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O FORMULĂ DE DERIVARE NUMERICĂ ŞI APLICAREA EI LA INTEGRAREA NUMERICĂ A UNEI ECUAŢII DIFERENŢIALE DE ORDINUL TREI

PARASCHIVA PAVEL

1. Ne propunem să determinăm o formulă de derivare numerică de forma:

$$\Delta^{n-1}f(x_1) = \sum_{i=1}^{n} A_i f'''(x_i) + R[f] \tag{1}$$

unde

$$\Delta^{n-1}f(x_1) = f(x_n) - C^1_{n-1}f(x_{n-1}) + \ldots + (-1)^{n-1}C^{n-1}_{n-1}f(x_1) \tag{2}$$

$x_1, x_2, \ldots, x_n$ fiind noduri în progresie aritmetică cu raţia h, $f \in C^{n+3}[x_1, x_n]$, iar $R[f]$ este nul cînd funcţia f este înlocuită cu un polinom oarecare de gradul $n + 2$.

Urmărind ,,metoda funcţiei φ" ca în lucrările [3], [4], [5] ataşăm intervalelor $[x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$ funcţiile $\varphi_1, \varphi_2, \ldots, \varphi_{n-1}$ soluţii ale ecuaţiilor diferenţiale

$$\varphi_i^{(n+2)}(x) = (-1)^{i-1}C^{i-1}_{n-2}, \; i = 1, 2, \ldots, n-1 \tag{3}$$

ce satisfac condiţiilor la limite

$$\begin{aligned} &\varphi_1^{(r)}(x) = 0, \; r = 0, 1, \ldots, n-2, n, n+1. \\ &\varphi_i^{(r)}(x_i) = \varphi_{i-1}^{(r)}(x_i), \; r = 0, 1, \ldots, n-2, n, n+1; i = 2, 3, \ldots, n-1 \\ &\varphi_{n-1}^{(r)}(x_n) = 0, \; r = 0, 1, \ldots, n-2, n, n+1 \end{aligned} \tag{4}$$

În aceste condiţii, aplicînd integralelor

$$\int_{x_i}^{x_{i+1}} \varphi_i^{(n+3)}(x)f(x)dx = 0; \quad i = 1, 2, \ldots, n-1 \tag{5}$$

formula generalizată de integrare prin părţi, însumîndu-le şi ţinînd seama de ecuaţiile diferenţiale (3) şi condiţiile la limite (4), obţinem formula de derivare numerică (1), în care

$$\begin{aligned} A_1 &= (-1)^{n+1}\varphi_1^{(n-1(}(x_1) \\ A_i &= (-1)^{n+1}[\varphi_i^{(n-1)}(x_i) - \varphi_{i-1}^{(n-1)}(x_i)]; \; i = 2, 3, \ldots, n-1. \\ A_n &= (-1)^n\varphi_{n-1}^{(n-1)}(x) \end{aligned} \tag{6}$$

şi

$$R[f] = \int_{x_1}^{n} \varphi(x) f^{(n+3)}(x)\,dx \tag{7}$$

funcţia φ coincizînd pe fiecare subinterval $[x_i, x_{i+1}]$ cu φ_i, $i = 1, 2, \ldots, n-1$.

2. Soluţia problemei la limite (3) + (4) este

$$\varphi(x) = \sum_{i=1}^{n-1} \left[(-1)^{i-1} C_{n-1}^{i-1} \frac{(x-x_i)_+^{n+2}}{(n+2)!} + \lambda_i \frac{(x-x_i)_+^{n-1}}{(n-1)!} \right] \tag{8}$$

unde

$$u_+ = \begin{cases} u & \text{dacă } u > 0 \\ 0 & \text{dacă } u \leqslant 0 \end{cases}$$

λ_i, $i = 1, 2, \ldots, n-1$ sînt constante

Cerînd ca funcţia φ să satisfacă şi condiţiile (4) din punctul x_n şi ţinînd seama de formulele

$$\sum_{i=0}^{n-2} (-1)^i C_{n-1}^i (n-1-i)^r = \begin{cases} 0 \ \text{dacă } r = 1, 2, \ldots, n-2 \\ (n-1)! \ \text{dacă } r = n-1 \\ (n-1)! \dfrac{n(n-1)}{2} \ \text{dacă } r = n \\ (n-1)! \dfrac{n(n-1)(n+1)(3n-2)}{24}, \ \text{dacă } r = n+1, \\ (n-1)! \dfrac{(n-1)^2 n^2 (n+1)(n+2)}{48}, \ \text{dacă } r = n+2 \end{cases} \tag{9}$$

se obţine următorul sistem de $n-1$ ecuaţii lineare, cu necunoscutele λ_i, $i = 1, 2, \ldots, n-1$.

$$\sum_{i=0}^{n-2} \lambda_{i+1} (n-1-i)^r = \begin{cases} 0, \ \text{dacă } r = 1, 2, \ldots, n-5 \\ -(n-4)! \ \text{dacă } r = n-4 \\ -(n-3)! \dfrac{n-1}{2} h^3, \ \text{dacă } r = n-3 \\ -\dfrac{(n-1)!(3n-2)}{24} h^3, \ \text{dacă } r = n-2 \\ -\dfrac{(n-1)!(n-1)^2 n}{48} h^3, \ \text{dacă } r = n-1 \end{cases} \tag{10}$$

al cărui determinant este

$$\Delta = (n-1)!\,V(n-1, n-2, \ldots, 2, 1) \neq 0$$

(V fiind determinantul lui Vandermonde al numerelor $n-1$, $n-2$, ..., 2, 1) Sistemul de ecuaţii (10) are soluţia

$$\lambda_k = (-1)^{k-1} \frac{h^3(k-1)(n-2k+1)}{2(n-2)(n-3)} C_{n-2}^{k-1} = (-1)^{k-1} h^3 \frac{n-2k+1}{2(n-k-1)} C_{n-4}^{k-2}$$

$$\text{pentru } k = 2, 3, \ldots, n-1 \qquad (11)$$

$$\lambda_1 = 0$$

Coeficienţii formulei au expresiile

$$A_k = (-1)^{n+k} \frac{h^3}{2(n-3)} C_{n-3}^{k-2}(n-2k+1),\ k = 2, 3, \ldots, n-1 \qquad (12)$$

$A_n = 0$, $A_1 = 0$

deoarece $A_n = (-1)^n \cdot \sum_{i=1}^{n-1} \lambda_i = -\sum_{i=1}^{n-1} A_i = 0$

iar

$$\sum_{i=2}^{k} \lambda_i = (-1)^{k-1} \frac{(n-5)(n-6)\ldots(n-k-1)(n-2k)}{2(k-2)!},\ n > 4 \qquad (13)$$

Coeficienţii A_k se mai bucură de proprietatea $A_k = (-1)^n A_{n-k+1}$. În cazurile $n = 4, 5, 6, 7, 8$ formula (1) devine

$$\Delta^3 f(x_1) = \frac{1}{2} h^3[f'''(x_3) + f'''(x_2)] + \int_{x_1}^{x_4} \varphi(x) f^{(7)}(x) dx$$

$$\Delta^4 f(x_1) = \frac{1}{2} h^3[f'''(x_4) - f'''(x_2)] + \int_{x_1}^{x_5} \varphi(x) f^{(8)}(x) dx \qquad (14)$$

$$\Delta^5 f(x_1) = \frac{1}{2} h^3[f'''(x_2) - f'''(x_3) - f'''(x_4) + f'''(x_5)] + \int_{x_1}^{x_6} \varphi(x) f^{(9)}(x) dx$$

$$\Delta^6 f(x_1) = h^3\left[-\frac{1}{2} f'''(x_2 + f'''(x_3) - f'''(x_5) + \frac{1}{2} f'''(x_6)\right] + \int_{x_1}^{x_7} \varphi(x) f^{(10)}(x) dx$$

$$\Delta^7 f(x_1) = \frac{h^3}{2} [f'''(x_2) - 3f'''(x_3) + f'''(x_4) + f'''(x_6) - 3f'''(x^6)] + \int_{x_1}^{x_8} \varphi(x) f^{(11)}(x) dx.$$

Observaţie: Integrînd ecuaţiile diferenţiale (3) cu condiţiile (4) în ordine inversă, se obţine

$$\varphi(x) = \sum_{i=1}^{n-1} \left[(-1)^{i-1} C_{n-1}^{i-1} \frac{(x_{n-i+1} - x)_+^{n+2}}{(n+2)!} + \lambda_i \frac{(x_{n-i+1} - x)_+^{n-1}}{(n-1)!} \right] \qquad (15)$$

3. Semnalăm următoarele proprietăţi ale funcţiei φ:

Proprietatea 1. Funcţia φ este simetrică faţă de dreapta $x = \frac{x_1 + x_n}{2}$. Într-adevăr, în formula (8) punînd $x = \alpha - y$, unde $\alpha = \frac{1}{2}(x_1 + x_n)$ obţinem

$$\varphi(\alpha - y) = \sum_{i=1}^{n-1}\left[(-1)^{i-1}C_{n-1}^{i-1}\frac{(\alpha - x_i - y)_+^{n+2}}{(n+2)!} + \lambda_i\frac{(\alpha - x_i - y)_+^{n-1}}{(n-1)}\right]$$

iar în formula (15), punînd $x = \alpha + y$ avem

$$\varphi(\alpha + y) = \sum_{i=1}^{n-1}\left[(-1)^{i-1}C_{n-1}^{i-1}\frac{(x_{n-i+1} - \alpha - y)_+^{n-2}}{(n+2)!} + \lambda_i\frac{(x_{n-i+1} - \alpha - y)_+^{n-1}}{(n-1)!}\right]$$

şi deoarece $\alpha - x_k = x_{n-k+1} - \alpha$ rezultă că $\varphi(\alpha - y) = \varphi(\alpha + y)$

CONSECINŢA 1. $\varphi'(\alpha) = 0$, *adică funcţia* φ *are un extremum în punctul* $x = \alpha$

Proprietatea 2. Are loc relaţia $\varphi^{(s)}(\alpha - y) = (-1)^s\varphi^{(s)}(\alpha + y)$; $s = 0, 1, \ldots, n + 2$. Acest rezultat se obţine uşor dacă se ţine seama de formulele (8) şi (15).

CONSECINŢA 2. $\varphi^{(2s-1)}(\alpha) = 0$, $s = 1, 2, \ldots, \left[\frac{n}{2} + 1\right]$.

Proprietatea 3. Funcţiile φ_1 şi φ_n sînt pozitive pe intervalele (x_1, x_2), respectiv (x_{n-1}, x_n). Ţinînd seama de expresiile acestor funcţii date de formulele (8) şi respectiv (15) ,proprietatea este imediată.

Proprietatea 4. Funcţia φ este pozitivă pe intervalul (x_1, x_n) Vom arăta că funcţia φ are un singur extremum în intervalul (x_1, x_n). S-a văzut că funcţia φ este continuă în intervalul (x_1, x_n), împreună cu derivatele sale pînă la ordinul $n - 2$ şi satisface condiţiilor

$$\varphi^{(j)}(x_1) = 0; \quad \varphi^{(j)}(x_n) = 0,\ j = 0, 1, \ldots, n - 2$$

Dacă am presupune că funcţia φ ar avea trei zerouri în intervalul (x_1, x_n) şi am aplica succesiv teorema lui Rolle, ar rezulta că $\varphi^{(n-2)}$ ar avea n zerouri în intervalul considerat, dar printr-un calcul simplu se vede că $\varphi^{(n-2)}$ are numai $n - 2$ zerouri în intervalul (x_1, x_n) şi anume pentru n par are cîte un zerou în fiecare subinterval (x_i, x_{i+1}), $i = 2, 3, \ldots, n - 2$, iar în intervalul din mijloc are două zerouri. Pentru n impar, are un zerou în nodul din mijloc (consecinţa 2) şi cîte un zerou în fiecare din intervalele (x_i, x_{i+1}), $i = 2, 3, \ldots, n - 2$.

4. Vom aplica acum formula (1) la rezolvarea numerică a problemei lui Cauchy

$$y'''(x) = g(x, y) \tag{16}$$

$$y(x_0)) = y_0,\ y'(x_0) = y_0',\ y''(x_0) = y_0'' \tag{17}$$

în ipoteza că soluţia acestei probleme există şi este unică în intervalul $[x_0, x_0 + a[$

Pentru integrarea numerică a problemei (16) + (17) considerăm în intervalul $[x_0, x_0 + a]$ nodurile $x_0, x_1, \ldots, x_p = x_0 + a$ în progresie aritmetică cu rația h. Presupunem că soluția a fost în prealabil calculată (folosind, de exemplu, o metodă de tip Runge-Kutta) pe nodurile $x_1, x_2, \ldots, x_{n-1}$; $n \leqslant p$, adică numerele $y(x_1), \ldots, y(x_{n-1})$ sînt cunoscute,.

Pentru a obține pe $y(x_n)$ aplicăm formula (1) și obținem

$$y(x_n) = C_{n-1}^1 y(x_{n-1}) + \ldots + (-1)^n C_{n-1}^n y(x_1) + \\ + h^3 \sum_{k=2}^{n-1} (-1)^{n+k} \frac{n-2k+1}{2(n-3)} C_{n-2}^{k-3} \cdot g[x_k, y(x_k)] + R_n \tag{18}$$

unde

$$R_n = \int_{x_1}^{x_n} \varphi(x) g^{(n)}(x, y(x) dx \tag{19}$$

Notînd cu G_n o margine superioară a lui $|g^{(n)}(x, y)|$ pe domeniul său de definiție, obținem

$$| R_n | \leqslant G_n A_n h^{n+3} \tag{20}$$

unde

$$A = \int_{x_1}^{x_n} \varphi(x) dx =$$

$$= \sum_{k=2}^{n-1} h^3(-1)^{n+k} \frac{n-2k+1}{2(n-3)} C_{n-3}^{k-3} \left[\frac{(x-x_1)^4 (x-x_2) \ldots (x-x_n)}{(n+3)!} \right]^{n} \Bigg|_{x=x_k}$$

Notînd cu y_k valorile calculate ale lui $y(x_k)$, $k = 0, 1, \ldots, n-1$ și prin

$$\delta_n = \Delta^{n-1} y_1 - h^3 \sum_{k=2}^{n-1} (-1)^{n+k} \frac{n-2k+1}{2(n-3)} C_{n-3}^{k-2} g(x_k, y_k) \tag{22}$$

δ_n depinzînd de metoda folosită pentru calculul lui y_k, $k = 0, 1, \ldots, n-1$ iar cu $r_k = y(x_k) - y_k$, $k = n, n+1, \ldots$ avem evaluarea

$$\Delta^{n-1} r_1 - h^3 \sum_{k=2}^{n-1} (-1)^{n+k} \frac{n-2k+1}{2(n-3)} C_{n-3}^{k-2} [g(x_k, y(x_k)) - g(x_k, y_k)] = R_n - \delta_n$$

sau

$$\Delta^{n-1} r_1 - L h^3 \sum_{k=2}^{n-1} (-1)^{n+k-1} C_{n-3}^{k-2} r_k = \delta_n - R_n$$

unde L este o margine superioară a lui $|g_y'(x, y)|$ în domeniul său de definiție.

5. Punîndu-ne problema convergenţei acestei metode de integrare numerică în sensul lui Dalhquist şi Henrici [2] şi [3] se observă că singura dintre formulele de derivare numerică (1) care ne furnizează o metodă convergentă de integrare numerică este cea obţinută în cazul $n = 4$.

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UNE FORMULE DE DÉRIVATION NUMÉRIQUE ET SON APPLICATION A L'INTÉGRATION NUMÉRIQUE D'UNE ÉQUATION DIFFÉRENTIELLE DE TROISIÈME ORDRE

(Résumé)

L'auteur du présent travail étudie la formule de dérivation numérique (1) à l'aide de la méthode de la „fonction φ" [3].

Les coefficients sont donnés par les formules (12) et le reste $R[f]$ se met sous la forme (7), ce qui démontre que la fonction φ, donnée par les formules (8), est positive en (x_1, x_n).

La formule de dérivation numérique (1) conduit à la méthode d'intégration numérique (18) pour le calcule de la solution du problème de Cauchy (16) + (17).

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OBSERVAŢII ASUPRA ECUAŢIEI DE TIP FUCHS ÎN ALGEBRE BANACH (I)

M. FRENKEL

În lucrarea profesorului D. V. Ionescu, intitulată *Le théoreme de Fuchs* [3] se face un studiu în legătură cu teorema lui Fuchs asupra ecuaţiei diferenţiale:

$$z^n w^{(n)} + p_1(z) z^{n-1} w^{(n-1)} + \ldots + p_n(z) w = 0, \tag{1}$$

scoţind în evidenţă ideea că proprietăţile esenţiale ale soluţiilor în vecinătatea punctului singular sînt acelea care apar şi în cazul ecuaţiei lui Euler corespunzătoare

$$z^n v^{(n)} + p_1(0) z^{n-1} v^{(n-1)} + \ldots + p_n(0) v = 0 \tag{2}$$

Asupra acestei idei, după cum se arată în lucrarea amintită, a insistat şi prof. Th. Angheluţă.

Prezenta notă are ca scop de a arăta că proprietăţile amintite sînt adevărate şi în cazul unei algebre Banach oarecare şi că datorită acestui fapt se pot face anumite observaţii asupra soluţiilor, cu ajutorul ecuaţiei lui Euler corespunzătoare. În continuare se deduc din cazul general proprietăţile corespunzătoare în cazul algebrei matricilor patratice precum şi cazul ecuaţiei diferenţiale (1).

§1. Se consideră ecuaţia diferenţială

$$zw'(z) = P(z)w(z) \tag{3}$$

în algebra Banach B, cu elementul unitate e, adică $P(z) \in B$ şi $P(z)$ este olomorfă în $z = 0$.

La această ecuaţie se asociază ecuaţia

$$zv'(z) = P(0)v(z), \tag{4}$$

care generalizează ecuaţia lui Euler (2).

Pentru a compara soluţiile celor două ecuaţii (3) şi (4) este necesar să enunţăm următoarele două teoreme cunoscute în teoria ecuaţiilor diferenţiale din domeniul complex [2].

TEOREMA 1. *Dacă $a_0 = P(0)$ satisface una din condiţiile*

(i) a_0 aparţine centrului algebrei
(ii) oricare două valori ale spectrului $\sigma(a_0)$, nu diferă printr-un întreg, atunci ecuaţia diferenţială (3) admite soluţie de forma

$$w(z) = \sum_{m=1}^{\infty} c_m z^{m+a_0}, \quad c_m \in B,$$

într-o vecinătate a punctului $z = 0$.*

* Prin definiţie $z^a = e^{a \operatorname{Log} z}$, $a \in B$, iar $z^{m+a_0} = z^{me+a_0}$

TEOREMA 2. *Dacă* $a_0 = P(0)$ *satisface condițiile*
(*i*) *numere întregi pozitive, în mulțimea diferențelor a două valori spectrale, sînt puncte izolate ale acestei mulțimi, și pentru orice număr întreg pozitiv n ecuația*

$$\alpha - \beta = n$$

are cel mult un număr finit de soluții α, β *din spectrul* $\sigma(a_0)$
(*ii*) *oricare asemenea* α *și* β *este pol pentru* $R(\lambda\,;\, a_0)$**
atunci ecuația diferențială (3) *admite soluție de forma*

$$w(z) = \sum_{j=1}^{p} (\mathrm{Log}\, z)^j \sum_{m=0}^{\infty} c_{mj}\, z^{m+a_0}\,;\quad c_{mj} \in B$$

într-o vecinătate a punctului $z = 0$; p *fiind un număr întreg pozitiv*

Se observă că în ambele cazuri soluția conține pe z^{a_0}.

Să considerăm ecuația lui Euler asociată, adică ecuația (4). Aceasta admite soluția

$$v(z) = z^{a_0}$$

Într-adevăr, din $v(z) = z^{a_0}$, rezultă $v'(z) = \frac{1}{z}\, a_0 \exp\,(a_0\, \mathrm{Log}\, z) = \frac{1}{z}\, a_0 z^{a_0}$. Deducem de aici că singularitățile soluției ecuației lui Euler (4) vor fi singularități și pentru ecuația (3) ; anume acelea care provin din termenul z^{a_0}. Funcția $z^{a_0} = \exp(a_0 \mathrm{Log}\, z)$ poate fi în $z = 0$, olomorfă, poate avea pe $z = 0$ punct singular pol, singularitate de natura z^α, $\alpha \in \mathbf{C}$ (deci punct critic), singularitate logaritmică etc.

§ 2. În cazul matricial se poate obține cu mai multe amănunte, decît în cazul general, legătura dintre natura singularității funcției z^{a_0} și spectrul a_0, ceea ce conduce la stabilirea legăturii dintre ecuațiile (3) și (4) în cazul matricial.

Fie deci $\mathfrak{M}_n$ algebra matricilor patratice și $a_0 \in \mathfrak{M}_n$. Se știe [1] că matricea z^{a_0} are în general termeni de forma

$$z^\lambda(\alpha_0 + \alpha_1\, \mathrm{Log}\, z + \ldots + \alpha_k (\mathrm{Log}\, z)^k)$$

unde $\lambda \in \sigma(a_0)$, sau combinații liniare de termeni de această formă.

Putem deosebi următoarele situații:

a) Matricea a_0 este scalară, adică

$$a_0 = \begin{pmatrix} \alpha & 0 & \ldots & 0 \\ 0 & \alpha & 0 \ldots & 0 \\ \cdot & \cdot & \cdot \cdot \cdot & \cdot \\ 0 & \ldots & & 0\ \alpha \end{pmatrix}$$

În acest caz putem scrie $z^{a_0} = z^\alpha\, I \in I = \begin{pmatrix} 1 & 0 & & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \ldots & & 0\ 1 \end{pmatrix}$

** $R(\lambda\,;\, a_0) = (\lambda e - a_0)^{-1}$

Funcţia z^{a_0} este olomorfă în $z = 0$, admite în acest punct singularitate pol, sau punct critic, după cum α este număr întreg pozitiv, întreg negativ sau număr complex oarecare. Nu conţine singularitate logaritmică.

b) Matricea a_0 are n valori proprii distincte $\lambda_1, \lambda_2, \ldots, \lambda_n$, atunci elementele matricii z^{a_0} vor fi formate din combinaţii liniare de termeni de forma z^{λ_i}, $i = \overline{1, n}$.

În consecinţă, funcţia z^{a_0} va fi olomorfă în $z = 0$, va admite pe $z = 0$ punct singular pol sau punct critic de natura $z^{\alpha}(\alpha \in \mathbf{C})$, după cum toate valorile proprii sînt numere întregi pozitive, numere întregi, sau numere complexe oarecare. z^{a_0} în acest caz nu conţine termen logaritmic.

c) Matricea a_0 are cel puţin o valoare proprie multiplă. Fie deci valoarea proprie λ_i de ordinul de multiplicitate k_i, $i = \overline{1, p}$, $p < n$, atunci se ştie [1] că dacă pentru $i = \overline{1, p}$ avem

$$A_i - \lambda_i I_i = \theta,$$

unde θ este elementul nul din spaţiu E_i, unde $E_i = \operatorname{Ker}(a_0 - \lambda I)^{k_i}$ şi $A_i = a_0/E_i$, atunci z^{a_0} nu conţine singularitate logaritmică.

d) Matricea a_0 are cel puţin o valoare proprie multiplă λ_i de ordinul de multiplicitate k_i, $i = \overline{1, p}$, $p < n$ şi pentru un i avem

$$A_i - \lambda_i I_i \neq \theta$$

în acest caz [1] z^{a_0} are termen logaritmic.

Din aceste observaţii deducem legătura dintre forma soluţiilor ecuaţiilor

$$zw'(z) = P(z)w(z) \tag{5}$$

$$zv'(z) = P(0)v(z) \tag{6}$$

unde $P(z) \in \mathfrak{M}_n$ este o funcţie olomorfă în $z = 0$, şi $a_0 = P(0)$.

În cele ce urmează vom ţine seama de teoremele 1 şi 2 din §1 şi de observaţiile a)—d) din acest §.

(i) Dacă $a_0 = \begin{pmatrix} \alpha & 0 & \ldots & 0 \\ . & . & . & . \\ 0 & \ldots & 0 & \alpha \end{pmatrix}$ sau dacă oricare două valori proprii ale spectrului $\sigma(a_0)$ nu diferă printr-un întreg, atunci soluţiile ecuaţiilor (5) şi (6) au aceleaşi singularităţi şi nu au singularitate logaritmică.

(ii) Dacă toate valorile proprii ale matricei a_0 sînt distincte sau dacă are loc situaţia c), atunci soluţia ecuaţiei lui Euler (6) are singularităţi provenite din z^{a_0} şi nici una din aceste singularităţi nu este logaritmică. Soluţia ecuaţiei (5) are toate singularităţile soluţiei ecuaţiei lui Euler, dar la acestea se pot adăuga şi singularităţi logaritmice.

(iii) Dacă are loc situaţia d), atunci atît soluţia ecuaţiei Euler (6) cît cea a ecuaţiei (5) au singularităţi logaritmice provenite din z^{a_0}.

§ 3. Ecuaţia diferenţială liniară de ordinul n

$$z^n v^{(n)} + z^{n-1} p_1(z) v^{(n-1)} + \ldots + p_n(z) v = 0 \tag{7}$$

se poate obţine în cazul cînd avem

$$P(z) = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & \ldots & & & \\ 0 & \ldots & & & & & & \\ & \ldots & & & & & n-2 & 1 \\ -p_n(z) & \ldots & & & & -p_3(z) & -p_2(z) & n-1-p_1(z) \end{pmatrix}$$

şi considerînd ecuaţia vectorială asociată celei matriciale

$$zw'(z) = P(z)w(z)$$

În acest caz matricea a_0 nu poate fi de forma $\begin{pmatrix} \alpha & 0 & \ldots & 0 \\ 0 & & \ldots & 0 \\ \ldots & & & \ldots \\ 0 & & 0 & \alpha \end{pmatrix}$ şi nu poate avea loc situaţia c).

În consecinţă deducem:

(i) Dacă oricare două valori proprii ale matricii $a_0 = P(0)$ nu diferă printr-un întreg, atunci toate soluţiile ecuaţiei lui Euler

$$z^n v^{(n)} + z^{n-1} p_1(0) v^{(n-1)} + \ldots + p_n(0) v = 0 \tag{8}$$

şi ale ecuaţiei (7) au aceleaşi singularităţi şi nu au singularitate logaritmică.

(ii) Dacă valorile proprii ale matricii a_0 sînt două cîte două distincte, atunci ecuaţia lui Euler (8) are singularităţi provenite din z^{a_0} de forma z^{λ_i}. Toate aceste singularităţi apar şi în soluţia ecuaţiei (7), care pe lîngă acestea poate avea şi singularităţi logaritmice.

(iii) Dacă matricea a_0 are valori proprii multiple, avem soluţii atît pentru ecuaţia (7) cît şi pentru ecuaţia (8) cu singularitate logaritmică.

Am regăsit în felul acesta proprietăţile soluţiilor ecuaţiei (7) menţionate în [3].

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REMARQUES SUR L'ÉQUATION DE TYPE FUCHS DANS LES ALGEBRES BANACH (I)

(Résumé)

Dans le présent travail on considère l'équation différentielle

$$zw'(z) = P(z)w(z)$$

où $P(z)$ est une fonction de variable z holomorphe dans le voisinage de $z = 0$, avec les valeurs dans une algèbre Banach. On fait des remarques concernant la liaison entre cette équation et l'équation correspondante de Euler.

ON PERFECT NUMBERS

MIHÁLY BENCZE

Perfect numbers are called those natural numbers which are equal to the sum of all their natural divisors being smaller than the number itself, they were studied as far back as in antiquity.

Even Euclide indicated the following method for finding all even perfect numbers: we calculate successively all partial sums of the following geometrical series: $1 + 2 + 2^2 + 2^3 + \ldots$ If such a sum is a prime number then multiplying it by the next term of the series we obtain a perfect number. Euler proved that this method allows us to find all even perfect numbers.

In other words, this shows us that all even perfect numbers are like: $2^{p-1} M_p$ where $M_p = 2^{p-1}$ is a prime number. The numbers of the following from $M_n = 2^n - 1$ are called Mersenne's numbers. From this follows that there are as much even perfect numbers as Mersenne's prime numbers. Odd perfect numbers are not known yet. It is known, if they exist then they should be very great (greater than 10^{20}, see also [3])

Throughont this paper we generalize on perfect numbers and we prove some characteristics relating to these numbers.

DEFINITION: Natural number N is called perfect-k if $\sigma(N) = k \cdot N$ where $\sigma(N)$ is the sum of all positive divisors of natural number N, and k is a natural number, greater than or equal to the unit.

For $k = 2$ we have the "classic" definition of perfect numbers. An interesting problem should be that of finding out the greater value, that may take k when N is chosen feom the sets of natural numbers.

Further on, we suppose $N \geqslant 3$.

THEOREM. *If natural number* $N = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$ *written in canonical form is a perfect-k then:*

$$\begin{cases} n\left(\sqrt[n]{\frac{3}{2}} - 1\right) & \text{if } N\text{-even} \\ n\left(\sqrt[2n]{k^2} - 1\right) & \text{if } N\text{-odd} \end{cases} < \sum_{i=1}^{n} \frac{1}{p_i} < \begin{cases} n\left(1 - \sqrt[n]{\frac{6}{k\pi^2}}\right) & \text{if } N\text{-even} \\ n\left(1 - \sqrt[n]{\frac{8}{k\pi^2}}\right) & \text{if } N\text{-odd} \end{cases}$$

Proof

$$\prod_{i=1}^{n} \frac{p_i}{p_i - 1} = \prod_{i=1}^{n} \frac{p_i^{\alpha_i+1} - 1}{(p_i - 1)p_i^{\alpha_i}} \prod_{i=1}^{n} \frac{p_i^{\alpha_i+1}}{p_i^{\alpha_i+1} - 1} = k \prod_{i=1}^{n} \frac{1}{1 - \frac{1}{p_i^{\alpha_i+1}}} =$$

$$= k \prod_{i=1}^{n} \left(\sum_{j=0}^{\infty} \left(\frac{1}{p_i^{\alpha_i+1}} \right)^j \right) \leqslant k \prod_{i=1}^{n} \left(\sum_{j=0}^{\infty} \frac{1}{p_i^{2j}} \right) <$$

$$< \begin{cases} k \sum_{i=1}^{\infty} \frac{1}{m^2} & \text{if } N\text{-even} \\ k \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} & \text{if } N\text{-odd} \end{cases} = \begin{cases} \frac{k\pi^2}{6} & \text{if } N\text{-even} \\ \frac{k\pi^2}{8} & \text{if } N\text{-odd} \end{cases} \tag{1}$$

Using Cauchy's inequality of harmonious and geometrical means it follows:

$$\prod_{i=1}^{n} \frac{p_i}{p_i - 1} \geqslant \left(\frac{n}{\sum_{i=1}^{n} \frac{p_i - 1}{p_i}} \right)^n = \left(\frac{n}{n - \sum_{i=1}^{n} \frac{1}{p_i}} \right)^n \tag{2}$$

Taking into consideration (1) and (2) we obtain the following inequalities:

$$\sum_{i=1}^{n} \frac{1}{p_i} < \begin{cases} n\left(1 - \sqrt[n]{\frac{6}{k\pi^2}}\right) & \text{if } N\text{-even} \\ n\left(1 - \sqrt[n]{\frac{8}{k\pi^2}}\right) & \text{if } N\text{-odd} \end{cases}$$

Now can we prove the inequalities which give an inferior delimitation If N is even then it follows:

$$\prod_{i=1}^{n} \frac{p_i + 1}{p_i} > \frac{3}{2} \tag{3}$$

If $x \geqslant 3$ then

$$\frac{x+1}{x} \geqslant \sqrt[4]{\frac{x^2}{(x-1)^2}} \tag{4}$$

(see also [5])

If N is odd then according to inequality (4) we have

$$\prod_{i=1}^{n} \frac{p_i + 1}{p_i} > \sqrt[3]{\prod_{i=1}^{n} \frac{p_i^2}{(p_i - 1)^2}} > \sqrt[3]{k^2} \tag{5}$$

because of

$$\prod_{i=1}^{n} \frac{p_i}{p_i - 1} = k \prod_{i=1}^{n} \frac{p_i^{\alpha_i+1}}{p_i^{\alpha_i+1} - 1} > k$$

Using Cauchy's inequality of arithmetical and geometrical means we have:

$$\prod_{i=1}^{n} \frac{p_i + 1}{p_i} \leqslant \left(\frac{\sum\limits_{i=1}^{n} \frac{p_i + 1}{p_i}}{n} \right)^n = \left(\frac{n + \sum\limits_{i=1}^{n} \frac{1}{p_i}}{n} \right)^n \tag{6}$$

From the inequalities (3), (5) and (6) there follows:

$$\sum_{i=1}^{n} \frac{1}{p_i} > \begin{cases} n\left(\sqrt[n]{\frac{3}{2}} - 1\right) & \text{if } N\text{-even} \\ n\,(\sqrt[3n]{k^2} - 1) & \text{if } N\text{-odd} \end{cases}$$

Q.E.D.

Lemma. *If m is a nonnegative real number and x a positive real number then:*

$$(x+1)m^{\frac{1}{x+1}} - xm^{\frac{1}{x}} \leqslant 1$$

Proff. For $m = 1$ we have the equality.

Let $0 < m < 1$. Since function $f(x) = xm^{\frac{1}{x}}$ is continous and derivable we can apply Lagrange's theorem and we obtain:

$$\frac{(x+1)m^{\frac{1}{x+1}} - xm^{\frac{1}{x}}}{(x+1) - x} = \frac{f(x+1) - f(x)}{(x+1) - x} = f'(z) \text{ where } x < z < x + 1$$

hence we have the inequality $m^{\frac{1}{z}}\left(1 - \frac{1}{z}\ln m\right) < 1$ or $1 - \frac{1}{z}\ln m < m^{-\frac{1}{z}}$.

Developing $m^{-\frac{1}{z}}$ into Mc Lauren's series it results: $1 - \frac{1}{z}\ln m < 1 - \frac{1}{1!\,z}\ln m + \frac{1}{2!\,z^2}\ln^2 m - \frac{1}{3!\,z^3}\ln^3 m + \ldots$ or $\sum\limits_{k=2}^{\infty} \frac{(-1)^k \ln^k m}{k!\,z^k} > 0$ or $\sum\limits_{k=2}^{\infty} \frac{\ln^k \frac{1}{m}}{k!\,z^k} > 0$ that is obvions because of $\ln \frac{1}{m} > 0$ and due to $\frac{1}{m} > 1$

Let $m > 1$. Then it is enough to show that function $g(x) = x(m^{\frac{1}{x}} - 1)$ is decreasing. $g'(x) = m^{\frac{1}{x}} - m^{\frac{1}{x}} \cdot \frac{1}{x}\ln m - 1 = -\sum\limits_{k=2}^{\infty} \frac{\ln^k m}{x^k(k-1)!}\left(1 - \frac{1}{k}\right) < 0$.

Since g is decreasing we may say that $g(x+1) < g(x)$ hence and from it follows the inequality of the enunciation.

Q.E.D.

COROLLARY 1. *If* $N = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$ *is one perfect-k number then*

$$\begin{cases} \ln \frac{3}{2} & \text{if } N\text{-even} \\ \frac{2}{3} \ln k & \text{if } N\text{-odd} \end{cases} < \sum_{i=1}^{n} \frac{1}{p_i} < \begin{cases} \ln \frac{k\pi^2}{6} & \text{if } N\text{-even} \\ \ln \frac{k\pi^2}{8} & \text{if } N\text{-odd} \end{cases}$$

Proof. Using the preceding Lemma it is proved that the following series are decreasing; $\left(n\left(\sqrt[n]{\frac{3}{2}} - 1\right)\right)_{n \in N}$ and $(n(\sqrt[3n]{k^2} - 1))_{n \in N}$ and the following series are growing: $\left(n\left(1 - \sqrt[n]{\frac{6}{k\pi^2}}\right)\right)_{n \in N}$ and $\left(n\left(1 - \sqrt[n]{\frac{8}{k\pi^2}}\right)\right)_{n \in N}$. It means that the minimum and maximum are eached only when $n \to \infty$. Since $n \to \infty$ we have $0 \cdot \infty$. That is why l'Hopital's rule and so we find again the results of the enunciation.

Q.E.D.

Remark 1. For $k = 2$ we have M. Perisastri's inequality (see [2]) namely $\sum_{i=1}^{n} \frac{1}{p_i} < 2 \ln \frac{\pi}{2}$.

COROLLARY 2. *Let* $N = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$ *perfect-k number and*
$p_{\max} = \max \{p_1, p_2, \dots, p_n\}$ *and*
$p_{\min} = \min \{p_1, p_2, \dots, p_n\}$ *then:*

$$p_{\min} < \begin{cases} \dfrac{1}{\sqrt[n]{\frac{3}{2}} - 1} & \text{if } N\text{-even} \\ \dfrac{1}{\sqrt[3n]{k^2} - 1} & \text{if } N\text{-odd} \end{cases} \quad \text{and}$$

$$p_{\max} > \begin{cases} \dfrac{1}{1 - \sqrt[n]{\frac{6}{k\pi^2}}} & \text{if } N\text{-even} \\ \dfrac{1}{1 - \sqrt[n]{\frac{8}{k\pi^2}}} & \text{if } N\text{-odd} \end{cases}$$

Proof. Considering that $n \cdot \frac{1}{p_{\max}} < \sum_{i=1}^{n} \frac{1}{p_i}$ respective $\sum_{i=1}^{n} \frac{1}{p_i} < n \cdot \frac{1}{p_{\min}}$ and from the theorem it follows the inequality of the enunciation.

Remark 2. Let $N = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_n^{\alpha_n}$ perfect-k number and $p_{\min} = \min\{p_1, p_2, \ldots, p_n\}$ then $p_{\min} < \frac{2n}{k^2 - 1} + 2$ (see the method of M. Perisastri's proof).

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ASUPRA NUMERELOR PERFECTE

(Rezumat)

În lucrare se generalizează noţiunea de număr perfect şi se stabiles inegalităţi pentru suma $\sum_{i=1}^{n} 1/p_i$ unde $N = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots p_n^{\alpha_n}$ este un număr k-perfect, obţinînd astfel şi o finisare şi o generalizare a unei teoreme dată de M. Perisastri.

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ON HEAT TRANSFER IN UNSTEADY BOUNDARY LAYER IN A ROTATING FLOW WITH VARIABLE SUCTION

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1. **Introduction.** Rotating flows (or Ekman layer) are of importance in cosmical and geophysical fluid dynamics. An analysis of steady Ekman layer in rotating fluids has been made by Prandtl [1] and Batchelor [2] for a non-porous plate whereas the effects of suction and injection on Ekman layer were discussed by Gupta [3]. The unsteady flow in an Ekman layer due to the oscillations of the plate was studied by Pop and Soundalgekar [4]. Purushothaman [5] resolved the problem of Ref. [4] by modifying the method of solution. In both Refs. [4,5] the energy equation for a rotating fluid was not solved. In spite of our earlier work it is of special interest to study the heat transfer problem in a rotating fluid past an infinite porous plate rotating with the fluid as a solid body rotation.

2. **Mathematical analysis.** We consider a Cartesian co-ordinate system (x^*, y^*, z^*) rotating uniformly with the fluid with an angular velocity Ω^* about the z^* — axis normal to the plate. x^* — and y^* — axes lie in the plane of the plate at $z^* = 0$. The infinite porous plate is assumed to oscillate in its own plane. The unsteady motion of the fluid is governed by the following equations:

$$\frac{\partial w^*}{\partial z^*} = 0 \tag{1}$$

$$\frac{\partial u^*}{\partial t^*} + w^* \frac{\partial u^*}{\partial z^*} - 2\Omega^* v^* = \nu \frac{\partial^2 u^*}{\partial z^{*2}} \tag{2}$$

$$\frac{\partial v^*}{\partial t^*} + w^* \frac{\partial v^*}{\partial z^*} + 2\Omega^* u^* = \nu \frac{\partial^2 v^*}{\partial z^{*2}}. \tag{3}$$

The mathematical expression for the oscillation of the plate is given by

$$u^* + iv^* = U_0^*(1 + \varepsilon e^{i\omega^* t^*})$$

where (u^*, v^*, w^*) are the components of velocity in the (x^*, y^*, z^*) directions, t^* the time, ν the kinematic viscosity of the fluid, ω^* the frequency of the oscillation, U_0^* a reference velocity, $i = \sqrt{-1}$, εU_0^* the amplitude of the oscillation of the plate with ε a positive and small constant.

If the suction velocity w^* is assumed to be a function of t^* then equation (1) on integration gives

$$w^* = -w_0^* \left[1 + \frac{\varepsilon A}{2} (e^{i\omega^* t^*} + e^{-i\omega^* t^*})\right] \tag{4}$$

where w_0^* is the constant suction velocity and A is a constant such that $\varepsilon A < 1$.

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The initial and boundary conditions of (2,3) are

$$t^* \leqslant 0: \quad u^* + iv^* = 0 \text{ for all } z^*$$
$$t^* > 0: \quad u^* + iv^* = U_0^*(1 + \varepsilon e^{i\omega^* t^*}) \text{ at } z^* = 0$$
$$u^* + iv^* \to 0 \qquad \text{as } z^* \to \infty.$$

These equations are solved in [5] and the solution in non-dimensional form is given by

$$p_0 = e^{-mz}, \quad p_1 = Se^{-nz} + (1 - S)e^{-mz}, \quad p_2 = (1 - S)(e^{-lz} - e^{-mz})$$

where

$$p(z,t) = p_0(z) + \varepsilon e^{i\omega t} p_1(z) + \varepsilon e^{-i\omega t} p_2(z) \tag{5}$$

and the non-dimensional quantities are defined as

$$z = z^* |w_0|/\nu, \quad t = w^* t^*/4\nu, \quad E = \nu\Omega^*/w_0^{*2}$$
$$p = p^*/U_0^*, \quad \omega = 4\nu\omega^*/w_0^{*2}. \tag{6}$$

We have also noted

$$l = \frac{1}{2}(1 + \sqrt{1 + 4i(2E - \omega/4)}), \quad m = \frac{1}{2}(1 + \sqrt{1 + 8iE})$$
$$n = \frac{1}{2}(1 + \sqrt{1 + 4i(2E + \omega/4)}), \quad S = 1 - \frac{2Aim}{\omega}.$$

We now solve the energy equation. If the dissipative terms are taken into account it is given by

$$c_p\left(\frac{\partial T^*}{\partial t^*} + w^* \frac{\partial T^*}{\partial z^*}\right) = \frac{k}{\rho}\frac{\partial^2 T^*}{\partial z^{*2}} + \nu\left[\left(\frac{\partial u^*}{\partial z^*}\right)^2 + \left(\frac{\partial v^*}{\partial z^*}\right)^2\right] \tag{7}$$

where c_p is the specific heat at constant pressure, T^* is the temperature in the boundary layer, k the thermal conductivity and ρ is the fluid density. In terms of the non-dimensional variables (6) and in view of (4), the equation (7) reduces to the following form

$$\frac{P}{4}\frac{\partial\theta}{\partial t} - P\left[1 + \frac{\varepsilon A}{2}(e^{i\omega t} + e^{-i\omega t})\right]\frac{\partial\theta}{\partial z} = \frac{\partial^2\theta}{\partial z^2} + PEc\left(\frac{\partial p}{\partial z}\right)\left(\frac{\partial \bar{p}}{\partial z}\right) \tag{8}$$

where

$$P = \mu c_p/k \text{ (Prandtl number)}, \quad \theta = (T^* - T_\infty^*)/T_\infty^*$$
$$Ec = U_0^{*2}/c_p T_\infty^* \text{ (Eckert number)}$$

and the bar denotes the complex conjugate quantity. Assuming an insulated plate the boundary conditions of (8) are

$$\frac{d\theta}{dz} = 0 \text{ at } z = 0 \text{ and } \theta \to 0 \text{ as } z \to \infty. \tag{9}$$

To solve the linear equation (8) we take

$$\theta(z,t) = \theta_0(z) + \varepsilon e^{i\omega t}\theta_1(z) + \varepsilon e^{-i\omega t}\theta_2(z). \tag{10}$$

Substituting (5) and (10) in (8), equating harmonic and non-harmonic terms, neglecting the coefficients of ε^3, we obtain the following ordinary differential equations:

$$\theta_0'' + P\theta_0' = -PEc\left[[p_0'\bar{p}_0' + \varepsilon^2(p_1'\bar{p}_1' + p_2'\bar{p}_2') - \frac{\varepsilon^2 AP}{2}(\theta_1' + \theta_2')\right] \tag{11}$$

$$\theta_1'' + P\theta_1' - \frac{i\omega P}{4}\theta_1 = -PEc\left(p_0'\bar{p}_2' + p_1'\bar{p}_0' - \frac{PA}{2}\theta_0'\right) \tag{12}$$

$$\theta_2'' + P\theta_2' - \frac{i\omega P}{4}\theta_2 = -PEc\left(p_0'\bar{p}_1' + p_2'\bar{p}_0' - \frac{PA}{2}\theta_1'\right) \tag{13}$$

where the prime denotes the derivation with respect to z. These are still coupled equations and not easily solvable. So we again assume

$$\theta_0(z) = \theta_{00}(z) + A\theta_{01}(z), \qquad \theta_1 = \theta_{10}(z) + A\ \theta_{11}(z)$$
$$\theta_2 = \theta_{20}(z) + A\theta_{21}(z). \tag{14}$$

Substituting (14) in the equations (11) to (13), comparing the coefficients of different powers of A, neglecting those of A^2, we get

$$\theta_{00}'' + P\theta_{00}' = -PEc\,[p_0'\bar{p}_0' + \varepsilon^2(p_1'\bar{p}_1' + p_2'\bar{p}_2')] \tag{15}$$

$$\theta_{01}'' + P\theta_{01}' = -\frac{\varepsilon^2 P}{2}(\theta_{10}' + \theta_{20}') \tag{16}$$

$$\theta_{10}'' + P\theta_{10}' - \frac{i\omega P}{4}\theta_{10} = -PEc(p_0'\bar{p}_2' + p_1'\bar{p}_0') \tag{17}$$

$$\theta_{11}'' + P\theta_{11}' - \frac{i\omega P}{4}\theta_{11} = -\frac{P}{2}\theta_{00}' \tag{18}$$

$$\theta_{20}'' + P\theta_{20}' + \frac{i\omega P}{4}\theta_{20} = -PEc(p_0'\bar{p}_1' + p_2'\bar{p}_0') \tag{19}$$

$$\theta_{21}'' + P\theta_{21}' + \frac{i\omega P}{4}\theta_{21} = -\frac{P}{2}\theta_{00}'. \tag{20}$$

From (9) and (14) it follows that the boundary conditions of the above equations are

$$\theta_{00}' = \theta_{01}' = \theta_{10}' = \theta_{11}' = \theta_{20}' = \theta_{21}' = 0 \text{ at } z = 0$$
$$\theta_{00},\ \theta_{01},\ \theta_{10},\ \theta_{11},\ \theta_{20},\ \theta_{21} \to 0 \text{ as } z \to \infty. \tag{21}$$

The solutions of the equations (14) to (20) subject to the boundary conditions (21) have been analytically derived. Since they are too complicated we shall give here only the expression of θ_0 as follows:

$$\theta_0 = \frac{PEc|m|^2}{M_4}\left[1 + \frac{\varepsilon^2 A^2 |m|^2}{4}\left(\frac{|M_1|^2 + |M_2|^2}{|M_1|^2|M_2|^2}\right)\right]\left(\frac{2M_r}{P} e^{-Pz} - e^{-2M_r z}\right) +$$

$$+ \frac{\beta|N|^2}{N_4}\left[1 - \frac{A}{2}\left(\frac{\overline{m}}{\overline{M}_1} + \frac{m}{M_1}\right) + \frac{A^2}{4}\frac{|m|^2}{|M_1|^2}\right]\left(\frac{2N_r}{P} e^{-Pz} - e^{-2N_r z}\right) +$$

$$+ \frac{A\overline{m}N}{2\alpha_1}\left(\frac{\overline{M}}{\overline{M}_1} - \frac{A}{2}\frac{|m|^2}{|M_1|^2}\right)\left(\frac{\overline{m} + N}{P} e^{-Pz} - e^{-(\overline{m}+N)z}\right) +$$

$$+ \frac{Am\overline{N}}{2\alpha_2}\left(\frac{m}{M_1} - \frac{A}{2}\frac{|m|^2}{|M_1|^2}\right)\left(\frac{m + \overline{N}}{P} e^{-Pz} - e^{-(\overline{m}+N)z}\right) +$$

$$+ \frac{\beta_1|L|^2}{4L_1}\left(\frac{2L_r}{P} e^{-Pz} - e^{-2L_r z}\right) - \frac{\beta_1 L\overline{m}}{4\alpha_3}\left(\frac{L + \overline{m}}{P} e^{-Pz} - e^{-(L+\overline{m})z}\right) -$$

$$- \frac{\beta_1 \overline{L} m}{4\alpha_4}\left(\frac{\overline{L} + m}{P} e^{-Pz} - e^{-(\overline{L}+m)z}\right) +$$

$$+ A\left\{\frac{P\beta}{2H(H-P)}\left[\frac{A(\overline{L}+m)\overline{L}|m|^2}{2\left(\alpha_4 - \frac{i\omega P}{4}\right)\overline{M}_1} - \frac{(\overline{m}+N)\overline{m}N}{\alpha_5 - \frac{i\omega P}{4}}\left(1 - \frac{Am}{2M_1}\right) +\right.\right.$$

$$\left.+ \frac{AM_r|m|^2}{M_5}\left(\frac{m}{M_1} + \frac{\overline{M}}{\overline{M}_2}\right)\right]\left(\frac{H}{P} e^{-Pz} - e^{-Hz}\right) -$$

$$- \frac{P\beta A(\overline{L}+m)\overline{L}|m|^2}{4\alpha_4\left(\alpha_4 - \frac{i\omega P}{4}\right)\overline{M}_1}\left(\frac{\overline{L}+m}{P} e^{-Pz} - e^{-(\overline{L}+m)z}\right) +$$

$$+ \frac{P\beta(\overline{m}+N)\overline{m}N}{2\alpha_5\left(\alpha_5 - \frac{i\omega P}{4}\right)}\left(1 - \frac{Am}{2M_1}\right)\left(\frac{\overline{m}^2 + N}{P} e^{-Pz} - e^{-(\overline{m}+N)z}\right) +$$

$$+ \frac{P\beta(m+\overline{N})m\overline{N}}{2\alpha_6\left(\alpha_6 + \frac{i\omega P}{4}\right)}\left(1 - \frac{A^m}{2M_1}\right)\left(\left(\frac{\overline{m}+N}{P} e^{-Pz} - e^{-(m+\overline{N})z}\right) -\right.$$

$$- \frac{P\beta A(L+\overline{m})L\overline{m}}{4\alpha_3\left(\alpha_3 + \frac{i\omega P}{4}\right)M_3}\left(\frac{L+m}{P} e^{-Pz} - e^{-(L+\overline{m})z}\right) +$$

$$+ \frac{P\beta AM_r|m|^2}{2M_4}\left[\frac{1}{M_5}\left(\frac{m}{M_1} + \frac{m}{\overline{M}_2}\right) + \frac{1}{M_6}\left(\frac{-}{\overline{M}_1} + \frac{m}{M_2}\right)\right]\left(\frac{2M_r}{P} e^{-Pz} - e^{-2M_r z}\right) -$$

$$- \frac{P\beta}{2R(R-P)}\left[\frac{(m+\bar{N})m\bar{N}}{\alpha_6 + \frac{i\omega P}{4}}\left(1 - \frac{A\bar{m}}{2\bar{M}_1}\right) - \right.$$

$$\left. - \frac{A(L+\bar{m})L|m|^2}{2\left(\alpha_3 + \frac{i\omega P}{4}\right)M_3} + \frac{AM_r|m|^2}{\bar{M}_4}\left(\frac{\bar{m}}{\bar{M}_1} + \frac{m}{M_2}\right)\right]\left(\frac{R}{P}e^{-Py} - e^{-Ry}\right)\Bigg\}$$

where

$$L = \frac{1}{2}\left(1 + \sqrt{1 + i(8E - \omega)}\right), \quad L_r = \text{Real part of } L$$

$$L_1 = 4L_r^2 - 2PL_r, \quad M_r = \text{Real part of } m$$

$$M_1 = m^2 - m - i(2E + \omega/4), \quad M_2 = m^2 - m - i(2E - \omega/4)$$

$$M_3 = \frac{M_1\bar{M}_1 + \bar{M}_2M_2}{M_1\bar{M}_1M_2\bar{M}_2}, \quad M_4 = 4M_r^2 - 2PM_r, \quad M_5 = 4M_r^2 - 2PM_r - i\omega P/4$$

$$M_6 = 4M_r^2 - 2PM_r + i\omega P/4$$

$$N = \frac{1}{2}\left(1 + \sqrt{1 + i(8E + \omega)}\right), \quad N_r = \text{Real part of } N$$

$$H = \frac{1}{2}\left(P + \sqrt{P^2 + i\omega P}\right), \quad R = \frac{1}{2}\left(P + \sqrt{P^2 - i\omega P}\right)$$

$$\alpha_1 = \bar{m}^2 + N^2 + 2\bar{m}N - P\bar{m} - PN$$

$$\alpha_2 = m^2 + \bar{N}^2 + 2m\bar{N} - Pm - P\bar{N}$$

$$\alpha_3 = L^2 + \bar{m}^2 + 2L\bar{m} - PL - P\bar{m}, \quad \alpha_4 = \bar{L}^2 + m^2 + 2\bar{L}m - P\bar{L} - Pm$$

$$\alpha_5 = \bar{m}^2 + N^2 + 2\bar{m}N - P\bar{m} - PN$$

$$\alpha_6 = m^2 + \bar{N}^2 + 2m\bar{N} - Pm - P\bar{N}$$

$$\beta = PEc\varepsilon^2, \quad \beta_1 = \frac{|m|^2}{|M_2|^2}PEc\varepsilon^2A^2.$$

3 Results. Substituting the obtained expressions of θ_0, θ_1 and θ_2 in equation (10) and taking real part we can obtain the temperature profiles. However it is more interesting to study the mean plate temperature $\theta_0(0)$, the amplitude $|Q_1| + |\bar{Q}_1|$ and the phase $\mathrm{tg}\gamma$ of the fluctuating part of the plate temperature which are given by

$$\theta(0) = \theta_0(0) + \varepsilon\{|Q_1| + |\bar{Q}_1|\}\cos(\omega t + \gamma)$$

$$Q_1 = Q_{1r} + iQ_{1i} = \theta_1(0), \quad \mathrm{tg}\gamma = Q_{1i}/Q_{1r}.$$

The numerical values of $\theta_0(0)$, $|Q_1| + |\bar{Q}_1|$ and $\mathrm{tg}\gamma$ are entered in the accompanying Table

Table 1

Values of $\theta_0(0)$, $|Q_1| + |\bar{Q}_1|$ and tg γ

A	Ec	P	E	ω	$\theta_0(0)$	$\|Q_1\| + \|\bar{Q}_1\|$	tg γ
0.3	0.01	0.71	0.2	5	0.0053	0.00249	−0.707
0.3	0.01	0.71	0.2	10	0.0053	0.00185	−0.850
0.3	0.01	0.71	0.4	5	0.0055	0.00268	−0.737
0.3	0.01	7.00	0.2	5	0.0053	0.00376	−0.511
0.3	0.02	0.71	0.2	5	0.0105	0.00499	−0.707
0,6	0.01	0.71	0.2	5	0.0052	0.00243	−0.639

We observe from this table that an increase in Ekman number E or greater viscous dissipative heat causes an increase in the mean plate temperature. But the mean plate temperature decreases with increasing A. The effect of Prandtl number P and the frequency ω on the mean plate temperature are not very significant. Also the amplitude of the plate temperature increases with increasing Ec, P, E or ω whereas it decreases with increasing A. The values of the phase tgγ of the fluctuating parts of the plate temperature being always negative we conclude that there is always a phase-lag.

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ASUPRA TRANSFERULUI DE CĂLDURĂ ÎN STRATUL LIMITĂ NESTAȚIONAR ÎNTR-O SCURGERE ÎN ROTAȚIE CU SUCȚIUNE VARIABILĂ

(Rezumat)

Se prezintă o soluție analitică aproximativă pentru problema transferului de căldură într-un fluid vâscos incompresibil în rotație rigidă împreună cu o placă plană poroasă infinită. Se consideră cazul unei viteze de sucțiune variabile în timp. Analiza rezultatelor obținute pune în evidență o creștere a temperaturii plăcii odată cu creșterea rotației fluidului (numărului Ekman E) și a disipației energiei (numărului Eckert Ec).

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SUR L'ALLURE DE QUELQUES COURBES D'APPROXIMATION STATISTIQUE

E. OANCEA, M. RĂDULESCU

On sait que les procédés statistiques d'approximation suivent la détermination analytique de la courbe de dépendance des deux caractéristiques statistiques, mais ils ne donnent aucune indication relativement à l'allure de cette courbe.

On revient maintenant sur un procédé considéré par nous dans [2], en le complétant avec certaines indications sur l'allure de la courbe d'approximation, obtenues de données d'observation.

Le procédé donné dans [2] se rapporte au cas oú la dépendance de deux caractéristiques statistiques est décrite par l'équation:

$$y = a_N x^N + a_{N-1} x^{N-1} + \dots + a_1 x + a_0,$$

qui a été présentée pour $N = 2, 3$, les cas les plus fréquents dans la pratique.

1. Soit X et Y deux caractéristiques statistiques pour lesquelles on connaît les donnees d'observation

$$(x_i, y_i) \quad i = \overline{1, k} \tag{1}$$

$$y_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}$$

On suppose que par la recherche statistique on a ètabli que la dépendance entre X et Y est de forme

$$y = ax^2 + bx + c \tag{2}$$

où les coefficients a, b, c ont été déterminés par une méthode d'approximation [2], comme il suit:

$$2a = \frac{1}{k-1} \sum_{i=1}^{k-1} u_i \tag{3}$$

où

$$u_i = \frac{\bar{y}_{i+1} - \bar{y}_i - \frac{1}{k-1}(\bar{y}_k - \bar{y}_1)}{k_i}, \quad i = \overline{1, k-1}$$

$$k_i = d^2\left(i - \frac{k}{2}\right), \quad d = x_{i+1} - x_i, \quad i = \overline{1, k-1}$$

et

$$b = \frac{1}{d(k-1)}(\bar{y}_k - \bar{y}_1) - 2a\bar{x}^* \tag{4}$$

$$\bar{x}^* = \frac{1}{k-1} \sum_{i=1}^{k-1} \frac{x_i + x_{i+1}}{2}.$$

En utilisant les données (1) on détermine un certain intervalle qui donne des indications relativement à la variation de la tangente à la courbe d'approximation dans un point d'abscisse $x \in \mathbf{R}$, en tenant compt du mode de la distribution des valeurs de y dans le voisinage de x.

Dans ce but, on associe à la dérivée de la fonction (2) une variable aléatoire convenablement choisie à l'aide de laquelle on obtient un intervalle de confiance.

Dans l'expression de la dérivée interviennent les coefficients $2a$ et b. On associe à ces coefficients des variables alétoires d'échantillon, pour $2a$:

$$\bar{U} = \frac{1}{k-1} \sum_{i=1}^{k-1} U_i \tag{5}$$

où

$$U_i = \frac{\bar{Y}_{i+1} - \frac{1}{k-1}(\bar{Y}_k - \bar{Y}_1) - \bar{Y}_i}{k_i}$$

dans l'hypothèse que $\bar{Y}_i$, $i = \overline{1, k}$ sont des variables aléatoires indépendantes normales $N(\bar{y}_i, \sigma_i)$. Alors $\bar{U}$ est une variable aléatoire normale $N(M_{\bar{U}}, \sigma_{\bar{U}})$ avec la valeur moyenne

$$M_{\bar{U}} = \frac{1}{k-1} \sum_{i=1}^{k-1} \frac{\bar{y}_{i+1} - \bar{y}_i - \frac{1}{k-1}(\bar{y}_k - \bar{y}_1)}{k_i} = 2a$$

et la variance

$$\sigma^2_{\bar{U}} = \frac{1}{(k-1)^2} \sum_{i=1}^{k-1} \frac{(k-1)^2(\sigma^2_{i+1} + \sigma^2_i) + \sigma^2_k + \sigma^2_1}{k^2_i}$$

Tenant compte que

$$y' = 2ax + b = 2a(x - \bar{x}^*) + \frac{1}{d(k-1)}(\bar{y}_k - \bar{y}_1)$$

on considère la variable aléatoire d'échantillon

$$V = \frac{1}{d(k-1)}(\bar{Y}_k - \bar{Y}_1) \tag{6}$$

qui est aussi normale $N(M_V, \sigma_V)$

$$M_V = \frac{1}{d(k-1)}(\bar{y} - \bar{y}_1)$$

$$\sigma^2_V = \frac{1}{d^2(k-1)^2}(\sigma^2_k + \sigma^2_1)$$

Alors à y' correspond la statistique

$$Z = \overline{U}(x - \bar{x}^*) + V$$

qui est normale $N(M_Z, \sigma_Z)$, avec la valeur moyenne et la variance, respectivement :

$$M_Z = M_{\overline{U}}(x - \bar{x}^*) + M_V$$

$$\sigma_Z^2 = (x - x^*)^2 \sigma_{\overline{U}}^2 + \sigma_V^2,$$

$x - \bar{x}^*$ est une constante.

Par conséquent, en utilisant la statistique

$$\bar{Z} = \frac{Z - M_Z}{\sigma_Z}$$

qui est normale $N(0, 1)$ et une probabilité de risque q, de la rélation

$$P(\bar{Z} \in (-\bar{z}_q, \bar{z}_q)) = 1 - q$$

on obtient pour un x fixe ($x \in R$) un intervalle de confiance pour y' :

$$M_Z - \sigma_Z \bar{z}_q < y' < M_Z + \sigma_Z z_q \tag{7}$$

ou

$$M_{\bar{U}}(x - \bar{x}^*) + M_V - \sqrt{(x - \bar{x}^*)^2 \sigma_{\overline{U}}^2 + \sigma_V^2}\, \bar{z}_q < y' < M_{\overline{U}}(x - \bar{x}^*) + $$
$$+ \sqrt{(x - \bar{x}^*)^2 \sigma_{\overline{U}}^2 + \sigma_V^2}\, \bar{z}_q \tag{8}$$

Remarques

1) On observe que l'intervalle de confiance (7) de y' dépend de la valeur $x - \bar{x}^*$. Quand x parcourt l'ensemble des nombres réels on obtient de fait une région de confiance associée à y'.

2) Dans le cas où σ_Z^2 est inconnue, c'est-à-dire σ_i^2, $i = \overline{1, k}$ sont inconnues, on peut utiliser une statistique de type Student, qui contient l'estimateur de la variance thèorique σ_Z^2. Cette variance peut être calculée avec les données d'observation analogue au procédé de [2].

3) Un intervalle de confiance pour y' dans un point d'abscisse $x = x_0$, donne une indication sur le mode de la variation de la tangente à la courbe d'approximation obtenue dans le voisinage du point considéré. Cette variation indique le mode de croissance ou de décroissance de la fonction (2) dans le voisinage du point x_0.

4) Parce que la courbe d'approximation est une parabole de deuxième degré, le problème de la concavité ne se pose pas, cela étant déterminé par la valeur de $2a$.

5) Pour établir la forme de la région de confiance associée à y', on obtient de (8) les courbes frontière de cette region :

$$y = M_{\bar{U}}(x - \bar{x}^*) + M_V \pm \bar{z}_q \sqrt{(x - \bar{x}^*)^2 \sigma_{\bar{U}}^2 + \sigma_V^2}$$

ou dans la forme rationale :

$$(x - \bar{x}^*)^2(\sigma_{\bar{U}}^2 \bar{z}_q^2 - M_{\bar{U}}^2 - y^2 + 2y(x - \bar{x}^*)M_{\bar{U}} - 2(x - \bar{x}^*)M_{\bar{U}}M_V +$$
$$+ 2M_V y + \bar{z}_q^2 \sigma_V^2 - M_V^2 = 0 \qquad (9)$$

On observe que l'équation (9) représente une hyperbole parce que

$$\begin{vmatrix} \sigma_{\bar{U}}^2 \bar{z}_q^2 - M_{\bar{U}}^2 & M_{\bar{U}} \\ M_{\bar{U}} & -1 \end{vmatrix} = -\sigma_{\bar{U}}^2 \bar{z}_q^2 < 0$$

Par conséquent, les deux branches de l'hyperbole (9) constituent les courbes frontière de la région de confiance (7).

6) L'intervalle de confiance pour chaque $x \in \mathbf{R}$, est symétrique relativement aux points de la droite correspondante à y', ayant les extrémités sur les branches de l'hyperbole (9).

7) L'intervalle de confiance qui donne la meilleure information relativement à la variation de la tangente correspond à $x = \bar{x}^*$.

8) Pour $x \to \infty$, les intervalles de confiance correspondants deviennent infinis. Mais pratiquement, intéresse seulement la variation de la tangente à la courbe d'approximation pour x appartenant à l'intervalle fini dans lequel varient les. données d'observation.

2. Dans le cas où la dépendance entre X et Y est :

$$y = \alpha x^3 + \beta x^2 + \gamma x + \delta \qquad (10)$$

pour obtenir des indications relativement à l'allure de la courbe d'approximation (10) on utilise les dérivées de y :

$$y' = 3\alpha x^2 + 2\beta x + \gamma \qquad (11)$$

$$y'' = 6\alpha x + 2\beta. \qquad (12)$$

Les coefficients 3α, 2β et γ ont été déterminés conformément à une méthode d'approximation [3] comme il suit :

$$3\alpha = \frac{1}{k-1} \sum_{i=1}^{k-1} u_i$$

où

$$u_i = \frac{(k-1)(\bar{y}_{i+1} - \bar{y}_i) - (\bar{y}_k - \bar{y}_1) - (\bar{y}_k - \bar{y}_{k-1} - \bar{y}_2 + \bar{y}_1)\,h_i}{(h_i^2 - \overline{M}_2 - \bar{x}^2)d(k-1)}$$

$h_i = x_i - \bar{x}$, $i = \overline{1, k-1}$, $d = x_{i+1} - x_i$, $i = \overline{1, k}$

$$\overline{M}_2 = \frac{1}{k-1} \sum_{i=1}^{k-1} x_i^2$$

$$\bar{x} = \frac{1}{k-1} \sum_{i=1}^{k-1} x_i$$

$$2\beta = \frac{\bar{y}_k - \bar{y}_{k-1} - \bar{y}_2 + \bar{y}_1}{d^2(k-1)} - 6\alpha\bar{x},$$

$$\gamma = \frac{1}{k-1}\Big[\bar{y}_k - \bar{y}_1 - 3\alpha\overline{M}_2(k-1) -$$

$$- \frac{1}{d^2}(\bar{y}_k - \bar{y}_{k-1} - \bar{y}_2 + \bar{y}_1)\bar{x} + 6\alpha(k-1)\bar{x}^2\Big]$$

L'expression de la dérivée

$$y' = 3\alpha x^2 + \left(\frac{\bar{y}_k - \bar{y}_{k-1} - \bar{y}_2 + \bar{y}_1}{d^2(k-1)} - 6\alpha\bar{x}\right) x +$$

$$+ \frac{1}{k-1}\left[\bar{y}_k - \bar{y}_1 - 3\alpha\overline{M}_2(k-1) - \frac{1}{d^2}(\bar{y}_k - \bar{y}_{k-1} - \bar{y}_2 + \bar{y}_1)x + 6\alpha(k-1)\bar{x}^2\right]$$

avec les variabiles d'echantillon U, V, W, devient

$$Y' = U(x^2 - 2\bar{x}x + 2\bar{x}^2 - \overline{M}_2) + V\frac{x - \bar{x}}{d^2(k-1)} + \frac{1}{k-1}W \qquad (13)$$

où

$$U = \frac{1}{k-1} \sum_{i=1}^{k-1} U_i$$

$$U_i = \frac{(k-1)(\overline{Y}_{i+1} - \overline{Y}_i) - (\overline{Y}_k - \overline{Y}_1) - (\overline{Y}_k - \overline{Y}_{k-1} - \overline{Y}_2 + \overline{Y}_1)h_i}{(h_i^2 - \overline{M}_2 - \bar{x}^2)d(k-1)}$$

Les variables aléatoires $\overline{Y}_i$, $\overline{i = 1, k}$ sont des variabiles aléatoires indépendantes, normales $N(\bar{y}_i, \sigma_i)$.

On obsèrve que U est une variable aléatoire normale $N(M_U, \sigma_U)$, où

$$M_U = \frac{1}{k-1}\sum_{i=1}^{k-1} M_{U_i}, \quad \sigma_U^2 = \frac{1}{(k-1)^2}\sum_{i=1}^{k-1}\sigma_{U_i}^2$$

$$M_{U_i} = \frac{(k-1)(\bar{y}_{i+1}-\bar{y}_i)-(\bar{y}_i-\bar{y}_1)-(\bar{y}_k-\bar{y}_{k-1}-\bar{y}_2+\bar{y}_1)h_i}{(h_i^2-\overline{M}_2-\bar{x}^2)d(k-1)}$$

$$\sigma_{U_i}^2 = \frac{(k-1)^2(\sigma_{i+1}^2+\sigma_i^2)+\sigma_k^2+\sigma_1^2+(\sigma_k^2+\sigma_{k-1}^2+\sigma_2^2+\sigma_1^2)h_i^2}{(h_i^2-\overline{M}_2-\bar{x}^2)^2 d^2(k-1)^2}$$

Et la variable aléatoire V est :

$$V = \overline{Y}_k - \overline{Y}_{k-1} - \overline{Y}_2 + \overline{Y}_1$$

qui est normale $N(M_V, \sigma_V)$:

$$M_V = \bar{y}_k - \bar{y}_{k-1} - \bar{y}_2 + \bar{y}_1$$
$$\sigma_V^2 = \sigma_k^2 + \sigma_{k-1}^2 + \sigma_2^2 + \sigma_1^2$$

et la variable aléatoire W est

$$W = \overline{Y}_k - \overline{Y}_1$$

qui est normale $N(M_W, \sigma_W)$:

$$M_V = \bar{y}_k - \bar{y}_1$$
$$\sigma_W^2 = \sigma_k^2 + \sigma_1^2.$$

De la rélation (13) il suit que Y' est une variabile aléatoire normale $N(M_{Y'}, \sigma_{Y'})$, avec

$$M_{Y'} = (x^2 - 2x\bar{x} + 2\bar{x}^2 - \overline{M}_2)M_U + \frac{x-\bar{x}}{d^2(k-1)}M_V + \frac{1}{k-1}M_W$$

$$\sigma_{Y''}^2 = (x^2 - 2x\bar{x} + 2\bar{x}^2 - \overline{M}_2)^2\sigma_U^2 + \frac{(x-x)^2}{d^4(k-1)^2}\sigma_V^2 + \frac{1}{(k-1)^2}\sigma_W^2$$

Alors la statistique.

$$Z = \frac{Y' - M_{Y'}}{\sigma_{Y'}}$$

est normale $N(0, 1)$. En choisissant une probabilité de risque q, on obtient pour un x fixé, $x \in \mathbf{R}$, un intervalle de confiance pour Y' :

$$M_{Y'} - z_q\sigma_{Y'} < Y' < M_{Y'} + z_q\sigma_{Y'} \tag{14}$$

où

$$P(|Z| < z_q) = 1 - q.$$

Remarques

1. On observe que l'intervalle de confiance (14) de y' dépend de la valeur $x - \bar{x}$. Quand x parcourt l'ensemble des nombres réels, on obtient en fait une région de confiance associée à y'.

2. Un intervalle de confiance pour, y' dans un point d'abscisse $x = x_0$ donne une indication sur le mode de la variation de la tangente à la courbe d'approximation obtenue, dans le voisinage du point considéré. Cette variation indique la manière de croissance ou de décroissance de la fonction (10) dans le voisinage du point x_0.

Pour établir la région de confiance associée à Y', de (14) on obtient les courbes frontière de cette région, c'est-à-dire:

$$y = y_1 \pm y_2 \tag{15}$$

où

$$y_1 = [(x - \bar{x})^2 + \bar{x}^2 - \overline{M}_2]M_U + \frac{x - \bar{x}}{d^2(k-1)} M_V + \frac{1}{k-1} M_W$$

$$y_2 = z_q \sqrt{[(x - \bar{x})^2 + \bar{x}^2 - \overline{M}_2]^2 \sigma_U^2 + \frac{(x - \bar{x}^2)}{d^4(k-1)^2} \sigma_V^2 + \frac{1}{(k-1)^2} \sigma_W^2}.$$

On obsèrve que tant y_1 que y_2 sont des courbes réelles et que y_1 répresente une parabole, et y_2 est une courbe de quatrième degré de types "hyperbolique". La courbe associée à y' est contenue en fait, dans la région de confiance.

En tenant compte de l'expression de la deuxième dérivée (12) de Y'', on lui associe la variable aléatoire d'echantillon

$$Y'' = 2(x - \bar{x})\, U + \frac{1}{d^2(k-1)} V$$

qui est normale $N(M_{Y''}, \sigma_{Y''})$, avec

$$M_{Y''} = 2(x - \bar{x})M_U + \frac{1}{d^2(k-1)} M_V$$

$$\sigma_{Y''}^2 = 4(x - \bar{x})^2 \sigma_U^2 + \frac{1}{d^4(k-1)^2} \sigma_V^2.$$

Par conséquent, en utilisant la statistique

$$Z^* = \frac{Y'' - M_{Y''}}{\sigma_{Y''}}$$

qui est normale $N(0, 1)$ et en choisissant une probabilité de risque q, de la rélation

$$P(|Z^*| < z_q^*) = 1 - q$$

on obtient l'intervalle de confiance pour la variabile aléatoire Y'':

$$M_{Y''} - z_q^* \sigma_{Y''} < Y'' < M_{Y''} + z_q^* \sigma_{Y''} \tag{16}$$

On observe que pour $x \in \mathbf{R}$, les inégalités (16) déterminent une région de confiance associée à Y'', dont les courbes frontière sont :

$$y = 2(x - \bar{x})M_U + \frac{M_V}{d^2(k-1)} \pm z_q^* \sqrt{4\sigma_U^2(x-\bar{x})^2 + \frac{\sigma_V^2}{d^4(k-1)^2}}$$

ou en forme rationale :

$$4(M_U^2 - \sigma_U^2 z_q^{*2})(x-\bar{x})^2 + y^2 - 4M_U(x-\bar{x})y + 4\frac{M_U M_V}{d^2(k-1)}(x-\bar{x}) - 2\frac{M_V}{d^2(k-1)}y + \frac{M^2 - z_q^* \sigma_V^2}{d^4(k-1)^2} = 0 \quad (17)$$

Cette courbe (17) est une hyperbole. On voit que la droite

$$y = 6\alpha x + 2\beta$$

est contenue dans la région de confiance donnée par (16).

Les intervalles (14) et (16) correspondant à une valeur x fixée (x appartenant à l'intervalle des données d'observation) donnent des indications sur la manière de croissance ou décroissance de la courbe d'approximation dans le voisinage du point considéré, et aussi sur la variation de la valeur de la convexité (concavité) de la courbe dans le voisinage de ce point.

Ces indications obtenues avec les données d'observation représentent un supplément qualitatif de l'étude de la courbe d'approximation considérée.

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ASUPRA ALUREI UNOR CURBE DE AJUSTARE STATISTICĂ

(Rezumat)

În lucrare se consideră procedeul de ajustare dat în [2], completîndu-l cu anumite indicaţii referitoare la alura curbei de ajustare obţinută prin datele de observaţie. În cazul cînd dependenţa dintre caracteristicile X şi Y este de forma (2), se construieşte un interval de încredere (7) pentru derivata y'. Dacă dependenţa dintre X şi Y este de forma (10) se construieşte un interval de încredere pentru y', (14) şi de asemenea un interval de încredere pentru y'', (16).

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXVI, 4, 1981

SUR LE THÉORÈME D'APPROXIMATION DE STONE-WEIERSTRASS

SORIN GH. GAL

Le théorème de Weierstrass est bien connu qui nous dit que chaque fonction continue $f: [a, b] \to \mathbf{R}$, $[a, b] \subset \mathbf{R}$, est la limite uniforme sur $[a, b]$ d'une suite de polynômes $(P_n)_{n\in\mathbf{N}}$.

Le but de cette note est de montrer que la suite $(P_n)_{n\in\mathbf{N}}$ peut être choisie monotone, c'est-à-dire $P_n(x) < P_{n+1}(x)$, $\forall x \in [a, b]$, $\forall x \in \mathbf{N}$ dans le cas croissant, respectivement $P_{n+1}(x) < P_n(x)$, $\forall x \in [a, b]$, $\forall n \in \mathbf{N}$ dans le cas décroissant. D'ailleurs, on démontre ce résultat dans le cadre plus général du théorème de Stone-Weierstrass, pour les cas réel et complexe.

Dans le cas d'approximation par des polynômes, on donne une démonstration simple, en utilisant les polynômes de Bernstein.

§1. Soient K un espace topologique compacte et l'algèbre réelle, $C(K; \mathbf{R}) = \{f: K \to \mathbf{R}\,;\, f \text{ — continue}\}$. Notons avec $A \subset C(K; \mathbf{R})$, une sous-algèbre de $C(K; \mathbf{R})$ ayant les propriétés:

I. *si* $f \in C(K; \mathbf{R})$ *est constante sur* K, *alors* $f \in A$,

II. $\forall x_1, x_2 \in K$, $x_1 \neq x_2$, $\exists F \in A$, $F(x_1) \neq F(x_2)$.

Avec ces notations, on sait que le théorème de Weierstrass-Stone (le cas réel) peut s'exprimer ainsi:

$$\text{„}\forall f \in C(K; \mathbf{R}),\ \forall \varepsilon > 0,\ \exists P \in A \text{ (qui dépend de } f \text{ et } \varepsilon\text{)},$$
$$\text{tel que } |f(x) - P(x)| < \varepsilon,\ \forall x \in K\text{''}.$$

On a:

THÉORÈME 1.1. *Si* $f, g \in C(K; \mathbf{R})$ *ont la propriété* $f(x) - g(x) \geq \rho > 0$, $\forall x \in K$, *alors il existe une fonction* $P \in A$, *tel que*

$$g(x) < P(x) < f(x),\ \forall x \in K.$$

Démonstration. En appliquant le théorème de Stone-Weierstrass pour la fonction $h \in C(K; \mathbf{R})$, $h(x) = \frac{f(x) + g(x)}{2}$ et $\varepsilon = \frac{\rho}{4}$, il existe une fonction $P \in A$ en vérifiant $|h(x) - P(x)| < \frac{\rho}{4}$, $\forall x \in K$, donc

$$P(x) - \frac{\rho}{4} < h(x) < P(x) + \frac{\rho}{4},\ \forall x \in K. \tag{1}$$

Mais $h(x) - g(x) = \frac{f(x) - g(x)}{2} \geq \frac{\rho}{2} > 0$, $\forall x \in K$, donc $h(x) \geq \frac{\rho}{2} + g(x)$, $\forall x \in K$.

Alors de (1) il résulte

$$\frac{\rho}{2} + g(x) \leqslant h(x) < P(x) + \frac{\rho}{4}, \quad \forall x \in K, \tag{1'}$$

d'où évidemment

$$g(x) < P(x), \ \forall x \in K.$$

De même, en tenant compte que

$$f(x) - h(x) = \frac{f(x) - g(x)}{2} \geqslant \frac{\rho}{2} > 0, \ \forall x \in K,$$

il est facile de voir que (1) devient

$$P(x) - \frac{\rho}{4} < h(x) \leqslant f(x) - \frac{\rho}{2}, \quad \forall x \in K, \tag{1''}$$

d'où $P(x) < f(x)$, $\forall x \in K$, c.q.e.d.

COROLLAIRE 1.2. *Pour chaque fonction* $f \in C(K\,;\,\mathbf{R})$, *il existe les suites de fonctions*

$$(P_n)_{n\in\mathbf{N}},\ (Q_n)_{n\in\mathbf{N}},\ P_n,\ Q_n \in A,\ \forall n \in \mathbf{N},$$

convergeant uniformément vers f *(sur* K*), monotone décroissante, respectivement monotone croissante.*

Démonstration. Soient $\varepsilon_n \searrow 0$, g_n, $h_n \in C(K\,;\,\mathbf{R})$, $g_n(x) = f(x) + \varepsilon_{n+1}$, $h_n(x) = f(x) + \varepsilon_n$, $\forall n \in \mathbf{N}$, $\forall x \in K$. En appliquant le théorème 1.1. pour g_n, h_n, il existe les fonctions $P_n \in A$, tel que

$$f(x) + \varepsilon_{n+1} < P_n(x) < f(x) + \varepsilon_n, \ \forall x \in K, \ \forall n \in \mathbf{N}. \tag{2}$$

Evidemment $P_n(x) > f(x)$, $\forall x \in K$, $\forall n \in \mathbf{N}$ et $(P_n)_{n\in\mathbf{N}}$ converge uniformément vers f, sur K. Puis, en écrivant la relation (2) pour ,,$n + 1$'', il résulte

$$f(x) + \varepsilon_{n+2} < P_{n+1}(x) < f(x) + \varepsilon_{n+1}, \ \forall x \in K, \ \forall n \in \mathbf{N}, \tag{2'}$$

et en tenant compte de (2), il résulte

$$P_{n+1}(x) < P_n(x), \ \ \forall x \in K, \ \ \forall n \in \mathbf{N}.$$

En appliquant le théorème 1.1. pour les fonctions

$$p_n(x) = f(x) - \varepsilon_n,\ q_n(x) = f(x) - \varepsilon_{n+1},\ \forall x \in K,\ \forall n \in \mathbf{N},\ \varepsilon_n \searrow 0,$$

on obtient sans difficulté la suite croissante $(Q_n)_{n\in\mathbf{N}}$ c.q.e.d.

§2. Considérons maintenant le cas complexe. Soit l'algèbre complexe $C(K\,;\,\mathbf{C}) = \{f : K \to \mathbf{C}\,;\, f - \text{continue}\}$, $\mathbf{C}$ étant le corp des nombres complexes, et $S \subset \mathbf{C}(K\,;\,\mathbf{C})$ une sous-algèbre ayant les propriétés :

a) *si* $f \in C(K\,;\,\mathbf{C})$, *est constante sur* K, *alors* $f \in S$,

b) $\forall x_1, x_2 \in K$, $x_1 \neq x_2$, $\exists\ F \in S$, $F(x_1) \neq F(x_2)$,

c) *si* $f \in S$, *alors* $\overline{f} \in S$.

Considérons dans $\mathbf{C}$ les relations, pour $u, z \in \mathbf{C}$, $u = v + w \cdot i$, $z = x + iy$, $v, x, w, y \in \mathbf{R}$:

$$u <_1 z \Leftrightarrow v < x, \quad w < y, \tag{3}$$

$$u <_2 z \Leftrightarrow v < x, \quad y < w, \tag{4}$$

$$u <_3 z \Leftrightarrow x < v, \quad w < y, \tag{5}$$

$$u <_4 z \Leftrightarrow x < v, \quad y < w. \tag{6}$$

À lieu:

THÉORÈME 2.1. *Pour chaque fonction f et chaque relation „$<_k$" ($k = \overline{1, 4}$), il existe une suite de fonctions $(F_n)_{n \in \mathbf{N}}$, $F_n \in S$, $\forall n \in \mathbf{N}$, convergeant uniformément vers f (sur K), „$<_k$" monotone décroissante, c'est-à-dire*

$$f(x) <_k F_{n+1}(x) <_k F_n(x), \quad \forall x \in K, \quad \forall n \in \mathbf{N} \quad (f \in C(K; \mathbf{C})).$$

Démonstration. Notons $S_{\mathbf{R}} \subset S$, $S_{\mathbf{R}} = \{f \in S; f : K \to \mathbf{R}\}$. Si $f \in S$, $f = \varphi + i \cdot \psi$, où φ, ψ sont des fonctions réelles, et évidemment $\varphi = \frac{1}{2}(f + \overline{f}) \in S$ et $\psi = \frac{1}{2i}(f - \overline{f}) \in S$.

Il résulte $\varphi \in S_{\mathbf{R}}$ et $\psi \in S_{\mathbf{R}}$. Puis, de la propriété b), il résulte que si $x_1, x_2 \in K$, $x_1 \neq x_2$, $\exists f = \varphi + i\psi \in S$, avec $f(x_1) \neq f(x_2)$, d'où resulte, evidemment, au moins une des inégalités: $\varphi(x_1) \neq \varphi(x_2)$ ou $\psi(x_1) \neq \psi(x_2)$, donc $S_{\mathbf{R}}$ sépare les points de K. Aussi, $S_{\mathbf{R}}$ contient les fonctions constantes réelles. Soit alors $f \in C(K; \mathbf{C})$ quelconque, $f(x) = \varphi(x) + i\psi(x)$. Il est évident que φ et ψ sont des fonctions réelles, continues. Supposons, par exemple, que la relation a lieu $<_1$. Il existe (d'après le corollaire 1.2.) les suites: $(P_n^{(1)})_{n \in \mathbf{N}}$, convergeant uniformément vers φ (sur K), $(P_n^{(2)})_{n \in \mathbf{N}}$, convergeant, uniformément vers ψ (sur K), monotones décroissantes, donc

$$\varphi(x) < P_{n+1}^{(1)}(x) < P_n^{(1)}(x), \quad \forall x \in K, \quad \forall n \in \mathbf{N},$$

$$\psi(x) < P_{n+1}^{(2)}(x) < P_n^{(2)}(x), \quad \forall x \in K, \quad \forall n \in \mathbf{N},$$

et $P_n^{(1)}, P_n^{(2)} \in S_{\mathbf{R}}$, $\forall n \in \mathbf{N}$. Alors la suite

$$(F_n)_{n \in \mathbf{N}}, \quad F_n(x) = P_n^{(1)}(x) + iP_n^{(2)}(x), \quad \forall x \in K, \quad \forall n \in \mathbf{N},$$

converge uniformément vers $\varphi + i\psi = f$ (sur K),

$$f(x) <_1 F_{n+1}(x) <_1 F(x), \quad \forall x \in K, \quad \forall n \in \mathbf{N}$$

et $F_n \in S$, $\forall n \in \mathbf{N}$ c.q.e.d.

Dans les cas des relations $<_2$, $<_3$, $<_4$, les démonstrations sont analogues.

Notons avec $M \subseteq C(K; \mathbf{C})$,

$$M = \{f \in C(K; \mathbf{C}) ; f(x) = \varphi(x) + i\psi(x),\ \varphi,\ \psi \in C(K; \mathbf{R}),\ \varphi(x) \geq 0$$
$$(\text{ou } \varphi(x) \leq 0)\ \ \forall x \in K \text{ et } \psi(x) \geq 0 \ (\text{ou } \psi(x) \leq 0)\ \forall x \in K\}.$$

Alors, a lieu :

COROLLAIRE 2.2. *Pour chaque fonction $f \in M$, il existe la suite $(F_n)_{n\in\mathbf{N}}$, $F_n \in S$, $\forall n \in \mathbf{N}$ convergeant uniformément vers f (sur K), ainsi que la suite de fonctions réelles $(|F_n|)_{n\in\mathbf{N}}$, converge uniformément, monotone décroissante, vers $|f|$. ($|f|(x) = |f(x)|$, $\forall x \in K$, où $|f(x)|$ représente le module du nombre complexe $f(x)$).*

Démonstration. Supposons, par exemple,

$$f(x) = \varphi(x) + i\psi(x), \text{ où } \varphi(x),\ \psi(x) \geq 0,\ \forall x \in K.$$

En appliquant le théorème 2.1., il existe une suite de fonctions

$$(F_n)_{n\in\mathbf{N}},\ F_n \in S,\ \forall n \in \mathbf{N},$$

convergeant uniformément vers f (sur K),

$$F_n(x) = P_n^{(1)}(x) + iP_n^{(2)}(x),\quad \forall x \in K,\ \forall n \in \mathbf{N},$$
$$0 \leq \varphi(x) < P_{n+1}^{(1)}(x) < P_n^{(1)}(x), \tag{7}$$
$$0 \leq \psi(x) < P_{n+1}^{(2)}(x) < P_n^{(2)}(x),\ \forall x \in K,\ \forall n \in \mathbf{N}$$

Il est claire que

$$(|F_n|)_{n\in\mathbf{N}},\ F_n = \sqrt{[P_n^{(2)}]^2 + [P_n^{(2)}]^2},$$

converge uniformément vers $\sqrt{\varphi^2 + \psi^2} = |f|$. Puis, de (7), il résulte

$$0 \leq \varphi^2(x) < [[P_{n+1}^{(1)}(x)]^2 < [P_n^{(1)}(x)]^2, \tag{7'}$$
$$0 \leq \psi^2(x) < [P_{n+1}^{(2)}(x)]^2 < [P_n^{(2)}(x)]^2,\ \forall x \in K,\ \forall n \in \mathbf{N},$$

donc

$$|f| = \sqrt{\varphi^2 + \psi^2} < \sqrt{[P_{n+1}^{(1)}]^2 + [P_{n+1}^{(2)}]^2} < \sqrt{[P_n^{(1)}] + [P_n^{(2)}]^2},$$

c.q.e.d.

Si, par exemple $\varphi(x) \geq 0$, $\forall x \in K$, et $\psi(x) \leq 0$, $\forall x \in K$ alors on choisit la suite $(P_n^{(1)})_{n\in\mathbf{N}}$ monotone décroissante et la suite $(P_n^{(2)})_{n\in\mathbf{N}}$ monotone croissante, c'est-à-dire

$$0 \leq \varphi(x) < P_{n+1}^{(1)}(x) < P_n^{(1)}(x),\ \forall x \in K,\ \forall n \in \mathbf{N},$$
$$P_n^{(2)}(x) < P_{n+1}^{(2)}(x) < \psi(x) \leq 0,\ \forall x \in K,\ \forall n \in \mathbf{N},$$

donc

$$0 \leqslant \varphi^2(x) < [P_{n+1}^{(1)}(x)]^2 < [P_n^{(1)}(x)]^2, \ \forall x \in K, \ \forall n \in \mathbf{N},$$

et

$$0 \leqslant \psi^2(x) < [P_{n+1}^{(2)}(x)]^2 < [P_n^{(2)}(x)]^2, \ \forall x \in K, \ \forall n \in \mathbf{N},$$

d'où évidemment $|f| < |F_{n+1}| < |F_n|, \ \forall n \in \mathbf{N}$ c.q.e.d.

§3. Soit $f \in C_{[0,1]}$, et notons avec

$$(B_n f)_{(x)} = \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k}, \ m \in \mathbf{N},$$

les polynômes de Bernstein.

On sait que nous avons

$$|f(x) - (B_m f)(x)| \leqslant K_0 \cdot \omega\left(f; \frac{1}{\sqrt{m}}\right), \tag{8}$$

où $K_0 = \dfrac{4\,306 + 837\sqrt{6}}{5\,832}$ est la constante de Sikkema et $\omega(f; \delta) = \max\limits_{|x'-x''| \leqslant \delta} |f(x') - f(x'')|$, est le module d'oscillation de la fonction f.

Aussi, on sait que $\lim\limits_{m \to \infty} \omega\left(f; \dfrac{1}{\sqrt{m}}\right) = 0$.

Alors, le resultat suivant est évident

LEMME 3.1. *Il existe une suite $(m_k)_{k \in \mathbf{N}}$, $m_k \in \mathbf{N}$, $m_k \to \infty$, (dépendant de f), ainsi que*

$$\omega\left(f; \frac{1}{\sqrt{m_k}}\right) \leqslant \frac{1}{k^2}, \quad \forall k \in \mathbf{N}. \tag{9}$$

Considérons maintenant la suite de polynômes $(\widetilde{B}_k f)_{k \in \mathbf{N}}$, $(\widetilde{B}_k f)_{(x)} = (B_{m_k} f)(x) + \alpha_k$, $k \in \mathbf{N}$, $x \in [0,1]$, où $(m_k)_{k \in \mathbf{N}}$ est la suite du lemme 3.1. et $\alpha_k = 2K_0 \cdot \sum\limits_{j=k}^{\infty} \dfrac{1}{j^2}$.

A lieu:

THÉORÈME 3.2. *La suite $(B_k f)_{k \in \mathbf{N}}$ converge uniformément vers f (sur $[0,1]$) monotone décroissante.*

Démonstration. En tenant compte de (8) et (9), nous avons:

$$|(B_{m_k} f)_{(x)} - (B_{m_{k+1}} f)(x)| \leqslant |(B_{m_k} f)_{(x)} - f(x)| + |f(x) - (B_{m_{k+1}} f)(x)| \leqslant$$

$$\leqslant K_0 \cdot \left[\omega\left(f; \frac{1}{\sqrt{m_k}}\right) + \omega\left(f; \frac{1}{\sqrt{m_{k+1}}}\right)\right] \leqslant K_0 \cdot \left(\frac{1}{k^2} + \frac{1}{(k+1)^2}\right) < \frac{2K_0}{k^2},$$

$$\forall x \in [0,1], \ \forall k \in \mathbf{N}.$$

Mais

$$(\widetilde{B}_k f)(x) - (\widetilde{B}_{k+1} f)(x) = (B_{m_k} f)(x) - (B_{m_{k+1}} f)(x) + \alpha_k - \alpha_{k+1} = (B_{m_k} f)(x) -$$

$$- (B_{m_{k+1}} f)(x) + \frac{2K_0}{k^2} > 0, \ \forall x \in [0,1], \ \forall k \in \mathbf{N}.$$

Comme α_k est le reste de la série convergeante $2K_0 \cdot \sum_{j=1}^{\infty} \frac{1}{j^2}$, il résulte $\alpha_k \searrow 0$, et le théorème est démontré.

COROLLAIRE 3.3. *La suite de polynômes* $(\bar{B}_k f)_{k\in\mathbf{N}}$, $(\bar{B}_k f)(x) = (B_{m_k} f)(x) - \alpha_k$ $\forall x \in [0,1]$, $\forall k \in \mathbf{N}$, *converge uniformément vers* f (*sur* $[0,1]$), *monotone croissante.*

Remarque. L'ordre d'approximation pour $\widetilde{B}_k f$ et $\bar{B}_k f$ est $o\left(\frac{1}{k-1}\right) + o\left(\frac{1}{k^2}\right)$, $k = 2, 3, \ldots$.

En effet, notons

$$s_k = 1 + \frac{1}{2^2} + \ldots + \frac{1}{(k-1)^2} + \frac{1}{k-1}, \ k = 2, 3, \ldots.$$

Évidemment que

$$\lim s_k = \frac{\pi^2}{6} \quad \left(\text{car} \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}\right).$$

Mais

$$s_k - s_{k+1} = \frac{k^2 - (k-1) - k(k-1)}{k^2(k-1)} = \frac{1}{k^2(k-1)} > 0, \ k = 2, 3, \ldots,$$

donc $s_k \searrow \frac{\pi^2}{6}$, quand $k \to \infty$.

D'ici, il résulte

$$\frac{1}{k-1} > \frac{\pi^2}{6} - \left(1 + \frac{1}{2^2} + \ldots + \frac{1}{(k-1)^2}\right) = \frac{\alpha_k}{2K_0}, \ k = 2, 3, \ldots.$$

Alors,

$$|(\widetilde{B}_k f)(x) - f(x)| = |(B_{m_k} f)(x) + \alpha_k - f(x)| \leqslant$$

$$\leqslant |(B_{m_k} f)(x) - f(x)| + |\alpha_k| \leqslant \frac{K_0}{k^2} + \frac{2K_0}{k-1}, \ k = 2, 3, \ldots$$

(Manuscrit reçu le 5 octombre 1980)

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ASUPRA TEOREMEI DE APROXIMARE A LUI STONE-WEIERSTRASS

(Rezumat)

În lucrare se arată că şirul polinoamelor de aproximare $(P_n)_{n\in N}$ al unei funcţii ce intervine în teorema lui Stone-Weierstrass poate fi ales monoton crescător respectiv descrescător.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXVI, 4, 1981

INTERPOLATION D'HERMITE-FEJÉR SUR DES NOEUDS QUADRUPLES — RACINES DES POLYNOMES D'HERMITE

FLORICA—ELENA CHISĂLIŢĂ

Soit f une fonction continue, définie sur R et $x_1, x_2, \ldots, x_n$ n nombres réels. Le polynôme (unique) de degré $\leqslant 4n-1$ qui vérifie les conditions

$$P_{4n-1}(f;\ x_k) = f(x_k), \qquad k = \overline{1, n}$$

$$\frac{\partial^e P_{4n-1}(f;\ x_k)}{\partial x^e} = 0 \qquad k = \overline{1, n} \text{ et } l = 1,\ 2,\ 3.$$

est défini par

$$P_{4n-1}(f;\ x) = \sum_{k=1}^{n} f(x_k) p_k(x) \tag{1}$$

où

$$p_k(x) = h_k(x) l_k(x)$$

$$l_k(x) = \frac{\omega(x)}{(x - x_k)\omega'(x_k)} \tag{2}$$

(polynômes fondamentaux de Lagrange)

$$h_k(x) = \left\{1 - \frac{2\omega''(x_k)}{\omega'(x_k)}(x - x_k) + \frac{1}{2}\left[5\left(\frac{\omega''(x_k)}{\omega'(x_k)}\right)^2 - \frac{1}{3}\,\frac{\omega'''(x_k)}{\omega'(x_k)}\right]\cdot(x - x_k)^2 + \right.$$
$$+ \frac{1}{6}\left[10\,\frac{\omega''(x_k)\omega'''(x_k)}{\omega'(x_k)\omega'(x_k)} - 15\left(\frac{\omega''(x_k)^3}{\omega'(x_k)}\right)^3 - \frac{\omega'''(x_k)}{\omega'(x_k)}\right]\cdot(x - x_k)^3 \tag{3}$$

où

$$\omega(x) = \prod_{k=1}^{n}(x - x_k)$$

et

$$\sum_{k=1}^{n} p_k(x) = 1. \tag{4}$$

Dans le cas où $f \in C[-1, 1]$ et $x_1, x_2, \ldots, x_n$ sont les racines du polynôme de Tchébychev de 1ère espèce, le prof. D. D. S t a n c u a établi la relation [3]

$$|f(x) - P_{4n-1}(f;\ x)| \leqslant 2\omega\left(\sqrt{\frac{2n^2 + 6n + 1}{3n^3}}\right)$$

Dans ce même cas nous avons obtenu [4]

$$|f(x) - P_{4n-1}(f;\ x)| \leqslant 0(1)\omega\left(\frac{\log n}{n}\right).$$

Dans le présent travail nous allons étudier le polynôme $P_{4n-1}(f;\ x)$ dans le cas où $x_1, x_2, \ldots, x_n$ sont les racines du polynôme d'Hermite $H_n(x)$, défini par

$$e^{-x^2}H_n(x) = (-1)^n \frac{d^n}{dx^n}(e^{-x^2})$$

On sait que $H_n(x)$ satisfait l'équation

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0 \tag{5}$$

et que ses racines ont les propriétés suivantes :

$$\left.\begin{aligned} &-\infty < x_{-m,n} < x_{-m+1,n} < \ldots < x_{-1,n} < 0 < x_{1,n} < x_{2,n} < \ldots < x_{m,n} < +\infty \\ &\qquad\qquad\text{pour } n = 2m, \\ &-\infty < x_{-m,n} < x_{-m+1,n} < \ldots < x_{-1,n} < x_{0n} \equiv 0 < x_{1,n} < x_{2,n} < \ldots < x_{m,n} \\ &\qquad\qquad\text{pour } n = 2m + 1, \end{aligned}\right\} \tag{6}$$

$$x_{kn} = -x_{-k,n} \quad \left(k = 1,\ 2,\ \ldots,\ \left[\frac{n}{2}\right]\right)$$

De (1), (2), (3), (5), (6) on déduit que $P_{4n-1}(f;\ x)$ a la forme suivante

$$P_{4n-1}(f;\ x) = \sum_{k=-m}^{m} f(x_{kn})p_{kn}(x), \tag{7}$$

où

$$p_{kn}(x) = h_{kn}(x)l_{kn}(x), \tag{8}$$

$$l_{kn}(x) = \frac{H_n(x)}{H_n'(x_{kn})(x - x_{kn})}, \tag{9}$$

$$\begin{aligned} h_{kn}(x) = 1 - 4x_{kn}(x - x_{kn}) + \frac{1}{3}\,[22x_{kn}^2 + 4(n-1)](x - x_{kn})^2 + \\ + \frac{1}{3}\,[-24x_{kn}^3 + 2x_{kn}(2n-3) - 20(n-1)](x - x_{kn})^3, \end{aligned} \tag{10}$$

$$k = (0),\ \pm 1,\ \pm 2,\ \ldots,\ \pm\left[\frac{n}{2}\right]$$

$$\sum_{k=-m}^{m} p_{kn}(x) \equiv 1 \quad (n = 1,\ 2,\ \ldots),\ x \in (-\infty,\ +\infty) \tag{11}$$

Notre but est d'évaleur la différence $P_{4n-1}(f;\ x) - f(x)$. Désignons par $\omega_\Delta(f;\ t)$ le module de continuité de f sur l'intervalle $[-\Delta - \varepsilon,\ \Delta + \varepsilon]$ et par

$\omega_\Delta(t)$ un module de continuité sur le même intervalle (v. [2]), Δ étant un no positif. On peut alors énoncer le théorème suivant :

THÉORÈME I. *Soit une fonction* $f \in C[-\Delta, \Delta]$. *Si*

$$\omega_\Delta(f;\, t) = 0[\omega_\Delta(t)]$$

alors, pour tout nombre x *de l'intervalle* $[-\Delta, \Delta]$ *a lieu la relation*

$$|f(x) - P_{4n-1}(f;\, x)| = 0(1) \sum_{i=1}^{[\sqrt{n}]} \omega_\Delta\left(\frac{i}{\sqrt{n}}\right) \frac{1}{i^2}$$

où 0 *dépend uniquement de* Δ *et* f, $|x_{kn}| < \Delta$

Pour la démonstration, donnons d'abord quelques relations co (v. Szegő [1])

$$\begin{cases} H_{2m}(x) = (-1)^m 2^{2m}\, m!\, L_m^{(-1/2)}(y) \\ H_{2m+1}(x) = (-1)^m 2^{2m+1}\, m!\, x\, L_m^{(1/2)}(y) \end{cases}$$

où $y = x^2$ et $L_m^{(\alpha)}(y)$ est le polynôme de Laguerre défini par

$$e^{-y} y^\alpha L_m^{(\alpha)}(y) = \frac{1}{m!} \cdot \frac{d^m}{dy^m} (e^{-y} y^{m+\alpha}),\ (\alpha > -1,\ y \geqslant 0)$$

Les racines $y_{km}^{(\alpha)}$, $k = \overline{1, m}$ de $L_m^{(\alpha)}(y)$

$$0 < y_{1m}^{(\alpha)} < y_{2m}^{(\alpha)} < \ldots < y_{mm}^{(\alpha)} < \infty$$

sont liées aux x_{kn} par les relations

$$\begin{cases} x_{kn} = \sqrt{y_{km}^{(-1/2)}},\ x_{-kn} = -\sqrt{y_{km}^{(-1/2)}} & \text{pour } n = 2m,\ k = 1, 2, \ldots, m \\ x_{kn} = \sqrt{y_{km}^{(1/2)}},\ x_{-kn} = -\sqrt{y_{km}^{(1/2)}} & \text{pour } n = 2m+1,\ k = 1, 2, \ldots, m \end{cases}$$

On sait qu'on a aussi :

$$L_m^{(\alpha)}(y) = 0\left(m^{-\frac{\alpha}{2} - \frac{1}{4}}\right),\ \alpha \leqslant -\frac{1}{2},\ 0 \leqslant y \leqslant \Omega$$

$$|L_m^{(\alpha)\prime}(y_{km}^{(\alpha)})| \sim m^{\frac{\alpha}{2} + \frac{1}{4}},\ (0 < y_{km}^{(\alpha)} < \Omega,\ m = 1, 2, \ldots)$$

$$|L_m^{(\alpha)}(y)| = \begin{cases} y^{-\frac{\alpha}{2} - \frac{1}{4}}\, 0\left(m^{\frac{\alpha}{2} - \frac{1}{4}}\right), & \alpha \geqslant -\frac{1}{2} \\ 0(m^\alpha) & 0 < y \leqslant \Omega \end{cases}$$

$$\sqrt{y_{km}^{(\alpha)}} = \frac{k\pi + 0(1)}{2\sqrt{m}} \qquad (0 < y_{km}^{(\alpha)} \leqslant \Omega,\ m = 1, 2, \ldots)$$

où $a_n \sim b_n$ signifie que $c_1 \leqslant \frac{a_n}{b_n} \leqslant c_2$ $(n \geqslant N)$, $0 < c_1 \leqslant c_2 < \infty$, b_n et Ω est une constante positive et arbitraire.

Évaluons la différence $|x - x_{kn}|$. Soit x_{jn} la racine la plus proche de x, c'est-à-dire $|x_{hn} - x| = |x_{k+1,n}, - x| \Rightarrow h = j$. L'on a [2]:

$$\begin{aligned} |j| = |j(n)| = 0(1) \quad & \text{pour } x = 0 \\ |j(n)| = 0(\sqrt{n}),\ |h| = 0(\sqrt{n}) \quad & \text{pour } |x| \leqslant \Omega \end{aligned} \tag{18}$$

et

$$|x_{kn}| \leqslant \Omega = \Omega_1 \text{ entraîne } |k| = 0(\sqrt{n}) \tag{19}$$

De (13) et (17) on a

$$|x_{kn} - x| = \left|\frac{k\pi - j\pi + 0(1)}{2\sqrt{m}}\right| = 0(1)\frac{i}{\sqrt{n}}, \text{ où si } k \neq j: k = j + i \text{ pour} \tag{20}$$

$k > j$ et $k = j - i$ pour $k < j$; l'on a aussi:

$$|x - x_{jn}| < |x_{j+1,n} - x_{j-1,n}| = 0\left(\frac{1}{\sqrt{n}}\right), \quad -\Delta \leqslant x,\ x_{kn} \leqslant \Delta. \tag{21}$$

Compte tenu des hypotheses, il s'ensuit que

$$\begin{cases} |f(x) - f(x_{kn})| = 0(1)\omega_\Delta\left(\dfrac{i}{\sqrt{n}}\right), & (k \neq j,\ k = j \pm i, \quad -\Delta \leqslant x,\ x_{kn} \leqslant \Delta) \\ |f(x) - f(x_{jn})| = 0(1)\omega_\Delta\left(\dfrac{1}{\sqrt{n}}\right). \end{cases} \tag{22}$$

Nous allons maintenant évaluer les polynômes fondamentaux de Lagrange $l_{kn}(x)$ pour $0 \leqslant |x|,\ |x_{kn}| \leqslant \Omega$. Soit $n = 2m$, $y = x^2$, alors de (9), (12), (13)—(17), (19) l'on a

$$|l_{k,2m}(x)| = \left|\frac{H_{2m}(x)}{H'_{2m}(x_{kn})(x - x_{kn})}\right| = \left|\frac{L_m^{(-1/2)}(y)}{2x_{kn}L_m'^{(-1/2)}(y_{km})(x - x_{kn})}\right| =$$

$$= 0(1)\frac{n^{-1/2}}{|kn^{-1/2}|\,|in^{-1/2}|} = 0\left(\frac{1}{i}\right), \quad k \neq j,\ 0 \leqslant |x|,\ |x_k| \leqslant \Omega_1$$

Pour $n = 2m + 1$, l'on a:

$$|l_{k,2m+1}(x)| = \left|\frac{H_{2m+1}(x)}{H'_{2m+1}(x_{kn})(x - x_{kn})}\right| = \left|\frac{xL_m^{(1/2)}(y)}{2x_{kn}^2 L_m'^{(1/2)}(y_{km})(x - x_{kn})}\right| =$$

$$= 0(1)\frac{xy^{-1/2}}{(k^2 n^{-1})(in^{-1/2})\,n^{1/2}} = 0\left(\frac{1}{i}\right), \quad 0 \leqslant |x|,\ |x_k| \leqslant \Omega,\ k \neq j$$

Pour $k = 0$ (v. [2]), $x \sim \dfrac{|j|}{n}$, $y \sim \dfrac{j^2}{n}$, donc

$$l_{2m+1}(x) = \left|\frac{H_{2m+1}(x)}{H'_{2m+1}(0)x}\right| = \left|\frac{xL_m^{(1/2)}(y)}{(2m+1)L_m^{(-1/2)}(0)x}\right| = 0(1)\frac{y^{-1/2}}{n} = 0\left(\frac{1}{i}\right)$$

L'on a donc,

$$|l_{kn}(x)| = 0\left(\frac{1}{i}\right), \; k \neq j, \; 0 \leqslant |x|, \; |x_{kn}| \leqslant \Omega_1, \; n = 1, 2, \ldots \qquad (2$$

Pour $k = j$, (voir [2])

$$|l_{jn}(x)| = 0(1), \qquad 0 \leqslant |x|, \; |x_{kn}| \leqslant \Omega_1, \qquad n = 1, 2, \ldots \qquad (2$$

Pour le polynôme $h_{kn}(x)$, tenant compte de (17), (19), (21) l'on obtient

$$|h_{kn}(x)| = 0(1) + 0(1)\frac{k^2}{n}\cdot\frac{i^2}{n} + 0(1)(n-1)\frac{i^2}{n\sqrt{n}} + 0(1)\frac{k^2}{n\sqrt{n}}\cdot\frac{i^2}{n\sqrt{n}} +$$

$$+ 0(1)\frac{k}{\sqrt{n}}\cdot\frac{(2n-3)i^2}{n\sqrt{n}} + 0(1)\frac{(n-1)i^2}{n\sqrt{n}} < 0(1)\, i^2, \qquad (2$$

De (11), (18), (19), (22)—(25) l'on a finalement

$$|f(x) - P_{4n-1}(f;\, x)| = \sum_{k=-m}^{m} |f(x) - f(x_{kn})|\, p_{kn}(x) =$$

$$= 0(1)\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta\left(\frac{i}{\sqrt{n}}\right)\frac{1}{i^2}.$$

3. Dans le cas où la fonction f remplit la condition

$|f(x)| = 0(x^{2t})$, $x \in (-\infty, +\infty)$ on peut énoncer le

THÉORÈME 2. *Si $|f(x)| = 0(x^{2t})$, $t \geqslant 0$ étant un entier fixé, alors dans conditions du Théoreme 1, l'évaluation*

$$|f(x) - P_{4n-1}(f;\, x)| = 0(1)\sum_{j=1}^{[\sqrt{n}]} \omega_\Delta\left(\frac{i}{\sqrt{n}}\right)\frac{1}{i^2}$$

a lieu pour tout $x \in [-\Delta, \Delta]$ et pour tout n. (si grand que soit x_{kn}).

Pour prouver ce théorème, on fera d'abord l'estimation suivante :

$$|f(x) - P_{4n-1}(f;\, x)| = \sum_{k=-m}^{m} |f(x) - f(x_{kn})|\, p_{kn}(x) =$$

$$= \sum_{\substack{k \\ x_{kn} \in [-\Delta,\Delta]}} |f(x) - f(x_{kn})|\, p_{kn}(x) + \sum_{x_{kn} \notin [-\Delta,\Delta]} |f(x) - f(x_{kn})|\, p_{kn}(x).$$

Ensuite, compte tenu de

$$|f(x) - f(x_{kn})| = 0(x_{kn}^{2t}) \quad \text{pour } n \geqslant N \text{ et } x_{kn} \text{ suffisamment éloigné de } \Delta$$

et de

$$\sum_{k=-m}^{m} x_{kn}^{2r} \{H_n'(x_{kn})\}^{-2} = \frac{\Gamma(2r+1)}{4^r\Gamma(r+1)} \cdot \frac{1}{2^{n+1} n!} \qquad (2r \leqslant 2n-1)$$

(voir [2]), en utilisant aussi (25), (16) et (15), l'on obtient

$$\sum_{\substack{k \\ x_{kn}^* \notin [-\Delta, \Delta]}} |f(x) - f(x_{kn})| \, p_{kn}(x) \leqslant (0(1) \sum_{-}^{m} x_{kn}^{2t+2} \frac{n\sqrt{n}}{\sqrt{n}} \, 0(n^2) \frac{1}{n} \{H_n'(x_{kn})\}^{-2} =$$

$$= 0(1) \frac{\Gamma(2t+3)}{4^{t+1}\Gamma(t+2)} \cdot \frac{n^2}{2^{n+1} n!} \underset{n \to \infty}{\longrightarrow} 0.$$

D'où, eu égard au Théorème 1 il s'ensuit

$$|f(x) - P_{4n-1}(f;\, x)| = 0(1) \sum_{i=1}^{[\sqrt{n}]} \omega_\Delta \left(\frac{i}{\sqrt{n}}\right) \frac{1}{i^2},$$

ce qui achève la démonstration.

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INTERPOLAREA LUI HERMITE—FEJÉR PE NODURI CVADRUPLE — RĂDĂCINI ALE POLINOMULUI LUI HERMITE

(Rezumat)

Lucrarea se ocupă de aproximarea uniformă a funcțiilor cu ajutorul polinomului de interpolare Hermite—Fejér, cu noduri cvadruple, rădăcinile polinomului ortogonal al lui Hermite de un grad egal cu numărul nodurilor folosite. În lucrare se extinde un rezultat al lui P.O.H. Vértesi [2], de la cazul nodurilor duble la cazul nodurilor cvadruple.

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LOCATING THE BUDDIES IN THE GENERAL BUDDY SYSTEMS

Z. KÁSA

In this paper we deal with a class of computer memory allocation methods: the general buddy systems. We give an algorithm for locating the buddies (which are adjacent memory blocks obtained by splitting a larger block).

1. **Buddy systems.** The *buddy system* is a dynamic storage allocation method, originally proposed by Knowlton [4] and Knuth [5]. In this method, if a free block of a particular size is not available to satisfy a request, a larger free block is split into smaller blocks (originally into two smaller blocks only), called *buddies*. The split continues until an appropiate block is found, which is allocated, and the other buddies are placed on the available (free) space list. Two or more free blocks (the number of which is determined by the particular system) may be merged together in one if and only if they are buddies.

Let us consider the buddy system proposed by Russel [6], in which block sizes satisfy the following recurrence equation:

$$u_m = a_1 u_{m-1} + a_2 u_{m-2} + \dots + a_r u_{m-r}, \quad r > 0 \tag{1}$$

where the first r values $u_0, u_1, \dots, u_{r-1}$ are given, a_i are nonnegative integers, and for large enough m, the sequence u_m is monotonically increasing. A block of size u_m is split into a_1 buddies of size u_{m-1}, a_2 blocks of size u_{m-2}, etc. This buddy system is called *simple buddy system*. For $a_1 = 1$, $a_2 = \dots = a_{r-1} = 0$, $a_r = 1$ we obtain the class of the buddy systems defined by Hirschberg [3]:

$$u_m = u_{m-r} + u_{m-1}, \quad r > 0 \tag{2}$$

In this class for $r = 1$ we obtain the *binary buddy system* [4, 5], and for $r = 2$ the *Fibonacci buddy system* [3].

Not all well-known buddy systems are simple buddy systems. For instance, the *weighted buddy method*, proposed by Shen and Peterson [7], is not a simple buddy system, because the block sizes instead of a recurrence equation, satisfy a system of two recurrence equations:

$$\begin{aligned} u_m &= v_m + v_{m-1} \\ v_m &= u_{m-1} + v_{m-2} \end{aligned} \tag{3}$$

with the initial conditions $v_0 = 1$, $v_1 = 2$ (u_0 of no inportance, u_1 may be calculated). The general solution of this system of equations is:

$$u_m = 3 \cdot 2^{m-1}, \quad v_m = 2^m$$

Blocks of size $3 \cdot 2^{m-1}$ are split into buddies of sizes 2^m and 2^{m-1} and blocks of size 2^m are split into buddies of sizes $3 \cdot 2^{m-2}$ and 2^{m-2}.

A *general buddy system* may be defined, in which block sizes satisfy a system of recurrence equations.

Let us consider a matrix $A = (a_{ij})_{i=\overline{1,k},\, j=\overline{0,r}}$ with a_{ij} nonnegative integers, and the sets:

$$U_{m-j} = \{u^{(1)}_{m-j}, u^{(2)}_{m-j}, \ldots, u^{(k)}_{m-j}\} \quad j = 0, 1, \ldots, r$$

with k natural, m nonnegative integer.

In the general buddy system block sizes satisfy the following system of recurrence equations:

$$u^{(i)}_m = \sum_{j=0}^{r} a_{ij} v^{(i)}_{m-j}, \qquad i = 1, 2, \ldots, k$$

with the initial conditions on $u^{(i)}_0, \ldots, u^{(i)}_{r-1}$, where

$$v^{(i)}_{m-j} \in U_{m-j} \text{ for } j = 1, \ldots, r$$

$$v^{(i)}_m \in U_m \setminus \{u^{(i)}_m\}$$

For $r = 2$ and $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ the weighted buddy system results:

$$u^{(1)}_m = u^{(2)}_m + u^{(2)}_{m-1}$$

$$u^{(2)}_m = u^{(1)}_{m-1} + u^{(2)}_{m-2}$$

with the initial conditions on $u^{(2)}_0$, $u^{(2)}_1$ (maybe on $u^{(1)}_1$, $u^{(1)}_1$), which is identical to (3).

2. Locating the buddies. In order to recombine a block from its buddies, which are all available, it is necessary to locate all buddies (to find their address), starting from one. We give a method to locate the buddies in the general buddy system, which method is a generalization of the one given in [1] for Fibonacci buddy system.

2.1. (*Simple buddy systems*) First, let us consider the case of the simple buddy system, satisfying the equation (1). Let us denote by $s_r = a_1 + \ldots + a_r$ and by $B_1, B_2, \ldots, B_{s_r}$ the s_r buddies. In order to locate the buddies, we define a buddy count, denoted by $BC(B_i)$ for every buddy B_i. We denote by $b_i(B)$ the ith bit in the binary representation of the $BC(B)$. A BC for any buddy has N bits. Then $BC(B) = b_1(B)b_2(B) \ldots b_N(B)$ (concatenation), or in a short form $BC(B) = b_1b_2 \ldots b_N(B)$.

When we split a parent block B into s_r buddies $B_1, B_2, \ldots, B_{s_r}$, we define the BC of the buddies in the following way:

$$b_N(B_i) = b_i(B) \qquad \text{for } i \leqslant N$$

$$b_N(B_i) = 0 \qquad \text{for } i > N$$

$$b_1b_2 \ldots b_{N-1}(B_k) = (k-1)_2 \quad \text{for } k = 1, 2, \ldots, s_r$$

where $(k-1)_2$ is the binary representation of the number $(k-1)$, (see fig. 1 and 2).

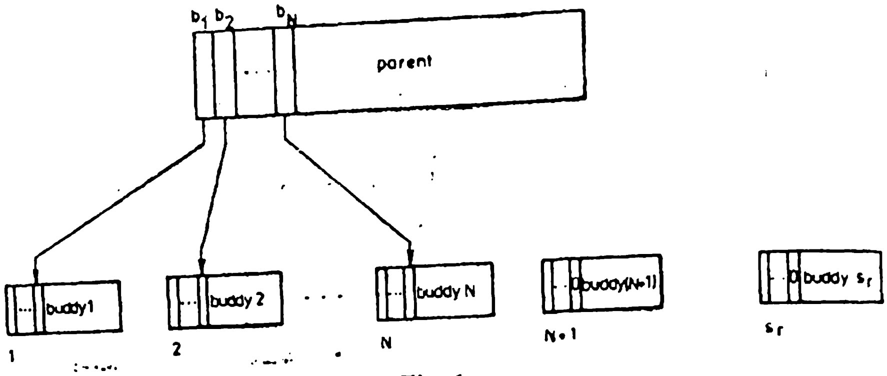

Fig. 1

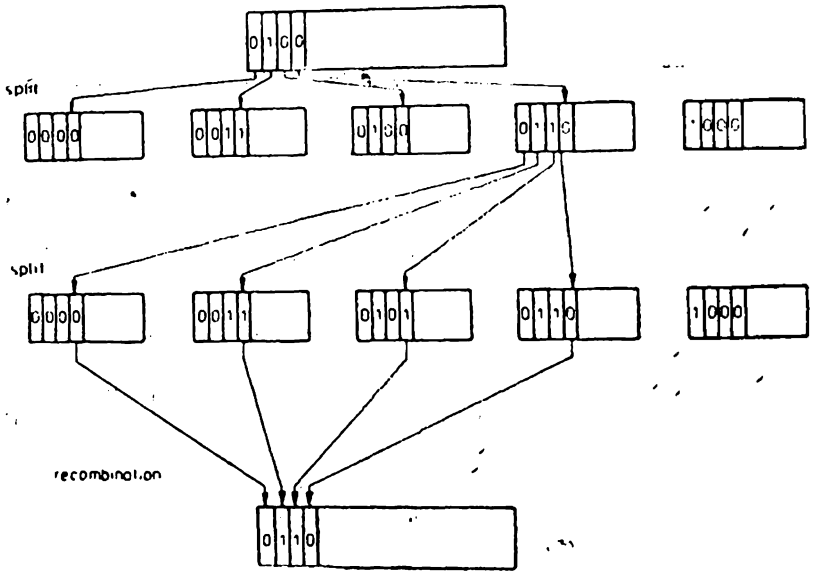

Fig. 2

When we recombine a block B from its buddies $B_1, \ldots, B_{s_r}$ we find the BC of the parent block B by the following formulas:

$$b_i(B) = b_N(B_i) \qquad \text{for } i = 1, 2, \ldots, N$$

(see fig. 2).

The buddy count BC needs N extrabits, $N = \lceil \log_2 s_r \rceil + 1$, where $\lceil x \rceil$ is the smallest natural number not less than x. Of course, besides the BC, every buddy needs a field to indicate the available/allocated status, and one to store the size index (i.e. m for u_m).

To locate all, buddies, starting from the ith buddy, we give the following algorithm:

Step 0. B_i is the ith buddy. We denote by $s_k = a_1 + \ldots + a_k$, $s_0 = 0$ and by $a(B)$ the address of the block B. Let k be so that

$$s_{k-1} < i \leqslant s_k$$

Step 1. If $i = s_r$ then go to step 5.

Step 2. $j := i$

Step 3. $s_k - i + 1$ times repeat:

  3.1. $j := j + 1$

  3.2. if $j < s_r$ then $a(B_j) := a(B_{j-1}) + u_{m-k}$.

Step 4. For $l = 1, 2, \ldots, r - k$ do:

  4.1. a_{k+l} times repeat:

    4.1.1. $j := j + 1$

    4.1.2. If $j < s_r$ then $a(B_j) := a(B_{j-1}) + u_{m-k-l}$

Step 5. $j := i$

Step 6. $i - s_{k-1} - 1$ times repeat:

  6.1. $j := j - 1$

  6.2. $a(B_j) := a(B_{j+1}) - u_{m-k}$

Step 7. For $l = 1, 2, \ldots, k - 1$ do:

  7.1. a_{k-l} times repeat:

    7.1.1. $j := j - 1$

    7.1.2. $a(B_j) := a(B_{j+1}) - u_{m-k+l}$

In the steps *4* and *7* the test is made before execution, as in *Algol 60* **for** statement.

In particular cases this algorithm may be essentially simplified. Thus, in the case of $a_1 = a_2 = \ldots = a_r = 1$ our algorithm becomes:

Step 1. For $j = i + 1, \ldots, r$ do:

$$a(B_j) := a(B_{j-1}) + u_{m-j-1}$$

Step 2. For $j = i - 1, \ldots, 1$ do:

$$a(B_j) := a(B_{j+1}) - u_{m-j}$$

If $r = 2$ and $a_1 = 1$, $a_2 = 1$ then this is the locating algorithm given in [1].

2.2. (*General buddy systems*) In the general buddy systems besides the N extrabits to store the BC, defined in section **2.1**, we need additional bits to fix the equation which is used to split a block at a moment. If $t = \max_{i=1,k} \{t_i | t_i$ is the number of $a_{ij} \neq 0$ in the ith row of the matrix $A\}$ a BC for a buddy in a general buddy system needs $k + \lceil \log_2 t \rceil$ extrabits. In the case of the weighted buddy system $k = 2$ and $t = 2$, thus any BC has 3 bits as in [7 Corrigendum].

3. Another approach to locate the buddies in the general buddy systems may be a generalization of Hinds' method [2]. In the case of the buddy system defined by the equation (2) H i n d s assigns a count LBC (Left-Buddy Count) to each buddy. The LBC of entire storage pool is zero at the beginning, the proper LBC of a buddy is zero for the block on the right, and LBC *of parent* $+ 1$ for the block on the left. Locating a buddy is a test for an $LBC = 0$ or $\neq 0$.

We may define the BC of each buddy in the general system as follows:

$$BC(B_1) := BC(B) + 1$$
$$BC(B_i) = 1 \text{ for } i = 2, 3, \ldots, s_r - 1$$
$$BC(B_{s_r}) = 0$$

But, this method provides a more complicated algorithm than the one given in section 2, and needs more extrabits to store the buddy count.

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LOCALIZAREA ZONELOR LIBERE ÎNTR-UN SISTEM GENERAL DE ALOCARE DINAMICĂ A MEMORIEI CALCULATOARELOR

R e z u m a t

Se definește un sistem general de alocare dinamică a memoriei, care cuprinde toate metodele de tip "buddy system". Se dă un algoritm pentru calculul adresei acelor zone libere de memorie (buddy) care pot fi comasate într-una singură..

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TOPOLOGICAL FUNCTORS AND INVARIANT OBJECTS

CSABA NÉMETHI

In this paper we establish some relations among initial sources, final sinks, and (co)reflections, and make a unified study of invariant objects in categories. A number of situations encountered in general topology and functional analysis can be treated more advantageously with this categorical apparatus.

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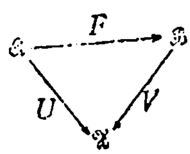
1. Preliminaries. If $\mathcal{A}$ is a category and A, $A' \in \text{ob}\ \mathcal{A}$, we shall denote by $[A, A']$ the set of all morphisms from A to A'. For basic notions and results of category theory we refer to [11].

In what follows let $U: \mathcal{A} \to \mathcal{X}$ be a functor. If A, $A' \in \text{ob}\ \mathcal{A}$, $U[A, A']$ will stand for the set $\{U(a) : a \in [A, A']\}$. We shall say that A is *finer* than A' and write $A' \leqslant A$ if $1_{A(U)} \in U[A, A']$. The equivalence relation generated in ob $\mathcal{A}$ by the quasiordering $\leqslant$ will be denoted by $\sim$. If U is faithful, then between any two equivalent objects in $\mathcal{A}$ there is an isomorphism which is sent by U to an identity in $\mathcal{X}$.

The following definitions are taken from [4], [5], [10], [20]. A source $(A, a_i : A \to A_i)_I$ in $\mathcal{A}$ is called *U-initial* if for each source $(A', a'_i: A' \to A_i)_I$ in $\mathcal{A}$ and each morphism $f: U(A') \to U(A)$ such that $U(a_i) \circ f = U(a'_i)$ for all $i \in I$, there exists a unique $a: A' \to A$ with $U(a) = f$ and $a_i \circ a = a'_i$ for each $i \in I$. A morphism $a: A \to A'$ in $\mathcal{A}$ is called *U-initial* if (A, a) is a U-initial source. A *U-initiality problem* consists of a family $(A_i)_I$ of objects in $\mathcal{A}$ together with a source $(X, f_i : X \to U(A_i))_I$ in $\mathcal{X}$; it will be denoted by $(X, f_i: X \to U(A_i), A_i)_I$. A *solution* to this problem is a U-initial source $(A, a_i: A \to A_i)_I$ such that $U(A) = X$ and $U(a_i) = f_i$ for all $i \in I$. U is called: *topological* if each U-initiality problem has a solution, and *transportable* if for each $X \in \text{ob}\ \mathcal{X}$, each $A \in \text{ob}\ \mathcal{A}$, and each isomoprhism $f: X \to U(A)$ there exists $A' \in \text{ob}\ \mathcal{A}$ and an isomorphism $a: A' \to A$ with $U(a) = f$. Any topological functor is transportable.

Dual notions: *U-final* sink and morphism, *U-finality problem, solution* to a U-finality problem. The notions of topological functor and transportable functor are self-dual. Any isomorphism in $\mathcal{A}$ is both a U-initial and U-final morphism.

In the rest of this section we consider a comutative diagram of categories and functors



THEOREM 1.1. *For the following three statements we have* (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3), *and if* U *is topological, then they are equivalent.*

(1) *For each* $B \in \mathrm{ob}\,\mathfrak{B}$ *there exists an* F*-universal map* (b_B, A_B) *such that* $V(b_B) = 1_{V(B)}$;

(2) F *has a left adjoint* $((b_B), (a_A)) : G \dashv F$ *such that* $U \circ G = V$, $V(b_B) = 1_{V(B)}$ *for all* $B \in \mathrm{ob}\,\mathfrak{B}$, *and* $U(a_A) = 1_{U(A)}$ *for all* $A \in \mathrm{ob}\,\mathfrak{A}$;

(3) F *sends* U*-initial sources to* V*-initial sources.*

The equivalence (1) $\Leftrightarrow$ (2) follows by a routine verification, while the relation between (2) and (3) is a particular case of O. Wyler's theorem on taut lifts [4], [22], [23], [24].

The proof of the following result is straightforward, hence omitted.

PROPOSITION 1.2. *Let* $B \in \mathrm{ob}\,\mathfrak{B}$, $A \in \mathrm{ob}\,\mathfrak{A}$, *and* $b : B \to F(A)$ *satisfy* $V(b) = 1_{V(B)}$. *Then for the following three statements we have* (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3), *and if* U *and* V *are faithful, then all the three are equivalent.*

(1) *The couple* (b, A) *is an* F*-universal map for* B;

(2) *For each* $A' \in \mathrm{ob}\,\mathfrak{A}$, $V[B, F(A')] \subseteq U[A, A']$;

(3) *For each* $A' \in \mathrm{ob}\,\mathfrak{A}$, $V[B, F(A')] = U[A, A']$.

2. Topological Functors and (Co)reflective Subcategories. Throughout this section $U : \mathfrak{B} \to \mathfrak{X}$ will be a functor and $\mathfrak{A}$ will be a subcategory of $\mathfrak{B}$, with inclusion functor J.

The next theorm shows that the construction of $U \circ J$-final sinks in many cases reduces to composing special $\mathfrak{A}$-reflections with U-final sinks.

THEOREM 2.1. *Assuming that* $\mathfrak{A}$ *is full, let* $(b_i : A_i \to B, B)_I$ *be a solution to the* U*-finality problem* $(A_i, f_i : U(A_i) \to X, X)_I$, *where* $A_i \in \mathrm{ob}\,\mathfrak{A}$ *for each* $i \in I$. *Then a sink* $(a_i : A_i \to A, A)_I$ *in* $\mathfrak{A}$ *is a solution to the* $U \circ J$*-finality problem* $(A_i, f_i, X)_I$ *iff there exists* $b : B \to A$ *with* $U(b) = 1_X$ *such that* (b, A) *is an* $\mathfrak{A}$*-reflection for* B *and* $b \circ b_i = a_i$ *for all* $i \in I$.

Proof. Suppose $(a_i, A)_I$ is a solution to the $U \circ J$-finality problem $(A_i, f_i, X)_I$. Having $1_X \circ U(b_i) = U(a_i)$ for each $i \in I$ and $(b_i, B)_I$ being a U-final sink, there exists a unique $b : B \to A$ with $U(b) = 1_X$ and $b \circ b_i = a_i$ for all $i \in I$. We claim that (b, A) is an $\mathfrak{A}$-reflection for B.

Let $b' : B \to A'$ be a morphism in $\mathfrak{B}$ with $A' \in \mathrm{ob}\,\mathfrak{A}$. Since $U(b') \circ U(a_i) = U(b' \circ b_i)$ for each $i \in I$ and $(a_i, A)_I$ is a $U \circ J$-final sink, there exists a unique $a : A \to A'$ such that $U(a) = U(b')$ and $a \circ a_i = b' \circ b_i$ for all $i \in I$. Then $U(a \circ b) = U(b')$ and $(a \circ b) \circ b_i = b' \circ b_i$ for each $i \in I$, which implies $a \circ b = b'$, because $(b_i, B)_I$ is a U-final sink.

If $a' : A \to A'$ is another morphism with $a' \circ b = b'$, then $U(a') = U(b') = U(a)$ and $a' \circ a_i = (a' \circ b) \circ b_i = b' \circ b_i = a \circ a_i$ for all $i \in I$, hence $a' = a$, because $(a_i, A)_I$ is a $U \circ J$-final sink.

Conversely, let (b, A) be an $\mathfrak{A}$-reflection for B with $U(b) = 1_X$ and $b \circ b_i = a_i$ for each $i \in I$. Then, obviously, $U(A) = X$ and $U(a_i) = f_i$ for all $i \in I$.

To prove that $(a_i, A)_I$ is a $U \circ J$-final sink, consider a sink $(a'_i : A_i \to A', A')_I$ in $\mathfrak{A}$ and a morphism $f: U(A) \to U(A')$ such that $f \circ U(a_i) = U(a'_i)$ for each $i \in I$. Then $f \circ U(b_i) = U(a'_i)$ for all $i \in I$. Since $(b_i, B)_I$ is a U-final sink, there exists a unique $b' : B \to A'$ with $U(b') = f$ and $b' \circ b_i = a'_i$ for each $i \in I$. (b, A) being an $\mathfrak{A}$-reflection for B, there exists a unique $a : A \to A'$ such that $a \circ b = b'$. Then $U(a) = U(b') = f$ and $a \circ a_i = (a \circ b) \circ b_i = b' \circ b_i = a'_i$ for all $i \in I$.

If $a' : A \to A'$ is another morphism with $U(a') = f$ and $a' \circ a_i = a'_i$ for each $i \in I$, then $U(a' \circ b) = f = U(b')$ and $(a' \circ b) \circ b_i = a' \circ a_i = a'_i = b' \circ b_i$ for all $i \in I$, which implies $a' \circ b = b'$, because $(b_i, B)_I$ is a U-final sink. Having $a' \circ b = b' = a \circ b$ and (b, A) being an $\mathfrak{A}$-reflection for B, we conclude that $a' = a$.

THEOREM 2.2. *For the following two statements we have* (1) $\Rightarrow$ (2) *if $\mathfrak{A}$ is closed under the equivalence* $\sim$, *and* (2) $\Rightarrow$ (1) *provided U is topological and $\mathfrak{A}$ is full.*

(1) *For each $B \in$ ob $\mathfrak{B}$ there exists an $\mathfrak{A}$-reflection (b_B, A_B) such that* $U(b_B) = 1_{U(B)}$;

(2) *$\mathfrak{A}$ is closed under the formation of U-initial sources in $\mathfrak{B}$, i.e., the domain B of each U-initial source $(B, b_i : B \to A_i)_I$ with $A_i \in$ ob $\mathfrak{A}$ for all $i \in I$, belongs to* ob $\mathfrak{A}$.

Proof. (1) $\Rightarrow$ (2). Let $(B, b_i : B \to A_i)_I$ be a U-initial source with $A_i \in$ ob $\mathfrak{A}$ for all $i \in I$, and consider an $\mathfrak{A}$-reflection (b_B, A_B) for B such that $U(b_B) = 1_{U(B)}$. Then for each $i \in I$ there exists a unique $a_i : A_B \to A_i$ in $\mathfrak{A}$ with $a_i \circ b_B = b_i$. Since $U(b_i) \circ 1_{U(B)} = U(a_i)$ for all $i \in I$ and $(B, b_i)_I$ is a U-initial source, there exists a unique $b : A_B \to B$ such that $U(b) = 1_{U(B)}$ and $b_i \circ b = a_i$ for each $i \in I$. This implies that $B \sim A_B$, hence $B \in$ ob $\mathfrak{A}$.

(2) $\Rightarrow$ (1). In view of Theorem 1.1 it suffices to show that if (2) holds, then $U \circ J$ is topological and J sends $U \circ J$-initial sources to U-initial sources. The first condition is trivial, while the second is an easy consequence of the fact that the solution of an initiality problem is determined up to an isomorphism of sources whose underlying $\mathfrak{X}$-morphism is an identity ([20], Proposition 1).

For example, the existence of the uniform modification of a semiuniformity [6] is just a particular case of this result. The dual of Theorem 2.2 implies, for instance, the existence of the barrelled and quasibarrelled modifications of a locally convex vector topology as well as the existence of the ultrabarrelled and quasiultrabarrelled modifications of a linear topology [13].

Ler $\mathfrak{A}^{\leftarrow}$ be the full subcategory of $\mathfrak{B}$ whose objects are the domains B of U-initial sources $(B, b_i : B \to A_i)_I$ with $A_i \in$ ob $\mathfrak{A}$ for all $i \in I$. The full subcategory $\mathfrak{A}^{\rightarrow}$ of $\mathfrak{B}$ is defined dually. It is esay to see that $\mathfrak{A}^{\leftarrow}$ is isomorphism-closed, hence if U is faithful, then $\mathfrak{A}^{\leftarrow}$ is also closed under the equivalence $\sim$. Since by [10] topological functors are faithful, Theorem 2.2 and the general associativity law for initial sources ([20, Proposition 5) immediately imply

COROLLARY 2.3. *If U is topological, then $\mathfrak{A}^{\leftarrow}$ is the least of all full and equivalence-closed subcategories $\mathfrak{A}'$ of $\mathfrak{B}$ that contain $\mathfrak{A}$ and have the property that for each $B \in$ ob $\mathfrak{B}$ there exists an $\mathfrak{A}'$-reflection (b_B, A'_B) with* $U(b_B) = 1_{U(B)}$.

Remark 2.4. If U is transportable and sends bimorphisms to isomorphisms, then any full and isomorphism-closed coreflective subcategory of $\mathfrak{B}$ whose object class contains a separating class for $\mathcal{B}$, possesses the dual of Property (1) in the statement of Theorem 2.2. Thus the preceding corollary implies

COROLLARY 2.5. *If U is topological and sends bimorphisms to isomorphisms, and ob $\mathcal{A}$ contains a separating class for $\mathcal{B}$, then $\mathcal{A}^{\to}$ is the least full and isomorphism-closed coreflective subcategory of $\mathcal{B}$ that contains $\mathcal{A}$, i.e., $\mathcal{A}^{\to}$ is the coreflective hull of $\mathcal{A}$.*

THEOREM 2.6. *If U is topological, for each $X \in \mathrm{ob}\,\mathfrak{X}$ $U^{-1}(X)$ is a set, $\mathfrak{X}$ has coproducts, and ob $\mathcal{A}$ contains a separating set for $\mathcal{B}$, then the objects of $\mathcal{A}^{\to}$ are precisely the codomains of U-final morphisms b whose domains are $\mathcal{B}$-coproducts of $\mathcal{A}$-objects and for which $U(b)$ is an epimorphism in $\mathfrak{X}$.*

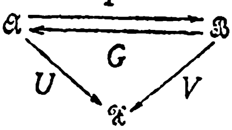
Proof. Let $B \in \mathrm{ob}\,\mathcal{A}^{\to}$. Then B is the codomain of a U-final sink $(b_i : A_i \to B, B)_I$ with $A_i \in \mathrm{ob}\,\mathcal{A}$ for all $i \in I$. Since U is topological and $U^{-1}(U(B))$ is a set, we may assume that I is a set, taking into account the associativity of final sinks. Let $\mathcal{S} \subseteq \mathrm{ob}\,\mathcal{A}$ be a separating set for $\mathcal{B}$, and consider the set $H = \bigcup_{S \in \mathcal{S}} [S, B]$. We may suppose that $H \cap I = \emptyset$. Let $J = H \cup I$, and consider the U-final sink $(b_j : A_j \to B, B)_J$, where for $j : S \to B$ with $S \in \mathcal{S}$, $A_j = S$ and $b_j = j$.

Let $(k_j, K)_J$ be a coproduct of the family $(U(A_j))_J$, and consider the U-finality problem $(A_j, k_j : U(A_j) \to K, K)_J$. If $(b'_j : A_j \to B', B')$ is a solution to this problem, it is also a coproduct of the family $(A_j)_J$ ([20]), Proposition 3). Now, $(b_j, B)_J$ is an epi-sink, hence so is $(U(b_j), U(B))_J$, because U, being topological, preserves colimits [10], By [20], Proposition 6 we conclude that B is the codomain of a U-final morphism b whose domain is B' and for which $U(b)$ is an epimorphism in $\mathfrak{X}$.

Conversely, if $B \in \mathrm{ob}\,\mathcal{B}$ is the codomain of a U-final morphism whose domain is a $\mathcal{B}$-coproduct of $\mathcal{A}$-objects, then one can easily see that $B \in \mathrm{ob}\,\mathcal{A}^{\to}$.

This theorem shows that $\mathcal{A}^{\to}$ often coincides with the monocoreflective hull of $\mathcal{A}$. Thus Theorems 2.2 and 3.1 in [21] are just particular cases of our Theorem 2.6.

3. Invariant Objects. In this section we consider a commutative diagram of categories and functors



with the property that $A \leqslant (G \circ F)(A)$ for all $A \in \mathrm{ob}\,\mathcal{A}$ and $(F \circ G)(B) \leqslant B$ for all $B \in \mathrm{ob}\,\mathcal{B}$.

Remark 3.1. It is obvious that the maps $A \mapsto F(A)$ and $B \mapsto G(B)$ are the components of a Galois correspondence between the quasiordered classes $(\mathrm{ob}\,\mathcal{A}, \leqslant)$ and $(\mathrm{ob}\,\mathcal{B}, \geqslant)$. Moreover, if U and V are faithful and for each $B \in \mathrm{ob}\,\mathcal{B}$ and $A \in \mathrm{ob}\,\mathcal{A}$, $b_B : B \to (F \circ G)(B)$ and $a_A : (G \circ F)(A) \to A$ are the unique morphisms with $V(b_B) = 1_{V(B)}$ and $U(a_A) = 1_{U(A)}$ respectively, then $((b_B), (a_A)) : G \dashv F$ is an adjoint situation. We do not assume, however, that U and V are faithful.

PROPOSITION 3.2. (1) *If* $B \in \mathrm{ob}\,\mathfrak{B}$, *then* $G(B) \sim (G \circ F \circ G)(B)$, *and* $G(B)$ *is a finest element of the class* $\{A \in \mathrm{ob}\,\mathfrak{A} : F(A) \leqslant B\}$.

(2) *If* $B \in \mathrm{ob}\,\mathfrak{B}$ *and* $A \in \mathrm{ob}\,\mathfrak{A}$, *then* $U[G(B), A] = V[B, F(A)]$.

(3) *If* $B, B' \in \mathrm{ob}\,\mathfrak{B}$, *then* $U[G(B), G(B')] = V[B, (F \circ G)(B')] = V[(F \circ G)(B), (F \circ G)(B')]$.

Proof. (1) is a general property of Galois correspondences.

(2) Let $f \in U[G(B), A]$. Then $f = U(a)$ with $a : G(B) \to A$, hence $f = V(F(a)) \in V[(F \circ G)(B), F(A)] \subseteq V[B, F(A)]$. Thus $U[G(B), A] \subseteq V[B, F(A)]$. The reverse inclusion follows similarly.

(3) By (2) we have $U[G(B), G(B')] = V[B, (F \circ G)(B')]$. On the other hand, (1) and (2) imply that $U[G(B), G(B')] = U[G((F \circ G)(B)), G(B')] = V[(F \circ G)(B), (F \circ G)(B')]$.

An object B of $\mathfrak{B}$ will be called *invariant* if $B \sim (F \circ G)(B)$. A dual notion can be defined for $\mathfrak{A}$. We shall denote by $\mathfrak{B}_{\mathrm{inv}}$ the full subcategory of $\mathfrak{B}$ determined by all invariant objects. $\mathfrak{B}_{\mathrm{inv}}$ is both isomorphism-and equivalence-closed in $\mathfrak{B}$.

Some examples of invariant objects are the following: topological closure spaces [6], principal and topological convergence spaces [9], sequential topological and closure spaces [8], [6], principal uniform convergence spaces [9], uniformizable and semiuniformizable closure spaces [6], uniformizable proximity spaces [6], proximally coarse semiuniform spaces [6], bornological and C-sequential locally convex spaces [12], [7], ultrabornological and Sc-sequential topological vector spaces [13], [14].

PROPOSITION 3.3. (1) *An object* B *of* $\mathfrak{B}$ *is invariant iff there exists* $A \in \mathrm{ob}\,\mathfrak{A}$ *such that* $B \sim F(A)$.

(2) *If* $B \in \mathrm{ob}\,\mathfrak{B}$, *then* $(F \circ G)(B)$ *is a finest element of the class* $\{B' \in \mathrm{ob}\,\mathfrak{B}_{\mathrm{inv}} : B' \leqslant B\}$.

(3) *If* $B, B' \in \mathrm{ob}\,\mathfrak{B}$, *then* $V[(F \circ G)(B), B'] \subseteq V[B, B'] \subseteq U[G(B), G(B')]$ *and if* B' *is invariant, then both inclusions turn to equalities.*

Proof. (1) and (2) are general properties of Galois correspondences.

(3) Both inclusions are obvious, and if $B' \sim (F \circ G)(B')$, then by Proposition 3.2. (3) they become equalities.

THEOREM 3.4. *For a* $B \in \mathrm{ob}\,\mathfrak{B}$ *the following three statements are equivalent:*

(1) B *is invariant:*

(2) *For each* $B' \in \mathrm{ob}\,\mathfrak{B}$, $V[(F \circ G)(B'), B] = V[B', B]$;

(3) *For each* $B' \in \mathrm{ob}\,\mathfrak{B}$, $V[B', B] = U[G(B'), G(B)]$.

Proof. In view of the preceding proposition we have (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3).

(2) $\Rightarrow$ (1). Choosing $B' = B$, by the hypothesis we have $V[(F \circ G)(B), B] = V[B, B]$, in particular $1_{V(B)} \in V[(F \circ G)(B), B]$, i.e., $B \leqslant (F \circ G)(B)$, which means that B is invariant.

(3) $\Rightarrow$ (1). Choosing $B' = (F \circ G)(B)$, the hypothesis and Proposition 3.2.(1) imply $V[(F \circ G)(B), B] = U[(G \circ F \circ G)(B), G(B)] = U[G(B), G(B)]$, in particular $1_{V(B)} = 1_{U(G(B))} \in V[(F \circ G)(B), B]$, i.e., B is invariant.

PROPOSITION 3.5. *If V is faithful, then for each $B \in \text{ob}\,\mathfrak{B}$, taking the unique morphism $b_B : B \to (F \circ G)(B)$ with $V(b_B) = 1_{V(B)}$, the couple $(b_B, (F \circ G)(B))$ is a $\mathfrak{B}_{\text{inv}}$-reflection for B.*

Proof. By Proposition 3.3. (2), $(F \circ G)(B) \in \text{ob}\,\mathfrak{B}_{\text{inv}}$. In addition, for each $B' \in \text{ob}\,\mathfrak{B}_{\text{inv}}$, $V[B, B'] = V[(F \circ G)(B), B']$ (Proposition 3.3. (3)). V being faithful, in view of Proposition 1.2 we infer that $(b_B, (F \circ G)(B))$ is a $\mathfrak{B}_{\text{inv}}$-reflection for B.

THEOREM 3.6. *If U and V are faithful, then for a $B \in \text{ob}\,\mathfrak{B}$ the following three statements are equivalent:*

(1) *B is invariant;*

(2) *If $B' \in \text{ob}\,\mathfrak{B}$ and $b : B \to B'$ are such that $(F \circ G)(b)$ is a V-initial morphism, then b is a V-initial morphism;*

(3) *If $B' \in \text{ob}\,\mathfrak{B}$ and $b : B \to B'$ are such that $G(b)$ is a U-initial morphism, then b is a V-initial morphism.*

Proof. (1) $\Rightarrow$ (2). If $B \sim (F \circ G)(B)$, then, since V is faithful, there exists an isomorphism $b_1 : B \to (F \circ G)(B)$ such that $V(b_1) = 1_{V(B)}$. Let $b_{B'} : B' \to (F \circ G)(B')$ be the unique morphism with $V(b_{B'}) = 1_{V(B')}$. Then $V(b_{B'} \circ b) = V((F \circ G)(b) \circ b_1)$, hence $b_{B'} \circ b = (F \circ G)(b) \circ b_1$, which is a V-initial morphism if $(F \circ G)(b)$ is supposed to be V-initial. This implies that b is a V-initial morphism.

(2) $\Rightarrow$ (3). If $G(b)$ is U-initial, then in view of Remark 3.1 and Theorem 1.1, $(F \circ G)(b)$ is V-initial. By the hypothesis this implies that b is a V-initial morphism.

(3) $\Rightarrow$ (1). Choose $B' = (F \circ G)(B)$ and $b =$ the unique $b_B : B \to (F \circ G)(B)$ with $V(b_B) = 1_{V(B)}$. Since by Proposition 3.2.(1) $G(B) \sim (G \circ F \circ G)(B)$, and since U is faithful, there exists an isomorphism $a : G(B) \to (G \circ F \circ G)(B)$ such that $U(a) = 1_{V(B)}$. Then $U(G(b)) = U(a)$, hence $G(b) = a$, which is U-initial. From the hypothesis it follows that b is a V-initial morphism. Having $V(b) \circ 1_{V(B)} = V(1_{(F \circ G)(B)})$, the V-initiality of b implies the existence of a unique $b_1 : (F \circ G)(B) \to B$ with $V(b_1) = 1_{V(B)}$ and $b_B \circ b_1 = 1_{(F \circ G)(B)}$. In conclusion $B \sim (F \circ G)(B)$, i.e., B is invariant.

From the propositions and theorems proved in this section one can reobtain a number of known results and deduce new ones for various categories of general topology and functional analysis. The works [1]—[3], [6]—[9], [12]—[19], [24] are just a few of those containing particular cases of our results or their duals.

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FUNCTORI TOPOLOGICI ŞI OBIECTE INVARIANTE

(Rezumat)

Lucrarea conţine unele rezultate privind legătura familiilor iniţiale şi finale cu subcategoriile (co)reflective. De asemenea, se face un studiu categorial al obiectelor invariante, intîlnite frecvent în topologia generală şi în analiza funcţională.

STUDIA UNIV. BABEŞ–BOLYAI, MATHEMATICA, XXVI, 4, 1981

LUNAR PERTURBATIONS IN ARTIFICIAL SATELLITES MOTION

VASILE MIOC and EUGENIA RADU

1. **Motion Equations.** The disturbed motion of an artificial satellite with respect of a geocentric frame is described by the following equations:

$$
\begin{aligned}
dp/du &= 2Zr^3T/\mu,\\
d\Omega/du &= Zr^3(B/D)W/\mu p,\\
di/du &= Zr^3BW/\mu p,\\
dq/du &= Zr^3kB(C/D)W/\mu p + Zr^2[r(A+q)/p + A]T/\mu + Zr^2BS/\mu,\\
dk/du &= -Zr^3qB(C/D)W/\mu p + Zr^2[r(B+k)/p + B]T/\mu - Zr^2AS/\mu,\\
dt/du &= Zr^2/\sqrt{\mu p},
\end{aligned}
\tag{1}
$$

where $q = e\cos\omega$, $k = e\sin\omega$, $Z = [1 - (r^2Cd\Omega/dt)/\sqrt{\mu p}]^{-1}$. We also used the notations: $A = \cos u$, $B = \sin u$, $C = \cos i$, $D = \sin i$, the other notations being usual.

The variations of the orbital elements in the interval $[u_0, u]$, i.e. between the initial and current positions, can be determined from the relations:

$$\Delta y = \int_{u_0}^{u} (dy/du)du, \tag{2}$$

where $y \in \{p, \Omega, i, q, k\}$. The integrals can be estimated from Equations (1) by the successive approximations method, with $Z \cong 1$ (C h e b o t a r e v, 1965), neglecting the superior order approximations.

2. **Disturbing Acceleration.** We shall neglect the lunar motion during a revolution of the satellite. Also, we shall consider the radii vectors Moon-satellite and Moon-Earth to be identical. The disturbing acceleration undergone by the satellite under the influence of the lunar attraction has the components (E l y a s b e r g, 1965):

$$
\begin{aligned}
S &= (\mu_L/r_L^3)r[3\cos^2\varphi\cos^2(v-\lambda) - 1],\\
T &= -3(\mu_L/r_L^3)r\cos^2\varphi\cos(v-\lambda)\sin(v-\lambda),\\
W &= 3(\mu_L/r_L^3)r\cos\varphi\sin\varphi\cos(v-\lambda),
\end{aligned}
\tag{3}
$$

where (φ, λ, r_L) are the spherical coordinates of the Moon in a geocentric right-handed frame $Oxyz$ (Ox-axis directed towards the satellite perigee, Oxy-plane being the satellite orbit plane), v is the true anomaly and $\mu_L = GM_L$ (where M_L = lunar mass, G = gravitation constant).

Passing from the frame $Oxyz$ to another geocentric right-handed frame $O\xi\eta\zeta$—($O\xi$axis directed towards the ascending node of the satellite orbit, $O\xi\eta \equiv Oxy$), by the transition relations:

$$L_1 = \cos\varphi \cos(\lambda + \omega),$$
$$L_2 = \cos\varphi \sin(\lambda + \omega), \qquad (4)$$
$$L_3 = \sin\varphi,$$

where L_1, L_2, L_3 (the cosines of the lunar radius vector r_L) can be calculated from the expressions:

$$L_1 = (\cos\Omega \cos\alpha_L + \sin\Omega \sin\alpha_L)\cos\delta_L,$$
$$L_2 = (\cos\Omega \sin\alpha_L - \sin\Omega \cos\alpha_L)C\cos\delta_L + D\sin\delta_L, \qquad (5)$$
$$L_3 = (\sin\Omega \cos\alpha_L - \cos\Omega \sin\alpha_L)D\cos\delta_L - C\sin\delta_L,$$

(α_L, δ_L) being the geocentric equatorial coordinates of the Moon, Equations (3) become:

$$S = 3\mu_L r_L^{-3} r(L_1A^2 + L_2B^2 + 2L_1L_2AB - 1/3),$$
$$T = -3\mu_L r_L^{-3} r[L_1L_2(B^2 - A^2) + (L_1^2 - L_2^2)AB], \qquad (6)$$
$$W = 3\mu_L r_L^{-3} r(L_1L_3A + L_2L_3B).$$

3. **Results.** We shall determine the perturbations caused by the lunar attraction in the orbital elements of the satellite during a nodal period of this one; so, the limits of integration in Equations (2) will be 0 and 2π. As for the geocentric radius vector of the satellite, we shall expand its expression: $r = p/(1 + e\cos v) = p/(1 + Aq + Bk)$ in series of powers of q and k, neglecting the terms in $q^m k^n$ for $m + n > 3$ (i.e. up to the third power of eccentricity):

$$r = p(1 - Aq - Bk + A^2q^2 + 2ABqk + B^2k^2 - A^3q^3 -$$
$$-3A^2Bq^2k - 3AB^2qk^2 - B^3k^3). \qquad (7)$$

Substituting Equations (6) and (7) in Equations (1) and integrating these ones between 0 and 2π with $Z \cong 1$, we obtain the variations of the orbital elements due to the lunar attraction:

$$\Delta p = 4X_0p_0(Y_{10}q_0^2 + Y_{40}q_0k_0 - Y_{10}k_0^2),$$
$$\Delta\Omega = X_0D_0(2Y_{30}/5 + Y_{30}q_0^2 + 2Y_{20}q_0k_0 + 3Y_{30}k_0^2),$$
$$\Delta i = X_0(2Y_{20}/5 + 3Y_{20}q_0^2 + 2Y_{30}q_0k_0 + Y_{20}k_0^2), \qquad (8)$$
$$\Delta q = X_0[-2Y_{10}q_0 + (2/5)Y_{60}k_0 - 5Y_{10}q_0^3 + Y_{60}q_0^2k_0 +$$
$$+ Y_{80}q_0k_0^2 + (Y_{60} + 2Y_{50})k_0^3],$$
$$\Delta k = -X_0[(2/5)Y_{70}q_0 - 2Y_{10}k_0 + Y_{70}q_0^3 + Y_{80}q_0^2k_0 +$$
$$+ (Y_{70} + 2Y_{50})q_0k_0^2 - 5Y_{10}k_0^3],$$

where the supplementary index "o" fixes the values of the respective quantities at the beginning of the considered revolution, namely in the ascending node ($t = t_0$, $u = u_0$). In Equations (8), $X_0 = (15/2)\pi(M_L/M)(p_0/r_{L0})^3$. M = Earth's mass, while Y_{j0}, $j = \overline{1, 8}$, have the following expressions:

$$\begin{aligned} Y_{10} &= L_{10}L_{20}; \ Y_{20} = L_{10}L_{30}; \ Y_{30} = L_{20}L_{30}; \\ Y_{40} &= L_{20}^2 - L_{10}^2; \ Y_{50} = (C_0/D_0)L_{20}L_{30}; \\ Y_{60} &= (C_0/D_0)L_{20}L_{30} + L_{10}^2 - 4L_{20}^2 + 1; \\ Y_{70} &= (C_0/D_0)L_{20}L_{30} - 4L_{10}^2 + L_{20}^2 + 1; \\ Y_{80} &= 2(C_0/D_0)L_{10}L_{30} - 5L_{10}L_{20}. \end{aligned} \tag{9}$$

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PERTURBAȚII LUNARE ÎN MIȘCAREA SATELIȚILOR ARTIFICIALI

(Rezumat)

Prin integrarea ecuațiilor mișcării unui satelit artificial al Pămîntului, folosind dezvoltări in serie pînă la puterea a treia a excentricității, se stabilesc expresii pentru variațiile elementelor orbitale, datorate atracției Lunii, pe durata unei perioade nodale a satelitului.

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UNELE PROPRIETĂŢI ALE TOPOLOGIEI INDUSE DE O G-2-METRICĂ

D. BORŞAN

Noţiunea de 2-metrică a fost introdusă de S. Gähler în 1963 [2]. În lucrarea [1] se generalizează această noţiune în modul următor:

Fie $(\mathfrak{M}, \leqslant)$ o mulţime parţial ordonată oarecare şi „$\preccurlyeq$" relaţia de ordonare parţială definită în mod natural în $\mathfrak{M} \times \mathfrak{M} \times \mathfrak{M}$ prin:

$$(a, b, c) \preccurlyeq (a_1, b_1, c_1) \underset{\text{def}}{\Leftrightarrow} a \leqslant a_1,\ b \leqslant b_1,\ c \leqslant c_1,$$

Fie $\varphi : \mathfrak{M} \times \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$ o operaţie ternară care satisface condiţiile:

(φ_1) $\qquad \varphi(a, b, c) = \varphi(a, c, b) = \varphi(b, c, a), \quad \forall a, b, c \in \mathfrak{M}$

(φ_2) $\qquad (a, b, c) \prec (a_1, b_1, c_1) \Rightarrow \varphi(a, b, c) < \varphi(a_1, b_1, c_1)$

În sfîrşit, fie $\mathcal{E} \subseteq \mathfrak{M}$.

DEFINIŢIA (1.1) [1]. *Prin g-2-metrică pe o mulţime X înţelegem o aplicaţie* $\rho : X \times X \times X \to \mathfrak{M}$, *satisfăcînd următoarele axiome*

$$(\rho_{1a})\ x, y \in X,\ x \neq y \Rightarrow \exists z \in X\ \exists e \in \mathcal{E} : \rho(x, y, z) \not< e;$$

$$(\rho_{1b})\ \forall e \in \mathcal{E} : \rho(x, y, z) < e \Leftrightarrow x = y \text{ sau } x = z \text{ sau } y = z;$$

$$(\rho_2)\ \rho(x, y, z) = \rho(x, z, y) = \rho(y, z, x);$$

$$(\rho_3)\ \rho(x, y, z) \leqslant \varphi[\rho(x, y, t), \rho(x, t, z), \rho(t, y, z)]$$

unde $x, y, z, t \in X$.

Cuplul (X, ρ) se numeşte *spaţiu g-2-metric*. Pentru $x, y \in X$ şi $e \in \mathcal{E}$, facem notaţia $V_e(x, y) = \{z \in X \mid \rho(x, y, z) < e\}$.

DEFINIŢIA (2.1). [1]. *Topologia care admite ca subbază familia*

$$\mathcal{S} = \{V_e(x, y)\quad x, y \in X,\ e \in \mathcal{E}\}$$

se numeşte topologia generată de g-2-metrica ρ *şi se notează cu* $\mathcal{T}_\rho$.

Introducem următoarele notaţii. Pentru o mulţime oarecare A, notăm cu $\mathfrak{L}_0(A)$ mulţimea submulţimilor finite ale lui A. Impunînd mulţimii $\mathcal{E}$ condiţia

(E_1) $\qquad e_0 \in \mathcal{E},\ a \in \mathfrak{M},\ a < e_0 \Rightarrow \exists e_1 \in \mathcal{E}\, \exists e_2 \in \mathcal{E} : \varphi(a, e_1, e_2) \leqslant e_0$

este valabilă [1]:

TEOREMA (1.1). *Dacă* (X, ρ) *este un spaţiu g-2-metric şi* $x \in X$, *familia*

$$\{W_\Sigma(x) = \bigcap_{(y, e) \in \Sigma} V_e(x, y) \mid \Sigma \in \mathfrak{L}_0(X \times \mathcal{E})\}$$

este o bază de vecinătăţi pentru punctul x.

2. Ne propunem să caracterizăm, în termenii g-2-metricii, noţiunea de limită a unui şir generalizat (Moore-Smith) de puncte din spaţiul topologic $(X, \mathcal{T}_\rho)$.

TEOREMA (2.1) *Fie $\rho : X \times X \times X \to \mathfrak{M}$ o g-2-metrică pe X şi $\mathcal{E} \subseteq \mathfrak{M}$ satisfăcînd ipoteza (E_1). Condiţia necesară şi suficientă pentru ca un şir generalizat $(x_\gamma, \gamma \in \Gamma)$ să tindă către $x \in X$, în spaţiul topologic $(X, \mathcal{T}_\rho)$ este ca pentru orice $e \in \mathcal{E}$ şi fiecare mulţime finită $M \subseteq X$ să existe $\gamma_0 \in \Gamma$ astfel încît, pentru $\gamma \geqslant_\Gamma \gamma_0$ să avem $\rho(x, y, x_\gamma) < e$, oricare ar fi $y \in M$.*

Demonstraţie. Avem de arătat că

$$x_\gamma \overset{\mathcal{T}_\rho}{\to} x \Leftrightarrow \forall e \in \mathcal{E} \forall M \in \mathfrak{L}_0(X) \exists \gamma_0 \in \Gamma : [\gamma \geqslant_\Gamma \gamma_0, y \in M \Rightarrow \rho((x, y, x_\gamma) < e]$$

Necesitatea. Presupunem că $x_\gamma \overset{\mathcal{T}_\rho}{\to} x$ şi fie $e \in \mathcal{E}$ şi $M \in \mathfrak{L}_0(X)$. Considerăm $\Sigma = \{(y, e) | y \in M\} = M \times \{e\}$ şi vecinătatea $W_\Sigma(x) = \bigcap_{y \in M} V_e(x, y)$, a punctului x. Cum $x_\gamma \overset{\mathcal{T}_\rho}{\to} x$, urmează că pentru $\Sigma \in \mathfrak{L}_0(X \times \mathcal{E})$ considerat, există un $\gamma_0 \in \Gamma$ depinzînd evident de Σ (deci de e şi de M) astfel ca, $x_\gamma \in W_\Sigma(x)$ pentru $\gamma \geqslant_\Gamma \gamma_0$, adică $\rho(x, y, x_\gamma) < e$ pentu $\gamma \geqslant_\Gamma \gamma_0$ şi $y \in M$.

Suficienţa. Presupunem că

$$\forall e \in \mathcal{E} \forall M \in \mathfrak{L}_0(X) \exists \gamma_0 \in \Gamma : [\gamma \geqslant_\Gamma \gamma_0, y \in M \Rightarrow \rho(x, y, x_\gamma) < e]$$

Vom arăta că $x_\gamma \overset{\mathcal{T}_\rho}{\to} x$. Ţinînd seama de teorema (1.1), va fi suficient să arătăm că oricare ar fi $\Sigma \in \mathfrak{L}_0(X \times \mathcal{E})$ există $\gamma_0 \in \Gamma$ astfel ca pentru $\gamma \geqslant_\Gamma \gamma_0$ să avem $x_\gamma \in W_\Sigma(x)$. Fie deci $\Sigma = \{(y_i, e_i) | i = \overline{1, n}\} \in \mathfrak{L}_0(X \times \mathcal{E})$. Notînd cu $M = \{y_i | i = \overline{1, n}\} \in \mathfrak{L}_0(X)$, pentru M şi e_i există prin ipoteză $\gamma_i \in \Gamma$ astfel ca $\gamma \geqslant_\Gamma \gamma_i$ să implice $\rho(x, y_i, x_\gamma) < e_i$. Cum mulţimea $(\Gamma, \geqslant_\Gamma)$ este dirijată (superior filtrantă), urmează că există $\gamma_0 \in \Gamma$, cu $\gamma_0 \geqslant_\Gamma \gamma_i$ pentru $i = \overline{1, n}$. Deducem că pentru $\gamma \geqslant_\Gamma \gamma_0$ avem $\rho(x, y_i, x_\gamma) < e_i$, oricare ar fi $i \in \{1, 2, \ldots, n\}$ deci $x_\gamma \in V_{e_i}(x, y_i)$ pentru $i = \overline{1, n}$ adică $x_\gamma \in \bigcap_{i=1}^{n} V_{e_i}(x, y_i) = W_\Sigma(x)$, pentru $\gamma \geqslant_\Gamma \gamma_0$ Cu aceasta teorema este demonstrată.

3. Să studiem acum proprietăţi de separaţie ale spaţiului $(X, \mathcal{T}_\rho)$.

TEOREMA (3.1). *Dacă ρ este o g-2-metrică pe X spaţiul $(X, \mathcal{T}_\rho)$ este un spaţiu T_1.*

Demonstraţie. Vom arăta că pentru $x, y \in X$, $x \neq y$, există o vecinătate a lui x care nu conţine punctul y. Fie $x \neq y$. Conform axiomei (ρ_{1a}) există $e \in \mathcal{E}$ şi $z \in X$, astfel ca $\rho(x, y, z) \nless e$. Mulţimea $V_e(x, z)$, ca element al subbazei topologiei $\mathcal{T}_\rho$ care conţine punctul x, este o vecinătate a lui x. Ea nu conţine punctul y (pentru că $\rho(x, z, y) \nless e$).

Observaţie (3.1). În absenţa axiomei (ρ_{1a}) spaţiul $(X, \mathcal{T}_\rho)$ nu este nici măcar spaţiu T_0.

În adevăr, dacă (ρ_{1a}) nu este satisfăcută, există $x, y \in X$ cu $x \neq y$, astfel ca $\rho(x, y, z) < e$, oricare ar fi $e \in \mathcal{E}$, şi pentru orice $z \in X$. Atunci în baza axiomei $(\rho_{1b})X = \{x, y\}$, iar punctele x şi y au cîte o singură vecinătate, spaţiul întreg X. Axioma T_0 nu este satisfăcută.

Pentru a asigura mai multă fineţe topologiei $\mathcal{T}_\rho$ introducem o nouă ipoteză asupra mulţimii $\mathcal{E}$, ipoteza (E').

(E') $$\forall e \in \mathcal{E}\, \exists e' \in \mathcal{E}, \quad e' < e.$$

TEOREMA (3.2). *Fie* $\rho : X \times X \times X \to \mathfrak{M}$ *o g-2-metrică pe* X *şi presupunem că* $\mathcal{E} \subseteq \mathfrak{M}$ *se bucură de proprietăţile* (E_1) *şi* (E'). *Spaţiul* $(X, \mathcal{T}_\rho)$ *este regular.*

Demonstraţie. Vom arăta că $x \in G \in \mathcal{T}_\rho \Rightarrow \exists G^* \in \mathcal{T}_\rho : x \in G^* \subseteq \overline{G^*} \subseteq G$. Fie deci $x \in X$ şi G o mulţime deschisă care conţine punctul x. Cum familia $\{W_\Sigma(x) = \bigcap_{(y, e) \in \Sigma} V_e(x, y) \mid \Sigma \in \mathfrak{L}_0(X \times \mathcal{E})\}$ este o bază de vecinătăţi pentru x, urmează că există $\Sigma \in \mathfrak{L}_0(X \times \mathcal{E})$ astfel ca $U = W_\Sigma(x) \subseteq G$. Fie $\Sigma = \{(y_i, e_i) \mid i = \overline{1, n}\}$. Considerăm $\Sigma' = \{(y_i, e_i') \mid i = \overline{1, n}\}$ cu $e_i' < e_i$, pentru fiecare $i \in \{1, \ldots, n\}$. Existenţa acestor elemente e_i' în $\mathcal{E}$ este asigurată prin ipoteza (E'). Mulţimea $W_{\Sigma'}(x) = \bigcap_{i=1}^{n} V_{e_i'}(x, y_i)$ este o vecinătate deschisă (element al subbazei topologiei $\mathcal{T}_\rho$) a punctului x şi avem $W_{\Sigma'}(x) \subseteq W_\Sigma(x)$ (deoarece $e_i' < e_i$, pentru $i = \overline{1, n}$). Notînd cu $G^* = W_{\Sigma'}(x)$, vom arăta că $\overline{G^*} \subseteq G$.

Avem

$$\overline{G^*} = \overline{W_{\Sigma'}(X)} = \overline{\bigcap_{i=1}^{n} V_{e_i'}(x, y_i)} \subseteq \bigcap_{i=1}^{n} \overline{V_{e_i'}(x, y_i)}.$$

Arătăm că $\overline{V_{e_i'}(x, y_i)} \subseteq V_{e_i}(x, y_i)$, pentru $i = \overline{1, n}$. Din $e_i' < e_i$ deducem, în baza ipotezei (E_1), că există elementele $f_i, g_i \in \mathcal{E}$ astfel ca $\varphi(e_i', f_i, g_i) \leqslant e_i$. Fie acum $z \in \overline{V_{e_i'}(x, y_i)}$. Urmează că, oricare ar fi o vecinătate V a punctului z, avem $V \cap V_{e_i'}(x, y_i) \neq \emptyset$. Alegem $V = V_{f_i}(x, y_i) \cap V_{g_i}(z, x)$ şi fie $z^* \in V \cap V_{e_i'}(x, y_i)$. Avem atunci (folosind axiomele g-2-metricii şi ţinînd seama că

$$\rho(x, y_i, z^*) < e_i', \ \rho(x, z^*, z) < g_i, \ \rho(y_i, z, z^*) < f_i)$$

$\rho(x, y_i, z) \leqslant \varphi[\rho(x, y_i, z^*), \rho(x, z^*, z), \rho(y_i, z, z^*)] < \varphi(e_i', f_i, g_i) \leqslant e_i$. Am arătat că $\rho(x, y_i, z) < e_i$, deci că $z \in V_{e_i}(x, y_i)$. În definitiv, pentru un i arbitrar din mulţimea $\{1, 2, \ldots, n\}$ am stabilit incluziunea $\overline{V_{e_i'}(x, y_i)} \subseteq V_{e_i}(x, y_i)$, adică

$$\bigcap_{i=1}^{n} \overline{V_{e_i'}(x, y_i)} \subseteq \bigcap_{i=1}^{n} V_{e_i}(x, y_i) = U \subseteq G.$$

Ţinînd seama de cele precedente avem $x \in G^* \subseteq \overline{G^*} \subseteq G$, ceea ce demonstrează teorema.

Observaţie (3.2). Dacă $(X, \mathcal{T}_\rho)$ este un spaţiu cu bază numărabilă, atunci în ipotezele (E_1) şi (E') asupra mulţimii $\mathcal{E}$, este chiar spaţiu complet normal conform unei cunoscute teoreme a lui Tychonoff [4].

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SOME PROPERTIES OF THE TOPOLOGY INDUCED BY A G-2-METRIC

(Summary)

The paper is devoted to some properties of the topology induced by a g-2-metric (defined in [1]).

It contains a characterization in terms of the g-2-metric of the concept of the limit of a net. Separation properties in a g-2-metric space are also discussed.

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A COINCIDENCE THEOREM FOR MULTIVALUED MAPPINGS IN METRIC SPACES

OLGA HADŽIĆ

In [1] Brian Fisher proved the following common fixed point theorem.

THEOREM 1. *Let S and T be continuous mappings of a complete metric space (X, d) into itself. Then S and T have a common fixed point in X if and only if there exists a continuous mapping A of X into $SX \cap TX$ which commutes with S and T and satisfies the inequality*

$$d(Ax, Ay) \leqslant qd(Sx, Ty), \text{ for all } x, y \in X,$$

where $0 < q < 1$. Indeed, S, T, and A then have a unique common fixed point.

Now, we shall prove a similar coincidence theorem for multivalued mappings. If (X, d) is a metric space by $CB(X)$ we shall denote the family of all closed and bounded subsets of X and if M_1 and M_2 are from $CB(X)$ then:

$$D(M_1, M_2) = \inf_{m_1 \in M_1, m_2 \in M_2} \{d(m_1, m_2)\}$$

and $T(M_1, M_2) = \max \{\sup_{m_1 \in M_1} \{D(m_1, M_2)\}, \sup_{m_2 \in M_2} \{D(M_1, m_2)\}\}$.

THEOREM 2. *Let (X, d) be a complete metric space, S and T continuous mappings from X into X, A a closed mapping from X into $CB(SX \cap TX)$ such that $ATx = TAx$, $ASx = SAx$, for every $x \in X$ and:*

$H(Ax, Ay) \leqslant q \cdot d(Sx, Ty)$, for every $x, y \in X$ where $q \in (0, 1)$. Then there exists a sequence $\{x_n\}_{n \in N}$ such that:

1. *For every $n \in N$, $Sx_{2n+1} \in Ax_{2n}$, $Tx_{2n} \in Ax_{2n-1}$.*
2. *There exists $z = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n+1}$.*
3. *$Tz \in Az$, $Sz \in Az$.*

Proof: Let x_0 be an arbitrary element from X. Since $Ax_0 \subseteq SX$ there exists $x_1 \in X$ so that $Sx_1 \in Ax_0$. Further:

$$H(Ax_1, Ax_0) \leqslant qd(Sx_1, Tx_0)$$

and since $Sx_1 \in Ax_0$ there exists $u^{(1)} \in Ax_1$ such that:

$$d(u^{(1)}, Sx_1) \leqslant qd(Sx_1, Tx_0) + q. \tag{1}$$

On the other hand $Ax_1 \subseteq TX$ and so there exists $x_2 \in X$ such that $u^{(1)} = Tx_2$ which implies that:

$$d(Tx_2, Sx_1) \leqslant qd(Sx_1, Tx_0) + q. \quad (2$$

Since $Tx_2 \in Ax_1$ we can obtain, similarly as above, that there exists $u^{(2)} \in Ax_2$ such that:

$$d(u^{(2)}, Tx_2) \leqslant qd(Tx_2, Sx_1) + q^2.$$

Further from $Ax_2 \subseteq SX$ and $u^{(2)} \in Ax_2$ we conclude that there exists $x_3 \in X$ so that $u^{(2)} = Sx_3$ and so we have that:

$$d(Sx_3, Tx_2) \leqslant qd(Tx_2, Sx_1) + q^2 = q[qd(Sx_1, Tx_0) + q] + q^2 = \\ = q^2 \cdot d(Sx_1, Tx_0) + 2q^2. \tag{3}$$

Similarly, it is easy to prove by induction that there exists a sequence $\{x_n\}_{n \in N} \subseteq \subseteq X$ with the following properties:

(i) $Sx_{2n+1} \in Ax_{2n}$, $Tx_{2n} \in Ax_{2n-1}$, for every $n \in N$

(ii) For every $n \in N$:

$$d(Sx_{2n+1}, x_{2n}) \leqslant q \cdot d(Tx_{2n}, Sx_{2n-1}) + q^{2n} \tag{4}$$

$$d(Tx_{2n}, Sx_{2n-1}) \leqslant q \cdot d(Sx_{2n-1}, Tx_{2n-2}) + q^{2n-1} \tag{5}$$

Let us prove that (ii) implies:

$$d(Sx_{2n+1}, Tx_{2n}) \leqslant q^{2n}d(Sx_1, Tx_0) + 2n \cdot q^{2n}, \text{ for every } n \in N \tag{6}$$

$$d(Tx_{2n}, Sx_{2n-1}) \leqslant q^{2n-1}d(Sx_1, Tx_0) + (2n-1)q^{2n-1}, \text{ for every } n \in N \tag{7}$$

Indeed, let us prove (6). For $n = 1$, (6) is proved. Suppose that $d(Sx_{2n+1}, Tx_{2n}) \leqslant \leqslant q^{2n} \cdot d(Sx_1, Tx_0) + 2n \cdot q^{2n}$. Then:

$$\begin{aligned} d(Sx_{2n+3}, Tx_{2n+2}) &\leqslant q \cdot d(Tx_{2n+2}, Sx_{2n+1}) + q^{2n+2} \leqslant \\ &\leqslant q[qd(Sx_{2n+1}, Tx_{2n+1}) + q^{2n+1}] + q^{2n+2} = \\ &= q^2 \cdot d(Sx_{2n+1}, Tx_{2n+1}) + 2q^{2n+2} \leqslant \\ &\leqslant q^2[q^{2n} \cdot d(Sx_1, Tx_0) + 2n \cdot q^{2n}] + 2q^{2n+2} = \\ &= q^{2n+2} \cdot d(Sx_1, Tx_0) + (2n+2)q^{2n+2}. \end{aligned}$$

Similarly, from $d(Tx_{2n}, Sx_{2n-1}) \leqslant q_{2n-1}d(Sx_1, Tx_0) + (2n-1)q^{2n-1}$ it follows that:

$$\begin{aligned} d(Tx_{2n+2}, Sx_{2n+1}) &\leqslant qd(Sx_{2n+1}, Tx_{2n}) + q^{2n+1} \leqslant \\ &\leqslant q[q^{2n}d(Sx_1, Tx_0) + 2n \cdot q^{2n}] + q^{2n+1} = \\ &= q^{2n+1}d(Sx_1, Tx_0) + 2n \cdot q^{2n+1} + q^{2n+1} = \\ &= q^{2n-1}d(Sx_1, Tx_0) + (2n+1)q^{2n+1} \end{aligned}$$

and so, using, (2), we conclude that (7) is valid, for every $n \in N$. Now from (6) and (7) we obtain:

$$d(Sx_{2n+1}, Sx_{2n-1}) \leqslant d(Sx_{2n+1}, Tx_{2n}) + d(Tx_{2n}, Sx_{2n-1}) \leqslant$$
$$\leqslant q^{2n}d(Sx_1, Tx_0) + 2n \cdot q^{2n} + q^{2n-1}d(Sx_1, Tx_0) + (2n-1) \cdot q^{2n-1} =$$
$$= (q^{2n} + q^{2n-1})d(Sx_1, Tx_0) + 2nq^{2n} - (2n-1)q^{2n-1}$$

and so it is obvious that $\{Sx_{2n+1}\}_{n \in N}$ is a Cauchy sequence and so there exists $z = \lim_{n \to \infty} Sx_{2n+1}$. Since:

$$\lim_{n \to \infty} d(Sx_{2n+1}, Tx_{2n}) = 0$$

it follows that $z = \lim_{n \to \infty} Tx_{2n}$.

Let us prove that $Tz \in Az$ and $Sz \in Az$. We shall prove only that $Tz \in Az$. The proof that $Sz \in Az$ is similar. Since T is continuous it follows that:

$$Tz = T(\lim_{n \to \infty} Sx_{2n+1}) = \lim_{n \to \infty} TSx_{2n+1}$$

Further, $Sx_{2n+1} \in Ax_{2n}$ and so we have $TSx_{2n+1} \in TAx_{2n} = A(Tx_{2n})$. The sequence Tx_{2n} is convergent and so from the fact that A is closed it follows that $Tz \in Az$ which completes the proof.

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● TEOREMĂ DE COINCIDENŢĂ PENTRU APLICAŢII MULTIVOCE ÎN SPAŢII METRICE

(Rezumat)

În lucrare se stabileşte o teoremă de puncte fixe comune pentru aplicaţii multivoce ce satisf ac o relaţie de forma $H(Ax, Ay) \leqslant qd(Sx, Ty)$, $0 < q < 1$ unde $S, T: X \to X$, $A: X \to CB(SX \cap \cap TX)$ iar (X, d) este un spaţiu metric complet.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXVI, 4, 1981

ON THE ISOMETRIES OF A MINKOWSKI PLANE OVER A FIELD OF CHARACTERISTIC 2

IOANA GROZE—CHIOREAN

Let K be any field with characteristic 2. We consider the following Minkowski plane over K: on the set K^2 the distance of any two points $P = (x, y)$ and $Q = (x', y')$ is defined by

$$\overline{PQ} = (x' - x)(y' - y).$$

DEFINITION. A bijective map $\tau: K^2 \to K^2$ is called an affine isometry if 1) $\overline{P^\tau Q^\tau} = \overline{PQ}$ for every $P, Q \in K^2$ and 2) f is an affin map. In the case of char $K \neq 2$ condition 2) is a consequence of condition 1).

If f is an automorphism of the field K, then the semilinear map $(x, y) \mapsto (f(x), f(y))$ is denoted by $\hat{f}: K^2 \to K^2$.

Generalizing a result of W. Benz [1], F. Radó [2] had proved the following theorem: if K is a field, char $K \neq 2$ and $\sigma: K^2 \to K^2$ a bijection such that $\overline{PQ} = 1 \Leftrightarrow \overline{P^\sigma Q^\sigma} = 1$, then there is a unique affine isometry $\tau: K^2 \to K^2$ and a unique $f \in$ Aut K such that $\sigma = \tau \circ \hat{f}$.

We shall establish a corresponding result for a filed K with char $K = 2$.

THEOREM. *Let K be a field,* char $K = 2$, $0 \in K \setminus \{0, 1\}$ *and* $\omega = \dfrac{1}{0^2 + \theta} + 1$. *Let $\sigma: K^2 \to K^2$ be a bijection such that $\overline{PQ} = 1 \Leftrightarrow \overline{P^\sigma Q^\sigma} = 1$ and $\overline{PQ} = \omega \Leftrightarrow \overline{P^\sigma Q^\sigma} = \omega$. Then there is a unique affine isometry $\tau: K^2 \to K^2$ and a unique automorphism f, such that $\sigma = \tau \circ \hat{f}$.*

First we prove:

LEMMA 1. *Let be $A, B \in K^2$, $A \neq B$, $\overline{AB} = a$ and $a, b, c \in K$. Then the following conditions are equivalent:*

1) *there is a unique point $P \in K^2$, such that $\overline{AP} = b$ and $\overline{BP} = c$,*

2) *$(a = 0$ and $b \neq c)$ or $(a \neq 0$ and $a = b + c)$.*

Proof: We may take $A = (0, 0)$, $B = (\alpha, \beta)$ and $P = (x, y)$. Condition 1) means that the system of equations

$$\begin{cases} xy = b \\ (x - \alpha)(y - \beta) = c \end{cases} \tag{1}$$

has a unique solution. We distinguish two cases:

Case 1: $a \neq 0$. Then $\alpha\beta = a \neq 0$ and the system (1) becomes:

$$\begin{cases} xy = b \\ xy - \alpha y - \beta x + a = c \end{cases} \Leftrightarrow \begin{cases} \beta x \cdot \alpha y = ab \\ \beta x + \alpha y = a + b - c \end{cases}$$

Thus, βx and αy are the roots of the equation.

$$z^2 - (a + b - c)z + ab = 0. \tag{2}$$

It is known (and easily seen) that an equation of form (2) has a unique solution in a field of characterist 2 if and only if $a + b - c = 0$ (or $a + b = c$).

Case 2: for $a = 0$ and $b \neq c$, the system (1) will have a unique solution because $a = 0 \Leftrightarrow \alpha\beta = 0 \Leftrightarrow \alpha = 0$ or $\beta = 0$, thus for $\alpha = 0$, we have $\beta \neq 0$ (since $A \neq B$) and

$$\begin{cases} x = \dfrac{b-c}{\beta} \\ y = \dfrac{b\beta}{b-c} \end{cases} ; \text{ for } \beta = 0,\ \alpha \neq 0 \begin{cases} x = \dfrac{b-c}{\alpha} \\ y = \dfrac{b\alpha}{b-c} \end{cases}.$$

If $a = 0$, $b = c$ the second equation (1) implies $x = 0$ or $y = 0$ contradicting the first equation (1). Thus lemma 1 is proved.

LEMMA 2. $\overline{AB} = 1 + \omega \Leftrightarrow \overline{A^\sigma B^\sigma} = 1 + \omega$.

Proof: Let $\overline{AB} = 1 + \omega$. Since $1 + \omega = 0$, we see by lemma 1 that there is a unique point $P \in K^2$, such that $\overline{AP} = 1$ and $\overline{BP} = \omega$. It follows that there is a unique point Q such that $\overline{A^\sigma Q} = 1$ and $\overline{B^\sigma Q} = \omega$. This implies by lemma 1 that $a = \overline{A^\sigma B^\sigma} = 0$ or $\overline{A^\sigma B^\sigma} = 1 + \omega$. We shall show that $\overline{A^\sigma B^\sigma} \neq 0$. For this purpose we have to see first if there is $M = (x; y)$ such that $\overline{AM} = \overline{BM} = 1$. This means

$$\begin{cases} xy = 1 \\ (x - \alpha)(y - \beta) = 1 \end{cases} \Leftrightarrow \begin{cases} \beta x \cdot \alpha y = 1 + \omega \\ \beta x + \alpha y = 1 + \omega \end{cases}$$

i.e. βx, αy are the solutions of $z^2 - (1 + \omega)z + 1 + \omega = 0$. Substituting $z = u(1 + \omega)$, we get

$$u^2 - u + \frac{1}{1 + \omega} = 0. \tag{3}$$

From the definition of ω it follows that $u = \theta$ verifies the equation (3). Since an equation of from (3) over a field of characteristic 2 has two solution or none, in our case there will exist two distinct points M and M' with $\overline{AM} = \overline{BM} = \overline{AM'} = \overline{BM'} = 1$. From the hipotesis of theorem we know that the distance 1 is preserved by the map σ; it follows that

$$\overline{A^\sigma M^\sigma} = \overline{B^\sigma M^\sigma} = 1. \tag{4}$$

Setting $A^\sigma = (\alpha_1, \alpha_2)$, $B^\sigma = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$ and $M^\sigma = (x, y)$, we have by

$$\overline{A^\sigma B^\sigma} = 0$$

$$\beta_1\beta_2 = 0 \tag{5}$$

and by (4)

$$(x - \alpha_1)(y - \alpha_2) = 1,\ (x - \alpha_1 - \beta_1)(y - \alpha_2 - \beta_2) = 1, \tag{6}$$

thus $\beta_2(x - \alpha_1) + \beta_1(y - \alpha_2) = 0$; $\beta_1 = 0$ yields $x - \alpha_1 = 0$ (since $\beta_2 \neq 0$), contradicting the first equation (6). Similary, $\beta_2 = 0$ also yields a contradiction. Hence $\overline{A^\sigma B^\sigma} \neq 0$ and thus by what have seen $\overline{A^\sigma B^\sigma} = 1 + \omega$, i.e.

$$\overline{AB} = 1 + \omega \Rightarrow \overline{A^\sigma B^\sigma} = 1 + \omega.$$

Applying this result to the map σ^{-1} we obtain the converse implication. Lemma 2 is proved.

LEMMA 3. $\overline{AB} = 0 \Leftrightarrow \overline{A^\sigma B^\sigma} = 0$.

Proof: $\overline{AB} = 0$ implies that there is a unique point $P \in K^2$, such that $\overline{AP} = 1$ and $\overline{BP} = \omega$. So, there is a unique P^σ for which $\overline{A^\sigma P^\sigma} = 1$ and $\overline{B^\sigma P^\sigma} = \omega$. Then $\overline{A^\sigma B^\sigma} = 0$ or $\overline{A^\sigma B^\sigma} = 1 + \omega$. If $\overline{A^\sigma B^\sigma} = 1 + \omega$, then $\overline{AB} = 1 + \omega$, which contradicts the hipotesis $\overline{AB} = 0$. Hence $\overline{AB} = 0 \Rightarrow \overline{A^\sigma B^\sigma} = 0$. Applying this to σ^{-1} we get the converse implication too.

We proved that the map σ preserves the zero distance. From this point of the proof of theorem is identical with the proof of the theorem given by F. R a d ó in [2].

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ASUPRA IZOMETRIILOR UNUI PLAN MINKOWSKI PESTE UN CÎMP DE CARACTERISTICĂ 2

(R e z u m a t)

Se extinde un rezultat al lui F. R a d ó [2] privind caracterizarea izometriilor unui plan Minkowski peste un cîmp K de caracteristică diferită de doi, pentru cazul car$K = 2$.

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SUR L'APPROXIMATION DES SOLUTIONS DES ÉQUATIONS DIFFÉRENTIELLES PAR DES FONCTIONS SPLINE A DÉFICIENCE

D. SOCEA

1. **Introduction.** On considère l'équation différentielle

$$y' = f(x, y) \tag{1}$$

ayant la condition initiale

$$y(x_0) = y_0 \tag{2}$$

où $f: [a, b] \times \mathbf{R} \to \mathbf{R}$ est une fonction suffisamment lisse, c'est-à-dire continue et dérivable continuellement, toutes les fois qu'il est nécessaire, et qui, en plus, satisfait, en rapport avec la variabile y, une condition Lipschitz à constante L. Le point x_0 satisfait la condition $a \leqslant x_0 \leqslant b$.

Dans ces conditions le problème (1)—(2) a une solution unique $y: I \to \mathbf{R}$. Le problème de l'approximation de cette solution par une fonction spline s de degré m et classe C^{m-1} a été étudié par Loscalzo — Tolbot [2].

Si le degré de la fonction spline d'approximation est grand ($m \geqslant 5$), alors les approximantes spline de classe C^{m-1} ne convergent pas. Pour obtenir une méthode convergente pour un spline s de degré $m (m \geqslant 2)$, les conditions de continuité doivent être relâchées. On obtiendra de la sorte une solution approximative qui sera une fonction spline avec déficience [1].

Dans cet ouvrage on donne une méthode d'approximation de la solution du problème (1)—(2) par une fonction spline polynomiale de déficience deux. Pour démontrer l'existence de l'unicité de la solution approximative on emploie le théorème de point fixe qui suit:

THÉORÈME 1. [9] *Soit (X, d_1), (Y, d_2) deux espaces métriques complets et $f: X \times Y \to X \times Y$, $f = (f_1, f_2)$ si: (i) Il existe, $\alpha, \beta \in \mathbf{R}$, $0 < \alpha < 1$, $\beta > 0$, de sorte que: $d_1(f_1(x_1, y_1), f_1(x_2, y_2)) \leqslant \alpha d_1(x_1, x_2) + \beta d_2(y_1, y_2)$, pour (x_1, y_1), $(x_2, y_2) \in X \times Y$*

(ii) Il existe, $\gamma, \delta \in \mathbf{R}$, $\gamma, \delta > 0, \dfrac{\gamma\beta}{1-\alpha} + \delta < 1$, de sorte que:

$d_2(f_2(x_1, y_1), f_2(x_2, y_2)) \leqslant \gamma d_1(x_1, x_2) + \delta d_2(y_1, y_2)$, *pour* $(x_1, y_1), (x_2, y_2) \in X \times Y$.

Alors f a un point fixe unique.

2. **Description de la méthode.** Considérons l'équation différentielle de premier ordre

$$y' = f(x, y) \tag{3}$$

avec la condition initiale

$$y(x_0) = y_0, \ x_0 \in I, \tag{4}$$

où $I \subset \mathbf{R}$ est un intervalle compact.

On fait les hypothèses suivantes :

(i) $f: I \times \mathbf{R} \to \mathbf{R}$, $f \in C^{m-3}(I \times \mathbf{R})$

(ii) il existe une constante Lipschitz L pour la fonction f, de sorte que :

$$|f(x, y) - f(x, Y)| \leqslant L|y - Y|,$$

quel que soit (x, y), $(x, Y) \in I \times \mathbf{R}$.

Ces hypothèses assurent l'existence et l'unicité de la solution $y: [a, b] \to \to \mathbf{R}$ du problème (3)—(4).

Soit $\Delta: a = x_0 < x_1 < \ldots < x_N < x_{N+1} = b$, $x_k = a + kh$, $0 \leqslant k \leqslant N+1$ une division uniforme de l'intervalle $[a, b]$.

Dans le premier sous-intervalle $[x_0, x_1]$ la fonction de spline s sera définie ainsi :

$$s(x) = \sum_{j=0}^{m-2} \frac{y^{(j)}(x_0)}{j!}(x - x_0)^j + \frac{a_0}{(m-1)!}(x - x_0)^{m-1} + \frac{b_0}{m!}(x - x_0)^m \tag{5}$$

où $y^{(j)}(x_0)$, $0 \leqslant j \leqslant m-2$ sont connus de (3)—(4).

Les paramètres a_0, b_0 se déterminent de sorte que dans le point $x = x_1$, la fonction s satisfasse l'équation (3) et l'équation obtenue par la dérivation de celle-ci :

$$y'' = f'_x(x, y) + f'_y(x, y)\, f(x, y) \tag{6}$$

C'est-à-dire

$$s'(x_1) = f(x_1, s(x_1))$$

$$s''(x_1) = f'_x(x_1, s(x_1)) + f'_y(x_1, s(x_1))f(x_1, s(x_1))$$

Dans le sous-intervalle suivant $[x_1, x_2]$, la fonction spline s est définie ainsi :

$$s(x) = \sum_{i=0}^{m-2} \frac{s^{(j)}(x_1)}{j!}(x - x_1)^j + \frac{a_1}{(m-1)!}(x - x_1)^{m-1} + \frac{b_1}{m!}(x - x_1)^m$$

où $s^{(j)}(x_1)$, $0 \leqslant j \leqslant m-2$, sont les limites de gauche, quand $x \to x_1$ des dérivées de s définie en (5), dans l'intervalle $[x_0, x_1]$, et les paramètres a_1, b_1 se déterminent des conditions :

$$s'(x_2) = f(x_2, s(x_2))$$

$$s''(x_2) = f'_x(x_2, s(x_2)) + f'(x_2, s(x_2))f(x_2, s(x_2))$$

Généralement, sur l'intervalle $[x_k, x_{k+1}]$, $0 \leqslant k \leqslant N$, l'approximation spline s sera définie par :

$$s(x) = \sum_{j=0}^{m-2} \frac{s^{(j)}(x_k)}{j!}(x - x_k)^j + \frac{a_k}{(m-1)!}(x - x_k)^{m-1} + \frac{b_k}{m!}(x - x_k)^m =$$

$$= A_k(x) + \frac{a_k}{(m-1)!}(x - x_k)^{m-1} + \frac{b_k}{m!}(x - x_k)^m, \tag{7}$$

où $s^{(j)}(x_k)$, $0 \leqslant j \leqslant m-2$ sont les limites de gauche quand $x \to x_2$ des dérivées de s définies dans l'intervalle $[x_{k-1}, x_k]$ et les paramètres a_k, b_k se déterminent des conditions:

$$s'(x_{k+1}) = f(x_{k+1}, x(x_{k+1}))$$

$$s''(x_{k+1}) = f'_x(x_{k+1}, s(x_{k+1})) + f'_y(x_{k+1}, s(x_{k+1}))f(x_{k+1}, s(x_{k+1}))$$

Il résulte de la construction que $s \in C^{m-2}[a, b]$.

THÉORÈME 2. *Il existe une constante h_0 de sorte que pour $h<h_0$ la fonction spline $s: [a, b] \to \mathbf{R}$, de degré m et de classe C^{m-2}, qui donne l'approximation de la solution du problème (3)—(4) existe et il est uniquement déterminé par la construction ci-dessus.*

Démonstration. Le théorème est démontré en montrant que les constantes a_k, b_k, $0 \leqslant k \leqslant N$ sont uniquement déterminées par (8). Pour la mise en évidence de tout cela, il faut supposer que: f a une dérivée partielle de premier ordre en rapport avec y bordée sur $[a, b] \times \mathbf{R}$ et elle a aussi la dérivée partielle mixte de deuxième ordre sur $[a, b] \times \mathbf{R}$ et soit M le bord supérieur de celle-là.

Remplaçant s de (7) en (8) on obtient le système

$$a_k = \frac{(m-1)!}{h^{m-2}}\left[f\left(x_{k+1}, A_k(x_{k+1}) + \frac{a_k}{(m-1)!}h^{m-1} + \frac{b}{m!}h^m\right) -\right.$$

$$- A'_k(x_{k+1}) - \frac{h}{m-1}\left(f'_x\left(x_{k+1}, A_k(x_{k+1}) + \frac{a_k}{(m-1)!}h^{m-1} + \frac{b_k}{m!}h^m\right) +\right.$$

$$+ f'_y\left(x_{k+1}, A_k(x_{k+1}) + \frac{a_k}{(m-1)!}h^{m-1} + \frac{b_k}{m!}h^m\right) \times$$

$$\left.\left.\times f\left(x_{k+1}, A_k(x_{k+1}) + \frac{a_k}{(m-1)!}h^{m-1} + \frac{b_k}{m!}h^m\right) - A''_k(x_{k+1})\right)\right]$$

$$b_k = \frac{(m-1)!(nm-2)}{h^{m-1}}\left[\frac{h}{m-2}\left(f'_x\left(x_{k+1}, A_k(x_{k+1}) + \frac{a_k}{(m-1)!}h^{m-1} + \frac{b_k}{m!}h^m\right) +\right.\right.$$

$$+ f'_y(x_{k+1}, A_k(x_{k+1}) + \frac{a_k}{(m-1)!}h^{m-1} + \frac{b_k}{m!}h^m)f\left(x_{k+1}, A_k(x_{k+1}) +\right.$$

$$\left.\left. + \frac{a_k}{(m-1)!}h^{m-1} + \frac{b_k}{m!}h^m\right) - A''_k(x_{k+1})\right) - f\left(x_{k+1}, A_k(x_{k+1}) +\right.$$

$$\left.\left. + \frac{a_k}{(m-1)!}h^{m-1} + \frac{b_k}{m!}h^m\right) + A'_k(x_{k+1})\right]$$

ou bien, en résumé:

$$a_k = g_1(a_k, b_k) \qquad (9)$$
$$b_k = g_2(a_k, b_k)$$

Nous définissons l'opérateur :

$$g : \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R},\ (a_k,\ b_k) \to (g_1(a_k,\ b_k),\ g_2(a_k,\ b_k))$$

et employons le premier théorème avec $X = Y = R,\ d_1 = d_2 = d$.

$$d(g_1(a_k^1,\ b_k^1),\ g_1(a_k^2,\ b_k^2)) \leqslant \left(hL + \frac{Mh^2}{m-1} + \frac{L^2h^2}{m-1}\right) d(a_k^1,\ a_k^2) +$$
$$+ \left(\frac{Lh^2}{m} + \frac{Mh^3}{m(m-1)} + \frac{L^2h^3}{m(m-1)}\right) d(b_k^1,\ b_k^2)$$

$$d(g_2(a_k^1,\ b_k^1),\ g_2(a_k^2,\ b_k^2)) \leqslant (Mh + L^2h + (m-2)L) d(a_k^1,\ a_k^2) +$$
$$+ \left(\frac{h^2M}{m} + \frac{h^2L^2}{m} + \frac{hL(m-2)}{m}\right) d(b_k^1,\ b_k^2)$$

Si nous notons :

$$\alpha = hL + \frac{Mh^2}{m-1} + \frac{L^2h^2}{m-1} \quad \beta = \frac{Lh^2}{m} + \frac{Mh^3}{m(m-1)} + \frac{L^2h^3}{m(m-1)}$$
$$\gamma = Mh + L^2h + (m-2)L \quad \delta = \frac{h^2M}{m} + \frac{h^2L^2}{m} + \frac{hL(m-2)}{m} \tag{10}$$

alors pour h, suffisamment petit, les conditions du premier théorème sont satisfaites. L'opérateur g a un point fixe unique, donc le système (9) a la solution unique $a_k,\ b_k$, pour $0 \leqslant k \leqslant N$, ce qui signifie que la fonction spline s existe uniquement.

Relativement à la convergence de la méthode décrite, on peut donner le :

THÉORÈME 3. *Si $f \in C^4([a,\ b] \times \mathbf{R})$ et s est la fonction spline de quatrième degré donne l'approximation de la solution du problème* (3)–(4), *alors il existe une constante K, indépendante de h, de sorte que, pour n'importe quels α, γ, définis en* (10) *qui satisfont les conditions du premier théoreme et $x \in [a,\ b]$, ont lieu les délimitations :*

$$|s^{(j)}(x) - y^{(j)}(x)| < Kh^{5-j},\ j = 0,\ 1,\ 2,\ 3,\ 4$$

à condition que $s^{(3)}(x_k)$, $s^{(4)}(x_k)$ soient calculés à l'aide de la moyenne arithmétique [3].

La démonstration de ce théorème se trouve en [7].

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ASUPRA APROXIMĂRII SOLUŢIILOR ECUAŢIILOR DIFERENŢIALE PRIN FUNCŢII SPLINE CU DEFICIENŢĂ

(R e z u m a t)

În această notă se dă o metodă de aproximare a soluţiei problemei lui Cauchy pentru ecuaţia diferenţială neliniară de ordinul întîi, prin o funcţie spline polinomială de deficienţă doi.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXVI, 4, 1981

THE INFLUENCE OF THE ASYMMETRICAL DIURNAL EFFECT ON AN EQUATORIAL CIRCULAR ORBIT

LUCIAN BURS

1. *Asymmetries of the Diurnal Effect.* The diurnal effect in the distribution of the atmospheric density for a given height and a given moment presents two main asymmetries [4], [5]:

— the curves characterizing the density Eastward the maximum of the diurnal effect show a more marked slope than those which characterize the density Westward the maximum;

— the minimum of the diurnal effect is deviated from the axis bulge — Earth's center with an angle τ (according to the model CIRA 1965, $\tau = 30°$).

In order to point out these asymmetries, we have proposed two empirical formulae [1], [2], [3]. The first formula has the form:

$$\rho = \rho_{\min}\left\langle 1 + A\begin{Bmatrix}\cos^{m_1}[(6/5)(\Phi/2)]\\ \cos^{m_2}[(6/7)(\Phi/2)]\end{Bmatrix}\right\rangle, \qquad (1)$$

where: ρ = density at the geocentric angular distance Φ from the maximum of the diurnal effect, $\rho_{\min}$ = nighttime minimum density, A = diurnal effect amplitude, m_1, m_2 = real numbers characterizing the asymmetry of the diurnal effect with respect to the axis bulge— Earth's center, while the factors 6/5 and 6/7 characterize the deviation of the minimum of the density from the above mentioned axis (6/5 for $0 \leqslant \Phi \leqslant 5\pi/6$ and 6/7 for $5\pi/6 \leqslant \Phi \leqslant 2\pi$) ≥ Generally, these factors have the form $6/p$, $6/q$ with $p + q = 12$.

By comparing the graph of the density diurnal variantion $(\rho - \rho_{\min})/\rho_{\min}$ with the graph of the density calculated by using the gamma distribution, a good agreement was remarked. Consequently, we have considered that the asymmetrical diurnal effect, for a revolution of the satellite, can be described by the law:

$$(\rho - \rho_{\min})/\rho_{\min} = S_1\Phi^{S_2}\exp(S_3\Phi), \qquad (2)$$

which leads to the second empirical formula considered by us:

$$\rho = \rho_{\min}[1 + S_1\Phi^{S_2}\exp(S_3\Phi)]. \qquad (3)$$

Here S_j, $j = 1, 3$, are constant parameters characterizing the asymmetrical diurnal effect ($S_1 > 0$, $S_2 > 1$, $S_3 < 0$), and $\Phi \in [0, 2\pi]$..

Table 1 contains the values of the parameters of Equations (1) and (3), for different heights. They were calculated on the basis of the model CIRA 1965, for a mean solar activity.

The degree of approximation for Equations (1) and (3) with respect to the model CIRA 1965 is shown in Table 2, which contains the scatterings $\sigma^2_{(1)}$, $\sigma^{(2)}_3$ and the mean relative errors $\delta_{(1)}$, $\delta_{(3)}$, corresponding to the two equations.

Table 1

h (km)	A	ρ_{min} (g/cm^3)	m_1	m_2	ln S_1	S_2	S_3
200	0.2366	2.557×10^{-13}	1	2	−1.0543	2.5101	−1.0706
400	2.3313	2.249×10^{-15}	3	3	0.8350	5.5912	−2.0510
600	6.9182	0.807×10^{-16}	3	5	2.5719	7.0423	−2.8237
800	10.4005	0.7069×10^{-17}	3.5	7	3.1795	8.4348	−3.4207

Table 2

h (km)	$\sigma^2_{(1)}$	$\delta_{(1)}$	$\sigma^2_{(1)}$	$\delta_{(3)}$
200	4.800×10^{-30}	0.62%	0.312×10^{-28}	1.27%
400	3.482×10^{-32}	3.45%	1.027×10^{-31}	6.90%
600	1.479×10^{-34}	4.76%	0.851×10^{-33}	10.96%
800	1.633×10^{-36}	5.50%	1.337×10^{-35}	12.45%

The values contained in Tables 1 and 2 were calculated by using a computer program; some improved values with respect to the corresponding ones given in [2] and [3] have resulted.

2. *Orbital Effects.* The influence of the asymmetrical diurnal effect on a satellite orbit will be studied in the following hypotheses:

(i) the orbit is circular ($e = 0$) and equatorial ($i = 0°$);

(ii) the declination of the maximum density point is zero;

(iii) for a given solar and geomagnetic activity, both the asymmetry of the diurnal effect with respect to the axis bulge—Earth's center ($m_1 \neq m_2$) and the deviation of the minimum from this axis ($p \neq q$) are considered;

(iv) m_1, m_2, S_2 are natural numbers.

Then in Equations (1) and (3) $\Phi = v$ (v = the true anomaly). The arguments of the cosines in Equation (1) will be $(6/p)(v/2)$ for $0 \leqslant v \leqslant 5\pi/6$ and $(6/q)(v/2)$ for $0 \leqslant v \leqslant 7\pi/6$, while in Equation (3) $v \in [0, 2\pi]$.

With our hypotheses, the equation for the orbital radius will be [1]:

$$da/dv = -a^2 \delta F \rho, \tag{4}$$

where: a = orbital radius, δ = drag parameter, F = a factor to allow for atmospheric rotation. By integrating Equation (4) with ρ given by Equation (1), we obtain the variation of the orbital radius for a revolution of the satellite:

$$\Delta a = -a^2 \delta F \rho_{\min} \left\{ 2\pi + A \left[\int_0^{5\pi/6} \cos^{m_1}(n_1 v) dv + \int_0^{7\pi/6} \cos^{m_2}(n_2 v) dv \right] \right\}, \tag{5}$$

where for the model CIRA 1972 $n_1 = 6/2p = 0.6$ and $n_2 = 6/2q = 3/7$. As m_1, m_2 are natural numbers and $n_1, n_2 \neq 0$, the integrals of Equation (5) can be calculated by using the recurrence formula:

$$\int \cos^m(nv)dv = \sin(nv)\cos^{m-1}(nv)/mn + [(m-1)/m] \int \cos^{m-2}(nv)dv. \tag{6}$$

Now, replacing the expression of ρ given by Equation (3) in Equation (4) and integrating this one for a revolution, we obtain the variation Δa in a first approximation:

$$\Delta a = -a^2 \delta F \rho_{\min} \left[2\pi + S_1 \int_0^{2\pi} v^{S_2} \exp(S_3 v) dv \right], \tag{7}$$

where, taking into account the fact that S_2 is a natural number, the integral can be calculated by using the recurrence formula:

$$\int v^k \exp(sv)dv = v^k \exp(sv)/s - (k/s) \int v^{k-1} \exp(sv)dv =$$

$$= \exp(sv)[v^k/s - kv^{k-1}/s^2 + k(k-1)v^{k-2}/s^3 - \ldots + (-1)^{k-1} k! v/s^k +$$

$$+ (-1)^k k!/s^{k+1}]. \tag{8}$$

3. *Numerical Example.* We shall consider a satellite moving on an equatorial circular orbit with a height of 600 km, and a mean solar activity of $150 \times 10^{-22} Wm^{-2}Hz^{-1}$. In these conditions, the model CIRA 1965 gives: $m_1 = 3$, $m_2 = 5$, $A = 6.9182$, $\rho_{\min} = 0.807 \times 10^{-16}$ g/cm³, $S_1 = 13.09$, $S_2 = 7$, $S_3 = -2.82$. With these data we have calculated, except a constant factor, the variations: Δa_{NI} (by numerical integration, using trapezia method), $\Delta a_{(5)}$ and $\Delta a_{(7)}$ (with Equations (5) and (7) respectively), and Δa_{sym} (by considering $m_1 = m_2 = 4$, i.e. the symmetrical diurnal effect, for which we took: $A_{sym} = 6.3575$, $\rho_{\min\,sym} = 0.8685 \times 10^{-16}$ g/cm³). The results are the following ones:

$$\Delta a_{NI} = -18.3056 \times 10^{-16}\ a^2 \delta F,$$

$$\Delta a_{(5)} = -18.2216 \times 10^{-16}\ a^2 \delta F,$$

$$\Delta a_{(7)} = -18.3346 \times 10^{-16}\ a^2 \delta F,$$

$$\Delta a_{sym} = -18.4667 \times 10^{-16}\ a^2 \delta F.$$

The relative errors of $\Delta a_{(5)}$, $\Delta a_{(7)}$ and Δa_{sym} with respect to Δa_{NI} are: 0.46% 0.16% and 0.88% respectively.

As it can be easily seen, the analytical solutions (5) and (7) are in a good enough agreement with the numerical integration and with the case of the symmetrical diurnal effect. So, the use of Equations (1) and (3) is justified.

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INFLUENŢA EFECTULUI DIURN ASIMETRIC ASUPRA UNEI ORBITE ECUATORIALE CIRCULARE

(R e z u m a t)

Sînt considerate două formule pentru densitatea atmosferei, ţinîndu-se cont de asimetriile efectului diurn pentru o înălţime dată şi un moment dat. Se studiază influenţa acestor asimetrii asupra unei orbite ecuatoriale circulare prin determinarea variaţiilor corespunzătoare ale razei orbitei pe durata unei revoluţii a satelitului.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXVI, 4, 1981

IN MEMORIAM

Profesorul GHEORGHE CHIŞ

În ziua de 19 mai 1981 s-a stins din viaţă profesorul universitar doctor docent Gheorghe Chiş, eminent dascăl şi om de ştiinţă clujean.

Născut la 8 august 1913 în comuna Santău, judeţul Satu-Mare, urmează şcoala primară în comuna natală şi liceul în oraşul Carei. Între anii 1931—1935 a urmat studiile universitare la Facultatea de ştiinţe a Universităţii din Cluj, Secţia matematică.

Începînd cu numirea în postul de preparator la Observatorul astronomic din Cluj, în anul 1936, profesorul Gheorghe Chiş trece prin toată ierarhia didactică: asistent (din 1943), şef de lucrări (din acelaşi an), conferenţiar (din 1950) şi profesor (din 1962) pînă la pensionarea sa (în 1977), după care îşi continuă activitatea în cadrul facultăţii ca profesor consultant.

În calitate de profesor la Facultatea de matematică a îndeplinit diferite funcţii de răspundere, ca: şef de catedră, decan şi director al Observatorului astronomic. A predat şi publicat numeroase cursuri de matematică, geodezie, topografie şi astronomie. Fiind în permanent contact cu tinerele generaţii, profesorul Gheorghe Chiş a reuşit să formeze o pleiadă de elevi, mulţi dintre ei desfăşurîndu-şi activitatea în învăţămîntul superior sau institute de cercetare din ţara noastră.

Omul de ştiinţă Gheorghe Chiş constituie un exemplu de spirit creator şi mobilizator. Obţinînd rezultate fundamentale în domeniul astronomiei şi astrofizicii, numele lui a devenit cunoscut în cercurile ştiinţifice din ţară şi străinătate. Acest fapt este confirmat prin alegerea sa în funcţiile de vicepreşedinte al Comitetului Naţional Român de Astronomie, membru al Comisiei pentru Activităţi Spaţiale, membru al Uniunii Astronomice Internaţionale şi membru al Comitetului pentru Cercetări Spaţiale (COSPAR).

Ca activist obştesc, ca director al Observatorului astronomic din Cluj-Napoca, ca şef de catedră şi decan al Facultăţii de matematică, profesorul Gheorghe Chiş a desfăşurat o bogată activitate pentru modernizarea învăţămîntului şi cercetării ştiinţifice în domeniul matematicii, mecanicii şi astronomiei din ţara noastră, pentru formarea şi educarea tinerei generaţii. De asemenea, el şi-a adus un aport deosebit de eficient, atît în cadrul Universităţii cultural-ştiinţifice al cărei rector a fost, cît şi în calitate de membru al Biroului Societăţii de Ştiinţe Matematice din R. S. România, la educarea ştiinţifică a oamenilor muncii.

Pentru meritele sale deosebite, profesorul Gheorghe Chiş a fost decorat cu ordine şi medalii ale R. S. România.

Imaginea dascălului şi omului de ştiinţă va rămîne veşnic vie în amintirea colaboratorilor, elevilor, tuturor celor care l-au cunoscut.

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