <$$

$$< 4B_0\eta_0 \left(\frac{2}{3}\right)^n \cdot \left(\frac{3}{4}\right)^{2^n} \cdot (3h_0)^{2^n-1}. \tag{11}$$

The space X being complete it results the existence of the limit of the sequence (x_n), and $\lim_{n\to\infty} x_n = x^* \in \overline{S(x_0, r)}$. For $p \to \infty$, the inequality (11) gives the rapidity of the conporgence of the sequence (x_n).

We shal prove that the point x^* is the solution of the equation (1). By the equality

$$[x_n, u_n; P] = [x_n, u_n, x^*; P](u_n - x^*) + [x_n, u^*, x_0; P](x_n - x_0) + [x^*, x_0; P],$$

using the hypothesis (iii), it results

$$\|[x_n, u_n; P]\| \leqslant 3Kr + \|[x^*, x_0; P]\| = M, \tag{12}$$

hence the set of linear lipschitzian mappings $\{[x_n, u_n; P] : n \in \mathbf{N}\}$ is uniformly bounded in quasinorm.

The equality (2), using (12) gives

$$\|P(x_n)\| = \|[x_n, u_n; P](x_{n+1} - x_n)\| \leqslant M\|x_{n+1} - x_n\|,$$

and for $n \to \infty$ we obtain

$$\lim_{n\to\infty} \|P(x_n)\| = \|P(x^*)\| = 0.$$

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METODA LUI STEFFENSEN ÎN SPAȚII FRÉCHET

(Rezumat)

Se consideră spațiul X real Fréchet cu o cvasinormă indusă de o distanță invariantă (vezi [4, p. 14]) și o funcție continuă $P: X \to X$. Pentru rezolvarea ecuației (1) $P(x) = x - \Phi(x) = 0$ se studiază metoda iterativă dată de formula (2) $x_{n+1} = x_n - \Gamma_n P(x_n)$, $n = 0, 1, 2, \ldots$, unde Γ_n este inversa diferenței divizate $[x_n, u_n : P]$, de ordinul întîi a funcției P în punctul $(x_n, u_n) = (x_n, \Phi(x_n)) \in X \times X$. În lucrare se demonstrează următoarea teoremă: *Dacă există un punct* $x_0 \in X$ *și numerele* B_0, η_0, K *astfel încît următoarele condiții să fie satisfăcute :* (i) $\|P(x_0)\| = \|x_0 - \Phi(x_0)\| = \|x_0 - u_0\| \leqslant \eta_0$; *(ii) există* $\Gamma_0 = [x_0, u_0; P]^{-1}$ *și* $\|\Gamma_0\| \leqslant B_0$; *(iii)* $\sup\{\|[x', x'', x'''; P]\| : x', x'', x''' \in S(x_0, r)\} \leqslant K$, *unde* $S(x_0, r) = \{x \in X : \|x - x_0\| \leqslant r\}$, $r = \eta_0(2B_0 + 1)$; *(iv)* $h_0 = B_0 K\left(\frac{4}{3} + 2B_0\right)\eta_0 < \frac{1}{3}$, *a unci egalitatea* (2) *definește prin recurență un șir de puncte* (x_n) *care are următoarele proprietăți :* (*j*) *există* $\lim_{n\to\infty} x_n = x^* \in S$ *și* $P(x^*) = 0$; (*jj*) *rapiditatea de convergență a șirului* (x_n) *este dată de inegalitatea :*

$$\|x_n - x^*\| \leqslant 4B_0\eta_0\left(\frac{2}{3}\right)^n\left(\frac{3}{4}\right)^{2^n}(3h_0)^{2^n-1}, \qquad n = 1, 2, 3, \ldots$$

O METODĂ VARIAŢIONALĂ CU POTENŢIAL LOCAL PENTRU PROBLEMA LUI MISES A STRATULUI LIMITĂ HIDRODINAMIC DE PE O PLACĂ PLANĂ

PETRE BRĂDEANU

Se aplică o metodă variaţională, bazată pe conceptul de potenţial local, cu ajutorul căreia se rezolvă problema la limită neliniară a stratului limită incompresibil, în raport cu variabilele lui Mises, care se formează pe o placă plană. Operatorul neliniar al lui Mises nu satisface condiţiile pentru existenţa unui potenţial (simetria derivatei în sens Fréchet a operatorului) şi în consecinţă nu se poate defini şi nici construi un potenţial al acestui operator după procedeele matematice clasice (Ritz, Mihlin, Galerkin, derivata Fréchet). Din acest motiv am introdus, ca şi alţi autori în probleme similare, folosind principii fizice — care pornesc de la principiul minimei produceri de entropie — un potenţial special, pentru operatorul lui Mises, denumit potenţial local (o funcţională de minim).

Se consideră un fluid vîscos incompresibil cu parametrii fizici $\vec{v} = (u, v)$ — viteza, ρ — densitatea, μ — coeficientul dinamic de viscozitate ($\nu = \mu/\rho$), în mişcare plană peste o semiplacă plană fixă. Fie Oxy sistemul de referinţă legat de placă: axa Ox este dirijată longitudinal pe placă (în sensul mişcării fluidului), axa Oy este normală la placă iar originea O este bordul de atac al plăcii (vîrful).

Ecuaţia mişcării staţionare a fluidului în stratul limită, în forma lui Mises, este

$$\rho \frac{\partial u}{\partial x} = \frac{\partial}{\partial \psi}\left(\mu u \frac{\partial u}{\partial \psi}\right), \; (x, \psi) \in \Omega = (0, L) \times (0, \psi_\infty) \; ; \tag{0.1}$$

$$u(x, 0) = 0, \; u(x, \infty) = u_\infty \; (= \text{const.}), \; 0 \leqslant x \leqslant L \; (\text{dat arbitrar}) \; ;$$

$$u(0, \psi) = u_\infty, \; 0 < \psi < \psi_\infty \; (\text{sau} < \infty),$$

unde $u(x, \psi)$ este funcţia necunoscută; ψ este funcţia de curent ($u = \partial\psi/\partial y$, 10—37
$v = -\partial\psi/\partial x$; $(x, y) \to (x, \psi)$).

Ecuaţia mişcării nestaţionare în stratul limită este

$$\rho \frac{\partial u}{\partial t} = -\rho u \frac{\partial u}{\partial x} + u \frac{\partial}{\partial \psi}\left(\mu u \frac{\partial u}{\partial \psi}\right) \tag{0.2}$$

1. **Construirea potenţialului local.** Pentru a construi potenţialul local corespunzător ecuaţiei stratului limită staţionar (0.1) vom porni de la ecuaţia (0.2) a mişcării nestaţionare. Se înmulţeşte această ecuaţie cu derivata locală $-\partial u/\partial t$ şi se introduce funcţia nepozitivă

$$f(x, \psi, t) = -\rho\left(\frac{\partial u}{\partial t}\right)^2 = \rho u \frac{\partial u}{\partial t} \frac{\partial u}{\partial x} - u \frac{\partial u}{\partial t} \frac{\partial}{\partial \psi}\left(\mu u \frac{\partial u}{\partial \psi}\right) \leqslant 0 \tag{1}$$

În planul (x, ψ), în domeniul Ω, să introducem funcția

$$F(t) = \iint_{\Omega} f(x, \psi, t)\, dx\, d\psi \leqslant 0 \qquad (\forall t > 0)$$

Avem, imediat, identitățile (indicele notează derivată parțială)

$$uu_t = (u^2 u_t)_x - uu_x u_t - u^2 u_{xt},$$

$$uu_t(uu_\psi)_\psi = (u^2 u_t u_\psi)_\psi - uu_t u_\psi^2 - u^2 uu_\psi u_{\psi t}$$

Dacă se aplică formula lui Green, se obține

$$F(t) = \iint_{\Omega} (-\rho u_x u_t - \rho u^2 u_{xt} + \mu u u_\psi^2 u_t + \mu u^2 u_\psi u_{\psi t})\, dx\, d\psi +$$

$$+ \int_C (\rho u^2 u_t\, d\psi + \mu u^2 u_t u_\psi\, dx) \leqslant 0, \qquad (C \equiv \partial\Omega) \tag{2}$$

Se presupune, acum, că mișcarea nestaționară reprezintă o abatere mică de la o mișcare staționară a fluidului caracterizată, la rîndul ei, prin funcția independentă de timp $u^0 = u^0(x, \psi)$. Atunci, se poate scrie relația

$$F(t) = \frac{\partial}{\partial t} J \leqslant 0 \tag{3}$$

în care funcționala $J : H^1(\Omega) \to R^1$, unde $H^1(\Omega)$ este spațiul funcțiilor u și u^0 (Sobolev), are forma

$$J(u\,;\, u_0) = \iint_{\Omega} \left[-\rho u^0 \frac{\partial u}{\partial x} - \rho u^0 \frac{\partial u^0}{\partial x} u + \mu u^0 \left(\frac{\partial u^0}{\partial \psi} \right)^2 u + \mu u^0 \frac{\partial u^0}{\partial \psi} \frac{\partial u}{\partial \psi} \right] dx\, d\psi +$$

$$+ \int_C \left(\rho u^0 u\, d\psi + \mu u^0 \frac{\partial u^0}{\partial \psi} u\, d\psi \right) \tag{4}$$

Această funcțională descrește în raport cu timpul devenind minimă pentru mișcarea staționară, adică atunci cînd $u = u^0$. Deci,

$$\delta_u J(u\,;\, u^0) = 0 \text{ cu condiția complementară } u = u^0$$

variația avînd loc numai în raport cu u (u^0 este o funcție fixată, dată de ecuația mișcării staționare). După ce se calculează variația întîi a funcționalei, $\delta_u J$, se va putea face $u^0 = u$.

Se poate arăta, și este necesar acest lucru, deoarece s-a admis o aproximație (abatere mică) care permite scoaterea operatorului $\partial/\partial t$ în fața integralei, că ecuația extremalei u pentru $J(u\,;\, u^0)$ este chiar (0.1). În acest scop se poate nota cu Φ funcția de sub semnul integralei duble și dovedi ușor că ecuația Euler-Lagrange

$$\Phi_u - (\Phi_{u_x})_x - (\Phi_{u_\psi})_\psi = 0 \tag{5}$$

se reduce la (0.1), la ecuația mișcării staționare.

2. Soluție de aproximație. Ecuația mișcării (0.1) este de natură parabolică. Cu transformarea $u = (u_\infty^2 - G)^{1/2}$ ecuația (0.1) devine $G_x = \mu u G_{\psi\psi}$. Se observă că pentru $u \to u_\infty$ (spre frontiera exterioară a stratului limită) ecuația mișcării se poate aproxima prin ecuația simplă a propagării căldurii ($\mu u_\infty = \text{const.}$):

$$G_x \approx \mu u_\infty \left(1 - \frac{G}{2U_\infty^2}\right) G_{\psi\psi}, \quad G_x = \mu u_\infty G_{\psi\psi} \text{ cu } \psi \to \psi_\infty$$

Folosind această proprietate, încercăm pentru problema de staționaritate (minimizare) a funcționalei $J(u\,;u^0)$ o soluție de aproximație de forma

$$u = u_\infty \, erf\left(\frac{\sqrt{\psi}}{2} z(x)\right) = \frac{2u_\infty}{\sqrt{\pi}} \int_0^{z\sqrt{\psi}/2} e^{-\lambda^2} d\lambda \tag{6}$$

$$\left(u_0 = u_\infty \; erf\left[\frac{\sqrt{\psi}}{2} z_0(x)\right]\right)$$

unde $z(x)$ este noua funcție necunoscută cu proprietățile

$$z : (0, L) \to R^1, \; z \in C^1(0, L], \; \lim_{x \to 0} z(x) = \infty \tag{7}$$

Funcția $z(x)$ se va determina cu îndeplinirea condiției de staționaritate a potențialului local $J(u\,;u^0)$, care devine o funcțională pe spațiul liniar al funcțiilor $z(x)$ și $z^0(x)$. Deci, avem condiția

$$J(z\,;z^0) = J_1 + J_2 + J_3 + J_4 + J_c = \text{minim}, \text{ (cu } z^0 \text{ dat)} \tag{7'}$$

unde J_k sînt cele patru integrale din (4) iar J_c este integrala curbilinie din (4).

Fie $\delta_z J(z\,;z^0)$ prima variație a funcționalei $J(z\,;z^0)$ în raport cu funcția $z(x)$; funcția $z^0(x)$ nu are variație. Din (7') rezultă condiția

$$\delta_z J(z\,;z^0) = \sum_{k=1}^{4} \delta_z J_k(z\,;z^0) + \delta_z J_c = 0, \quad \text{cu} \quad z^0 = z \tag{8}$$

care se va transforma, după calculul variațiilor $\delta_z J_k$, într-o ecuație diferențială pentru funcția $z(x)$.

Avem

$$J_1 = -\rho \iint_\Omega u^{0^2} \frac{\partial u}{\partial x} dx\, d\psi =$$

$$= -\frac{\rho u_\infty^3}{\sqrt{\pi}} \int_0^L \int_0^\infty \sqrt{\psi}\, erf^2 (z^0 \sqrt{\psi}/2)\, z' e^{-z^2\psi/4} dx\, d\psi$$

Folosind

$$\delta(z' e^{-z^2\psi/4}) = \left(\delta z' - \frac{\delta z}{2} zz' \psi\right) e^{-z^2\psi/4}, \; z' = \frac{dz}{dx}$$

cu o integrare prin părți și aplicarea, apoi, a condiției $z^0 = z$ găsim

$$\delta J_1(z) = \delta^{(1)} J_1(z) + T_1 \tag{9}$$

unde

$$\delta^{(1)} J_1(z) = \frac{64\,\rho u_\infty^3}{\pi} K_1 \int_0^L \frac{z'}{z^4}\,\delta z\,\delta x,$$

$$K_1 = \int_0^\infty \lambda^3 e^{-2\lambda^2} erf\,\lambda\,d\lambda = -\frac{1}{4}\int_0^\infty \lambda^2 erf\,\lambda\,d(e^{-2\lambda^2}) = \frac{1}{6\sqrt{3}} = 0{,}096225,$$

$$T_1 = -\frac{\rho u_\infty^3}{\sqrt{\pi}}\left(\delta z \cdot \int_0^\infty \sqrt{\psi}\,e^{-\lambda^2} erf^2\,\lambda^0\,d\psi\right)_0^L.$$

Se poate arăta că

$$\delta J_2(z) = -\frac{\rho u_\infty^3}{\pi}\iint_\Omega z'\,\psi\,e^{-2\lambda^2} erf\,\lambda\,\delta z\,dx\,d\psi = -\frac{1}{2}\,\delta^{(1)} J_1(z);$$

$$\delta J_3(z) = \frac{\mu u_\infty^4}{4\pi\sqrt{\pi}}\int_0^L\int_0^\infty z^2 e^{-3\lambda^2} erf\,\lambda\,\delta z\,dx\,d\psi = \frac{\pi u_\infty^4}{\pi\sqrt{\pi}} K_3 \int_0^L z(x)\,\delta z\,dx$$

unde

$$K_3 = \int_0^\infty e^{-3\lambda^2} erf\,\lambda\,d\lambda = \frac{2}{\sqrt{\pi}}\sum_{n=0}^\infty \frac{(-1)^n}{n!(2n+1)}\int_0^\infty \lambda^{2n+1} e^{-3\lambda^2}\,d\lambda;$$

$$K_3 = \frac{2}{\sqrt{\pi}} K_3' = \frac{2}{\sqrt{\pi}}(0{,}16666 - 0{,}01852 + 0{,}00370 - 0{,}00090 +$$

$$+ 0{,}00023 - 0{,}00006) = \frac{2}{\sqrt{\pi}} \cdot 0{,}15114$$

Să trecem la integrala J_4. Avem

$$J_4 = \mu\iint_\Omega u^{0^2}\frac{\partial u^0}{\partial\psi}\frac{\partial u}{\partial\psi}\,dx\,d\psi = \frac{\mu u_\infty}{4\pi}\iint_\Omega\left(\frac{z^0}{\psi}\,e^{-\lambda^{0^2}} erf^2\,\lambda^0\right) z\,e^{-\lambda^2}\,dx\,d\psi$$

Variaţia acestei funcţionale este

$$\delta J_4(z) = \frac{\mu u_\infty^4}{4\pi}\int_0^L\int_0^\infty z\,e^{-2\lambda^2}(1 - 2\lambda^2)\,erf^2\,\lambda\,\delta z\,dx\,\frac{d\psi}{\psi} = \frac{\mu u_\infty^4}{2\pi} K_4 \int_0^L z(x)\,\delta z\,dx$$

unde

$$K_4 = \int_0^\infty \left(\frac{1}{\lambda} - 2\lambda\right) e^{-2\lambda^2} erf^2\,\lambda\,d\lambda \qquad (10)$$

Aici, pentru calculul integralei, determinăm dezvoltarea

$$erf^2 \lambda = \frac{4}{\pi} \sum_{k=1}^{11} c(2k) \cdot \lambda^{2k}$$

unde

$$c(2) = 1 \,;\; c(4) = -0{,}666666 \,;\; c(6) = 0{,}311111 \,;\; c(8) = -0{,}114286 \,;$$
$$c(10) = 0{,}035132 \,;\; c(12) = -0{,}009363 \,;\; c(14) = 0{,}002211$$

Avem

$$K_4 = \frac{4}{\pi} \sum_{k=1}^{11} c(2k) \left[I(2k-1) - 2I(2k+1) \right]$$

unde

$$I(2k+1) = \int_0^\infty \lambda^{2k+1} e^{-2\lambda^2} d\lambda = \frac{k}{2} I(2k-1), \qquad k = \overline{1,11}$$

şi

$$I(1) = \frac{1}{4} \,;\; I(3) = \frac{1}{8} \,;\; I(5) = \frac{1}{8} \,;\; I(7) = \frac{3}{16} \,;\; I(9) = \frac{3}{8} \,;$$
$$I(11) = \frac{15}{16} \,;\; I(13) = \frac{45}{16} \,;\; I(15) = \frac{315}{32} \,;\; \ldots$$

Se alege valoarea aproximativă

$$K_4 = \frac{4}{\pi} K_4' = \frac{4}{\pi} [0{,}083333 - (0{,}077777 - 0{,}064286) -$$
$$- (0{,}052698 - 0{,}043889) - \ldots] \approx \frac{4}{\pi} \, 0{,}061033$$

Integrala curbilinie J_c, din (4), conduce, deoarece frontiera $x = L$ este liberă, la o variaţie $\delta J_c(z\,;z^0)$ care, însă, se anulează cu termenul la limită T_1 din expresia variaţiei $\delta J_1(z\,;z^0)$.

— Să revenim la formula (8) a variaţiei totale, în care se înlocuiesc toate variaţiile calculate mai sus. Condiţia (8) se reduce, atunci, la egalitatea

$$\delta J(z) = \frac{2U_\infty^3}{\pi} \int_0^L \left(16\,\rho\, K_1 \frac{z'}{z^4} + \frac{\mu U_\infty}{\pi} K z \right) \delta\, z(x)\, dx = 0 \tag{11}$$

$$(K = K_3' + K_4') \tag{12}$$

din care rezultă, $\delta z(x)$ fiind variaţie arbitrară, ecuaţia diferenţială a funcţiei necunoscute $z(x)\,[z \to \infty,\; x \to 0]$:

$$\frac{dz}{z^4} + \frac{\mu U_\infty}{16\pi\rho} \frac{K}{K_1} z(x) = 0 \Rightarrow z(x) = \left(\frac{4\pi\rho}{\mu U_\infty} \frac{K_1}{K} \frac{1}{x} \right)^{1/4} \tag{13}$$

Soluţia aproximativă, distribuţia aproximativă a vitezei, în problema lui Mises este

$$u(x, \psi) = u_\infty \, erf\left(\sqrt{\psi}\sqrt[4]{\frac{\pi}{4}\frac{K_1}{K}\frac{1}{\nu u_\infty x}}\right) = u_\infty \, erf\left(0{,}772\,\frac{\sqrt{\psi}}{\sqrt[4]{\nu U_\infty x}}\right) \tag{14}$$

— *Verificare.* Pentru verificarea exactităţii formulei aproximative (14) facem comparaţie cu soluţia lui Blasius [2] şi soluţia obţinută cu o metodă variaţională în planul fizic Oxy [1]. În acest scop introducem variabila lui Blasius $\eta = y\sqrt{u_\infty/\nu x}$ şi funcţia vitezei $f'(\eta) = u/u_\infty$. Pentru funcţia de curent ψ avem formula

$$\frac{u}{u_\infty} = f'(\eta) = \frac{1}{u_\infty}\sqrt{\frac{\rho u_\infty}{\mu x}} \Rightarrow \psi = \sqrt{\nu u_\infty x}\, f(\eta) \tag{a}$$

Prin urmare, formula (14) poate fi pusă în forma

$$\frac{u}{u_\infty} = erf\left(0{,}772\sqrt{f(\eta)}\right) \tag{b}$$

Formăm, acum, tabelul 1 în care este trecută soluţia lui Blasius şi valorile date de (b) ca şi soluţia aproximativă din planul coordonatelor (x, y) [1]. Valorile date de (b), care corespunde la (14), sînt în acord cu soluţia lui Blasius.

Tabel 1

η	$f(\eta)$	$f'(\eta) = u/u_\infty$	u/u_∞ (14)	u/u_∞ [1]
0,0	0	0	0	0
0,2	0,00664	0,06641	0,06762	0,0705
0,6	0,05974	0,19894	0,2009	0,209
1,0	0,16557	0,32979	0,3389	0,342
1,6	0,42032	0,511676	0,5205	0,521
2,0	0,65003	0,62977	0,6194	0,624
3,0	1,39682	0,84605	0,8019	0,816

— *Formula tensiunii de frecare.* Tensiunea de frecare a fluidului vîscos pe suprafaţa plăcii, $\tau_w(x)$, este dată de formula lui Newton

$$\tau_w(x) = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu\left(u\,\frac{\partial u}{\partial \psi}\right)_{\psi=0} \tag{15}$$

Se deduce, de aici, imediat că

$$\tau_w(x) = \frac{\mu u_\infty^2}{2\pi} z^2(x)\left[\frac{1}{\lambda e^{\lambda^2}}\int_0^\lambda e^{-\alpha^2}\, d\alpha\right]_{\lambda=0} = \frac{\mu u_\infty^2}{2\pi} z^2(x)$$

Formula tensiunii de frecare este

$$\tau_w(x) = \mu u_\infty \sqrt{\frac{K_1}{\pi K}} \sqrt{\frac{u_\infty}{\nu x}} = 0{,}379 \, \mu u_\infty \sqrt{\frac{u_\infty}{\nu x}} \tag{16}$$

Metoda lui Blasius introduce factorul 0,332. Procedeul variațional descris mai sus dă un rezultat, în problema determinării frecării, mai slab decît procedeul numeric al lui Blasius.

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A VARIATIONAL METHOD WITH LOCAL POTENTIAL FOR THE MISES PROBLEM OF THE HYDRODYNAMIC BOUNDARY LAYER OVER A FLAT PLATE

(Summary)

A variational method is used to solve the nonlinear boundary value problem of the incompressible boundary layer, in terms of the Mises coordinates, which forms over the flat plate. We have a problem which cannot be described by self-adjoint differential equations.

We start from the equation of the unsteady motion in order to construct a local potential (the functional (4)). This functional decreases with time and assumes a minimum value for the steady motion. The motion equation (0.1) is of parabolic type. Using these properties for the minimization problem of the functional we try the solution (6) which introduces the unknown function $z(x)$. The stationarity condition of the functional leads to the differential equation (13) and to the function $z(x)$, (13). For the approximate distribution of the flow velocity the formula (14) is obtained. In order to verify the exactitude of the approximate solution (14) we make a comparison with Blasius solution (Table 1). A formula, (16), for the skin friction on the surface of the plate is given. This formula has an accuracy smaller than the Blasius formula.

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OPERATIONS AVEC DES INTERVALLES DE CONFIANCE

E. OANCEA, M. RĂDULESCU

Dans une recherche statistique relative à deux caractéristiques statistiques X et Y, on se pose le problème de déterminer des intervalles de confiance correspondant respectivement à la variable aléatoire $X+Y$, XY ou X/Y. Aux caractéristiques statistiques X, Y correspondent, du point de vue théorique, des variables aléatoires X, Y. Le problème qu'on se pose est de déterminer un intervalle de confiance pour la variable aléatoire $X+Y$, XY, X/Y, en connaissant préalablement des intervalles de confiance pour X et Y.

1. L'opération somme. Soient X, Y des variables aléatoires indépendentes, continues avec la densité de probabilité respective f_1, f_2 et les intervalles de confiance (α_1, β_1) et (α_2, β_2) donnés par:

$$\left.\begin{aligned} P(X \in (\alpha_1, \beta_1)) &= 1 - q_1 \\ P(Y \in (\alpha_2, \beta_2)) &= 1 - q_2 \end{aligned}\right\} \qquad (1)$$

q_1, q_2 fixés préalablement.

On cherche un intervalle de confiance pour la variable aléatoire somme $Z = X + Y$ en utilisant l'intervalle de confiance pour X respectivement pour Y.

Quand on peut déterminer la densité de probabilité

$$f_z(z) = \int_R f_1(x) f_2(z - x)\, dx,$$

alors on a un intervalle de confiance pour Z donné par:

$$P(Z \in (\alpha, \beta)) = 1 - q, \quad q = \min(q_1, q_2).$$

Mais notre problème s'agit du cas où on ne peut pas déterminer f_z. Alors on choisit pour Z un intervalle de confiance $I_q = (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$ pour lequel on a:

$$P(Z \in I_q) \geqslant P(X \in (\alpha_1\, \beta_1) \cap Y \in (\alpha_2, \beta_2)) = (1 - q_1)(1 - q_2) = 1 - q, \quad (2)$$

c'est-à-dire $-q = q_1 q_2 - (q_1 + q_2)$.

Cet intervalle de confiance peut être toujours déterminé, mais généralement il n'est pas optimal.

On réalise l'amélioration de l'intervalle de confiance associé à Z soit par:

a) la majoration de la probabilité de confiance $1 - q$

b) la diminution de la longueur de l'intervalle de confiance I_q

a) Dans ce but, on choisit $0 < \bar{q} < q$ et l'intervalle

$$I_q(z_m - kh,\ z_m + kh)$$

où

$$z_m = \frac{\alpha_1 + \alpha_2 + \beta_1 + \beta_2}{2}$$

$h \in R_+$ suffisamment petit ainsi que

$$\begin{aligned} &P(X \in (x_m, x_m + h)) \simeq hf_1(x_m) \\ &P(Y \in (y_m, y_m + h)) \simeq hf_2(y_m) \\ &x_m = \frac{\alpha_1 + \beta_1}{2}, \qquad y_m = \frac{\alpha_2 + \beta_2}{2} \end{aligned} \tag{3}$$

On évalue la probabilité

$$P(Z \in (z_m, z_m + kh)), \quad k \in N \tag{4}$$

par la discrétisation:

$$P(Z \in (z_m + \overline{i-1}\, h,\ z_m + ih)) = \iint_{D_i} f_1(x) f_2(y)\, dx\, dy =$$

$$= \int_{x_m}^{x_m+h} f_1(x)\, dx \int_{z_m+\overline{i-1}h-x_m}^{z_m+ih-x_m} f_2(y)\, dy \simeq hf_1(x_m) \int_{y_m+\overline{i-1}h}^{y_m+ih} f_2(y)\, dy$$

où

$$D_i = \{(x, y),\ x_m \leqslant x \leqslant x_m + h,\ z_m + \overline{i-1}h \leqslant x + y \leqslant z_m + ih\}$$

Par conséquent

$$P\Big(Z \in (z_m, z_m + kh) \simeq P\Big[\bigcup_{j=0}^{k-1} Z \in (z_m + jh,\ z_m + \overline{j+1}h)\Big)\Big] =$$

$$= \sum_{j=0}^{k-1} hf_1(x_m) \int_{y_m+jh}^{y_m+\overline{j+1}h} f_2(y)\, dy = hf_1(x_m) \int_{y_m}^{y_m+kh} f_2(y)\, dy \tag{5}$$

De façon analogue

$$P(Z \in (z_m - kh,\ z_m)) \simeq hf_1(x_m) \int_{y_m-kh}^{y_m} f_2(y)\, dy \tag{6}$$

Alors

$$\begin{aligned} P(Z \in (z_m - kh,\ z_m + kh)) &= P(Z \in (z_m - kh,\ z_m)) + \\ &+ P(Z \in (z_m, z_m + kh)) = 1 - \bar{q} \end{aligned} \tag{7}$$

En utilisant les évaluations (5) et (6) on a

$$hf_1(x_m) \int_{y_m-kh}^{y_m+kh} f_2(y)\, dy = 1 - \bar{q} \tag{8}$$

d'où on peut déterminer k. C'est à dire, de (8) il résulte

$$hf_1(x_m)[F_2(y_m + kh) - F_2(y_m - kh)] = 1 - \bar{q} \tag{9}$$

F_2 étant la fonction de répartition de Y, ou

$$F_2(y_m + kh) - F_2(y_m - kh) = \frac{1 - \bar{q}}{hf_1(x_m)} \tag{10}$$

En considérant la valeur $F_2(y_m)$ de la fonction de répartition F_2 (on sait que pour les lois les plus usuelles de probabilité on a des tableaux), on détermine un intervalle approximativement centré sur $F_2(y_m)$

$$(F_2(y_m - \gamma),\ F_2(y_m + \gamma))$$

pour lequel on a la relation (10). Alors, en notant

$$\gamma = hk$$

il résulte k à qui on attribue la valeur

$$k = \left[\frac{\gamma}{h}\right].$$

b) Dans ce cas, on cherche la diminution de la longueur de l'intervalle de confiance initial I_q. Après l'évaluation de la probabilité (7) on choisit $\gamma \doteq kh$, h fixé préalablement en conformité à (3) ainsi que l'intervalle $I_{\bar{q}}$ vérifie la condition

$$2kh = \|I_q\| < \|I_{\bar{q}}\|$$

après quoi on détermine $\bar{q}$ de l'égalité

$$1 - [F_2(y_m + kh) - F_2(y_m - kh)]\, hf_1(x_m) = \bar{q}.$$

Remarques: 1) On peut procéder de manière analogue pour déterminer la valeur de k respectivement $\bar{q}$, en utilisant l'évaluation

$$hf_2(y_m) \int_{x_m - kh}^{x_m + kh} f_1(x)\, dx = 1 - \bar{q}$$

2) On peut choisir h la valeur qui représente le pas des tableaux des fonctions de répartition F_1, F_2 ou f_1, f_2 si celles-ci sont les usuelles (de la lois N, χ^2, Student).

3) Il est possible quelque fois d'améliorer par le procédé donné les deux sens: aussi la longueur de I_q ainsi que la probabilité de confiance q.

4) Le procédé donné est valable aussi dans le cas d'une somme de $n > 2$ variables aléatoires indépendantes, en l'appliquant consécutivement pour $X_1 + X_2$ et X_3 etc.

5) Le procédé peut être utilisé aussi pour

$$Z = aX + bY.$$

par l'amplification de l'intervalle de confiance pour de X et Y respectivement avec a et b.

6) On peut aussi choisir l'intervalle de confiance pour Z de la forme

$$I_{1\bar{q}} = (\alpha_1 + \alpha_2;\ \alpha_2 + \beta_1 + k_1),\ 0 \leqslant k_1 \leqslant \beta_2 - \alpha_2$$

où

$$I_{2\bar{q}} = (\alpha_1 + \alpha_2 + k_2;\ \beta_1 + \beta_2),\ 0 \leqslant k_2 \leqslant \beta_1 - \alpha_1$$

7) Si les variables aléatoires X, Y ne sont pas indépendantes, mais de type continu, pour déterminer un intervalle de confiance de $X + Y$ amélioré conformément au procédé donné; il est possible, si on connaît aussi la densité de probabilité du vecteur (X, Y).

8) Le procédé présenté est utile dans le cas où les variables aléatoires de la somme $X + Y$ sont de lois de probabilité de classes différentes, par exemple: X est $N(m, \sigma)$ et Y est $\chi^2(s, \sigma)$. Dans le cas où les deux variables appartiennent à la même classe, quelquefois il existe la propriété de stabilité et alors la densité de probabilité de la somme $X + Y$ est connue.

9) On observe que le procédé d'évaluation de la probabilité (4) est applicable généralement si les fonctions f_1, f_2 sont continues, ce qui arrive toujours pour les lois plus usuelles de probabilité.

2. L'opération produit. Soit X, Y les variables aléatoires du point 1, indépendantes, avec les densités de probabilité f_1, f_2 et les intervalles de confiance (1).

Le problème d'un intervalle de confiance pour $Z = XY$, si on peut en déterminer la densité de probabilité

$$f_z(z) = \int_R f_1(u) f_2\left(\frac{z}{u}\right) \frac{du}{u}$$

est simple, on a

$$P(Z \in (\alpha, \beta)) = 1 - q,\quad q = \min(q_1, q_2)$$

Dans le cas où f_z n'est pas connue on procède comme il suit:
On considère pour Z l'intervalle $I_q = (\alpha, \beta)$ où

$$\alpha = \min\{\alpha_1\alpha_2,\ \beta_1\beta_2\}$$

$$\beta = \max\{\alpha_1\alpha_2,\ \beta_1\beta_2\}.$$

Mais cet intervalle n'est pas toujours optimal, alors pour l'améliorer on peut:

a) majorer la probabilité de confiance $1 - q$ où

$$1 - q = (1 - q_1)(1 - q_2) = P(X \in (\alpha_1, \beta_1) \cap Y \in (\alpha_2, \beta_2))$$

b) diminuer la longueur de l'intervalle I_q.

a) On choisit $\bar{q} < q$ et l'intervalle

$$I_{\bar{q}} = (z_m - kh,\ z_m + kh)$$

où

$$z_m = \frac{\alpha_1\alpha_2 + \beta_1\beta_2}{2}$$

et h en conformité à (3) et

$$x_m = \frac{\alpha_1\alpha_2 + \beta_1\beta_2}{\alpha_2 + \beta_2}, \qquad y_m = \frac{\alpha_2 + \beta_2}{2}$$

ainsi que $z_m = x_m y_m$.

Alors

$$P(Z \in (z_m, z_m + kh)) = P\left[\bigcup_{j=0}^{k-1} Z \in (z_m + jh,\ z_m + \overline{j+1}h))\right] =$$

$$= \sum_{j=0}^{k-1} \iint_{D_j} f_1(x) f_2(y)\, dx\, dy,$$

où

$$D_j = \{(x, y)\ x_m \leqslant x \leqslant x_m + h,\ z_m + jh \leqslant xy \leqslant z_m + \overline{j+1}h\}$$

et

$$\iint_{D_j} f_1(x) f_2(y)\, dx\, dy = \int_{x_m}^{x_m+h} f_1(x)\, dx \int_{\frac{z_m+jh}{x}}^{\frac{z_m+\overline{j+1}h}{x}} f_2(y)\, dy \tag{11}$$

De (3) et (11) il résulte

$$P(Z \in (z_m, z_m + kh)) \simeq hf_1(x_m) \int_{y_m}^{y_m + \frac{kh}{x_m}} f_2(y)\, dy$$

et de façon analogue

$$P(Z \in (z_m - kh, z_m)) \simeq hf_1(x_m) \int_{y_m - \frac{kh}{x_m}}^{y_m} f_2(y)\, dy$$

Parce que

$$P(Z \in (z_m - kh,\ z_m + kh) = 1 - \bar{q} \tag{12}$$

on a

$$hf_1(x_m)\left[F_2\left(y_m + \frac{kh}{x_m}\right) - F_2\left(y_m - \frac{kh}{x_m}\right)\right] = 1 - \bar{q}$$

ou

$$F_2(y_m + \gamma) - F_2(y_m - \gamma) = \frac{1 - q}{hf_1(x_m)}$$

où $\gamma = \frac{kh}{x_m}$ et donc $k = \left[\frac{\gamma x_m}{h}\right]$.

b) Pour diminuer l'intervalle de confiance initial I_q après l'évaluation (12), on choisit $\gamma = kh$ et h fixé conformément à (3) et pour qu'on ait

$$2kh = \|I_{\bar{q}}\| < \|I_q\|$$

et on détermine $\bar{q}$ de l'égalité

$$1 - \left[F_2\left(y_m + \frac{kh}{x_m}\right) - F_2\left(y_m - \frac{kh}{x_m}\right)\right] hf_1(x_m) = \bar{q}$$

Les remarques 1, 2, 3, 7 et 9 du point 1 restent, par analogie, valables aussi dans ce cas.

3. L'opération quotient. Soit les variables X, Y indépendantes, comme celles du point 1 et les intervalles de confiance (1) si on peut déterminer la densité de probabilité de la variable aléatoire $Z = \frac{X}{Y}$, $Y \neq 0$,

$$f_z(z) = \int_R f_1(u) f_2(zu) \, |u| \, du$$

alors un intervalle de confiance pour Z est donné par :

$$P(Z \in (\alpha, \beta)) = 1 - q, \; q = \min(q_1, q_2)$$

Mais dans le cas où il n'est pas possible de déterminer f_z, on procède comme il suit :

On considère pour Z l'intervalle $I_q = (\alpha, \beta)$ où

$$\alpha = \min\left(\frac{\alpha_1}{\alpha_2}, \frac{\beta_1}{\beta_2}\right)$$

$$\beta = \max\left(\frac{\alpha_1}{\alpha_2}, \frac{\beta_1}{\beta_2}\right)$$

avec $\alpha_2 \neq 0$, $\beta_2 \neq 0$. Parce que $Y \neq 0$ il résulte que l'intervalle (α_2, β_2) ne peut par contenir l'origine, c'est à dire soit $0 < \alpha_2 < \beta_2$ soit $\alpha_2 < \beta_2 < 0$.

De même que dans les cas précédents, pour améliorer l'intervalle de confiance, on peut soit :

a) majorer la probabilité de confiance $1 - q$

b) diminuer la longueur de l'intervalle I_q.

a) On choisit $\bar{q} < q$ et l'intervalle $I_{\bar{q}}$:

$$(z_m - kh, z_m + kh)$$

où

$$z_m = \frac{1}{2}\left(\frac{\alpha_1}{\alpha_2} + \frac{\beta_1}{\beta_2}\right) = \frac{x_m}{y_m},$$

$$x_m = \left(\frac{\alpha_1}{\alpha_2} + \frac{\beta_1}{\beta_2}\right)\frac{1}{\alpha_2 + \beta_2}$$

$$y_m = \frac{2}{\alpha_2 + \beta_2}$$

et h en conformité à (3).

En considérant le cas où $y < 0$, on évalue la probabilité

$$P(Z \in (z_m, z_m + kh) = P\left(\bigcup_{j=0}^{k-1} Z \in (z_m + jh,\ z_m + \overline{j+1}h)\right) =$$

$$= \sum_{j=0}^{k-1} \iint_{D_j} f_1(x)\, f_2(y)\, dx\, dy$$

où

$$D_j = \left\{(x, y) \middle| y_m \leqslant y \leqslant y_m + h,\ z_m + jh \leqslant \frac{x}{y} \leqslant z_m + \overline{j+1}h\right\}$$

Alors

$$\iint_{D_j} f_1(x)\, f_2(y)\, dx\, dy = \int_{y_m}^{y_m+h} f_2(y)\, dy \int_{y(z_m+jh)}^{y(z_m+\overline{j+1}h)} f_1(x)\, dx$$

et

$$P(Z \in (z_m, z_m + kh)) \simeq hf_2(y_m) \int_{x_m}^{x_m+khy_m} f_1(x)\, dx$$

De façon analogue

$$P(Z \in (z_m - kh, z_m) \simeq hf_2(y_m) \int_{x_m-khy_m}^{x_m} f_1(x)\, dx$$

Donc l'égalité

$$P(Z \in (z_m - kh, z_m + kh)) = 1 - \bar{q}$$

devient :

$$hf_2(y_m)\,[F_1(x_m + khy_m) - F_1(x_m - khy_m)] = 1 - \bar{q} \qquad (13)$$

où

$$F_1(x_m + \gamma) - F_1(x_m - \gamma) = \frac{1-q}{hf_2(y_m)}$$

donc

$$\gamma = khy_m, \qquad k = \left[\frac{\gamma}{y_m h}\right]$$

b) Pour diminuer l'intervalle de confiance I_q après l'évaluation (13), on choisit $\gamma = kh$, h fixé comme dans (3), ainsi que

$$2kh = \|I_{\bar q}\| < \|I_q\|$$

et on détermine $\bar q$ de l'égalité

$$\bar q = 1 - hf_2(y_m)\,[F_1(x_m + khy_m) - F_1(x_m - khy_m)]$$

(Manuscrit reçu le 20 septembre 1981)

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OPERAŢII CU INTERVALE DE ÎNCREDERE

(Rezumat)

Într-o cercetare statistică relativ la două caracteristici statistice X şi Y se pune problema determinării unor intervale de încredere corespunzătoare variabilelor aleatoare $X+Y$, XY sau X/Y. În prezenta notă se dă un mod de determinare a acestor intervale în cazul în care X şi Y sînt variabile aleatoare de tip continuu şi în general, cînd X şi Y nu verifică aceeaşi lege de probabilitate.

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ON THE TRANSITIVITY OF AFFINE STRUCTURES

MIHAELA BERINDEANU-BÁNYAI

In the paper [2] F. Rado has proved that a planar affine Barbilian structure is L-transitive for any line L, whenever it is L_1-, L_2- and L_3-transitive, where L_i are cross lines two by two.

In the present note we shall extend this result for an arbitrary rudimentary affine plane. To this effect we have to admit two supplementary hypotheses:

1) there exists a line L_4 such that the six pairs L_i, L_j $(i \neq j)$ are cross lines;

2) for any line L, there are two lines L' and L'' such that L, L'; L, L''; L', L'' are cross lines.

Nothing about non-neighbouring pairs of points is supposed.

A rudimentary affine structure is a system $(\mathfrak{P}, \mathfrak{L}, \|)$, where $\mathfrak{P}$ is a set, $\mathfrak{L}$ is a non-empty family of non-empty subsets of $\mathfrak{P}$ and $\|$ is an equivalence relation on $\mathfrak{L}$, satisfying the following condition

$$\text{(E)}\ \forall p \in \mathfrak{P}, \forall L \in \mathfrak{L},\ \exists_1 L' \in \mathfrak{L} : p \in L' \| L.$$

The elements of $\mathfrak{P}$ will be called *points*, the elements of $\mathfrak{L}$ *lines*; if $L_1 \| L_2$, then L_1 and L_2 are said to be *parallel*; the line L' occuring in (E), determined uniquely by p and L, is denoted by $(p\|L)$. The lines L_1 and L_2 are said to be *cross lines* and denoted by $L_1 © L_2$ if any parallel line to L_1 intersects any parallel line to L_2 in exactly one point.

A map $\delta : \mathfrak{P} \to \mathfrak{P}$ is called a *dilatation* if $q \in (p\|L) \Rightarrow q^\delta \in (p^\delta\|L)$.

A bijection $\varphi : \mathfrak{P} \to \mathfrak{P}$ is called an *automorphism* of $(\mathfrak{P}, \mathfrak{L}, \|)$ if for each $S \subseteq \mathfrak{P}$ and each $L_1, L_2 \in \mathfrak{L}$

$$S \in \mathfrak{L} \Leftrightarrow \varphi(S) \in \mathfrak{L}$$

$$L_1\|L_2 \Leftrightarrow \varphi(L_1)\|\varphi(L_2).$$

The set of automorphismus of $(\mathfrak{P}, \mathfrak{L}, \|)$ is a group under composition denoted by Aut $(\mathfrak{P}, \mathfrak{L}, \|)$. Let $L \in \mathfrak{L}$; if $\tau \in$ Aut $(\mathfrak{P}, \mathfrak{L}, \|)$ is also a dilatation and $\tau(L') = L'$ for any $L'\|L$, then τ is called a *trace-translation* or, more precisely, an *L-translation*. If for all $p, q \in L$ there exists an *L-translation* such that $\tau(p) = q$, then $(\mathfrak{P}, \mathfrak{L}, \|)$ is said to be *L-transitive*.

Let $\mathbf{T}(L)$ be the set of all L-translations. Then $\mathbf{T}(L)$ is a subgroup of Aut $(\mathfrak{P}, \mathfrak{L}, \|)$. The subgroup of Aut $(\mathfrak{P}, \mathfrak{L}, \|)$ generated by all trace-translations is called the *translation group of* $(\mathfrak{P}, \mathfrak{L}, \|)$ and is denoted by $\mathbf{T}(\mathfrak{P}, \mathfrak{L}, \|)$; its elements are called *translations*.

In the paper [1] the following results are proved:

THEOREM 1. *Suppose that the rudimentary affine structure* $(\mathfrak{P}, \mathfrak{L}, \|)$ *is* L_1 *and* L_2*-transitive, where* $L_1 © L_2$ *and that there exists* L_3 *such that* $L_1 © L_3$, $L_2 © L_3$. *Then*

i) $\forall \tau_1 \in \mathbf{T}(L_1)$, $\forall \tau_2 \in \mathbf{T}(L_2) : \tau_1\tau_2 = \tau_2\tau_1$.

ii) $\mathbf{T}(L_1) \cdot \mathbf{T}(L_2)$ *is a group acting sharply simply transitive on the set* $\mathfrak{P}$.

THEOREM 2. *Admit the hypotheses of Theorem 1 and that* $\mathbf{T}(L_1)$ *is Abelian. Then*

i) $\mathbf{T}(L_1) \cdot \mathbf{T}(L_2)$ *is Abelian.*

ii) *If for any line* L, *there exist lines* L' *and* L'' *such that* $L \circledcirc L' \circledcirc L'' \circledcirc L$, *then for any* $L \in \mathfrak{L}$, $\mathbf{T}(L)$ *is sharply simply transitive on* L, *and the translation group* $\mathbf{T}(\mathfrak{A}, \mathfrak{L}, \|)$ *coincides with* $\mathbf{T}(L_1) \cdot \mathbf{T}(L_2)$.

In what follows we shall establish the following:

THEOREM 3. *Let* $(\mathfrak{A}, \mathfrak{L}, \|)$ *be a rudimentary affine structure and* L_1, L_2, L_3, L_4 *cross lines, two by two, such that* $(\mathfrak{A}, \mathfrak{L}, \|)$ *is* L_1-L_2-L_3-*transitive. Then*

i) $\mathbf{T}(L_i)$ *is Abelian for* $i = 1, 2, 3, 4$.

ii) *If for any* $L \in \mathfrak{L}$ *there exist* L' *and* L'' *such that* $L \circledcirc L' \circledcirc L'' \circledcirc L$, *then* $\mathbf{T}(L)$ *is sharply transitive on* L *for every line* L *and the translation group* $\mathbf{T}(\mathfrak{A}, \mathfrak{L}, \|)$ *coincides with* $\mathbf{T}(L_1) \cdot \mathbf{T}(L_2)$.

Proof: We may admit without loss of generality that L_1, L_2, L_3, L_4 have a common point o. Consider $p \in L_4$ and $p_2 := (p\|L_1) \cap L_2$, $p_1 : (p\|L_2) \cap L_1$. By hypothesis there exist $\tau_1 \in \mathbf{T}(L_1)$ and $\tau_2 \in \mathbf{T}(L_2)$ such that $\tau_1(o) = p_1$ $\tau_2(o) = p_2$. Denote $\tau := \tau_1\tau_2$; by theorem 1 $\tau = \tau_2\tau_1$. We have $\tau(o) = \tau_1\tau_2(o) = p$ and $\tau(L_4) = \tau(o\|L_4) = (o^{\tau}\|L_4) = (p\|L_4) = L_4$. Let $H\|L_4$ and prove $\tau(H) = H$. Let $q := H \cap L_3$. We know that there exists $\tau_3 \in \mathbf{T}(L_3)$ such that $\tau_3(o) = q$. Since $H = (q\|L_4)$, it follows that $\tau(H) = \tau(q\|L_4) = (q^{\tau}\|L_4)$. We have $\tau(q) = \tau_1\tau_2\tau_3(o) = \tau_3\tau_1\tau_2(o) = \tau_3(p)$. On the other hand $\tau_3 \in \mathbf{T}(L_3)$ implies

$$\tau_3(L_4) = \tau_3(o\|L_4) = (o^{\tau_3}\|L_4) = (q\|L_4) = H;$$

thus $\tau_3(p) \in H$, which implies

$$\tau(H) = \tau(q\|L_4) = (q^{\tau}\|L_4) = (p^{\tau_3}\|L_4) = H.$$

Hence $(\mathfrak{A}, \mathfrak{L}, \|)$ is L_4-transitive and in view of Theorem 1, ii each $\tau \in \mathbf{T}(L_4)$ can be written as $\tau_2\tau_1$ with $\tau_i \in \mathbf{T}(L_i)$, $i = 1, 2$.

Now we prove that $\mathbf{T}(L_4)$ is Abelian. Let $\tau_4, \tau_4' \in \mathbf{T}(L_4)$. By what we have just shown, there exist $\tau_1 \in \mathbf{T}(L_1)$ and $\tau_2 \in \mathbf{T}(L_2)$ such that $\tau_4 = \tau_1\tau_2$. We may write, using also Theorem 1:

$$\tau_4\tau_4' = \tau_1\tau_2\tau_4' = \tau_1\tau_4'\tau_2 = \tau_4'\tau_1\tau_2 = \tau_4'\tau_4.$$

Thus $\mathbf{T}(L_4)$ is Abelian. Similarly $\mathbf{T}(L_1)$, $\mathbf{T}(L_2)$, $\mathbf{T}(L_3)$ are also Abelian. Statement ii) follows from Theorem 2, ii).

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DESPRE TRANZITIVITATEA STRUCTURILOR AFINE

(Rezumat)

F. Rado a demonstrat că o structură afină Barbilian plană este L-tranzitivă în raport cu orice dreaptă L, dacă este L_1-, L_2- şi L_3-tranzitivă, dreptele L_1, L_2, L_3 fiind încrucişate două cîte două. În prezenta notă se extinde acest rezultat pentru un plan afin rudimentar cu următoarele proprietăţi: 1) există o dreaptă L_4 încrucişată cu L_1, L_2 şi L_3. 2) pentru fiecare dreaptă L se pot găsi drepte L' şi L'' astfel încît perechile de drepte L, L'; L', L''; L'', L să fie încrucişate.

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THE HEAT TRANSFER IN A VISCOUS UNSTEADY FLOW THROUGH CIRCULAR DUCTS

DOINA BRĂDEANU

1. **Formulation of the thermal problem.** a) *The problem equations.* Let us considered a viscous incompressible fluid, with thermal conductivity λ, in unsteady flow inside a circular cylinder (tube) of radius R and very long lenght L. It is assumed that at the initial moment $t = 0$ the fluid is at rest and is subjected to a pressure gradient $(p_0 - p_L)/L$ where p_0 is the pressure in the cross section $z = 0$ and $p_L < p_0$ is the pressure of the fluid in circular section $z = L$ (here, Oz is the axis of the duct). It is assumed also that $\tilde{T}^{(0)}$ is the temperature of the fluid at the moment $t = 0$ and that $\tilde{T}_w$ is the temperature of the duct. Suppose that the flow, which is produced in these conditions, is unsteady, axisymmetrical (straight lines). The thermal conductivity of the fluid is not neglected. In the domain of the flow, let us now introduce the cylindrical coordinates (r, z, φ) where r is the radial coordinate and φ is the polar angle (fig. 1). Then, the velocity and temperature field in the fluid is represented by the scalar functions $v_z(r, t)$ and $\tilde{T}(r, t)$ where t is time.

The momentum and energy equations (Poiseuille flow) are deduced from the equations of the Navier-Stokes type in the form [1], [3]

$$\rho \frac{\partial v_z}{\partial t} = -\frac{\partial p}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) \tag{1}$$

$$\rho c_p \frac{\partial \tilde{T}}{\partial t} = \mu\left(\frac{\partial v_z}{\partial r}\right)^2 + \frac{\lambda}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{T}}{\partial r}\right) \tag{2}$$

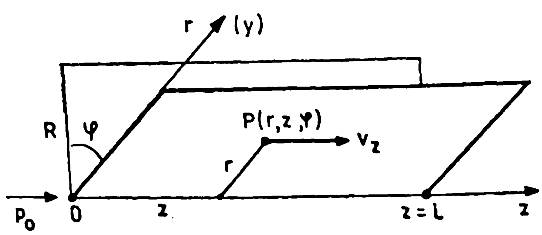

Fig. 1.

where the notations are in the usual form; the term $\mu(\partial v_z/\partial r)^2$ represents the dissipation function (a measure of the heat produced by the dissipation of the mechanical energy by friction) μ and λ are the constant coefficients of the viscosity and thermal conductivity, ρ — the density and c_p — the specific heat.

We can now make the following transformations for the independent variables and functions

$$y = \frac{r}{R}, \quad \tau = \frac{\mu t}{\rho R^2}, \quad U = \frac{v_z}{\left(-\frac{\partial p}{\partial z}\right)\frac{R^2}{4\mu}}, \quad \theta = \frac{\widetilde{T} - \widetilde{T}^{(0)}}{\widetilde{T}_w - \widetilde{T}^{(0)}} \tag{3}$$

and we introduce the notations (σ — Prandth number)

$$m = \frac{R^4(\partial p/\partial z)^2}{16\,\mu^2\, c_p(\widetilde{T}_w - \widetilde{T}^{(0)})}, \quad \sigma \equiv \frac{\mu c_p}{\lambda}$$

Then, the equations (1)—(2) take the form

$$\frac{\partial U}{\partial \tau} = 4 + \frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial U}{\partial y}\right), \quad (y, \tau) \in \Omega \tag{4}$$

$$\frac{\partial \theta}{\partial \tau} = m\left(\frac{\partial U}{\partial y}\right)^2 + \frac{1}{\sigma}\,\frac{1}{y}\,\frac{\partial}{\partial y}\left(y\frac{\partial \theta}{\partial y}\right), \quad (y, \tau) \in \Omega \tag{5}$$

$$U(y, 0) = 0, \; U(0, \tau) = \text{finite}\left(\frac{\partial \theta}{\partial y}(0, \tau) = 0\right), \; U(1, \tau) = 0 \tag{6}$$

$$(0 < y \leqslant 1\,;\; \tau > 0)$$

$$\theta(y, 0) = 0, \; \theta(0, \tau) = \text{finite}\left(\frac{\partial \theta}{\partial y}(0, \tau) = 0\right), \; \theta(1, \tau) = 1 \tag{7}$$

$$(0 < y < 1\,;\; \tau > 0)\,; \qquad \Omega = (0, 1) \times (0, \tau_1)$$

For this problem we seek the solutions $U, \theta \in C^{2,1}(\Omega)$ of which derivatives $\partial/\partial y = 0$ for $y = 0$ — by this condition the discontinuity in these equations is eliminated — and which, of course, must verify the boundary condition and the symmetry condition (6)—(7).

Let us consider the set of the functions

$$S = \{U, \theta \; U, \theta \in C^{2,1}(\Omega), \; U \text{ and } \theta \text{ verifies (6)}-(7)\}$$

In set S, [5], [4], the motion equation has the exact solution

$$U(y, \tau) = 1 - y^2 - 8\sum_{n=1}^{\infty} \frac{J_0(\alpha_n y)}{\alpha_n^3 J_1(\alpha_n)}\, e^{-\alpha_n^2 \tau}, \tag{8}$$

where J_0 is the Bessel function of the zero order and the first kind $(J_0(\alpha) = 0)$:

$$J_p(\alpha_n y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+p)!}\left(\frac{\alpha_n y}{2}\right)^{2k+p}, \qquad p = 0, 1, 2, \ldots \tag{9}$$

2. **The exact solution of the energy equation.** The energy equation, (5), is linear and nonhomogeneous under a nonhomogeneous boundary condition. For solving this problem we seek the solution in the form $\theta = \theta_0 + \theta_1$, where θ_0 verifies the homogeneous equation subjected to nonhomogeneous conditions (given in this problem) and θ_1 verifies the nonhomogeneous equation under

homogeneous conditions. Let $A(\theta)$ be the linear operator from the energy equation; thus, for the verification of the equation we have

$$A(\theta) = A(\theta_0) + A(\theta_1) = g(y, \tau)\,;\ g(y, \tau) = m\left(\frac{\partial U}{\partial y}\right)^2$$

$$\theta(y, 0) = \theta_0(y, 0) + \theta_1(y, 0) = 0,\ \theta(1, \tau) = \theta_0(1, \tau) + \theta_1(1, \tau) = 1$$

a) *The homogeneous energy equation under nonhomogeneous conditions.* Let $\theta_0(y, \tau)$ be the solution of the homogeneous energy equation under nonhomogeneous boundary conditions and let the change of the functions be

$$V = \theta_0(y, \tau) - 1\,; \qquad (a \equiv 1/\sigma)$$

Then, we have the problem

$$\frac{\partial V}{\partial \tau} = a\left(\frac{\partial^2 V}{\partial y^2} + \frac{1}{y}\frac{\partial V}{\partial y}\right) \tag{10}$$

$$V(y, 0) = -1,\ V(1, \tau) = 0\,;\ (V(0, \tau) = \text{finite})$$

We use the separation method of the variables by choosing

$$V(y, \tau) = Y(y)T(\tau)$$

and we obtain the differential system

$$\frac{dT}{T} = -\alpha^2 a\, d\tau$$

$$\frac{d^2Y}{dy^2} + \frac{1}{y}\frac{dY}{dy} + \alpha^2 Y = 0$$

where α is an unknown constant value. The solutions of these equations are

$$T(\tau) = C_0 e^{-\alpha^2 a\tau},\ Y(y) = C_1 J_0(\alpha y) + C_2 N_0(\alpha y)$$

where C_0, C_1 and C_2 are arbitrary integration constants and J_0, N_0 are Bessel functions of zero order and 1st and 2nd kind (the equation of the function $Y(y)$ is just the Bessel equation of $n = 0$ order).

The general solution has the following form

$$V(y, \tau) = C_0 e^{-a\alpha^2\tau}\,[C_1 J_0(\alpha y) + C_2 N_0(\alpha y)] \tag{11}$$

It is known, [4], that the Neumann function $N_0(x) \to -\infty$ when $x \to 0$. Consequently, in order to have a finite solution, imposed by the physical problem (17), we take $C_2 = 0$.

The boundary condition on the wall of the duct $V(1, \tau) = 0$ provides the equation

$$J_0(\alpha) = 0$$

Let $\alpha_1, \alpha_2, \ldots$ be the positive roots of this equation which at the same time are the eigenvalues of Bessel's ordinary differential operator. To thesse eigenvalues the eigenfunctions system corresponds $J_0(\alpha_n y)$, $n = 1, 2, 3, \ldots$

Consequently, the problem (10) has the particular solutions of the form (B_n are arbitrary constants)

$$V_n(y, \tau) = B_n e^{-a\alpha_n^2 \tau} J_0(\alpha_n y), \; n = 1, 2, 3, \ldots$$

The linear problem (10) has the solution

$$V(y, \tau) = \sum_{n=1}^{\infty} B_n e^{-a\alpha_n^2 \tau} J_0(\alpha_n y) \tag{11'}$$

in which the coefficients B_n must be determined.

Let us impose the initial condition $V(y, 0) = -1$, and then we obtain

$$-1 = \sum_{n=1}^{\infty} B_n J_0(\alpha_n y)$$

that is, the Fourier-Bessel series of the function $f(y) = -1$. In order to determine the B_n constants we use the orthogonality property on [0, 1] with the weight y of Bessel's function. We obtain

$$-\int_0^1 J_0(\alpha_m y)\, y\, dy = \sum_{n=1}^{\infty} B_n \int_0^1 J_0(\alpha_m y)\, J_0(\alpha_n y)\, y\, dy =$$

$$= B_m \int_0^1 [J_0(\alpha_m y)]^2 y\, dy = \frac{1}{2} B_m [J_0'(\alpha_m)]^2 \tag{12}$$

if we use the orthogonality property and the known formula from the theory of the Bessel functions

$$\int_0^R r[J_0(\alpha_m r)]\, dr = \frac{R^2}{2} [J_0'(\alpha_m R)]^2 \tag{13}$$

(α_m — eigenvalues, $J_0(\alpha_m r)$ — eigenfunctions)

We now calculate the integral from the left side of the equality (13). If we introduce the Bessel function J_0, in accordance with (9) and then integrate, we obtain

$$\int_0^1 J_0(\alpha_m y)\, y\, dy = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!} \left(\frac{\alpha_m}{2}\right)^{2k+1} \frac{1}{\alpha_m} = \frac{1}{\alpha_m} J_1(\alpha_m)$$

It is also known that $J_0'(x) = -J_1(x)$, [4]. Then, from (12) we obtain the values

$$B_m = -\frac{2}{\alpha_m J_1(\alpha_m)}$$

One can now write, by means of (11'), the solution $V(y, \tau)$.

The solution of the homogeneous energy equation under nonhomogeneous boundary conditions is

$$\theta_0(y, \tau) = 1 - 2\sum_{n=1}^{\infty} e^{-\frac{1}{a}\alpha_n^2\tau} \frac{J_0(\alpha_n y)}{\alpha_n J_1(\alpha_n)} \tag{14}$$

(α_n — are the positive roots of the equation $J_0(\alpha) = 0$; see (22))

b) *The solution of the nonhomogeneous energy equation subjected to homogeneous boundary conditions.* Let α_n, $n = 1, 2, \ldots$ be the eigenvalues of the operator from the homogeneous energy equation. The values α_n are the roots of the equation $J(\alpha) = 0$. Let $J_0(\alpha_1 y), J_0(\alpha_2 y), \ldots, J_0(\alpha_n y), \ldots$ be the eigenfunctions corresponding with the eigenvalues α_n. The functions θ_1 and g are developed in a series of $J_0(\alpha_n y)$ eigenfunctions (generalized Fourier series), introduced by the homogeneous equation, which forms an orthogonal system of functions (a system of linear independent functions; a complete system).

Let us introduce the Fourier-Bessel expansions by setting

$$\theta_1(y, \tau) = \sum_{n=1}^{\infty} c_n(\tau)\, J_0(\alpha_n y) \tag{15}$$

$$g(y, \tau) = \sum_{n=1}^{\infty} d_n(\tau)\, J_0(\alpha_n y) \tag{16}$$

$$(\theta_1(y, 0) = 0, \qquad \theta_1(1, \tau) = 0) \tag{16'}$$

where the Fourier coefficients $c_n(\tau)$ and $d_n(\tau)$ are the unknown functions which are to be determined.

The generalized Fourier expansions (15)—(16) are substituted in energy equation (5), which, reduces to the identity

$$\sum_n c_n' J_0(\alpha_n y) \equiv \sum_n d_n\, J_0(\alpha_n y) + a\left(\sum_n c_n \frac{d^2 J_0}{dy^2} + \frac{1}{y}\sum_n c_n \frac{dJ_0}{dy}\right)$$

with

$$\frac{d^2 J_0(\alpha_n y)}{dy^2} + \frac{1}{y}\frac{dJ_0(\alpha_n y)}{dy} + \alpha_n^2 J_0(\alpha_n y) = 0$$

This identity is further reduced to

$$\sum_n \left(\frac{dc_n}{d\tau} - d_n + a\, c_n\, \alpha_n^2\right) J_0(\alpha_n y) \equiv 0$$

From here the resulting nonhomogeneous ordinary equations are

$$\frac{dc_n(\tau)}{d\tau} + a\,\alpha_n^2\, c_n(\tau) = d_n(\tau), \; n = 1, 2, 3, \ldots \tag{17}$$

$$c_n(0) = 0, \; n = 1, 2, 3, \ldots$$

whose solutions are

$$c_n(\tau) = e^{-a\alpha_n^2 \tau} \int_0^\tau e^{a\alpha_n^2 t} d_n(t)\, dt$$

The solution of the nonhomogeneous energy equation (under homogeneous conditions) can be written in the following form

$$\theta_1(y, \tau) = \sum_{n=1}^{\infty} \int_0^\tau e^{-a\alpha_n^2(\tau - t)} d_n(t)\, J_0(\alpha_n y)\, dt \tag{18}$$

Now, we must calculate the Fourier coefficients $d_n(t)$. In order to do this, we set

$$y\, g(y, \tau) J_0(\alpha_m y) = \sum_{n=1}^{\infty} d_n(\tau)\, J_0(\alpha_n y)\, J_0(\alpha_m y) y$$

From here, by integrating on [0, 1], we find

$$\int_0^1 g(y, \tau)\, J_0(\alpha_m y)\, y\, dy = d_m(\tau) \int_0^1 [J_0(\alpha_m y)]^2\, y\, dy =$$

$$= \frac{1}{2} d_m(\tau)\, [J_0'(\alpha_m)]^2 = \frac{1}{2} d_m(\tau) [J_1(\alpha_m)]^2$$

if we take into account the orthogonal Bessel functions and formula (13). We obtain the formulas

$$d_n(\tau) = \frac{2}{[J_1(\alpha_n)]^2} \int_0^1 g(s, \tau)\, J_0(\alpha_n s)\, s\, ds, \qquad n = 1, 2, 3, \ldots$$

The solution of the problem, from (18), is

$$\theta_1(y, \tau) = \int_0^\tau \int_0^1 G(y, s, \tau - t)\, g(s, t)\, ds\, dt \tag{19}$$

if we introduce Green's function:

$$G(y, s, \tau - t) = \sum_{n=1}^{\infty} \frac{2s}{[J_1(\alpha_n)]^2} J_0(\alpha_n y)\, J_0(\alpha_n s)\, e^{-a\alpha_n^2(\tau - t)} \tag{20}$$

c) *The solution of the energy equation.* The solution of the energy equation (5) under initial and boundary conditions (7) is

$$\theta(y, \tau) = \theta_0(y, \tau) + \theta_1(y, \tau) = 1 - 2\sum_{n=1}^{\infty} e^{-\frac{1}{\tau}\alpha_n^2 \tau} \frac{J_0(\alpha_n y)}{\alpha_n J_1(\alpha_n)} + \int_0^\tau \int_0^1 G(y, s, \tau - t)\, g(s, t)\, ds\, dt. \tag{21}$$

where:
— The Green function $G(y, s, \tau - t)$ for the Bessel operator has the expression (20).
— The function $g(y, t)$ has the expression

$$g(y, t) = m\left(\frac{\partial U}{\partial y}\right)^2$$

with m given in (3) and U given in (8)
— The coefficients α_n are the positive roots (found in Tables) of the equation

$$J_0(\alpha) = 0$$

$$(\alpha_1 = 2{,}4048\,;\ \alpha_2 = 5{,}5201\,;\ \alpha_3 = 8{,}6537\,;\ \alpha_4 = 11{,}7915\,;\ \alpha_5 = 14{,}9309\,;\ \ldots) \qquad (22)$$

— J_0 and J_1 are the Bessel functions of the first kind and of the zero and first order, given in (9).

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TRANSFERUL DE CĂLDURĂ ÎN MIȘCAREA NESTAȚIONARĂ A UNUI FLUID VÎSCOS PRIN CONDUCTE CILINDRICE CIRCULARE

(Rezumat)

Se presupune că mișcarea nestaționară în conductă este axialsimetrică pe traiectorii rectilinii și că în momentul inițial (t = 0) fluidul este în repaos și este supus la un gradient de presiune. Conductibilitatea termică a fluidului și disipația nu sînt neglijabile dar termenul de convecție este eliminat din ecuația energiei. Ecuația neomogenă a energiei are condiții la limită neomogene și este rezolvată cu ajutorul funcțiilor Bessel și a funcției Green.

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A METHOD FOR THE CALCULATION OF EXOSPHERIC TEMPERATURES FROM OBSERVED UPPER ATMOSPHERE DENSITIES

IRINA PREDEANU*

1. Some peculiarities of "observed" exospheric temperatures. The exospheric temperature is a basic parameter of the high atmosphere, from which other parameters (concentration, density, etc.) can be deduced, according to the relations established for a given atmospheric model. The knowledge of this parameter from observations is very important; the comparison of its values with the theoretical ones yields an improvement of the models. By comparison with the density, the exospheric temperature (being independent of the height) has the advantage that its values determined at different heights can be directly compared, while the density values, in order to become comparable, must be normalized to a standard height; this supplementary operation can alter the results. So, the existence of some errors in the model can lead to errors of about 50–100% when the density is extrapolated from a height of about 1500 km to a reference one of 2300 km [4].

The passing from density values (determined from satellite orbital drag data or from direct measurements with accelerometers or mass spectrometers) to exospheric temperature values cannot be made by close calculations, because in the frame of the atmospheric models one determines the parameters in the following succession: exospheric temperature, temperature in the considered point, concentration of constituents and afterwards the density. An interpolation (linear or using a polynomial with coefficients determined by the least squares method) between the model data is possible but unpractical. Moreover, above 1000 km, where the concentration of the atomic hydrogen increases and the biunique correspondence density-temperature vanishes, the errors of the values provided by this method can reach 100% [3].

The density variation with the exospheric temperature for constant heights between 120–2000 km [2] is plotted in Figure 1. It is obvious that above 800 km there are two temperature values corresponding to a single density value. Also, the temperature T_m corresponding to the minimum of the function $\rho(T)$ lies between 500–1000 K and shifts toward higher values when the considered altitude increases. So, it is possible that above 1000 km to a single density it would correspond two temperatures differing each other by a factor larger than 2.

Therefore, it is useful to find a method which would remove these uncertainties, able to provide correct temperature values with a small calculation effort.

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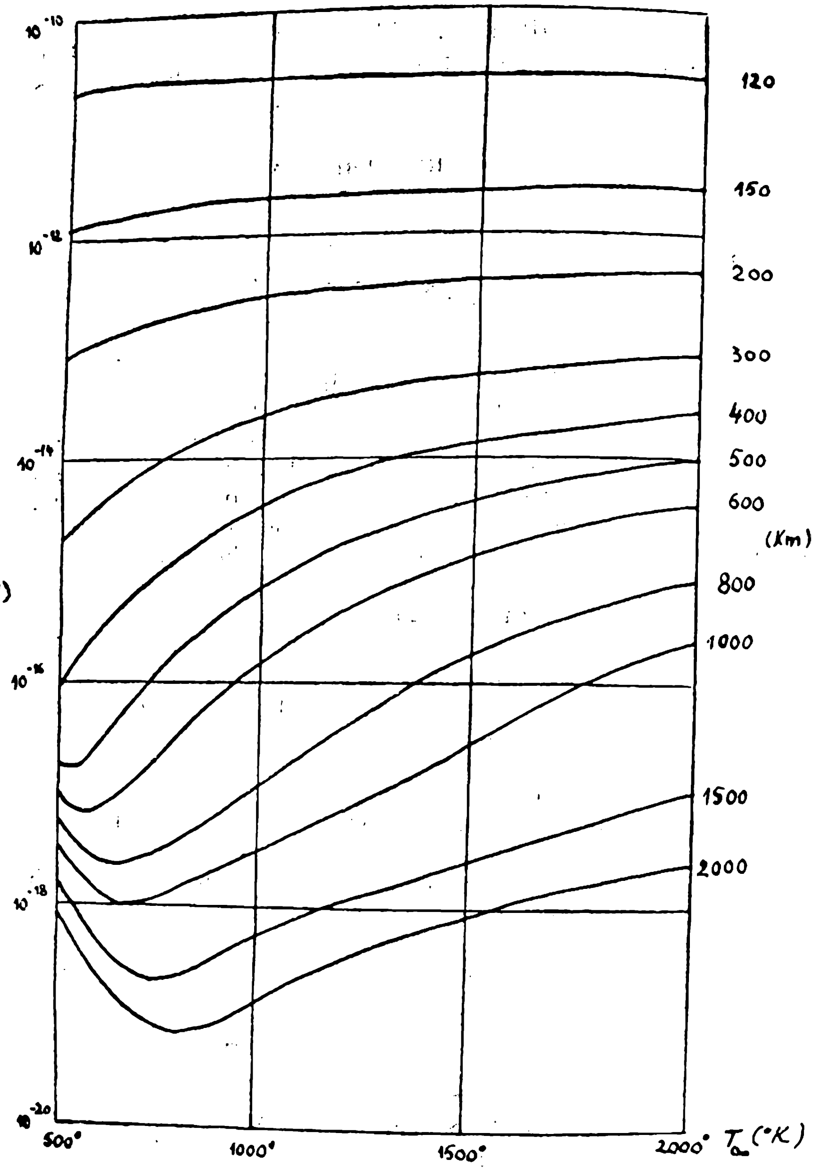

Fig. 1.

2. The method. Let ρ_0 be the density — considered as known — of the neutral atmosphere at the height z_0, obtained from orbital drag data. In order to determine the corresponding exospheric temperature T_0, we have considered the function:

$$F(T) = [\log \rho(T) - \log \rho_0]^2, \tag{1}$$

where $\rho(T)$ corresponds to the temperature T at the height z_0, according to the adopted model.

The needed value (T_0) is the solution of the equation $F(T) = 0$, which cannot be directly solved. Consequently, in order to find an approximate solution T_a for which $F(T)$ would acquire a small enough value, we have used the successive approximations method of the „faster slope" [1].

The minimum of $F(T)$ is to be found first by calculating successive values of T for points T_k having the form:

$$T_k = T_{k-1} - h_k(dF/dT)_{T_1}, \tag{2}$$

with $h_k = 2h_{k-1}$ and:

$$(dF/dT)_{T_1} = \{[F(T + \delta) - F(T)]/\delta\}_{T=T_1}, \tag{3}$$

an appropriate increase δ being chosen.

When we obtain for $F(T)$ a value greater than the previous one, the seeking will be performed in the last interval, at points T_j determined with the relation:

$$T_j = T_{j-1} - \bar{h}_j(dF/dT)_{T_1}, \tag{4}$$

where $\bar{h}_j = \bar{h}_{j-1}/2$, till $\bar{h}_j$ reaches a fixed lower limit. In this case, one calculates again the derivative of F and repeats the proceeding till $F(T)$ reaches a value smaller than a given $\varepsilon > 0$.

The rapid convergence of this proceeding is ensured by a suitable choice of δ, ε, h_1 and of the last value $\bar{h}_d$, according to the considered accuracy.

3. **Practicability.** The method was successfully tested to recalculate temperatures from densities using Jacchia's (1971) model [2]. The uncertainties related to the possibility of two solutions was restricted to a vicinity of about 100° around the minimum T_m of the functions $\rho(T)$ (Figure 1). In the neighbourhood of this point we stated one of the boundary values of the exospheric temperature, in order to have a biunique correspondence density-temperature. The other boundary value will be at 500 K or at 1900 K, as the theoretical temperature (including diurnal, solar and geomagnetic variations) is lower or higher than an estimated value of T_m.

As initial value of the exospheric temperature in the successive approximations process, a value close to the theoretical one for the considered point and moment will be chosen. In this manner, the probability to choose the correct solution of $F(T) = 0$ between the two possible ones is maximal.

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O METODĂ PENTRU CALCULAREA TEMPERATURILOR EXOSFERICE DIN DENSITĂŢI OBSERVATE ALE ATMOSFEREI ÎNALTE

(R e z u m a t)

Este propus un procedeu practic şi precis pentru determinarea de valori ale temperaturii exosferice din densităţi observate, bazat pe „metoda pantei celei mai rapide" de minimizare a unei funcţii prin aproximaţii succesive. Metoda a fost testată cu ajutorul modelului atmosferic Jacchia 1971. Rezultatele obţinute justifică utilizarea acestei metode, în special pentru înălţimi mai mari de 1 000 km.

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SOME APPLICATIONS OF A COMMON FIXED POINT THEOREM

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In [1] the following common fixed point theorem is proved.

THEOREM A. *Let S and T be continuous mappings of a complete metric space* (X, d) *into itself. The mappings S and T have a common fixed point if and only if there exists a continuous mapping $A : X \to SX \cap TX$, which commutes with S and T, such that:*

$$d(Ax, Ay) \leqslant q \cdot d(Sx, Ty), \text{ for every } x, y \in X$$

where $q \in [0, 1)$. Then, there exists one and only one fixed point of the mappings A, S and T.

Remark: B. Fisher has proved that the assumption of the continuity of the mapping A can be dropped.

Using this theorem and fixed point theorem from [7] we shall prove the existence of a common fixed point for mappings S, T and $A + F$, where F is a compact mapping.

First, we shall give some notions and a result from [7]. Let E be a vector space over the real or complex number field $\mathcal{K}$. The mapping $\| \|^* : E \to [0, \infty)$ is paranorme if and only if the following conditions are satisfied:

1. $\|x\|^* = 0 \Leftrightarrow x = 0$.
2. $\|-x\|^* = \|x\|^*$, for every $x \in E$.
3. $\|x + y\|^* \leqslant \|x\|^* + \|y\|^*$, for every $x, y \in E$.
4. If $\|x_n - x_0\|^* \to 0$ and $r_n \to r_0$ $(r_n, r_0 \in \mathcal{K}, n \in N)$ then

$$\|r_n x_n - r_0 x_0\|^* \to 0.$$

Then $(E, \| \|^*)$ is a paranormed space. Every paranormed space is a topological vector space in which the fundamental system of neighbourhoods of zero $\{V_s\}_{s>0}$ is defined by:

$$V_s = \{x | x \in E, \|x\|^* < s\} \qquad (s > 0)$$

In [7] the following fixed point theorem is proved.

THEOREM B. *Let $(E, \| \|^*)$ be a paranormed space, K be a convex and closed subset of E and $f: K \to K$ be a compact mapping so that there exists $C(f(L)) > 0$ such that:*

$$\|tx\|^* \leqslant C(f(K))t\|x\|^*, \text{ for every } t \in [0, 1]$$

and every $x \in f(K) - f(K)$. Then there exists $x \in K$ such that $x = fx$.

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DEFINITION. *Let $(E, \| \|^*)$ be a paranormed space and $K \subseteq E$. The set K is of Z type if and only if there exists $C(K) > 0$ so that:*

$$\|tx\|^* \leqslant C(K)\, t\|x\|^*, \text{ for every } t \in [0, 1] \text{ and every } x \in K - K.$$

In [7] is given an example of E and K, where $C(K) = 3$ and also an application on a system of integral equations. If $(E, \| \|^*)$ is a normed space then $C(K) = 1$, for every $K \subseteq E$ and so Theorem B is a generalization of Schauder fixed point theorem.

Some generalizations of Theorem B are proved in [4] and [6]. First, we shall prove the following Lemma.

LEMMA. *Let S and T be continuous mappings of a complete metric space (X, d) into itself, U be a topological space, for every $u \in U$ $A_u : X \to SX \cap TX$ be continuous mapping so that:*

$A_uSx = SA_ux$, $A_uTx = TA_ux$, for every $x \in X$ and every $u \in U$. Suppose that the following conditions are satisfied:

1. *For every $x \in X$ the mapping $u \mapsto A_ux$ is a continuous mapping from U into $SX \cap TX$.*
2. *$d(A_ux, A_uy) \leqslant qd(Sx, Ty)$, for every $x, y \in X$ and $q \in [0, 1)$.*

Then there exists one and only one continuous mapping $\tilde{z}: u \mapsto z(u)$ such that:

$$z(u) = A_u(z(u)) = S(z(u)) = T(z(u)),\quad u \in U. \tag{1}$$

Proof: From the conditions of Lemma it follows that all the conditions of Theorem A are satisfied and so for every $u \in U$ there exists one and only one element $z(u) \in SX \cap TX$ such that (1) holds. It remains to prove that the mapping $\tilde{z}: u \mapsto z(u)$ is continuous. Let u_0 be an arbitrary element from U. We shall prove that the mapping $\tilde{z}: u \mapsto z(u)$ is continuous at the point u_0. Let $\varepsilon > 0$. Then we have:

$$\begin{aligned} d(z(u), z(u_0)) &\leqslant d(z(u), A_u(z(u_0))) + d(A_u(z(u_0)), z(u_0)) = \\ &= d(A_u(z(u)), A_u(z(u_0))) + d(A_u(z(u_0)), A_{u_0}(z(u_0))) \leqslant \\ &\leqslant q \cdot d(S(z(u)), T(z(u_0))) + d(A_u(z(u_0)), A_u\ (z(u_0))) = \\ &\leqslant q \cdot d(S(z(u)), T(z(u_0))) + d(A_u(z(u_0)), A_{u_0}(z(u_0))) = \\ &= qd(z(u), z(u_0)) + d(A_u(z(u_0)), A_{u_0}(z(u_0))) \end{aligned}$$

and so:

$$d(z(u), z(u_0)) \leqslant \frac{d(A_u(z(u_0)),\ A_{u_0}(z(u_0)))}{1 - q} \tag{2}$$

Since the mapping $u \mapsto A_u(z(u_0))$ is continuous there exists a neighbourhood $V(u_0) \subseteq U$ of the element u_0 so that the following implication holds: $u \in V(u_0) \Rightarrow$ $\Rightarrow d(A_u(z(u_0)), A_{u_0}(z(u_0))) < (1-q)\varepsilon$ and from (2) it follows that $u \in V(u_0)$ implies $d(z(u), z(u_0)) < \varepsilon$.

THEOREM 1. *Let $(X, \| \|^*)$ be a complete paranormed space, $K \subseteq X$ be a closed and convex subset, S and $T : X \to X$ be continuous additive mappings,*

$A: K \to X$, $F: K \to X$ be continuous mappings so that $\overline{F(K)}$ is compact, $S|F(K) = T|F(K) = Id|F(K)$, $AK + FK \subseteq SK \cap TK$, $SK \cup TK \subseteq K$ and:

$$\|Ax - Ay\|^* \leqslant q\|Sx - Ty\|^*, \ q \in [0, 1), \ \textit{for every} \ x, y \in K.$$

If S and T are commutative with A and $SK \cap TK$ is of Z type there exists $x \in K$ so that:

$$x = Ax + Fx = Sx = Tx.$$

Proof: For every $u \in K$ let $A_u x = Ax + Fu$, $x \in K$. Let us prove that the family $\{A_u\}_{u \in K}$ satisfies all the conditions of Lemma. First $\|A_u x - A_u y\|^* \leqslant q\|Sx - Ty\|^*$, for every $u \in K$ and every $x, y \in K$. Since X is a topological vector space and F is continuous it follows that $u \mapsto A_u x$ is continuous for every $x \in K$. From $AK + FK \subseteq SK \cap TK$ it follows that $A_u K \subseteq SK \cap TK$, for every $u \in K$. Further, $SA_u x \Leftrightarrow S(Ax + Fu) = SAx + SFu = ASx + Fu = A_u Sx$ and similarly $TA_u x = A_u Tx$, for every $x, u \in K$. From the Lemma it follows that for every $u \in K$ there exists $Ru \in SK \cap TK$ so that:

$$Ru = ARu + Fu = SRu = TRu, \text{ for every } u \in K$$

and the mapping $R: K \to SK \cap T(K)$ is continuous. We shall prove that K and R satisfies all the conditions of Theorem B. Since $R(K) \subseteq S(K) \cap T(K)$ and the set $S(K) \cap T(K)$ is of Z type it follows that the set $R(K)$ is of Z type also. Let us prove that $\overline{R(K)}$ is compact. Since X is complete it is enough to prove that $R(K)$ is precompact. For every $z_1, z_2 \in K$ we have that:

$$Rz_1 = ARz_1 + Fz_1 = SRz_1 = TRz_1, \ Rz_2 = ARz_2 + Fz_2 = SRz_2 = TRz_2$$

and so:

$$\|Rz_1 - Rz_2\|^* \leqslant q\|S(Rz_1) - T(Rz_2)\|^* + \|Fz_1 - Fz_2\|^* \leqslant \leqslant q\|Rz_1 - Rz_2\|^* + \|Fz_1 - Fz_2\|^*$$

and so:

$$\|Rz_1 - Rz_2\|^* \leqslant \frac{\|Fz_1 - Fz_2\|^*}{1 - q} \tag{3}$$

Let $r > 0$. Then there exists a finite set $\{Fz_1, Fz_2, \ldots, Fz_n\}$ (z_i n K, $i \in \{1, 2, \ldots, n\}$) so that $FK \subseteq \bigcup_{i=1}^{n} \{Fz_i + U_{r(1-q)}\}$. Then from (3) it follows that $RK \subseteq \subseteq \bigcup_{i=1}^{n} \{Rz_i + U_r\}$. So, all the conditions of Theorem B are satisfied and there exists $x \in K$ such that $x = Rx$. Then $x = Rx = Ax + Fx = Sx = Tx$.

The following common fixed point theorem is a generalization of the known result for nonexpansive mappings in Banach spaces

THEOREM 2. *Let $(X, \|\ \|^*)$ be a complete paranormed space, $K \subseteq X$ be a closed and convex subset, $S, T: X \to X$ be linear continuous mappings so that $SK \cap TK$ is bounded, $A: K \to X$ be a continuous demicompact mapping, $F: K \to X$ be compact mapping, x_0 be a star point for SK and TK, $AK + FK \subseteq$*

$\subseteq SK \cap TK$, $SK \cup TK \subseteq K$, $S\,F(K) \cup \{x_0\} = T\,F(K) \cup \{x_0\} = Id\,F(K) \cup \cup \{x_0\}$ *and* $\|Ax - Ay\|^* \leqslant \|Sx - Ty\|^*$, *for every* $x, y \in K$. *If* S *and* T *are commutative with* A *and* $SK \cap TK$ *is of* Z *with* $C(SK \cap TK) \leqslant 1$ *type then there exists* $x \in K$ *so that:*

$$x = Ax + Fx = Sx = Tx.$$

Proof: Let $\{r_n\}_{n\in N} \subseteq (0, 1)$ be such that $\lim_{n\to\infty} r_n = 1$. For every $n \in N$ let $A_n x = r_n Ax + (1 - r_n)x_0$, $F_n x = r_n Fx$, for every $x \in K$. We shall prove that for every $n \in N$, A_n, F_n, S and T satisfy all the conditions of Theorem 1. Since S and T are linear it follows that:

$SF_n x = S(r_n Fx) = r_n SFx = r_n Fx = F_n x = Id|F_n(x)$, for every $x \in K$ and similarly $TF_n x = Id|F_n x$, for every $x \in K$ and for every $n \in N$. So $S|F_n(K) = T|F_n(K) = Id|F_n(K)$, for every $n \in N$. Further, $SA_n x = S(r_n Ax + (1 - r_n)x_0) = r_n SAx + (1 - r_n)x_0 = r_n ASx + (1 - r_n)x_0 = A_n Sx$, for every $x \in K$ and every $n \in N$ and similarly T and A_n are commutative. Let us show that $A_n K + F_n K \subseteq SK \cap TK$. Since for every $n \in N$, $A_n K + F_n K = r_n(AK + FK) + (1 - r_n)x_0$, $AK + FK \subseteq SK \cap TK$ and x_0 is a star point for SK and TK it follows that $A_n K + F_n K \subseteq SK \cap TK$. From Theorem 1 it follows that for every $n \in N$ there exists $x_n \in K$ so that:

$$x_n = A_n x_n + F_n x_n = Sx_n = Tx_n. \tag{4}$$

Then:

$$x_n - Ax_n - Fx_n = (r_n - 1)(Ax_n + Fx_n) + (1 - r_n)x_n, \qquad n \in N$$

and since $\lim_{n\to\infty} r_n = 1$, $Ax_n + Fx_n \subseteq SK \cap TK$ and the set $SK \cap TK$ is bounded it follows from (4) that:

$$\lim_{n\to\infty} x_n - Ax_n - Fx_n = 0. \tag{5}$$

The set $\overline{F(K)}$ is compact and so there exists a subsequence $\{Fx_{n_k}\}_{k\in N}$ such that $\lim_{k\to\infty} Fx_{n_k} = z \in X$. Then from (5) it follows that:

$$\lim_{k\to\infty} x_{n_k} - Ax_{n_k} = z. \tag{6}$$

The mapping A is demicompact and so from (6) it follows that there exists a subsequence $\{x_{n_{k_r}}\}_{r\in N}$ such that $\lim_{r\to\infty} x_{n_{k_r}} = u \in K$. From (5) it follows that $u = Au + Fu$ and since $x_n = Sx_n = Tx_n$ and S and T are continuous it follows that $u = Au + Fu = Su = Tu$.

Remark: It is obvious that we can take $x_0 = 0$ if 0 is a star point of the set K, since S and T are linear mappings.

COROLLARY 1. *Let* $(X, \|\ \|)$ *be a Banach space,* K *be a closed and convex subset of* X, x_0, SF *and* T *be as in Theorem 2,* $A: K \to X$, S *and* T *be commutative with* A, A *be a continuous* Φ*-densifying mapping, where* $\Phi: U \to G$, U *be the family of all bounded subsets of* X, $(G, \leqslant)$ *be a totally ordered set and the* *measure of noncompactness Φ is monotone, 2-regular and algebraically semiadditive.* If $AK + FK \subseteq SK \cap TK$ *and* $\|Ax - Ay\| \leqslant \|Sx - Ty\|$, *for every* x, $y \in K$ *then there exists* $x \in K$ *so that* $x = Ax + Fx = Sx = Tx$.

Proof: It remains to prove, similarly as in Theorem 3, that from (6) it follows that there exists a convergent subsequence $\{x_{n_k}\}_{n\in N}$ of the sequence $\{x_n\}_{n\in N}$. Let $y_n = x_n - Ax_n$, for every $n \in N$. Then $x_n = y_n + Ax_n$, for every $n \in N$ and so $\{x_n | n \in N\} \subseteq \{y_n | n \in N\} + \{Ax_n | n \in N\}$. Since $x_n \in SK \cap TK$ and $SK \cap TK$ is bounded we conclude that $\{x_n\ n \in N\} \in U$.
So we have; since Φ is algebraically semiadditive, that:

$$\Phi[\{x_n | n \in N\}] \leqslant \Phi[\{y_n | n \in N\}] + \Phi[\{Ax_n | n \in N\}].$$

Since the measure Φ is 2-regular it follows that $\Phi[\{y_n | n \in N\}] = 0$ and so:

$$\Phi[\{x_n | n \in N\}] \leqslant \Phi[\{Ax_n | n \in N\}].$$

Since the mapping A is Φ densifying it follows that there exists a convergent subsequence $\{x_{n_k}\}_{n\in N}$ from the sequence $\{x_n\}_{n\in N}$. As in Theorem 2 it follows that there exists $u \in K$ such that $u = Au + Fu = Su = Tu$.

Remark: It is easy to see that in Theorem A we can suppose that instead of $d(Ax, Ay) \leqslant q \cdot d(Sx, Ty)$, for every $x, y \in X$ we have the existence of $m \in N$ so that:

$$d(A^m x, a^m y) \leqslant qd(Sx, Ty), \text{ for every } x, y \in X,\ q \in [0, 1). \tag{7}$$

Indeed, in the case that (7) holds we conclude, from Theorem A that there exists one and only one element $x \in X$ such that:

$$x = A^m x = Sx = Tx \tag{8}$$

From (8) it follows that $Ax = A^m(Ax) = S(Ax) = T(Ax)$ and so Ax is the common fixed point for A^m, S and T. Since there exists one and only one common fixed point for mappings A^m, S and T we obtain that $x = Ax = Sx = Tx$.

Using this generalization it is easy to see that in Lemma we can suppose, instead the condition 2. that there exists $m \in N$ such that:

$$d(A_u^m x, a_u^m y) \leqslant qd(Sx, Ty), \text{ for every } x, y \in X\ (q \in [0, 1))$$

and for some $m \in N$. So, if in Theorem 1, the mapping A is additive, there exists $m \in N$ such that:

$$d(A^m x, A^m y) \leqslant q \cdot d(Sx, Ty), \text{ for every } x, y \in K$$

and S and T are as in Theorem 1, then there exists $x \in K$ such that

$$x = Ax + Fx = Sx = Tx.$$

The Kuratowski's measure of noncompactness $\alpha: U \to [0, \infty)$ defined by:

$$\alpha(A) = \inf\{\varepsilon | \varepsilon > 0, \text{ there exists a finete cover } \mathcal{A} \text{ of the set } A \text{ such that } \operatorname{diam}(B) < \varepsilon, \text{ for every } B \in \mathcal{A}\}$$

is 2-regular ($\alpha(Q) = 0$ if $\bar{Q}$ is compact), monotone ($Q_1, Q_2 \in U$, $Q_1 \subseteq Q_2$ implies that $\alpha(Q_1) \leqslant \alpha(Q_2)$) and algebraically semiadditive ($\alpha(Q_1 + Q_2) = \alpha(Q_1) + \alpha(Q_2)$) and so we have the following Corollary.

COROLLARY 2. *Let X, K, S, F, T and x_0 be as in Corollary 1 and $A : K \to X$ be continuous α densifying mapping such that $\|Ax - Ay\| \leqslant \|Sx - Ty\|$, for every $x, y \in K$, S and T be commutative with A, $AK + FK \subseteq SK \cap TK$ then there exists $x \in K$ so that $Ax + Fx = Sx = Tx = x$.*

Now, we shall prove a generalization of Theorem 1 if X is a topological vector space, using the following fixed point theorem proved by Rzepecki [6].

THEOREM C. *Let X be a Hausdorff topological vector space, $\mathfrak{U}$ be the fundamental system of neighbourhoods of zero in X, $F : K \to Z$ be continuous mapping, Z be a compact subset of K so that for every $V \in \mathfrak{U}$ and $x \in Z$ there exists $U \in \mathfrak{U}$ so that:*

$$\mathrm{co}((x + U) \cap Z) \subseteq x + V \tag{9}$$

Then there exists $x \in K$ so that $x = Fx$.

It is easy to see that Zima's fixed point follows from Rzepecki's fixed point theorem.

PROPOSITION. *Let X be a Hausdorff topological vector space, $\mathfrak{U}$ be the fundamental system of neighbourhoods of zero in X, K be a closed and convex subset of X, A, P, S and T be continuous mappings from K into X so that $AK + FK \subseteq SK \cap TK \subseteq K$ and for every $V \in \mathfrak{U}$ and every $x \in \overline{SK \cap TK}$ there exists $U \in \mathfrak{U}$ so that:*

$$\mathrm{co}((x + U) \cap (\overline{SK \cap TK}) \subseteq x + V.$$

If for every $u \in K$ there exists one and only one element $Ru \in SK \cap TK$ such that $Ru = ARu + Fu = SRu = TRu$ and the set $\overline{\{Ru\}}_{u \in K}$ is compact there exists $x \in K$ so that:

$$x = Ax + Fx + Sx = Tx.$$

Proof: As in Theorem 1 we shall prove that the mapping $R : K \to SK \cap TK$ has a fixed point. Let us prove that the mapping R is continuous. Suppose that $\{x_\alpha\}_{\alpha \sim \mathfrak{A}}$ is a net from K and $\lim\limits_{\alpha \in \mathfrak{A}} x_\alpha = x$. We shall prove that $\lim\limits_{\alpha \in \mathfrak{A}} Rx_\alpha = Rx$. Since the set $\overline{\{Ru\}}_{u \in K}$ is compact there exists a subnet $\{x_{\alpha_\beta}\}_{\beta \in \mathfrak{B}}$ such that $\lim\limits_{\beta \in \mathfrak{B}} Rx_{\alpha_\beta} = y$. Further, for every $\beta \in \mathfrak{B}$: $Rx_{\alpha_\beta} = ARx_{\alpha_\beta} + Fx_{\alpha_\beta} = SRx_{\alpha_\beta} = TRx_{\alpha_\beta}$ and since A, F, S and T are continuous it follows that:

$$\lim_{\beta \in \mathfrak{B}} Rx_{\alpha_\beta} = A(\lim_{\beta \in \mathfrak{B}} Rx_{\alpha_\beta}) + F(\lim_{\beta \in \mathfrak{B}} x_{\alpha_\beta}) = S(\lim_{\beta \in \mathfrak{B}} x_{\alpha_\beta}) = T(\lim_{\beta \in \mathfrak{B}} Rx_{\alpha_\beta})$$

So we have that $y = Ay + Fx = Sy = Ty$ which implies that $y = Rx$. So, every subnet of $\{Rx_{\alpha'}\}_{\alpha' \in \mathfrak{A}}$ has a convergent subnet with the same limit Rx and so $\lim\limits_{\alpha \in \mathfrak{A}} Rx_\alpha = Rx$. Since $RK \subseteq SK \cap TK$ the mapping R on K satisfies all the conditions of Theorem C and so $\mathrm{Fix}(R) \neq \Phi$. From $x \in \mathrm{Fix}(R)$ it follows that $x \in \mathrm{Fix}(A + F) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$ and so

$$x = Ax + Fx = Sx = Tx.$$

Now, we shall give a Corollary of the above Proposition. First, we shall give some definitions.

We shall subsequently denote the set of all real numbers by $\mathbf{R}_\Delta$. Furthermore, let E be a vector space over $\mathbf{K}$ (real or complex number field) and $\mathbf{R}_\Delta$ be the set of all mappings from Δ into $\mathbf{R}$. The Tihonov product topology and the operations of $+$ and scalar multiplications as usual. If $f, g \in \mathbf{R}_\Delta$ we say that $f \leqslant g$ if and only if $f(t) \leqslant g(t)$, for every $t \in \Delta$ and by $\mathbf{P}_\Delta$ we shall denote the cone of nonnegative elements in $\mathbf{R}_\Delta$.

In [5] S. Kasahara introduced the following notion of paranormed space, which we shall call Φ paranormed space.

DEFINITION. *The triplet $(E, \| \|, \Phi)$ is a Φ paranormed space if and only if $\| \| : E \to \mathbf{P}_\Delta$ and Φ is a linear, continuous, positive mapping from $\mathbf{R}_\Delta$ into $\mathbf{R}_\Delta$ such that the following conditions are satisfied:*

1. $\|x\| = 0 \Leftrightarrow x = 0$.
2. $\|x\| + \|y\| \leqslant \Phi(\|x\|) + \Phi(\|y\|)$
3. $\|tx\| \leqslant |t| \, \|x\|$, for every $x \in E$ and every $t \in \mathbf{K}$.

Let us denote by $\mathcal{U}$ the family of all neighbourhoods of zero in $\mathbf{R}_\Delta$. Then E is a topological vector space in which $\{V_U\}_{U \in \mathcal{U}}$ is the family of neighbourhoods of zero in E, and:

$$V_U = \{x \mid x \in E, \ \|x\| \in U\}$$

In [5] it is proved that every Hausdorff topological vector space is a Φ paranormed space $(E, \| \|, \Phi)$ over a topological semifield. In [3] the following definition is given.

DEFINITION. *Let $(E, \| \|, \Phi)$ be a Φ paranormed space over a topological semifield $\mathbf{R}_\Delta$ and $K \subseteq E$. If for every $n \in N$, every $u_i \in K - K$ $(i = 1, 2, \ldots, n)$ and $(s_1, s_2, \ldots, s_n) \in R^n$ such that $s_i \in [0, 1]$ $(i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} s_i = 1$,*

$$\left\| \sum_{i=1}^{n} s_i u_i \right\| \leqslant \sum_{i=1}^{n} s_i \Phi(\|u_i\|)$$

we say that the set K is of Φ-type.

Now, we have the following Corollary.

COROLLARY. *Let $(X, \| \|, \Phi)$ be a Φ — paranormed space, K be a closed and convex subset of X, $A, F, S, T : K \to X$ be continuous mappings, $AK + FK \subseteq \subseteq SK \cap TK \subseteq K$ and $SK \cap TK$ is of Φ type. If for every $u \in K$ there exists one and only one element $Ru \in SK \cap TK$ such that:*

$$Ru = ARu + Fu = SRu = TRu$$

and the set $\overline{\{Ru\}}_{u\in K}$ *is compact there exists* $x \in K$ *so that:*

$$x = Ax + Fx = Sx + Tx.$$

Proof: In [2] is proved that for every $V \in \mathcal{U}'$ there exists $U \in \mathcal{U}'$ such that for every $x \in SK \cup TK$

$$co((x + U) \cup (SK \cup TK)) \subseteq x + V(\mathcal{U}' = \{V_\sigma\}_{\sigma\in\mathcal{U}})$$

and so all the conditions of the Proposition are satisfied.

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UNELE ALPICAŢII A UNEI TEOREME DE PUNCT FIX COMUN

(Rezumat)

În [1] este demonstrată următoarea teoremă de punct fix comun.

Teoremă. Fie S şi T aplicaţii continue a unui spaţiu metric complet (X, d) pe el însăşi. Aplicaţiile S şi T au un punct fix comun dacă şi numai dacă există o aplicaţie continuă $A : X \to SX \cap TX$, care comută cu S şi T, astfel că

$$d(Ax, Ay) \leqslant qd(Sx, Ty),$$

pentru orice $x, y \in X$, unde $q \in [0, 1)$. Atunci există unul şi numai un punct fix al aplicaţiilor A, S şi T.

Folosind o teoremă a lui B. Fisher [1] şi a lui K. Zima [7] se demonstrează o teoremă de punct fix comun pentru aplicaţiile S, T şi $A + F$ unde F este o aplicaţie compactă.

L'OEUVRE SCIENTIFIQUE ET DIDACTIQUE DU PROFESSEUR TIBERIU MIHĂILESCU

(80 années depuis sa naissance)

M. ŢARINĂ et P. ENGHIŞ

Une personnalité marquante de l'Université de Cluj, après la deuxième guerre mondiale, a été le professeur Tiberiu Mihăilescu, dont l'apport à l'organisation de l'enseignement et de la recherche dans le domaine de la géometrie a été inestimable.

Il est né le 6 février 1902 (19 février d'aprés l'ancien style du calendrier) dans une modeste famille de la ville de Bucarest. Après de brillantes études aux anciens collèges „Sf. Sava" et „G. Lazăr" de sa ville natale, il a suivi les cours de la Faculté de Sciences de l'Université de Bucarest, section de mathématiques. Il a passé son examen de licence en 1927 enseignat tout d'abord comme professour de mathématiques aux lycées de Ploieşti, Tîrgovişte et Bucarest. En 1942 il a soutenu sa thèse de docteur ès mathématiques à l'Université de Bucarest et puis il a fonctionné comme chef de travaux à la chaire de géometrie de cette institution. Plus tard, en 1948, il est nommé professeur de géométrie à l'Université „V. Babeş" de Cluj, à la faculté de mathématiques et physique. Au temps du dévellopement de cette faculté, comme de la chaire de géometrie, il a déployé une intense activité scientifique et didactique. En 1962 le professeur T. Mihăilescu a été transféré, à sa demande, comme chef de la chaire de mathématiques à l'Institut de Pétrol, Gaz et Géologie de Bucarest. Il y a fonctionné jusqu'en 1967, année de sa retraite. Malgré d'une grave maladie, il a gardé jusqu'à sa fin une extraordinaire finesse d'esprit. Il est mort, le 25 mars 1979 étant enterré dans le cimitière Bellu (évangelique) de Bucarest.

L'oeuvre scientifique. Les premiers travaux scientifiques de T. Mihăilescu datent de 1939. Son domaine préféré a été la géometrie différentielle projective, abordée aussi par son maître, le professeur Georges Ţiţeica. L'outil qu'il a employé systématiquement dans son oeuvre a été la méthode du repère mobile, élaborée pour le cas des variétés générales par E. Cartan. L'étude approfondie des nouvelles techniques introduites par le mathématicien francais, surtout la théorie des systèmes extérieurs en involution, l'ont conduit à des résultats remarquables. Nous allons présenter quelques directions d'évolution de la recherche mathématique entreprise par T. Mihăilescu, en rappellant quelques-uns de résultats les plus importants.

La théorie de réseaux conjugués a été le chapitre de la géometrie différentielle projective dans lequel s'inscrivent les premiers travaux et surtout sa thèse [5]. T. Mihăilescu étudie des réseaux conjugués à propriétés spéciales. Ainsi, il a introduit les classes des réseaux harmoniques et non harmoniques, en déterminant pour chacune le degré de généralité. Parmi d'autres contributions, on a son étude des réseaux (M) — dénommés aussi réseaux Darboux-Ţiţeica, dont il a demontré qu'ils étaient les mêmes que les réseaux considerés par M. Jonas. On a montré aussi que les réseaux (R) de Ţiţeica et ceux de Terracini-Pantazi sont contenus dans la classe plus générale des réseaux de Laplace en correspondance asymptotique. Des résultats substantiels concernent les réseaux isothermes conjugués. Ainsi, par exemple, il a montré que sur une surface de courbure projective nulle il existe toujours un tel réseau. De plus, sur une telle surface, il existe un fascicule de réseaux conjugués dont l'axe cuspidale coïncide avec la première droite de Wilczynski. Dans [26] il a adapté la notion de réseau isotherme conjugué au cas d'une surface réglée non développable.

La théorie de variétés non-holonomes constitue une autre direction des recherches du professeur T. Mihăilescu. Son mémoire [7] est en fait le premier exposé systématique de la théorie de ces variétés qui utilise la méthode du repère mobile. Les étapes successives de l'installation du repère local sur une variété non holonome V_2^3 sont présentées d'une manière systématique en mettant en évidence le rôle de la polarité de Bompiani-Pantazi et des cônes de Malus. D'autres résultats se rapportent à des classes remarquables des variétés non-holonomes. Ainsi, en ce qui concerne les variétés paraboliques, T. Mihăilescu a montré l'existence de trois types de ces variétés, dont il détermine le degré de généralité. Il démontre l'existence des variétés paraboliques à des lignes asymptotiques curvilignes. De même, il a étudié et a donné la classification des variétés non-holonomes de Ţiţeica-Wilczynski qui représentent en fait les sphères projectives. Dans un ample mémoire [32] T. Mihăilescu étudie les systèmes triples non-holonomes, en complétant la théorie par la donnée de l'interprétation géométrique des invariants. Pour d'autres détails concernant le sujet, voir l'article [2] de la bibliographie.

En dehors des problèmes liés aux variétés non holonomes de l'espace projectif, T. Mihăilescu a considéré aussi des hypersurfaces non-holonomes d'un espace de Riemann [16], en définissant la courbure extérieure de celles-ci et en donnant quelques propriétés concernant les parallelismes de type Vrănceanu et Synge.

La théorie de la correspondance dans les travaux de T. Mihăilescu est liée de manière naturelle à l'étude des réseaux, des systèmes triples et des variétés non-holonomes [32], [34], [44].

D'autres résultats se réfèrent à une certaine généralisation de l'équation différentielle de Riccati [13], [14], à des théorèmes projectifs de type Gauss-Bonnet [21] et à diverses questions de géométrie différentielle affine et projective.

Le professeur T. Mihăilescu est bien connu par ses livres sur la géométrie différentielle projective [43], [44], qui contiennent d'ailleurs beaucoup de ses propres résultats. Ils ont été référés très favorablement dans les revues de pécialité. De même, ses mémoires sur la géométrie différentielle affine des courbes et des surfaces [24], [36], [37] constituent des synthèses originales remarquables.

Dans son activité scrientifique T. Mihăilescu a entretenu une correspondance très étendue avec de grands géomètres de son domaine de recherche, comme par exemple : E. Cartan, E. Bompiani, L. Godeaux, B. Segre, M. Villa, G. Boll, O. Boruvka, P. S. Finikov, L. Santalo, Su Buchin etc.

L'activité didactique. Le professeur T. Mihăilescu a été renommé pour ses leçons d'une éloquence brillante. Pendant son séjour à Cluj, il a donné divers cours de géométrie et d'autres disciplines mathématiques comme par exemple : géométrie analytique, géométrie projective, géométrie différentielle, fondements de la géométrie, équations différentielles et à dérivées partielles, équations de la physique mathématique, calcul variationnel etc. De même, il a donné des cours spéciaux de géométrie pour les étudiants des dernières années avec une thématique variée, en y exposant la théorie des groupes de transformations, et diverses questions de géométrie différentielle affine et projective à l'aide de la méthode du repère mobile. Parfois, ces cours contenaient aussi des problèmes liés à ses recherches.

Certaines de ses leçons ont été multipliées pour l'usage des étudiants de l'Université et de l'Institut Polytechnique de Cluj, ainsi que de l'Institut de Petrol, Gaz et Géologie de Bucarest.

Le professeur T. Mihăilescu a été un homme d'une grande érudition, intéressé aussi par des questions d'histoire, de philologie, de philosophie, de physique, chimie, de technique et biologie. Ces préoccupations se sont finalisées dans sa collaboration à la rédaction du Lexicon technique roumain. Il a été un bibliophile passionné et, avant sa mort, il a disposé la donation de sa bibliothèque de mathématiques à la faculté de Cluj-Napoca qu'il a servi avec tant de ferveur.

La figure distinguée du professeur Tiberiu Mihăilescu restera toujours vivante dans l'esprit de ses élèves et de ses collaborateurs.

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RECENZII

R. Ahlswede, I. Wegener, **Suchprobleme**, Studienbücher, Teubner Stuttgart, 1979, 328 pag.

Lucrarea trece în revistă diferite aspecte legate de problema căutării, problemă de mare importanţă şi actualitate în diverse domenii ale ştiinţei. Ea se adresează matematicienilor, informaticienilor, statisticienilor, inginerilor, cercetătorilor din domeniul ştiinţelor naturii şi celor din domeniul literaturii.

Materialul este organizat în patru părţi şi completat de o bogată listă bibliografică. În prima parte se dau o serie de noţiuni şi definiţii, precum şi un model de căutare. Partea a doua tratează problema căutării cu teste de eroare, şi anume: problema căutării în binar, coduri alfabetice, problema sortării, probleme vagi şi probleme de geometrie, probleme speciale de căutare cu teste de eroare.

Partea a treia este dedicată problemei căutării cu teste aleatoare. După o introducere despre aproximări stochastice, se prezintă problema căutării cu răspunsuri aleatoare şi cu legături feed beack, problema identificării şi ierarhizării. Partea a patra tratează despre probleme de căutare cu inspecţie, prezentînd cele mai generale modele de acest tip.

Materialul prezentat în lucrare are o formă ştiinţifică riguroasă, este bine organizat şi sistematizat, de mare interes atît teoretic cît şi aplicativ în toate domeniile de cercetare.

E. OANCEA

W. Schempp, B. Dressler, **Einführung in die harmonische Analyse**, B. G. Teubner, Stuttgart, 1980, 300 pag.

Cartea tratează Analiza armonică din punctul de vedere al teoriei grupurilor. Prima parte cuprinzînd capitolele: Analiza armonică pe grupul toroidal n-dimensional T^n şi Analiza armonică pe spaţiul real n-dimensional euclidian R^n, cuprinde rezultatele clasice fundamentale din teoria seriilor şi integralelor Fourier. Partea a doua este constituită din capitolele: Măsura Haar pe grupuri topologice local compacte, Analiza armonică pe grupuri compacte şi Analiza armonică pe perechi Ghelfand. Ea este consacrată analizei armonice necomutative şi implicaţiilor acesteia în teoria dezvoltării după funcţii speciale.

Lucrarea conţine un mare număr de exemple ilustrative şi probleme.

ERVIN SCHECHTER

An Introduction to Convex Polytopes, by Arne Brondsted. Graduate Texts in Mathematics. Springer–Verlag, New York–Heidelberg–Berlin, 1983. VIII + 160 pp.

Professor Brondsteds very atractive book is directed to the study of the combinatorial aspects of the theory of convex polytopes. It provides a complete and self-contained exposure of the main theorems in this field. Prerequisites are only standard linear algebra and elementary topology in $\mathbf{R}^d$. Nevertheless, it takes the reader to most interesting recent achievements.

Chapter 1 contains basic topics as the affine structure of $\mathbf{R}^d$, convex hull, relative interior of a convex set, supporting hyperplanes, the facial structure of a closed convex set and polarity.

Chapter 2 is devoted to the general theory of convex polytopes and to the study of simple, simplicial, cyclic and neighbourly polytopes.

The combinatorial theory of convex polytopes is developed in Chapter 3. The first two section treat Euler's relation in the d-dimensional case and the Dehn–Sommerville relations, valid for simple and simplicial polytopes. The following sections are devoted to the celebrated Upper Bound Theorem and Lower Bound Theorem, which solve important extremum problems involving the number of faces of simple and simplicial polytopes. A final section reports on recent results. Many exercises have been provided throughout.

There are three appendices dealing with latices, graphs and combinatorial identities, respectively.

This beautiful book is well suited for graduate students. It succeed admirably to introduce the readers in a vigorously flourishing field.

F. RADÓ

I. P. Cluj, Municipiul Cluj-Napoca, C-da nr. 3017/83

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