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60 DE ANI
DE ACTIVITATE A ŞCOLII MATEMATICE CLUJENE

Şcoala matematică clujeană are vechi şi bogate tradiţii. Încă de la înfiinţarea Universităţii româneşti din Cluj (1919), învăţămîntul şi cercetarea în domeniul matematicii au fost stimulate de prezenţa marelui matematician român Dimitrie Pompeiu, care a organizat şi a condus primul seminar de matematică, reuşind să mobilizeze o seamă de matematicieni români şi străini, într-o atmosferă de cordială colaborare. În acest cadru s-au publicat mai multe cărţi şi monografii, inclusiv revista de prestigiu internaţional „Mathematica", fondată în 1929. Sub auspiciile acestei reviste, s-a desfăşurat la Cluj, în acelaşi an 1929, primul congres al matematicienilor români, cu remarcabile participări de peste hotare. În perioada interbelică, învăţămîntul matematic clujean a fost slujit cu mult devotament de dascăli valoroşi şi entuziaşti ca Nicolae Abramescu, Aurel Angelescu, Theodor Angheluţă, Gheorghe Bratu, Gheorghe Iuga şi Petre Sergescu.

După 23 August 1944, sub impulsul noilor transformări sociale din ţara noastră, conduse de Partidul Comunist Român, cercetările de matematică şi învăţămîntul matematic în cadrul Universităţii clujene a cunoscut o mare dezvoltare. La consolidarea şcolii matematice clujene un aport substanţial l-au adus profesorii Gheorghe Călugăreanu, Tiberiu Popoviciu şi Dumitru V. Ionescu, care timp de mai multe decenii s-au impus printr-o activitate prodigioasă, imprimînd un spirit modern în cercetarea matematică, prin abordarea unor probleme noi de mare însemnătate teoretică şi practică. Gh. Călugăreanu s-a impus prin cercetări în domeniul teoriei funcţiilor şi topologiei. El a creat teoria invarianţilor de prelungire analitică şi a obţinut rezultate remarcabile în teoria funcţiilor univalente şi în teoria nodurilor. T. Popoviciu a creat la Cluj prima şcoală românească de analiză numerică. A obţinut rezultate fundamentale în teoria aproximării funcţiilor şi a stimulat cercetările matematice aplicative, în calitatea sa de director al fostului Institut de calcul al Academiei R. S. România. Profesorul D. V. Ionescu a iniţiat la Cluj cercetările de teoria ecuaţiilor diferenţiale şi integrale. Este creatorul unei metode generale de construire a formulelor de cuadratură. Cercetările de geometrie au fost impulsionate de profesorii Tiberiu Mihăilescu (în geometria diferenţială şi proiectivă) şi Eugen Gergely (în geometria neeuclidiană). Cercetările de algebră modernă au fost iniţiate la Universitatea clujeană de profesorul Gheorghe Pic, iar cele de mecanică de academicianul Caius Iacob. Activitatea în cadrul Observatorului astronomic, înfiinţat de Gh. Bratu şi Gh. Demetrescu, a fost dezvoltată de profesorii Ioan Armeanca, Ioan Curea şi Gheorghe Chiş.

În ultimii ani cercetarea matematică clujeană s-a dezvoltat şi diversificat. La domenii de cercetare cu bogată tradiţie s-au adăugat domenii noi, ca cele de cercetări operaţionale, de mecanică cerească şi de informatică. În cadrul celor 19 seminarii de cercetare care funcţionează în prezent pe lîngă colectivele de catedră şi Centrul de calcul al Universităţii, sînt abordate cu succes o serie de teme de cercetare fundamentală şi teme legate de producţie. Avînd un rol bine cunoscut, seminariile de cercetare mobilizează toate forţele matematice din componenţa Institutului de matematică Cluj-Napoca, precum şi doctoranzi şi studenţi.

Rezultatele cercetărilor clujeni sînt valorificate atît în cele două reviste proprii („Studia Universitatis Babeş-Bolyai, Mathematica" şi „Mathematica – Revue d'analyse numérique et de théorie de l'approximation", cu cele două serii: Mathematica şi L'analyse numérique et la théorie de l'approximation), cît şi în alte publicaţii de specialitate din ţară şi de peste hotare. Tot în cadrul Facultăţii de matematică este redactată revista „Matematikai Lapok" – ediţia în limba maghiară a revistei „Gazeta Matematică".

Monografiile şi culegerile de lucrări, prezentate la diferitele manifestări ştiinţifice organizate de facultate, sînt o altă formă de valorificare a cercetărilor proprii.

Activitatea contractuală, legată direct de aplicaţiile cercetărilor matematice în producţie, a cunoscut în ultimul timp o mare dezvoltare.

Rezultatele cercetării au intrat în circuitul internaţional atît prin publicaţiile amintite, cît şi prin organizarea numeroaselor manifestări ştiinţifice clujene cu prestigioase participări internaţionale.

Toate aceste realizări, ca şi rezultatele obţinute în pregătirea atîtor promoţii de absolvenţi, dau şcolii clujene de matematică un binemeritat prestigiu.

Aceste realizări au fost posibile datorită condiţiilor create şi sprijinului acordat de către conducerea de partid şi de stat dezvoltării învăţămîntului şi cercetării ştiinţifice din ţara noastră.

Alături de întregul popor român, cadrele didactice şi studenţii facultăţii noastre, împreună cu toţi cercetătorii Institutului de matematică Cluj-Napoca, omagiază evenimentul deosebit pe care-l trăim, împlinirea a 60 de ani de la crearea Partidului Comunist Român, şi se angajează să-şi dedice toată capacitatea şi puterea de muncă creşterii prestigiului şcolii matematice româneşti.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXVI, 1, 1981

SOME PROPERTIES OF PROJECTORS IN FINITE π-SOLVABLE GROUPS

RODICA COVACI

1. The purpose of this paper is to prove the existence and the conjugacy of projectors in finite π-solvable groups. The same properties were proved in [2] for covering subgroups in finite π-solvable groups.

All groups considered are finite.

We resume the notions of the paper in the following definitions:

DEFINITION 1.1. *a)* A class $\mathfrak{X}$ of groups is a *homomorph* if $\mathfrak{X}$ is closed under homomorphisms.

b) A group G is *primitive* if there is a maximal subgroup W of G with $\mathrm{core}_G\, W = 1$, where $\mathrm{core}_G\, W = \bigcap\{W^g / g \in G\}$.

c) A homomorph $\mathfrak{X}$ is a *Schunck class* if it is *primitively closed*, i.e. if any group G, all of whose primitive factor groups are in $\mathfrak{X}$, is itself in $\mathfrak{X}$.

d) Let $\mathfrak{X}$ be a class of groups, G a group and H a subgroup of G. H is called $\mathfrak{X}$-*maximal* in G if $H \in \mathfrak{X}$ and for any H^* with $H \leqslant H^* \leqslant G$, $H^* \in \mathfrak{X}$, we have $H = H^*$. The subgroup H is an $\mathfrak{X}$-*projector* of G if for any $N \trianglelefteq G$, HN/N is $\mathfrak{X}$-maximal in G/N. At last, H is an $\mathfrak{X}$-*covering subgroup* of G if $H \in \mathfrak{X}$ and $H \leqslant K \leqslant G$, $K_0 \trianglelefteq K$, $K/K_0 \in \mathfrak{X}$ implies $K = HK_0$.

DEFINITION 1.2. Let π be a set of primes and π' the complement to π in the set of all primes.

a) A group is π-*solvable* if every chief factor is either a solvable π-group or a π'-group. If π is the set of all primes, we obtain the notion of solvable group.

b) We shall call π-*homomorph* (π-*Schunck class*) a π-*closed* homomorph (Schunck class), i.e. a homomorph (Schunck class) $\mathfrak{X}$ with the property:

$$G/O_{\pi'}(G) \in \mathfrak{X} \Rightarrow G \in \mathfrak{X},$$

where $O_{\pi'}(G)$ denotes the largest normal π'-subgroup of G.

We use in our considerations some important theorems of R. BAER [1] which we give below.

THEOREM 1.3. *A solvable minimal normal subgroup of a group is abelian.*

THEOREM 1.4. *If the group G is primitive and G has a $\neq 1$ normal solvable subgroup, then G has one and only one minimal normal subgroup.*

THEOREM 1.5. *If W is a maximal subgroup of G with $\mathrm{core}_G\, W = 1$ and N is a minimal normal subgroup of G, then $G = WN$ and $W \cap N = 1$.*

THEOREM 1.6. *If G has a $\neq 1$ normal solvable subgroup and a maximal subgroup S with $\mathrm{core}_G\, S = 1$, then:*

a) *the existence of a $\neq 1$ normal solvable subgroup of S implies the existence of a normal subgroup $N \neq 1$ of S with $(|N|, |G : S|) = 1$;*

b) *if S has a normal subgroup $N \neq 1$ with $(|N|, |G : S|) = 1$, then S is conjugate to any maximal subgroup T of G with $\mathrm{core}_G\, T = 1$.*

2. In this section, we give some properties of projectors.

Let $\mathfrak{X}$ be a homomorph and G a group. It is easy to see that any $\mathfrak{X}$-covering subgroup of G is an $\mathfrak{X}$-projector of G, but conversely not. Thus, the notion of projector is more general than the notion of covering subgroup, therefore poorer in properties.

Definition 1.1. d) implies two immediate properties of projectors, given in:

PROPOSITION 2.1. *Let $\mathfrak{X}$ be a homomorph, G a group and H an $\mathfrak{X}$-projector of G. Then:*

a) *for any $N \trianglelefteq G$, HN/N is an $\mathfrak{X}$-projector of G/N;*

b) *for any $x \in G$, H^x is an $\mathfrak{X}$-projector of G.*

Henceforth, every group considered will be finite and π-solvable. In these conditions, we can prove further properties of projectors.

The conjugacy and existence of convering subgroups in finite π-solvable groups are already established in [2]. If $\mathfrak{X}$ is a π-Schunck class, any π-solvable group has $\mathfrak{X}$-covering subgroups (see [2], 2.2.). Because an $\mathfrak{X}$-covering subgroup is also an $\mathfrak{X}$-projector, we have:

THEOREM 2.2. *If $\mathfrak{X}$ is a π-Schunck class, any π-solvable group has $\mathfrak{X}$-projectors.*

It remains an open problem, namely if the π-Schunck classes are the only π-homomorphs relative to which the π-solvable groups have projectors.

The main result of this paper is the proof of conjugacy of projectors in π-solvable groups. In preparation for this result we give a lemma which generalizes [3], 5.11.

LEMMA 2.3. *Let $\mathfrak{X}$ be a π-Schunck class, G a π-solvable group and A an abelian normal subgroup of G with $G/A \in \mathfrak{X}$. Then:*

(1) *there is a subgroup S of G with $S \in \mathfrak{X}$ and $AS = G$;*

(2) *if S_1 and S_2 are $\mathfrak{X}$-maximal subgroups of G with $AS_1 = G = AS_2$ then S_1 and S_2 are conjugate in G.*

Proof. (1). Let $\mathfrak{S} = \{S^*/S^* \leqslant G, AS^* = G\}$. Since $G \in \mathfrak{S}$, $\mathfrak{S} \neq \emptyset$. Considering $\mathfrak{S}$ ordered by inclusion and applying ZORN's lemma, $\mathfrak{S}$ has a minimal element S. We shall prove that $S \in \mathfrak{X}$.

Put $D = S \cap A$. Then $D \trianglelefteq G$. Let W be a maximal subgroup of S. We have $D \leqslant W$. Indeed, if we suppose that is not subgroup of W, we obtain $DW = S$, hence $AW = ADW = AS = G$, which means $W \in \mathfrak{S}$, in contradiction with the minimality of S in $\mathfrak{S}$. Put $N = \mathrm{core}_G\, W$. Clearly, $D \leqslant N$. Then $S/N \simeq (S/D)/(N/D)$. Because $S/D = S/S \cap A \simeq AS/A = G/A \in \mathfrak{X}$, we deduce, $\mathfrak{X}$ being a homomorph, $S/N \in \mathfrak{X}$.

For any primitive factor group S/N of S, we can find a maximal subgroup W of S such that $N = \mathrm{core}_S\, W$. But this means that any primitive factor group of S is in $\mathfrak{X}$, which implies, by the primitively closure of $\mathfrak{X}$, that $S \in \mathfrak{X}$.

(2). We prove by induction on $|G|$. We distinguish two cases:

a) $G \in \mathfrak{X}$. S_1 and S_2 being $\mathfrak{X}$-maximal subgroups of G, $S_1 = S_2 = G$ and the lemma is proved.

b) $G \notin \mathfrak{X}$. It means that there is a primitive factor group G/N with $G/N \notin \mathfrak{X}$. We have $NS_1 \neq G$ and $NS_2 \neq G$. It is not difficult to see that AN/N is a minimal normal subgroup of G/N. Put $M = AN$. Clearly, $(NS_i)M = G$, $i = 1, 2$.

M/N is a solvable π-group. Indeed, M/N being a minimal normal subgroup of G/N, M/N is a chief factor of the π-solvable group G, hence M/N is a solvable π-group or a π'-group. If M/N is a π'-group, $M/N \leqslant 0_{\pi'}(S/N)$ and $(G/N)\ /\ 0_{\pi'}(G/N) \simeq ((G/N)/(M/N))/(0_{\pi'}(G/N)/(M/N)$; but $(G/N)/(M/N) \simeq G/M = ANS_1/AN \simeq S_1/S_1 \cap (AN) \in \mathfrak{X}$; it follows that $(G/N)\ /\ 0_{\pi'}(G/N) \in \mathfrak{X}$, which implies, by the π-closure of $\mathfrak{X}$, the contradiction $G/N \in \mathfrak{X}$. Thus, M/N is a solvable π-group.

By 1.3., M/N is abelian. This and $NS_i < G$, $i = 1,2$, imply that NS_1/N and NS_2/N are maximal subgroups of G/N. Further, $\mathrm{core}_{G/N}NS_i/N=1$, $i = 1,2$, because, by 1.4., G/N has one and only one minimal normal subgroup M/N, and so, the assumption $\mathrm{core}_{G/N}NS_i/N \neq 1$ implies $M/N \leqslant \mathrm{core}_{G/N}NS_i/N$, hence $G/N = NS_i/N \cdot M/N = NS_i/N \cdot \mathrm{core}_{G/N}NS_i/N = NS_i/N$, in contradiction with $NS_i \neq G$.

Let us prove that NS_1/N and NS_2/N are conjugate in G/N. If $NS_1/N=1$, we have $G/N = NS_1/N \cdot M/N = M/N$; but $G/N = NS_2/N \cdot M/N$; hence $NS_2/N \cdot M/N = M/N$, that is $NS_2/N \leqslant M/N$; it follows that $NS_2/N \cap M/N = NS_2/N$; but, on the other side, 1.5. implies $NS_2/N \cap M/N = 1$; we conclude that $NS_2/N = 1$. This shows that $NS_1/N = 1$ implies $NS_2/N = 1$ and so, NS_1/N and NS_2/N are conjugate in G/N in this case. Let us suppose now that $NS_1/N \neq 1$. We shall use 1.6. We know that $M/N \neq 1$ is a normal solvable subgroup of G/N and NS_1/N is a maximal subgroup of G/N with $\mathrm{core}_{G/N}NS_1/N = 1$. Let us prove that NS_1/N has a normal subgroup $L/N \neq 1$ with $(|L/N|, |G/N : NS_1/N|) = 1$. Indeed, NS_1/N being $\neq 1$, let K/N be a minimal normal subgroup of NS_1/N. K/N is either a solvable π-group, or a π'-group. If K/N is a solvable π-group, then, by 1.6.a), there is a normal subgroup $L/N \neq 1$ of NS_1/N with $(|L/N|, |G/N : NS_1/N|) = 1$. If K/N is a π'-group, then even $K/N \neq 1$ is a normal subgroup of NS_1/N with $(|K/N|, |G/N : NS_1/N|) = 1$. Applying now 1.6.b), NS_1/N and NS_2/N are conjugate in G/N. Hence NS_1 and NS_2 are conjugate in G.

Put $G^* = NS_1 = (NS_2)^g = NS_2^g$, where $g \in G$, and $A^* = A \cap G^*$. We apply the induction for G^*. Let us notice that A^* is an abelian normal subgroup of G^*, with $G^*/A^* \in \mathfrak{X}$ and S_1, S_2^g are $\mathfrak{X}$-maximal subgroups of G^*, with $A^*S_1 = (A \cap G^*)S_1 = S_1(A \cap G^*) = (S_1A) \cap G^* = G \cap G^* = G^*$ and $A^*S_2^g = (A \cap G^*)S_2^g = S_2^g(A \cap G^*) = (S_2^g A) \cap G^* = G \cap G^* = G^*$. By induction, S_1 and S_2^g are conjugate in G^*, hence S_1 and S_2 are conjugate in G.

And now, the main result of the paper:

THEOREM 2.4. *If $\mathfrak{X}$ is a π-Schunck class, then any two $\mathfrak{X}$-projectors of a π-solvable group G are conjugate in G.*

Proof. By induction on $|G|$.

Let S_1 and S_2 be two $\mathfrak{X}$-projectors of G and M a minimal normal subgroup of G. We put $\bar{S}_1 = MS_1$ and $\bar{S}_2 = MS_2$.

$\bar{S}_1$ and $\bar{S}_2$ are conjugate in G. Indeed, by 2.1.a), $\bar{S}_1/M$ and $\bar{S}_2/M$ are $\mathfrak{X}$-projectors in G/M and hence, by induction, they are conjugate in G/M. But this means that $\bar{S}_1$ and $\bar{S}_2$ are conjugate in G, i.e. $MS_1 = \bar{S}_1 = \bar{S}_2^g = MS_2^g$, with $g \in G$.

In order to prove that S_1 and S_2 are conjugate in G, let us notice that, M being a minimal normal subgroup of the π-solvable group G, two cases can happen:

1) M is a solvable π-group. By 1.3., M is abelian. Let us show that we are in the hypotheses of lemma 2.3.(2): $\mathfrak{X}$ is a π-Schunck class, $\bar{S}_1$ is a π-solvable group, M is an abelian normal subgroup of S_1 with $S_1/M = MS_1/M \simeq S_1/M \cap S_1 \in \mathfrak{X}$ and we have $\bar{S}_1 = MS_1 = MS_2^g$, where S_1 and S_2^g are $\mathfrak{X}$-maximal subgroups in $\bar{S}_1$. It follows that S_1 and S_2^g are conjugate in $\bar{S}_1$, hence S_1 and S_2 are conjugate in G.

2) M is a π'-group. This leads to $M \leqslant 0_{\pi'}(\bar{S}_1)$. Then $\bar{S}_1/0_{\pi'}(\bar{S}_1) \simeq (\bar{S}_1/M)/(0_{\pi'}(\bar{S}_1)/M)$. But $\bar{S}_1/M \in \mathfrak{X}$. We deduce that $\bar{S}_1/0_{\pi'}(S_1) \in \mathfrak{X}$, hence, $\mathfrak{X}$ being π-closed, $\bar{S}_1 \in \mathfrak{X}$. By the $\mathfrak{X}$-maximality of S_1 and S_2^g in $\bar{S}_1$, $S_1 = \bar{S}_1 = S_2^g$, where $g \in G$. The theorem is completely proved.

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UNELE PROPRIETĂŢI ALE PROIECTORILOR ÎN GRUPURI FINITE π-RESOLUBILE

(Rezumat)

În lucrare se arată că dacă $\mathfrak{X}$ este o π-clasă Schunck de grupuri finite π-resolubile, atunci în orice grup finit π-resolubil există $\mathfrak{X}$-proiectori şi doi $\mathfrak{X}$-proiectori sînt conjugaţi. Se generalizează astfel la grupurile π-resolubile existenţa şi conjugarea proiectorilor în grupuri finite resolubile demonstrate de W. Gaschutz în [3]. Proprietăţi similare pentru subgrupurile acoperitoare ale grupurilor finite π-resolubile, subgrupuri mai particulare decît proiectorii, au fost deja demonstrate în [2].

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PROPRIETĂŢI ALGEBRICE ALE UNEI CLASE DE OPERATORI CONVOLUTIVI POZITIVI

GR. MOLDOVAN

1. Operatori convolutivi pozitivi pentru funcţii de o singură variabilă. Polinoamele lui Bernstein care au servit la demonstarea celebrei teoreme a lui Weierstrass au stîrnit mult interes în rîndul matematicienilor, care le-au consacrat numeroase studii.

Teoremele lui Korovkin de convergenţă a şirurilor de operatori liniari şi pozitivi au folosit în demonstrarea lor, prin analogie, proprietăţi remarcabile ale polinoamelor lui Bernstein considerate acum ca şir de operatori liniari şi pozitivi.

Construcţia polinoamelor lui Bernstein poate fi considerată ca plecînd de la identitatea simplă pe care o exprimă formula binomului lui Newton. Putem construi, astfel, pornind de la anumite identităţi, operatori liniari.

Într-un cadru mai general, pe mulţimea polinoamelor putem defini nişte convoluţii discrete. Fie $\mathfrak{L}(E)$, $E \subseteq R$, R— mulţimea numerelor reale, mulţimea tuturor şirurilor $P = (P_n)_{n\in N}$, N — mulţimea întregilor nenegativi, de polinoame; $P_n : E \to R$. Această mulţime formează un spaţiu liniar peste R în raport cu operaţiile obişnuite.

Fie $P^{(1)}$, $P^{(2)} \in \mathfrak{L}(E)$. Cu cele două şiruri formăm un şir de funcţii $(Q_n)_{n\in N}$, $Q_n : E \times E \to R$ în modul următor:

$$Q_n(u, v) = \sum_{k=0}^{n} P_k^{(1)}(u)\, P_{n-k}^{(2)}(v), \quad n \in N, \ u, v \in E \tag{1}$$

Definiţie. Relaţia (1) determină o operaţie de convoluţie „$*$" pe mulţimea $\mathfrak{L}(E)$:

$$Q = P^{(1)} * P^{(2)}, \ Q = (Q_n)_{n\in N}. \tag{2}$$

O clasă particulară de convoluţie o constituie convoluţia binomă care are proprietatea

$$P_n(u + v) = \sum_{k=0}^{n} P_k(u)\, P_{n-k}(v), \quad u, \ v \in E \text{ şi } u + v \in E. \tag{3}$$

Dacă notăm familia de polinoame cu această proprietate cu $P_1(E)$, avem $\mathfrak{L}_1(E) \subset \mathfrak{L}(E)$.

Operaţiei de convoluţie (2)

$$* : \mathfrak{L}P(E) \times \mathfrak{L}(E) \to \mathfrak{F}(E^2)$$

pentru o funcţie $f: X \to R$, $X \subseteq R$ şi $\Delta = (x_k^{(n)})$, $k = \overline{0, n}$, $n = 0, 1, \ldots$ îi asociem următoarea operaţie

$$\underset{(f,\Delta)}{*} : \mathfrak{L}(E) \times \mathfrak{L}(E) \to \mathfrak{F}_1(E^2)$$

care face ca perechii $(P^{(1)}, P^{(2)})$ de elemente din $\mathfrak{X}(E)$ să-i corespundă un element în $\mathfrak{F}(E^2)$ prin intermediul următoarei relații convolutive

$$Q_n(f;\ u,\ v) = \sum_{k=0}^{n} f(x_k^{(n)}) P_k^{(1)}(u) P_{n-k}^{(2)}(v), \tag{4}$$

$$Q_n(1\ ;\ u,\ v) \neq 0$$

DEFINIȚIE. Operatorii

$$L_n(\cdot\ ;\ P^{(1)},\ P^{(2)}\ ;\ x) : R^X \to R^X$$

$$L_n(f;\ P^{(1)},\ P^{(2)}\ ;\ x) = A_n(x) \sum_{k=0}^{n} f(x_k^{(n)}) P_k^{(1)}(u(x)) P_{n-k}^{(2)}(v(x)),$$

unde $u : X \to E$ și $v : X \to E$ sînt două funcții arbitrare, iar

$$A_n(x) = [Q_n(1\ ;\ u(x),\ v(x))]^{-1} < \infty$$

îi numim operatori *convolutivi pozitivi.*

Observație. 1) Operatorul (5) este un operator liniar.

2) Dacă operatorii convolutivi (5) îi considerăm pozitivi, atunci pe aceștia îi numim *operatori convolutivi pozitivi.*

Particularizînd șirurile de polinoame $P^{(1)}$ și $P^{(2)}$ obținem diferite exemple de operatori convolutivi. Astfel, dacă

$$P^{(1)} = P^{(2)} = (P_n(x))_{n \in N}\ ;\ P_n(x) = \frac{x^n}{n!} \tag{6}$$

iar $u(x) = x$, $v(x) = 1 - x$, $x_k^{(n)} = \frac{k}{n}$ obținem operatorii lui Bernstein, pentru altă distribuție de noduri alți operatori. Apoi, dacă

$$P^{(1)} = P^{(2)} = (P_n(x))_{n \in N}$$

$$P_n(x) = \binom{x}{n} = \frac{x(x-1)\ \ldots\ (x-n+1)}{n!} \tag{7}$$

iar $u(x) = x$, $v(x) = 1 - x$, $x_k^{(n)} = \frac{k}{n}$ obținem operatorii lui D. D. S t a n c u [5]. În sfîrșit, dacă $P^{(1)} = P^{(2)} = (P_n(x,\ y))_{n \in N}$

$$P_n\ (x,\ y) = \frac{x}{x + yk} \binom{x + yk}{k} \tag{8}$$

obținem o altă clasă de operatori convolutivi pozitivi considerați în [1, 2]. Aceşt operatori generalizează operatorii lui Bernstein și Stancu.

Se pot considera și alte exemple de operatori liniari și pozitivi de acest fel. Una din principalele probleme ce se studiază în legătură cu acești operatori este problema convergenţei. Această problemă a fost studiată în multe lucrări a diferiţilor autori.

În cele ce urmează ne propunem să studiem cîteva proprietăţi algebrice a operatorilor convolutivi pozitivi care au o anumită particularitate.

2. **O proprietate de automorfism a unei clase de operatori convolutivi pozitivi de tip polinomial.** Vom considera în acest paragraf următorul operator convolutiv pozitiv

$$L_n(f;\ x) = A_n \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} P_k(x) P_{n-k}(1 - x) \tag{9}$$

asociat relaţiei convolutive

$$P_n(u + v) = \sum_{k=0}^{n} \binom{n}{k} P_k(u) P_{n-k}(v), \text{ deci } A_n = [P_n(1)]^{-1} \tag{10}$$

în care s-a luat $u(x) = x$ şi $v(x) = 1 - x$.

Presupunem că $P_k(u)$ sînt polinoame de grad k cu rădăcini reale şi coeficientul lui x la puterea cea mai mare este 1.

LEMĂ. *Dacă*

$$P_k(u) = P_i(u) \cdot P_{k-i}(u), \quad i = 0,\ 1,\ \ldots,\ k$$

atunci avem următoarea relaţie

$$\sum_{k=0}^{n} k^p \binom{n}{k} P_k(u) P_{n-k}(v) = \tag{11}$$

$$= \sum_{i=1}^{p} S_p^i n(n - 1)\ \ldots\ (n - i + 1) P_i(u) P_{n-i}(u + v), \quad 0 \leqslant p \leqslant n$$

unde S_p^i sînt numerele lui Stirling de speţa a doua.

Demonstraţie. Numerele lui Stirling de speţa a doua se definesc prin intermediul relaţiei

$$k^p = \sum_{i=1}^{p} S_p^i k(k - 1)\ \ldots\ (k - i + 1). \tag{12}$$

Aceste numere se bucură de o serie întreagă de proprietăţi interesante. Calculul lor se poate face folosind o formulă de recurenţă

$$S_{n+1}^i = S_n^{i-1} + i S_n^i. \tag{13}$$

Observăm că se poate scrie relaţia

$$k(k - 1) \ldots (k - i + 1) \binom{n}{k} = n(n - 1) \ldots (n - i + 1) \binom{n - i}{k - i}. \tag{14}$$

Pentru $i = 3$ relaţia se verifică uşor

$$k(k-1)(k-2)\frac{n(n-1)\ldots(n-k+1)}{k!} =$$

$$= n(n-1)(n-2)\frac{(n-3)(n-4)\ldots(n-k+1)}{(k-3)!}.$$

Înlocuind în membrul stîng al relaţiei (11) pe care-l notăm cu S obţinem

$$S = \sum_{k=0}^{n} k^{p}\binom{n}{k} P_k(u)P_{n-k}(v) =$$

$$\sum_{i=1}^{p} S_p^i n(n-1)\ldots(n-i+1)\sum_{k=i}^{n}\binom{n-i}{k-i} P_k(u)\ P_{n-k}(v) =$$

$$= \sum_{i=1}^{p} S_p^i n(n-1)\ \ldots\ (n-i+1)\ P_i(u)\sum_{k=i}^{n}\binom{n-i}{k-i} P_{k-i}(u)P_{n-i-(k-i)}(v) =$$

$$= \sum_{i=1}^{p} S_p^i n(n-1)\ \ldots\ (n-i+1)P_i(u)\sum_{k=0}^{n-i}\binom{n-i}{k} P_k(u)\ P_{n-i-k}(v).$$

În ultimul termen ţinînd seama de relaţia (10), adică

$$\sum_{k=0}^{n-i}\binom{n-i}{k} P_k(u)\ P_{n-i-k}(v) = P_{n-i}(u+v)$$

rezultă

$$S = \sum_{i=1}^{p} S_p^i n(n-1)\ \ldots\ (n-i+1)\ P_i(u)P_{n-i}(u+v)$$

ceea ce demonstrează lema enunţată.

Să notăm cu $\mathfrak{L}_p$ mulţimea polinoamelor de grad cel mult p. Această mulţime formează un spaţiu vectorial finit dimensional ($p+1$ dimensional) cu baza $e_k = x^k$, $k = \overline{0, p}$.

Operatorul convolutiv pozitiv (9) este de tip polinomial în ipoteza că $P_n(x)$ este un polinom. El este definit pe clasa funcţiilor continue pe intervalul $[0, 1]$, adică $C[0, 1]$ şi cu valori în aceeaşi mulţime. Avem deci $L_n : C[0, 1] \to C[0, 1]$.

Considerăm acum cazul cînd L_n este definit pe o submulţime a mulţimii $C[0, 1]$ şi anume mulţimea polinoamelor $\mathfrak{L}_r$ de grad cel mult r, $\mathfrak{L}_r \subset C[0, 1]$. Restricţia operatorului convolutiv pozitiv (9) la mulţimea $\mathfrak{L}_r$ este tot un polinom.

TEOREMA *Fie restricţia operatorului liniar L_n definit de (9) pentru mulţimea $\mathfrak{L}_r$, $0 \leqslant r \leqslant n$. Acest operator liniar polinomial realizează un automorfism în mulţimea $\mathfrak{L}_r$, adică*

$$L_n\mathfrak{L}_r = \mathfrak{L}_r \tag{14}$$

Demonstrație. Ținînd seama de lema precedentă avem pentru $0 \leqslant p \leqslant r$

$$L_n(t^p, x) = A_n \sum_{k=0}^{n} \frac{k^p}{n^p} \binom{n}{k} P_k(x) P_{n-k}(1-x) = \frac{A_n}{n^p} \sum_{k=0}^{n} k^p \binom{n}{k} \cdot$$
$$\cdot P_k(x) P_{n-k}(1-x)$$

deci

$$L_n(t^p; x) = \frac{A_n}{n^p} \sum_{i=1}^{p} S_p^i n(n-1) \ldots (n-i+1) P_i(x) P_{n-i}(1). \tag{15}$$

Cum $P_{n-i}(1)$ este o constantă iar $P_i(x)$ este un polinom de grad cel mult i rezultă că $L_n(t^p; x)$ este un polinom de grad p. Pe de altă parte, ținînd seama de liniaritatea operatorului L_n avem incluziunea

$$L_n \mathfrak{L}_r \subseteq \mathfrak{L}_r \tag{16}$$

unde prin $L_n\mathfrak{L}_r$ am notat mulțimea valorilor operatorilor L_n restrînsă la mulțimea $\mathfrak{L}_r$ a polinoamelor de grad cel mult r.

Pentru a demonstra incluziunea inversă este suficient să demonstrăm că matricea asociată operatorului L_n pentru baza $e_0, e_1, \ldots, e_r$; $e_k = x^k$, $k = 0, 1, \ldots, r$ este nesingulară.

Din lema precedentă și din (15) rezultă

$$L_n(t^p; x) = \frac{A_n}{n^p} \sum_{e=1}^{p} a_{ep} x^e$$

Coeficientul lui x^p adică $\frac{A_n}{n^p} a_{pp}$ este $\frac{A_n}{n^p} S_p^p n(n-1) \ldots (n-p+1) P_{n-p}(1)$

adică

$$\frac{A_n}{n^p} a_{pp} = A_n P_{n-p}(1) S_p^p \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \ldots \left(1 - \frac{p-1}{n}\right) \neq 0 \tag{17}$$

deci $L_n(t^p; x)$ este un polinom de grad efectiv p.

Matricea asociată operatorului L_n este o matrice triunghiulară, după cum rezultă de mai sus, iar elementele de pe diagonală sînt coeficienții lui x la puterea cea mai mare din $L_n(t^p; x)$. Or, acești coeficienți sînt toți diferiți de 0 după cum rezultă din (17), deci determinantul matricei patratice asociat operatorului L_n este diferit de zero, ceea ce implică faptul că aplicația L_n este bijectivă și în consecință avem:

$$L_n \mathfrak{L}_r \supseteq \mathfrak{L}_r$$

Această incluziune și cea stabilită la (16) demonstrează teorema enunțată.

Exemple. 1. *Operatorii lui Bernstein*. Dacă în (9) considerăm $P_k(x) = x^k$ acești operatori devin operatorii lui Bernstein; B_n, în consecință

$$B_n \mathfrak{L}_r = \mathfrak{L}_r$$

deci polinoamele lui Bernstein determină un automorfism pe mulțimea polinoamelor de grad cel mult r.

2. *Operatorii lui D. D. Stancu.* Dacă în (9) considerăm $P_k(x) = x(x-1) \ldots (x-k+1)$ obținem operatorii lui D. D. Stancu S_n și observînd că $P_k(x) = P_i(x)\, P_{k-i}(x)$, deci este realizată condiția din lemă, rezultă:

$$S_n\mathfrak{L}_r = \mathfrak{L}_r$$

deci operatorii lui Stancu determină un automorfism pe mulțimea polinoamelor de grad cel mult r.

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PROPRIETÉS ALGÉBRIQUES D'UNE CLASSES DES OPÉRATEURS CONVOLUTIFS POSITIFS

(Résumé)

Les résultats principaux de ce travail sont les suivants:

1. Si $P_k(u) = P_i(u) \cdot P_{k-i}(u)$ alors nous avons (11), où S_p^i, $i = 0, 1, \ldots, k$, sont les nombres de Stirling.

2. (Théorème). La restriction des opérateurs linéaires polynomials L_n, (9), à l'ensemble $\mathfrak{L}_r$, $0 < r < n$ réalise un automorphisme dans l'ensemble $\mathfrak{L}_r$, $L_n\mathfrak{L}_r = \mathfrak{L}_r$

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ASUPRA IMPLEMENTĂRII ALGORITMULUI DE ANALIZĂ SINTACTICĂ AL LUI DÖMÖLKI

Z. KÁSA

0. **Introducere**. În articolul de față descriem trei algoritmi, cu ajutorul cărora se poate genera matricea care reprezintă suportul de bază al analizei sintactice după algoritmul lui Dömölki [1]. De asemenea descriem acest algoritm de analiză sintatică pentru o clasă de limbaje în care se poate defini o relație de prioritate între anumite simboluri, care se vor numi operatori.

Algoritmul lui Dömölki realizează o analiză sintatică ascendentă, comparînd deodată simbolul citit cu părțile drepte ale tuturor producțiilor. Orice potrivire se marchează într-o secvență de biți prin cifra 1. După ce se marchează ultima potrivire într-o producție, se reduce partea dreaptă respectivă la simbolul neterminal care reprezintă partea stîngă a producției folosite.

1. **Operații cu secvențe de biți** Fie $B = \{0, 1\}$ şi $B^n = B \times B \times \ldots \times B$ (produsul cartezian al mulțimii B cu ea însăşi de n ori). Un element $\mathbf{x}$ din B^n se va numi secvență de biți de n componente, şi se va nota:

$$\mathbf{x} = (x_1, x_2, \ldots, x_n) \quad \text{sau } \mathbf{x} = x_1x_2 \ldots x_n.$$

Definim următoarele operații:

1°. Disjuncție:

$$\vee : B^n \times B^n \to B^n, \ \mathbf{z} = \mathbf{x} \vee \mathbf{y} \text{ unde } z_i = \begin{cases} 0 \text{ dacă } x_i = y_i = 0 \\ 1 \text{ în caz contrar} \end{cases}$$

2°. Conjuncție:

$$\wedge : B^n \times B^n \to B^n, \ \mathbf{z} = \mathbf{x} \wedge \mathbf{y} \text{ unde } z_i = \begin{cases} 1 \text{ dacă } x_i = y_i = 1 \\ 0 \text{ în caz contrar} \end{cases}$$

3°. Deplasare la stînga:

$\ll : B^n \times \mathbf{N} \to B^n$, unde $\mathbf{N}$ reprezintă mulțimea numerelor întregi nenegative. $\mathbf{z} = \mathbf{x} \ll m$ (deplasare la stînga cu m biți) unde

$$z_i = \begin{cases} x_{i+m} \text{ dacă } i = 1, 2, \ldots, n - m \\ 0 \quad \text{ dacă } i = n - m + 1, n - m + 2, \ldots, m \end{cases} \qquad m \leqslant n$$

4°. Deplasare la dreapta:

$\gg : B^n \times \mathbf{N} \to B^n$, $\mathbf{z} = \mathbf{x} \gg m$ (deplasare la dreapta cu m biți) unde

$$z_i = \begin{cases} 0 \quad \text{ dacă } i = 1, 2, \ldots, m \\ x_{i-m} \text{ dacă } i = m + 1, m + 2, \ldots, n \end{cases}, \qquad m \leqslant n$$

Secvența nulă se va nota cu $\mathbf{0}$. Deplasările la stînga şi la dreapta au aceeaşi prioritate în evaluarea expresiilor şi sînt mai prioritare față de disjuncție şi conjuncție. Conjuncția este prioritară față de disjuncție.

2. Analizorul sintactic al lui Dömölki. Producțiile gramaticii, care generează limbajul de analizat, se împart în două clase: producțiile care au în partea lor dreaptă un singur simbol (clasa producțiilor SLP — single-length-production) și producțiile care au mai mult de un simbol în partea lor dreaptă (clasa producțiilor MLP — multi-length-production).

Ilustrăm algoritmul lui Dömölki prin gramatica următoare:

$G = (N, T, P, E)$ unde

$N = \{E, T\}$ (neterminale)

$T = \{+, -, \times, i\}$ (terminale)

$P = \{E : : = E + T, \ E : : = E - T, \ E : : = T, \ E : : = T \times i,$

$T : : = i\}$ (producții)

E reprezintă simbolul start al gramaticii

Cu ajutorul producțiilor SLP construim matricea S (fig. 1a). Pentru orice producție SLP de forma $k : : = j$ punem $S_{jk} : = 1$ (simbolurile gramaticii se codifică prin numere naturale). Cu algoritmul lui W a r s h a l l [2] se calculează închiderea tranzitivă a matricei S (fig. 1b). Noua matrice S se completează cu coloanele nule corespunzătoare simbolurilor $+$, $-$, $\times$, i și elementele de pe diagonala principală se modifică în 1. Astfel se obține matricea S finală (fig. 1c).

	E	T
E	0	0
T	1	0
$+$	0	0
$-$	0	0
$\times$	0	0
i	0	1

F i g. 1a.

	E	T
E	0	0
T	1	0
$+$	0	0
$-$	0	0
$\times$	0	0
i	1	1

F i g. 1b.

	E	T	$+$	$-$	$\times$	i
E	1	0	0	0	0	0
T	1	1	0	0	0	0
$+$	0	0	1	0	0	0
$-$	0	0	0	1	0	0
$\times$	0	0	0	0	1	0
i	1	1	0	0	0	1

F i g. 1c.

Cu ajutorul acestei matrice S se completează matricea **M** corespunzătoare producțiilor MLP, matricea care va fi folosită în analiza sintactică. Fiecărui simbol din partea dreaptă a unei producții MLP îi corespunde o coloană în matricea

M, se copiază coloana corespunzătoare din S. Matricea **M** obținută este cea din fig. 2.

	$E ::=$			$E ::=$			$T ::=$		
	E	$+$	T	E	$-$	T	T	$\times$	i
E	1	0	0	1	0	0	0	0	0
T	1	0	1	1	0	1	1	0	0
$+$	0	1	0	0	0	0	0	0	0
$-$	0	0	0	0	1	0	0	0	0
$\times$	0	0	0	0	0	0	0	1	0
i	1	0	1	1	0	1	1	0	1

Fig. 2.

Fie m numărul liniilor, iar n numărul coloanelor din **M**. Fie $h_1, h_2, \ldots, h_p$ numărul simbolurilor din părțile drepte ale producțiilor MLP, producții considerate în ordinea în care apar în matricea **M**. Fie $r_k = 1 + \sum_{i=1}^{k} h_i$ pentru $k = 0, 1, \ldots, p - 1$ unde p reprezintă numărul producțiilor MLP. Prin convenție $r_0 = 1$. Algoritmul lui Dömölki folosește următoarele trei secvențe de biți:

$$\mathbf{b} = b_1 b_2 \ldots b_n \quad \text{unde} \quad b_j = \begin{cases} 1 & \text{dacă există un } k \text{ astfel încît } j = r_k \\ 0 & \text{în caz contrar} \end{cases}$$

$$\mathbf{e} = e_1 e_2 \ldots e_n \quad \text{unde} \quad \mathbf{e} = (\mathbf{b} \ll 1) \vee \underbrace{000 \ldots 01}_{n \text{ biți}}$$

$\mathbf{g} = q_1 q_2 \ldots q_n$ unde $q_j = 1$ dacă simbolul citit j s-a potrivit cu elementul $\mathbf{M}_{ji}$.

Algoritmul lui Dömölki:

1. Se pornește cu $\mathbf{g} = \mathbf{0}$.
2. Pentru fiecare simbol citit a se calculează

$$\mathbf{g} := ((\mathbf{g} \gg 1) \vee \mathbf{b}) \wedge \mathbf{M}[a, *]$$

unde $\mathbf{M}[a, *]$ reprezintă linia din **M** care corespunde simbolului a.

3. Dacă $\mathfrak{g} = 0$ atunci şirul de intrare este eronat.
Dacă $\mathfrak{g} \wedge \mathfrak{e} = 0$ atunci analiza se continuă cu citirea simbolului următ
Dacă $\mathfrak{g} \wedge \mathfrak{e} \neq 0$ atunci se poate face o reducere. Producţia în cauză e
determinată de poziţia cifrei 1 din $\mathfrak{g} \wedge \mathfrak{e}$. (există o singură cifră 1 în $\mathfrak{g} \wedge$

Exemplu: Să analizăm şirul de intrare $i + i - i$

$\mathfrak{b} = 100\,100\,100$ şi $\mathfrak{e} = 001\,001\,001$

La început: $\mathfrak{g} = 000\,000\,000$
S-a citit i: $\mathfrak{g} = 100\,100\,100$
S-a citit $+$: $\mathfrak{g} = 010\,000\,000$
S-a citit i: $\mathfrak{g} = 101\,100\,100$

$\mathfrak{g} \wedge \mathfrak{e} = 001\,000\,000$ şi $i + i$ se reduce la E, rămînc $E - i$, s-a folo prima producţie MLP. Se continuă cu $\mathfrak{g}$, care corespunde simbolului ante rior lui E în şirul de intrare. În cazul nostru nu există un astfel de simb E fiind primul din şir, deci $\mathfrak{g} = 000\,000\,000$

S-a citit E: $\mathfrak{g} = 100\,100\,100$
S-a citit $-$: $\mathfrak{g} = 000\,010\,000$
S-a citit i: $\mathfrak{g} = 100\,101\,100$

$\mathfrak{g} \wedge \mathfrak{e} = 000\,001\,000$ şi $E - i$ se reduce la E, s-a folosit producţia a dou Şirul de intrare s-a redus la E (simbolul start al gramaticii), analiza sinta tică s-a terminat cu succes.

Dacă aplicăm algoritmul de mai sus expresiei $i + i \times i$, se va reduce $i +$ la E şi expresia obţinută $E \times i$ nu se va putea reduce la E, algoritmul se va termina semnalînd eroare, cu toate că expresia iniţială este corectă.

Pentru a se evita astfel de situaţii, la orice semnalare de eroare se revin la ultima reducere, se schimbă în $\mathfrak{g}$ cifra 1, care a provocat reducerea, în 0 şi se continuă analiza mai departe.

Exemplu: Se analizează $i + i \times i$.

1. Iniţial: $\mathfrak{g} = 000\,000\,000$
2. S-a citit i: $\mathfrak{g} = 100\,100\,100$
3. S-a citit $+$: $\mathfrak{g} = 010\,000\,000$
4. S-a citit i: $\mathfrak{g} = 101\,100\,100$
5. $\mathfrak{g} \wedge \mathfrak{e} \neq 0$ deci $i + i \times i$ se reduce la $E \times i$.
6. S-a citit E: $\mathfrak{g} = 000\,000\,000$; $\mathfrak{g} = 100\,100\,100$
7. S-a citit $\times$: $\mathfrak{g} = 000\,000\,000$

4a. $\mathfrak{g} = 0$ reprezintă o eroare, se revine la 4, se modifică $\mathfrak{g}$.
5a. S-a citit $\times$: $\mathfrak{g} = 100\,100\,100$
6a. S-a citit i: $\mathfrak{g} = 000\,000\,010$; $\mathfrak{g} = 100\,100\,101$

$g \wedge e \neq 0$ şi se reduce expresia iniţială la $i + T$, noul g va fi cel din etapa 3.

7a. $g = 010\ 000\ 000$

8. S-a citit T: $g = 101\ 100\ 100$

$g \wedge e \neq 0$ şi se reduce expresia $i + T$ la E, şi algoritmul se termină cu succes.

În [1] se arată cum se poate evita revenirea în cazul limbajelor $LR(1)$. Ţinînd seamă de următorul simbol de intrare, în majoritatea cazurilor se poate evita revenirea. În acest caz trebuie marcate acele producţii care pot provoca reducere eronată. În cazul nostru aceste producţii sînt $E ::= E + T$ şi $E ::= E - T$. În cursul analizei, dacă reducerea ar trebui făcută cu o producţie marcată, ea nu se mai efectuează, ci ultimul simbol citit se consideră din nou ca simbol de intrare şi se continuă analiza. Astfel expresia $i + i + i$ se va reduce mai întîi la $i + E$ şi apoi la E.

În articolul de faţă dăm o variantă a algoritmului lui Dömölki pentru acele limbaje în care se poate defini o prioritate a operatorilor. Vom numi operator orice simbol a cărui neconsiderare în cursul analizei poate duce la o reducere eronată. Pentru exemplul nostru se poate defini următoarea prioritate a operatorilor: $+\ 1, -\ 1, \times\ 2$. Înaintea unei reduceri, se compară prioritatea ultimului operator citit cu prioritatea simbolului de intrare următor. Dacă prioritatea acestui simbol este mai mare, reducerea nu se efectuează. Dacă simbolul următor nu este operator, comparaţia este ineficientă şi se efectuează reducerea.

3. **Descrierea algoritmilor**. Algoritmii vor fi descrişi într-un limbaj asemănător limbajului Algol 60. Acest limbaj este uşor de înţeles, totuşi pentru a evita orice confuzie, descriem pe scurt deosebirile principale faţă de Algol 60.

a. **for** $i = 1$ **to** n **do** ... este echivalent cu **for** $i := 1$ **step** 1 **until** n **do** ...

b. **begin** ... **end repeat until** *cond*. Se repetă execuţia blocului delimitat de **begin** şi **end**, dacă condiţia *cond* este falsă, şi se trece la următoarea instrucţiune în caz contrar.

c. **while** *cond* **do** ... Se execută intrucţiunea sau blocul următor lui **do** atît timp cît condiţia *cond* este adevărată, şi se trece la instrucţiunea imediat următoare celui urmat de **do**, în caz contrar.

ALGORITMUL A. Creează matricea S iniţială (ca în fig. 1a). Simbolurile gramaticii sînt codificate cu valori naturale în ordinea în care sînt considerate în S. Pentru exemplul nostru avem codificarea:

E	T	$+$	$-$	$\times$	i
1	2	3	4	5	6

Producţiile SLP de forma: $k ::= k_1, k ::= k_2, \ldots, k ::= k_r$ sînt date la intrare sub forma: $k : k_1, k_2, \ldots, k_r$. Toate producţiile SLP sînt alăturate fără a lăsa spaţiu între ele; şirul lor se termină cu ; . În cazul nostru şirul de intrare va fi: 1 : 2,2 : 6. S[*, l] reprezintă coloana l a matricei S, $\mathbf{u} = 10 \ldots 0$ (m biţi), unde m reprezintă numărul liniilor din S, în cazul nostru $m = 6$, iar r ne dă numărul coloanelor nenule din S.

Descrierea algoritmului:

```
r: = 0;
begin read (a, b);
    if b = ':' then
            begin
                r: = r + 1;
                l: = a;
                S[*, l]: = 0
            end
        else
            S[*, l]: = (u » (a − 1)) ∨ S[*, l]
end
repeat until b = ';';
```

ALGORITMUL B. Realizează închiderea tranzitivă a matricei S. r obţinut din algoritmul A, reprezintă numărul coloanelor din S iniţială, m reprezintă numărul liniilor din S, **u** este identic cu cel din algoritmul A. *Descrierea algoritmului* (algoritmul lui Warshall)

```
for j = 1 to r do
  for i = 1 to m do
    if(u » (i − 1)) ∧ S[*, j] ≠ 0 then
                          S[*, j]: = S[*, i] ∨ S[*, j];
```

ALGORITMUL C. Construieşte matricea **M** din S şi din producţiile MLP. Cele din urmă se dau ca date de intrare sub forma: $i_1 i_2 \ldots i_{h_1} j_1 j_2 \ldots j_{h_2} \ldots$ unde $i_1 i_2 \ldots i_{h_1}$ reprezintă partea dreaptă a primei producţii MLP, $j_1 j_2 \ldots j_{h_2}$ a celei de a doua producţie MLP ş.a.m.d. Sfîrşitul se marchează cu ;. (Părţile stîngi vor fi memorate într-un tabel). $\mathbf{u} = 10\ldots0$ (m biţi) iar $\mathbf{v} = 10\ldots0$ (n biţi). În cazul nostru $m = 6$, $n = 9$, şirtul de intrare este 132142256; reprezentînd $E + TE - TT \times i$

Descrierea algoritmului:

```
for i = 1 to m do M[i, *] = 0;
h: = 1;
read(a);
while a ≠ ';' do begin
            for i = 1 to m do
                if ((u » (a − 1)) ∨ S[*, a]) ∧ (u»
                » (i − 1)) ≠ 0
                then M[i, *]: = M[i, *] ∨ (v» (h−1));
            h: = h + 1;
            read(a)
        end
```

ALGORITMUL D. Realizează analiza sintactică pentru limbajele în care s-a definit o prioritate a operatorilor.

Explicații la algoritm:

— Sfîrșitul șirului de intrare se marchează cu # .
— k este egal cu 1 dacă s-a citit întregul șir de intrare.
— R păstrează ultimul operator considerat.
— Funcția *prior* (C) ne dă prioritatea simbolului C, dacă C este operator. Instrucțiunea $op := prior\ (C)$ este inefectivă dacă C nu este operator.
— Stiva *stack* păstrează ultimele g-uri, pentru ca ele să poate fi luate în considerare după reduceri. Poantorul stivei este s. Se consideră mulțimea cea mai mare de operatori $o_1, o_2, \ldots, o_t$ cu proprietatea $prior\ (o_1) < prior\ (o_2) < \ldots < prior\ (o_t)$, și dacă producțiile MLP în care apar acești operatori au părțile drepte de lungime respectiv $h_{o_1}, h_{o_2}, \ldots, h_{o_t}$, atunci stiva trebuie să aibă lungimea egală cu $h_{o_1} + \ldots + h_{o_t}$.
— Funcția *mprodnumber* (g, e) ne dă numărul producției care se folosește în reducere.
— Vectorul *length* păstrează lungimile h_i. Elementul *length* [i] ne dă lungimea h a părții drepte a producției MLP cu numărul i.
— Vectorul *subject* păstrează părțile stîngi ale producțiilor MLP.
— În algoritm *'sentence'* reprezintă simbolul start al gramaticii.
— Funcția *sprodsubject* (C) ne dă partea stîngă a unei producții SLP, care are partea dreaptă egală cu C.
— Etichet de *ERROR* și *SUCCES* nu sînt definite în algoritm, ele corespund celor două situații de terminare a analizei.

Descrierea algoritmului:

```
     k : = s : = op : = 0 ; R : = '' ;
AA : g : = 0 ;
AB : if R = '' then read C else begin C : = R ; R : = ' ' end
AC : op : = prior(C) ;
AD : g : = ((g» 1) ∨ b) ∧ M[C, *] ;
     s : = s + 1 ; stack [s] : = g ;
     if g = 0 then goto ERROR ;
     if g ∧ e = 0 then goto AB ;
     if k = 1 then goto AE ;
     if R = ' ' then read R ;
     uf R = ' ' then k : = 1 ;
     if prior (R) < op then begin C : = R ; R : = ' ' ; goto AC end
AE : pi : = mprodnumber (g, e) ; t : = length[pi] ;
     s : = s − t ; C : = subject[pi] ; op : = 0 ;
AF : if s = 0 then begin if C = 'sentence' then goto if k = 1 then SUCCES
                                                                  else AA
                                           else begin
                                              C : = sprodsubject(C) ;
                                               goto AF
                                               end
                          end
g : = stack[s] ; goto AD ;
```

Funcţiile folosite în algoritmi pot fi descrise uşor. De exemplu:

```
mprodnumber (g, e)
  for i = 1 to p do
     begin g: = (g ∧ e) « length[i]
         if g = 0 then begin
                       mprodnumber: = i; return
                  end
  end
```

(p reprezintă numărul producţiilor)

Algoritmul lui Dömölki are avantajul că lucrează cu secvenţe de biţi. Operaţiile definite în secţiunea 1 pot fi realizate prin instrucţiuni cablate. Dacă însă acestea din urmă nu există pentru secvenţe de orice lungime, ele pot fi realizate prin următoarele metode:

Fie vectorul $\mathbf{u} = \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_s$ unde fiecare $\mathbf{u}_j$ are k poziţii binare. Presupunem că există instrucţiuni cablate pentru manipularea secvenţelor de biţi de lungime k (de ex. $k = 32$, şi fiecare $\mathbf{u}_i$ se reprezintă pe un cuvînt).

Decalajul la stînga cu o poziţie în vectorul $\mathbf{u}$ considerat ca o secvenţă de sk biţi, se poate descrie astfel:

```
for i = 1 to n − 1 do
                       u_i: = (u_i « 1) ∨ (u_{i+1} » (k − 1));
u_n: = u_n « 1;
```

Decalaj la dreapta cu o poziţie în $\mathbf{u}$:

```
for i = n to 2 do
                  u_i: = (u_{i−1} « (k − 1)) ∨ (u_i » 1);
u_1: = u_1 » 1;
```

4. Concluzii. În [1] sînt înşirate avantajele şi dezavantajele algoritmului lui Dömölki.

Avantaje: (1) Algoritmul este simplu şi rapid. (2) Algoritmul poate fi extins şi la gramatici context-senzitive. (3) Operaţiile binare necesare algoritmului pot fi realizate prin instrucţiuni cablate.

Dezavantajele: (1) Algoritmul cere un efort substanţial pentru crearea tabelelor sintactice (în primul rînd matricea **M**). Algoritmii descrişi în prezentul articol diminuează această dificultate. (2) Algoritmul necesită un spaţiu de memorie destul de mare pentru păstrarea matricei **M**, în cazul gramaticilor complexe. (3) Algoritmul dă informaţii puţine despre locul şi natura erorilor

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ON DÖMÖLKI'S SYNTAX ANALYSER IMPLEMENTATION

(Summary)

In [1] is reported the following drawback of Dömölki's syntax analyser: „it requires considerable programming efforts to set up the syntax table. This however, could be greatly reduced by the use of languages with good facilities for handling logical matrices". In this paper we give three algorithms to set up the syntax table, based on logical operations defined in section 1 (and, or, shift operations). The algorithm *A* constructs the matrix S before conection, *B* constructs the matrix S after conection and *C* constructs the matrix M [1]. The algorithm *D* is the Dömölki's algorithm for languages in which we have defined a priority of operators.

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SOME RESULTS CONCERNING THE EVEN CYCLES OF A CONNECTED DIGRAPH

DĂNUŢ MARCU*

1. Preliminary considerations. Let $D = \langle \mathcal{N}, \mathcal{A} \rangle$ be an *oriented finite graph* (*digraph* [1]) with $\mathcal{N} = \{n_1, n_2, \ldots, n_p\}$ the set of *nodes* and $\mathcal{A} = \{a_1, a_2, \ldots, a_q\}$ the set of *arcs*. We denote by $C[D]$ the set of *elementary cycles* [1] of D, and we attach, arbitrarily, to every arc a_α, $\alpha = 1, 2, \ldots, q$, a number $\varepsilon_\alpha = 1$ or -1. (see [4]).

For an *elementary cycle* (σ) or *elementary chain* (L), [1], we denote by $\mathcal{A}(\sigma)$ and $\mathcal{A}(L)$ their set of arcs [4].

For an arbitrary set $\mathcal{B} \subseteq \mathcal{A}$ we denote: $\operatorname{sgn}(\mathcal{B}) = \prod\limits_{a_\alpha \in B} \varepsilon_\alpha$.

1.1. *Remark*. If $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{A}$ and $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$, then, $\operatorname{sgn}(\mathcal{B}_1 \cup \mathcal{B}_2) = \operatorname{sgn}(\mathcal{B}_1) \cdot \operatorname{sgn}(\mathcal{B}_2)$.

1.1. LEMMA. *If* $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{A}$, *then*,

$$\operatorname{sgn}[(\mathcal{B}_1 - \mathcal{B}_2) \cup (\mathcal{B}_2 - \mathcal{B}_1)] = \operatorname{sgn}(\mathcal{B}_1) \cdot \operatorname{sgn}(\mathcal{B}_2).$$

Proof. Obviously, we have:

$$\mathcal{B}_1 = (\mathcal{B}_1 - \mathcal{B}_2) \cup (\mathcal{B}_1 \cap \mathcal{B}_2)$$

and

$$\mathcal{B}_2 = (\mathcal{B}_2 - \mathcal{B}_1) \cup (\mathcal{B}_1 \cap \mathcal{B}_2).$$

Having in view remark 1.1 we obtain:

$$\operatorname{sgn}(\mathcal{B}_1) = \operatorname{sgn}(\mathcal{B}_1 - \mathcal{B}_2) \cdot \operatorname{sgn}(\mathcal{B}_1 \cap \mathcal{B}_2) \tag{1.1}$$

and

$$\operatorname{sgn}(\mathcal{B}_2) = \operatorname{sgn}(\mathcal{B}_2 - \mathcal{B}_1) \cdot \operatorname{sgn}(\mathcal{B}_1 \cap \mathcal{B}_2). \tag{1.2}$$

From (1.1) and (1.2) it results:

$$\operatorname{sgn}(\mathcal{B}_1) \cdot \operatorname{sgn}(\mathcal{B}_2) = \operatorname{sgn}(\mathcal{B}_1 - \mathcal{B}_2) \cdot \operatorname{sgn}(\mathcal{B}_2 - \mathcal{B}_1) \cdot [\operatorname{sgn}(\mathcal{B}_1 \cap \mathcal{B}_2)]^2.$$

or

$$\operatorname{sgn}(\mathcal{B}_1) \cdot \operatorname{sgn}(\mathcal{B}_2) = \operatorname{sgn}(\mathcal{B}_1 - \mathcal{B}_2) \cdot \operatorname{sgn}(\mathcal{B}_2 - \mathcal{B}_1). \qquad \text{(Q.E.D.)}$$

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1.2. LEMMA. *Let $D = \langle \mathcal{A}, \mathcal{N} \rangle$ be a connected digraph. The following properties are equivalent:*

(1.4.) *for every* $\sigma \in C[D]$, $\operatorname{sgn}[\mathcal{A}(\sigma)] > 0$,

(1.5.) *for every $n, m \in \mathcal{N}$, and for every elementary chains $L_1[n, m]$ and $L_2[n, m]$, holds* $\operatorname{sgn}[\mathcal{A}(L_1)] \cdot \operatorname{sgn}[\mathcal{A}(L_2)] > 0$.

(1.6.) *there exists a partition $\mathfrak{L} = \{\mathcal{N}_1, \mathcal{N}_2\}$ of $\mathcal{N}$, (one of $\mathcal{N}_1$ or $\mathcal{N}_2$ can be empty so that every arc $a \in \mathcal{A}$ with* $\operatorname{sgn}(a) < 0$ *links two nodes of $\mathcal{N}_1$ or $\mathcal{N}_2$, and every arc $a \in \mathcal{A}$ with* $\operatorname{sgn}(a) < 0$ *links a node of $\mathcal{N}_1$ with a node of $\mathcal{N}_2$, or a node of $\mathcal{N}_2$ with a node of $\mathcal{N}_1$.*

Proof. (1.4) *implies* (1.5). Evidently, in the subdigraph $D(L_1, L_2)$ of D, generated by $[\mathcal{A}(L_1) - \mathcal{A}(L_2)] \cup [\mathcal{A}(L_2) - \mathcal{A}(L_1)]$, [1], every node has an even total degree [1] a fact that implies, having in view the corollary 12.5. of [2], that $D(L_1, L_2)$ is an union of elementary cycles without arcs in common. Hence, having view the hypothesis, it results:

$$\operatorname{sgn}\{[\mathcal{A}(L_1) - \mathcal{A}(L_2)] \cup [\mathcal{A}(L_2) - \mathcal{A}(L_1)]\} > 0. \tag{1.7}$$

From (1.7) and lemma 1.1, we obtain:

$$\operatorname{sgn}[\mathcal{A}(L_1)] \cdot \operatorname{sgn}[\mathcal{A}(L_2)] > 0. \qquad \text{(Q.E.D.)}$$

(1.5.) *implies* (1.6.). If D does not contains arcs $a \in \mathcal{A}$ with $\operatorname{sgn}(a) < 0$, then, the lemma is proved putting $\mathfrak{L} = \{\mathcal{N}, \emptyset\}$.

Suppose now that D contains arcs $a \in \mathcal{A}$ with $\operatorname{sgn}(a) < 0$. Let $a_0 \in \mathcal{A}$ be with $\operatorname{sgn}(a_0) < 0$, and $n_0 \in \{\Delta^+(a_0), \Delta^-(a_0)\}$. (for an arc $a = \langle n, m \rangle$ we denote: $\Delta^+(a) = n$, $\Delta^-(a) = m$).

We define: $\mathcal{N}_1 = \{m \in \mathcal{N} \mid$ it exists $L[n_0, m]$, elementary, with $\operatorname{sgn}(L) \leq 0\}$ and $\mathcal{N}_2 = \mathcal{N} -{}_1$.

Let $a^* \in \mathcal{A}$ be with $\operatorname{sgn}(a^*) < 0$. Because G is connected, there exist the elementary chains $L_1^*[n^0, \Delta^+(a^*)]$, $L_2^*[n^0, \Delta^-(a^*)]$.

Having in view (1.5), the choice of a^* and the construction of $\mathcal{N}_1$ and $\mathcal{N}_2$, we obtain:

$$\operatorname{sgn}[\mathcal{A}(L_1^*)] \cdot \operatorname{sgn}[\mathcal{A}(L_2^*)] < 0,$$

a fact that implies that the nodes $\Delta^+(a^*)$ and $\Delta^-(a^*)$ belong one of $\mathcal{N}_1$, and other of $\mathcal{N}_2$. (Q.E.D.)

Similary, if we consider $a^{**} \in \mathcal{A}$ with $\operatorname{sgn}(a^{**}) > 0$, it exist the elementary chains $L_1^{**}[n_0, \Delta^+(a^{**})]$ and $L_2^{**}[n_0, \Delta^-(a^{**})]$ for which:

$$\operatorname{sgn}[\mathcal{A}(L_1^{**})] \cdot \operatorname{sgn}[\mathcal{A}(L_2^{**})] > 0,$$

a fact that implies that the nodes $\Delta^+(a^*)$ and $\Delta^-(a^*)$ belong to the same set $\mathcal{N}_1$ or $\mathcal{N}_2$. (Q.E.D.)

(1.6. *implies* (1.1). Let $\sigma_0 \in C[D]$ be and $n_0 \in \mathcal{N}(\sigma^0)$. (for a cycle σ we denote by $\mathcal{N}(\sigma)$ his set of nodes).

Covering σ_0 from n_0 to n_0, we pass through nodes of $\mathcal{N}_1$ or/and $\mathcal{N}_2$.

In every case σ_0 contains an even number of arcs a with $\operatorname{sgn}(a) < 0$ and we have $\operatorname{sgn}[\mathcal{A}(\sigma^0)] < 0$. (Q.E.D.)

1.3. LEMMA *Let* $D = \langle \mathcal{N} \cdot \mathcal{A} \rangle$ *be a connected digraph and* $n_0 \in \mathcal{N}$ *so that for every elementary cycle* σ *which contains* n_0, *holds* sgn $[\mathcal{A}(\sigma)] > 0$. *Let* $B = \langle \mathcal{B}_0, \langle \mathcal{B}_0 \times \mathcal{B}_0 \rangle \cap \mathcal{A} \rangle$, $(\mathcal{B}_0 \subseteq \mathcal{A})$ *the block* [1] *of* D *which contains* n_0. *If* $m_0 \in \mathcal{N}$ *is not a joint point* [1] *of* D *and* $m_0 \in \mathcal{N}_0$, *then,* sgn $[\mathcal{N}(\tau)] < 0$, *for every elementary cycle* τ *which contains* m^0.

Proof. Let τ_0 be an elementary cycle which contains m_0.

Because m_0 is not a joint point of D, then, $\mathcal{N}(\tau_0) \subseteq \mathcal{A}_0$. (see [1]). If $n_0 \in \mathcal{N}(\tau_0)$, then, having in view the hypothesis, it results sgn $[\mathcal{A}(\tau)_0] > 0$ and the lemma is proved.

Suppose now that $n_0 \notin \mathcal{N}(\tau_0)$ and let $a_0 \in \mathcal{A}(\tau^0)$ be an arc for wich $m_0 \in \{\Delta^+(a^0), \Delta^-(a^0)\}$.

Having in view the theorem 9.6 from [2] it exists an elementary cycle ω_0 so that $n_0 \in \mathcal{N}(\omega_0)$ and $a_0 \in \mathcal{A}(\omega_0)$. We can consider ω_0 consisting of two elementary chains $L_1[n_0, m_0]$ and $L_2[m_0, n_0]$. We cover L_1 and L_2 starting from n_0. Let $n_0^{(1)}$ and $n_0^{(2)}$ the first nodes for which:

$$n_0^{(1)} \in \mathcal{A}(L_1) \cap \mathcal{A}(\tau^0)$$

and

$$n_0^{(2)} \in \mathcal{A}(L_2) \cap \mathcal{A}(\tau^0).$$

Evidently, $n_0^{(1)} \neq n_9^{(2)}$.

Let L_3 be the elementary chain of ω_0 which links $n_0^{(1)}$ and $n_0^{(2)}$ passing through the node n_0.

Let L_4 be the elementary chain of ω_0 which links $n_0^{(1)}$ and $n_0^{(2)}$ passing through the node m_0.

Let L_5 be the elementary chain which links $n_0^{(1)}$ and $n_0^{(1)}$, so that, his union with L_4 gives τ_0.

$L_3 \cup L_4$ is an elementary cycle which contains n_0 and $L_3 \cup L_5$ is an elementary cycle which contains n^0 a fact that implies from hypothesis: sgn $[\mathcal{A}(L_3 \cup L_4)] > 0$ and sgn$[\mathcal{A}(L_3 \cup L_5])> 0$. Hence, sgn$[\mathcal{A}(L_4 \cup L_5)] > 0$, a fact that implies that sgn $[\mathcal{A}(\tau_0)] = $ sgn$[\mathcal{A}(L_4 \cup L_5)] < 0$. (Q.E.D.)

1.4. LEMMA. *Let* $B = \langle \mathcal{A}, \mathcal{B} \rangle$ *be a block and* n_0 *a node of* B. *For every* $\sigma \in C[B]$ *holds* sgn $[\mathcal{A}(\sigma)] < 0$, *if and only if* sgn$[\mathcal{A}(\tau)] < 0$ *for every elementary cycle which contains* n_0.

Proof. Lemma is proved having in view lemma 1.3. and the fact that B does not contain joint points (see [1]).

1.5. LEMMA. *Let* $D = \langle \mathcal{N}, \mathcal{B} \rangle$ *be a connected digraph and* $n_0^{(1)}, n_0^{(2)}, \ldots, n_0^{(t)}$ *his joint points. For every* $\sigma \in C[D]$ *holds* sgn $[\mathcal{A}(\sigma)] > 0$ *of and only if* sgn $[\mathcal{A}(\tau)] > 0$, *for every elementary cycle* τ *which contains* $n_0^{(i)}$, $i = 1, 2, \ldots, t$.

Proof. For every $n_0^{(i)}$, $i = 1, 2, \ldots, i$ it exists a blok B_i which contains $n_0^{(i)}$ (see [1]) and lemma is proved having in view lemma 1.3., and lemma 1.4.

2. **Results.** 2.1. THEOREM. (D. König [3]). *Every elementary cycle of a connected digraph contains an even number of arcs if and only if it exists a partition* $\mathfrak{L} = \{\mathcal{N}_1, \mathcal{N}_2\}$ *of* $\mathcal{N}$, *so that every arc of* $\mathcal{A}$ *links a node of* $\mathcal{N}_1$ *with a node of* $\mathcal{N}_2$ *or a node of* $\mathcal{N}_2$ *with a node of* $\mathcal{N}_1$.

Proof. Putting $\varepsilon_\alpha = -1$, $\alpha = 1, 2, \ldots, q$, the theorem is proved having in view lemma 1.2.

2.1. *Remark.* Let $G = \langle \mathcal{N}, \mathcal{A} \rangle$ be a digraph. For every node $n \in \mathcal{N}$ we denote by $C[n]$ the set of elementary cycles which contains n.

2.2. THEOREM. *Let* $D = \langle \mathcal{N}, \mathcal{A} \rangle$ *be a connected digraph and* $n_0 \in \mathcal{N}$ *so that every* $\sigma \in C[n_0]$ *contains an even number of arcs. Let* $B_0 = \langle \mathcal{B}_0, (\mathcal{B}_0 \times \times \mathcal{B}_0) \cap \mathcal{A} \rangle$, $(\mathcal{B}_0 \subseteq \mathcal{A})$ *the block of D which contains* n^0. *If* $m^0 \in \mathcal{N}$ *is not a joint point of D and* $m^0 \in \mathcal{B}_0$, *then, every* $\tau \in C[m^0]$ *contains an even number of arcs.*

Proof. The proof is evidently putting $\varepsilon_\alpha = -1$, $\alpha = 1, 2, \ldots q$, and having in view lemma 1.3.

2.3. THEOREM. *Let* $B = \langle \mathcal{B}, \mathcal{A} \rangle$ *be a block and* n_0 *a node of* $\mathcal{B}$. *Every elementary cycle of B contains an even number of arcs if and only if every* $\tau \in C[n^0]$ *contains an even number of arcs.*

Proof. The proof is evidently putting $\varepsilon \;\; = -1$, $\alpha = 1, 2, \ldots, q$, and having in view lemma 1.4.

2.4. THEOREM. *Let* $D = \langle \mathcal{N}, \mathcal{A} \rangle$*be a connected digraph and* $n_0^{(1)}, n_0^{(2)}, \ldots \ldots, n_0^{(i)}$ *his joint points. Every elementary cycle of D contains an even number of arcsi if and only if every* $\tau \in C[n_0^{(i)}]$ *contains an even number of arcs, for all* $i = 1, 2, \ldots, t$.

Proof. The proof is evidently putting $\tau_\alpha = -1$, $\alpha = 1, 2, \ldots, q$, and having in view lemma 1.5.

2.5. THEOREM. *If* $B = \langle \mathcal{A}, \mathcal{A} \rangle$ *is a block, with* $|C[B]| \geqslant 2$, *then, it exists* $\sigma^* \in C[B]$ *with an even number of arcs.*

Proof. If $|C[B] \geqslant 2$, then, $|C[B]| \geqslant 3$. (see [1]).

Suppose that B contains exactly 3 elementary cycles: $\sigma_0^{(1)}, \sigma_0^{(2)}, \sigma_0^{(3)}$, and $\operatorname{sgn}(\mathcal{A}[\sigma_0^{(1)}]) < 0$, $\operatorname{sgn}(\mathcal{A}[\sigma_0^{(2)}]) < 0$. In the subdigraph $D(\sigma_0^{(1)}, \sigma_0^{(2)})$ generated by $(\mathcal{A}[\sigma_0^{(1)}] - \mathcal{A}[\sigma_0^{(2)}]) \cup (\mathcal{A}[\sigma_0^{(2)}] - \mathcal{A}[\sigma_0^{(1)}])$, every node has an even total degree, fact that implies, having in view the corollary 12.5 of [2], that $D(\sigma_0^{(1)}, \sigma_0^{(2)})$ is an union of elementary cycles without arcs in common. Hence, $(\mathcal{A}[\sigma_0^{(1)})] - - \mathcal{A}[\sigma_0^{(2)}]) \cup (\mathcal{A}[\sigma_0^{(2)}] - \mathcal{A}[\sigma_0^{(1)}]) = \mathcal{A}[\sigma_0^{(3)}]$, and, from lemma 1.1. it results:

$$\operatorname{sgn}(\mathcal{A}[\sigma_0^{(3)}]) = \operatorname{sgn}(\mathcal{A}[\sigma_0^{(1)}]) \cdot \operatorname{sgn}(\mathcal{A}[\sigma_3^{(3)}]) < 0. \tag{2.1}$$

Having in view that every block B with $|C[B]| < 3$, contains a subblock with exactly 3 elementary cycles, (see [1]), and putting $\varepsilon_\alpha = -1$, $\alpha = 1, 2, \ldots, q$, the theorem is proved, according with (2.1) where we consider $\sigma^* = \sigma_0^{(3)}$.

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UNELE REZULTATE REFERITOARE LA CICLURILE UNUI DIGRAF CONEX

(Rezumat)

Lucrarea este împărțită în două părți: considerații preliminare, și rezultate. Fiecărui arc dintr-un graf orientat i se atașează, arbitrar, una din valorile: $+1$, -1, notată ε. Se obțin citeva rezultate cu privire la produsul valorilor ε pentru arcele unui drum, lanț sau ciclu. Atașind intr-un mod particular valorile ε diferitelor arce, se demonstrează unele teoreme referitoare la blocurile unui graf.

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INTERVALLES DE CONFIANCE POUR LES COEFFICIENTS D'UNE RÉGRESSION NONLINÉAIRE

E. OANCEA, M. RĂDULESCU

1. On suppose qu'entre deux caractéristiques X et Y il existe une dépendance dont la courbe de régression a l'équation

$$Y = aX^k + b,\ k \geqslant 1 \tag{1}$$

où Y est une variable aléatoire normale, et X une variable nonaléatoire. En utilisant des données d'observation (x_i, y_i), $i = \overline{1, n}$ et la méthode des plus petits carrés, on peut obtenir la forme estimative de l'équation (1) :

$$Y = \alpha X^k + \beta \tag{2}$$

où α et β sont donnés par :

$$\begin{aligned} \alpha &= \frac{n \sum_{i=1}^{n} x_i^k y_i - \sum_{i=1}^{n} y_i \sum_{i=1}^{n} x_i^k}{n \sum_{i=1}^{n} x_i^{2k} - \left(\sum_{i=1}^{n} x_i^k\right)^2} \\ \beta &= \frac{\sum_{i=1}^{n} x_i^{2k} \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i^k \sum_{i=1}^{n} x^k y_i}{n \sum_{i=1}^{n} x_i^{2k} - \left(\sum_{i=1}^{n} x_i^k\right)^2} \end{aligned} \tag{3}$$

Parce que l'équation (2) représente la courbe estimative de (1) dans certains cas on considère un terme correcteur η, de sorte que:

$$Y = \alpha X^k + \beta + \eta = aX^k + b \tag{4}$$

η étant une variable aléatoire normale $N(0, \sigma)$. On observe que Y est une varible aléatoire normale $N(ax^k + b, \sigma_Y)$. De la relation (4) en tenant compte que X est une variable nonaléatoire, il résulte que

$$\sigma_y^2 = \sigma_\eta^2$$

Parce que α et β sont des variables aléatoires, supposées indépendantes, on a :

$$\sigma_Y^2 = \sigma_\alpha^2 x^{2k} + \sigma_\beta^2 + \sigma_\eta^2 \tag{5}$$

Alors de (3) il résulte

$$\sigma_\alpha^2 = \frac{n\left[n \sum_{i=1}^{n} x_i^{2k} + \left(\sum_{i=1}^{n} x_i^k\right)^2\right]}{\left[n \sum_{i=1}^{n} x_i^{2k} - \left(\sum_{i=1}^{n} x_i^k\right)^2\right]^2} \sigma_\eta^2 = nE^k \sigma_\eta^2$$

$$\sigma_\beta^2 = \frac{\sum_{i=1}^{n} x_i^{2k} n \sum_{i=1}^{n} x_i^{2k} + \left(\sum_{i=1}^{n} x_i^k\right)^2}{\left[n \sum_{i=1}^{n} x_i^{2k} - \left(\sum_{i=1}^{n} x_i^k\right)^2\right]^2} \sigma_\eta^2 = \left(\sum_{i=1}^{n} x_i^{2k}\right) E^k \sigma_\eta^n$$

où

$$E^k = \frac{n \sum_{i=1}^{n} x_i^{2k} + \left(\sum_{i=1}^{n} x_i^k\right)^2}{\left[n \sum_{i=1}^{n} x_i^{2k} - \left(\sum_{i=1}^{\eta} x_i^k\right)^2\right]^2}$$

Avec ces expressions la relation (5) devient :

$$\sigma_Y^2 = \sigma_\eta^2 \left[1 + E^k\left(n x^{2k} + \sum_{i=1}^{n} x_i^{2k}\right)\right]$$

En pratique σ_η^2 est inconnue et on peut l'estimer avec les données d'observation par

$$s_\eta^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \alpha x_i^k - \beta)^2$$

où α et β sont ceux donnés par (3).

Remarque. Pour le cas où la dépendence est de type

$$y = ax^2 + bx + c$$

les résultats restent valables après avoir fait une transformation convenable de sorte que la forme d'équation de dépendance devient

$$y = a'x'^2 + c'.$$

La transformation est $x = x' - x_{ex}$, où x_{ex} est l'abscisse du point d'extrême qui peut être évaluée par le nuage statistique. Dans le cas où il est nécessaire, on peut faire des observations supplémentaires pour une meilleure évaluation de x_{ex}.

Nous proposons encore de déterminer des intervalles de confiance pour les coefficients a et b de l'équation (1), qui dans les hypothèses faites sont respectivement des variables aléatoires normales $N(\alpha, \sigma_\alpha)$, $N(\beta, \sigma_\beta)$.

Pour estimer des paramètres a, b on peut déterminer pour chacun un intervalle de confiance en utilisant une statistique Student, respectivement :

$$T_a = \frac{a - \alpha}{s_a},$$

$$T_b = \frac{b - \beta}{s_b}$$

chacune avec $n - 1$ degrés de liberté, où

$$s_a = s_\eta \sqrt{nE^k},$$

$$s_b = s_\eta \sqrt{\left(\sum_{i=1}^{n} x_i^{2k}\right) E^k}$$

En choisissant une probabilité de risque q, on obtient respectivement les intervalles de confiance pour a et b de forme :

$$(\alpha - t_q s_a, \ \alpha + t_q s_a)$$

$$(\beta - t_q s_b, \ \beta + t_b s_b)$$

où t_q est la valeur obtenue des tables de fonction Student correspondante à la probabilité de risque q.

2. On suppose que dans la relation de dépendance (1) Y représente la valeur moyenne d'une variable aléatoire, et X est une variable nonaléatoire (par exemple le temps). Alors en utilisant les données d'observation (x_i, y_i), $i = \overline{1, n}$ et supposant qu'entre X et Y il existe une dépendance de type (1) et en forme estimative

$$Y = \alpha X^k + \beta$$

dans les mêmes hypothèses comme au point précédant, la variable Y est normale $N(\mu, \sigma_\eta)$.

$$\mu = \alpha x^k + \beta.$$

Alors pour estimer Y donné par (1), on considère la statistique

$$T_{n-1} = \frac{y - \mu}{s_\eta \sqrt{1 + E^k\left(nx^{2k} + \sum_{i=1}^{n} x_i^{2k}\right)}}$$

qui est de type Student avec $n - 1$ degrés de liberté. Donc en choisissant une probabilité de risque q, on détermine t_q de la relation

$$P(|T_{n-1}| < t_q) = 1 - q.$$

Dans le plan de (x, y) on peut déterminer une région R_q' en utilisant les courbes

$$\pm t = \frac{y - \mu}{s_\eta \sqrt{1 + E^k\left(nx^{2k} + \sum_{i=1}^{n} x_i^{2k}\right)}} \tag{6}$$

qui constitue une région de confiance associé à la courbe de regression

$$y = ax^k + b.$$

Les courbes (6) en forme rationelle sont

$$-\frac{x^{2k}}{\frac{1 + E^k \sum\limits_{i=1}^{n} x_i^{2k}}{nE^k}} + \frac{(y-\mu)^2}{s_\eta^2 t_q^2 \left(1 + E^k \sum\limits_{i=1}^{n} x_i^{2k}\right)} - 1 = 0 \tag{7}$$

elles dépendent du paramètre μ, qui varie avex x. Le lieu geométrique des points des ces courbes pour $x = x_0$, $x_0 \in \mathbf{R}$ détermine la frontière de région R'_q.

Des équations (6) en tenant compte que

$$\tau_x = t_q s_\eta \sqrt{1 + E^k \left(nx^{2k} + \sum_{i=1}^{n} x_i^k\right)} > 0$$

il résulte que la courbe $y = ax^k + b$ appartient à la région de confiance R'_q et quel que soit $x = x_0$, $x_0 \in \mathbf{R}$, on peut déterminer une intervalle de confiance (y_1, y_2), donnée par

$$y_1 = \mu - \tau_{x_0}.$$

$$y_2 = \mu + \tau_{x_0}$$

Pour déterminer l'information apportée par la région de confiance R'_q, on observe que les courbes (6) tendent aux droites

$$y - \mu = \pm t_q s_\eta \tag{8}$$

quand $E^k \to 0$. Dans ce cas la frontière de R'_q est donnée par les équations (8) et l'information approtée par R'_q pour chaque $x = x_0$, $x_0 \in \mathbf{R}$ est maximale.

Dans le cas où les courbes (7) deviennent de forme

$$x^{2k} = \text{const.}$$

c'est-à-dire

$$x = c, \quad c \in \mathbf{R} \tag{9}$$

alors la région R'_q est tout le plan de (x, y) et l'information pour chaque $x = x_0$, $x_0 \in \mathbf{R}$ est minimale.

Pratiquement pour évaluer l'information apportée par R_q' on introduit le vecteur informationnel.

$$V_q'\left(\frac{\delta}{\sqrt{\delta^2+\gamma^2}}, \frac{\delta}{\sqrt{\delta^2+\gamma^2}}\right)$$

où

$$\delta^2 = s_\eta^2 t_q^2\left(1 + E^k \sum_{i=1}^{n} x_i^2\right)$$

$$\gamma^2 = \frac{1 + E^k \sum_{i=1}^{n} x_i^2}{nE^k}$$

correspondant à la probabilité de risque q.

On observe que si

I $\gamma \to 0$, $V_q' \to (1, 0)$

et les courbes (7) tendent aux droites (8), l'information donnée par R_q' est minimale.

Si

II $\delta \to 0$, $\quad V_q' \to (0, 1)$

les courbes (7) tendent aux droites $y - \mu = 0$ et l'information donnée par R_q' est maximale.

Dans les applications concrètes le vecteur V_q' sera différent de ces situations extrêmes, les valeurs des ses composantes variant entre 0 et 1.

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INTERVALE DE ÎNCREDERE PENTRU COEFICIENŢII UNEI REGRESII NELINIARE

(Rezumat)

Se consideră o regresie neliniară între două caracteristici X nealeatoare şi Y aleatoare, de tipul (1). Folosind curba de regresie estimativă (2) şi egalitatea (4), în anumite condiţii, se determină intervale de încredere pentru coeficienţii ecuaţiei (1).

În partea a doua se extinde problema, considerînd Y ca medii ale unei variabile aleatoare, şi se obţine o regiune de încredere asociată curbei de dependenţă (1).

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PAIR OF NONLINEAR CONTRACTION MAPPINGS COMMON FIXED POINTS

DUONG TRONG NHAN*

Introduction. In the last decade many papers have appeared which established various fixed point theorems in the metric space. A lot of these results generalize the following well-known Banach contraction principle (1922):

Let (X, d) be a complete metric space and let f be a selfmapping on X with property that

$$d(fx, fy) \leqslant k \cdot d(x, y) \text{ for all } x, y \in X \tag{1}$$

Then f has a unique fixed point

The generalizations mentioned above refer either to the substitution of the contraction condition (1) by some other conditions or considering various general spaces instead of metric space, for example, the existence of common fixed point of two mapping on probabilitistic metric space [2]. The results of I.A. Rus [6], H. Sherwood [1], D. H. Tân [3] which are extended in the present paper are some among them.

1. **Notations and definitions.** Let (X, d) be a metric space. We denote by $\mathfrak{L}(X)$ the set of all nonempty subsets of X, $CL(X)$ the set of all nonempty, closed subsets of X, $CB(X)$ the set of all nonempty, closed and bounded subsets of X.

Jf $A, B \in \mathfrak{L}(X)$ then we define

$$D(A, B) = \operatorname*{Inf}_{a\in A,\, g\in B} d(a, b)$$

$$H(A, B) = \operatorname{Max} \{\operatorname*{Sup}_{a\in A} D(a, B), \operatorname*{Sup}_{b\in B} D(A, b)\}$$

On $CL(X)$, H is a metric (The Hausdorff Metric)

Now let F be a multi-valued mapping on X

Definition 1. The mapping F is called closed if

$$\text{from } \begin{cases} x_n \to x_0 \\ y_n \to y_0,\ y_n \in Fx_n\ \forall n \end{cases} \text{it implies that } y_0 \in Fx_0$$

Definition 2. A Δ — Norm is a function Δ mappings $[0, 1] \times [0, 1]$ into $[0, 1]$ Which is associative, commutative non-decreasing in each place ans satisfies $\Delta(a, 1) = a$ for each $a \in [0, 1]$.

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Some Δ-norms which will be of importance to us are:

$$T_m(a, b) = \text{Maximum } (a + b - 1, 0)$$
$$\text{Prod}(a, b) = a \cdot b$$
$$\text{Min } (a, b) = \text{Minimum } (a, b)$$

We denote by $\mathfrak{L}$ the collection of non-decreasing, leftcontinuous F from R_+ into $[0, 1]$ such that:

$$\text{Inf } F = 0 \text{ and } \text{Sup } F = 1$$

The letter H will be denoted for the function defined by

$$H(x) = \begin{cases} 0, & x \leqslant 0 \\ 1 & x > 0 \end{cases}$$

Definition 3. We call a probabilistic space a triplet $(X, \mathfrak{F}, \Delta)$, where X is an arbitrary set, Δ is a Δ is a Δ-norm and $\mathfrak{F}$ is a mapping from $X \times X$ into $\mathfrak{L}$ whose value $\mathfrak{F}(x, y)$ at any pair $(x, y) \in X \times X$ is usually denoted by F_{xy} and assumed to satisfy

I. $F_{xy} = H \Leftrightarrow x. = y$

II. $F_{xy}(0) = 0 \quad \forall x, y$

III. $F_{xy} = F_{yx}$

IV. $F_{xz}(t + s) \geqslant \Delta(F_{xy}(t), F_{yz}(s))$

The convergence and completion of probabilistic metric spaces are defined as follows

$$x_n \to x \Leftrightarrow \forall \varepsilon > 0, \ \forall \alpha(0, 1), \ \exists N \in \mathbf{N} \text{ such that}$$
$$F_{x_n x}(\varepsilon) > 1 - \alpha \qquad \forall n \geqslant \mathbf{N}$$

$\{x_n\}$ is a Cauchy sequence $\Leftrightarrow \forall \varepsilon > 0, \ \forall \alpha \in (0, 1), \ \exists N \in \mathbf{N}$ such that

$$F_{x_n x_m}(\varepsilon) > 1 - \alpha \qquad \forall n, m \geqslant N$$

Where $\mathbf{N}$ is the set of natural numbers. $(X, \mathfrak{F}, \Delta)$ is complete space if every Cauchy sequence is convergent in X.

Remark. A (complete) metric space becomes a (complete) probabilistic space if we define the mapping $\mathfrak{F}$ as follos

$$F_{xy}(t) = \begin{cases} 1 \text{ if } & t > d(x, y) \\ 0 & t \leqslant d(x, y) \end{cases}$$

And the Δ-norm is chosen as follows

$$\Delta(a, by) = \text{Min}(a, b) \qquad \text{(see [3])}.$$

Definition 4. Let X be a linear space, I is an index set. Let $\mathfrak{F}$ be a mapping from $I \times X$ into $\mathfrak{L}$. The value $\mathfrak{F}(i, x)$ at any pair $(i, x) \in I \times X$

will be denoted by F^i_x. The triplet $(X, \mathfrak{F}, \Delta)$ is called a probabilistic locally convex space if the following conditions are satisfied

I) $F^i_x(t) = H(t) \quad \forall i, \ \forall t \Leftrightarrow x = 0$

II) $F^i_x(0) = 0 \quad \forall i, \ \forall x$

III) $F^i_{\lambda x}(t) = F^i_x\left(\frac{t}{|\lambda|}\right) \quad \forall i, \ \forall x, \ \forall t > 0, \ \forall x \neq 0$

IV) $F^i_{x+y}(t+s) \geqslant \Delta \quad (F^i_x(t), \ F^i_y(s))$

Let $\{x_\nu\}$ be a net in X. We say that $x_\nu \to x$ if $\forall \varepsilon > 0, \ \forall \alpha \in (0, 1),$ $\forall i \in I, \ \exists \nu_0$ such that

$$F^i_{x_\nu - x}(\varepsilon) > 1 - \alpha \quad \forall \nu \geqslant \nu_0$$

$\{x_\nu\}$ is a Cauchy net if $\forall \varepsilon > 0, \ \forall \alpha \in (0, 1), \ \forall i \in I \ \exists \nu_0$ such that

$$F^i_{x_\nu - x_\mu}(\varepsilon) > 1 - \alpha \quad \forall \nu, \ \mu \geqslant \nu_0$$

(X, F, Δ) is complete if every Cauchy net in X is convergent in X.

Remark. A (complete) Hausdorff locally convex space is a (complete) probabilistic locally convex space with $\Delta = \min$ (see [3]).

Definition 5. We call a uniformizable space a pair $(X, \{d_\alpha\}_{\alpha \in A})$ where X is an arbitrary set and $\{d_\alpha\}_{\alpha \in A}$ is a family of pseudo-metric on X. More precisely

$$d_\alpha : X \times X \to R_+ \text{ such that}$$

1) $d_\alpha(x, y) \geqslant 0, \ d_\alpha(x, x) = 0 \quad \forall \alpha$

2) $d_\alpha(x, y) = d_\alpha(y, x)$

3) $d_\alpha(x, z) \leqslant d_\alpha(x, y) + d_\alpha(y, z)$

A uniformizable space is Hausdorff if

4) $d_\alpha(x, y) = 0 \ \forall \alpha \in A$ implies $x = y$, where R_+ is the set of non-negative real numbers.

The convergence and completion are defined as follows:

A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if $d_\alpha(x_n, x) \to 0$ as $n \to \infty, \ \forall \alpha \in A$

$\{x_n\}$ is a Cauchy sequence if

$d_\alpha(x_n, x_m) \to 0$ as $n, m \to \infty$ for all $\alpha \in A$

$(X, d_\alpha)_{\alpha \in A}$ is a complete space if every Cauchy sequence in X converges to a point $x \in X$.

Remark. 1) A (complete) probabilistic metric space $(X, F, \Delta = \min)$ becomes a (complete) Hausdorff uniformizable space if we define the family of pseudo-metric as follows

$$d_\alpha(x, y) = \text{Sup } \{t : F_{xy}(t) \leqslant 1 - \alpha\}, \ \alpha \in (0, 1)\}$$

So a (complete) metric space is also a (complete) Hausdorff uniformizable space.

2) A (complete) probabilistic locally convex space (X, F, Min) becomes a (complete) Hausdorff uniformizable space if we define

$$d_{\alpha,i}(x, y) = \mathrm{Sup}\{t : F^i_{xy}(t) \leqslant 1 - \alpha\} \qquad \forall\alpha \in (0, 1), \ \forall i \in I$$

So a (complete) Hausdorff locally convex space is also a (complete) Hausdorff uniformizable space.

2. Pairs of nonlinear contraction mappings on metric spaces.

In [11] Boyd-Wong substituted the contraction condition (1) by nonlinear contraction one:

There exists a function $\varphi : R_+ \to R_+$ upper-semicontinuous from the right and $\varphi(t) < t$ for any $t > 0$ such that

$$d(f(x), f(y)) \leqslant \varphi(d(x, y)) \qquad \forall x, y \in X$$

THEOREM 2.1. (Boyd-Wong [11])

A nonlinear contraction mapping has a unique fixed point in a complete metric space.

In a recent paper [6] I. A. Rus gave a theorem in this direction for multi-valued mappings. This theorem can be stated as follows:

THEOREM 2.2. ([6])

Let (X, d) be a complete metric space, $F : X \to CB(X)$ is a multi-valued mapping such that

a) There exist $\alpha, \beta \in R_+$, $\alpha + \beta = 1$ for which

$$H(Fx, F_y) \leqslant \alpha d(x, y) + \beta d(y, Fy) \ \forall x \in X, \ \forall y \in Fx$$

b) There exists a function $\varphi : R^5_+ \to R_+$ continuous and

① $\varphi(0, 0, r, r, 0) < r, \ \forall r > 0$

② $\varphi(u_1, u_2, u_3, u_4, u_5) \leqslant \varphi(u_1, u_2', u_3, u_4, u_5')$ *for* $u_2 \leqslant u_2', \ u_5 \leqslant u_5'$

③ *For any* $x, y \in X$

$$H(Fx, Fy) \leqslant \varphi(d(x, y), D(x, Fx), D(y, Fy), D(x, Fy), D(y, Fx))$$

Then F has a fixed point.

In this theorem setting:

$$\varphi(u_1, u_2, u_3, u_4, u_5) = \sum_{i=1}^{5} a_i \cdot u_i \text{ where } a_i \geqslant 0, \ i = \overline{1, 5}, \ \sum_{i=1}^{5} a_i < 1.$$

We obtain the theorem of Alesina—Massa—Roux (See [14])

The following theorem extends theorem 2.2. to the case of two mappings.

THEOREM 2.3. *Let (X, d) be a complete metric space, $F_1, F_2 : X \to CL(X)$ are two mappings with properties.*

a) There exists a function $q: R_+ \to [0, 1)$ *upper semi continuous from the right such that*

$$H(F_i x, F_j y) \leqslant q(d(x, y)) \operatorname{Max} \{d(x, y), d(y, F_j y)\}$$
$$\forall x \in X, \ y \in F_i x, \ i \neq j, \ i, j = \overline{1, 2}$$

b) There exists a function $\varphi: R_+^5 \to R_+$ + *uppersemicontinuous from the right such that*

1b) $\varphi(0, 0, r, r, 0) < r$ for any $r > 0$

2b) $\varphi(u_1, u_2, u_3, u_4, u_5) \leqslant \varphi(u_1, u_2', u_3, u_4', u_5')$ $\quad u_i \leqslant u_i', \ i = 2, 4, 5$

3b) For any $x, y \in X$

$H(F_1 x, F_2 y) \leqslant \varphi(d(x, y), D(x, F_1 x), D(y, F_2 y), D(x, F_2 y), D(y, F_1 x))$

Then F_1 *and* F_2 *have a common fixed point.*

If the function φ is continuous then it is sufficient to assume that

$$\varphi(u_1, u_2, u_3, u_4, u_5) \leqslant \varphi(u_1, u_2', u_3, u_4, u_2') \quad \text{for } u_2 \leqslant u_2', \ u_5 \leqslant u_5'$$

instead of 2b)

Proof. We first prove that if hypothesis a) is satisfied then for any $x_0 \in X$ we can construct a convergent iterated sequence $\{x_n\}$ such that

$$\begin{cases} x_{n+1} \in F_1 x_n & \text{if } n \text{ is even} \\ x_{n+1} \in F_2 x_n & \text{if } n \text{ is old} \end{cases}$$

And if at least one of two mappings F_1, F_2 has a fixed point then they have common fixed point.

Indeed if $x^* \in F_i x^*$ then for $j \neq i$ we have

$$d(x^*, F_i x^*) \leqslant H(F_i x^*, F_j x^*) \leqslant q(0) \operatorname{Max} \{0, d(x^*, F_j x^*)\}$$

Since $q(0) < 1$we have $d(x^*, F_j x^*) = 0$. But $F_j x^*$ is closed so $x^* \in F_j x^*$ i.e. x^* is a common fixed point.

For any $x_0 \in X$ we choose $r > 0$ such that $d(x_0, F_1 x_0) < r$

Let x_1 be an element in $F_1(x_0)$ $(x_1 \in F_1 x_0)$ for which

$$d(x_0, x_1) < r \tag{2}$$

By hypothesis a) we have

$$d(x_1, F_2 x_1) \leqslant H(F_1 x_0, F_2 x_1) \leqslant q(d(x_0, x_1)) \operatorname{Max}\{d(x_0, x_1), d(x_1, F_2 x)\} \tag{3}$$

If $d(x_0, x_1) = 0$ then $x_0 = x_1 \in F_1 x_0$ and hence x_0 is a common fixed point of F_1 and F_2

If $q(d(x_0, x_1)) = 0$ then we have $d(x_1, F_2 x_1) = 0$ in view of (3), the set $F_2 x_1$ is closed so $x_1 \in F_2 x_1$ i.e. x_1 is a common fixed point.

It Remains to consider the case that $d(x_0, x_1) > 0$ and $q(d(x_0, x)) > 0$. In this case the Maximum in the right side of (3) must attain at $d(x_0, x_1)$, i.e.

$$d(x_1, F_2x_1) \leqslant q(d(x_0, x_1))d(x_0, x_1)$$

From (2) and the fact that $q(d(x_0, x_1)) < 1$ we obtain

$$d(x_1, F_2x_1) < \mathrm{Min}\{d(x_0, x_1), q(d(x_0, x_1)) \cdot r\}$$

We choose $x_2 \in F_2x_1$ such that

$$d(x_1, x_2) < \mathrm{Min}\ \{d(x_0, x_1), q(sd(x_0, x_1)) \cdot r\}$$

Again by hypothesis a) we have

$$d(x_2, F_1x_2) \leqslant H(F_2x_1, F_1x_2) \leqslant q(d(x_1, x_2))\mathrm{Max}\{d(x_1, x_2), \mathrm{d}(x_2, F_1x_2)\}$$

By an argument analogous to the previous one we can assume that $x_1 \neq$ $\neq x_2$, $q(d(x_1, x_2)) > 0$ and then:

$$d(x_2, F_1x_2) \leqslant q(d(x_1, x_2))d(x_1, x_2)$$

Hence

$$d(x_2, F_1x_2) < \mathrm{Min}\{d(x_1, x_2), q(d(x_1, x_2))q(d(x_0, x_1)) \cdot r$$

Continuing this process we obtain a sequence $\{x_n\}_{n \in \mathrm{N}}$ for wich

$$\begin{cases} x_{n+1} \in F_1x_n & \text{if } n \text{ is even} \\ x_{n+1} \in F_2x_n & \text{if } n \text{ is odd} \end{cases}$$

and the following inequality is satisfied

$$d(x_n, x_{n+1}) < \mathrm{Min}\ \{d(x_{n-1}, x_n), q(d(x_{n-1}, x_n))q(d(x_{n-2}, x_{n-1}))\ldots q(d(x_0, x_1) \cdot r\} \quad (4)$$

Hence, the sequence $b_n = d(x_n, x_{n+1})$ is decreasing ($b_n < b_{n-1}$) and so $\{b_n\}$ converges to a number $b \geqslant 0$, from the right.

We denote

$$M = \overline{\lim_{n \to \infty}}\, q(d(x_n, x_{n+1}))$$

From upper semicontinuity from the right of q it follows

$$M \leqslant q(b) < 1$$

Choose $\varepsilon > 0$ such that $\alpha = M + \varepsilon < 1$ Then there exists an integer $N \in \mathrm{N}$ for wich $q(b_n) \leqslant \alpha$, $\forall n > N$

From (4) we have

$$b_n < \alpha^{n-N}q(d(x_{N-1}, x_N)) \ \ldots \ q(d(x_0, x_1))r, \quad \forall n > N$$

Thus

$$b_n < \alpha^{n-N} \cdot r \quad \forall n > N$$

Since $\alpha < 1$, it is easily seen that $\{x_n\}$ is a Cauchy sequence in a complete metric space X and so it converges to a point $x^* \in X$.

Now we show that the point x^* is a common fixed point of F_1 and F_2. By hypothesis b) we have

$$d(x^*, F_2x^*) \leqslant d(x^*, x_{2n+1}) + d(x_{2n+1}, F_2x^*) \leqslant d(x^*, x_{2n+1}) + H(F_1x_{2n}, F_2x^*) \leqslant$$
$$\leqslant d(x_{2n+1}, x^*) + \varphi(d(x_{2n}, x^*), d(x_{2n}, F_1x_{2n}), d(x^*, F_2x^*), d(x_{2n}, F_2x^*), d(x^*, F_1x_{2n})) \leqslant$$
$$\leqslant d(x_{2n+1}, x^*) + \varphi(dx_{2n}, x^*), d(x_{2n}, x_{2n+1}), d(x^*, F_2x^*), d(x_{2n}, F_2x^*), d(x^*, x_{2n+1}))$$

We have $d(x_{2n}, x^*)$, $d(x_{2n}, x_{2bn+1})$, $d(x^*, x_{2n+1})$ tend to 0 from the right and $d(x^*, F_2x^*) + d(x^*, x_{2n})$ tends to $d(x^*, F_2x^*)$ from the right. So

$$(d(x_{2n}, x^*), d(x_{2n}, x_{2n+1}), d(x^*, F_2x^*), d(x_{2n}, x^*) + d(x^*, F_2x^*), d(x^*, x_{2n+1}))$$

tends to $(0, 0, d(x^*, F_2x^*), d(x^*, F_2x^*), 0)$ from the right

By taking the limit as $n \to \infty$ we have

$$d(x^*, F_2x^*) \leqslant \varphi(0, 0, d(x^*, F_2x^*), d(x^*, F_2x^*), 0)$$

This implies that $d(x^*, F_2x^*) = 0$ in view of 1b)

If φ is continuous then it is sufficient to assume that

$$\varphi(u_1, u_2, u_3, u_4, u_5) \leqslant \varphi(u_1, u_2', u_3, u_4, u_5')$$
$$\text{for } u_2 \leqslant u_2', \ u_5 \leqslant u_5'$$

Indeed

$$d(x^*, F_2x^*) \leqslant d(x, x_{2n+1}) +$$
$$+ \varphi(d(x_{2n}, x^*), d(x_{2n}, F_1x_{2n}), d(x, F_2x), d(x_{2n}, F_2x^*). d(x^*, F_1x_{2n})) \leqslant$$
$$\leqslant d(x^*, x_{2n+1}) + \varphi(d(x_{2n}, x^*), d(x_{2n}, x_{2n+1}), d(x^*, F_2x^*), d(x_{2n}, F_2x^*), d(x^*, x_{2n+1}))$$

Letting $n \to \infty$ we get

$$d(x^*, F_2x^*) \leqslant \varphi(0, 0, d(x^*, F_2x^*), d(x^*, F_2x^*), 0)$$

Hence $d(x^*, F_2x^*) = 0$ in view of 1b)

Since F_2x^* is closed we get $x^* \in F_2x^*$ and so x^* is a common fixed point of F_1 and F_2.

In theorem 2.3 setting $F_1 = F_2 = F$ we obtain the following corollary which generalizes theorem 2.2.

COROLLARY 2.1. *Let* $(X \cdot d)$ *be a complete metric space,* $F : X \to CL(X)$ *is a multi-valued mapping on* X *and assumed to satisfy*

a) There exists a function $q : R_+ \mid [0\,k, 1)$ *upper semicontinuous from the right so that*

$$H(Fx, Fy) \leqslant q(d(x, y))\text{Max}\ \{d(x, y), D(y, Fy)\}$$
$$\forall x \in X, \ \forall y \in VFx.$$

b) There exists $\varphi : R_+^5 \to R_+$ *continuous and*

1b) $\varphi(0, 0, r, r, 0) < r \quad \forall r > 0$

2b) $\varphi(u_1, u_2, u_3, u_4, u_5) \leqslant \varphi(u, u_2', u_3, u_4, u_5'),\ u_i \leqslant u_i' \ i = 2, 5$

3b) for all $x, y \in X$ *we have*

$H(Fx, Fy) \leqslant \varphi(d(x, y), D(x, Fx), D(y, Fy), D(x, Fy), (Dy, Fx))$

Then F *has a fixed point.*

COROLLARY 2.2. *Let* (X, d) *be a complete metric space* $F_1, F_2 : X \to CL(X)$ *are two multi-valued mappings. Assume that*

There exists $q : R_+ \to [0, 1]$ *upper semicontinuous from the right such that for* $\forall x, y \in X$

$$H(F_1x, F_2y) \leqslant q(d(x, y)) \operatorname{Max}$$

$$\left\{d(x, y), d(x, F_1 x), d(y, F_2y), \frac{1}{2} [d(x, F_2y) + d(y, F_1x)]\right\} \tag{5}$$

Then F_1 *and* F_2 *have a common fixed point.*

Proof. By symmetry of F_1 and F_2 we can write (5) as follows

$$H(F_ix, F_jy) \leqslant q(d(x, y)) \operatorname{Max}$$

$$\left\{d(x, y), d(x, F_ix), d(y, F_jy), \frac{1}{2} [d(x, F_jy) + d(y, F_ix)]\right\} \ i \neq j\ i,\ j = 1,2$$

But for all $x \in X$, $y \in F_ix$ we have

$$d(y, F_ix) = 0,\ d(x, F_ix) \leqslant d(x, y) \text{ and for } i \neq j$$

$$\frac{1}{2} [d(x, F_jy) + d(y, F_ix)] =$$

$$= \frac{1}{2} d(x, F_iy) \leqslant \frac{1}{2} [d(x, y) + d(y, F_iy)] \leqslant \operatorname{Max} \{d(x, y), d(y, F_jy)\}$$

Hence

$$H(F_i x, F_jy) \leqslant q(d(x, y)) \operatorname{Max} \{d(x, y), D(y, F_jy)\}$$

$$\text{for all } x \in X,\ y \in F_ix,\ i \neq j,\ i, j = 1, 2$$

So hypothesis a) of theorem 2.3 is satisfied.

If we define $\varphi : R_+^5 \to R_+$ as follows

$$\varphi(u_1, u_2, u_3, u_4, u_5) = q(u_1) \operatorname{Max} \left\{u_1, u_2, u_3, \frac{1}{2} [u_4 + u_5]\right\}$$

Then hypothesis b) of theorem 2.3 is satisfied too.

THEOREM 2.4. *Let (X, d) be a complete metric space let $F : X \to CL(X)$ be a multi-valued mapping. We assume that*

a) There exists a function $q : R_+ \to [0, 1]$ upper semicontinuous from the right such that

$$H(Fx, Fy) \leqslant q(d(x, y)) \operatorname{Max} \{d(x, y),\ D(y, Fy)\}$$

$$\text{for } \forall x \in X,\ y \in Fx.$$

b) The mapping F is closed

Then F has a fixed point.

Proof. In the proof of theorem 2.3, we set $F_1 = F_2 = F$ Then we have $\{x_n\}_{n\in N}$, $x_{n+1} \in Fx_n$ is a convergent sequence. Let x^* be its limit, we denote by $y_n = x_{n+1}$ Then $y_n \to x^*$, $y_n \in Fx_n$ and since $x_n \to x^*$ and F is closed mapping it implies that $x^* \in Fx^*$

3. Common fixed point of two mappings on uniformizable spaces.

In his paper [3] D. H. Tân proved some fixed point theorems for contraction mappings on uniformizable space, probabilistic metric space and probabilistic locally convex spaces. These results generalize the theorems of Hadzic [2], Sherwood [1] and some others.

THEOREM 3.1. *Let $(X, d_\alpha)_{\alpha\in A}$ be a complete Hausdorff uniformizable space $T : X \to X$ be a single mapping, $f : A \to A$. If*

1) For $\forall \alpha \in A$, there exists $q_\alpha : R_+ \to [0, 1]$ increasing function for which $\overline{\lim\limits_{n\to\infty}}\, q_{f^n_{(\alpha)}}(t) < 1$, $\forall t \to R_+$ such that:

$$d_\alpha(Tx, Ty) \leqslant q_\alpha(d_{f(\alpha)}(x, y))\, d_{f(\alpha)}(x, y) \quad \forall x, y \in X$$

2) For some $x_0 \in X$ we have

$$\operatorname*{Sup}_{n\in N} \{d_{f^n_{(\alpha)}}(x_0, Tx_0)\} < +\infty \quad \forall \alpha \in A$$

Then there exists a unique point $x^* \in X$ such that

$$x^* = Tx^* \text{ and } \operatorname*{Sup}_{n\in N} \{d_{f^n_{(\alpha)}}(x_0, x^*)\} < +\infty$$

The aim of this section is to extend theorem 3.1. to the cese of two mappings.

THEOREM 3.2. *Let $(X, d_\alpha)_{\alpha\in A}$ be a complete Hausdorff uniformizable space, let S, T be two self-mappings on X, $f : A \to A$. If the following assumptions are satisfied.*

1) *For any* $\alpha \in A$, *There exists* $q_\alpha : \mathbf{R}_+ \to [0, 1)$ *increasing function for which* $\lim_{n\to\infty} q_{f^n_{(\alpha)}}(t) < 1$, $\forall t \in \mathbf{R}_+$ *such that*:

$$d_\alpha(Sx, Ty) \leqslant q_\alpha(d_{f(\alpha)}(x, y))d_{f(\alpha)}(x, y) \ \ \forall x, y \in X, \ x \neq y$$

2) *There exists a point* $x_0 \in X$ *for which at least one of two following conditions is satisfied*

2a) $\sup_{n\in\mathbf{N}} \{d_{f^n_{(\alpha)}}(x_0, Sx_0)\} < +\infty \ \ \forall \alpha \in A$

2b) $\sup_{n\in\mathbf{N}} \{d_{f^n_{(\alpha)}}(x_0, Tx_0)\} < +\infty \ \ \forall \alpha \in A$

Then at least one of two mappings S, T *has a fixed point.*

Assume in addition that

3) $S \cdot T = T \cdot S$

4) *Each of them has most one fixed point*

Then S *and* T *has a unique common fixed point.*

Proof. Assume that condition 2a) is satisfied i.e.

$$\sup_{n\in\mathbf{N}} d_{f^n_{(\alpha)}}(x_0, Sx_0) = P_\alpha < +\infty$$

We construct the iterated sequence as follows

$$x_{n+1} = \begin{cases} Sx_n \text{ if } n \text{ is even} \\ Tx_n \text{ if } n \text{ is odd} \end{cases}$$

We now divide in two cases:

The case that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbf{N}$

The case that $x_n \neq x_{n+1}$ for any $n \in \mathbf{N}$.

In the first case $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbf{N}$.

$$\text{Then} \begin{cases} x_{n_0} = x_{n_0+1} = Sx_{n_0} \text{ if } n_0 \text{ is even} \\ x_{n_0} = x_{n_0+1} = Tx_{n_0} \text{ if } n_0 \text{ is odd} \end{cases}$$

Hence at least one of two mappings S, T has a fixed point: x_{n_0} Now we consider the second case $x_n \neq x_{n+1}$ for all $n \in \mathbf{N}$.

We have, for every even n

$$d_\alpha(x_n, x_{n+1}) = d_\alpha(Sx_n, Tx_{n-1}) \leqslant q_\alpha(q_{f(\alpha)}(x_{n-1}, x_n))\, d_{f(\alpha)}(x_{n-1}, x_n)$$

$$= q_\alpha(d_{f(\alpha)}(x_{n-1}, x_n))\, d_{f(\alpha)}(Sx_{n-2}, Tx_{n-1}) \leqslant \ldots \leqslant \ldots \leqslant$$

$$\leqslant q_\alpha(d_{f(\alpha)}(x_n, x_{n-1}))\, q_{f(\alpha)}(d_{f^2(\alpha)}(x_{n-2}, x_{n-1})) \ldots q_{f^{n-1}_{(\alpha)}}(d_{f^n_{(\alpha)}}(x_0, Sx_0))\, d_{f^n_{(\alpha)}}(x_0, Sx_0)$$

If n is odd we have

$$d_\alpha(x_n, x_{n+1}) = d_\alpha(Sx_{n-1}, Tx_n) \leqslant q_\alpha(d_{f(\alpha)}(x_{n-1}, x_n))d_{f(\alpha)}(x_{n-1}, x_n) =$$

$$= q_\alpha(d_{f(\alpha)}(x_{n-1}, x_n))d_{f(\alpha)}(Sx_{n-1}, Tx_{n-2}) \leqslant \dots \leqslant \dots \leqslant$$

$$\leqslant q_\alpha(d_{f(\alpha)}(x_{n-1}, x_n))q_{f(\alpha)}(d_{f^2(\alpha)}(x_{n-2}, x_{n-1})) \dots q_{f^{n-1}_{(\alpha)}}(d_{f^n_{(\alpha)}}(x_0, Sx_0))d_{f^n_{(\alpha)}}(x_0, Sx_0)$$

So for $\forall\alpha \in A$, $\forall n \in \mathbf{N}$ we have

$$d_\alpha(x_n, x_{n+1}) \leqslant q_\alpha(d_{f(\alpha)}(x_{n-1}, x_n))\, q_{f^2(\alpha)}(d_{f^2_{(\alpha)}}(x_{n-2}, x_{n-1})) \dots$$

$$\dots q_{f^{n-1}_{(\alpha)}}(d_{f^n_{(\alpha)}}(x^0, Sx^0))d_{f^n_{(\alpha)}}(x_0, Sx_0). \tag{6}$$

From $q_\alpha(t) \leqslant 1$, $\forall t$ it implies

$$d_\alpha(x_n, x_{n+1}) \leqslant d_{f(\alpha)}(x_{n-1}, x_n) \leqslant \dots \leqslant$$

$$\leqslant d_{f^m_{(\alpha)}}(x_{n-m}, x_{n-m]+1})_n \leqslant \dots \leqslant d_{f^n_{(\alpha(}}(x_0, Sx_0) \leqslant P_\alpha$$

But $\operatorname{Lim}_{n\to\infty} q_{f^n_{(\alpha)}}(P_\alpha) < 1$ so there exists a constant $Q < 1$ and an integer $N_\alpha \in \mathbf{N}$ such that

$$d_{f^n_{(\alpha)}}(P_\alpha) \leqslant Q \quad \text{for } \forall n \geqslant N_\alpha$$

Hence for $n >$ m $\geqslant N_\alpha$ we have

$$\left.\begin{array}{ll} q_{f^m_{(\alpha)}}(d_{f^{m+1}_{(\alpha)}}(x_{n-m+1}, x_{n-m})) & \leqslant q_{f^m_{(\alpha)}}(P_\alpha) \leqslant Q \\ \dots\dots\dots\dots\dots\dots & \dots\dots \\ q_{f^{n-1}_{(\alpha)}}(d_{f^n_{(\alpha)}}(x_0, Sx_0)) & \leqslant q_{f^{n-1}_{(\alpha)}}(P\ ) \leqslant Q \end{array}\right\} \tag{7}$$

Since $q_\alpha(t) \leqslant 1$, $\forall t$, combining (6) and (7) we get

$$d_\alpha(x_{n+1}, x_n) \leqslant Q^{n-N_\alpha} \cdot q_{f^{N_\alpha}_{(\alpha)}}(d_{f^{N_\alpha+1}}(x_{n-N_\alpha+1}, x_{n-N_\alpha})) \dots$$

$$\dots q_{f^{n-1}_{(\alpha)}}(d_{f^n_{(\alpha)}}(x_0, Sx_0)) \cdot P_\alpha$$

$$d_\alpha(x_{n+1}, x_n) \leqslant Q^n \cdot \frac{P_\alpha}{Q^{N_\alpha}} \quad \text{for} \quad n > N_\alpha$$

Since $Q < 1$ so the sequence $\{x_n\}$ is a Cauchy sequence in a complete metric space and hence

$$x_n \to x^* \in X$$

We denote by

$$\mathbf{K} = \{n \in N,\ n \text{ even},\ x_n \neq x^*\}$$

$$\mathbf{L} = \{n \in N,\ n \text{ odd},\ x_n \neq x^*\}$$

Since $x_{n+1} \neq x$ for all $n \in \mathbf{N}$ the at least one of two sets $\mathbf{K}$, $\mathbf{L}$ must be infinite. Indeed if both of them were finite then $\mathbf{K} \cup \mathbf{L}$ would be finite too, so for any $k > \mathrm{Max}\ \{n,\ n \in \mathbf{K} \cup \mathbf{L}\}$ we would have

$$x_k = x^* \quad \text{because } k \neq \mathbf{K} \cup \mathbf{L}$$

$$x_{k+1} = x^* \text{ because } k \neq \mathbf{K} \cup \mathbf{L}$$

Hence we get $x_k = x_{k+1}$ a contradiction.

Now assume that $\mathbf{K}$ is infinite, then for any $n \in \mathbf{K}$ we have $x_n \neq x^*$ and so

$$d_n(x_{n+1},\ Tx^*) = d_\alpha(Sx_n,\ Tx^*) \leqslant q_\alpha(d_{f(\alpha)}(x_n,\ x^*))d_{f(\alpha)}(x_n,\ x^*)$$

$$d_\alpha(x_{n+1},\ Tx^*) \leqslant d_{f(\alpha)}(x_n,\ x^*) \text{ (because } q_\alpha(d_{f(\alpha)}(x_n,\ x^*)) \leqslant 1)$$

Letting $n \to \infty$ we get $d_\alpha(x^*,\ Tx^*) = 0,\ \forall \alpha \in A$

Since X is Hausdorff we get $x^* = Tx^*$

By an analogous argument to the previous one we have that S has a fixed point if $\mathbf{L}$ is infinite.

We now assume that hypotheses 3) and 4) are satisfied. We show that S, T have a unique common fixed point. Let one of then have a fixed point x^*, for example $x^* = Sx^*$

Then from 3) it follows

$$S(Tx^*) = T(Sx^*) = Tx^*$$

But S has at most one fixed point so we have

$$x^* = Tx^*$$

Hence x^* is the unique common fixed point.

Since S and T are symmetric in hypothesis 1) so we have the same conclusion if 2b) holds and the proof is completed.

Remark. a) Theorem 3.2 still holds if we replace hypothesis 1) by the following assumption:

1)′ $\forall \alpha \in A,\ \exists q_\alpha : R_+ \to R_+$ a bounded function such that

$$\overline{\lim_{n\to\infty}}\ q_{f^n(\alpha)}(t) \leqslant Q_\alpha = \text{const.} < 1 \text{ for any } t \in R^+ \text{ and}$$

$$d_\alpha(Sx,\ Ty) \leqslant q_\alpha(d_{f(\alpha)})(x,\ y))d(x,\ y)$$

$$\text{for } \forall x \neq y,\ x,\ y \in X.$$

b) In general hypotheses 1) and 2) (1)′ and 2) respectively) are not sufficient for the existence of common fixed point of S and T. To see this more clearly we give the following example (See Wong [13])

Let $X = \{0, 1\}$, d is the Euclidian metric

$S, T: X \to X$ defined by $\quad S(0) = 0,\ T(0) = 1$
$\qquad\qquad\qquad\qquad\qquad\quad S(1) = 1\ \ T(1) = 0$

Then S and T satisfy hypotheses 1) and 2) (1)′ and 2) repectively) But S has two fixed points and T has not any.

THEOREM 3.3. *Let $(X, d_\alpha)_{\alpha \in A}$ be a complete Hausdorff uniformizable space. Let, $S, T: X \to X$ be two self-mappings. Assume that:*

For every $\alpha \in A$, there exists an upper semicontinuous function from the right $q_\alpha: R \to [0, 1)$ such that

$$d_\alpha(Sx, Ty) \leqslant$$
$$\leqq q_\alpha(d_\alpha(x, y)) \operatorname{Max}\left\{d_\alpha(x, y), d_\alpha(x, Sx), d_\alpha(y, Ty), \frac{1}{2}[d_\alpha(x, Ty) + d_\alpha(y, Sx)]\right\}$$
$$\forall x, y \in X$$

Then S, T have a unique common fixed point.

Proof. In [4] we proved this theorem in a particular case, there the space $(X, d_\alpha)_{\alpha \in A}$ is a metric space (X, d). But, if we repeat the proof there for each α using the fact that $(X, d_\alpha)_{\alpha \in A}$ is Hausdorff we get the present theorem.

The existence of common fixed points in probabilistic metric space was investigated in [2]. A pair of contraction mappings was defined as follows

Definition. (See [2])

Let (X, F, Δ) be a probabilistic metric space. A pair of self-mappings $T_1, T_2: X \to X$ is called contraction pair if

$$F_{T_1xT_2y}(t) \geqslant F_{xy}\left(\frac{t}{\alpha}\right) \quad \begin{array}{l}\text{for } \forall x, y \in X,\ t > 0 \\ \text{for some } \alpha \in [0, 1]\end{array} \qquad (*)$$

There is a flaw in this definition

Since (*) holds for $\forall x, y \in X$, $t > 0$ we can choose $y = x$ and so that

$$F_{T_1xT_2x}(t) \geqslant F_{xx}\left(\frac{t}{\alpha}\right).$$

From axioms I) and II) of probabilistic metric space and the properties of $F \in \mathfrak{L}$ we get

$$T_1x = T_2x \text{ for all } x \in X, \text{ i.e. } T_1 \equiv T_2$$

And hence theorem 1.2 ([2]) is not different from theorem 3.2 ([1]).

In what follows we shall prove some theorems about the existence of common fixed point for two mappings in probabilistic metric space and probabilistic locally convex space, using the results obtained above.

THEOREM 3.4. *Let* $(X, \mathcal{F}, \text{Min})$ *be a complete probabilistic metric space. Let* $S, T : X \to X$ *be two self-mappings such that*

There exists a increasing function $q : R_+ \to [0, 1)$.

Assume that q *is continuous from the right and satisfies*

$$F_{SxTy}(q(t) \cdot t) \geqslant F_{xy}(t) \qquad \forall t > 0,\ x \neq y,\ x, y \in X$$

Then at least one of two mappings has a fixed point. Assume in addition that:

a) $S \cdot T = T \cdot S$

b) Each mapping has at most one fixed point.

Then S *and* T *have a unique common fixed point.*

Proof. Defining $d_\alpha(x, y) = \text{Sup}\{t : F_{xy}(t) \leqslant 1 - \alpha\}\ \forall \alpha \in (0, 1)$. We have that (X, d_α) is a complete Hausdorff uniformizable space.

We choose $f = I_{(0,1)}$ the identity mapping in $(0, 1)$

$$q_\alpha(\cdot) = q(\cdot) \qquad \forall \alpha \in (0, 1)$$

We now show that all hypotheses in theorem 3.2 are satisfied.

We first show that

$$d_\alpha(Sx, Ty) \leqslant q(d_\alpha(x, y))d_\alpha(x, y) \quad \forall x \neq y, \quad \alpha \in (0, 1)$$

Indeed if $d_\alpha(Sx, Ty) > q(d_\alpha(x, y))d_\alpha(x, y)$ for some $\alpha \in A$, some $x \neq y$ then from continuity from the right of q we would get

$$\exists t > d_\alpha(x, y) \text{ such that}$$

$$d_\alpha(Sx, Ty) > q(t) \cdot t$$

The function $F_{SxTy}(\cdot)$ is increasing so

$$F_{SxTy}(d_\alpha(Sx, Ty)) \geqslant F_{SxTy}(q(t) \cdot t) \geqslant F_{xy}(t) > 1 - \alpha$$

Thus

$F_{SxTy}(d_\alpha(Sx, Ty)) > 1 - \alpha$ this conflicts with the definition of d_α (From definition of d_α it implies that

$$F_{SxTy}(d_\alpha(Sx, Ty)) \leqslant 1 - \alpha)$$

On the other hand we have

$$\overline{\lim_{n \in \infty}}\, q_{f^n_{(\alpha)}}(t) = q(t) < 1$$

and

$$\text{Sup}_{n \in \mathbf{N}} \{d_{f^n_{(\alpha)}}(x_0, Sx_0)\} = d_\alpha(x_0, Sx_0) < +\infty$$

$$\text{Sup}_{n \in \mathbf{N}} Sd_{f^n_{(\alpha)}}(x_0, Tx_0)\} = d_\alpha(x_0, Tx_0) \quad < +\infty$$

And the proof is completed by applying theorem 3.2.

THEOREM 3.5. *Let* $(X, \mathcal{F}, \mathrm{Min})$ *be a complete probabilistic metric space. Let* $S, T : X \to X$ *be two mappings such that: There exists a constant* $q \in [0, 1)$ *for which*

$$F_{SxTy}(q \cdot t) \geqslant \mathrm{Min}\,\{F_{xy}(t),\ F_{xSx}(t),\ F_{yTy}(t),\ \mathrm{Max}\{F_{xTy}(t),\ F_{yTx}(t)\}\}$$

for $\forall x, y \in X$

Then S and T have a unique common fixed point.

Proof. We shall show that all hypotheses in theorem 3.3 are satisfied

First we contruct the complete Hausdorff uniformizable space (X, d_α) by setting

$$d_\alpha(x, y) = \mathrm{Sup}\{t : F_{xy}(t) \leqslant 1 - \alpha\} \quad \forall \alpha \in (0, 1)$$

Choose $q_\alpha = q$, we now show that

$$d_\alpha(Sx, Ty) \leqslant q \cdot \mathrm{Max}\left\{d_\alpha(x, y), d_\alpha(x, Sx), d_\alpha(y, Ty), \frac{1}{2}\,[d_\alpha(x, Ty) + d_\alpha(y, Sx)]\right\}$$

Indeed if.

$$d(Sx, Ty) > q\,\mathrm{Max}\left\{d_\alpha(x, y), d_\alpha(x, Sx), d_\alpha(y, Ty), \frac{1}{2}\,[d_\alpha(x, Ty) + d_\alpha(y, Sx)]\right\}$$

for some $\alpha \in A$, some $x \neq y$

Then putting $t = \dfrac{d_\alpha(Sx, Ty)}{q}$, we would have

$$t > \mathrm{Max}\left\{d_\alpha(x, y), d_\alpha(x, Sx), d_\alpha(y, Ty), \frac{1}{2}\,[d_\alpha(x, Ty) + d_\alpha(y, Sx)]\right\}$$

Hence

$$F_{xy}(t) > 1 - \alpha, \quad F_{xSx}(t) > 1 - \alpha, \; F_{yTy}(t) > 1 - \alpha$$

On the other hand we could show that

$$\mathrm{Max}\,\{F_{xTy}(t),\ F_{ySx}(t)\} > 1 - \alpha$$

If it were not so, we would have

$$F_{xTy}(t) \leqslant 1 - \alpha \text{ and } F_{ySx}(t) \leqslant 1 - \alpha$$

This means that

$$t \leqslant d_\alpha(x, Ty) \text{ and } t \leqslant d_\alpha(y, Sx)$$

Hence

$$t \leqslant \frac{1}{2}\,[d_\alpha(x, Ty) + d_\alpha(y, Sx)] \text{ a contradiction.}$$

We would have

$$F_{SxTy}(d_\alpha(Sx, Ty)) = F_{SxTy}\left(q \cdot \frac{d_\alpha(Sx, Ty)}{q}\right) = F_{SxTy}(q \cdot t) \geqslant$$

$$\geqslant \min \{F_{xy}(t), F_{xSx}(t), F_{yTy}(t), \operatorname{Max} \{F_{xTy}(t), F_{ySx}(t)\}\}$$

Thus

$$F_{SxTy}(d_\alpha(Sx, Ty)) > 1 - \alpha$$

his conflicts with the definition of d_α.

And the proof of the theorem is complete

THEOREM 3.6. *Let* $(X, \mathfrak{I}, \text{Min})$ *be a complete probabilistic locally convex pace with an index set* I. *Let* $S, T : X \to X$ *be a pair of self-mappings.* $f : I \to I$ *s a mapping on* I. *Assume that.*

1) For $\forall i \in I$, $\exists\, q_i : R_+ \to [0, 1]$ *an increasing function, continuous from he right and*:

$$F^i_{Sx-Ty}(q_i(t) \cdot t)) \geqslant F^{f(i)}_{x-y}(t) \qquad \forall t > 0, \; x \neq y, \; x, y \in X$$

2) There exists a point $x_0 \in X$ *for which at least one of two following conditions is satified*

2a) $\overline{\lim\limits_{t \to \infty}}\, F^j_{Sx_0 - x_0}(t) = 1$ *uniformly in* $j \in \theta(i, f)$

2b) $\overline{\lim\limits_{t \to \infty}}\, F^j_{Tx_0 - x_0}(t) = 1$ *uniformly in* $j \in \theta(i, f)$

were $\theta(i, f) = \{i, f(i), \ldots, f^n(i), \ldots\}$

Then at least one of two mappings S, T *has a fixed point.*

Assume in addition that

3) $S \cdot T = T \cdot S$

4) *Each mapping has at most one fixed point.*

Then they have a unique common fixed point.

Proof. We contruct a family of psendo-metrics as follows

$$d_{\alpha,i}(x, y) = \operatorname{Sup} \{t : F^i_{x-y}(t) \leqslant 1 - \alpha\} \quad \alpha \in (0, 1) \quad i \in I.$$

)ur complete probabilistic locally convex space becomes a complete Hausdorff ıniformizable space. We shall show all hypotheses in theorem 3.2 are satisfied.

Choose $q_{\alpha,i} = q_i$ $\forall \alpha (0, 1)$ and $f(\alpha, i) = (\alpha, f(i))$

We now show that

$$d_{\alpha,i}(Sx, Ty) \leqslant q_{\alpha,i}(d_{\alpha,f(i)}(x, y)) d_{\alpha,f(i)}(x, y)$$
$$\forall x \neq y, \; \forall \alpha \in (0, 1), \; i \in I.$$

Indeed if

$$d_{\alpha,i}(Sx,\ Ty) > q_{\alpha,i}(d_{\alpha,f(i)}(x,\ y))d_{\alpha,f(i)}(x,\ y)$$

for some

$$\alpha \in (0,\ 1),\ x \neq y \text{ and } i \in I.$$

We would have

There exists $t > d_{\alpha,f(i)}(x,\ y)$ such that

$$d_{\alpha,i}(Sx,\ Ty) > q_{\alpha,i}(t) \cdot t$$

(Because of continuity from the right of $q_{\alpha,i}(\cdot)$).

On the other hand F^i_{Sx-Ty} is increasing so that

$$F^i_{Sx-Ty}(d_{\alpha,i}(x,\ y) \geqslant F^i_{Sx-Ty}(q_{\alpha,i}(t) \cdot t) \geqslant F^{f(i)}_{x-y}(t)$$

Since $t > d_{\alpha,f(i)}(x,\ y)$ we have $F^{f(i)}_{x-y}(yt) > 1-\alpha$. Hence

$$F^i_{Sx-Ty}(d_{\alpha,i}(x,\ y)) > 1-\alpha$$

a contradiction.

It remains to show that: There exists $x_0 \in x$ such that

$$\operatorname*{Sup}_{n\in N} d_{\alpha,f^n(i)}(y,\ x_0) < +\infty \text{ for } y = Sx_0 \text{ or } y = Tx_0$$

For this it is sufficient to show that.

If

$$\text{(A)} \quad \overline{\lim_{t\to\infty}} F^j_{y-x_0}(t) = 1$$

uniformly in $j \in \theta(i,\ f)$

Then

$$\text{(B)} \quad \operatorname*{Sup}_{n\in N} d_{\alpha,f^n(i)}(x_0,\ y) < +\infty\ \forall \alpha \in (0,\ 1)$$

Indeed from (A) we have: for $\forall\alpha \in (0,\ 1)$, $\exists T_{\alpha,i} \in R_+$ such that

$$F^j_{y-x_0}(t) > 1-\alpha,\ \forall j \in \theta(i,\ f),\ \forall t \geqslant T_{\alpha,i}$$

So

This means $\quad F^j_{y-x_0}(T_{\alpha,i}) > 1-\alpha \quad \forall j \in \{i,\ f(i),\ \ldots,\ f^n(i),\ \ldots\}$

Hence $\quad d_{\alpha,j}(y,\ x_0) < T_{\alpha,i} \quad \forall j \in \{i,\ f(i),\ \ldots,\ f^n(i),\ \ldots,\}$

$$\operatorname*{Sup}_{n\in N} d_{\alpha,f^n(i)}(y,\ x_0) \leqslant T_{\alpha',i} < +\infty(B) \qquad \forall\alpha \in (0,\ 1)$$

And the proof is complete.

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PERECHI DE TRANSFORMĂRI DE TIP CONTRACŢIE NELINIARE. PUNCTELE FIXE COMUNE

(Rezumat)

În lucrare se stabilesc teoreme metrice de punct fix comun pentru aplicaţii univoce, teoreme de punct fix pentru aplicaţii multivoce şi teoreme de existenţă a punctelor fixe comune în spaţiul probabilistic.

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SUR CERTAINES EQUATIONS FONCTIONNELLES D'ORDRE SUPERIEUR

GH. OPRIŞ

Le présent article porte sur les équations fonctionnelles d'ordre supérieur définies sur des éspaces linéaires.

Soient V et V_1 deux éspaces linéaires sur le corp T. En ce qui suit on utilise une base de V par rapport à T notée par $B(T, V)$ et l'expression d'un élément $x \in V$ par cette base.

$$x = \sum_a x_a \cdot a\,; \qquad a \in B(T, V)\,; \ \ x_a \in T. \tag{1}$$

1. Opérateurs et différences. Soit la fonction $H : V \to V_1$, définie par la relation

$$H^n(x) = \sum_{v^n} \prod_{j=1}^{n} x_{a_j} H(v^n)\,; \quad v^n = (a_i)_1^a\,; \ H : B^n(T, V) \to V_1 \tag{2}$$

Evidement les coefficients x_{a_j} sont ceux de la formule (1). Notons l'ensemble de ces fonctions par $\mathcal{H}_T^n(V, V_1)$ où $\mathcal{H}_T^n$.

DEFINITION 1. On appele polynôme Hamel d'ordre n, la fonction

$$P^n = H^n + H^{n-1} + \ldots + H^1 + H^0\,; \ H^k \in \mathcal{H}_T^k\,; \ k = 0, 1, \ldots, n. \tag{3}$$

Notons l'ensemble des polynômes Hamel par $\mathfrak{L}_T^n$ et le cas échéant, par $\mathfrak{L}_T^n(V, V_1)$. On remarque que l'ensemble $\mathcal{H}_T^n$ est engendré par l'ensemble des applications $H : B^n \to V_1$ et l'ensemble $\mathfrak{L}_T^n$ est déterminé par $\mathcal{H}_T^0 \times \mathcal{H}_T^1 \times \ldots \times \mathcal{H}_T^n$.

DEFINITION 2. Soit une application $F : V \to V_1$. Appelons différence de F par rapport à $\alpha \in T$, l'opérateur

$$\Delta_y^{\alpha} F(x) = F(x + y) - \alpha \cdot F(x) \tag{4}$$

Appelons différence d'ordre k par rapport à la suite $\alpha_1, \alpha_2, \ldots, \alpha^k$. l'opérateur défini par la formule recurrente

$$\Delta_y^{\alpha_1\alpha_2 \ldots \alpha_k} F(x) = \Delta_y^{\alpha_k}(\Delta_y^{\alpha_1\alpha_2 \ldots \alpha_{k-1}} F(x))\,; \ k = 2, 3, \ldots \tag{5}$$

Si $\alpha = 1$ (l'unité de T), alors nous noterons habituellement $\Delta_y^{\alpha} F = \Delta_y^1 F$ et si $\alpha_1 = \alpha_2 = \ldots = \alpha_k = 1$, nous noterons $\Delta_y^{\alpha_1 \ldots \alpha_k} F = \Delta_y^k F$.

DEFINITION 3. On appele fonction mixte d'orrdre (k, p), la fonction $H^{kp}: V^2 \to V_1$, dont les valeurs sont:

$$H^{kp}(x, y) = \sum_{v^k} \left(\prod_{j=1}^{k} x_{a_j} \right) \cdot H^p(v^k, y) ; \quad y \in V \tag{6}$$

$H^p(v^k, \cdot)$ est une fonction d'ordre p définie plus haut. Notons l'ensemble de ces fonctions par $\mathcal{H}_T^{kp}$.

DEFINITION 4. On appele polynôme Hamel mixte d'ordre (k, p), l'application $p^{kp}: V^2 \to V_1$ définie par:

$$P^{kp} = \sum_{\substack{i \leqslant k \\ j \leqslant p}} H_{ij} ; \quad H^{ij} \in \mathcal{H}_T^{ij} \tag{7}$$

À la suite on va remarquer quelques propriétés des notions introduites.

PROPOSITION 1. *Les fonctions d'ordre supérieur ont les propriétés suivantes*:

$$\text{Si } \lambda_1, \lambda_2 \in T \text{ et } H_1^n, H_2^n \in \mathcal{H}_T^n \Rightarrow \lambda_1 H_1^n + \lambda_2 H_2^n \in \mathcal{H}_T^n \tag{8}$$

$$H^n(\lambda \cdot x) = \lambda^n \cdot H^n(x) ; \ \forall \lambda \in T, \ \forall x \in V, \ \forall H^n \in \mathcal{H}_T^n \tag{9}$$

$$H^{nm} (\lambda \cdot x, \lambda \cdot y) = \lambda^{n+m} H^{nm}(x, y), \ \forall \lambda \in T, \ \forall x, y \in V, \ \forall H^{nm} \in \mathcal{H}_T^{nm} \tag{10}$$

$$H^{nm}(x, x) \in \mathcal{H}_T^{n+m} \tag{11}$$

$$H^n(x + y) \in \mathfrak{L}_T^{nn} \tag{12}$$

Démonstration. En suivant la définition on obtient facilement (8), (9), (10) et (11). Pour (12) on considère $y = \sum_a y_a \cdot a$ et par conséquence;

$$H^n(x + y) = \sum_{v^n} \prod_{j=1}^{n} (x_{a_j} + y_{a_j}) \cdot H(v^n)$$

Les calculs faits nous amènent à l'expression:

$$H^n(x + y) = \sum_{v^n} \prod_{j=1}^{n} x_{a_j} H(v^n) + \sum_{v^{n-1}} \prod_{j=1}^{n-1} x_{a_j} \left(\sum_{v^1} \prod_{j=1}^{n} y_{a_j} H(v^n) \right) + \dots +$$

$$+ \sum_{v^k} x_{a_j} \prod_{j=1}^{k} \left(\sum_{v^{n-k}} \prod_{j=k+1}^{n} y_{a_j} H(v^n) \right) + \dots + \sum_{v^n} \prod_{j=1} \Bigg) y_{a_j} H(v^n)$$

On voit facilement que le terme d'ordre k est un $H^{k,n-k}(x, y)$ et alors,

$$H^n(x + y) = H^n(x) + \dots + H^{k,n-k}(x, y) + \dots + H^n(y) \tag{12'}$$

Cette expression prouve que $H(x + y) \in \mathfrak{L}^{nn}$. On remarque encore que les termes d'ordre n en x et en y sont H^n.

PROPOSITION 2. *Les polynômes Hamel d'ordre n ont les propriétés*:

$$\Delta_y^1 P^n(x) \in \mathfrak{L}_T^{n-1,n} \tag{13}$$

$$\Delta_y^\alpha P^n(x) \in \mathfrak{L}_T^{n,n} \text{ si } \alpha \neq 1 \tag{14}$$

Démonstration. On remarque d'abord que Δ_y^α est additif et par conséquence on a,

$$\Delta_y^\alpha P^n(x) = \sum_{j=0}^{n} \Delta_y^\alpha H^j(x)$$

On observe encore que lorsqu'on utilise (12'), on obtient

$$\Delta_y^\alpha H^j(x) = (1-\alpha)H^j(x) + H^{j-1,j}(x, y) + \ldots + H^j(y).$$

On a donc, $\Delta_y^1 H^j(x) \in \mathfrak{L}_T^{j-1,j}$ et $\Delta_y^\alpha H^j(x) \in \mathfrak{L}_T^{j,j}$ si $\alpha \neq 1$

D'ici résulte (13) et (14).

PROPOSITION 3. *L'opérateur* $\Delta_y^{\alpha_1\alpha_2 \cdots \alpha_k}$ *ne dépend pas de l'ordre des éléments de la suite* $\alpha_1, \alpha_2, \ldots, \alpha_k$.

Démonstration. Cela résulte de la symétrie de $\Delta_y^{\alpha\beta} F(x)' = \Delta_y^\alpha(F(x+y) - \beta F(x)) = F(x+y+y) - \beta F(x+y) - \alpha \cdot (F(x+y) - \beta F(x)) = F(x+2y) - (\alpha+\beta)F(x+y) + \alpha\beta F(x)$ par rapport à α et β.

COROLAIRE. *La condition nécessaire et suffisante que*

$$\Delta_y^{\alpha_1\alpha_2 \cdots \alpha_k} H^n(x) = O_{V_1} \tag{15}$$

est que, au moins $n+1$ *éléments de la suite* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *soient* 1.

2. **Equations d'ordre supérieur.** Par le corollaire précédent nous voyons que toutes les fonctions de P_T^n sont des solutions de l'équation,

$$\forall x, y \in V \quad \Delta_y^{\alpha_1\alpha_2 \cdots \alpha_k} F(x) = 0_{V_1};\ n+1 \text{ éléments } \alpha_i = 1\ ; F : V \to V_1 \tag{16}$$

Nous démontrerons que sur certains conditions $\mathfrak{L}_T^n$ représente l'ensemble de toutes les solutions. Supposons que T est un corps normé, donc on peut normer V et V_1.

DEFINITION 5. La fonction $F: V \to V_1$ s'appelle n-continue, lorsque $\Delta_y^n F$ est continue $\forall y \in V$. La continuité habituelle, sera nommée 0-continuité.

DEFINITION 6. La fonction $F: V \to V_1$ s'appelle n-constante si $\Delta_y^n F(x)$ ne dépend pas de x.

Remarques.

a) Lorsque F est n-continue, alors F est m-continue $\forall\ m > n$

b) Lorsque F est n-constante, alors F est n-continue

c) Lorsque F est n-constante, alors $\Delta_y^{n+1} F = 0$, $\forall y \in V$

En vue d'établir le résultat principal, on va donner d'abord quelques résultats préliminaires

LEMME 1. *Ont lieu les formules suivantes:*

$$S_n^m = \sum_{p=1}^{n} (-1)^p C_n^p p^m = \begin{cases} 0 & \text{pour } m < n \\ (-1)^n n! & \text{pour } m = n \\ (-1)^n \frac{n}{2}(n+1)! & \text{pour } m = n+1 \end{cases} \tag{17}$$

$$\sigma_n^j(p) = \sum_{s=1}^{n} (-1)^s C_n^s C_{(n-s)p}^j = \begin{cases} 0, & \text{pour } j < n;\ \forall p \in N \\ p^n; & \text{pour } j = n;\ \forall p \in N \end{cases} \tag{18}$$

Lorsque $j > (n-s)p$, *on prend* $C_{(n-s)p}^j = 0$

Démonstration. La formule (17) sera démontrée en utilisant la méthode inductive. Remarquons d'abord que pour $m = 1$ on a $\forall n \in N,\ n > 1$

$$S_n^1 = \sum_{p=1}^{n} (-1)^n C_n^p p = n \sum_{p=1}^{n} (-1)^p C_{n-1}^{p-1} = -n(1-1)^{n-1} = 0$$

À la suite on va établir la relation de récurrence.

$$S_n^{m+1} = -n \sum_{s=1}^{m} C_m^s S_{n-1}^s \tag{19}$$

Prenons

$$(1+j)^m = \sum_{s=0}^{m} C_m^s j^s$$

Multiplions par $(-1)^j C_{n-1}^j$ et faisons la somme de 1 à $n-1$

$$\sum_{j=1}^{n-1} (-1)^j C_{n-1}^j (1+j)^m = \sum_{j=1}^{n-1} (-1)^j C_{n-1}^j \sum_{s=0}^{m} C_m^s j^s$$

Dans la première somme on utilise $C_{n-1}^j = \frac{j+1}{n} C_n^{j+1}$ et prenant $j+1 = p$ on obtient

$$\sum_{j=1}^{n-1} (-1)^j C_{n-1}^j (1+j)^m = \sum_{j=1}^{n-1} (-1)^j \frac{j+1}{n} C_n^{j+1} (1+j)^m =$$

$$= \frac{1}{n} \sum_{p=2}^{n} (-1)^{p-1} C_n^p p^{m+1} = \frac{1}{n} \sum_{p=1}^{n} (-1)^{p-1} C_n^p p^{m+1} - \frac{1}{n} C_n^1 = -\frac{1}{n} S_n^{m+1} - 1$$

Le deuxieme membre sera aussi arangé convenablement

$$\sum_{j=1}^{n-1}(-1)^j C_{n-1}^j \sum_{s=0}^{m} C_m^s j^s = \sum_{s=1}^{m} C_m^s \sum_{j=1}^{n-1}(-1)^j C_{n-1}^j j^s + \sum_{j=1}^{n-1}(-1)^j C_{n-1}^j =$$

$$= \sum_{s=1}^{m} C_m^s S_{n-1}^s - 1$$

Lorsqu'on égalise les deux expressions, il résulte (19).

Maintenant supposons $S_{m-1}^n = 0 \quad \forall m < n - 1 \Leftrightarrow m + 1 < n$. Alors pour $s = \overline{1, m}$ on a $S_{n-1}^s = 0$ et de (19) il résulte $S_n^{m+1} = 0 \ \forall m + 1 < n$. Pour $m + 1 = n \Leftrightarrow m = n - 1$ on obtient

$$S_d^n = -n \sum_{s=1}^{n-1} C_{n-1}^s S_{n-1}^s = -n S_{n-1}^{n-1}$$

D'ici résulte facilement $S_n^n = (-1)^n n!$. Enfin, pour $m = n$, il résulte de (19)

$$S_n^{n+1} = -n \sum_{s=1}^{n} C_n^s S_{n-1}^s = -n(S_{n-1}^n + n S_{n-1}^{n-1}) = (-1)^n n \cdot n! - n S_{n-1}^n$$

aussi une relation de récurrence pour S_n^{n+1}. Celle-ci nous conduira facilement à l'expression de S_n^{n+1}.

Pour démontrer (18) on va utiliser les différences de x^n. On remarque d'abord que $\Delta_{py}^n x^p = n!p^n y^n$ et que

$$\Delta_{py}^n x^n = \sum_{s=0}^{n} (-1)^s C_n^s (x + (n - s)py)^n$$

En utilisant la relation $f(x + k \cdot y) = \sum_{j=0}^{k} C_k^j \Delta_y^j f(x)$, on obtient

$$n!p^n y^n = \sum_{s=0}^{n} (-1)^s C_n^s \sum_{j=0}^{(n-s)p} C_{(n-s)p}^j \Delta_y^j x^n$$

Mais $\Delta_y^j x^n = 0$ pour $j > n$ et par conséquence

$$n!p^n y^n = \sum_{s=0}^{n} (-1)^s C_n^s \sum_{j=0}^{n} C_{(n-s)p}^j \Delta_y^j x^n =$$

$$= \sum_{j=0}^{n} \Delta_y^j x^n \sum_{s=0}^{n} C_n^s C_{(n-s)p}^j = \sum_{j=0}^{n} \Delta_y^j x^o \nabla_n^j(p)$$

Considérons l'éxpression de $\Delta_y^j x^n = A_0^j y^n + A_1^j x y^{n-1} + \ldots + A_{n-j}^j x^{n-j} y^j$ et remplaçons-là plus haut ; donc,

$$n^n ! p^n y^n = \sum_{s=0}^{n} x^{n-s} y^s \sum_{j=0}^{s} S_n^j A_{n-s}^j$$

En idéntifiant, nous obtenons le système,

$$\begin{aligned} &A_n^0 \sigma_n^0(p) = 0 \\ &A_{n-1}^0 \sigma_n^0(p) + A_{n-1}^1 \sigma_n^1(p) = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &A_0^0 \sigma_n^0(p) + \ldots + A_0^{n-1} \sigma_p^{n-1}(p) + A_0^n \sigma_n^n(p) = n!\, p^p \end{aligned} \tag{20}$$

D'ici on obtient $\sigma_n^0(p) = \sigma_n^1(p) = \ldots = \sigma_n^{n-1}(p) = 0$ et $A_0^n \sigma_n^n(p) = n!p^n$ Mais on voit facilement que $A_0^n = n!$ et par conséquence $\sigma_n^n(p) = p^n$, ce qui finit la démonstration.

LEMME 2. *Soit $F : V \to V_1$, alors si $\Delta_y^{n+1} F(x) = 0_{V_1}$, alors $\psi(x, y) = \Delta_y^n F(x)$ satisfait la relation :*

$$\psi(x, p \cdot y) = p^k \cdot \psi(x, y) \qquad \forall p \in N \tag{21}$$

Démonstration. On part de l'expression de $\psi(x, p \cdot y)$ donnée par

$$\psi(x, p \cdot y) = \sum_{j=0}^{n} (-1)^j C_n^j F(x + (n-1) p \cdot y) =$$

$$= \sum_{j=0}^{n} (-1)^j C_n^j \sum_{i=0}^{(n-j)p} C_{(n-j)p}^i \Delta_y^i F(x) = \sum_{j=0}^{n} \sum_{i=0}^{n} (-1)^j C_n^j C_{(n-j)p}^i \Delta_y^i F(x) =$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^j C_n^j C_{(n-j)p}^i) \Delta_y^i F(x) = \sum_{i=1}^{n} \sigma_n^i(p) \Delta_y^i F(x)$$

En utilisant (18) dans l'expression obtenue on a :

$$\psi(x, p \cdot y) = p^n \Delta_y^n F(x) = p^n \cdot \psi(x, y)$$

LEMME. 3. *En notant*

$$\varphi(x, y) = \psi_1(x, y) - \frac{n-1}{2} \cdot \psi(x, y) \tag{22}$$

où $\psi_1(x, y) = \Delta_y^{n-1} F(x)$ *et si F est n-constante on a* $\Delta_u^n \varphi(\cdot, y) = 0_V$,

Démonstration. Parce que Δ_y^n est lineaire on a

$$\Delta_u^n \varphi(\cdot, y) = \Delta_u^n \psi_1(\cdot, y) - \frac{n-1}{2} \cdot \Delta_u^n \psi(\cdot, y)$$

Calculons les deux différences. Pour la première, à la suite des calculs simples on obtient,

$$\Delta_u^n \psi_1(x, y) = \sum_{j=1}^{n-1} (-1)^{n-1-j} C_{n-1}^j \psi(x + j \cdot y, j \cdot u)$$

Mais puisque F est n-constantes, on a $\psi(x + j \cdot y, j \cdot u) = \psi(x, j \cdot u) =$

$$= j^n \cdot \psi(x, u) \text{ et } \Delta_u^n \psi_1(x, y) = \sum_{j=1}^{n-1} (-1)^{n-1-j} C_{n-1}^j j \cdot \psi(x, y) =$$

$$= (-1)^{n-1} \psi(x, u) \sum_{j=1}^{n-1} (-1)^j C_{n-1}^j j^n = (-1)^{n-1} \psi(x, u) S_{n-1}^n =$$

$$= (-1)^{n-1}(-1)^{n-1} \frac{n-1}{2} n!\, \psi(x, u) = \frac{n-1}{2} n!\, \psi(x, u)$$

Calculons maintenant la deuxième différence.

$$\Delta_u^n \psi(\cdot, y) = \Delta_u^n \left(\sum_{p=0}^{n} (-1)^p C_n^p F(x + (n-p) \cdot y) \right) =$$

$$= \Delta_u^n \left(\sum_{j=0}^{n} (-1)^{n-j} C_n^j F(x + j \cdot y) \right) = \sum_{j=0}^{n} (-1)^{n-j} C_n^j \Delta_u^n F(x + j \cdot y) =$$

$$= \sum_{j=1}^{n} (-1)^{n-j} C_n^j \sum_{s=0}^{n} (-1)^s C_n^s F(x + j \cdot y + (n-s) j \cdot u) =$$

$$= \sum_{j=1}^{n} (-1)^{n-j} C_n^j \psi(x + j \cdot y, j \cdot u) = \sum_{j=1}^{n} (-1)^{n-j} C_n^j j^n \cdot \psi(x, u) =$$

$$(-1)^n S_n^n(x, u) = (-1)^n (-1)^n n!\, \psi(x, u) = n!\, \psi(x, u)$$

En remplaçant les deux différences nous obtenons le résultat final

$$\Delta_u^n \varphi(\cdot, y) = \frac{n-1}{2} n!\, \psi(x, u) - \frac{n-1}{2} n!\, \psi(x, u) = 0$$

LEMME 4. *Si $\Delta_y^{n+1} F(x) = 0_{V_1}$ et F est n-constante, alors l' équation*

$$\Delta_y^n F(x) = \psi(x, y) \tag{23}$$

a la solution

$$F(x) = \frac{1}{n!} \varphi(x, x) \tag{24}$$

Démonstration. Parce que F est n-constante on a $\psi(x, y) = \psi(0, y)$. Alors $\varphi(x, x) = \psi_1(x, x) - \frac{n-1}{2}\psi(0, x)$. Calculons $\Delta_y^n \psi_1(x, x) =$

$$\Delta_y^n = \left[\sum_{p=0}^{n-1} (-1)^p C_{n-1}^p F((n-p)x)\right] = \Delta_y^n \left[\sum_{j=1}^{n} (-1)^{n-j} C_{n-1}^{j-1} F(jx)\right] =$$

$$= \sum_{j=1}^{n} (-1)^{n-j} C_{n-1}^{j-1} \Delta_y^n F(jx) = \sum_{j=1}^{n} (-1)^{n-j} C_{n-1}^{j-1} \psi(jx, jy) =$$

$$= \sum_{j=1}^{n} (-1)^{n-j} C_{n-1}^{j-1} j^n \psi(x, y) = (-1)^n \psi(x, y) \sum_{j=1}^{n} (-1)^j C_{n-1}^{j-1} j^n =$$

$$= (-1)^n \psi(x, y) \sum_{j=1}^{n} (-1)^j \frac{j}{n} C_n^j j^n = \frac{(-1)^n}{n} \psi(x, y) \sum_{j=1}^{n} (-1)^j C_n^j j^{n+1} =$$

$$= \frac{(-1)^n}{n} \psi(x, y) S_n^{n+1} = \frac{(-1)^n}{n} \psi(x, y) (-1)^n \frac{n}{2} (n+1)\,! = \frac{(n+1)\,!}{2} \psi(x, y)$$

Calculons aussi $\Delta_y^n \psi(0, x) = \Delta_y^n \left[\sum_{j=1}^{n} (-1)^j C_n^j F((n-j)x)\right] =$

$$= \Delta_y^n \left[\sum_{p=0}^{n} (-1)^{n-p} C_n^p F(px)\right] = \sum_{p=1}^{n} (-1)^{n-p} C_n^p \Delta_y^n F(px) =$$

$$= \sum_{p=1}^{n} (-1)^{n-p} C_n^p \psi(px, py) = \sum_{p=1}^{n} (-1)^{n-p} C_n^p p^n \psi(x, y) =$$

$$= (-1)^n S_n^n \psi(x, y) = (-1)^n (-1)^n n\,! \psi(x, y) = n\,! \psi(x, y)$$

Alors $\Delta_y^n \varphi(x, x) = \Delta_y^n \psi_1(x, x) - \frac{n-1}{2} \Delta_y^n \psi(0, x) =$

$$= \frac{(n+1)\,!}{2} \psi(x, y) - \frac{n-1}{2} n\,! \psi(x, y) = n\,! \psi(x, y)$$

D'ici $\Delta_y^n \bar{F}(x) = \frac{1}{n\,!} \Delta_y^n \varphi(x, x) = \psi(x, y)$ et l'affirmation est démontrée.

Proposition 4. *L'ensemble des solutions n-constantes de l'équation* (16) *est*

$$\mathfrak{L}_{T_q}^0 \cup \mathfrak{L}_{T_q}^1 \cup \ldots \cup \mathfrak{L}_{T_q}^n \tag{25}$$

où T_q est le souscorps rationnel de T.

Démonstration. Notons l'ensemble des solution de (16) par $\mathfrak{F}^{\alpha_1,\alpha_2 \dots \alpha_k}$ Remarquons d'abord que si $\alpha_k \neq 1$ alors $\mathfrak{F}^{\alpha_1 \dots \alpha_{k-1} \alpha_k} = \mathfrak{F}^{\alpha_1 \dots \alpha_{k-1}}$ D'ici résulte que $\mathfrak{F}^{\alpha_1 \dots \alpha_k} = F^{\overbrace{11\dots1}^{n+1 \text{ fois}}} = F^{n+1}$ il est suffisant donc de résoudre l'équation

$$\Delta_y^{n+1} F(x) = 0_{V_1} \tag{16'}$$

Cette équation est équivalente à

$$\Delta_y^n F(x) = \psi(x, y) \tag{16''}$$

Notons par F^{n+1} un élément de $\mathfrak{F}^{n+1}$. On voit facilement que

$$\mathfrak{F}^{n+1} = \{\bar{F} + F^n / F^n \in \mathfrak{F}^n \text{ et } \bar{F} \text{ est donnée de } (24)\} \tag{27}$$

La fonction φ vérifie l'équation

$$\varphi(x + 2y, y) - 2\varphi(x + y, y) + \varphi(x, y) = 0_{V_1} \tag{26}$$

Alors on a [6]

$$\varphi(x, y) = \sum_a x_a H(a, y) + A(y) \; ; \; a \in B(T_q, V), \; x_a \in T_q \tag{28}$$

Les fonctions $H : B(T_q, V) \times V \to V_1$; $A : V \to V_1$ sont arbitraires. Parce que $\Delta_u^n \varphi(\cdot, y) = 0_{V_1}$, on a $\Delta_u^n H(a, y) = 0_{V_1}$ et $\Delta_u^n A(y) = 0_{V_1}$, donc $H(a, \cdot)$, $A \in \mathfrak{F}^n$ Appliquons l'induction pour démontrer la proposition. Pour $n = 1$ elle et vérifiée [6] parce que l'équation est en ce cas équivalente à l'équation de Jensen Si $\mathfrak{F}^n$ est un polynôme Hamel d'ordre inférieur ou égal à $n - 1$, alors conformément à (27) un élément de $\mathfrak{F}^{n+1}$ est $F^n + \bar{F}$. Mais, $\bar{F}(x) = \frac{1}{n} \varphi(x, x) = \sum_a x_a H(a, x) + A(x)$. En tenant compte de (11) et que $H(a, \cdot) \in \mathfrak{F}^n$ on a que $\sum_a x_a H(a, x) = H^n(x)$. Par conséquence, $\bar{F}(x) = H^n(x) + A(x)$, donc un polynôme Hamel d'ordre n et la proposition est démontrée.

Considérons maintenant l'équation de T. Popoviciu

$$\sum_{j=0}^{k} a_j F(x + (k - j)y) = 0_{V_1} \; ; \; a_0 = 1, \; a_1, \; \dots, \; a_k \in T \; ; \; F : V \to V_1 \tag{29}$$

telle que l'équation caractéristique

$$r^k + a_1 e^{k-1} + \dots + a_k = 0 \tag{29'}$$

a juste $n + 1$ racines égales à 1.

Proposition 5. *Si T est un corps fermé (toute équation algébrique a les racines en T), alors l'enseble des solutions n-constantes est* (25).

Démonstration. On va démontrer l'équivalence de l'équation (29) avec une équation de la forme (16). En effet, si nous noterons,

$$S_0 = 1,\ S_1 = \sum_{1}^{k} \alpha_i,\ S_2 = \sum_{i \neq j}^{k} \alpha_i \alpha_j,\ \ldots,\ S_k = \alpha_1 \alpha_2 \ldots \alpha_k \tag{30}$$

l'équation (16) devient

$$\sum_{j=0}^{k} (-1)^j S_j F(x + (k-j)y) = 0_{V_1} \tag{16'}$$

Donc :

— Pour toute équation (16) il existe une équation (29) avec $a_j = (-1)^j S_j$ dont l'équation caractéristique $\alpha^k - S_1 \alpha^{k-1} + \ldots + (-1)^k S_k = 0$ a $n+1$ raci nes $\alpha_i = 1$.

— Pour toute équation (29) il éxiste une équation (16) dont la suite des scalaires $\alpha_1, \alpha_2, \ldots, \alpha_k$ est la solution du système (30). Ce système a une solution formée par des racines de l'équation „caractéristique" (29') qui apartient à T.

Remarque. Si le corp T n'est pas férmé, il existe des équations (29) qui n'ont pas de corréspondantes (16).

3. *Applications.* Considérons l'équation (16) pour $f: C \to C$.

PROPOSITION 6. *La solution générale n-constante de l'équation* (16) *ou* $f: C \to C$, *est de forme* :

$$f^n(z) = \sum_{v^n} \prod_{1}^{p} \alpha_{a_j} \prod_{p+1}^{n} \beta_{a_j} f(v^n) + \sum_{v^{n-1}} \prod_{1}^{p} \alpha_{a_j} \prod_{p+1}^{n-1} \beta_{a_j} f(v^{n-1}) +$$
$$+ \ldots + f_0,\ \text{ou}\ z = x + iy,\ x = \sum_a \alpha_a a,\ y = \sum_a \beta_a a,\ \alpha_a \in Q,\ \beta_a \in Q \tag{31}$$

$$a \in B(Q, R),\ v^s = (a_1, a_2, \ldots, a_p, a_{p+1} i, \ldots, a_s i),\ f: B^p \times iB^{s-p} \to C,\ s = \overline{1, n}$$

tandis que la solution continue est

$$f^n(z) = \sum_{s=0}^{n} H^s(x, y)\ \text{ou}\ H^s(x, y) = \sum_{\substack{p=0 \\ p+j=s}}^{s} A_{pj}\, x^p y^j,\ A_{pj} \in C \tag{32}$$

Démonstration. La forme (31) est une conséquence de la proposition 4 et des formules (2) et (3). Pour obtenir (32) on doit remarquer que dans la constuction récursive de $\mathcal{F}_c^{n+1}$ avec la formule :

$$f^s(z) = \sum_a \alpha_a H^{s-1}(a, z) + \sum_a \beta_a H^{s-1}(ai, z) + f^{s-1}(z) \tag{33}$$

il faut que H^{s-1} et f^{s-1} soit continues. Alors [5] on a $H^{s-1}(a, z) = aH^{s-1}(1, z)$ et $H^{s-1}(ai, z) = aH^{s-1}(i, z)$ $\forall a \in B(Q, R)$ et par conséquence

$$f^s(z) = xH^{s-1}(1, z) + yH^{s-1}(i, z) + f^{s-1}(z) \tag{33'}$$

Cette expression de f sert à la construction de (32).

PROPOSITION 7. *La solution générale n-constante de l'équation* (29) *avec* $a_j \in C$ *et* $F: C \to C$ *est de la forme* (31), *tandis que la solution continue est* (32).

Démonstration. En utilisant les deux propositions précédentes, il est suffisant de remarquer que C est un corp fermé.

Remarque. Lorsqu'on considère les coefficients de l'équation (29) $a_j \in R$ et $f: R \to R$ on retrouve l'équation de T. Popoviciu [8] et son résultat contenu dans la proposition suivante :

PROPOSITION 8. *La solution générale n-continue de l'équation de T. Popoviciu est de forme* :

$$f^n(x) = \sum_{v^n} \prod_1^n \alpha_{n_j} f(v^n) + \sum_{v^{n-1}} \prod_1^{n-1} \alpha_{a_j} f(v^{n-1}) + \ldots + f^0 \tag{34}$$

ou $v^s = (a_1, \ldots, a_s) \in B^s(Q, R)$, $f: B^s \to R$, $\alpha_{a_i} \in Q$, $s = \overline{1, n}$ *et la solution continue est le polynome n-ème degré.*

Démonstration. On va montrer que si f est n-continue, il résulte que f est n-constante et puis la proposition est une conséquence aux propositions précédentes.

Si f est n-continue, alors $\psi(x, y) = \Delta_y^n f(x)$ est continue par rapport à x. Mais on a $\psi(x, py) = p^n\psi(x, y)$ $\forall x, y \in R$, $\forall p \in Z$. En prenant $u = py$ on obtient $\psi(x, u) = p^n\psi\left(x, \frac{u}{p}\right)$ et puis $\psi\left(x, \frac{1}{p} u\right) = \left(\frac{1}{p}\right)^n \psi(x, u)$ $\forall x,u \in R$ et $\forall p \in Z$. On peut dire alors que $\psi(x, ry) = r^n\psi(x, y) \forall x, y \in R$ et $\forall r \in Q$. Si f satisfait (16) alors ψ satisfait $\psi(x + ry, ry) = \psi(x, ry)$. On obtient alors $r^n\psi(x + ry, y) = r^n\psi(x, y)$ $\forall x,y \in R$, $r \in Q$. Soit maintenant $x,y \neq 0$, $u \in R$; Il existe alors $\{r_n\} \to \frac{u}{y}$. On a donc, $\psi(x + r_n y, y) = \psi(x, y)$. On peut passer à la limite en obtenant $\psi(x + u, y) = \psi(x, y)$ $\forall x, y, u \in R$ d'où résulte que ψ est constante par rapport à x, c'est-à-dire f est n-constante, ce qu'on devait montrer pour démontrer la proposition.

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ASUPRA UNOR ECUAȚII FUNCȚIONALE DE ORDIN SUPERIOR

(Rezumat)

Articolul studiază unele ecuații funcționale numite de mulți autori ca fiind de ordin superior.

Se introduc, în primul paragraf, funcții și polinoame Hamel de ordin superior prin formulele (2), (3), (6) și (7), precum și diferențe ,,strîmbe'' (5), în raport cu un șir de scalari. În continuare, prin propozițiile 1, 2 și 3, se studiază proprietățile acestor noțiuni. Printre acestea se remarcă aceea că polinoamele Hamel satisfac ecuația (16).

Cu ajutorul lemelor 1, 2, 3 și 4 se demonstrează propoziția 4, care dă soluțiile ecuației (16) în ipoteza că F este n-constantă.

Prin propoziția 5 se rezolvă în condiții generale ecuația (29), studiată de T. Popoviciu.

În încheierea articolului se rezolvă ecuațiile (16) și (29) pentru $f: C \to C$.

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BOUNDARY LAYER GROWTH ON A SPINNING BODY

IOAN POP

1. Introduction. In recent years there has been a good deal of discussion of the boundary layer growth problem on a body. Thus Illingworth [3] has initiated the study of this problem for an axisymmetric body which is instantaneously given a constant velocity along its axis and a constant angular veloity about it in an incompresible viscous fluid of infinite extent. After the initiation by Illingworth have followed many papers for the same problem under a wide variety of boundary conditions. Mention may be made of the papers of Wadhwa [11], Durić [1], Pop [4—7], and Pop and Soundalgekar [8]. Several other papers have made useful contributions to this topic by using the method of matched inner and outer expansion. Thus following this technique Srivastava [10], Sozou [9] and Zapryanov [13] have considered the impulsively started spinning sphere problem for finite Reynolds numbers. In particular their analysis includes that of Illingworth as a special case, which is given in the limit of Reynolds number tends to infinity.

Recently Durić [2] has developed a method of the universalisation of the unsteady two dimensional incompressible boundary layer equations in the sense that neither equations nor boundary conditions depend on particular problem. In the present paper a study on the lines of Durić's method of the growth of the boundary layer on an axisymmetric body spinning about its axis is presented. The theory is applied to the sphere for which the time of separation is determined.

2. Equations of the problem. Using orthogonal curvilinear coordinates (x, y, z), with x, measured along the surface of the body, y in the normal direction to the body and z on the direction of the arc of the transversal crossection, the incompessible boundary layer equations which govern unsteady flow on a body of revolution spinning about its axis, which is parallel to the stream are

$$\frac{\partial^2\psi}{\partial t\partial y} + \frac{1}{r}\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{1}{r^2}\frac{dr}{dx}\left(\frac{\partial\psi}{\partial y}\right)^2 - \frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} - w^2\frac{dr}{dx} = = r\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}\right) + \nu\frac{\partial^3\psi}{\partial y^3}, \tag{1}$$

$$\frac{\partial w}{\partial t} + \frac{1}{r}\frac{\partial\psi}{\partial y}\frac{\partial w}{\partial x} - \frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial w}{\partial y} + \frac{w}{r^2}\frac{dr}{dx}\frac{\partial\psi}{\partial y} = \nu\frac{\partial^2 w}{\partial y^2} \tag{2}$$

where $\psi(x, y, t)$ is the stream function, $w(x, y, t)$ is the velocity component in the z direction, $U(x, t)$ is the main stream velocity, $r(x)$ is the radius of the transversal crosection and ν is the kinematic viscosity of the fluid. The boundary conditions of the problem are the following:

$$
\begin{aligned}
t \leqslant 0: &\quad \frac{1}{r}\frac{\partial\psi}{\partial y} = U(x, t),\ w = 0 \text{ everywhere,} \\
t > 0: &\quad \psi = \frac{\partial\psi}{\partial y} = 0,\ w = r\omega(t) \text{ on } y = 0, \\
&\quad \frac{1}{r}\frac{\partial\psi}{\partial y} \to U(x, t),\ w \to 0 \text{ as } y \to \infty.
\end{aligned}
\tag{3}
$$

We will be interested im those cases where similarity solutions are possible, i.e.

$$
U(x, t) = \Omega(t) V(x), \qquad \omega(t) = \omega_0 \Omega(t)
$$

where ω_0 is a constant. Next we make use of the momentum equation for a flat plate

$$
\frac{dz_p}{dt} = 2R \tag{4}
$$

where

$$
R = \xi_p - \alpha_1,\ \xi_p = \left[\frac{\partial(u/\Omega)}{\partial(y/\delta_p)}\right]_{y=0},\ \alpha_1 = \frac{1}{\Omega}\frac{d\Omega}{dt} z_p
$$

$$
z_p = \frac{\delta_p^2}{\nu}, \qquad \delta_p(t) = \int_0^\infty \left(1 - \frac{u}{U}\right) dy
$$

$\delta_p(t)$ being the displacement thickness for a flat plate.

Let us now consider the new variable $\eta = Ay/\delta_p$ where A is a norming constant, and the sets of parameters

$$
\begin{aligned}
\alpha_i &= \frac{1}{\Omega^i}\frac{d^i\Omega}{dt^i} z_p^i, \quad \beta_i = \Omega^i z_p^i V^{i-1}\frac{d^i V}{dx^i}, \\
\gamma_i &= \frac{1}{\Omega^i} z_p^i \frac{V^i}{r}\frac{d^i r}{dx^i},\ i = 1, 2, \ldots
\end{aligned}
\tag{5}
$$

In view of (4) and (5) we have

$$
\frac{\partial}{\partial t} = \frac{1}{z_p}\sum_{i=1}^{n}\left(a_i\frac{\partial}{\partial\alpha_i} + b_i\frac{\partial}{\partial\beta_i} + c_i\frac{\partial}{\partial\gamma_i}\right)
$$

$$
\frac{\partial}{\partial x} = \frac{1}{\Omega V z_p}\sum_{i=1}^{n}\left(d_i\frac{\partial}{\partial\beta_i} + e_i\frac{\partial}{\partial\gamma_i}\right)
$$

where
$$a_i = (-\alpha_1 + 2iR)\alpha_i + \alpha_{i+1},\ b_i = (i\alpha_1 + 2iR)\beta_i,$$
$$c_i = (i\alpha_1 + 2iR)\gamma_i,\ d_i = (i-1)\beta_1\beta_i + \beta_{i+1},$$
$$e_i = (i\beta_1 - \gamma_1)\gamma_i + \gamma_{i+1}.$$

By putting
$$\psi(x,\ y,\ t) = \frac{r\delta_p}{A}\,\Omega V F(\eta,\ \alpha_i,\ \beta_i,\ \gamma_i)$$
$$w(s,\ y,\ t) = \omega_0 r \Omega H(\eta,\ \alpha_i,\ \beta_j,\ \gamma_i)$$

in equations (1), (2) and (3), it follows
$$A^2\frac{\partial^3 F}{\partial\eta^3} + R\eta\frac{\partial^2 F}{\partial\eta^2} + \beta_1\left[1 - \left(\frac{\partial F}{\partial\eta}\right)^2 + F\frac{\partial^2 F}{\partial\eta^2}\right] +$$
$$+ \alpha_1\left(1 - \frac{\partial F}{\partial\eta}\right) + \gamma_1\left(F\frac{\partial^2 F}{\partial\eta^2} + \gamma H^2\right) =$$
$$= \sum_{i=1}^{n} d_i\left(\frac{\partial^2 F}{\partial\eta\partial\eta_i}\frac{\partial F}{\partial\eta} - \frac{\partial F}{\partial\beta_i}\frac{\partial F}{\partial\tau_i}\right) + c_i\left(\frac{\partial^2 F}{\partial\eta\partial\gamma_i}\frac{\partial F}{\partial\eta} - \frac{\partial F}{\partial\gamma_i}\frac{\partial^2 F}{\partial\eta^2}\right)\Bigg] + \tag{6}$$
$$+ \sum_{i=1}^{n}\left(a_i\frac{\partial^2 F}{\partial\alpha_i\partial\eta} + b_i\frac{\partial^2 F}{\partial\beta_i\partial\eta} + c_i\frac{\partial^2 F}{\partial\gamma_i\partial\eta}\right),$$

$$A^2\frac{\partial^2 H}{\partial\eta^2} + R\eta\frac{\partial H}{\partial\eta} + \beta_1 F\frac{\partial H}{\partial\eta} - \alpha_1 H + \gamma_1\left(F\frac{\partial H}{\partial\eta} - 2H\frac{\partial F}{\partial\eta}\right) =$$
$$= \sum_{i=1}^{n}\left[d_i\left(\frac{\partial H}{\partial\beta_i}\frac{\partial F}{\partial\eta} - \frac{\partial F}{\partial\beta_i}\frac{\partial H}{\partial\eta}\right) + e_i\left(\frac{\partial H}{\partial\gamma_i}\frac{\partial F}{\partial\eta} - \frac{\partial F}{\partial\gamma_i}\frac{\partial H}{\partial\eta}\right)\right] + \tag{7}$$
$$+ \sum_{i=1}^{n}\left(a_i\frac{\partial H}{\partial\alpha_i} + b_i\frac{\partial H}{\partial\beta_i} + c\frac{\partial H}{\partial\gamma_i}\right)$$

where $\gamma = (\omega_0 r/V)^2$ is the spin parameter held constant for the sake of universalisation of these equations. The boundary conditions of the above equations are

$$t \leqslant 0: \quad \frac{\partial F}{\partial\eta} = 1, \quad H = 0 \quad \text{for all } \eta,$$
$$t > 0: \quad F = \frac{\partial F}{\partial\eta} = 0, \quad H = 1 \text{ at } \eta = 0,$$
$$\frac{\partial F}{\partial\eta} \to 1, \quad H \to 0 \text{ as } \eta \to \infty.$$

The equations (6) and (7) are universal equations of unsteady boundary layer on a spinning body. They do not contain values characteristics of some given particular problem. These equations can be integrated numerically by means of the electronic computer or analytically by writting the functions F and H as the series

$$F = F_0(\eta,\ \alpha_i) + \beta_1 F_{01}(\eta,\ \alpha_i) + \gamma_1 F_{1a}(\eta,\ \alpha_i) +$$
$$+ \beta_1^2 F_{02}(\eta,\ \alpha_i) + \gamma_1^2 F_{1a}(\eta,\ \alpha_i) + \beta_1\gamma_1 F_{12}(\eta,\ \alpha_i) + \dots$$
$$H = H_0(\eta,\ \alpha_i) + \beta_1 H_{01}(\eta,\ \alpha_i) + \gamma_1 H_{1a}(\eta,\ \alpha_i) +$$
$$+ \beta_1^2 H_{02}(\eta,\ \alpha_i) + \gamma_1^2 H_{1a}(\eta,\ \alpha_i) + \beta_1\gamma_1 H_{12}(\eta,\ \alpha_i) + \dots$$

where each F_i and H_i may be expanded as a series of α_i according to the expressions

$$\begin{aligned}
F_0 &= f_{0,0}(\eta) + \alpha_1 f_{0,1}(\eta) + \alpha_1^2 f_{0,11}(\eta) + \alpha_2 f_{0,2}(\eta) + \dots \\
F_{01} &= f_{01,0}(\eta) + \alpha_1 f_{01,1}(\eta) + \alpha_1^2 f_{01,11}(\eta) + \dots \\
F_{1a} &= f_{1a,0}(\eta) + \alpha_1 f_{1a,1}(\eta) + \alpha_1^2 f_{1a,11}(\eta) + \dots \\
H_0 &= h_{0,0}(\eta) + \alpha_1 h_{0,1}(\eta) + \alpha_1^2 h_{0,11}(\eta) + \dots \\
H_{01} &= h_{01,0}(\eta) + \alpha_1 h_{01,1}(\eta) + \alpha_1^2 h_{01,11}(\eta) + \dots \\
H_{1a} &= h_{1a,0}(\eta) + \alpha_1 h_{1a,1}(\eta) + \alpha_1^2 h_{1a,11}(\eta) + \dots\,.
\end{aligned} \tag{8}$$

A suitable form of the function $R(\alpha_i)$ is the same as that of the function $F_0(\eta,\ \alpha_i)$ namely

$$R = R_0 + \alpha_1 R_1 + \alpha_1^2 R_{11} + \alpha_2 R_2 + \dots \tag{9}$$

where R_i are constants.

The functions f_i and h_i are found to satisfy

$$\begin{aligned}
D_0(f'_{0,0}) &= 0 \\
D_1(f'_{0,1}) &= -\frac{1}{A^2}\,(R_1\eta f''_{0,0} - f'_{0,0} + 1) \\
D_2(f'_{0,11}) &= -\frac{1}{A^2}\,(R_1\eta f''_{0,1} + R_{11}\,\eta f''_{0,0} - 2R_1 f'_{0,1}) \\
&\dots \qquad \dots \qquad \dots \qquad \dots \\
D_1(f'_{01,0}) &= -\frac{1}{A^2}\,(1 - f'^2_{0,0} + f_{0,0} f''_{0,0}) \\
D_2(f'_{01,1}) &= \frac{1}{A^2}\,[2(R_1 + 1)f'_{01,1} - R_1\eta f''_{01,0} + 2f'_{0,0}f'_{0,1} - f_{0,0}f''_{0,1} - f''_{0,0}f_{0,1}]
\end{aligned} \tag{10}$$

$$D_1(f'_{1a,0}) = -\frac{1}{A^2}(f_{0,0}f''_{0,0} + \sigma h^2_{0,0})$$

$$D_2(f'_{1a,1}) = -\frac{1}{A^2}[R_1\eta f''_{1a,0} - 2(R_1+1)f'_{1a,0} + f_{0,0}f''_{0,1} + f''_{0,1}f_{0,1} + 2\sigma h_{0,0}h_{0,1}];$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$D_0(h_{0,0}) = 0$$

$$D_1(h_{0,1}) = -\frac{1}{A^2}(R_1\eta h'_{0,0} - h_{0,0})$$

$$D_2(h_{0,11}) = \frac{1}{A^2}(2R_1h_{0,1} - R_{11}\eta h'_{0,0} - R_1\eta h'_{0,1})$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$D_1(h_{01,0}) = -\frac{1}{A^2}f_{0,0}h'_{0,0}$$

$$D_2(h_{01,1}) = \frac{1}{A^2}[2(R_1+1)h_{01,0} - R_1\eta h'_{01,0} - h'_{0,0}f_{0,1} - f_{0,0}h'_{0,1}] \qquad (11$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$D_1(h_{1a,0}) = \frac{1}{A^2}(2h_{0,0}f'_{0,0} - f_{0,0}h'_{0,0})$$

$$D_2(h_{1a,1}) = \frac{1}{A^2}[2(R_1+1)h_{1a,0} - R_1\eta h'_{1a,0} + 2h_{0,0}f'_{0,1} - f_{0,0}h'_{0,1} + 2h_{0,1}f'_{0,0} - h'_{0,0}f_{0,1}$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

where the primes denote differentiation with respect to η and for convenien we have defined the operator

$$D_i = \frac{d^2}{d\eta^2} + \frac{R_0}{A^2}\eta\frac{d}{d\eta} - 2\frac{R_0}{A^2}i.$$

The boundary conditions of (10) and (11) are

$$f_{0,0}(0) = f'_{0,0}(0) = 0,\ h_{0,0}(0) = 1,\ f'_{0,0}(\infty),\ h_{0,0}(\infty) \to 0,$$

$$f_{0,1}(0) = f'_{0,1}(0) = h_{0,1}(0) = f_{01,0}(0) = f'_{01,0}(0) = h_{01,0}(0) =$$

$$= f_{1a,0}(0) = f'_{1a,0}(0) = h_{1a,0}(0) = \ldots = 0,$$

$$f'_{0,1}(\infty),\ h_{0,1}(\infty),\ h'_{01,0}(\infty),\ f'_{1a,0}(\infty),\ h_{1a,0}(\infty),\ \ldots, \to 0.$$

From the equation

$$R = A\left(\frac{\partial^2 F_0}{\partial\eta^2}\right) = -\alpha_1$$

taking into account (8) and (9) we have

$$R_0 = Af''_{0,0}(0),\ R_1 = Af''_{0,1}(0) - 1,\ R_{11} = Af''_{0,11}(0), \ldots$$

A further condition used in the subsequent analysis is

$$\frac{R_0}{A^2} = 2. \tag{14}$$

This is obtained requisteng that the ordinary differential equations (10) and (11) become of parabolique type. Consequently the equations (10) and (11) subject to (12) can be solved in a straightforward manner in terms of Gauss error function (see W a t s o n [12]), giving

$$\begin{aligned}
f_{0,0}(\eta) &= \eta + g_{1/2}(\eta) - \frac{1}{\sqrt{\pi}} \\
f'_{0,1}(\eta) &= \frac{1}{A^2}\left[\frac{1}{4}g_0(\eta) + \frac{R_1}{16}g_{-1}(\eta) - g_1(\eta)\right] \\
f'_{0,11}(\eta) &= \frac{1}{A^2}\left[\frac{1}{24}\left(R_{11} - \frac{R_1}{4A^2} - \frac{R_1^2}{2}\right)g_{-1}(\eta) - \frac{R_1^2}{512}g_{-2}(\eta)\right] \\
f'_{01,0}(\eta) &= \frac{1}{2A^2}\left[g_0(\eta) - \left(3 + \frac{4}{3\pi}\right)g_1(\eta) - \frac{1}{3\sqrt{\pi}}g_{-1/2}(\eta) + \right. \\
&\quad \left. + \frac{1}{8}g_{-1}(\eta) + g_{1/2}^2(\eta) - g^0(\eta)g_1(\eta)\right] \\
f'_{1a,}(\eta)_0 &= \left(\frac{\pi}{2} + \frac{4}{3} + 2\sigma\right)g_1(\eta) + \frac{\pi}{16}g_{-1}(\eta) - \frac{\pi}{6\sqrt{\pi}}g_{-1/2}(\eta) - \\
&\quad - \frac{\pi}{2}g_0(\eta)g_1(\eta) - \frac{\pi}{2}\sigma g_{1/}^2(\eta)\,;
\end{aligned} \tag{15}$$

. . . .

$$\begin{aligned}
h_{0,0}(\eta) &= g^0(\eta) \\
h_{0,1}(\eta) &= \frac{1}{A^2}\left[g_1(\eta) - \frac{1}{4}g^0(\eta) - \frac{R_1}{16}g_{-1}(\eta)\right] \\
h_{01,0}(\eta) &= \frac{1}{A^2}\left[-\left(\frac{1}{2} + \frac{4}{3\pi}\right)g_1(\eta) - \frac{1}{16}g_{-1}(\eta) + \right. \\
&\quad \left. + \frac{1}{2}g_0(\eta)g_1(\eta) + \frac{1}{6\sqrt{\pi}}g_{-1/2}(\eta)\right] \\
h_{1a,0}(\eta) &= \frac{1}{A^2}\left[\left(\frac{3}{2} + \frac{8}{3\pi}\right)g_1(\eta) - \frac{1}{16}g_{-1}(\eta) + \frac{1}{6\sqrt{\pi}}g_{-1/2}(\eta) - \right. \\
&\quad \left. - \frac{1}{2}g_0(\eta) + \frac{1}{2}g_0(\eta)g_1(\eta) - g_{1/2}^2(\eta)\right]
\end{aligned} \tag{16}$$

Now from (13)—(16) we otain

$$A=\frac{1}{\sqrt{\pi}},\quad R_0=\frac{2}{\pi},\quad R_1=-\frac{2}{3},\quad R_{11}=\frac{\pi}{72}.$$

By considering the parameter α_1 sufficiently small from equation (4) it follow

$$z_p(t)=\frac{4}{\pi}\,\Omega^{-4/3}\int_0^t \Omega^{4/3}dt.$$

As is known the separation flow from the wall occurs when $(\partial u/\partial y)_{y=0}$ becomes zero. This gives the equation

$$f''_{0,0}(0)+\alpha_1 f''_{0,1}(0)+\beta_1 f''_{01,0}(0)+\gamma_1 f''_{1a,0}(0)=0 \tag{17}$$

where

$$f''_{0,0}(0)=\frac{2}{\sqrt{\pi}},\quad f''_{0,1}(0)=\frac{\sqrt{\pi}}{3},\quad f''_{01,0}(0)=\frac{\sqrt{\pi}}{2}\left(1+\frac{4}{3\pi}\right)$$

$$f''_{1a,0}(0)=\frac{1}{\sqrt{\pi}}\left[\frac{\pi}{2}-\frac{4}{3}+(\pi-2)\sigma\right].$$

3. Application. Let the sphere of radius a put into spiral motion by an impulsively jerk (fig. 1). Then we have

$$r(x)=a\sin\frac{x}{a},\quad V(x)=\frac{3}{2}\,U\sin\frac{x}{a},\quad \Omega(t)=1$$

and hence

$$z_p=\frac{4}{\pi}t,\quad \alpha_1=\frac{1}{\Omega}\frac{d\Omega}{dt}z_p=0,\quad \beta_1=\Omega z_p\frac{dV}{dx}=\frac{6}{\pi a}tU_\infty\cos\frac{x}{a},$$

$$\gamma_1=\Omega z_p\frac{V}{r}\frac{dr}{dx}=\frac{6}{\pi a}tU_\infty\cos\frac{x}{a},\quad \sigma=\frac{4}{9}\bar{\sigma}$$

where $\bar{\sigma}=(\omega_0 a/U\infty)^2$. If we take into account that the separation correspond to having $\cos\frac{x}{a}=-1$, i.e. the las stagnation point, the times t_s at which separation occurs using (17) is given by

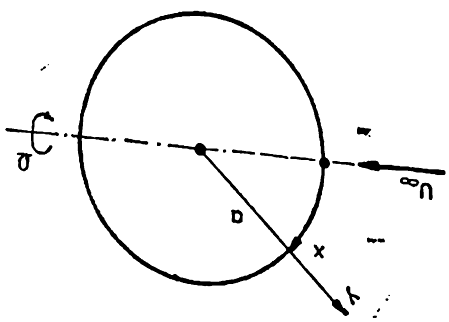

Fig. 1.

$$\bar{t}_s=\frac{1}{9\pi-6+4(\pi-2)\bar{\sigma}}= \tag{18}$$

$$=\frac{1}{2{,}3636+0{,}4845\bar{\sigma}}$$

which coincides with that of Illingworth [3].

In the case of a uniform accelerated motion of the same sphere we have

$$\Omega(t) = t, \quad z_p = \frac{12}{7\pi}\, t, \quad \alpha_1 = \frac{12}{7\pi}$$

$$\beta_1 = \gamma_1 = \frac{18}{7\pi a}\, t^2 U_\infty \cos\frac{x}{a}$$

and the separation time is

$$\bar{t}_s^2 = \frac{9}{9\pi - 6 + 4(\pi - 2)\bar{\sigma}} = \frac{1}{0{,}78769 + 0{,}16136\bar{\sigma}} \tag{19}$$

which agrees with that of Wadhwa [10].

Some numerical values of the time t_s at which separation takes place, calculated from (18) and (19), are listed in the accompanying table. It is noted that the time of earliest separation decreases with the increase in the angular velocity.

Table 1

Some values of t_s calculated from (18) and (19)

$\bar{\sigma}$	$t_s^{\text{imp.}}$	$t_s^{\text{accel.}}$
0	0.4231	1,1267
1	0.3511	1.0264
5	0.2089	0.7260
10	0.1387	0.6453
100	0.0197	0.2431

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CREŞTEREA STRATULUI LIMITĂ PE UN CORP ÎN MIŞCARE DE SPIN

(R e z u m a t)

Se studiază formarea stratului limită pe un corp axial-simetric în mişcare de spin (transla şi rotaţie în jurul aceleiaşi axe). Prima dată se universalizează ecuaţiile stratului limită, în sen că atît ecuaţiile cît şi condiţiile la limită nu depind de condiţiile unei probleme particulare. Ast universalizarea este realizată transformînd seturile de parametri, care exprimă influenţa timpu şi a condiţiilor la limită, în variabile independente. În continuare se caută soluţia ecuaţiilor obţin în forma dezvoltărilor în serie după diferiţi parametri. Rezultatele găsite sînt aplicate la cazul u sfere, calculîndu-se timpul de separare al stratului limită de suprafaţa ei. Evaluările stabilite coin pe deplin cu cele găsite de alţi autori, pe alte căi.

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EXTENSION OF COLLINEATIONS DEFINED ON SUBSETS OF A PAPPIAN PROJECTIVE PLANE CONTAINING A CONIC

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1. **Introduction.** A series of papers deals with the extension to the entire projective plane of a collineation defined on certain subset of the plane [1], [2], [4], [5], [6], [7]. In [6] F. Radó has proved the following theorem: Let $\mathfrak{L}$, $\mathfrak{L}'$ be the point sets of two Desarguesian projective planes and $\mathfrak{C}$ a subset of $\mathfrak{L}$ containing three non-concurrent lines and a point non-incident with them; then every collineation $\Phi : \mathfrak{C} \to \mathfrak{L}'$ can be extended to a collineation defined on the whole plane $\mathfrak{L}'$. This result was generalized by the author in the sense that the set $\mathfrak{C}$ is supposed to contain only certain sets of points lying on three lines and one point more [4], [5].

The purpose of the present paper is to generalize the above theorem of F. Rado in an other direction. Using a method of functional equation, we shall extend collineations defined on sets containing a coinc, a line and a point not belonging to them.

2. **Theorem.** *Let $\mathfrak{L}$ and $\mathfrak{L}'$ be the point-sets of two Pappian projective planes over fields K and K' having characteristics different from 2) and let $\mathfrak{C} \subset \mathfrak{L}$ contain all the points of a non-degenerate conic Γ, all the points of a line l and a point $A \notin \Gamma \cup l$. If A is not the pole of the line l with respect to Γ and $\Gamma \cap l \neq \emptyset$, then for every injective collineation $\Phi : \mathfrak{C} \to \mathfrak{L}'$ with $\Phi(\Gamma) = \Gamma'$, where Γ' is a non degenerate conic, there exists a unique collineation $f : \mathfrak{L} \to \mathfrak{L}'$ such that $f|\mathfrak{C} = \Phi$.*

Proof. Since $\Gamma \cap l \neq \emptyset$, we have to distinguish only two cases: 1^0 the line l is a tangent to Γ, 2^0 l and Γ have two common points.

Case 1^0. Let X be the points of contact of the line l with Γ and let Y be the second point in which the line AX interesects the conic Γ. Denote by O the point of intersection of l with the tangent to Γ in the point Y. Choose a point $E \in \Gamma\setminus\{X, Y\}$.

Consider in the plane $\mathfrak{L}$ a projective coordinate system assigning to the point O, X, Y, E $(0, 0, 1)$, $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 1)$ respectively. A point $M \in l\setminus\{X\}$ has coordinates of the form $(\lambda, 0, 1)$, $\lambda \in K$. The point $N(x_1, x_2, x_3)$ lies on Γ if and only if $x_1x_2 = x_3^2$, thus $N \in \Gamma\setminus\{X\}$ is described by $(t^2, 1, t)$, $t \in K$.

We choose in the plane $\mathfrak{L}'$ a projective coordinate system in such a way that the points $X' := \Phi(X)$, $Y' := \Phi(Y)$, $0' := \Phi(0)$, $E' := \Phi(E)$ get the same coordinates as X, Y, O, E. Then the coordinates of the points $\Phi(M)$ will be $[\psi_1(\lambda), 0, 1]$, where $\psi_1 : K \to K'$ is a certain map. It follows from the hypothesis of the theorem that the tangents at X and Y to the conic $\Gamma' = \Phi(\Gamma)$ coincide with the lines $\Phi(l)$ and $O'Y'$; thus the image $\Phi(N)$ of the point $N \in \Gamma\backslash\{X\}$ will have coordinates $[\psi^2(t),\ 1,\ \psi(t)]$, where $\psi: K \to K'$. Sino $(0,\ 0,\ 1) \xrightarrow{\Phi} (0,\ 0,\ 1)$ and $(1,\ 0,\ 0) \xrightarrow{\Phi} (1,\ 0,\ 0)$, we have

$$\psi_1(0) = \psi(0) = 0. \tag{1}$$

The collineation Φ takes the point $YE \cap OX$ in $Y'E' \cap O'X'$, hence $(1,\ 0,\ 1) \xrightarrow{\Phi} (1,\ 0,\ 1)$ and

$$\psi_1(1) = 1. \tag{2}$$

From $E' \in \Phi(\Gamma)$ we deduce

$$\psi(1) = 1. \tag{3}$$

Let $N(t^2,\ 1,\ t)$, $N_1(t'^2,\ 1,\ t')$ two distinct points on $\Gamma\backslash\{X\}$ and let $M(\lambda, 0, 1) \in l\backslash\{X\}$. The points N, N_1 and M are collinear if and only if

$$\lambda = t + t' \tag{4}$$

The collinearity of N, N_1, M implies the collinearity of $\Phi(N)$, $\Phi(N_1)$, $\Phi(M)$ hence (4) implies

$$\psi_1(\lambda) = \psi(t) + \psi(t'). \tag{5}$$

Setting $t' = 0$ in (5), we get by (1) and (4) :

$$\psi_1 = \psi, \tag{6}$$

and therefore (5) becomes

$$\psi(t + t') = \psi(t) + \psi(t'), \qquad \forall t,\ t' \in K. \tag{7}$$

We have $A(a,\ 1,\ 0)$ with $a \in K\backslash\{0\}$. The points $N(t^2,\ 1,\ t)$ $N_1(t'^2, 1, t')$ and A are collinear if and only if $tt' + a = 0$. This equality implies $\psi(t)\psi(t') + a' = 0$, where $(a',\ 1,\ 0)$ are coordinates of $\Phi(A)$, $a' \in K'\backslash\{0\}$. We get for $t \in K\backslash\{0\}$

$$\psi(t)\psi\left(-\frac{a}{t}\right) = -a'. \tag{8}$$

It follows from (1) and (7) that $\psi(-t) = -\psi(t)$, hence we have by (8) and (3) $a' = \psi(a)$ and

$$\psi\left(\frac{a}{t}\right) = \frac{\psi(a)}{\psi(t)}. \tag{9}$$

Put in (9) $t = \dfrac{a}{x-1} - \dfrac{a}{x}$, $x \neq 0,\ 1$ and consider (7) and (9) to write successively:

$$\psi[x(x-1)] = \psi(a)\{\psi[a(x-1)^{-1} - ax^{-1}]\}^{-1} = \psi(a)\{\psi[a(x-1)^{-1}] - \psi[ax^{-1}]\}^{-1} =$$
$$= \{[\psi(x) - 1]^{-1} - [\psi(x)]^{-1}\}^{-1} = \psi(x)[\psi(x-1)].$$

Hence, by (7),

$$\psi(x^2) = \psi^2(x), \qquad \forall x \in K. \tag{10}$$

In view of (10) and (7) we can write for any $x, y \in K$:

$$\psi[(x+y)^2] = \psi(x^2 + 2xy + y^2) = \psi(x^2) + 2\psi(xy) + \psi(y^2) =$$
$$= [\psi(x+y)]^2 = [\psi(x) + \psi(y)]^2 = \psi^2(x) + 2\psi(x)\psi(y) + \psi^2(y),$$

yielding for all $x, y \in K$

$$\psi(xy) = \psi(x)\psi(y). \tag{11}$$

Case 2^0. Let X, Y be the common points of l and Γ and O the pole of the line l with respect to Γ. Since $A \neq O$, at least one of the lines XA, YA has a second point of intersection with Γ, say $\{E\} = XA \cap \Gamma$, $E \neq X, Y$. Choose a projective coordinate system in $\mathfrak{L}$ such that O, X, Y, E receive coordinates $(0, 0, 1)$, $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 1)$ respectively. Then $(1, \lambda, 0)$, $\lambda \in K$ represents a point $M \in l\backslash\{Y\}$, and $(1, t^2, t)$ a point $N \in \Gamma\backslash\{Y\}$ (the equation of Γ being $x_1x_2 = x_3^2$). Denote the coordinates of A by $(a, 1, 1)$, $a \in K\backslash\{1\}$.

The line $l' = \Phi(l)$ intersects the conic Γ' in $X' = \Phi(X)$ and $Y' = \Phi(Y)$. Let O' be the pole of l' with respect to Γ and let N_1, N_2, N_3, N_4 be points on Γ such that $N_1N_2 \cap N_3N_4 = \{O\}$. Then two diagonal points M_1 and M_2 of the complete quadrangle $N_1N_2N_3N_4$ lie on l, while the third is the point O. Since the map Φ is a collineation on $\Gamma \cup l$, the quadrangle $N_1'N_2'N_3'N_4'$ ($N_i' = \psi(N_i)$, $i = 1, \ldots, 4$) will have two diagonal points, $\Phi(M_1)$ and $\Phi(M_2)$ on l' and the third is the pole of l' with respect to Γ', i.e. O'. Thus we have

LEMMA 1. *If* $N_1, N_2 \in \Gamma$ *and* O, N_1, N_2 *are collinear, then the points* O', $\Phi(N_1)$, $\Phi(N_2)$ *are collinear.*

LEMMA 2. *If the points* $M_1, M_2 \in l$ *are conjugate with respect to* Γ, *then the points* $\Phi(M_1), \Phi(M_2) \in l'$ *are conjugate with respect to* Γ'.

Let $A' := \Phi(A)$ and $E' := \Phi(E)$. Choose a coordinate system in $\mathfrak{L}'$ such that the points O', X', Y', E' receive the same coordinates as the points O, X, Y, E have in $\mathfrak{L}$. Then the coordinates of $\Phi(M)$, $\Phi(N)$ writes $(1, \psi_1(\lambda), 0)$, $(1, \psi^2(t), \psi(t))$ respectively, with $\psi_1, \psi: K \to K'$.

From $X(1, 0, 0) \xrightarrow{\Phi} X'(1, 0, 0)$ and $E(1, 1, 1) \xrightarrow{\Phi} E'(1, 1, 1)$ it follows that

$$\psi_1(0) = \psi(0) = 0 \tag{12}$$

and

$$\psi(1) = 1. \tag{13}$$

If $E_1 := OE \cap l$, $E_1' := O'E' \cap l'$, we have by lemma 1:

$E_1(1, 1, 0) \xrightarrow{\Phi} E_1'(1, 1, 0)$ and

$$\psi_1(1) = 1. \tag{14}$$

By lemma 2, $(1, -1, 0) \xrightarrow{\Phi} (1, -1, 0)$, hence

$$\psi_1(-1) = -1 \tag{15}$$

The points $N(1, t^2, t)$, $N_1(1, t'^2, t') \in \Gamma\setminus\{Y\}$ and $M(1, \lambda, 0) \in l\setminus\{Y\}$ a collinear if and only if $\lambda = tt'$. Since Φ is a collineation

$$\lambda = tt' \Rightarrow \psi_1(\lambda) = \psi(t)\psi(t'),$$

i.e.

$$\psi_1(tt') = \psi(t)\psi(t'), \qquad \forall t,\ t' \in K. \tag{16}$$

For $t' = 1$ we get by (13)

$$\psi_1(t) = \psi(t), \qquad \forall t \in K, \tag{17}$$

hence

$$\psi(tt') = \psi(t)\psi(t'), \qquad \forall t,\ t' \in K. \tag{18}$$

Set in (18) $t' = t^{-1}$, then $t' = -1$, we obtain by (14), (15) and (17)

$$\psi(t^{-1}) = [\psi(t)]^{-1}, \qquad \forall t,\ t' \in K\setminus\{0\}, \tag{19}$$

$$\psi(-t) = -\psi(t), \qquad \forall t,\ t' \in K. \tag{20}$$

The points A and $A' = \Phi(A)$ have $(a, 1, 1)$ and $(a', 1, 1)$ as coordinates, where $a \in K\setminus\{1\}$, $a' \in K\setminus\{1\}$. The points $A(a, 1, 1)$, $N(1, t^2, t)$, $N'(1, t'^2, t')$, $t \neq t'$, are collinear if and only if

$$t + t' - att' - 1 = 0. \tag{21}$$

Relation (21) implies the collinearity of $\Phi(A)$, $\Phi(N)$, $\Phi(N')$, hence (21) implies $\psi(t) + \psi(t') - a'\psi(t)\psi(t') - 1 = 0$. Thus we have

$$\psi\left(\frac{t-1}{at-1}\right) = \frac{\psi(t)-1}{a'\psi(t)-1}, \qquad \forall t \in K\setminus\left\{\frac{1}{a}\right\}. \tag{22}$$

The line AY also intersects the conic Γ in the point $\left(1, \frac{1}{a^2}, \frac{1}{a}\right)$, which is taken by Φ in the second intersection point of $A'Y'$ with Γ', i.e. in $\left(1, \frac{1}{a'^2}, \frac{1}{a'}\right) = \left(1, \psi\left(\frac{1}{a^2}\right), \psi\left(\frac{1}{a}\right)\right)$. Using (19) we get $a' = \psi(a)$, hence by (22)

$$\psi[(t-1)(at-1)^{-1}] = [\psi(t)-1][\psi(a)\psi(t)-1]^{-1}, \ \forall t \in K\setminus\{a^{-1}\} \tag{23}$$

Setting $t = a(x-1)(ax-1)^{-1}$ in (23) we can write by (19) and (23)

$$\begin{aligned}
&\psi\{[a(x-1)(ax-1)^{-1}-1][a^2(x-1)(ax-1)^{-1}-1]^{-1}\} = \psi(a+1-ax)^{-1} = \\
&= [\psi(a+1-ax)]^{-1} = \psi[a(x-1)(ax-1)^{-1}-1]\{\psi(a)\psi[a(x-1)(ax-1)^{-1}-1]\}^{-1} = \\
&= \{\psi(a)[\psi(x)-1][\psi(a)\psi(x)-1]^{-1}-1\}\{\psi^2(a)[\psi(x)-1][\psi(a)\psi(x)-1]^{-1}-1\}^{-1} = \\
&= [\psi(a)+1-\psi(a)\psi(x)]^{-1},
\end{aligned}$$

hence

$$\psi(a+1-ax) = \psi(a) + 1 - \psi(a)\psi(x) \tag{24}$$

which, in view of (18), can be written as

$$\psi(-x + 1 + a^{-1}) = -\psi(x) + 1 + [\psi(a)]^{-1}, \tag{25}$$

yielding $\psi(1 + a^{-1}) = 1 + [\psi(a)]^{-1}$. Set $x = -(1 + a^{-1})y$ in (25) to deduce

$$\psi(y + 1) = \psi(y) + 1, \quad \forall y \in K. \tag{26}$$

Using (18), (19) and (26) we get for all $x \in K$ and all $y \in K\backslash\{0\}$

$$\psi(x + y) = \psi[y(xy^{-1} + 1)] = \psi(y)\psi(xy^{-1} + 1) =$$
$$= \psi(y)\{\psi(x)[\psi(y)]^{-1} + 1\} = \psi(x) + \psi(y),$$

hence

$$\psi(x + y) = \psi(x) + \psi(y), \quad \forall x, y \in K. \tag{27}$$

Thus in both cases 1° and 2° we have shown that the map $\psi : K \to K'$, induced by the collineation $\Phi : \mathfrak{C} \to \mathfrak{L}'$ has the following properties: $\psi(x + y) = \psi(x) + \psi(y)$ and $\psi(xy) = \psi(x)\psi(y)$, $\forall x, y \in K$.

Then the map $f : \mathfrak{L} \to \mathfrak{L}'$, defined by

$$K(x_1, x_2, x_3) \to K'[\psi(x_1), \psi(x_2), \psi(x_3)]$$

s a collineation of $\mathfrak{L}$ into $\mathfrak{L}'$ and clearly $f|_{\mathfrak{C}} = \Phi$.

The collineation $f : \mathfrak{L} \to \mathfrak{L}'$, with $f|_{\mathfrak{C}} = \Phi$, is unique, because we can draw through any point $P \in \mathfrak{L}\backslash\Gamma$ two distinct lines each intersecting in two distinct points. Thus the proof is complete.

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PRELUNGIREA COLINIAŢIILOR DEFINITE PE O SUBMULŢIME A UNUI PLAN PROIECTIV PAPPUSIAN CARE CONŢINE O CONICĂ

(Rezumat)

În lucrare se arată că orice coliniaţie definită pe o parte a unui plan proiectiv pappusian care conţine o conică Γ, o dreaptă $d(d \cap \Gamma \neq \emptyset)$ şi un punct O (diferit de polul lui d în raport cu Γ) se poate extinde într-un mod unic pe întregul plan.

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