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SUMAR — CONTENTS — SOMMAIRE

Profesorul Dumitru V. Ionescu la a 80-a aniversare ● Le professeur D. V. Ionescu — pour son 80e anniversaire 3

G. MICULA, The „D. V. Ionescu" method of constructing approximation formulas ● Metoda D. V. Ionescu de construire a formulelor de aproximare a analizei 6

P. BLAGA, GH. COMAN, Multivariate interpolation formulas of Birkhoff type ● Formule de interpolare de tip Birkhoff pentru funcţii de mai multe variabile 14

P. BRĂDEANU, Asupra unei metode variaţionale cu potenţial local pentru mişcarea unui fluid viscos incompresibil în conducte circulare ● Sur une méthode variationnelle à potentiel local pour l'écoulement d'un fluide incompressible visqueux dans les tubes cylindriques 23

A. COŢIU, Noi procedee de integrare numerică a ecuaţiilor diferenţiale de ordinul întîi ● Nouveaux procédés d'intégration numérique des équations différéntielles du premier ordre 32

D. D. STANCU, A generalization of the Schoenberg approximating spline operator ● O generalizare a operatorului spline de aproximare al lui Schoenberg 37

I. A. RUS, On a problem of Darboux-Ionescu ● Asupra unei probleme a lui Darboux-Ionescu 43

P. PAVEL, Une relation entre les différences divisées et quelques formules de dérivation numérique ● O relaţie între diferenţele divizate şi unele formule de derivare numerică 46

D. RĂDULESCU, Sur un probleme de D. V. Ionescu ● Despre o problemă a lui D. V. Ionescu 51

M. MICULA, The Runge-Kutta-Fehlberg procedures for the numerical solution of Volterra integral equations ● Procedee de tip Runge-Kutta-Fehlberg pentru rezolvarea numerică a ecuaţiilor integrale Volterra 56

P. T. MOCANU, On a differential inequality for analytic functions in the unit disc ● Asupra unei inegalităţi diferenţiale pentru funcţiile analitice în discul unitate 62

I. MARUŞCIAC, Generalized Chebyshev polynomials and reversed geometric programming ● Polinoamele lui Cebîşev generalizate şi programarea geometrică reversă . . . .

S. GROZE, A method analogous to the Chebyshev method for the solving of the operator equations ● O metodă analoagă cu metoda lui Cebîşev pentru rezolvarea ecuaţiilor operatoriale . . . .

Á. PÁL, T. OPROIU, On the determination of circular orbits of artificial satellites from optical visual observations ● Asupra determinării orbitelor circulare ale sateliţilor artificiali din observaţii optice vizuale . . . .

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXVI, 2, 1981

PROFESORUL DUMITRU V. IONESCU LA A 80-A ANIVERSARE

Publicînd acest volum, ne îndeplinim o plăcută îndatorire, aceea de a adresa profesorului doctor docent D.V. Ionescu un cuvînt de omagiu la cea de-a 80-a sa aniversare.

Cadrele didactice şi studenţii facultăţii noastre, toţi cercetătorii Institutului de matematică Cluj-Napoca, îşi exprimă cu această ocazie sentimentele lor de admiraţie şi recunoştinţă pentru remarcabila activitate pe care profesorul D.V. Ionescu a desfăşurat-o şi continuă să o desfăşoare, neobosit, de mai bine de cinci decenii în cadrul facultăţii noastre.

Este greu, fie chiar să rezumăm prodigioasa activitate a profesorului D. V. Ionescu. Ne vom mărgini, deci, numai la cîteva dintre aspectele activităţii sărbătoritului.

Personalitatea profesorului D.V. Ionescu pune în evidenţă, în primul rînd, tripla sa vocaţie: aceea de dascăl, care a contribuit în mod hotărîtor la formarea a numeroase generaţii de absolvenţi; aceea de cercetător ştiinţific, situat printre primele rînduri ale creatorilor unor metode noi în matematică, creator de şcoală în domeniul analizei numerice; şi aceea de activist obştesc, îndrumător de catedră, secţie de cercetare şi facultate — la a căror organizare şi dezvoltare a contribuit temeinic.

Să urmărim schematic aceste diverse aspecte.

Născut la Bucureşti, la 14 mai 1901, D. V. Ionescu şi-a făcut studiile la Liceul „Sfîntul Sava". Încă în cursul inferior al liceului, a început să rezolve probleme la „Gazeta Matematică" (înfiinţată în 1895), al cărei colaborator şi susţinător a rămas pentru totdeauna.

După examenul de bacalaureat, în septembrie 1919, D. V. Ionescu s-a înscris la Facultatea de ştiinţe a Universităţii din Bucureşti, Secţia de matematică, unde a avut ca profesori pe Georghe Ţiţeica, Anton Davidoglu, David Emmanuel, Nicolae Coculescu, Traian Lalescu, Theodor Angheluţă. A absolvit facultatea cu rezultate strălucite, obţinînd ca student premiul „Hillel".

În anul 1923, D. V. Ionescu pleacă la Paris, ca bursier, intrînd la celebra „École Normale Supérieure". La „Sorbonne" şi la „Collège de France" a urmat cursurile marilor matematicieni E. Picard, P. Montel, E. Goursat, E. Vessiot etc. La 7 iunie 1927 şi-a susţinut cu mare succes teza de doctorat în matematică intitulată „Sur une classe d'équations fonctionelles", în care face o generalizare a unor rezultate ale lui G. Darboux, E. Picard şi E. Goursat pentru anumite ecuaţii cu derivate parţiale.

Întors în ţară, în toamna anului 1928, D. V. Ionescu este numit conferenţiar la Universitatea din Cluj, de care s-a legat pentru toată viaţa. La 15 mai 1931 devine profesor agregat şi apoi titularizat la 1 iulie 1934. În perioada grea a celui de-al doilea război mondial, cînd Facultatea de ştiinţe din Cluj a funcţionat la Timişoara, D.V. Ionescu a fost decanul acestei facultăţi, reuşind să învingă vicisitudinile vremii. În anii 1949–1955, D.V. Ionescu a fost profesor şef de catedră şi la Politehnica din Cluj, iar din 1955 devine şeful catedrei de ecuaţii diferenţiale din cadrul facultăţii noastre, pînă la pensionarea sa, în 1971, dată de la care îşi continuă activitatea ca profesor consultant.

De la înfiinţarea Filialei din Cluj a Academiei R.P.R., D. V. Ionescu a activat ca şef de secţie la Institutul de Calcul.

Continuator al înaintaşilor săi N. Abramescu, A. Angelescu, Th. Angheluţă, Gheorghe Bratu şi P. Sergescu, la Universitatea clujeană, profesorul D.V. Ionescu a contribuit temeinic la formarea a peste 50 de promoţii de absolvenţi.

Predînd numeroase cursuri de matematică şi mecanică, D-sa a elaborat şi publicat totodată valoroase cursuri şi manuale, care sînt folosite şi astăzi de studenţi, doctoranzi şi profesori.

Prin lecţiile sale magistral dezvoltate, prin eleganţa demonstraţiilor din aceste lecţii, D. V. Ionescu a fost întotdeauna fascinant şi de neîntrecut. A reuşit să formeze în jurul său o pleiadă de elevi — ajutîndu-i să se descopere pe ei înşişi — mulţi dintre ei devenind cadre didactice universitare, cercetători ştiinţifici sau specialişti în alte instituţii sau întreprinderi productive din ţară.

Omul de ştiinţă D.V. Ionescu constituie pentru noi un exemplu de elan ştiinţific, creator şi mobilizator. Obţinînd rezultate remarcabile în domeniul ecuaţiilor funcţionale, ecuaţiilor diferenţiale şi cu derivate parţiale, ecuaţiilor integrale, algebrei, geometriei, mecanicii raţionale, dar mai cu seamă în domeniul analizei numerice, numele lui a devenit cunoscut în cercuri ştiinţifice largi din ţară şi străinătate.

Ocupîndu-se, timp de peste 30 de ani, de cercetarea formulelor de aproximare ale analizei matematice, D. V. Ionescu a creat o metodă generală de construire efectivă a acestor formule, metodă numită de autorul ei „metoda funcţiei φ”, dar care astăzi îi poartă numele.

Prin metoda sa, D.V. Ionescu obţine rezultate remarcabile în construirea formulelor de aproximare ale analizei, cum ar fi : formule de cuadratură, formule de interpolare, formule de derivare numerică, reprezentarea integrală a diferenţelor divizate, aproximarea soluţiilor ecuaţiilor diferenţiale etc. Numeroasele sale rezultate matematice sînt materializate în cele peste 250 de articole şi memorii, şi într-o serie de tratate şi manuale, printre care se disting monografiile „Cuadraturi numerice”, publicată în Editura tehnică, Bucureşti, 1957, şi „Diferenţe divizate”, publicată în Editura Academiei, Bucureşti, 1978, ultima fiind distinsă cu prestigiosul premiu „Gh. Lazăr” al Academiei R.S.R.

Ca activist obştesc, ca şef de catedră, şef de secţie de cercetare şi decan al facultăţii noastre, profesorul D. V. Ionescu a desfăşurat o bogată activitate pentru dezvoltarea învăţămîntului şi cercetării ştiinţifice în domeniul matematicii din ţara noastră, şi cu deosebire din centrul nostru universitar.

Toate aceste aspecte poartă marca unui dascăl şi unui om de ştiinţă de prestigiu. Pentru calităţile şi meritele sale deosebite, cîştigate în îndelungata sa carieră, profesorului D.V. Ionescu i s-a conferit titlul suprem de profesor universitar emerit al R.S. România.

Noi, cadrele didactice, cercetătorii şi studenţii Facultăţii de matematică, toţi matematicienii Institutului de matematică Cluj-Napoca, profităm de această fericită împrejurare pentru a adresa profesorului D.V. Ionescu, veneratului dascăl, urările noastre cele mai călduroase de sănătate, fericire şi noi succese pentru continua înflorire a învăţămîntului şi cercetării matematice din ţara noastră.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXVI, 2, 1981

To Professor D. V. Ionescu on his 80 th birthday

THE „D.V. IONESCU" METHOD OF CONSTRUCTING APPROXIMATION FORMULAS

G. MICULA

In more than 50 years of intense research in approximation formulas of Analysis, D.V. Ionescu created a new general method of constructing such formulas called by its author *„the method of function* φ".

Our aim here is to present this method in its main lines. The method of D.V. Ionescu distinguishes itself through its general character being applicable to all linear approximating formulas of Analysis in one or more variables, such as: Quadrature formulas, Interpolation formulas, Formulas of numerical differentiation, Divided differences, Numerical solution of ordinary and partial differential equations, etc.

The D.V. Ionescu method consists mainly in associating to any approximating formula, of a boundary-value problem on an ordinary or partial differential equation, the boundary conditions being suitable chosen according to the formula to be established.

The D.V. Ionescu method is presented in detail mainly in the monographs [1], [2] which are unique in mathematical literature.

1. The D.V. Ionescu method and the divided differences. Let $f \in C^n[x_0, x_n]$ be a given function and

$$x_0 < x_1 < \ldots < x_n \tag{1.1}$$

a system of simple knots of the interval $[x_0, x_n]$. One denotes usually by $[x_0, x_1, \ldots, x_n; f]$ the divided difference of order n of the function f on the knots (1.1).

To the intervals $[x_0, x_1]$, $[x_1, x_2)$, $\ldots$, $[x_{n-1}, x_n]$ one associates the polynoms $\varphi_1, \varphi_2, \ldots, \varphi_n$, respectively, of degree $\leqslant n-1$, i.e. the solutions of the differential equations:

$$\varphi_1^{(n)} = 0,\ \varphi_2^{(n)} = 0, \ldots, \varphi_n^{(n)} = 0 \tag{1.2}$$

By the formula of integrating by parts one gets

$$\int_{x_{i-1}}^{x_i} \varphi_i f^{(n)}\, ds = [\varphi_i f^{(n-1)} - \varphi_i' f^{(n-2)} + \ldots + (-1)^n \varphi_i^{(n-1)} f]_{x_{i-1}}^{x_i},\quad i = \overline{1,n} \tag{1.3}$$

Adding the equalities (1.3) and denoting the left-hand side by $\mathfrak{L}[f]$ we observe that this functional depends on the values of $f, f', \ldots f^{(n-1)}$ on the knots $x_0, x_1, \ldots, x_n$.

To the equations (1.2) we attach the following boundary conditions:

$\varphi_1^{(k)}(x_0) = 0, \quad \varphi_n^{(k)}(x_n) = 0 \qquad k = 0, 1, \ldots n-2$

$\varphi_j^{(k)}(x_i) = \varphi_{j-1}^{(k)}(x_i), \quad j = 2, 3, \ldots, n-1, \; i = 1, 2, \ldots, n-1, \; k = 0, 1, \ldots, n-2 \qquad (1.4)$

$\mathfrak{L}[x^n] = 1$

By a through computation one deduces that the boundary value problem (1.2), (1.4) has a unique solution φ such that $\varphi\,|_{[x_{i-1},\,x_i]} = \varphi_i,\ i = \overline{1,n}$.

THEOREM 1. *If $f \in C^n[x_0, x_n]$, then the divided difference of order n of the function f on the knots $x_0, x_1, \ldots, x_n$ can be expressed by formula:*

$$[x_0, x_1, \ldots, x_n; f] = \int_{x_0}^{x_n} \varphi(s)\, f^{(n)}(s)\, ds \tag{1.5}$$

where the function φ coincides on each subinterval $[x_{i-1}, x_i]$ with the functions $\varphi_i(i = \overline{1,n})$ which are the solutions of the boundary value problem (1.2), (1.4).

The function φ from (1.5) can be effectively written as:

$$\varphi(s) = c_0 \frac{(x - x_0)_+^{n-1}}{(n-1)!} + c_1 \frac{(x - x_1)_+^{n-1}}{(n-1)!} + \ldots + c_n \frac{(x - x_n)_+^{n-1}}{(n-1)!} \tag{1.6}$$

where

$$u_+ := \max\{u, 0\},\ c_i := (-1)^i \frac{V(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)}{V(x_0, \ldots, x_n)}$$

$i = 0, 1, \ldots n$, and V is the usual Vandermonde determinant.

It is clear from (1.6) that the function φ is a *polynomial spline function* of degree $n-1$ and of class C^{n-2}. So it follows that the D.V. Ionescu method is also a constructing method of spline functions.

THEOREM 2. *The spline function from* (1.6) *is positive on the interval* $[x_0, x_n]$ *and*

$$\int_{x_0}^{x_n} \varphi(s)\, ds = \frac{1}{n!}$$

COROLLARY. *The divided difference from* (1.5) *can be written by a meon theorem as*

$$[x_0, x_1, \ldots, x_n; f] = \frac{f^{(n)}_{(c)}}{n!}, \qquad c \in\ ]x_0, x_n[$$

Remark. The D.V. Ionescu method can be used in a similar way to express the divided difference of the function f on the multiple knots $x_0, x_1, \ldots, x_k$, with the orders of multiplicity respectively, $n_0, n_1, \ldots, n_k$. One obtains

$$[\underbrace{x_0, x_0, \ldots, x_0}_{n_0}, \underbrace{x_1, x_1, \ldots, x_1}_{n_1}, \ldots \underbrace{x_k, x_k, \ldots, x_k}_{n_k}; f] = \int_{x_0}^{x_k} \varphi(s)\, f^{(n)}(s)\, ds \tag{1.7}$$

where $n_0 + n_1 + \ldots + n_k = n$ and φ is here a *polynomial spline function* of degree n with the multiple knots of orders $n_0, n_1, \ldots, n_k$ respectively, and φ is also positive on $[x_0, x_k]$.

2. The D. V. Ionescu method and quadrature formulas. To illustrate the application of D.V. Ionescu method to construct any quadrature formula, we shall present here only the case of the quadrature formula of Gauss.

Let $n \in N$ be fixed and $f \in C^{2n}[a,b,]$ a given function and let (1.1) be the unknown knots inside of the interval $[a,b]$.

The quadrature formula of Gauss is the following formula

$$\int_a^b f(x)dx = c_1 f(x_1) + C_2 f(x_2) + \ldots + C_n f(x_n) + R[f] \tag{2.1}$$

where the coefficient $C_1, C_2, \ldots, C_n$ and the knots $x_1, x_2, \ldots, x_n$ are to be determined such that the formula (2.1) has the degree of exactness $2n$-1 i.e. $R[f]=0$ for $f(x) = x^j$, $j = 0,1, \ldots, 2n-1$.

The D.V. Ionescu method leads to the determination of the coefficients of the Gauss quadrature formula and also its remainder.

One considers the divided difference

$$[a,x_1,x_1,x_2,x_2, \ldots, x_n,x_n, b ; g] \tag{2.2}$$

of the function $g \in C^{2n+1}[a, b]$ on the simple knots a,b and the duble knots $x_1, x_2, \ldots, x_n$ which are the roots of Legendre polynom on the interval $[a,b]$. From (1.7) follows that this divided difference can be written as

$$[a, x_1, x_1, \ldots, x_n, x_n, b ; g] = \int_a^b \overline{\varphi}(s)\, g^{(2n+1)}(s)\, ds \tag{2.3}$$

or

$$a_0 g(a) + \sum_{=1}^{n} [A_i g(x_i) + A_i' g'(x_i)] + A_{n+1} g(b) = \int_a^b \overline{\varphi}(s) g^{(2n+1)}(s)\, ds \tag{2.4}$$

It can be shown that $A_i = 0$, $i = \overline{1,n}$, $A_0 + A_{n+1} \neq 0$.

If we write in (2.4) $g'(x) = f(x)$ one obtains the quadrature formula of Gauss (2.1) with the coefficients

$$C_1 = -\frac{A_1'}{A_{n+1}}, \ldots, C_n = -\frac{A_n'}{A_{n+1}} \tag{2.5}$$

and with the remainder

$$R[f] = \int_a^b \psi(x) f^{(2n)}(x)\, dx, \quad \psi(x) := \frac{1}{A_{n+1}} \overline{\varphi}(x) \tag{2.6}$$

So, the following theorem is proved :

THEOREM 3. *The coefficients and the remainder of the quadrature formula of Gauss* (2.1) *are given in* (2.5) *and* (2.6) *where the function* ψ *is defined in the integral representation of divided difference* (2.3) *and the knots* $x_1, x_2, \ldots, x_n$ *are the roots of the Legendre poliynom of degree* n *on the interval* $[a, b]$.

Remark. Similarly, can be used the D.V. Ionescu method in order to obtain the more general quadrature formulas of Gauss and Turan type and also to determine any kind of quadrature formulas.

D.V. Ionescu [3], [4], [5] extended his method for the several variables and he constructs also many kind of practical cubature formulas of the form:

$$\iint\limits_{D} f(x, y)\, dx\, dy = \sum_{i=0}^{n} \sum_{j=0}^{m} A_{ij} f(x_i, y_j) + R[f; x, y]$$

where the remainder is represented by a double definite integral on the domain D.

3. **The D. V. Ionescu method and numerical differentiation.** V.A. Fadeeva gave the following numerical differentiation formula:

$$\Delta^2 f(x_1) = \frac{h}{2}\,[f'(x_3) - f'(x_1)] + R[f]$$

where the knots x_1, x_2, x_3 are equidistant with the step-size h.

Here, we shall apply the D.V. Ionescu method for the obtaining of the following numerical differentiation formula:

$$\Delta^{n-1} f(x_1) = A_1 f'(x_1) + A_2 f'(x_2) + \ldots + A_n f'(x_n) + R[f] \tag{3.1}$$

where the knots $x_1, x_2, \ldots, x_n$ are equidistant with the step-size h, and the coefficients $A_1, A_2, \ldots A_n$ will be determined such that the remainder $R[f]$ vanishes for $f(x) = x^j$, $j = 0, 1, \ldots n$. By $\Delta^{n-1} f(x_1)$ we denote the finite difference of order $n - 1$ of a given function $f \in C^n[a,b]$ in the point x_1 on the equidistant knots $x_1, x_2, \ldots, x_n$.

We attach to the intervals $[x_1, x_2]$, $[x_2, x_3]$, $\ldots$, $[x_{n-1}, x_n]$ the functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ which are solutions of the following differential equations:

$$\varphi_1^{(n+1)} = 0,\ \varphi_2^{(n+1))} = 0,\ \ldots,\ \varphi_n^{(n+1)} = 0 \tag{3.2}$$

Supposing $f \in C^{n+1}[x_1, x_n]$, on each subinterval $[x_i, x_{i+1}]$ integrating by parts it follows

$$\int_{x_i}^{x_{i+1}} \varphi_i(x) f^{(n+1)}(x)\, dx = [\varphi_i f^{(n)} - \varphi_i' f^{(n-1)} + \ldots + (-1)^n \varphi_i^{(n)} f]_{x_i}^{x_{i+1}} \tag{3.3}$$

Adding these equalities (3.3) for $i = 1, 2, \ldots, n - 1$ and denoting the left-hand side by $\mathfrak{L}[f]$ one gets

$$\mathfrak{L}[f] = \int_{x_1}^{x_n} \varphi(x) \overset{(n+1)}{f}(x)\, dx \tag{3.4}$$

where the kernel function φ coincides on each subinterval $[x_i, x_{i+1}]$ with the function φ_i $(i = 1, 2, \ldots, n-1)$ and $\mathfrak{L}[f]$ depends on the values of $f, f', \ldots, f^{(n)}$ on the knots $x_1, x_2, \ldots, x_n$.

Now one formulates the following boundary value problem: To determine the functions $\varphi_1, \varphi_2, \ldots, \varphi_n$, which are the solutions of the differential equations (3.2) and so that all the coefficients of $f'', f''', \ldots, f^{(n)}$ in $\mathfrak{L}[f]$ vanish and in addition the coefficients of $f(x_1), f(x_2), \ldots, f(x_n)$ to be exactly the coefficients of $\Delta^{n-1}(x_1)$ from (3.1).

This problem needs the boundary conditions on the functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ similar with the conditions (1.4). It can be proved that this boundary value problem has a unique solution $\varphi_1, \varphi_2, \ldots, \varphi_n$, which are all the special polynoms.

Thus, inserting in (3.3) the polynomials $\varphi_1, \varphi_2, \ldots, \varphi_n$ which are the solutions of the above boundary value problem, we get the numerical differentiation formula (3.1) with the coefficients:

$$A_k = (-1)^{n+k-1} \frac{(n+2k-1)}{2(n-1)} C_{n-1}^{k-1}, \quad k = 1, 2, \ldots, n \tag{3.5}$$

and with the remainder

$$R[f] = \int_{x_1}^{x_n} \varphi(x) f^{(2n+1)}(x)\, dx, \quad \varphi|_{[x_i, x_{i+1}]} = \varphi_i \tag{3.6}$$

It can be shown that the function φ is a *spline function* which is negative on the $[x_1, x_n]$, and also that

$$\int_{x_1}^{x_n} \varphi(s)\, ds = -\frac{h^{n+1}}{12} \text{ and } R[f] = -\frac{h^{n+1}}{12} f^{(n+1)}(c), \quad c \in]x_1, x_n[$$

Remark. The D.V. Ionescu method can be used in order to obtain any others numerical differentiation formulas.

4. The extension of the D.V. Ionescu method to several variable functions. In more detail, these extensions are presented in [2], [3], [4].

Shortly, we insert here some of the most important results concerning the divided differences of the functions in two, and p variables.

One considers the rectongle $D: x_0 \leqslant x \leqslant x_n$; $y_0 \leqslant y_n \leqslant y_u$ and the knots $M_{ik}(x_i, y_k)$ where

$$x_0 < x_1 < \ldots < x_{n-1} < x_n;\ y_0 < y_1 < \ldots < y_{n-1} < y_n \tag{4.1}$$

Let $f \in C^{2n}(D)$ be a given function and $n \in N$, fixed.

DEFINITION. *The divided difference of order n of the function f on the knots $M_{ik}(i, k = 0, 1, \ldots, n)$ is difined by*

$$\left[\begin{matrix} x_0, x_1, \ldots, x_n \\ y_0, y_1, \ldots, y_n \end{matrix}; f\right] := \sum_{i=0}^{n} \sum_{k=0}^{n} C_{ik} f(x_i, y_k) \tag{4.2}$$

where the coefficients C_{ik} are defined by:

$$C_{ik} = (-1)^{i+k} \frac{V(x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)}{V(x_0, x_1, \ldots, x_n)} \times \frac{V(y_0, y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n)}{V(y_0, y_1, \ldots, y_n)}$$

THEOREM 4. *The divided difference of order n of the function f on the knots $M_{ik}(i, k = 0, 1, \ldots, n)$ is represented by the formula:*

$$\left[\begin{matrix} x_0, x_1, \ldots, x_n \\ y_0, y_1, \ldots, y_n \end{matrix}; f\right] = \iint_D \varphi(x, y) \frac{\partial^{2n} f(x, y)}{\partial x^n \partial y^n} dx\, dy$$

where the kernel function $\varphi: D \to R$ is a spline function of two variables, which coincides on each rectangle $D_{ik}: \{(x,y) \mid x_i \leqslant x \leqslant x_{i+1};\ y_k \leqslant y \leqslant y_{k+1}\}$ with the function φ_{ik} which is the unique solution of the following boundary value problem:

$$\frac{\partial^{2n} \varphi_{i,k}}{\partial x^n \partial y^n} = 0, \qquad i,k = 0, 1, \ldots, n$$

$$\frac{\partial^{2(n-r)}(\varphi_{ik} - \varphi_{i,k-1})}{\partial x^{n-r} \partial y^{n-r-1}}(x, y_k) = 0 \qquad \frac{\partial^{2(n-1)-1}(\varphi_{i,k} - \varphi_{i-1,k})}{\partial x^{n-r-1} \partial y^{n-r}} x_i\, y) = 0$$

$$\frac{\partial^{2(n-r-1)}(\varphi_{ik} + \varphi_{i-1,k} - \varphi_{i,k-1} - \varphi_{i-1,k-1})}{\partial x^{n-r-1} \partial y^{n-r-1}}(x_i, y_k) = \begin{cases} C_{ik}, & r = 0 \\ 0, & r \neq 0 \end{cases}$$

$r = 0,1,\ldots, n-1;\ i,k = 0,1, \ldots, n.$

The proof of this theorem is in [2] where it is written effectivly the funtion φ and it is shown that this function is positive in the rectagle D.

For the first time was defined by D.V. I o n e s c u [2] *the divided difference of order n of a given function f' of p variables $x_1, x_2, \ldots, x_p$ on the prescribed $M_{i_1, i_2, \ldots, i_p}$.*

It is proved, in a convenient assumption on the function f that the following integral representation of the divided difference holds:

$$\left[\begin{matrix} x_1^{(0)}\ x_1^{(1)}, \ldots, x_1^{(n)} \\ x_2^{(0)}\ x_2^{(1)}, \ldots, x_2^{(n)} \\ \cdots\cdots \\ x_p^{(0)}\ x_p^{(1)}, \ldots, x_p^{(n)} \end{matrix}; f\right] = \iint_D \ldots \int \varphi(x_1, \ldots, x_p) \frac{\partial^{pn} f(x_1, \ldots, x_p)}{\partial x_1^n \ldots \partial x_p^n} dx_1\, dx_2 \ldots dx_n$$

where the domain D is defined by: $x_1^0 \leqslant x_1 \leqslant x_1^{(n)}, \ldots, x_p^0 \leqslant x_p \leqslant x_p^{(n)}$ One proves that the function φ is also the unique solution of a boundary value problem and it is the *spline function of p variable*, positive on the domain D and it vanishes on the boundary of D.

The divided difference theory for the function of several variables is in detail studied in [2].

5. The D. V. Ionescu method and the numerical solution of differential equations. The D.V. Ionescu method is also applicable to obtain new important methods of numerical solution of differential equations with given conditions.

Let consider an initial value problem

$$y' = f(x,y),\ y(x_0) = y_0, \tag{5.1}$$

where the function $f: D \to R$, $D \subset R^2$ satisfies the conditions of existence and unicity of a solution $y: I \to R$ of the problem (5.1)

Any numerical method to solve the problem (5.1) has the starting point the partition of the interval $I := [a,b]$ by the knots

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b \tag{5.2}$$

and the seeking an algoritm in order to find the solution of problem (5.1) in the points $x_1, x_2, \dots, x_n$ when it is known this solution in the point x_0, (or even in another points).

Supposing the knots (5.2) to be equidistants with the stepsize h, the problem (5.1) can be written as:

$$y(x + ih) = y(x) + \int_x^{x+ih} f(s, y(s))\, ds,\ x \in [a, b],\quad i = 1, 2, \dots \tag{5.3}$$

Let k,p,q be three integers numbers. If the argument x takes a fixed value x_{k-q}, from (5.3) it follows that the algorithm to find y_{k+p} it is

$$y_{k+p} = y_{k-q} + \int_{x_{k-q}}^{x_{k+p}} f(s, y)(s))\, ds \tag{5.4}$$

If we denote $f(s,y(s)) := g(s)$, then any quadrature formula approximating the definite integral $\int_{x_{k-q}}^{x_{k+p}} g(s)\, ds$ furnishes a numerical method for the problem (5,1), a method depending of the chosen quadrature formula.

The D.V. Ionescu method is useful to get general procedures for the numerical solution of differential equations as: the Adams-Bashforth algorithm, the Nyström method, the Milne-Simpson procedure, etc.

It is very important to underline here that, D.V. Ionescu gave a general procedure to obtain the Runge-Kutta methods of any order for the numerical solution of differential equations.

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METODA D. V. IONESCU DE CONSTRUIRE A FORMULELOR DE APROXIMARE A ANALIZEI

(R e z u m a t)

În aceasta lucrare se prezintă metoda „D. V. Ionescu" de construire a formulelor generale de aproximare ale analizei. Pentru a ilustra generalitatea şi avantajele metodei D. V. Ionescu, in lucrare se prezintă construirea formulelor de cuadratură, de derivare numerică, reprezentarea integrală a diferenţelor divizate a funcţiilor de una şi mai multe variabile, precum şi rezolvarea numerică a ecuaţiilor diferenţiale şi integrale.

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To Professor D. V. Ionescu on his 80 th birthday

MULTIVARIATE INTERPOLATION FORMULAS OF BIRKHOFF TYPE

P. BLAGA and GH. COMAN

0. **Introduction.** Taking into account its practical utility, in the last time, the punctual and blending interpolation problem for functions of several variables defined on variaous domains, usually for bivariate functions defined on a rectangle or a triangle, was largely studied.

To this end, there were used, first of all, interpolation operators of Taylor, Lagrange and Hermite type [1,3,5,7]. More recently [4], some interpolation formules on triangle using Birkhoff' s type operators were constructed.

In this paper new interpolation formulas of Birkhoff type for rectangular and triangular domains are constructed.

For simplicity, let us consider the standard domains

$$D_h = [0,h]X[0,h] \text{ and } T_h = \{(x,y)\,|\,(x,y) \in \mathbf{R}^2,\ x \geqslant 0,\ y \geqslant 0,\ x+y \leqslant h\},$$

$h > 0$, any other rectangle and triangle can be obtained by D_h respectively T_h by an affine transformation.

1. **Interpolation formulas on rectangle.** First, the necessary unidimensional operators will be constructed.

Let f: $[0,h] \to R$ be given.

LEMMA 1. 1) *If there exists $f'(h)$ then the operator B_1 defined by*

$$(B_1f)(x) = f(0) + xf'(h) \tag{1.1}$$

satisfies the conditions

i) $(B_1f)(0) = f(0)$; $(B_1f)'(h) = f'(h)$.

ii) $B_1f = f$, $\forall f \in \mathbf{P}_1$ (*the polynomials of the degree at most* 1).

2) *if $f \in C^2[0,h]$ then the remainder of the interpolation formula*

$$f = B_1f + R_1f$$

has the expression

$$(R_1f)(x) = x\left(\frac{x}{2} - h\right)f''(\xi_1),\ \ \xi_1 \in [0,h] \tag{1.2}$$

and

$$|(R_1f)(x)| \leqslant \frac{h^2}{2} M_2f,\ \forall x \in [0,h] \text{ with } M_2f = \sup_{0\leqslant x\leqslant h} |f''(x)|. \tag{1.3}$$

The proof of the first part is a simple verification.

By Peano's theorem, using ii), one obtaines

$$(R_1f)(x) = \int_0^h \varphi_1(x,t)f''(t)\,dt \tag{1.4}$$

where

$$\varphi_1(x,t) = R_1\,[(x-t)_+] = (x-t)_+ - x \leqslant 0,\ x,t \in [0,h].$$

Applying to (1.4) the mean value theorem one obtaines (1.2) and taking into account that

$$\max_{0\leqslant x\leqslant h}\left| x\left(\frac{x}{2} - h\right)\right| = \frac{1}{2}h^2$$

it follows (1.3).

LEMMA 2. 1) *If there exist $f'(0)$ and $f'(h)$, then the operator B_2 defined by*

$$(B_2f)\ (x) = \alpha\,(x)f'\,(0) + f\left(\frac{h}{2}\right) + \beta(x)f'\,(h) \tag{1.5}$$

with

$$\alpha(x) = \frac{(2x-h)(3h-2x)}{8h},\qquad \beta(x) = \frac{4x^2-h^2}{8h}$$

possesses the properties

i) $(B_2f)'(0) = f'(0),\ (B_2f)\left(\frac{h}{2}\right) = f\left(\frac{h}{2}\right),\ (B_2f)'(h) = f'(h)$

ii) $B_2f = f,\ \forall f \in \mathbf{P}_2$.

2) *If $f \in C^3\,[0,h]$, then the remainder term of the interpolation formula*

$$f = B_2f + R_2f$$

has the representation

$$(R_2f)(x) = \frac{(2x-h)(2x^2-2hx-h^2)}{24}f'''(\xi_2),\ \xi_2 \in [0,h] \tag{1.6}$$

and

$$|(R_2f)(x)| \leqslant \frac{h^3}{24}M_3f,\quad \forall x \in [0,h]. \tag{1.7}$$

The proof is a straightforward computation. It is more difficult to study the sign of the function φ_2 in the integral representation of the remainder term

$$(R_2f)(x) = \int_0^h \varphi_2(x,\,t)\,f'''(t)\,dt$$

obtained by using Peano's theorem. We have $\varphi_2(x, t) \geqslant 0$, $x \in \left[0, \frac{h}{2}\right]$ and $\varphi_2(x; t) \leqslant 0$, $x \in \left[\frac{h}{2}, h\right]$ for any $t \in [0,h]$. Such, (1.6) follows by the mean value theorem. Because

$$\max_{0 \leqslant x \leqslant h} |(2x - h)(2x^2 - 2hx - h^2)| = h^3$$

we also have (1.7).

Using these operators, interpolation formulas on D_h are generated.

Let be $f: D_h \to \mathbf{R}$. One denotes by P^x the operator which is applied to f with regard to the variable x and by R^x the corresponding remainder operator.

THEOREM 1. A. *If there exist* $f^{(1,0)}(h,0)$, $f^{(0,1)}(0,h)$, $f^{(1,1)}(h,h)$, *then the operator* $P_{11} = B_1^x B_1^y$, i.e.

$$(P_{11}f)(x,y) = f(0,0) + xf^{(1,0)}(h,0) + yf^{(0,1)}(0,h) + \\ + xyf^{(1,1)}(h,h)$$

has the properties

i) $(P_{11}f)(0,0) = f(0,0)$, $(P_{11}f)^{(1,0)}(h,0) = f^{(1,0)}(h,0)$,
$(P_{11}f)^{(0,1)}(0,h) = f^{(0,1)}(0,h)$, $(P_{11}f)^{(1,1)}(h,h) = f^{(1,1)}(h,h)$.

ii) $P_{11}f = f$, $\forall f \in \mathbf{P}_1$.

iii) *if, futhermore* $f \in C^{2,2}(D_h)$, *then the remainder operator* $R_{11} = R_1^x \oplus R_1^y$ *of the formula*

$$f = P_{11}f + R_{11}f \tag{1.8}$$

has the representation

$$(R_{11}f)(x,y) = x\left(\frac{x}{2} - h\right)f^{(2,0)}(\xi, y) + y\left(\frac{y}{2} - h\right)f^{(0,2)}(x, \eta) - \\ - xy\left(\frac{x}{2} - h\right)\left(\frac{y}{2} - h\right)f^{(2,2)}(\xi_1, \eta_1)$$

where $\xi, \eta, \xi_1, \eta_1 \in [0,h]$ *and*

$$|(R_{11}f)(x,y)| \leqslant \frac{h^2}{2} M_{20}f + \frac{h^2}{2} M_{02}f + \frac{h^4}{4} M_{22}f, \ \forall (x, y) \in D_h.$$

B. *If there exist the necessary derivatives then the operator* $P_{22} = B_2^x B_2^y$, i.e.

$$(P_{22}f)(x,y) = \alpha(x)\left[\alpha(y)f^{(1,1)}(0,0) + f^{(1,0)}\left(0, \frac{h}{2}\right) + \beta(y)f^{(1,1)}(0,h)\right] + \\ + \alpha(y)f^{(0,1)}\left(\frac{h}{2}, 0\right) + f\left(\frac{h}{2}, \frac{h}{2}\right) + \beta(y)f^{(0,1)}\left(\frac{h}{2}, h\right) + \\ + \beta(x)\left[\alpha(y)f^{(1,1)}(h,0) + f^{(1,0)}\left(h, \frac{h}{2}\right) + \beta(y)f^{(1,1)}(h, h)\right]$$

possesses the properties

i) *it interpolates the corresponding derivatives of the function f on the vertices* of D_h *an the value of f on the center of D_h.*

ii) $P_{22}f = f, \ \forall f \in \mathbf{P}_2$.

iii) *if $f \in C^{3,3}(D_h)$, then the remainder of the interpolation formula*

$$f = P_{22}f + R_{22}f \tag{1.9}$$

has the representation

$$(R_{22}f)\ (x,y) = \psi(x)f^{(3,0)}(\xi,y) + \psi\ (y)\ f^{(0,3)}(x,\eta) - \\ - \psi(x)\ \psi\ (y)f^{(3,3)}(\xi_1,\ \eta_1),$$

where $\xi,\ \eta,\ \xi_1,\ \eta_1\ \in [0,h]$ *and* $\psi(x) = \frac{1}{24}(2x - h)(2x^2 - 2hx - h^2)$.

We also have

$$|(R_{22}f)\ (x,y)| \leqslant \frac{h^3}{24} M_{30} f + \frac{h^3}{24} M_{03} f + \frac{h^6}{576} M_{33} f.$$

The proof is based on the following decompositions of the identic operator

$$I = B_p^x B_q^y + R_p^x \oplus R_q^y,\ p,q = 1,2,\ p = q$$

and on the results of lemmas 1.2,

Remark 1. Analogous formulas can be obtained using the operators

$$P_{12} = B_1^x B_2^y \text{ or } P_{21} = B_2^x B_1^y.$$

THEOREM 2. *A. If there exist* $f^{(1,0)}\ (h,y),\ f^{(0,1)}\ (x,h),\ \forall\ x,y \in [0,h]$, *then the operator* $S_{11} = B_1^x \oplus B_1^y$, *i. e.* $(S_{11}f)(x,\ y) = f(x,\ 0) + f(0,\ y) - f(0,\ 0) + x[f^{(1,0)}(h,\ y) - f^{(1,0)}\ (h,\ 0)] + y[f^{(0,1)}x,\ h) - f^{(0,1)})(0,\ h)] - xyf^{(1,1)}(h,\ h)$ *has the properties*

i) $(S_{11}f)\ \ (x,0) = f\,(x,\ 0),\ \ (S_{11}f)\ \ (0,\ y) = f\,(0,\ y)$.

$$(S_{11}f)^{(1,0)}(h,\ y) = f^{(1,0)}(h,\ y),$$
$$(S_{11}f)^{(0,1)}(x,\ h) = f^{(0,1)})\ (x,\ h),\ \forall\ x,\ y\ \in [0,\ h].$$

ii) $S_{11}f = f,\ \ \forall f \in \mathbf{P}_3$.

iii) *if $f \in C^{2,2}(D_h)$, then for the remainder term of the blending interpolation formula*

$$f = S_{11}f + \widetilde{R}_{11}f, \tag{1.10}$$

where $\widetilde{R}_{11} = R_1^x R_1^y$, *we have*

$$(\widetilde{R}_{11}f)\ (x,\ y) = xy\left(\frac{x}{2} - h\right)\left(\frac{y}{2} - h\ f^{(2,2)}(\xi,\ \eta),\ \xi,\ \eta \in [0,\ h]\right.$$

and

$$|(\widetilde{R}_{11}f)\ (x,y)| \leqslant \frac{h^4}{2} M_{22}f, \quad \forall (x,y) \in D_h.$$

B. *If there exist* $f^{(1,0)}(0,y)$, $f^{(0,1)}(x,0)$, $f^{(1,0)}(h,y)$, $f^{(0,1)}(x,h)$, $\forall\, x,y \in [0,h]$ *then the operator* $S_{22} = B_2^x \oplus B_2^y$ *possesses the properties*

i) $S_{22}f$ *interpolates the function* f *on*

$$\left\{(x,y) \mid (x,y) \in \mathbf{R}^2,\ x = \frac{h}{2},\ 0 \leqslant y \leqslant h\right\} \cup \left\{(x,y) \mid (x,y) \in \mathbf{R}^2,\ y = \frac{h}{2},\ 0 \leqslant x \leqslant h\right\}$$

and the corresponding normal derivatives on the D_h *edges.*

ii)
$$S_{22}f = f, \quad \forall f \in \mathbf{P}_5$$

iii) *if* $f \in C^{3,3}(D_h)$, *the remainder term of the blending interpolation formula*

$$f = S_{22}f + \widetilde{R}_{22}f \tag{1.11}$$

has the representation

$$(\widetilde{R}_{22}f)\ (x,y) = \psi(x)\,\psi(y)\,f^{(3,3)}(\xi,\eta), \quad \xi,\eta \in [0,h]$$

and

$$|(\widetilde{R}_{22}f)\ (x,y)| \leqslant \frac{h^6}{576} M_{33}f, \ \forall\,(x,y) \in D_h.$$

The proof follows from the above lemmas and the identity decompositions

$$I = B_p^x \oplus B_q^y + R_p^x R_q^y, \quad p,q = 1,2,\ p = q.$$

Remark 2. Analogous results are obtained taking other combinations of the operators B_1 and B_2.

Remark 3. The approximation order of a blending interpolation formula is the product of the approximation order of the coresponding unidimensional operators while the approximation order of the punctual interpolation formula is the minimum of these orders. For example, $(R_{11}f)\ (x,y) = 0\,(h^2)$ and $(\widetilde{R}_{11}f)(x,y) = 0(h^4)$. On the other hand, a punctual interpolation formula uses a finite set of information about of the function f while a blending interpolation formula implies the informations about f on a infinite set of points, i.e. it is not a numerical approximation formula. This deficiency can be eliminated by applying new operators to those terms which contain functions of a variable, such that the approximation order be preserved.

As an example, one considers the formula (1.10), whose approximation order is 4. In order to preserve the approximation order, let us use in the second level of approximation Lagrange's operator L_3 corresponding to the nodes 0, $h/3$, $2h/3$, h.

One obtains

$$f = Pf + Rf$$

where

$$P = B_1^x L_3^y + L_3^x B_1^y - B_1^x B_1^y$$

and

$$R = B_1^x R_3^y + B_1^y R_3^x + \widetilde{R}_{11},$$

i.e.

$$(Pf)(x,y) = \sum_{i=0}^{3} \{l_i(y)\,[f(0,y_i) + xf^{(1,0)}(h,y_i)] + l_i(x)[f(x_i,0) + yf^{(0,1)}(x_i,h)]\} -$$
$$- f(0,0) - xf^{(1,0)}(h,0) - yf^{(0,1)}(0,h) - xyf^{(1,1)}(h,h)$$

where l_i are the fundamental Lagrange interpolation polynomials.

It is easily seen that

$$(Rf)(x,y) = 0(h^4),\ \forall (x,y) \in D_h.$$

2. **Interpolation formulas on triangle.** THEOREM 3. *If* $f\colon T_h \to \mathbf{R}$ *and there exist* $f^{(1,0)}(h,0)$, $f^{(0,1)}(0,h)$, $f^{(1,0)}\left(\frac{h}{2}, \frac{h}{2}\right)$, $f^{(0,1)}\left(\frac{h}{2}, \frac{x}{2}\right)$ *then the operator defined by*

$$(Pf)(x,y) = \frac{(h-x-y)}{h^2} f(0,0) + \frac{x(2h-x-h)}{h^2} f(h,0) + \frac{y(2h-x-y)}{h^2} f(0,h) -$$
$$- \frac{x(h-x)}{h} f^{(1,0)}(h,0) - \frac{y(h-y)}{h} f^{(0,1)}(0,h) + \frac{xy}{h}\left(f^{(1,0)} + f^{(0,1)}\right)\left(\frac{h}{2}, \frac{h}{2}\right) \quad (2.1)$$

possesses the properties

i) $(Pf)(V_i) = f(V_i)$, $i = \overline{0,2}$; $(Pf)^{(1,0)}(V_1) = f^{(1,0)}(V_1)$,

$$(Pf)^{(0,1)}(V_2) = f^{(0,1)}(V_2),\ \frac{\partial Pf}{\partial n}\left(\frac{h}{2}, \frac{h}{2}\right) = \frac{\partial f}{\partial n}\left(\frac{h}{2}, \frac{h}{2}\right)$$

where $V_0 = (0,0)$, $V_1 = (h,0)$, $V_2 = (0,h)$ *and* $\frac{\partial}{\partial n}$ *is the normal derivative.*

ii) $Pf = f$, $\forall f \in \mathbf{P}_2$.

iii) *if* $f \in C^3(T_h)$, *then for the remainder term of the formula*

$$f = Pf + Rf$$

we have

$$|(Rf)(x,y)| \leqslant \left[\frac{27 + 7\sqrt{21}}{1728}(M_{30}f + M_{03}f) + \frac{35 + 13\sqrt{13}}{1728} M_{12}f + \frac{1}{8} M_{21}f\right] h^3.$$

Proof. The first two assertions follow by a direct verification. For the remainder term, using Peano theorem, one obtains

$$(Rf)(x,y) = \int_0 K_{30}(x,y\,;s)f^{(3,0)}(s,0)\,ds + \int_0^h K_{03}(x,y\,;t)f^{(0,3)}(0,t)\,dt$$

$$+\int_0^h K_{12}(x,y\,;t)f^{(1,2)}(0,t)\,dt + \iint_{T_h} K_{21}(x,y\,;s,t)f^{(2,1)}(s,t)\,ds\,dt$$

where

$$K_{30}(x,y\,;s) = \frac{(x-s)_+^2}{2} - \frac{x(h-s)}{2h^2}\,[(2h-x-y)(h-s)-2h(h-x)] -$$

$$-\frac{xy}{2h}\left(\frac{h}{2}-s\right)_+$$

$$K_{03}(x,y\,;t) = K_{30}(y,x\,;t)$$

$$K_{12}(x,y\,;t) = x(y-t)_+ - \frac{xy(h-t)}{h}\left(\frac{h}{2}-t\right)_+^0$$

$$K_{21}(x,y\,;s,t) = (x-s)_+(y-t)_+^0 - \frac{xy}{h}\left(\frac{h}{2}-s\right)_+^0\left(\frac{h}{2}-t\right)_+^0.$$

The functions K_{30} and K_{03} do not change the sign on $T_h \times [0,h]$. Hence, by mean value theorem, one obtaines

$$(Rf)(x,y) = \varphi(x,y)f^{(3,0)}(\xi,0) + \varphi(y,x)f^{(0,3)}(0,\eta) +$$

$$+\int_0^h K_{12}(x,y\,;t)f^{(1,2)}(0,t)\,dt + \iint_{T_h} K_{21}(x,y;s,t)\,f^{(2,1)}(s,t)\,ds\,dt,$$

$$\xi,\eta \in [0,h],$$

where

$$\varphi(x,y) = \frac{x[4(x-h)^2+hy]}{24} \quad \text{and} \quad \max_{T_h} |\varphi(x,y)| = \frac{27+7\sqrt{21}}{1721}.$$

It follows that

$$|(Rf)(x,y)| \leqslant \frac{27+7\sqrt{21}}{1721}\,h^3(M_{30}f + M_{03}f) + M_{12}f\int_0^h |K_{12}(x,y\,;t)|\,dt +$$

$$+M_{21}f\iint_{T_h} |K_{21}(x,y\,;s,t)|\,ds\,dt.$$

Taking into account that

$$\int_0^h |K_{12}(x,y\,;t)|\,dt \leqslant \frac{35+13\sqrt{13}}{1728}h^3, \quad \iint_{T_h} |K_{21}(x,y\,;s,t)|\,ds\,dt \leqslant \frac{h^3}{8}$$

the proof follows.

THEOREM 4. *Let be* $f\colon T_h \to \mathbf{R}$ *and* P_1, P_2 *operators defined by*

$$(P_1f)(x,y) = f(0,0) + xf^{(1,0)}\left(\frac{hx}{x+y}, \frac{hy}{x+y}\right) + yf^{(0,1)}\left(\frac{hx}{x+y}, \frac{hy}{x+y}\right)$$

$$(P_2f)(x,y) = f(x,0) + f(0,y) - f(0,0).$$

If there exist: $f^{(1,0)}, f^{(0,1)}$ *on the hypotenuse of* T_h, $f^{(1,0)}(x,0)$, $x \in [0,h]$ *and* $f^{(0,1)}(0,y)$, $y \in [0,h]$, *then the operator* $P = P_1 \oplus P_2$, *i.e.*

$$(Pf)(x,y) = f(x,0) + f(0,y) - f(0,0) + x\left[f^{(1,0)}\left(\frac{hx}{x+y}\ \frac{hy}{x+y}\right) - f^{(1,0)}\left(\frac{hx}{x+y},0\right)\right] +$$

$$+ y\left[f^{(0,1)}\left(\frac{hx}{x+y}, \frac{hy}{x+y}\right) - f^{(0,1)}\left(0, \frac{hy}{x+y}\right)\right]$$

possesses the properties:

i) $(Pf)(x,0) = f(x,0), \ x \in [0,h]$

$(Pf)(0,y) = f(0,y), \ y \in [0,h]$

$$\frac{\partial Pf}{\partial r}\left(\frac{hx}{x+y}, \frac{hy}{x+y}\right) = \frac{\partial f}{\partial r}\left(\frac{hx}{x+y}, \frac{hy}{x+y}\right), \quad x,y \in [0,h]$$

where r *is the direction of the position vector of* $M\left(\frac{hx}{x+y}, \frac{hy}{x+y}\right)$.

ii) $Pf = f, \ \forall f \in \mathbf{P}_1$.

iii) *if* $f \in C^{1,1}(T_h)$, *then* $Rf = f - Pf$ *has the expression*

$$(Rf)(x,y) = xyf^{(1,1)}(\xi,\eta), \ (\xi,\eta) \in T_h \tag{2.2}$$

and

$$|(Rf)(x,y)| \leqslant \frac{h^2}{4} M_{11}f, \ \forall (x,y) \in T_h. \tag{2.3}$$

The proof follows easily. For the remainder we have

$$(Rf)(x,y) = \iint_T K_{11}(x,y\,;s,t) f^{(1,1)}(s,t)\,ds\,dt$$

where

$$K_{11}(x,y\,;s,t) = (x-s)_+^0 (y-t)_+^0.$$

By the mean value theorem, it follows (2.2), and $\max\limits_{T_h}(xy) = \frac{h^2}{4}$ implies (2.3).

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FORMULE DE INTERPOLARE DE TIP BIRKHOFF PENTRU FUNCŢII DE MAI MULTE VARIABILE

(Rezumat)

În această lucrare sînt construite formule de interpolare punctuală, respectiv blending pentru funcţii de mai multe variabile pe domenii rectangulare şi pe simplexe, folosind îndeosebi operatori de interpolare de tip Birkhoff. Se insistă, în primul rînd, asupra unor formule simple din punct de vedere al aplicaţiilor.

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Profesorului D. V. Ionescu
la a 80-a aniversare

ASUPRA UNEI METODE VARIAŢIONALE CU POTENŢIAL LOCAL PENTRU MIŞCAREA UNUI FLUID VÎSCOS INCOMPRESIBIL ÎN CONDUCTE CIRCULARE

PETRE BRĂDEANU

1. Construirea unui potenţial local pentru mişcarea staţionară în conducte circulare. Se consideră un fluid incompresibil cu viscozitatea μ şi conductibilitatea termică λ variabile (dependente de temperatura fluidului) în mişcare axial-simetrică pe traiectorii rectilinii în conducte cilindrice circulare (mişcare Poiseuille). Ecuaţiile mişcării, considerată nestaţionară şi cu gradient longitudinal de temperatură, sînt scrise în forma (mişcări lente)

$$\rho r \frac{\partial u}{\partial t} = - r \frac{\partial p}{\partial z} + \frac{\partial}{\partial r}\left(r\mu \frac{\partial u}{\partial r}\right), \quad \text{în } \Omega_1 \tag{1}$$

$$\rho r c_p \frac{\partial T}{\partial t} = - \rho r c_p u \frac{\partial T}{\partial z} + r\mu\left(\frac{\partial u}{\partial r}\right)^2 + \frac{\partial}{\partial r}\left(r\lambda \frac{\partial T}{\partial r}\right), \quad \text{în } \Omega_1 \tag{2}$$

$$(\Omega_1 = \Omega \times (0,T) \text{ cu } \Omega = \{(r,z) \mid 0 \leqslant R_1 \leqslant r \leqslant R_2, 0 \leqslant z \leqslant z_0)$$

$$(u \geqslant 0,\ T > 0,\ \mu > 0,\ \lambda > 0 \text{ în } \Omega_1)$$

unde (r,z,φ) sînt coordonatele cilindrice, $u(r,z,t)$ — viteza fluidului în direcţia axei conductei (Oz), $T(r,z,t)$ — temperatura absolută a fluidului, $p(z,t)$ — presiunea, $\rho(=\text{const.})$ — densitatea, c_p — căldura specifică (pentru fluidul vîscos incompresibil).

Funcţiile principale necunoscute sînt viteza $u(r,z,t)$ şi temperatura $T(r,z,t)$; celelalte mărimi se consideră cunoscute.

Pentru a deduce potenţialul local — o funcţională variaţională de minim care se asociază acestor ecuaţii — se procedează, în mod formal, după cum urmează.

Se înmulţeşte (1) cu $-2\,\partial u/\partial t$ şi (2) cu $-(1/T)\,\partial T/\partial t$ şi se construieşte funcţia nepozitivă (locală)

$$\begin{aligned}\psi = -2\frac{\rho r}{T}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{\rho r c_p}{T}\left(\frac{\partial T}{\partial t}\right)^2 &= -\frac{\partial}{\partial r}\left(2\mu r \frac{\partial u}{\partial r}\frac{\partial u}{\partial t} + \lambda r \frac{\partial T}{\partial r}\cdot\frac{1}{T}\frac{\partial T}{\partial t}\right) + \\ &+ 2\mu r \frac{\partial u}{\partial r}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r}\right) + \lambda r \frac{\partial T}{\partial r}\frac{\partial}{\partial r}\left(\frac{1}{T}\frac{\partial T}{\partial t}\right) + 2r\frac{\partial p}{\partial z}\frac{\partial u}{\partial t} + \\ &+ \frac{\rho r u c_p}{T}\frac{\partial T}{\partial z}\frac{\partial T}{\partial t} - \frac{\mu r}{T}\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial T}{\partial t} \leqslant 0\end{aligned} \tag{3}$$

În expresia funcției ψ să facem următoarele modificări:

$$\frac{\partial}{\partial r}\left(\frac{1}{T}\frac{\partial T}{\partial t}\right) = -\frac{1}{T^2}\frac{\partial T}{\partial r}\frac{\partial T}{\partial t} + \frac{1}{T}\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial r}\right) = \frac{\partial}{\partial t}\left(\frac{1}{T}\frac{\partial T}{\partial r}\right),$$

$$\lambda\frac{\partial T}{\partial r}\frac{\partial}{\partial r}\left(\frac{1}{T}\frac{\partial T}{\partial t}\right) = \lambda T\cdot\frac{1}{T}\frac{\partial T}{\partial r}\frac{\partial}{\partial t}\left(\frac{1}{T}\frac{\partial T}{\partial r}\right) = \frac{1}{2}\lambda T\frac{\partial}{\partial t}\left(\frac{1}{T}\frac{\partial T}{\partial r}\right)^2, \tag{4}$$

$$2\mu\frac{\partial u}{\partial r}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r}\right) - \frac{\mu}{T}\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial T}{\partial t} = \mu T\frac{\partial}{\partial t}\left[\frac{1}{T}\left(\frac{\partial u}{\partial r}\right)^2\right].$$

Să considerăm, acum, funcția ($d\tau = rdr\, d\varphi\, dz$)

$$F(t) = \iiint\limits_{\Omega}\frac{1}{r}\psi(r,z,t)d\tau = 2\pi\int\limits_{R_1}^{R_2}\int\limits_{0}^{z_0}\psi(r,z,t)\ dr\ dz \leqslant 0. \tag{5}$$

Se substituie funcția (3), ținînd seama de identitățile (4), în (5), se presupune că mișcarea are condiții la limită independente de timp pe suprafața conductei (unde, potrivit aderenței fluidului vîscos la suprafața rigidă, viteza fluidului este nulă) și, aplicînd formula lui Green, se obține

$$F(t) = 2\pi\int\limits_{0}^{z_0}\int\limits_{R_1}^{R_2}\left\{\mu T\frac{\partial}{\partial t}\left[\frac{1}{T}\left(\frac{\partial u}{\partial r}\right)^2\right] + \frac{1}{2}\lambda T\frac{\partial}{\partial t}\left(\frac{1}{T}\frac{\partial T}{\partial r}\right)^2 + \right.$$
$$\left. + 2\frac{\partial p}{\partial z}\frac{\partial u}{\partial t} + \rho c_p\frac{u}{T}\frac{\partial T}{\partial z}\frac{\partial T}{\partial t}\right\}r\,dr\,dz \leqslant 0$$

Să presupunem că starea nestaționară este foarte aproape de o stare de mișcare staționară cu parametrii u_0, T_0, μ_0, λ_0 astfel că pot avea loc următoarele aproximații ale coeficienților derivate ∂t:

$$\lambda T \approx \lambda_0 T_0,\ \mu T \approx \mu_0 T_0,\ \frac{\partial p}{\partial z} \approx \frac{\partial p_0}{\partial z}\text{etc.} \tag{6'}$$

Atunci, funcția $F(t)$, deoarece u_0, T_0 etc. nu depind de timpul t, se poate reduce la forma

$$F(t) = 2\pi\frac{\partial}{\partial t}J(u, T; u_0, T_0) \leqslant 0$$

unde $J : C^1[\Omega] \to R^1$ dată prin formula

$$J(u, T; u_0, T_0) = \int\limits_{R_1}^{R_2}\int\limits_{0}^{z_0}\left[\frac{\lambda_0 T_0}{2T^2}\left(\frac{\partial T}{\partial r}\right)^2 + \frac{\mu_0 T_0}{T}\left(\frac{\partial u}{\partial r}\right)^2 + 2\frac{\partial p}{\partial z}u + \right.$$
$$\left. + \rho c_p\frac{u_0}{T_0}\frac{\partial T_0}{\partial z}T\right]rdrdz \tag{8}$$

este o funcțională care descrește în raport cu timpul devenind minimă pentru mișcarea staționară, adică atunci cînd $u = u_0$, $T = T_0$; funcționala J este potențialul local asociat ecuațiilor (1) — (2) corespunzătoare regimului staționar de mişcare.

Prin urmare, avem

$$\delta_u J(u, T; u_0, T_0) = 0, \quad \delta_T J(u, T; u_0, T_0) = 0$$

cu condițiile complementare $u = u_0$, $T = T_0$, unde variația are loc numai în raport cu u şi, respectiv, T; u_0 şi T_0 se consideră funcții fixate (variațional), date de ecuațiile staționare (1) — (2) cu $\partial/\partial t = 0$.

Observație. Se poate arăta că ecuațiile extremalelor u şi T pentru funcționala J sînt chiar ecuațiile (1) — (2) cu $\partial/\partial t = 0$. În acest scop se folosesc ecuațiile lui Euler-Lagrange

$$\frac{\partial}{\partial r}\left(\frac{\partial \Phi}{\partial \alpha_r}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \Phi}{\partial \alpha_z}\right) - \frac{\partial \Phi}{\partial \alpha} = 0 \text{ cu } \alpha = u\,;\,T, \tag{8'}$$

unde Φ este funcția de sub semnul integralei în (8) iar indicii r şi z notează derivate parțiale. După ce se calculează derivatele parțiale respective şi se fac substituțiile în (8′) se aplică condițiile $u = u_0$, $T = T_0$, $\mu = \mu_0$ etc.

Calculele efectuate mai sus şi potențialul local (8) corespund la mai multe tipuri de mişcări staționare ale unui fluid vîscos cu conductibilitate termică în conducte circulare:

— Mişcare staționară în regiunea de intrare între doi cilindri coaxiali de raze R_1 şi R_2 cu ecuațiile şi condițiile

$$(1) - (2) \text{ cu } \partial/\partial t = 0$$

$$u(r,0) = \tilde{u}(r),\ u(R_1,z) = u\ (R_2,z) = 0,\ u(r,z_0) = u_0\ (r)\,;$$

$$T(r,0) = \tilde{T}\ (r),\ T(R_1,z) = T_{w1},\ T(R_2,z) = T_{w2} \text{ (sau flux termic dat)}.$$

Potențialul local este funcționala (8).

— Mişcare staționară complet dezvoltată ($z \to \infty$) între doi cilindri coaxali de raze R_1 şi R cu ecuațiile şi condițiile

$$(1) - (2) \text{ cu } \frac{\partial}{\partial t} = 0,\quad \frac{\partial}{\partial z} = 0$$

$$u(R_1) = u\ (R_2) = 0\,;\ T(R_1) = T_{w1},\ T(R_2) = T_{w2}$$

(condițiile pentru temperatură pot fi specificate şi în raport cu fluxul termic pe suprafețele cilindrilor). Potențialul local va fi funcționala (8) în care se elimină derivarea în raport cu z (ultimul termen) şi de asemenea integrarea în raport cu z.

— Mişcarea staționară în regiunea de intrare a unei conducte circulare şi mişcare staționară complet dezvoltată într-o conductă circulară (fără gradient de temperatură în direcția conductei). Studiem ultima mişcare considerînd că coeficienții μ şi λ sînt liniar dependenți de temperatura fluidului.

2. Mişcare Poiseuille cu viscozitate şi conductivitate termică variabile. Această mişcare este descrisă, după (1) — (2), de ecuaţiile diferenţiale ordinare neliniare de forma (mişcare Poiseuille):

$$\frac{d}{dr}\left(\mu r\frac{du}{dr}\right)=r\frac{dp}{dz},\ \frac{d}{dr}\left(\lambda r\frac{dT}{dr}\right)=-\mu r\left(\frac{du}{dr}\right)^2,\ 0<r<R$$
$$u(0)=u_m,\ u(R)=0;\ T(0)=T_m,\ T(R)=T_w \tag{9}$$
$$\left(\frac{\partial u}{\partial r}(0)=\frac{\partial T}{\partial r}(0)\ \text{— condiţie de simetrie}\right)$$

cu

$$\mu(T)=\mu^*\left[1+\tilde{\alpha}(T-T^*)\right],\ \lambda(T)=\lambda^*\left[1+\tilde{\beta}(T-T^*)\right]; \tag{10}$$
$$\frac{dp}{dz}=\text{const}=-\frac{\Delta p}{l}<0,\ (\Delta p=p^0-p_l>0),$$

T_w şi u_w — const. (valori necunoscute), $T_w=\text{const}$ (valoare dată) ($R=$ = raza conductei), unde $u(r)$ — viteza longitudinală în conductă şi $T(r)$ – temperatura absolută a fluidului sînt funcţiile necunoscute, continue şi pozitive pe $[0,R]$, cu $u(R)=0$ şi $T>0$.

Potenţialul local al acestei mişcări staţionare este dat de funcţionala (8) în care se elimină variabila z (adică, integrala după z şi $\partial/\partial z$).

Se trece la variabile adminsionale punînd

$$r=RY,\ u=U\left(\frac{\lambda^* T_w}{\mu^*}\right)^{1/2},\quad T=\theta\, T_w,$$
$$\gamma=-\frac{\Delta p}{l}\frac{R^2}{(\mu^*\lambda^* T_w)^{1/2}}<0,\ \alpha=\tilde{\alpha}T_w,\ \beta=\tilde{\beta}T_w.$$

Potenţialul local al acestei mişcări staţionare, în noile variabile, are forma

$$J(U,\theta;\theta_0)=\int_0^1\Big\{\left[1+\beta(\theta^0-\theta^*)\right]\frac{\theta_0}{2\theta^2}\left(\frac{d\theta}{dY}\right)^2+ \tag{11}$$
$$+\left[1+\alpha(\theta_0-\theta^*)\right]\frac{\theta_0}{\theta}\left(\frac{dU}{dY}\right)^2+2\gamma U\Big\}Y\,dY$$

definită pe subspaţiul

$$V=\{U,\theta\in C^1[0,1]\mid U(1)=0,\ U(0)=U_m>0;\ \theta(1)=1,\ \theta(0)=\theta_m; \tag{11'}$$
$$U'(0)=\theta'(0)=\text{o simetrie}\}$$

Potenţialul local (11), în cazul $\beta=0$ este introdus prin alte procedee (de natură fizică) în lucrările [1] şi [2].

Pentru problema de minim a funcţionalei J se încearcă, din subspaţiul V, soluţiile

$$U(Y) = b(1 - Y^2),\ \theta(Y) = 1 + a(1 - Y^4) \tag{12}$$

unde a şi b sînt două constante necunoscute, care devin aproximaţii pentru soluţiile exacte ale problemei la limită diferenţială considerată iniţial.
Se determină constantele a şi b cu condiţiile de staţionaritate ale funcţionalei

$$\left\{\frac{\partial}{\partial b} J(U(b), \theta(a); \theta_0)\right\}_{\theta_0=\theta} = 0, \tag{13}$$

$$\left\{\frac{\partial}{\partial a} J(U(b), \theta(a); \theta_0)\right\}_{\theta_0=\theta} = 0.$$

Se obţin, pentru a şi b, ecuaţiile $(' = d/dY)$

$$2\int_0^1 \left[1 + \alpha(\theta - \theta^*)U' \frac{dU'}{db} + \gamma \frac{dU}{db}\right] Y dY = 0$$

$$\int_0^1 \left\{[1 + \beta(\theta - \theta^*)]\left[\frac{\theta'}{\theta}\frac{d\theta'}{da} - \left(\frac{\theta'}{\theta}\right)^2 \frac{d\theta}{da}\right] - [1 + \alpha(\theta - \theta^*)]\frac{U'^2}{\theta}\frac{d\theta}{da}\right\} Y dY = 0 \tag{14}$$

Pentru rezolvarea acestui sistem, ca şi în celelalte metode variaţionale aproximative, integralele definite:

$$I_1 = = \int_0^1 \frac{\theta'}{\theta}\frac{d\theta'}{da} Y dY = 16a\int_0^1 \frac{Y^7 dY}{1 + a(1 - Y^4)} = -4\left(1 - \frac{1+a}{a}\ln(1 + a)\right);$$

$$\left(I_* = \int_0^1 \frac{Y^{11} dY}{1 + a(1 - Y^4)} = -\frac{1}{4a^3}\left(\frac{3a+2}{2} - \frac{(1+a)^2}{a}\ln(1 + a)\right)\right)$$

$$I_2(a) = -\int_0^1 \left(\frac{\theta'}{\theta}\right)^2 \frac{d\theta}{da} Y dY = 4\left(2 - \frac{a+2}{a}\ln(1 + a)\right); \tag{15}$$

$$I_3(a) = \beta\int_0^1 (\theta - \theta^*)\frac{\theta'}{\theta}\frac{d\theta'}{d\theta} Y dY = 2\beta\left[2a + 2 - \frac{a^2 + 3a + 2}{a}\ln(1 + a)\right];$$

$$I_4(a) = -\beta\int_0^1 (\theta - \theta^*)\left(\frac{\theta'}{\theta}\right)^2 \frac{d\theta}{da} Y dY = -2\beta\left[3a + 6 - \frac{a^2 + 6a + 6}{a}\ln(1 + a)\right],$$

$$I_5(a) = -\int_0^1 [1 + \alpha(\theta - \theta^*)] \frac{U'^2}{0} \frac{d\theta}{da} Y dY =$$

$$= \frac{b^2}{a}(\alpha - 1) - \frac{b^2(\alpha a + 2\alpha - 2)}{2a^2} \ln(1 + a).$$

$$\left(\text{se alege } \theta^* = \frac{1}{2}(1 + \theta_m) = \frac{1}{2}(2 + a)\right)$$

Substituind aceste integrale în sistemul (14) obţinem ecuaţiile

$$b = -\frac{\gamma}{4},$$

$$\frac{1}{a} \ln(1 + a) = \frac{-4\beta a^2 + 8(1 - 2\beta)a - \frac{\gamma^2}{8}(1 - \alpha)}{-12\beta a^2 + \left(8 - 16\beta + \frac{1}{16}\alpha\gamma^2\right)a - \frac{\gamma^2}{8}(1 - \alpha)} \tag{16}$$

în care α, β şi γ pot lua diferite valori în raport cu modul cum se aleg funcţiile $\mu(T)$, $\lambda(T)$, căderea de presiune $\Delta p/l$ şi temperatura T_w a conductei.

Să observăm, mai întîi, că ecuaţia transcendentă (16) are, pentru orice valori (α, β, γ) soluţia $a = 0$ care nu convine problemei variaţionale.

Dacă $\alpha = 0$ şi $\beta = 0$, adică coeficienţii μ şi λ sînt constanţi, ecuaţia precedentă se reduce la ecuaţia

$$\ln(1 + a) = a$$

cu soluţia $a = 0$ care nu poate fi acceptată. În acest caz problema este liniară — cu coeficienţii fizici constanţi — şi poate fi rezolvată prin cuadraturi elementare.

Soluţiile problemei, în cazul $\alpha = \beta = 0$, sînt (Poiseuille)

$$u(Y) = v_m(1 - Y^2), \; \theta(Y) = 1 + \frac{\mu v_m^2}{4\lambda T_w}(1 - Y^4) \tag{17}$$

$$\left(v_m = -\frac{dp}{dz}\frac{R^2}{4\mu} \text{ şi } \frac{\mu v_m^2}{4\lambda T_w} = \left(\frac{\gamma}{8}\right)^2\right)$$

sau

$$U(Y;\gamma) = -\frac{\gamma}{4}(1 - Y^2), \quad \theta(Y;\gamma) = 1 + \left(\frac{\gamma}{8}\right)^2(1 - Y^4)$$

Prin urmare, dacă $\alpha = \beta = 0$, potrivit cu notaţia folosită în soluţia de încercare, avem $a = (\gamma/8)^2$ şi $b = -\gamma/4$.

Valorile acestor coeficienţi, a şi b, pentru diferite valori γ sînt date în tabelul 1 (prima coloană a valorilor lui a şi coloana lui b). Rezolvarea ecuaţiei (16) s-a făcut prin procedeul înjumătăţirii intervalului. Soluţiile aproximative, pentru a, sînt date în tabelul 1 împreună cu reziduul (eroarea) corespunzătoare pentru diferite valori ale parametrilor fizici α, β, γ. Tot în tabelul 1 sînt prezentate şi valorile coeficientului b (independent de α şi β). Să mai remarcăm, în fine, că ecuația (16) se reduce, pentru $\beta = 0$ (λ = const, conductibilitate termică constantă) la ecuația dată și rezolvată în lucrarea [1]. Tabelul 1 pune în evidență influența dependenței coeficientului de conductibilitate termică asupra distribuției temperaturii în mișcarea Poiseuille. Pentru a estima această influență se poate face comparație cu rezultatele din [1] unde, de exemplu, pentru $\alpha = -0{,}5$ și $\gamma = -5{,}0$ ($\beta = 0$) s-a găsit $a = 0{,}37958$. Tabelul 1 dă valoarea $a = 0{,}35500$ (cu $\beta = -1$).

Se poate verifica exactitatea soluției, cînd β este variabil (liniar), prin considerarea soluției pentru $\alpha = 0$ (adică μ = const.). Într-adevăr, dacă $\alpha = 0$ și β variază liniar cu temperatura, ecuațiile mișcării și energiei,(9), se integrează prin cuadraturi simple. Calcule elementare conduc la soluțiile exacte

$$U(Y) = -\frac{Y}{4}(1 - Y^2), \qquad (A)$$

$$\theta(Y\,;\,\beta,\gamma) = \theta^* + \frac{1}{\beta}\left[-1 + \sqrt{1 + 2\beta\left\{\left(\frac{Y}{8}\right)^2(1 - Y^4) + (1-\theta^*)\left[1 + \frac{\beta}{2}(1-\theta^*)\right]\right\}}\right]$$

Să determinăm, acum, soluția aproximativă (12) în cazul $\alpha = 0$. Dacă, luînd $\alpha = 0$, se aleg valorile $\beta = -1/2$ și $\gamma = -4{,}0$ ecuația (16) are soluția $a = 0{,}2443$ (cu o eroare de ordinul 3.10^{-6}). Soluția aproximativă θ ($Y\,;a$) dată de (12), în aceste supoziții, va fi

$$\theta_a(Y) = 1 + 0{,}2443(1 - Y^4) \qquad (B)$$

$$(\alpha = 0\,;\,\beta = -1/2\,;\,\gamma = -4{,}0\,;\,a = 0{,}2443) \qquad (C)$$

Să alegem în soluția exactă (A), ca și în deducerea soluției (B), valoarea $\theta^* = 1 + a/2$. Atunci, soluția exactă (A) cu valorile (C) primește forma

$$\theta_e(Y\,;\,-1/2,-4) = 3{,}12215 - 2\sqrt{1{,}12588 - 0{,}25(1 - Y^4)} \qquad (D)$$

În tabelul 2 se pot compara valorile aproximative $\theta_a(Y)$, deduse prin metoda variațională aproximativă (pentru μ = const.) cu valorile exacte θ_e obținute prin integrarea exactă prin cuadraturi a ecuației energiei cînd β variază liniar cu temperatura (iar μ = const.). Aproximația este suficient de bună (în special spre peretele conductei), după cum arată și valorile erorii relative $\varepsilon_r = |\theta_e - \theta_a|/|\theta_e|$ (în %).

Evident că pe lîngă eroarea de calcul ε care se introduce în rezolvarea ecuației (16), consemnată în tabelul 1, mai apar și erori în etapele metodei variaționale: prin alegerea tipului de potențial local și adoptarea soluției aproximative simple (aici cu un singur parametru, a pentru θ și b pentru U). Eroarea de aproximație, introdusă prin (12) este evaluată în [1] prin comparație cu rezultatele date de metoda diferențelor finite (în cazul $\beta = 0$). Rezultatele date de cele două metode coincid destul de bine pentru diferite valori γ (considerate și în tabelul 1). Această constatare garantează și acceptarea aproximației, determinată în tabelul 1, pentru soluția exactă a problemei (9) — (10) în care conductibilitatea termică se consideră variabilă liniar cu temperatura (ca și viscozitatea).

Tabel 1

Reziduu	β / γ \ α	*a*				*b*
	β	0	−1,0	−0,5	−0,5	
	α	0	−0,5	−1,0	−2,0	
	−1	0,015625	0,01556	0,015564	0,015525	0,250
ε			$1\cdot10^{-6}$	$-3\cdot10^{-6}$	$3\cdot10^{-7}$	
	−2	0,062500	0,06153	0,061543	0,060933	0,500
ε			$-5\cdot10^{-8}$	$-1\cdot10^{-6}$	$-4\cdot10^{-7}$	
	−3	0,140625	0,13577	0,135905	0,133050	0,750
ε			$-2\cdot10^{-7}$	$-8\cdot10^{-7}$	$3\cdot10^{-7}$	
	−4	0,250000	0,234995	0,235600	0,227436	1,000
ε			$3\cdot10^{-6}$	$8\cdot10^{-6}$	$-2\cdot10^{-7}$	
	−5	0,390626	0,35500	0,356967	0,339264	1,250
ε			$4\cdot10^{-6}$	$-1\cdot10^{-6}$	$1\cdot10^{-7}$	

Tabel 2

Y	0_a	0_e	$\varepsilon_r\%$
0	1,2443	1,250381	0,5
1/4	1,243346	1,249934	0,5
1/2	1,229031	1,233769	0,4
3/4	1,167002	1,167687	0,06
1	1	1	0

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SUR UNE MÉTHODE VARIATIONNELLE A POTENTIEL LOCAL POUR L'ÉCOULEMENT D'UN FLUIDE INCOMPRESSIBLE VISQUEUX DANS LES TUBES CYLINDRIQUES

(R é s u m é)

Dans le présent travail on déduit, par un procédé formel, des potentiels locaux (les fonctionnelles variationnelles (8), (10)) d'un mouvement fluide incompressible visqueux lent, à travers un tube cylindrique (l' écoulement Poiseuille), en supposant que le gradient longitudinal de la température est non nul, la viscosité μ et la conductivité thermique λ de fluide dépendent linéairement de la température.

On cherche pour le probléme variationnel de la détermination de la vitesse U (et u) et de la température θ (et T), le long du rayon du tube $r = RY$, une solution approchée simple, (12), à un paramètre (à et, respectivement, b) dans l'espace V, (11'). On déduit, pour a et b, les équations (16). L'équation transcendante (16) est résolue par la méthode de bipartition: les solutions approchées, avec le résidu ε (c'est-à-dire l'erreur) obtenu par substitution, sont données dans le Tableau 1.

L'exactitude des résultats obtenus, pour le problème hydrotermodynamique, peut être vérifieé par comparaison avec les résultats du travail [1] et ceux obtenus en utilisant l'intégration exacte, (A), dans le cas $\alpha = 0$ et $\beta \neq 0$,(10). Le Tableau 2 consigne les valeurs exactes θ_e et approchées θ_a de la température θ, données par les formules (B) et (D).

STUDIA UNIV. BABEŞ–BOLYAI, MATHEMATICA, XXVI, 2, 1981

Profesorului D. V. Ionescu
la a 80-a aniversare

NOI PROCEDEE DE INTEGRARE NUMERICĂ A ECUAŢIILOR DIFERENŢIALE DE ORDINUL ÎNTÎI

A. COŢIU

În prima parte a acestei lucrări se dă o nouă generalizare relativă la transformarea lui E. Fehlberg [5], pentru integrarea numerică a ecuţaiilor diferenţiale de ordinul întîi.

Ţinîndu-se seamă de transformarea obţinută, în partea a doua a lucrării se arată cum se pot stabili procedee de ordinul 7 de exactitate [1], pe trei noduri, relative la integrarea numerică a ecuaţiilor diferenţiale de ordinul întîi. Metoda de lucru aplicată în această parte a lucrării, a mai fost folosită de autor [2], [3], [4], şi de aceea nu sînt prezentate, în mod detaliat, toate calculele. Această metodă, în literatura de specialitate, este cunoscută sub denumirea de *metoda lui Runge-Kutta*, şi se datoreşte lui *C.* Runge [9] şi *W.* Kutta [8]. Preocupări de această natură, la noi în ţară, au fost iniţiate şi susţinute de D. V. Ionescu [6], [7].

1. Să considerăm ecuaţia diferenţială de ordinul întîi

$$z' = \varphi(x, z), \tag{1}$$

şi fie $z(x)$ soluţia ei, care satisface la condiţia iniţială

$$z(x_0) = z_0 . \tag{2}$$

Presupunem că sînt satisfăcute condiţiile care asigură existenţa şi unicitatea soluţiei $z(x)$, pe intervalul $[x_0, x]$, unde $x = x_0 + h$.

În locul funcţiei $z(x)$, soluţia ecuaţiei diferenţiale (1), cu condiţia iniţială (2), să introducem, printr-o transformare, o nouă funcţie $y(x)$, astfel încît:

a) $y(x)$ să fie soluţie a ecuaţiei diferenţiale transformate

$$y' = f(x, y)\ ; \tag{3}$$

b) $y(x)$ să satisfacă la aceeaşi condiţie iniţială

$$y(x_0) = y_0 = z_0\ ; \tag{4}$$

c) soluţia $y(x)$ şi funcţia $f(x,y)$, să satisfacă pe nodul x_0, la condiţiile

$$y_0' = 0, \quad y_0'' = 0, \tag{5}$$

$$(f_y')_0 = 0, \ (f_{xy}'')_0 = 0, \ (f_{x^2y}''')_0 = 0. \tag{6}$$

Se arată că relaţiile (5) atrag după ele condiţia

$$(f_x')_0 = 0. \tag{7}$$

Se verifică ușor că condițiile (4) și (5) sînt satisfăcute, dacă între funcțiile $z(x)$ și $y(x)$ avem relația

$$z = \theta(x, y) = y + z_0'(x - x_0) + \frac{1}{2} z_0''(x - x_0)^2 + A(x - x_0)(y - y_0) +$$
$$+ B(x - x_0)^2 (y - y_0) + C(x - x_0)^3 (y - y_0), \tag{8}$$

oricare ar fi constantele A, B, C.

Din relația (8), prin derivare în raport cu x, și ținînd seamă de egalitățile (1) și (3), rezultă

$$\varphi(x, z) = [1 + A(x - x_0) + B(x - x_0)^2 + C(x - x_0)^3] f(x, y) + z_0' +$$
$$+ z_0''(x - x_0) + A(y - y_0) + 2B(x - x_0)(y - y_0) + 3C(x - x_0)^2(y - y_0). \tag{9}$$

Prin raționamente și calcule analoage cu cele din lucrările [2], [3], [4], [5], se obțin următoarele valori pentru constantele A, B și C:

$$A = (\varphi_z')_0, \quad B = \frac{1}{2}[(\varphi_{xz}'')_0 + (\varphi_z')_0^2],$$
$$C = \frac{1}{6}[(\varphi_{x^2z}''')_0 + 3(\varphi_z')_0 (\varphi_{xz}'')_0 + (\varphi_z')_0^3]. \tag{10}$$

Din egalitatea (9), în care A, B și C sînt dați de relațiile (10), obținem expresia funcției $f(x, y)$, care figurează în ecuația diferențială transformată (3):

$$f(x, y) = \frac{1}{1 + A(x - x_0) + B(x - x_0)^2 + C(x - x_0)^3} \{\varphi[x, \theta(x, y)] -$$
$$- z_0' - z_0''(x - x_0) - A(y - y_0) - 2B(x - x_0)(y - y_0) - 3C(x - x_0)^2(y - y_0)\}, \tag{11}$$

unde $\theta(x,y)$ este dat de egalitatea (8). Relația (8), în care A, B și C sînt dați de (10), definește transformarea pe care o avem în vedere.

Din cele arătate mai sus rezultă că integrarea ecuației diferențiale (1), cu condiția inițială (2), se reduce, prin transformarea (8), la integrarea ecuației (3), cu condiția inițială (4), unde funcția $f(x,y)$ din membrul al doilea al ecuației transformate (3), este dată de relația (11), constantele A, B și C, fiind date de (10).

Soluția $y(x)$ și funcția $f(x,y)$, satisfac, pe nodul x_0, la condițiile (5), (6), (7).

Dacă se integrează numeric ecuația diferențială (3), și se obține soluția ei aproximativă $\tilde{y}(x)$, atunci relația (8), unde A, B și C sînt dați de (10), ne dă soluția aproximativă $\tilde{z}(x)$ a ecuației diferențiale inițiale (1), înlocuind în (8) pe $y(x)$ prin $\tilde{y}(x)$.

2. Să arătăm acum cum se stabilesc procedee de integrare numerică pentru ecuația diferențială transformată (3).

Soluția $y(x)$ a ecuației diferențiale (3), dezvoltată după puterile lui h, într-o vecinătate a nodului x_0, potrivit formulei lui Taylor, este

$$y(x) = y(x_0 + h) = y_0 + \frac{y_0'''}{3!} h^3 + \frac{y_0^{IV}}{4!} h_4 + \ldots + \frac{y_0^{(7)}}{7!} h^7 + \ldots \tag{12}$$

În relația (12) s-a ținut seamă de condițiile (5); de asemenea, avem în v... dere procedee de ordinul 7 de exactitate [1], pe trei noduri.

Pentru aplicarea procedeelor, pe care le avem în vedere, la integrarea nu... merică a ecuației diferențiale (3), să scriem următoarele formule:

$$\tilde{y}(x) = y(x_0 + h) \doteq y_0 + c_1 k_1 + c_2 k_2 + c_3 k_3, \tag{1...}$$

unde

$$\begin{aligned} k_1 &= f\,(x_0 + \alpha_1\, h, y_0)\, h, \\ k_2 &= f\,(x_0 + \alpha_2\, h, y_0 + \beta k_1)\, h, \\ k_3 &= f\,(x_0 + \alpha_3\, h,\, y_0 + \gamma k_1 + \delta k_2)\, h. \end{aligned} \tag{1...}$$

Aceste formule conțin 9 constante : $\alpha_1, \alpha_2, \alpha_3, c_1, c_2, c_3, \beta, \gamma, \delta$. Vom determin... aceste constante, comparînd coeficienții lui $h^3, h^4, \ldots, h^7$, din dezvoltarea dup... formula lui Taylor, a membrului al doilea al relațiilor (13) și (14), într-o vecin... tate a lui x_0, cu coeficienții corespunzători din egalitatea (12).

Ținînd seamă de condițiile (5), (6) și (7), sîntem conduși la următor... sistem de ecuații

$$\begin{gathered} c_1\, \alpha_1^i + c_2\, \alpha_2^i + c_3\, \alpha_3^i = \frac{1}{i+1}, \\ i = 2,3,4,5,6 \\ c_2\, \alpha_1^2\, \alpha_2^3\, \beta + c_3\, \alpha_1^2\, \alpha_3^3\, \gamma + c_3\, \alpha_2^2\, \alpha_3^3\, \delta = \frac{1}{21}, \\ c_2 \alpha_1^4\, \beta^2 + c_3\, \alpha_1^4\, \gamma^2 + c_3\, \alpha_2^4\, \delta^2 + 2c_3\, \alpha_1^2\, \alpha_2^2\, \gamma\delta = \frac{1}{63}. \end{gathered} \tag{15}$$

Daca scriem că primele 5 ecuații sînt compatibile, în raport cu c_1, c_2, c_3 unde α_j sînt presupuși diferiți de zero, și diferiți între ei, obținem relațiile

$$\begin{aligned} &\frac{1}{3}\, \alpha_1\, \alpha_2\, \alpha_3 - \frac{1}{4}\,(\alpha_1\, \alpha_2 + \alpha_2\, \alpha_3 + \alpha_3\, \alpha_1) + \frac{1}{5}\,(\alpha_1 + \alpha_2 + \alpha_3) = \frac{1}{6}, \\ &\frac{1}{4}\, \alpha_1\, \alpha_2\, \alpha_3 - \frac{1}{5}\,(\alpha_1\, \alpha_2 + \alpha_2\, \alpha_3 + \alpha_3\, \alpha_1) + \frac{1}{6}\,(\alpha_1 + \alpha_2 + \alpha_3) = \frac{1}{7}. \end{aligned} \tag{16}$$

Dacă se ia $\alpha_3 = 1$, cum se obișnuiește în formulele lui Runge-Kutta, din ecuațiile (16) obținem

$$\alpha_1 = \frac{4 - \sqrt{2}}{7}, \quad \alpha_2 = \frac{4 + \sqrt{2}}{7}, \quad \alpha_3 = 1. \tag{17}$$

Din primele 3 ecuații ale sistemului (15), rezultă

$$c_1 = \frac{184 + 11\sqrt{2}}{480}, \quad c_2 = \frac{184 - 11\sqrt{2}}{480}, \quad c_3 = \frac{1}{15}. \tag{18}$$

Din ultimele două ecuații ale sistemului (15), în care $\alpha_j, c_j, j = 1,2,3$ se înlocuiesc cu valorile lor, și în care se ia $\beta = \frac{1}{2}$, cum se obișnuiește în formulele lui Runge-Kutta, se obțin constantele γ și δ.

Observații. 1° Transformarea dată în această lucrare este mai complicată decît transformările din lucrările [5] și [2], dar are avantajul că permite obținerea de procedee de ordinul 7 de exactitate, pe cînd transformările stabilite în lucrările [5] și [2] permit obținerea de procedee de ordinul 6 de exactitate.

2° Ne exprimăm părerea că merită atenție cercetarea faptului dacă nu cumva transformarea dată în această lucrare permite să se stabilească procedee de ordinul 7 de exactitate pe 2 noduri.

3° În ultimele două ecuații ale sistemului (15), în care α_j și c_j, $j = 1,2,3$ se înlocuiesc cu valorile lor, se poate alege de asemenea $\beta = \frac{1}{3}$, sau $\beta = \frac{1}{4}$, cum se obișnuiește în formulele lui Runge-Kutta. În aceste cazuri, vom obține valori corespunzătoare pentru γ și δ.

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NOUVEAUX PROCÉDÉS D'INTÉGRATION NUMÉRIQUE DES ÉQUATIONS DIFFÉRENTIELLES DU PREMIER ORDRE

(Résumé)

Dans la première partie de ce travail on donne une nouvelle généralisation de la transformation de E. Fehlberg [5] relative à l'intégration numérique des équations différentielles du premier ordre. Cette nouvelle généralisation est donnée par l'égalité (8), où les constantes A, B et C sont données par les relations (10). Moyennant la transformation (8), l'intégration de l'équation différentielle (1) avec la condition initiale (2), se réduit à l'intégration de l'équation différentielle (3) et la fonction $f(x, y)$ vérifient les conditions (5), (6) et (7) sur le noeud x_0. La fonction $f(x, y)$ qui figure dans le second membre de l'équation différentielle transformée (3) est donnée par l'égalité (11). Dans la seconde partie de ce travail on présente la manière dont on peut construire des procédés du septième ordre d'exactitude

[1] sur trois noeuds, pour l'intégration numérique de l'équation différentielle transformée (3). L
forme de ces procédés s'exprime par les formules (13) et (14), où les constantes α_j, c_j $(j = 1, 2, 3)$,
γ, δ, se déterminent par la résolution du système (15). Puis, on intègre numériquement l'équatio
différentielle transformée (3). Le passage de la solution approchée $\widetilde{y(x)}$ de l'équation diférentielle (3
à la solution $\tilde{z}(x)$ de l'équation différentielle initiale (1), s'exécute par le remplacement de $y(x)$ pa
$\tilde{y}(x)$ dans la formule de transformation (8).

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To Professor D. V. Ionescu
on his 80th birthday

A GENERALIZATION OF THE SCHOENBERG APPROXIMATING SPLINE OPERATOR

D. D. STANCU

It is known that in 1965 I. J. Schoenberg [9] has presented a spline type generalization of the Bernstein operator, which has been further investigated in a joint paper by M. J. Marsden and I. J. Schoenberg [5] and later by M. J. Marsden [6], [7].

In this paper we shall present a slight generalization of the Schoenberg operator by using some equally spaced nodes depending on two non-negative parameters which can be chosen so that the corresponding approximation formula has the degree of exactness -1, 0 or 1. After establishing a uniform convergence theorem, one uses a decomposition formula for divided differences, in order to deduce some calculating explicit formulas and one insists on the case when there are no knots, case which leads to a Bernstein type operator investigated earlier in [10] and [11].

Considering two integers m $(m \geqslant 1)$, n $(n \geqslant 0)$ and two real parameters α and β satisfying the condition: $0 \leqslant \alpha \leqslant \beta$, we construct a spline-type linear positive operator, in the sense of Schonenberg, defined for any f: $[0,1] \to \mathbf{R}$, by:

$$(S_{m,n}^{\alpha,\beta} f)(x) := \sum_{j=0}^{m+n} f(\xi_{m,j}^{\alpha,\beta})\, N_{m,n,j}(x), \tag{1}$$

where we have

$$0 = \underbrace{x_{-m} = \ldots = x_{-1} = x_0}_{m+1} < x_1 < \ldots < x_n < \underbrace{x_{n+1} = \ldots = x_{n+m+1}}_{m+1} = 1,$$

$$\xi_{m,j}^{\alpha,\beta} := \frac{1}{m+\beta}\,[x_{-m+1+j} + \ldots + x_{-1} + x_0 + x_1 + \ldots + x_j + \alpha], \tag{2}$$

$$N_{m,n,j}(x) := (x_{j+1} - x_{-m+j})\ [x_{-m+j}, \ldots, x_{-1}, x_0, x_1, \ldots, x_{j+1}\,;\ (t-x)_+^m\,]. \tag{3}$$

Here $x_1, x_2, \ldots, x_n$ are the *knots*, $\xi_{m,j}^{\alpha,\beta}$ are the *nodes* and $N_{m,n,j}$ — the *fundamental spline functions* — are the *normalized B-splines.*

We have the nodes

$$\xi_{m,0}^{\alpha,\beta} = \frac{\alpha}{m+\beta},\quad \xi_{m,1}^{\alpha,\beta} = \frac{x_1+\alpha}{m+\beta},\quad \xi_{m,2}^{\alpha,\beta} = \frac{x_1+x_2+\alpha}{m+\beta},\ \ldots,\ \xi_{m,m+n-1}^{\alpha,\beta} =$$
$$= \frac{x_n+m-1+\alpha}{m+\beta},\quad \xi_{m,m+n}^{\alpha,\beta} = \frac{m+\alpha}{m+\beta}$$

and it is obvious that

$$0 \leqslant \xi_{m,0}^{\alpha,\beta} < \xi_{m,1}^{\alpha,\beta} < \ldots < \xi_{m,m+n-1}^{\alpha,\beta} < \xi_{m,m+n}^{\alpha,\beta} \leqslant 1.$$

It can be readily shown that

$$(S_{m,n}^{\alpha,\beta} f)(0) = f\left(\frac{\alpha}{m+\beta}\right), \quad (S_{m,n}^{\alpha,\beta} f)(1) = f\left(\frac{m+\alpha}{m+\beta}\right), \tag{4}$$

so that if $0 = \alpha < \beta$ then the operator defined by (1) — (3) is interpolatory at the left side of $[0,1]$ and if $0 < \alpha = \beta$ then it is interpolatory at the right side only.

If $\alpha = \beta = 0$ then we have the Schoenberg original operator $S_{m,n} = S_{m,n}^{0,0}$, which is interpolatory at both sides of $[0,1]$.

It should be observed that if $0 < \alpha < \beta$ then this operator is not interpolatory at any point of $[0,1]$ and in this case the degree of exactnes of the approximation formula

$$f(x) = (S_{m,n}^{\alpha,\beta} f)(x) + (R_{m,n}^{\alpha,\beta} f)(x) \tag{5}$$

equals -1 (i.e., the remainder vanishes if and only if f is a polynomial identically zero on $[0,1]$), while if $0 = \alpha < \beta$ or $0 < \alpha = \beta$ this degree is 0 ;the highest degree of exactness is 1 and it is achieved if and only if $\alpha = \beta = 0$.

Because

$$\xi_{m,j}^{\alpha,\beta} = \frac{1}{1+\frac{\beta}{m}}\left(\xi_{m,j} + \frac{\alpha}{m}\right),$$

where

$$\xi_{m,j} = \xi_{m,j}^{0,0} = \frac{1}{m}(x_{-m+1+j} + \ldots + x_1 + x_0 + x_1 + \ldots + x_j),$$

and the Schoenberg operator $S_{m,n}$ reproduces the linear functions, we can write

$$(S_{m,n}^{\alpha,\beta} e_0)(x) = 1, \quad (S_{m,n}^{\alpha,\beta} e_1)(x) = \frac{mx+\alpha}{m+\beta} = x + \frac{\alpha - \beta x}{m+\beta},$$

$$(S_{m,n}^{\alpha,\beta} e_2)(x) = \left[1 + \frac{\beta}{m}\right]^{-2}\left[(S_{m,n} e_2)(x) + \frac{2\alpha x}{m} + \frac{\alpha^2}{m^2}\right],$$

where: $e_j(x) := x^j$ $(j = 0,1,2)$ for $x \in [0,1]$.

According to a theorem of Marsden [6], a necessary and sufficient condition that $S_{m,n} e_2 \to e_2$ uniformly on $[0,1]$ is: $\frac{1}{m} \| \Delta \| \to 0$, where $\| \Delta \|$ is the norm of the partition of the interval $[0,1]$ by the points x_j. It will occur if either $m \to \infty$, or m bounded and $\| \Delta \| \to 0$.

In our case we should assume that $m \to \infty$ in order to have

$$S_{m,n}^{\alpha,\beta} e_j \to e_j \quad (j = 0, 1,2)$$

uniformly on $[0,1]$, and according to the well known theorem of Bohman-Korovkin we can state the following

THEOREM. *If the parameters* α *and* β *satisfy the conditions* $0 \leqslant \alpha \leqslant \beta$ *then for any* $f \in C[0,1]$ *and a giwen* $n \in N$, *we have*

$$\lim_{m \to \infty} S^{\alpha,\beta}_{m,n} f = f,$$

uniformly on $[0,1]$.

Assuming that

$$m \geqslant n \geqslant 0,$$

we can write

$$(S^{\alpha,\beta}_{m,n} f)(x) = \sum_{j=0}^{n-1} f\left(\frac{x_1 + x_2 + \ldots + x_j + \alpha}{m+\beta}\right) x_{j+1} [\underbrace{0, \ldots 0}_{m+1-j}, x_1, \ldots, x_{j+1}; (t-x)^m_+] +$$

$$+ \sum_{j=n}^{m} f\left(\frac{x_1 + \ldots + x_n + j - n + \alpha}{m+\beta}\right) [\underbrace{0, \ldots, 0}_{m+1-j}, x_1, \ldots, x_n, \underbrace{1, \ldots, 1}_{j+1-n}; (t-x)^m_+] +$$

$$+ \sum_{j=m+1}^{m+n} f\left(\frac{x_{-m+1+j} + \ldots + x_n + j - n + \alpha}{m+\beta}\right) (1 - x_{-m+j}) [x_{-m+j}, \ldots, x_n, \underbrace{1, \ldots, 1}_{j+1-n}; (t-x)^m_+].$$

If we use the following decomposition formula for divided differences

$$[a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_n; f(t)] =$$

$$= \left[a_0, a_1, \ldots, a_m; \frac{f(t)}{(t-b_0)\ldots(t-b_n)}\right] + \left[b_0, b_1, \ldots, b_n; \frac{f(t)}{(t-a_0)\ldots(t-a_m)}\right]$$

and take into account that

$$(t-x)^m_+ = (t-x)^m + (-1)^{m+1}(x-t)^m_+,$$

we can write the preceding formula in the following form

$$(S^{\alpha,\beta}_{m,n} f)(x) = \sum_{j=0}^{n-1} f\left(\frac{x_1 + \ldots + x_j + \alpha}{m+\beta}\right) x_{j+1} \left[x_1, \ldots, x_{j+1}; \frac{(t-x)^m_+}{t^{m+1-j}}\right] +$$

$$+ \sum_{j=n}^{m} f\left(\frac{x_1 + \ldots + x_n + j - n + \alpha}{m+\beta}\right) [\underbrace{0, \ldots, 0}_{m+1-j}, x_1, \ldots, x_n, \underbrace{1, \ldots, 1}_{j+1-n}; (t-x)^m_+] +$$

$$+ \sum_{j=m+1}^{m+n} f\left(\frac{x_{-m+1+j} + \ldots + x_n + j - n + \alpha}{m+\beta}\right) (1 - x_{-m+j}) \times$$

$$\times \left[x_{-m+j}, \ldots, x_n; \frac{(t-x)^m_+}{(1-t)^{j+1-n}}\right]. \tag{6}$$

An important special case is $n = 0$, when we have no knots and from the second part of the preceding equality we should retain only the second sum and we have

$$(S_{m,0}^{\alpha,\beta} f)(x) = \sum_{j=0}^{m} f\left(\frac{j+\alpha}{m+\beta}\right) [\underbrace{0, \ldots, 0}_{m+1-j}, \underbrace{1, \ldots, 1}_{j+1}; (t-x)_+^m]. \tag{7}$$

We consider now the following integral representation for divided differences with multiple nodes a and b:

$$[\underbrace{a, \ldots, a}_{p+1}, \underbrace{b, \ldots, b}_{q+1}; f] = \frac{1}{(b-a)^{p+q+1}} \int_a^b \frac{(x-a)^q (b-x)^p}{p!\, q!} f^{(p+q+1)}(x)\, dx,$$

which can be proved by using a representation by integrals, due to Hermite, of a divided difference.

By taking $a = 0$, $b = 1$, $p = \mathrm{m} - j$ and $q = j$, we obtain

$$[\underbrace{0, \ldots, 0}_{m+1-j}, \underbrace{1, \ldots, 1}_{j+1}; f] = \frac{1}{m!} \int_0^1 \binom{m}{j} x^j (1-x)^{m-j} f^{(m+1)}(x)\, dx.$$

On the other hand, according to the well known theorem of Peano on the integral representation of a linear functional having a certain degree of exactness, we can write

$$[\underbrace{0, \ldots, 0}_{m+1-j}, \underbrace{1, \ldots, 1}_{j+1}; f] = \frac{1}{m!} \int_0^1 [\underbrace{0, \ldots, 0}_{m+1-j}, \underbrace{1, \ldots, 1}_{j+1}; (t-x)_+^m] f^{(m+1)}(x)\, dx.$$

It may actually be shown that we have

$$[\underbrace{0, \ldots, 0}_{m+1-j}, \underbrace{1, \ldots, 1}_{j+1}; (t-x)_+^m] = p_{m,j}(x), \tag{8}$$

where $x \in [0,1]$ and

$$p_{m,j}(x) = \binom{m}{j} x^j (1-x)^{m-j}.$$

This result, presented first by S c h o e n b e r g in [9], has interest also from a probabilistic point of view, because we have here a representation by a divided difference of the binomial probability distribution.

According to (7) and (8) we have

$$(S_{m,0}^{\alpha,\beta} f)(x) = (B_m^{\alpha,\beta} f)(x) = \sum_{j=0}^{m} p_{m,j}(x) f\left(\frac{j+\alpha}{m+\beta}\right).$$

We have arrived in this way at a Bernstein-type polynomial, investigated earlier in our papers [10], [11].

We remark further that more explicitely formule (6) can be written:

$$(S^{\alpha,\beta}_{m,n}f)(x) = \frac{(x_1 - x)^m_+}{x_1^m} f\left(\frac{\alpha}{m+\beta}\right) + \sum_{j=1}^{n-1} x_{j+1}\left[x_1, \ldots, x_{j+1}; \frac{(t-x)^m_+}{t^{m+1-j}}\right] \times$$

$$\times f\left(\frac{x_1 + \ldots + x_j + \alpha}{m+\beta}\right) + \left[x_1, \ldots, x_n, 1; \frac{(t-x)^m_+}{t^{m+1-n}}\right] f\left(\frac{x_1 + \ldots + x_n + \alpha}{m+\beta}\right) +$$

$$+ \sum_{j=n+1}^{m-1} [\underbrace{0, \ldots, 0}_{m+1-j}, x_1, \ldots, x_n, \underbrace{1, \ldots, 1}_{j+1-n}; (t-x)^m_+] f\left(\frac{x_1 + \ldots + x_n + j - n + \alpha}{m+\beta}\right) +$$

$$+ (-1)^n \left[0, x_1, \ldots, x_n; \frac{(x-t)^m_+}{(1-t)^{m+1-n}}\right] f\left(\frac{x_1 + \ldots + x_n + m - n + \alpha}{m+\beta}\right) +$$

$$+ \sum_{j=m+1}^{m+n-1} (1 - x_{-m+j}) \left[x_{-m+j}, \ldots, x_n; \frac{(x-t)^m_+}{(1-t)^{j+1-n}}\right] \times$$

$$\times f\left(\frac{x_{-m+1+j} + \ldots + x_n + j - n + \alpha}{m+\beta}\right) + \frac{(x - x_n)^m_+}{(1-x_n)^m} f\left(\frac{m+\alpha}{m+\beta}\right).$$

Now it is easy to see that the relations (4) are true.

Assuming that $\alpha = \beta = 0$, the approximation formula (5) becomes the Schoenberg approximation formula

$$f(x) = (S_{m,n}f)(x) + (R_{m,n}f)(x),$$

having the degree of exactness one.

The remainder of this formula has been investigated by D. Leviatan [4] and by Gh. Coman [1]. This remainder can be expressed in an integral form by using either a known theorem of Peano or „the method of function φ" of D.V. Ionescu [2], [3], as well as by means of a second order divided difference if one uses a theorem of T. Popoviciu [8].

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O GENERALIZARE A OPERATORULUI SPLINE DE APROXIMARE AL LUI SCHOENBERG

(Resumat)

În această lucrare se generalizează operatorul spline de aproximare al lui I. J. Schoenberg prin introducerea unor noi noduri, care depind de doi parametri ne-negativi α şi β, care satisfac condiţiile: $0 \leq \alpha \leq \beta$. Se demonstrează mai întîi o teoremă de convergenţă uniformă, iar apoi prin utilizarea unei formule de descompunere a diferenţelor divizate se dau expresii explicite pentru calculul valorii operatorului studiat în lucrare. În cazul special cînd nu avem puncte de racord se obţine un operator de tip Bernstein studiat anterior în lucrarea autorului [10].

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXVI, 2, 1981

To Professor D. V. Ionescu
on his 80th birthday

ON A PROBLEM OF DARBOUX-IONESCU

IOAN A. RUS

1. The present paper deals with the following problem of Darboux-Ionescu ([5]; see also [1] — [4], [6] — [7]).

$$\frac{\partial^2 u}{\partial x\, \partial y} = f(x, y, u(g(x, y), h(x, y)), \; x, y \in I_2 \tag{1}$$

$$\begin{aligned} u(x, y) &= \varphi(x), \; x \in [0, a] \\ u(0, y) &= \psi(y), \; y \in [0, b] \end{aligned} \tag{2}$$

where

$$I_2 = [0, a] \times [0, b], \; f \in C(I_2 \times \mathbb{R}),$$
$$\varphi \in C^1[0, a], \; \psi \in C^1[0, b], \; (g, h) \in C(I_2, I_2)$$

and

$$\varphi(0) = \psi(0)$$

Using the same method as that used by Bielecki ([4]), we prove some existence and uniqueness theorems for the problem (1) + (2), considering in $C(I_2)$ the following norms ($\tau > 0$)

$$\|u\|_C = \max_{(x,y) \in I_2} |u(x, y)|,$$

$$\|u\|_B = \max_{(x,y) \in I_2} |u(x, y)e^{-\tau(x+y)}|$$

$$\|u\|_{B_1} = \max_{(x,y) \in I_2} |u(x, y)e^{-\tau x}|,$$

$$\|u\|_{B_2} = \max_{(x,y) \in I_2} |u(x, y)e^{-\tau y}|$$

2. We have

THEOREM 1. *Assume that*

(i) $f \in C(I_2 \times \mathbb{R})$

(ii) $\varphi \in C^1[0, a], \; \psi \in C^1[0, b], \; \varphi(0) = \psi(0) = u_0$

(iii) $g \in C(I_2, [0, a]), \; h \in C(I_2, [0, b])$

(iv) *there exists* $L \in C(I_2)$ *such that*

$$|f(x, y, u) - f(x, y, v)| \leqslant L(x, y)\, |u - v|,$$

for all $(x, y) \in I_2$, $u, v \in \mathbf{R}$

(v) *there exist* $\tau > 0$ *and* $0 < q < 1$ *such that*

$$\int_0^x \int_0^y L(s, t) e^{\tau[g(s,t)+h(s,t)-x-y]} \, ds \, dt \leqslant q$$

for all $(x, y) \in I_2$

Then the problem (1) + (2) *of Darboux-Ionescu has in* $C(I_2)$ *a unique solution.*

Proof. The problem (1) + (2) is equivalent to the integral equation

$$u(x, y) = \varphi(x) + \psi(y) - u_0 + \int_0^x \int_0^y f(s, t), u(g(s, t), h(s, t)) \, ds \, dt$$

We define a mapping $A : C(I_2) \to C(I_2)$ by

$$(Au)(x, y) = \varphi(x) + \psi(y) - u_0 + \int_0^x \int_0^y f(s, t, u(g(s, t), h(s, t))) \, ds \, dt$$

Now the proof follows by using Banach's fixed point theorem for the mapping

$$A : (C(I_2), \ \|\cdot\|_B) \to (C(I_2), \ \|\cdot\|_B)$$

Theorem 2. *Theorem 1 will remain true if condition* (v) *is replaced by one of the following conditions*:

(v′) *there exist* $\tau > 0$ *and* $0 < q < 1$ *such that*

$$\int_0^x \int_0^y L(s, t) e^{\tau(g(s,t)-x)} \, ds \, dt \leqslant q$$

for all $(x, y) \in I_2$

(v″) *there exist* $\tau > 0$ *and* $0 < q < 1$ *such that*

$$\int_0^x \int_0^y L(s, t) e^{\tau(h(s,t)-y)} \, ds \, dt \leqslant q$$

(v‴) $\displaystyle\int_0^a \int_0^b L(s, t) \, ds \, dt < 1$

Proof. We consider the mapping A as in the proof of thoerem 1, but in the following Banach spaces

$$A : (C(I_2), \ \|\cdot\|_{B_i}) \to (C(I_2), \ \|\cdot\|_{B_i}), \ i = 1, 2$$

and

$$A : (C(I_2), \ \|\cdot\|_C) \to (C(I_2), \ \|\cdot\|_C)$$

3. *Remarks*

(i) From theorem 2 (v‴) we have a theorem of D. V. Ionescu ([5]).

(ii) If $g(x, y) + h(x, y) \leqslant x + y$, then from the theorem 1 we have a theorem of M. C. Alicu [2].

(iii) If L is a constant function we have the results given in [9].

(iv) From theorem 2 we have some results given by R. P. Agorwal and E. Thandapani [1].

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ASUPRA UNEI PROBLEME A LUI DARBOUX-IONESCU

(Rezumat)

În prezenta lucrare se stabilesc teoreme de existenţă şi de unicitate pentru problema Darboux-Ionescu (1) + (2), folosind teorema de punct fix a lui Banach şi alegînd norme convenabile în $C(I_2)$

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXVI, 2, 1981

Au professeur D. V. Ionescu — pour son 80e anniversaire

UNE RELATION ENTRE LES DIFFERENCES DIVISÉES ET QUELQUES FORMULES DE DÉRIVATION NUMERIQUE

PARASCHIVA PAVEL

1. L'analyse ci-dessous donne une méthode pour obtenir les formules de dérivation numèrique

$$\Delta^2 f(x_0) = h^2 f''(x_1) + R \tag{1}$$

et

$$\Delta^3 f(x_0) = \frac{h^3}{2}\,[f'''(x_1) + f'''(x_2)] + R \tag{2}$$

et nous établissons une relation entre les différences divisées et les formules de dérivation numèrique (1) et (2).

Les formules (1) et (2) sont des cas particuliers des formules de dérivation numérique:

$$\Delta^{n-1} f(x_0) = \sum_{i=2}^{n-1} A_i f''(x_i) + R$$

$$\Delta^{n-1} f(x_0) = \sum_{i=1}^{n-1} A_i f'''(x_i) + R$$

où

$$\Delta^{n-1} f(x_0) = f(x_n) - C_n^1 f(x_{n-1}) + \ldots + (-1)^n C_n^n f(x_0)$$

formules qui ont été étudiées dans [3] et [4]

2. D. V. Ionescu [1] a demontré que si la fonction $f \in C^2\,[a,b]$ sa différence divisée sur les noeuds x_0, x_1, x_2 où $a \leqslant x_0 < x_1 < x_2 \leqslant b$ peut être représentée par une intégrale définie, donc nous avons une formule de la forme

$$[x_0, x_1, x_2; f] = \int_{x_0}^{x_2} \varphi(x) f''(x)\,dx \tag{3}$$

la fonction φ étant positive sur l'intervalle (x_0, x_2) et égale à

$$\varphi_1(x) = \frac{x - x_0}{(x_1 - x_0)(x_2 - x_0)}, \qquad \varphi_2(x) = \frac{x_2 - x}{(x_2 - x_0)(x_2 - x_1)}$$

sur les intervalles $[x_0, x_1]$, $[x_1, x_2]$

En appliquant à l'intégrale de la formule (3) une formule de quadrature avec le noeud x_1, si $f \in C^4\,[a,b]$, on obtient

$$[x_0, x_1, x_2\,;\, f] = A\, f''(x_1) + R_1 \tag{4}$$

où

$$A_1 = \frac{1}{2} \quad \text{et} \quad R_1 = \int_{x_0}^{x_2} \psi(x)\, f^{\mathrm{IV}}(x)\, dx$$

La fonction ψ égale à ψ_1 et ψ_2 sur les intervalles $[x_0,\, x_1]$, $[x_1,\, x_2]$ étant la solution du problème aux limites

$$\psi_1'' = \varphi_1\,; \quad \psi_2'' = \varphi_2 \tag{5}$$

$$\psi_1\,(x_0) = \psi_2\,(x_2) = 0\,;\; \psi_1'(x_0) = \psi_2'(x_2) = 0\,;\; \psi_1\,(x_1) = \psi_2\,(x_1) \tag{6}$$

Si nous prenons

$$x_1 = \frac{x_0 + x_2}{2}$$

la solution du problème (5) + (6) est unique:

$$\psi_1(x) = \int_{x_0}^{x} (x - s)\, \varphi_1(s)\, ds, \quad \psi_2\,(x) = \int_{x}^{x_2} (s - x)\, \varphi_2(s)\, ds \tag{7}$$

La formule (4) devient la formule (1) avec

$$R = 2h^2 \int_{x_0}^{x_2} \psi(x)\, f^{\mathrm{IV}}(x)\, dx \tag{8}$$

où la fonction ψ este donnée par les formules (7) avec $x_1 = \dfrac{x_0 + x_2}{2}$ et elle est positive pour $x \in (x_0, x_2)$. Pour le reste on obtient l'évaluation

$$|\, R\, | \leqslant \frac{h^5}{12} M_4,\; M_4 = \sup_{[x_0,x_2]} |f^{\mathrm{IV}}(x)\,|\,;\; h = \frac{x_2 - x_0}{2} \tag{9}$$

3. Si $f \in C^3$ [a, b], D. V. Ionescu [1] a démontré aussi que

$$[x_0,\; x_1,\; x_2,\; x_3\,;\, f] = \int_{x_0}^{x_3} \varphi(x)\, f'''(x)\, dx \tag{10}$$

où $a \leqslant x_0 < x_1 < x_2 < x_3 \leqslant b$ et la fonction φ est positive sur l'intervalle

(x_0, x_3) et égale à

$$\varphi(x) = \frac{1}{(x_3 - x_0)(x_2 - x_0)(x_1 - x_0)} \cdot \frac{(x - x_0)_+^2}{2!} - \frac{1}{(x_1 - x_0)(x_2 - x_1)(x_3 - x_1)} \cdot$$
$$\cdot \frac{(x - x_1)_+^2}{2!} + \frac{1}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} \cdot \frac{(x - x_2)_+^2}{2!}$$

où

$$u_+ = \begin{cases} u\,; & \text{si } u > 0 \\ 0\,; & \text{si } u \leqslant 0 \end{cases}$$

Si $f \in C^6\ [a,b]$, en appliquant à l'intégrale de la formule (10) une formule de quadrature à noeuds simples x_1, x_2, on obtient

$$[x_0, x_1, x_2, x_3\,; f] = A_1 f'''(x_1) + A_2 f'''(x_2) + R_2 \tag{11}$$

avec les coefficients

$$A_1 = \frac{3x_2 - x_0 - x_1 - x_3}{24(x_2 - x_1)}\,; \qquad A_2 = \frac{x_0 + x_2 - x_3 - 3x_1}{24(x_2 - x_1)}$$

et le reste

$$R_2 = -\int_{x_0}^{x_3} \theta(x) f^{(6)}(x)\, dx$$

la fonction θ égale à θ_1, θ_2 et θ_3 sur les intervalles $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$ étant la solution du problème aux limites:

$$\theta_i''' = \varphi_i, \quad i = 1,2,3.$$
$$\theta_j^{(p)}(x_s) = 0\,;\ j = 1,3\,;\ p = 0,1,2\,;\ s = 0,3.$$
$$\theta_j^{(p)}(x_s) = \theta_{j+1}^{(p)}(x_s)\,;\quad j = 1,2\,;\ p = 0,1\,;\ s = 1,\ 2.$$

On voit sans difficulté que si les noeuds vérifient la condition

$$\delta = (x_2 - x_1)(2x_0^2 - 3x_1^2 - 3x_2^2 + 2x_3^2 - 3\,x_0 x_1 - 3x_0\,x_2 + 2x_0 x_2 + $$
$$+ 12x_1x_2 - 3x_1x_3 - 3x_2x_3) = 0 \tag{12}$$

le problème aux limites a une solution unique.

On observe que pour les noeuds équidistants

$$x_1 = \frac{2x_0 + x_3}{3}, \qquad x_2 = \frac{x_0 + 2x_3}{3} \tag{13}$$

nous avons $\delta = 0$. Dans ce cas la fonction θ change de signe quand $x \in [x_0, x_3]$ et

$$\int_{x_0}^{x_3} \theta(x)\, dx = 0$$

ce qui veut dire que le degré d'éxactitude de la formule (15) est égale à 6.

En supposant que $f \in C^7 [a,b]$, on obtient également une formule de forme (15) avec les noeuds (13) et les coeficients

$$A_1 = a_2 = \frac{1}{12} \text{ et le reste } R_2 = \int_{x_0}^{x_3} \theta(x) f^{(7)}(x)\, dx$$

où la fonction θ est la solution du problème aux limites

$$\theta_i^{IV} = \varphi_i,\ i = 1,2,3.$$

$$\theta_j^{(p)}(x_s) = 0\,;\ p = 0,1,2,3\,;\ j = 1,3\,;\ s = 0,3.$$

$$\theta_j^{(p)}(x_s) = \theta_{j+1}^{(p)}(x_s)\,;\ j = 1,2\,;\ p = 0,1,2\,;\ s = 1,2$$

La solution de ce probléme est

$$\theta(x) = \frac{49}{4320}\left[(x - x_0)_+^6 - (x - x_1)_+^6 + (x - x_2)_+^6\right] - \frac{1}{72}\left[x - x_1)_+^3 + (x - x_2)_+^3\right]$$

et la formule (11) devient la formule (2) avec

$$R = 6h^3 \int_{x_0}^{x_3} \theta(x) f^{(7)}(x)\, dx, \qquad h = \frac{x_3 - x_0}{4}$$

La fonction θ garde le signe sur l'intervalle $[x_0, x_3]$ elle est positive. Pour le reste nous avons l'évaluation

$$|R| \leqslant \frac{3}{2 \cdot 51} h^7 M^7\,;\ M = \sup_{[x_0, x_3]} |f^{(7)}(x)| \tag{14}$$

Note. Si nous prenons d'une autre manère les noeuds satisfaisant la condition (12), nous obtenons d'autres formules de dérivation numérique.

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O RELAŢIE ÎNTRE DIFERENŢELE DIVIZATE ŞI UNELE FORMULE DE DERIVARE NUMERICĂ

(R e z u m a t)

În lucrare se stabileşte o legătură între formulele de derivare numerică (1) şi (2) şi formulele (3) şi (4) care dau reprezentarea diferenţelor divizate $[x_0, x_1, x_2 ; f]$ respectiv $[x_0, x_1, x_2, x_3 ; f]$ sub formă de integrală definită.

Aplicînd integralelor din formulele (3) respectiv (10)o formulă de cuadratură cu nodurile simple x_1, respectiv x_1, x_2 oarecare şi cerînd ca noile formule obţinute să coincidă cu formulele de derivare numerică (1) şi (2) se deduce că acest lucru este posibil în primul caz pentru $x_1 = \frac{x_0 + x_2}{2}$, iar în al doilea caz pentru x_2 şi x_3 daţi de (13). Restul acestor formule este pus sub formă de integrală definită şi se dau evaluări ale lui prin inegalităţile (9) respectiv (14).

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXVI, 2, 1981

Au professeur D. V. Ionescu
— pour son 80e anniversaire

SUR UN PROBLEME DE D.V. IONESCU

D. RĂDULESCU

1. En 1927, dans sa thèse de doctorat [3], D.V. Ionescu ètudie pour la première fois dans la littérature mathématique des équations du type:

$$\frac{\partial^2 u}{\partial x \partial y} = \sum_{i=1}^{m} \left[a_i(x, y)\, u(g_i\ (x, y)) + b_i(x, y) \frac{\partial u(g_i(x, y))}{\partial x} + \right.$$
$$\left. + c_i\ (x, y) \frac{\partial u(g_i(x, y))}{\partial y} \right] + d(x, y).$$

Nous allons étudier dans l'analyse subséquente des équations de forme

$$u_{xy}(x, y) = f(x, y, u\,(x, y),\ u(g_1(x, y),\ h_1(x, y)),\ u_x(x, y),$$
$$u_x(g_2(x, y),\ h_2(x, y)),\ u_y(x, y),\ u_y(g_3(x, y),\ h_3(x, y))) \qquad (1)$$

où

$$-\tau_1 \leqslant g_i(x, y) \leqslant a,\ i = 1,2,3\ \ \forall\,(x, y) \in [s, a] \times [s, b]$$
$$-\tau_2 \leqslant h_i(x, y) \leqslant b,\ i = 1,2,3\ \ \forall\,(x, y) \in [s, a] \times [s, b].$$

Pour simplifier l'écriture, nous notons

$$D_0 = \{(x, y)/0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b\}$$
$$D_1 = \{(x, y)/\ 0 \leqslant x \leqslant a, -\tau_2 \leqslant y \leqslant 0\}$$
$$D_2 = \{(x, y)/ -\tau_1 \leqslant x \leqslant 0,\quad 0 \leqslant y \leqslant b\}$$
$$D_3 = \{(x, y)/ -\tau_1 \leqslant x \leqslant 0,\quad -\tau_2 \leqslant y \leqslant 0\}$$
$$D = D_1 \cup D_2 \cup D_3$$

On cherche les solutions u de l'équation (1) pour lesquelles

$$u(x, y) = \psi(x, y)\ ,\ \ \forall\,(x, y) \in D,$$

$\psi : D \to$ R étant une fonction continue donnée.

2. Nous avons

THEOREME *Si*

(i) $f, g_i, h_i,\ i = \overline{1,3}$ *sont continues sur* D_0

(ii) f *est lipshitzienne*

$$|f(x, y, q_1, \ldots, q_6) - f(x, y, \bar{q}_1, \ldots, \bar{q}_6)| \leqslant \sum_{i=1}^{6} L_i |q_i - \bar{q}_i|$$
$$\forall (x, y, q_1, \ldots, q_6),\ (x, y, \bar{q}_1, \ldots, \bar{q}_6) \in D^0 \times R^6$$

(iii) *il existe une fonction* $F : \mathrm{R} \times \mathrm{R} \times (0, \infty) \to (0, \infty)$, *continue et*

$$\lim_{\tau \to \infty} F(x, y, \tau) = 0$$

avec les propriétés:

a)

$$\lim_{\tau \to \infty} \frac{F(x, y, \tau)}{F(g_i(s, t), h_i(s, t), \tau)} = 0 = \lim_{\tau \to \infty} \frac{F(x, y, \tau)}{F(s, t, \tau)}$$

$i = 1,2,3$ *et pour tout* $(x, y) \in D_0$

b) *les fonctions* $F(\cdot, y, \tau)$ *et* $F(x, \cdot, \tau)$ *sont croissantes.*

alors le problème (1) *a une solution unique dans* $C(D)$.

Démonstration. Le problème (1) est èquivalent à l'équation intégrale

$$u(x, y) = \int_0^x \int_0^y \theta(x, y) f(s, t, u(s, t), u(g_1(s, t), h_1(s, t)), u_x(s, t),$$
$$u_x(g_2(s, t), h_2(s, t)), u_y(s, t), u_y(g_3(s, t), h_3(s, t)))\, dsdt + p(x, y)$$

où

$$p(x, y) = \begin{cases} \psi(x, y) & (x, y) \in D \\ \psi(x, 0) + \psi(0, y) - \psi(0,0) & (x, y) \in D_0 \end{cases}$$

et

$$\theta(x, y) = \begin{cases} 1 & (x, y) \in D_0 \\ 0 & (x, y) \in D \end{cases}$$

On définit la famille suivante de normes de type Bielecki

$$\|u\|_F = \max \{ \max_{D_0 \cup D} F(x, y, \tau) \,|u(x, y)|, \max_{D_0 \cup D} F(x, y, \tau) \,|u_x(x, y)|,$$
$$\max_{D_0 \cup D} F(x, y, \tau) \,|u_y(x, y)| \}$$

où la fonction F vérifie les conditions (iii). On définit de même l'opérateur $A : C(D) \to C(D)$

$$Au(x, y) = \int_0^x \int_0^y \theta(x, y) f(s, t, u(s, t), u(g_1(s, t), h_1(s, t)), u_x(s, t),$$
$$u_x(g_2(s, t), h_2(s, t)), u_y(s, t), u_y(g_3(s, t), h_3(s, t)))\, dsdt + p(x, y)$$

On va démontrer que A est une contraction. Nous avons

$$|Au(x,y) - Av(x,y)| \leqslant \int_0^x \int_0^y \theta(x,y)\,[L_1|u(s,t) - v(s,t)| + \dots$$

$$+ L_6|u_y(g_3(s,t), h_3(s,t)) - v_y(g_3(s,t), h_3(s,t))|]\,dsdt \leqslant$$

$$||u-v||_F \int_0^x \int_0^y \theta(x,y)\left[\frac{L_1}{F(s,t,\tau)} + \dots + \frac{L_6}{F(g_3(s,t), h_3(s,t), \tau)}\right] dsdt$$

D'où il résulte

$$|Au(x,y) - Av(x,y)|\,F(x,y,\tau) \leqslant ||u-v||_F \int_0^x \int_0^y \theta(x,y)\left[\frac{L_1F(x,y,\tau)}{F(s,t,\tau)} + \right.$$

$$\left. \dots + \frac{L_6F(g_3(x,y), h_3(x,y), \tau)}{F(g_3(s,t), h_3(s,t), \tau)}\right] dsdt$$

Analogue

$$|(Au)_x(x,y) - (Av)_x(x,y)|\,F(x,y,\tau) \leqslant ||u-v||_F \int_0^y \theta(x,y)\left[\frac{L_1F(x,y,\tau)}{F(s,t,\tau)} + \right.$$

$$\left. \dots + \frac{L_6F(g_3(x,y), h_3(x.,y), \tau)}{F(g_3(s,t), h_3(s,t), \tau)}\right] dsdt$$

$$|(Au)_y(x,y) - (Av)_y(x,y)|\,F(x,y), \tau)_F \leqslant ||u-v||_F \int_0^x \theta(x,y)\left[\frac{L_1F(x,y,\tau)}{F(s,t,\tau)} + \right.$$

$$\left. \dots + \frac{L_6F(g_3(x,y), h_3(x.,y), \tau)}{F(g_3(s,t), h_3(s,t), \tau)}\right] dsdt$$

Les hypothèses imposées sur la fonction F nous pérmettent de choisir τ ainsi que A soit une contraction.

Remarques.

1) Pour les équations de la forme

$$\frac{\partial^2 u}{\partial x \partial y} = f(x, y, u(g(x,y), h(x,y)))$$

la condition (iii) b) du théorème est superflue.

2) Si

$$g(x,y) + h(x,y) \leqslant x + y$$

on peut choisir

$$F(x, y, \tau) = e^{-\tau(x+y)}$$

En ce cas on obtient des résultats de [2], [4].

3) Si

$$g(x, y) \leqslant x$$

pour tout $(x,y) \in D$, en prenant

$$F(x, y, \tau) = e^{-\tau x}$$

on obtient un théorème du [4].

4) Si

$$g(x, y) \leqslant y$$

pour tout $(x, y) \in D$, en prenant

$$F(x, y, \tau) = e^{-\tau y}$$

on obtient un théorème du [4].

5) Si $f \in C(R_+^2 \times R^3)$ vérifie les conditions

$$|f(x, y, 0,0,0)| \leqslant L(x, y)$$

$$|f(x, y, p_1, q_1, r_1) - f(x, y, p_2, q_2, r_2)| \leqslant L(x, y) \{|p_1 - p_2| + |q_1 - q_2| + |r_1 - r_2|\}$$

où L, L_x, L_y sont des fonctions continues et non-negatives et $g \in C(R_+^2, R_+^2)$ $g(x,y) = (h(x,y), k(x,y))$ où $h(x,y) \leqslant x$, $k(x,y) \leqslant y$ en prenant

$$F(x, y, \tau) = e^{-\tau}\Phi(x, y)$$

où

$$\Phi(x, y) = \int_0^x \int_0^y L(u, v)\, du dv + \int_0^y L(x,v)\, dv + \int_0^x L(u,y)\, du$$

on obtient un théorème du [1].

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DESPRE O PROBLEMĂ A LUI D. V. IONESCU

(Rezumat)

În această notă se studiază o generalizare a problemei lui D. V. Ionescu analizată în lucrarea [3] utilizînd o normă de tip Bielecki. În particular se obțin mai multe rezultate cunoscute referitoare la existența și unicitatea soluțiilor ecuațiilor integrale de tip Volterra cu argument modificat.

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To Professor D. V. Ionescu on his 80th birthday

THE RUNGE-KUTTA-FEHLBERG PROCEDURES FOR THE NUMERICAL SOLUTION OF VOLTERRA INTEGRAL EQUATIONS

MARIA MICULA

The Runge-Kutta methods for the numerical solution of integral equations developed parallel with the same kind of procedures for the differential equations.

D. V. Ionescu [3], [4] established a general method to obtain the Runge-Kutta procedures of any order for the numerical solution of differential equations.

The Runge-Kutta-Fehlberg procedures to approximate the solutions of nonlinear Volterra integral equations were investigated by A. N. Lomakovich [5], who gave procedurea of order $0(h^{m+5})$, $m = 0, 1, \ldots, 8$ with several substitutions in the transformed integral equations.

In our approach we shall construct the Runge-Kutta-Fehlberg procedures of order $0(h^4)$ with two substitutions and also of order $0(h^{m+3})$ $(m \in \mathbf{N},\ m \geqslant 2)$ with only three substitutions in the transformed integral equation [6].

Let be the nonlinear Volterra integral equation of the second kind:

$$\varphi(x) = \int_{x_0}^{x} G[x, s, \varphi(s)]\, ds \tag{1}$$

where $G: I^2 \times \mathbf{R} \to \mathbf{R}$ is a given function satisfying all the conditions for the existence of a unique solution $\varphi \in C(I)$ of the equation (1).

Supposing that $I := [x_0, x_0 + M]$, $x_0 \in \mathbf{R}$, $M \in \mathbf{R}_+$ we shall give some Runge-Kutta-Fehlberg procedures to approximate this solution.

Let $\Delta_h := \{x_i = x_{i-1} + h \mid i = 1, 2, \ldots, n,\ x_n = x_0 + M,\ h > 0\}$ be a partition of the interval I.

To approximate the solution of the integral equation (1) by Runge-Kutta-Fehlberg procedures we shall transforme this equation using the Fehlberg transformation [5].

One considers the function $y: \mathbf{R} \to \mathbf{R}$ which satisfies a conditions:

$$\begin{aligned} &\text{(i)} \quad y(x_0) = \varphi(x_0) = 0 \\ &\text{(ii)} \quad y^{(k)}(x_0) = 0,\ k = 1, 2, \ldots, m,\ m \in \mathrm{N} \end{aligned} \tag{2}$$

The transformation satisfying these conditions is the following:

$$\varphi(x) = \Phi(x, y) = y(x) + \sum_{k=1}^{m} \frac{(x - x_0)^k}{k!} \left(\frac{\partial^{k-1} G[x, s, \varphi(s)]}{\partial s^{k-1}}\right)_{s=x_0} \tag{3}$$

From (3) and taking in consideration the equation (1) one gets for the function y the expression:

$$y(x) = \int_{x_0}^{x} \left\{ G[x, s, \Phi(s, y)] - \sum_{k=1}^{m} \frac{(s - x_0)^{k-1}}{(k-1)!} \left(\frac{\partial^{k-1} G[x, s, \Phi(s, y)]}{\partial s^{k-1}} \right)_{s=x_0} \right\} ds \qquad (4)$$

.e.

$$y(x) = \int_{x_0}^{x} f(x, s, y(s))\, ds \qquad (5)$$

Easily, it follows that the function $f\colon I^2 \times \mathbf{R} \to \mathbf{R}$ from (5) has the properties:

$$\begin{aligned}
&\text{(i)} \quad \left(\frac{\partial^{k+j} f(x,s,y(s))}{\partial x^k\, \partial s^j} \right)_{s=x_0} = 0, \quad j = 0, 1, \ldots, m-1, \; k = 1, 2, \ldots \\
&\text{(ii)} \quad \left(\frac{\partial^j f}{\partial x^j} \right) = 0, \; j = 1, 2, \ldots, m, \; f(x, x_0, y_0) = 0 \qquad (6)\\
&\text{(iii)} \quad (f_{s^k})_0 = 0, \; k = 1, 2, \ldots, m-1
\end{aligned}$$

Here, by $(f_s)_0$ we denote the value of the derivative of the function f in the point $x = s = x_0$.

The approximation of the solution of the integral equation (1) is reduced to the approximation of the integral equation (5) by the transformation (3).

In what follows we shall deduce the Runge-Kutta-Fehlberg procedures of order $0(h^4)$ with two substitutions and also of order $0(h^{m+3})$, $(m \geqslant 2)$ only with three substitutions in order to approximate the solution of the integral equation (5).

1. The Runge-Kutta-Fehlberg procedures of order $0(h^4)$. Let $\bar{y}_i$ be the approximating values of the exact solution y of the integral equation (5) calculated in the point x_i of the partition Δ_h.

First, one calculates the value $\bar{y}_1 = \bar{y}(x_0 + h)$.

To obtain the Runge-Kutta-Fehlberg procedures of order $0(h^4)$, the function y will satisfy the conditions (i,) (ii) from (2) for $m = 2$.

The approximating solution of (5) is required under the form:

$$\bar{y}(x_0 + h) = c_1 k_1 + c_2 k_2$$

where

$$\begin{aligned}
k_1 &= hf(x_0 + h,\; x_0 + \alpha_1 h,\; y_0) \\
k_2 &= hf(x_0 + \alpha_2 h,\; x_0 + \alpha_1 h,\; y_0 + \beta_0 k_1)
\end{aligned} \qquad (7)$$

Expanding in Taylor series in the point x_0 the exact solution y and also the approximating solution $\bar{y}$, by identification of the coefficients of the same power of the h (until h^4) one gets for parameters c_i, α_i, $(i = 1, 2)$, β_0

the following equation:

$$c_1\alpha_1^2 + c_2\alpha_1^2 = \frac{1}{3}$$

or

$$c_1 + c_2 = \frac{1}{3\alpha_1^2} \tag{8}$$

Because the parameters α_2, β_0 are arbitrary, we take for these parameters the values zero.

From the equation (8) it follows that we have here a family of Runge-Kutta-Fehlberg procedures of order $0(h^4)$.

One obtains a very suitable procedure from this family by taking $c_2=0$, $\alpha_1 = \frac{1}{3}$, $c_1 = \frac{1}{3}$, $\alpha_2 = 1$, $\beta_0 = 0$. This procedures is of order $0(h^4)$ and needs only one substitution in the transformed integral equation.

2. **The Runge-Kutta-Fehlberg procedures of order $0(h^{m+3})$, $m \geqslant 2$.** Keeping the same conditions (2) we can find a family of Runge-Kutta-Fehlberg procedures of order $0(h^{m+3})$, $(m \geqslant 2)$, which need only three substitutions in the transformated equation (5).

To obtain these procedures one uses the following Runge-Kutta scheme:

$$\bar{y}(x_0 + h) = c_1k_1 + c_2k_3$$

where

$$\begin{aligned} k_1 &= hf(x_0 + h,\ x_0 + \alpha_1h,\ y_0) \\ k_2 &= hf(x_0 + \alpha_2h,\ x_0 + \alpha_1h,\ y_0 + \beta_0k_1) \\ k_3 &= hf(x_0 + h,\ x_0 + \alpha_2h,\ y_0 + \beta_1k_1 + \beta_2k_2) \end{aligned} \tag{9}$$

The parameters c_i, α_i, $i = 1, 2$, β_j, $j = 0, 1, 2$, will be determined from the following system of three equations with six unknowns:

$$\begin{aligned} c_1\alpha_1^m + c_2\alpha_2^m &= \frac{1}{m+1} \\ c_1\alpha_1^{m+1} + c_2\alpha_2^{m+1} &= \frac{1}{m+2} \\ \alpha_1^m(\beta_1 + \beta_2)c_2 &= \frac{1}{(m+1)(m+2)} \end{aligned} \tag{10}$$

This system has the solution

$$c_1 = \frac{(m+2)\,\alpha_2 - (m+1)}{(m+1)(m+2)\,\alpha_1^m(\alpha_2 - \alpha_1)}, \qquad c_2 = \frac{m + 1 - (m+2)\,\alpha_1}{(m+1)(m+2)\,\alpha_2^m(\alpha_2 - \alpha_1)}$$

$$\beta_1 + \beta_2 = \frac{\alpha_2^m(\alpha_2 - \alpha_1)}{\alpha_1^m\,[(m+1) - (m+2)\,\alpha_1]}$$

and $\beta_0 = 0$ because it does not appear in the system (10).

A very suitable choise of parameters is $\alpha_2 = 1$, $\alpha_1 = \frac{1}{2}$, $\beta_1 = 0$. Then

$$c_1 = \frac{2^{m+1}}{(m+1)(m+2)}, \quad c_2 = \frac{m}{(m+1)(m+2)}, \quad \beta_2 = \frac{2^m}{m} \tag{11}$$

To find the approximating values $\bar{y}_i$, $i \geqslant 2$, the integral equation (1) will be written under the form:

$$\varphi(x) = \sum_{j=1}^{i-1} \int_{x_{j-1}}^{x_j} G[x, s, \varphi(s)]\, ds + \int_{x_{i-1}}^{x} G[x, s, \varphi(s)]\, ds \tag{12}$$

Denoting

$$\mu_j(x) = \int_{x_{j-1}}^{x_j} G[x, s, \varphi(s)]ds, \quad \sigma_{i-1}(x) = \sum_{j=1}^{i-1} \mu_j(x) \tag{13}$$

$$\varphi_{i-1}(x) = \int_{x_{i-1}}^{x} G[x, s, \varphi(s)]\, ds \tag{14}$$

this equation becames:

$$\varphi(x) = \sigma_{i-1}(x) + \varphi_{i-1}(x) \tag{12'}$$

Substituing the function φ from (12′) in (14), we have

$$\varphi_{i-1}(x) = \int_{x_{i-1}}^{x} G[x, s, \sigma_{i-1}(s) + \varphi_{i-1}(s)]\, ds$$

From (12′) it follows that $\varphi(x_i) = \sigma_{i-1}(x_i) + \varphi_{i-1}(x_i)$, therefore to know the values of the solution φ in the knots x_i is necesary to know the values $\sigma_{i-1}(x_i)$ and $\varphi_{i-1}(x_i)$.

Supposing to be known the approximating values of the solution φ_k, $k = 1, 2, \ldots, i-1$, let $\bar{\mu}_k$ be the known approximating values of μ_k ($k=1, 2, \ldots, i-1$) and then:

$$\bar{\sigma}_{i-1}(x) = \sum_{j=1}^{i-1} \bar{\mu}_j(x)$$

Thus, we have the following integral equation

$$\varphi^*_{i-1}(x) = \int_{x_{i-1}}^{x} G[x, s, \bar{\sigma}_{i-1}(s) + \varphi^*_{i-1}(s)]\, ds \tag{15}$$

whose solution we want to find ($\varphi^*_{i-1}(x_{i-1}) = 0$).

For the integral equation (15) a transformations as (3) in the point x_{i-1} is used and thus one obtains the integral equation

$$(xy)^{*_{i-1}} = \int_{x_{i-1}}^{x} f[x, s, y^{*}_{i-1}(s)]\, ds \tag{16}$$

where f verifies the conditions (6) for $x = s = x_{i-1}$.

The approximation of the solution of the integral equations (16) is made by the aplication of one of the Runge-Kutta-Fehlberg procedures described above.

In order to find $\mu_j(x_i)$ we replace in (13) the value of φ, obtained from (12′) and we have

$$\mu_i(x) = \int_{x_{j-1}}^{x_j} G[x, s, \bar{\sigma}_{i-1}(s) + \varphi_{i-1}(s)]\, ds$$

or replacing φ_{i-1} by φ^{*}_{i-1}

$$\mu^{*}_{j}(x) = \int_{x_{j-1}}^{x_j} G[x, s, \bar{\sigma}_{i-1}(s) + \varphi^{*}_{i-1}(s)]\, ds$$

Denote

$$\sigma^{*}_{j}(x) = \int_{x_{j-1}}^{x_j} f[x, s, y^{*}(s)]\, ds$$

The connection between μ^{*}_{j} and σ^{*}_{j} is given by the relation:

$$\mu^{*}_{j}(x) = \sigma^{*}_{j}(x) + \sum_{k=1}^{m} \frac{h^k}{k!} \left(\frac{\partial^{k-1} G[x, s, \varphi(s)]}{\partial s^{k-1}}\right)_{s = x_{i-1}}$$

The approximation $\bar{\sigma}_j$ of the σ^{*}_{j} is made by one of the procedures (7) or (9), starting with the point x_{i-1}.

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PROCEDEE DE TIP RUNGE-KUTTA-FEHLBERG PENTRU REZOLVAREA NUMERICĂ A ECUAŢIILOR INTEGRALE VOLTERRA

(Rezumat)

În această lucrare se deduc cu ajutorul unei transformări de tip Fehlberg procedee de tip Runge-Kutta-Fehlberg de ordinul $0(h^4)$ cu doua substituţii, şi de ordinul $0(h^{m+3})$, $(m \geqslant 2)$ cu trei substitutii pentru rezolvarea aproximativă a ecuaţiilor integrale de tip Volterra.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXVI, 2, 1981

To Professor D. V. Ionescu on his 80th birthday

ON A DIFFERENTIAL INEQUALITY FOR ANALYTIC FUNCTIONS IN THE UNIT DISC

PETRU T. MOCANU

1. In some previous papers [3], [4], [5], [6], a general method to obtain information on the solutions of certain differential inequalities in the complex plane, was elaborated. A particular form of this method is given by the following

LEMMA A. *Let u and v denote complex variables $u = u_1 + iu_2$, $v = v_1 + iv_2$ and let $F(u, v)$ be a complex-valued function that satisfies the following conditions:*

(a) *F is continuous in a domain $D \subset C^2$;*

(b) *$(1, 0) \in D$ and $\mathrm{Re}\, F(1, 0) > 0$;*

(c) *$\mathrm{Re}\, F(iu_2, v_1) \leqslant 0$, whenever $(iu_2, v_1) \in D$ and $v_1 \leqslant -\frac{1}{2}(1 + u_2^2)$.*

If $p(z) = 1 + p_1 z + \ldots$ is an analytic function in the unit disc $U = \{z \in C; |z| < 1\}$, such that $(p(z), zp'(z)) \in D$ and $\mathrm{Re}\, F(p(z), zp'(z)) > 0$ hold for all $z \in U$, then $\mathrm{Re}\, p(z) > 0$ in U.

2. We shall apply Lemma A to obtain the following result concerning a differential inequality for analytic functions in the unit disc.

THEOREM 1. *Let $f(z) = z + a_2 z^2 + \ldots$ be an analytic function in U, with $f'(z) \neq 0$ for $z \in U$ and let $\alpha \geqslant 0$. If*

$$\mathrm{Re}\left[(1-\alpha)\frac{f(z)}{z} + \alpha\sqrt{f'(z)}\right] > \frac{1}{2}, \quad \text{for all } z \in U, \tag{1}$$

then

$$\mathrm{Re}\,\frac{f(z)}{z} > \frac{1}{2}, \quad \text{for all } z \in U. \tag{2}$$

This result is sharp.

Proof. If we denote

$$p(z) = 2\frac{f(z)}{z} - 1,$$

then

$$f(z) = \frac{z}{2}[p(z) + 1].$$

Let

$$F(p(z), zp'(z)) = \frac{1-\alpha}{2}\,[p(z) + 1] + \alpha\sqrt{\frac{1}{2}\,[zp'(z) + p(z) + 1]} - \frac{1}{2}\,.$$

From (1) we deduce

$$\operatorname{Re} F(p(z), zp'(z)) > 0, \quad z \in U.$$

We shall show that the function

$$F(u, v) = \frac{1-\alpha}{2}\,(u + 1) + \alpha\sqrt{\frac{1}{2}\,(1 + u + v)} - \frac{1}{2}$$

satisfies the conditions (a), (b), (c) of Lemma A. Since $f'(z) \neq 0$, i.e. $u + v + 1 \neq 0$, we deduce that F is continuous in the domain $D = \{(u, v) \in C^2 ;\ u + v + 1 \neq 0\}$. Moreover $(1, 0) \in D$ and $\operatorname{Re} F(1, 0) = 1/2 > 0$. Next we have

$$\operatorname{Re} F(iu_2, v_1) = -\frac{\alpha}{2} + \alpha\operatorname{Re}\sqrt{\frac{1}{2}\,(1 + v_1 + iu_2)}.$$

Let $\zeta = \xi + i\eta = (1 + v_1 + iu_2)/2$. If $v_1 \leqslant -(1 + u_2^2)/2$ then

$$\xi \leqslant \frac{1}{4} - \eta^2,$$

which shows that ζ lies in the interior of the parabola $\xi = 1/4 - \eta^2$, and we easily deduce $\operatorname{Re}\sqrt{\zeta} \leqslant 1/2$. Hence we obtain

$$\operatorname{Re} F(iu_2, v_1) = -\frac{\alpha}{2} + \alpha\operatorname{Re}\sqrt{\zeta} \leqslant 0, \text{ for } v_1 \leqslant -\frac{1}{2}\,(1 + u_2^2),$$

which is the condition (c). Applying Lemma A, from (3) we deduce $\operatorname{Re} p(z) > 0$, for $z \in U$, which is (2). The function $f(z) = z/(1 + z)$ shows that (2) is sharp. This completes the proof of Theorem 1.

Remark. If we denote by R_α, $\alpha \geqslant 0$, the class of all functions $f(z) = z + a_2z^2 + \ldots$ analytic in U, with $f'(z) \neq 0$, which satisfy (1), Theorem 1 shows that $R_\alpha \subset R_0$. Moreover, since (1) is linear with respect to α, from Theorem 1 we deduce $R_\alpha \subset R_\beta$, for all $\alpha > \beta \geqslant 0$.

3. For $\alpha = 1$, from Theorem 1 we obtain

COROLLARY 1.1. *If $f(z) = z + a_2z^2 + \ldots$ is analyitc in U, then*

$$\operatorname{Re}\sqrt{f'(z)} > \frac{1}{2}\ (z \in U) \Rightarrow \operatorname{Re}\frac{f(z)}{z} > \frac{1}{2} \quad (z \in U).$$

or, equivalently,

$$f'(z) \prec \frac{1}{(1+z)^2} \Rightarrow \frac{f(z)}{z} \prec \frac{1}{1+z}\,.$$

This result, which is due to Y. Komatu [2], is a particular case of the following Theorem, which is a generalization of a result of D. J. Hallenbeck and S. Ruscheweyh, [1], concerning the subordination by convex functions.

THEOREM 2. [7]. *Let f and g be two analytic functions in U, such that $f(0) = g(0) = 1$, $g'(0) \neq 0$ and*

$$\operatorname{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) > -\frac{1}{2}, \text{ for all } z \in U.$$

If $f(z) \prec g(z)$, then

$$\frac{1}{z}\int_0^z f(t)dt \prec \frac{1}{z}\int_0^z g(t)dt.$$

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ASUPRA UNEI INEGALITĂŢI DIFERENŢIALE PENTRU FUNCŢIILE ANALITICE ÎN DISCUL UNITATE

(Rezumat)

Se demonstrează că dacă $\alpha \geqslant 0$ şi $f(z) = z + a_2z^2 + \ldots$ este o funcţie analitică cu $f'(z) \neq 0$ în discul unitate U, care verifică inegalitatea diferenţială (1), atunci f verifică şi inegalitatea (2), iar acest rezultat este cel mai bun posibil.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXVI, 2, 1981

To Professor D .V. Ionescu
on his 80th birthday

GENERALIZED CHEBYSHEV POLYNOMIALS AND REVERSED GEOMETRIC PROGRAMMING

I. MARUŞCIAC

1. Introduction. Let $\nu_1, \nu_2, \ldots, \nu_n$ be fixed natural numbers. Set $m = \nu_1 + \nu_2 + \ldots + \nu_n$. It is well known that the remainder term $r_{m-1}(f; x)$ in Newton's interpolation formula

$$f(x) = L_{m-1}(f;\ x) + r_{m-1}(f;\ x)$$

on the nodes $(x_i)_1^n$, $a \leqslant x_1 < x_2 < \ldots < x_n \leqslant b$, with multiplicities $(\nu_i)_1^n$ respectively, satisfies the following inequality

$$||r_{m-1}|| \leqslant ||f^{(m)}|| \max_{x \in [a,b]} |(x - x_1)^{\nu_1}(x - x_2)^{\nu_2} \ldots (x - x_n)^{\nu_n}|, \tag{1}$$

where $||f||$ is the Chebyshev norm i.e. $||f|| = \max \{|f(x)| : x \in [a, b]\}$. From (1) it is seen that the following extremal problem arises: determine $x_1^*, x_2^*, \ldots, x_n^* \in [a, b]$ such that

$$\max_{x \in [a,b]} |(x - x_1^*) \ldots (x - x_n^*)| = \inf_{a \leqslant x_1 < \ldots < x_n < b} \max_{x \in [a,b]} |(x - x_1)^{\nu_1} \ldots (x - x_n)^{\nu_n}|. \tag{2}$$

At this problem have arrived T. Popoviciu [5] and L. Tschakaloff [6] by studying quadrature formulae of highest degree of precision. They proved the existence of the extremal nodes for the L_2-norm case.

In the complex plane the similar problem was studied by the author in 1968 [4]. When the problem is studied on a compact set of the complex plane, the existence of such a least deviation from zero polynomial is allways assured but the uniqueness does not hold, as it was shown in [4].

In the real case, when the compact set is a finite closed interval, the uniqueness of such an extremal polynomial was established recently by B. D. Bojanov [2].

In this note we inted to show that in order to compute the extremal nodes, i.e. the extremal polynomial, we can use the reversed geometric programming technic [3]. More explicitely, we shall show that the problem of finding the extremal polynomial (2) is equivalent to solve a reversed geomertic programming problem.

2. Generalized Chebyshev polynomial and a nonlinear programming problem. We shall need the following result established in [2]:

THEOREM 1. *Let $(\nu_k)_1^n$ be a fixed system of arbitrary natural numbers and $[a, b]$ a given interval in R. There exists a unique system of points $(x_x^*)_1^n \subset [a, b]$*

such that

$$||(\cdot - x_1^*)^{\nu_1} \ldots (\cdot - x_n^*)^{\nu_n}|| = \inf_{a \leq x_1 < x_2 < \ldots < x_n \leq b} ||(\cdot - x_1)^{\nu_1} \ldots (\cdot - x_n)^{\nu_n}||.$$

Moreover, $a < x_1^* < x_2^* < \ldots < x_n^* < b$. *The extremal polynomial* $t_m(\nu; \cdot)$,

$$t_m(\nu; x) = (x - x_1^*)^{\nu_1}(x - x_2^*)^{\nu_2} \ldots (x - x_n^*)^{\nu_n},$$

$m = \nu_1 + \nu_2 + \ldots + \nu_n$, *is uniquely determined by the condition that there exist* $n - 1$ *points* $(t_k)_1^{n-1}$, $a = t_0 < x_1^* < t_1 < x_2^* < \ldots < t_{n-1} < x_n^* < t_n = b$, *such that*

$$t_m(\nu; t_k) = (-1)^{m-\nu_1-\ldots-\lambda_k}||t_m(\nu; \cdot)||, \; k = 0, 1, \ldots, n. \tag{3}$$

Consider two system $(x_k)_1^n \subset [a, b]$, $(t_k)_0^n \subset [a, b]$ such that

$$a = t^0 < x_1 < t_1 < \ldots < x_k < t_k < \ldots < x_n < t_n = b \tag{4}$$

and denote by $\pi_m(\nu)$ the class of all algebraic polynomials of degree m of the form

$$p_m(x) = (x - x_1)^{\nu_1}(x - x_2)^{\nu_2} \ldots (x - x_n)^{\nu_n},$$

$\nu_1 + \nu_2 + \ldots + \nu_n = m$.

Set

$$\begin{aligned} u_{2k+1} &= x_{k+1} - t_k, \; k = 0, 1, \ldots, n - 1 \\ u_{2k} &= t_k - x_k, \; k = 1, 2, \ldots, n \end{aligned} \tag{5}$$

Then for $p_m \in \pi_m(\nu)$ we have

$$t_m(t_k) = (t_k - x_1)^{\nu_1}(t_k - x_2)^{\nu_2} \ldots (t_k - x_k)^{\nu_k}(t_k - x_{k+1})^{\nu_{k+1}} \ldots (t_k - x_n)^{\nu_n} =$$

$$= (u_2 + u_3 + \ldots + u_{2k})^{\nu_1}(u_4 + u_5 + \ldots + u_{2k})^{\nu_2} \ldots u_{2k}^{\nu_k}(-u_{2k+1})^{\nu_{k+1}} \ldots$$

$$\ldots (-u_{2k+1} - \ldots - u_{2n-1})^{\nu_n} = (-1)^{\nu_{k+1}+\ldots+\nu_n}(u_2 + u_3 + \ldots + u_{2k})^{\nu_1} \cdot$$

$$(u_4 + u_5 + \cdots + u_{2k})^{\nu_2} \ldots u_{2k}^{\nu_k} \times u_{2k+1}^{\nu_{k+1}} \ldots (u_{2k+1} + \ldots + u_{2n-1})^{\nu_n}.$$

In view of $\nu_1 + \nu_2 + \ldots + \nu_n = m$, it follows that

$$\nu_{k+1} + \ldots + \nu_n = m - \nu_1 - \nu_2 - \ldots - \nu_k$$

and, therefore, conditions (3) become

$$\begin{aligned} &(u_2 + u_3 + \ldots + u_{2k})^{\nu_1}(u_4 + \ldots + u_{2k})^{\nu_2} \ldots u_{2k}^{\nu_k}u_{2k+1}^{\nu_{k+1}} \ldots (u_{2k+1} + \cdots \\ &\qquad \cdots + u_{2n-1})^{\nu_n} = \mu \qquad k = 1,0, \ldots , \end{aligned} \tag{3'}$$

where $\mu = ||p_m||$.

System (3′) can be written under the form:

$$v_{k1}^{\nu_1}v_{k2}^{\nu_2}\dots v_{kn}^{\nu_n}=\mu,\ k=0,\ 1,\ \dots,\ n, \tag{3″}$$

where

$$v_{kj}=u_{2j}+u_{2j+1}+\dots+u_{2k},\ 1\leqslant j\leqslant k,\ k=1,\ 2,\ \dots,\ n$$

$$v_{kj}=u_{2k+1}+u_{2k+2}+\dots+u_{2j-1},\ k<j\leqslant n,\quad k=0,1,\dots,$$

Now we consider the following nonlinear programming problem:

$$\text{minimize } \mu \tag{4}$$

subject to

$$v_{k1}^{\nu_1}v_{k2}^{\nu_2}\dots v_{kn}^{\nu_n}-\mu=0,\ k=0,\ 1,\ \dots,\ n \tag{5}$$

$$\sum_{i=1}^{k}\nu_i v_{k1}^{\nu_1}v_{k2}^{\nu_2}\dots v_{ki}^{\nu_i-1}\dots v_{kn}^{\nu_n}-\sum_{i=k+1}^{n}v_{k1}^{\nu_1}\dots v_{ki}^{\nu_i-1}\dots v_{kn}^{\nu_n}=0 \tag{6}$$

$$v_{kj}=u_{2j}+u_{2,j+1}+\dots+u_{2k},\ 1\leqslant j\leqslant k,\ k=1,\ 2,\ \dots,\ n \tag{7}$$

$$v_{kj}=u_{2k+1}+u_{2k+2}+\dots+u_{2j-1},\quad k<j\leqslant n,\ k=0,\ 1,\ \dots,\ n$$

$$u_1+u_2+\dots+u_{2n}=b-a \tag{8}$$

$$u_j>0,\ j=1,\ 2,\ \dots,\ 2n.$$

THEOREM 2. *If $(u^*,\ \mu^*)\in R_+^{2n}\times R^+$ is an optimal solution to the nonlinear programming problem (4) — (8), then polynomial $p_m^*\in\pi_m(\nu)$,*

$$p_m^*(x)=(x-x_1^*)^{\nu_1}(x-x_2^*)^{\nu_2}\dots(x-x_n^*)^{\nu_n}, \tag{9}$$

where

$$x_m^*=t_0^*+u_1^*,\ t_1^*=x_1^*+u_2^*$$

$$x_2^*=t_1^*+u_3^*,\ t_2^*=x_2^*+u_4^*$$

$$\dots\dots\dots\dots\dots$$

$$x_n^*=t_{n-1}^*+u_{2n-1}^*,\ t_n^*=x_n^*+u_2^*$$

is the extremal polynomial to the problem (2).

Proof. Let $t_m\in\pi_m(\nu)$,

$$t_m(x)=(x-x_1^0)^{\nu_1}(x-x_2^0)^{\nu_2}\dots(x-x_n^0)^{\nu_n}$$

be the extremal polynomial to the problem (2). Then in view of Theorem 1. there exist $(t_k^0)_0^n\subset[a,\ b]$,

$$a=t_0^0<x_1^0<t_1^0<x_2^0<\dots<t_{n-1}^0<x_n^0<t_n^0=b$$

such that

$$t_m(t_k^0) = (t_k^0 - x_1^0)^{\nu_1} \ldots (t_k^0 - x_n^0)^{\nu_n} = (-1)^{m-\nu_1-\ldots-\nu_k} \, ||t_m|| \tag{11}$$

Consider

$$u_{2k+1}^0 = x_{k+1}^0 - t_k^0, \; k = 0, 1, \ldots, n-1$$
$$u_{2k}^0 = t_k^0 - x_k^0, \; k = 1, 2, \ldots, n$$
$$v_{kj}^0 = u_{2j}^0 + \ldots + u_{2k}^0, \; 1 \leqslant j \leqslant k, \; k = 1, 2, \ldots, n$$
$$v_{kj}^0 = u_{2k+1}^0 + \ldots + u_{2j-1}^0, \; k < j \leqslant n, \; k = 0, 1, \ldots, n$$

we have

$$\sum_{k=1}^{2n} u_k^0 = \sum_{k=0}^{n-1} (x_{k+1}^0 - t_k^0) + \sum_{k=1}^{n} (t_k^0 - x_k^0) = t_n^0 - t_0^0 = b - a$$

i.e. $u_j^0 > 0$, $j = 1, 2, \ldots, 2n$, and (8) is satisfied.

From (11) it is seen that

$$(v_{k1}^0)^{\nu_1} \ldots (v_{kn}^0)^{\nu_n} = \mu^0, \; \mu^0 = ||t_m||$$

Since $t_1^0, \ldots, t_{n-1}^0$ are extreme points for t_m, it follows that

$$t_m'(t_k^0) = \sum_{i=1}^{n} \nu_i (t_k^0 - x_k^0)^{\nu_1} \ldots (t_k^0 - x_i^0)^{\nu_i - i} \ldots (t_k^0 - x_n^0)^{\nu_n} = 0, \; k = 1, 2, \ldots, n-1$$

i.e. (6) are also satisfied.

Therefore $(u_1^0, u_2^0, \ldots, u_{2n}^0, \mu^0)$ is a feasible solution to the problem (4) – (8). So

$$\mu^* \leqslant \mu^0. \tag{12}$$

But from (6) and (10) we conclude that

$$(p_m^*)'(t_k^*) = 0, \; k = 1, 2, \ldots, n-1.$$

Clearly $p_m^{*\prime}$ has $q \leqslant 2n-1$ distinct real zeros: $t_1^*, t_2^*, \ldots, t_{n-1}^*$ and some of x_k^*. That means

$$\mu^* = \max_{0 \leqslant k \leqslant n} |p_m(t_k^*)| = ||p_m^*||$$

and, therefore

$$\mu^* \geqslant \mu^0 \tag{13}$$

Inequalities (12) – (13) show that $\mu^* = \mu^0$.

Because of the unicity of the extremal polynomial to the problem (2), we conclude that $p_m^* = t_m$, and so theorem is proved.

3. A reversed geometric programming formulation.

Remark. In view of the special structure of the programming problem (4) — (8), it is clear that (8) can be replaced by

$$u_1 + u_2 + \dots + u_{2n} \geqslant b - a$$

and (5), (7) by

$$v_{k1}^{\nu_1} v_{k2}^{\nu_2} \dots v_{kn}^{\nu_n} \leqslant \mu, \ k = 0, 1, \dots, n$$

$$u_{2j} + u_{2j+1} + \dots + u_{2k} \leqslant v_{kj}, \ 1 \leqslant j \leqslant k, \ k = 1, 2, \dots, n$$

$$u_{2k+1} + \dots + u_{2j-1} \leqslant v_{kj}, \ k < j \leqslant n, \ k = 0, 1, \dots, n$$

respectively.

Thus if equality (6) is replaced by two inequalities we arrive at the following complementary geometric programming problem [1]:

Problem CGP

$$\textit{minimize } \mu$$

subject to the constraints

$$v_{k1}^{\nu_1} v_{k2}^{\nu_2} \dots v_{kn}^{\nu_n} \mu^{-1} \leqslant 1, \ k = 0, 1, \dots, n$$

$$\sum_{i=1}^{k} \nu_i v_{k1}^{\nu_1} \dots v_{ki}^{\nu_i - 1} \dots v_{kn}^{\nu_n} - \sum_{i=k+1}^{n} \nu_i v_{k1}^{\nu_1} \dots v_{ki}^{\nu_i - 1} \dots v_{kn}^{\nu_n} = 0, \tag{14}$$

$$k = 1, 2, \dots, n - 1$$

$$u_{2j} v_{kj}^{-1} + u_{2j+1} v_{kj}^{-1} + \dots + u_{2k} v_{kj}^{-1} \leqslant 1, \ 1 \leqslant j \leqslant k, \ k = 1, 2, \dots, n$$

$$u_{2k+1} v_{kj}^{-1} + \dots + u_{2j-1} v_{kj}^{-1} \leqslant 1, \ k < j \leqslant n, \ k = 0, 1, \dots, n$$

$$\frac{1}{b-a} u_1 + \frac{1}{b-a} u_2 + \dots + \frac{1}{b-a} u_{2n} \geqslant 1,$$

$$u_j > 0, \ j = 1, 2, \dots, 2n; \ v_{kj} > 0, \ k = 0, 1, \dots, n; \ j = 1, 2, \dots, n$$

It is known (see [4], p. 173) that equality (14) is equivalent to

$$\sum_{i=1}^{k} \nu_i v_{k1}^{\nu_1} \dots v_{ki}^{\nu_i - 1} \dots v_{kn}^{\nu_n} \leqslant u_{2n+1} \leqslant \sum_{i=k+1}^{n} \nu_i v_{k1}^{\nu_1} \dots v_{ki}^{\nu_i - 1} \dots v_{kn}^{\nu_n},$$

where $u_{2n+1} > 0$. So program CGP can be reduced to the following reversed geometric programming problem:

Program RGP:

$$\textit{minimize } \mu$$

subject to

$$v_{k1}^{\nu_1} v_{k2}^{\nu_2} \dots v_{kn}^{\nu_n} \leqslant 1, \qquad k = 0, 1, \dots, n$$

$$\sum_{i=1}^{k} \nu_i v_{k1}^{\nu_1} \dots v_{ki}^{\nu_i - 1} \dots v_{kn}^{\nu_n} u_{2n+1}^{-1} \leqslant 1, \qquad k = 1, 2, \dots, n-1$$

$$u_{2j} v_{kj}^{-1} + u_{2j+1} v_{kj}^{-1} + \dots + u_{2k} v_{kj}^{-1} \leqslant 1, \ 1 \leqslant j \leqslant k, \qquad k = 1, 2, \dots, n$$

$$u_{2k+1} v_{kj}^{-1} + \dots + u_{2j-1} v_{kj}^{-1} \leqslant 1, \ k < j \leqslant n, \ k = 0, 1, \dots, n$$

$$\sum_{i=k+1}^{n} \nu_i v_{k1}^{\nu_1} \dots v_{ki}^{\nu_i - 1} \dots v_{kn}^{\nu_n} u_{2n+1}^{-1} \geqslant 1, \qquad k = 1, 2, \dots, n-1$$

$$\sum_{i=1}^{2n} \frac{1}{b-a} u_i \geqslant 1, \ u_j > 0, \ j = 1, 2, \dots, 2n+1$$

$$v_{kj} > 0, \ k = 0, 1, \dots, n; \ j = 1, 2, \dots, n.$$

To reduce the reversed geometric program RGP to a prototype polynomial program we can use the condensation method described in [4].

To conclude, we see that extremal problem (2) of finding the generalized Chebyshev polynomial with multiple roots can be reduced to the problem of solving a sequence of standard geometric programs.

Example. Consider $[a, b] = [0, 1)$, $\nu_1 = 3$, $\nu_2 = 2$, $m = \nu_1 + \nu_2 = 5$. Denote

$$\mu = u_0, \ v_{02} = u_6, \ v_{21} = u_7,$$

the corresponding reversed geometric program is the following:

$$\text{minimize } u_0$$

subject to :

$$u_0^{-1} u_1^3 u_6^2 \leqslant 1$$

$$u_0^{-1} u_2^3 u_3^2 \leqslant 1$$

$$u_0^{-1} u_4^2 u_7^3 \leqslant 1$$

$$3 u_2^2 u_3^2 u_5^{-1} \leqslant 1$$

$$u_1 u_6^{-1} + u_2 u_6^{-1} + u_3 u_6^{-1} \leqslant 1$$

$$u_2 u_7^{-1} + u_3 u_7^{-1} \leqslant 1$$

$$2 u_2^3 u_2 u_5^{-1} \geqslant 1$$

$$u_1 + u_2 + u_3 + u_4 \geqslant 1, \ u_j > 0, \ j = 1, 2, \dots, 7.$$

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POLINOAMELE LUI CEBÎŞEV GENERALIZATE ŞI PROGRAMAREA GEOMETRICĂ REVERSĂ

(Rezumat)

În studiul formulelor de integrare numerică de tip Gauss sîntem conduşi la următoarea problemă de cea mai bună aproximare: să se determine $x_1^*, x_2^*, \ldots, x_n^* \in [a, b]$ astfel încît

$$\max_{x\in[a,b]} |(x - x_1^*)^{\nu_1} \ldots (x - x_n^*)^{\nu_n}| = \inf_{a\le x_1<\ldots<x_n\le b} \max_{x\in[a,b]} |(x - x_1)^{\nu_1} \ldots (x - x_n)^{\nu_n}| \qquad (2)$$

unde $\nu_1, \nu_2, \ldots, \nu_n$ sînt numere naturale date. Polinomul din membrul întîi al formulei (2) se numeşte polinomul lui Cebîşev cu rădăcini multiple pe intervalul $[a, b]$. În prezenta lucrare se arată că problema determinării polinomului lui Cebîşev cu rădăcini multiple se poate reduce la o problemă de programare geometrică reversă. La rîndul său, o asemenea problemă se reduce la un şir de probleme de programare geometrică standard.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXVI, 2, 1981

To Professor D. V. Ionescu
on his 80th birthday

A METHOD ANALOGOUS TO THE CHEBYSHEV METHOD FOR THE SOLVING OF THE OPERATOR EQUATIONS

SEVER GROZE

1. In this paper we present an iterative method which, using high order divided differences, allow us to construct a sequence which approximates the solution of the equation

$$P(x) = \theta \tag{1}$$

where $P: X \to Y$ is a continuous application of the Fréchet space X in the Fréchet space Y, θ being the null element of the space Y.

The necessity to contruct such a method, with a high convergence order and which do not use Fréchet derivatives, follows by the fact that in the Fréchet spaces a mean theorem [1] cannot be given ; hence the known iterative methods are not applicable.

2. To approximate the solution of the equation (1) we shall use the algorithm

$$x_{n+1} = x_n - \Lambda_n P(x_n) - \Lambda_n P_{x_n, x_{n-1}, x_{n-2}} \Lambda_{n-1} P(x_{n-1}) \bar{\Lambda}_n P(x_n), \quad (n = 0, 1, 2, \ldots) \tag{2}$$

where x_{-2}, x_{-1}, x_0 are given elements in X

$$\Lambda = [P_{x_n, x_{n-1}}]^{-1}, \quad \bar{\Lambda}_n = [P_{x_n, x_{n-2}}]^{-1}.$$

With respect to the existence and the uniqueness of the solution of the equation (1) we have:

THEOREM. 1° *If it exists* $\Lambda = [P_{x', x''}]^{-1}$, $\forall x', x'' \in S(x_0, R_0)$, *and* $)|\Lambda|(\leqslant \beta$, by $)|\cdot|($ *we denote the cvasinorm of an element of a Fréchet space* [2]

2° $)|P(x_i)|(\leqslant \eta_i$, $i = -2, -1, 0$ *and* $\eta_0 \leqslant \eta_{-1} \leqslant \eta_{-2}$

3° *For any* $x', x'', x''', x^{IV} \in S_0$, *we have*

$$)|P_{x', x'', x'''}|(\leqslant M, \quad)|P_{x', x'', x''', x^{IV}}|(\leqslant N$$

4° $h_{-2}E_{-2} < 1$, *where* $h_{-2} = B^2 M \eta_{-2}$,

$$E_{-2}^2 = \frac{1}{H_{-2}}\left\{3 + \left(\frac{N}{BM^2} + 1\right)(1 - h_{-2})^3\right\} \quad \text{and} \quad H_2 = 1 - h_{-2}(1 + h_{-2}) > 0$$

then the equation (1) *has, in* $S_0(x_0, R_0)$ *with* $R_0 = \dfrac{2B\,\eta_{-2}}{1 - (h_{-2}E_{-2})^2}$, *a solution* x^* *and only one, which is the limit of the sequence* (2), *the convergence order being given* *by the inequality*

$$\rho_X(x^* - x_n) \leqslant \frac{2B\eta_{-2}}{1-(h_{-2}E_{-2})^4}(h_{-2}E_{-2})^{2\sum\limits_{i=2}^{n-1} t_i} \leqslant R_0\,(h_{-2}E_{-2})^{2\sum\limits_{i=2}^{n-1} t_i} \tag{3}$$

where $t_i = t_{i-3} + t_{i-2} + t_{i-1}$ $(i = 2, 3, \ldots)$ *and* $t_{-1} = -1,\ t_0 = t_1 = 1$

Remark. 1) From the conditions 4° it follows that $h_{-2} < 1$.

2) The condition $E_{-2}h_{-2} < 1$ is satisfied, for exemple, if $h_{-2} < \frac{1}{4}$ and $\frac{N}{BM^2} \leqslant \frac{378}{125}$.

Proof. From the theorem conditions, it follows $x_{-1}, x_{-2} \in S_0$. Indeed

$$)|\,x_{-2} - x_0\,|(=)|\,\bar{\Lambda}_0 P_{x_0, x_{-1}}(x_{-2} - x_0\,|(\leqslant)|\,\Lambda_0\,|(\,()|\,P(x_{-2})\,|(+)|\,P(x_0)\,|() \leqslant$$
$$\leqslant 2B\eta_{-2} < R_0$$

and in the same way we can obtain that $x_{-1} \in S_0$.

Hence, using the conditions $1^\circ, 2^\circ, 3^\circ$ and (2), it follows the posibility to construct x_1 and

$$)|\,x_1 - x_0\,|(\leqslant \beta\eta_0(1 + h_{-2}) \leqslant R_0$$
$$)|\,Px_1, x_0, x_{-11}, x_{-2}\,|(\leqslant N.$$

Now we show that x_{-1}, x_0, x_1 satisfies the theorem hypotheses: the condition 1° is evidently verified.

To prove 2°, we consider the application $F_n : X \to X$, given by

$$F_n(x) = x - \gamma_n\,P(x) - \nu_n[P(x)]^2 \tag{4}$$

where

$$\gamma_n u = \Lambda_n u + \Lambda_n\,P_{x_n, x_{n-1}, x_{n-2}}\,\Lambda_{n-1}\,P(x_{n-1})\,\bar{\Lambda}_n u + \Lambda_n\,P_{x_n, x_{n-1}, x_{n-2}}\,\Lambda_{n-1}\,u\,\Lambda_n\,P(x_n)$$

$$\nu_n u_1 u_2 = -\,\Lambda_n\,P_{x_n, x_{n-1}, x_{n-2}}\,u_1\,\Lambda_{n-1}\,\bar{\Lambda}_n u_2 \qquad n = 0, 1, 2, \ldots$$

we obtain

$$F_n(x_{n-2}) = F_n(x_{n-1}) = F_n(x_n) = x_{n+1},\ F_{x_n, x_{n-1}} = \theta^*$$
$$F_{x_n, x_{n-1}, x_{n-2}} = \theta^{**}$$

where θ^* is the null element in $(X, X)^*$, and θ^{**} is the null element in $(X \times X, X)^*$, and $(A, B)^*$ denotes the set of linear and bounded applications defined on A and with values in B.

We also have

$$F_{x_{n+1}, x_n, x_{n-1}, x_{n-2}}\,u_1 u_2 u_3 = \gamma_n\,P_{x_{n+1}, x_n, x_{n-1}, x_{n-2}}\,u_1 u_2 u_3 -$$
$$-\,\nu_n\,P_{x_{n+1}, x_n, x_{n-1}, x_{n-2}}\,u_1 u_2 u_3\,P(x_n) - \nu_n\,P_{n+1, x_{n-1}, x_{n-2}}\,u_1 u_2 \cdot P_{x_{n+1},}\ u_3 -$$
$$-\,\nu_n\,P_{x_{n-1}, x_{n-2}}\,u_1\quad P_{x_{n+1}, x_n, x_{n-2}}\,u_2 u_3 - \nu_n\,P_{(x_{n-1})}\,P_{x_{n+1}, x_n, x_{n-1}, x_{n-2}}\,u_1 u_2 u_3.$$

Applying to F_0 for $x = x_1$ the Newton type formula

$$P(x) = P(x_1) + P_{x_1,x_2}(x - x_1) + P_{x_1,x_2,x_3}(x - x_2)(x - x_1) + \\ + P_{x_1,x_2,x_3,x}(x - x_3)(x - x_2)(x - x_1) \tag{5}$$

we obtain

$$\| P(x_1) \| (1 - h_{-2}(1 + h_{-2})) \leqslant (N + M^2B) \| x - x_{-2} \| \cdot \| x - x_{-1} \| \cdot \\ \cdot \| x - x_0 \| + 3h^2_{-2}\eta_0 .$$

Using the property $F_0(x_{-2}) = F_0(x_{-1}) = F_0(x_0) = x_1$, it follows

$$\| P(x_1) \| \leqslant (h_{-2}E_{-2})^2\eta_0 = \eta_1 < \eta_0 \tag{6}$$

hence 2° is verified.

The hypothesis 3° is evidently verified.

For the hypothesis 4° we have

$$h_{-1} = B^2M\eta_{-1} \leqslant h_{-2},\ H_{-1} = 1 - h_{-1}(1 + h_{-1}) \geqslant H_{-2}$$

from where $E_{-1} \leqslant E_{-2}$, hence $h_{-1}E_{-1} < 1$.

It follows that we can construct x_2 and that

$$\| x_2 - x_0 \| \leqslant B\eta_0 \ (1 + (E_{-2}h_{-2})h_{-1}) \leqslant 2B\eta_{-2} < R_0 .$$

Next we show that x_0, x_1, x_2 verfies the conditions from the theorem. Indeed 1° and 3° are fulfilled. Taking in (4) $n = 1$ and aplying (5) it follows

$$\| P(x_2) \| \leqslant (E_{-1}h_{-1})^2\eta_1 = \eta_2 < \eta_1 . \tag{7}$$

For 4° we have $h_0 = B^2M\eta_0 \leqslant h_{-1}$, $H_0 \geqslant H_{-1}$, $E_0 \leqslant E_{-1}$ and $E_0h_0 < 1$.

By induction, it can be proved that the properties are verified for any x_n and that

$$\eta_{n+3} \leqslant (h_nE_n)^2 \ \eta_{n+2} \tag{8}$$

$$h_{n+3} \leqslant \frac{(h_n E_n)^3}{E_n} \tag{9}$$

$$h_nE_n < 1 \tag{10}$$

where $\| P(x_n) \| \leqslant \eta_n$, $h_n = B^2M\eta_n$, and

$$E_n^2 = \frac{1}{H_n}\left(3 + \left(\frac{N}{BM^2} + 1\right)(1 + h_n)^3\right), \tag{11}$$

where $H_n = 1 - h_n(1 + h_n)$.

Since (8) it follows $x_n \in S_0(x_0, R_0)$, $x_n \in S_0'(x_{-1}, R_0)$ and $x_n \in S_0''(x_{-2}, R_0)$. Indeed we have, for example

$$\| x_n - x_0 \| \leqslant 2B(\eta_{n-1} + \cdots + \eta_1 + \eta_0) \leqslant \frac{2B\eta_0}{1 - (E_nh_n)^2} \leqslant R_0 .$$

From (8) and (9) we obtain

$$\eta_{n+1} \leqslant (E_{-2}\, h_{-2})^{2\sum_{i=2}^{n} t_i} \tag{12}$$

Hence

$$\| x_{n+p} - x_n \| \leqslant \frac{2B\eta_{-2}}{1 + (h_{-2}E_{-2})^2} (E_{-2}\, h_{-2})^{2\sum_{i=2}^{n} t_i} \leqslant R_0 (E_{-2} h_{-2})^{2\sum_{i=2}^{n} t_i} \tag{13}$$

By (10) and (13) it follows that the sequence (x_n) is fundamental, hence convergent to an element $x^* \in X$. Also $x^* \in S_0(x_0, R_0)$, $x^* \in S_0'(x_{-1}, R_0)$, $x^* \in S_0''(x_{-2}, R_0)$.

If in (13) we take $p \to \infty$, it follows (3).

Because from (12) it follows.

$$\| P(x_n) \| \leqslant (h_{-2}\, E_{-2})^{2\sum_{i=2}^{n-1} t_i} \eta_0$$

for n $\to \infty$, P being continuous, we have

$$P(x^*) = \theta.$$

If for a $\tilde{x} \in S_0$ we have $P(\tilde{x}) = \theta$, then $P_{x^*, \tilde{x}}(\tilde{x} - x^*) = 0$ hence $\tilde{x} - x^* \in \text{Ker}\,(P_{x^*, \tilde{x}}) \cap S_0$. Taking into account that 1° implies ker $(P_{x^*, \tilde{x}}) \cap S_0 = \{\theta\}$, we have $\tilde{x} - x^* = 0$, hence x^* is the unique solution of the equation (1) in S_0.

Remark. A similar theorem, for Banach spaces, was given by M. Balázs and G. Goldner in [3].

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O METODĂ ANALOAGĂ CU METODA LUI CEBÎŞEV PENTRU REZOLVAREA ECUAŢIILOR OPERATORIALE

Rezumat)

Folosindu-se o metodă de rezolvare aproximativă analoagă cu a lui Cebîşev, este demonstrată o teoremă de existenţă şi unicitate pentru soluţia unei ecuaţii operatoriale $P(x) = \theta$, unde $P: X \to Y$ este o aplicaţie continuă, X şi Y fiind spaţii Fréchet.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXVI, 2, 1981

To Professor D. V. Ionescu on his 80th birthday

ON THE DETERMINATION OF CIRCULAR ORBITS OF ARTIFICIAL SATELLITES FROM OPTICAL VISUAL OBSERVATIONS

ÁRPÁD PÁL and TIBERIU OPROIU

1. **Introduction.** The Station 1132 of the Cluj-Napoca Astronomical Observatory usually performs optical visual observations on artificial Earth satellites, by using a balloon-theodolite [1]. The accuracy of these observations ($0^{\circ}.1$ in position and $0^{s}.1$ in time) is not suitable for the determination of a preliminary elliptic orbit. As most of the observed satellites have quasi-circular orbits, we have tried to determine circular orbits, which are less susceptible to the accuracy of the observations.

Among the classical methods of orbit calculation, the problem of the determination of circular orbits is the simplest. It needs only two position observations and allows the determination of four orbital parameters (instead of six): a = circular orbit radius, i = inclination of the orbital plane to the equatorial plane, Ω = longitude of the ascending node, and u = argument of the latitude, corresponding to the moment t.

It is obvious that the parameters determined in this manner represent only roughly the real motion of the satellite. However, their values can be used as initial data for a differential improvement of the elliptic orbit for a long enough time interval. The orbital elements obtained in this manner can be compared with similar values resulted from other sources and can constitute valuable informations for the identification of the respective satellite. Also, these elements can be used as initial data in the calculation of satellite ephemeris for a not too large time interval.

2. **Basic Equations.** The determination of the parameters of a circular orbit was performed according to Zeinalov's algorithm [2]. The satellite motion is considered with respect to an inertial geocentric right-handed frame $0xyz$, having the origin 0 in the Earth's mass center, $0x$ - axis directed towards the equinox point and $0z$-axis directed towards the celestial North pole.

The geocentric position of the satellite in this frame is given by the vectorial relation:

$$\vec{r} = \vec{R} + \vec{\rho}, \tag{1}$$

where $\vec{r}(x, y, z)$ is the geocentric radius vector of the satellite ($|\vec{r}| = a$), $\vec{R}(X, Y, Z)$ is the geocentric radius vector of the tracking station, while $\vec{\rho}(\rho_x, \rho_y, \rho_z)$ represents the topocentric radius vector of the satellite. If the geographical coordinates of the tracking station are known (φ = the latitude, λ = the longitude, measured from Greenwich towards East, H = the altitude in meters above sea level), then the coordinates X, Y, Z are given by the relations:

$$
\begin{aligned}
X &= X_0 \cos \theta - Y_0 \sin \theta, \\
Y &= X_0 \sin \theta + Y_0 \cos \theta, \\
Z &= Z_0,
\end{aligned} \tag{2}
$$

where θ is the Greenwich sidereal time corresponding to a given moment t. The quantities X_0, Y_0, Z_0 represent the coordinates of the tracking station in a geocentric rectangular frame $0XYZ$, similar to $0xyz$, but with $0X$ — axis directed towards the Greenwich meridian (a fixed frame with respect to the Earth). They are given by:

$$
\begin{aligned}
X_0 &= R \cos \varphi' \cos \lambda, \\
Y_0 &= R \cos \varphi' \sin \lambda, \\
Z_0 &= R \sin \varphi',
\end{aligned} \tag{3}
$$

where:

$$
\begin{aligned}
R \cos \varphi' &= (C + H/a_e) \cos \varphi, \\
R \sin \varphi' &= (S + H/a_e) \sin \varphi, \\
C &= (1 - e^2 \sin^2 \varphi)^{-1/2}, \\
S &= (1 - e^2)\, C.
\end{aligned} \tag{4}
$$

In these relations, R and φ' represent the geocentric radius vector and the geocentric latitude of the tracking station, respectively, a_e is the equatorial radius of the reference ellipsoid, and e^2 is a constant connected with the Earth's oblateness (Krassovsky's ellipsoid has: $a_e = 6378.245$ km, $e^2 = 0.66934210 \times \times 10^{-2}$). We shall take the quantity a_e as length unit.

The observations we use are given in horizontal (A,h) or equatorial ($\alpha_{1950,0}$, $\delta_{1950.0}$) coordinates. In both cases we must pass to the equatorial coordinates corresponding to the epoch t. Hence, the observations give:

$$
t_i,\ \alpha_i,\ \delta_i,\ i = 1,2. \tag{5}
$$

Denoting by:

$$
\begin{aligned}
\lambda &= \cos \delta \cos \alpha, \\
\mu &= \cos \delta \sin \alpha, \\
\nu &= \sin \delta,
\end{aligned} \tag{6}
$$

the cosines of the direction $\vec{\rho}$ ($|\vec{\rho}| = \rho$), Equation (1) can be written in a scalar form:

$$
\begin{aligned}
x &= \lambda \rho + X, \\
y &= \mu \rho + Y, \\
z &= \nu \rho + Z.
\end{aligned} \tag{7}
$$

From (7) we obtain:

$$a^2 = \rho^2 + 2\,\rho G + R^2, \tag{8}$$

where:

$$\begin{aligned} a^2 &= x^2 + y^2 + z^2,\\ R^2 &= X^2 + Y^2 + Z^2,\\ G &= R\cos\psi,\\ R\cos\psi &= \lambda X + \mu Y + \nu Z. \end{aligned} \tag{9}$$

Equation (8) binds three unknown quantities: a, ρ_1, ρ_2 (corresponding to the moments t_1, t_2, respectively). In Equation (9), ψ is the angle between the topocentric direction towards the satellite (whose cosines are λ, μ, ν) and the direction towards the observation point (whose cosines are X/R, Y/R, Z/R).

Solving Equation (8) with respect to ρ, we obtain:

$$\rho = -G + \sqrt{a^2 - F^2}, \tag{10}$$

where, as in [2], F^2 denotes the difference $R^2 - G^2$. The radical of (10) has been given with its positive sign in order to have $\rho > 0$.

If the satellite position at the moments t_1, t_2 is determined by the angles u_1, u_2, respectively, then the arc $u_2 - u_1$ (usually denoted by $2f$) represents the ngle between the geocentric radii vectors $\vec{r}_1(x_1, y_1, z_1)$ and $\vec{r}_2(x_2, y_2, z_2)$, where $|\vec{r}_1| = |\vec{r}_2| = a$. This arc can be geometrically determined from the relation:

$$2f_g = \arccos\,[a^{-2}\,(x_1x_2 + y_1y_2 + z_1z_2)], \tag{11}$$

or, in a dynamical way, from the relation:

$$2f_d = n\,(t_2 - t_1), \tag{12}$$

where:

$$n = k\,a^{-3/2} \tag{13}$$

is the mean motion of the satellite. The determination of $2f$ in two different manners gives us the possibility to calculate the radius of the circular orbit (a) by the successive approximations method. So, taking an arbitrary value for a, we find with Equation (10) the quantities ρ_1 and ρ_2. Then, from (7) we find the geocentric coordinates of the satellite and finally, from Equations (11) — (13), we determine $2f_g$ and $2f_d$. In the case of a circular orbit, $2f_g = 2f_d$; therefore the value we require for a must satisfy the relation:

$$F(a) = 2f_d - 2f_g = 0. \tag{14}$$

As in [2], [3], we shall use the following recurrent formula for a:

$$a_{n+2} = a_{n+1} + \Delta a, \tag{15}$$

where:

$$\Delta a = -\,(a_{n+1} - a_n)\,[\Delta f_{n+1}/(\Delta_{n+1} - \Delta f_n)], \tag{16}$$

and:

$$\Delta f_n = F(a_n) = 2f_d - 2f_g. \tag{17}$$

This iterative process ends when the condition:

$$|\Delta f_{n+1}| < \varepsilon \tag{18}$$

is satisfied. Here ε is a previously given positive quantity (for example, $\varepsilon = 10^{-9}$).

The correction Δa can be calculated with (16) only if two approximations of a are known. For the first approximation we shall consider $a_1 = 1.00$, and for the second one we take (as in [2]) $\Delta a = 0.01$; so, $a_2 = 1.01$. Then, the calculations continue according to Equations (15) — (18).

The orbital elements i and Ω can be determined from the relations [4]:

$$\begin{aligned} [a_1a_2] \sin i \sin \Omega &= y_1z_2 - y_2z_1, \\ [a_1a_2] \sin i \cos \Omega &= x_1z_2 - x_2z_1, \\ [a_1a_2] \cos i &= x_1y_2 - x_2y_1, \end{aligned} \tag{19}$$

where:

$$[a_1a_2] = a^2 \sin (2f). \tag{20}$$

It can be easily seen that the circular orbits become undetermined if $i = 0°$ (the case of equatorial orbits) or if $2f \in \{0°, 180°\}$ i.e. the two observations are coinciding or are performed in diametrically opposed directions. If we use observations from only one station and from only one transit, then we can consider $0° < 2f < 180°$ nearly always.

Finally, the argument of latitude $u_1 = u(t_1)$ can be determined from the relations:

$$\begin{aligned} a \sin u_1 &= z_1 \operatorname{cosec} i, \\ a \cos u_1 &= x_1 \cos \Omega + y_1 \sin \Omega. \end{aligned} \tag{21}$$

On the basis of the above described algorithm, we have elaborated a program in FØRTRAN IV for the computer FELIX C—256 of the Cluj-Napoca University Calculation Center.

3. **Numerical Applications.** The program was elaborated on the basis of Zeinalov's algorithm; it was often applied to the observations of satellites with quasi-circular orbits, performed at the tracking station 1132 Cluj-Napoca. The orbital elements we obtained are in a good agreement with the similar values contained in the ephemerides received by the Station from Appleton Laboratory in Slough and University of Aston in Birmingham (England).

We give here an example of such a determination. The satellite we have chosen is 1978—64 *A* (Seasat), because of its quasi-circular orbit (e $\cong$ 0.00028).

The geographical coordinates of the station are:

$$\begin{aligned} \varphi &= 46°42'48''.0 \text{ North}; \\ \lambda &= 23°35'52''.5 \text{ East}; \\ H &= 750 \ m. \end{aligned}$$

The observations were performed on 18 September 1980. The two positions are:

$$t_1 = 02^h26^m57^s \cdot 4\ UT; \quad A_1 = 346°55'; \quad h_1 = 35°25';$$
$$t_2 = 02^h30^m29^s \cdot 5\ UT; \quad A_2 = 256°15'; \quad h_2 = 29°00'.$$

The orbital elements, calculated by using the above presented method are:

Epoch: MJD 44500.102053 (18.IX.1980, $02^h26^m57^s \cdot 4\ UT$);

$$a = 1.123075\ a_e;$$
$$i = 108°1;$$
$$\Omega = 207°003;$$
$$u_1 = 121°.079.$$

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ASUPRA DETERMINĂRII ORBITELOR CIRCULARE ALE SATELIŢILOR ARTIFICIALI DIN OBSERVAŢII OPTICE VIZUALE

(Rezumat)

În lucrare se arată posibilitatea folosirii observaţiilor optice vizuale la determinarea orbitelor circulare ale sateliţilor artificiali. În acest scop este recomandat algoritmul lui Zeinalov, pe baza căruia s-a elaborat un program de calcul pentru calculatorul FELIX C−256 al Centrului de calcul al Universităţii „Babeş-Bolyai" din Cluj-Napoca. Ca exemplu numeric sînt date elementele orbitale ale satelitului Seasat (1978−64 *A*), determinate pe bază de observaţii optice vizuale efectuate la Staţia nr. 1132 din Cluj-Napoca.

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Lei 20