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In memoriam

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXV, 2, 1980

ON THE PROPERTY W FOR THE MULTIPLICATIVE GROUP OF THE QUATERNIONS ALGEBRA

JAN AMBROSIEWICZ

We say that a group G has the property W if for each w is: $K_w K_w \leqslant G$, where w is the order of element $g \in G$ and $K_w K_w = \{g \in G : 0(g) = w\}$ (see [1]).

In the works [2], [3] it was proved, among other things, that the abelian groups, modular and regular groups have the property W.

Now we prove (see theorem 1) that the multiplicative group of quaternions algebra has not the property W.

First of all we establish the orders of elements of this group. The order of quaternions with norm $\neq 1$ is ∞. We establish the order of quaternion $a + bi + cj + dk$ with norm 1. Since algebra of the quaternions is isomorphic to the set of the matrices of the form

$$A = \begin{bmatrix} a + bi & c + di \\ - c + di & a - bi \end{bmatrix}$$

it is enough to establish the order of A.

The matrix A has the eigenvalue $\lambda_1 = a + i\sqrt{1 - a^2}$ and $\lambda_2 = a - i\sqrt{1 - a^2}$; therefore for a $\neq \pm 1$, $\lambda_1 \neq \lambda_2$ and then there is such an invertible matrix T that $T^{-1}AT = \text{diag}\,(\lambda_1, \lambda_2) = D$. The orders of conjugate elements, are equal, therefore $0(A) = 0(D)$.

$D = \text{diag}\,(\lambda_1, \lambda_2) = \text{diag}\,(\cos\alpha + i \sin\alpha, \cos\alpha - i \sin\alpha)$, therefore

$$D^n = E \Leftrightarrow (\cos\alpha + i \sin\alpha)^n = 1 \Leftrightarrow \alpha = \frac{2\pi S}{n}.$$

If $0(D) = n$ then it must be $(s, n) = 1$. Let $s_1, \ldots, s_m$ be such natural numbers that for $h = 1, \ldots, m$, $(s_h, n) = 1$, $0 \leqslant s_h \leqslant n - 1$.

Let $a_{s_h} = \cos \frac{2\pi Sh}{n}$, $\varphi(n)$ denote Euler function, $0(a_{s_h} + bi + cj + dk) = n$.

Then

$$K_n = \{a_{s_h} + bi + cj + dk,\ a_{s_h}^2 + b^2 + c^2 + d^2 = 1,\ h = 1, \ldots, \varphi(n)\}.$$

has a finite number of series of elements, where each series beigns with a_{s_h}.

LEMMA. *The quaternion $\alpha + \beta i + \gamma j + \delta k$ with norm 1 is a product of two quaternions of order n iff there are such r and $t \in \{1, \ldots, \varphi(n)\}$ that the following holds*

$$a_{s_r}a_{s_t} - \sqrt{a_{s_r}^2 a_{s_t}^2 - a_{s_r}^2 - a_{s_t}^2 + 1} \leqslant \alpha \leqslant a_{s_r}a_{s_t} + \sqrt{a_{s_r}^2 + a_{s_t}^2 - a_{s_r}^2 - a_{s_t}^2 + 1} \qquad (1)$$

Proof. The factoring of quaternion $\alpha + \beta i + \gamma j + \delta k$ with norm 1 into the product of two elements of order n is equivalent to solvability in real numbers of equations

$$\begin{cases} (a_{s_r} + x_r i + y j + Z_r k)(a_{s_t} + x_t i + y_t j + z_t k) = \alpha + \beta i + \gamma j + \delta k \\ a_{s_r}^2 + x_r^2 + y + z_r^2 = 1 \\ a_{s_t}^2 + x_t^2 + y + z_t^2 = 1 \end{cases} \tag{2}$$

From 2/ we have

$$a_{s_t} + x_t i + y_t j + z_t k = (a_{s_r} - x_r i - y_r j - z_r k)(\alpha + \beta i + \gamma j + \delta k)$$

Hence

$$\begin{cases} a_{s_r}\alpha + x_r\beta + y_r\gamma + z_r\delta = a_{s_t} \\ a_{s_r}^2 + x_r^2 + y_r^2 + z_r^2 = 1 \\ a_{s_t}^2 + x_t^2 + y_t^2 + z_t^2 = 1 \\ x_t = a_{s_r}\beta - x_r\alpha - y_r\delta + z_r\gamma \\ y_t = a_{s_r}\gamma + x_r\delta - y_r\alpha - z_r\beta \\ z_t = a_{s_r}\delta - x_r\gamma + y_r\beta - z_r\alpha \end{cases} \tag{3}$$

The solvability of (3) is equivalent to the non-contradiction system of equation

$$\begin{cases} x_r\beta + y_r\gamma + z_r\delta = a_{s_t} - a_{s_r}\alpha \\ a_{s_r}^2 + x_r^2 + y_r^2 + z_r^2 = 1 \end{cases} \tag{3'}$$

and hence we have the condition

$$\alpha^2 - 2a_{s_r}a_{s_t}\alpha + a_{s_r}^2 + a_{s_t}^2 - 1 \leqslant 0,$$

which, as it is easy to calculate, is equivalent to the condition (1).

Let G be the multiplicative group of algebra quaternions and G_1 be the subgroup of G, whose elements are quaternions with norm 1. We know that for each finite w the K_w is the subset of G_1 and $G - G_1$ is a section of the set K_∞ of group G.

From previous lemma we have

COROLLARY 1. *If intervals* (1) *cover the whole close interval* $\langle -1, 1 \rangle$, *then* $K_n K_n \leqslant G_1$.

For certain n, $a_{s_r} = a_{s_t} = p$ for each $r, t \in \{1, \ldots, \varphi \mid (n)\}$ (for example $n = 3$ or $n = 6$). Then the condition (1) is reduced to

$$2p^2 - 1 \leqslant \alpha \leqslant 1. \tag{4}$$

THEOREM 1. *In the multiplicative group* G_1 *the square of set* K_n ($n \neq 1, 2, 4$), *which has only one series of elements, is not a subgroup of group* G_1.

Proof. It is enough to prove the existance of two quaternions $q = \alpha + \beta i$ and $q_1 = \alpha_1 + \beta_1 i$, which are reducible into the product of two elements of order n and such that their product does not possess reducible into product of two elements of order n. Since K_n is formed, in this case, of quaternions which begin with $a_{s_i} = p$, then from (4) the problem resolves itself to showing non-contradiction of system

$$\begin{cases} -1 \leqslant \alpha\alpha_1 - \beta\beta_1 < 2p^2 - 1 \\ 2p^2 - 1 \leqslant \alpha_1 \leqslant 1 \\ 2p^2 - 1 \leqslant \alpha \leqslant 1 \\ \alpha^2 + \beta^2 = 1 \\ \alpha_1^2 + \beta_1^2 = 1 \end{cases} \tag{5}$$

If we put additionally $\alpha_1 = \alpha$, $\beta_1 = \beta = \sqrt{1-\alpha^2}$ then (5) reduces to

$$\begin{cases} |\alpha| < |p| \\ 2p^2 - 1 \leqslant \alpha \leqslant 1 \end{cases} \tag{6}$$

If the intervals (6) were separable then we would have $2p^2 - 1 > |p|$ in other words $|p| > 1$ or $|p| < -\frac{1}{2}$, but it is impossible. So there are quaternions q and q_1 which fulfil necessary conditions.

THEOREM 2. *If in set K_n determined by p there are series of elements, which begin also with $-p$ and $p \in \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, then $K_n K_n = G_1$.*

Proof. From lemma the quaternion $\alpha + \beta i + \gamma j + \delta k$ is factoring into the product of two quaternions beginning with p if

$$2p^2 - 1 \leqslant \alpha \leqslant 1, \tag{7}$$

and into the product of two quaternions from which one begins with p and the second with $-p$ if

$$-1 \leqslant \alpha \leqslant -2p^2 + 1. \tag{8}$$

Since for $p \in \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ we have

$$-2p^2 + 1 \geqslant 2p^2 - 1$$

then these intervals (7) and (8) will cover the interval $\langle -1, 1 \rangle$ and each quaternion with norm 1 (see corollary 1) will be reducible into the two quaternions of order n.

COROLLARY 2. *$K_{2^n} K_{2^n} = G_1$ for $n \neq 1$.*

Indeed, let in set K_{2^n} be a series of elements, which begins with

$$p = \cos \frac{2\pi k}{2^n}, \ (k, 2^n) = 1.$$

Then a quaternion which begins with

$$p_1 = -\cos\frac{2\pi k}{2^n} = -p = \cos\left(\pi - \frac{2\pi k}{2^n}\right) = \cos\frac{2^n\pi - 2\pi k}{2^n} = \cos\frac{2\pi(2^{n-1} - k)}{2^n}$$

is also from this set, because

$$(2^{n-1} - k, 2^n) = 1 \text{ for } n - 1 \geqslant 1.$$

We must yet prove that there are such k that $p \in \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ but this follows from inequality

$$\frac{\pi}{4} \leqslant \frac{2k\pi}{2^n} \leqslant \frac{\pi}{2} \Leftrightarrow 2^{n-2} \leqslant 2k \leqslant 2^{n-1} \Leftrightarrow 2^{n-3} \leqslant k \leqslant 2^{n-2}$$

and from the fact that between these numbers 2^{n-3} and 2^{n-2} is always an even number.

COROLLARY 3. *$K_\infty K_\infty = G_1$, where K_∞ means a set of elements of order ∞ whose norm is* 1.

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DESPRE PROPRIETATEA *W* PENTRU GRUPUL MULTIPLICATIV AL ALGEBREI CUATERNIONILOR

(Rezumat)

În mai multe lucrări [2], [3] a fost definită o proprietate a unui grup numită proprietatea *W*. S-a arătat că grupurile modulare, abeliene și regulare se bucură de această proprietate. În prezenta lucrare se arată că grupul multiplicativ al unei algebre cvaternionice nu se bucură de această proprietate.

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STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXV, 2, 1980

ON Z-BIRECURRENT SPACES

C. NIŢESCU*

1. Let V_n be an n-dimensional Riemannian space of arbitrary signature. Let R_{hijk}, R_{ij} and R denote the curvature tensor, Ricci tensor and scalar curvature respectively.

A Riemannian space is called recurrent [1] if its curvature tensor satisfies

$$R_{hijk,m} = \varphi_m R_{hijk} \tag{1.1}$$

for some non-zero vector φ , where comma denotes the covariant derivative.

This concept has been generalized by A. Lichnerowicz [2] who says, that a non-flat Riemannian space is second order recurrent or briefly birecurrent if

$$R_{hijk,mn} = a_{mn} R_{hijk} \tag{1.2}$$

Roter [3] has shown that the tensor a_{mn} appearing in equation (1.2) is symmetric.

Spaces, whose Ricci-tensor R_{ij} satisfies the equation

$$R_{ij,mn} = a_{mn} R_{ij} \tag{1.3}$$

for some tensor a_{mn}, where $R_{ij} \neq 0 \neq a_{mn}$, are called 2-Ricci-recurrent spaces.

The so-called concircular curvature tensor

$$Z_{hijk} = R_{hijk} - \frac{R}{n(n-1)} (g_{hk}g_{ij} - g_{hj}g_{ik}) \tag{1.4}$$

introduced by K. Yano [4] gives $g^{hk}Z_{hijk} = G_{ij}$ where

$$G_{ij} = R_{ij} - R\frac{g_{ij}}{n}. \tag{1.5}$$

If V_n admits a tensor field $T\ldots$ such that $T\ldots,_{mn} = a_{mn}T\ldots$ where a_{mn} is a non-zero tensor field of V_n, then V_n is called a T birecurrent space and it is denoted by BT_n. Therefore we shall call the Riemannian space V_n to be a BZ_n space if its concircular curvature tensor satisfies

$$Z_{hijk,mn} = a_{mn} Z_{hijk} \tag{1.6}$$

with a_{mn} a non-zero tensor in V_n, and we shall call it BG_n if the identity

$$G_{hk,mn} = a_{mn} G_{hk} \tag{1.7}$$

is satisfied.

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2. LEMMA 1. *The concircular curvature tensor of a V_n-space satisfies Walker's identity* .

$$Z_{hijk,[mn]} + Z_{jkmn,[hi]} + Z_{mnhi,[jk]} = 0 \tag{2.1}$$

Proof.

$$Z_{hijk,[mn]} = R_{hijk,[mn]} - \frac{q_{hk}q_{ij} - q_{hj}q_{ik}}{n(n-1)} R,[mn]$$

where $R = g^{ij} R_{ij}$ is the curvature invariant or scalar curvature of the space, therefore $R_{,mn} = R_{,nm}$ and

$$Z_{hijk,[mn]} = R_{hijk,[mn]}$$

Using Lemma 1 of [6] we easily obtain from (1.4) the identity (2.1).

LEMMA 2. *The recurrence tensor of BZ_n is symmetric.*

Proof. If the space is a BZ_n-space, then Walker's identity (2.1) gives

$$A_{mn}Z_{hijk} + A_{hi}Z_{jkmn} + A_{jk}Z_{mnhi} = 0 \tag{2.2}$$

where $A_{mn} = a_{mn} - a_{nm}$. Thus, using Lemma 2 of [6] we deduce from (2.2) that $A_{mn} = 0$, since $Z_{hijk} \neq 0$ and $Z_{hijk} = Z_{jkhi}$.

LEMMA 3. *The concircular curvature tensor of a BZ_n-space satisfies the equation*

$$Z_{pijk}R^p_{hmn} + Z_{hpjk}R^p_{imn} + Z_{hipk}R^p_{jmn} + Z_{hijp}R^p_{kmn} = 0 \tag{2.3}$$

Proof. It follows from Lemmas 1 and 2 that

$$Z_{hijk,mn} - Z_{hijk,nm} = 0.$$

Applying now Ricci's identity, we immediately obtain (2.3).

LEMMA 4. *The equations*

$$G_{pk}R^p_{hmn} + G_{hp}R^p_{kmn} = 0 \tag{2.4}$$

and

$$G_{pk}G^p_n + G^m_p Z^p_{kmn} + \frac{R}{n-1} G_{nk} = 0 \tag{2.5}$$

hold for BZ_n-spaces.

Proof. Contracting (2.3) with g^{ij}, we get

$$G_{pk}R^p_{hmn} + Z_{hpjk}R^{pj}_{mn} + Z_{hipk}R^{pi}_{mn} + G_{hp}R^p_{kmn} = 0$$

But $Z_{hplk}R^{pl}_{mn} = - Z_{hplk}R^{lp}_{mn}$ because of the antisymmetry of Riemann's tensor and hence (2.4) holds. If we replace Riemann's tensor in (2.4) in terms of concircular curvature tensor we get

$$G_{pk}\left[Z^p_{hmn} + \frac{R}{n(n-1)}(\delta^p_n g_{hm} - \delta^p_m g_{hn})\right] + G_{hp}\left[Z^p_{kmn} + \frac{R}{n(n-1)}(\delta^p_n g_{km} - \delta^p_m g_{kn})\right] = 0$$

or

$$G_{pk}Z^p_{kmn} + G_{hp}Z^p_{kmn} + \frac{R}{n(n-1)}(G_{nk}g_{hm} - G_{mk}g_{hn} + G_{hn}g_{km} - G_{hm}g_{kn}) = 0$$

Transvecting it with g^{hm} we have

$$G_{pk}G^p_n + G^m_p Z^p_{kmn} + \frac{R}{n(n-1)}(nG_{nk} - G_{nk} + G_{nk}) = 0$$

and thus (2.5) holds too.

THEOREM 1. *A Ricci birecurrent space is BG_n. A BG_n-space with recurrence tensor field a_{mn} is Ricci birecurrent if its scalar curvature R satisfies*

$$R_{mn} = a_{mn}R \tag{2.6}$$

Proof. The first part of the theorem is trivial. Suppose V_n is a BG_n-space. Then from (1.5) we have (1.7). Therefore

$$a_{mn}G_{ij} = R_{ij,mn} - \frac{R_{,mn}}{n}g_{ij}$$

$$a_{mn}R_{ij} - R_{ij,mn} = (a_{mn}R - R_{,mn})\frac{g_{ij}}{n}$$

The use of (2.6) in the above equation shows that V_n is Ricci-birecurrent.

THEOREM 2. *A BZ_n-space is BG_n with the same recurrence tensor.*

Proof. Let V_n be BZ_n with the birecurrence tensor a_{mn}. Transvecting (1.6) with g^{ij} we get (1.7).

THEOREM 3. *A V_n birecurrent space is BZ_n with the same recurrence tensor. A BZ_n-space with recurrence tensor field a_{mn} is a V_n birecurrent space if it is a space with birecurrent scalar curvature with recurrence tensor field a_{mn}.*

Proof. It is trivial to check that a birecurrent space is a BZ_n-space. Suppose V_n is a BZ_n-space. From (1.4) and (1.6) we get

$$a_{mn}Z_{hijk} = R_{hijk,mn} - \frac{R_{,mn}}{n(n-1)}(g_{hk}g_{ij} - g_{hj}g_{ik})$$

The above equation can be written as

$$a_{mn}\left[R_{hijk} - \frac{R}{n(n-1)}(g_{hk}g_{ij} - g_{hj}g_{ik})\right] = R_{hijk,mn} - \frac{R_{,mn}}{n(n-1)}(g_{hk}g_{ij} - g_{hj}g_{ik})$$

or

$$R_{hijk,mn} - a_{mn}R_{hijk} = \frac{g_{hk}g_{ij} - g_{hj}g_{ik}}{n(n-1)}(R_{,mn} - a_{mn}R)$$

and hence the theorem results.

Deriving twice recurrently the identity (2.5) we get

$$G_{pk,l}G_n^p + G_{pk}G_{n,l}^p + G_{p,l}^m Z_{kmn}^p + G_p^m Z_{kmn,l}^p + \frac{1}{n-1}(R_{,l}G_{nk} + RG_{nk,l}) = 0$$

$$G_{pk,lq}G_n^p + G_{pk,l}G_{n,q}^p + G_{pk,q}G_{n,l}^p + G_{pk}G_{n,lq}^p + G_{p,lq}^m Z_{kmn}^p + G_{p,l}^m Z_{kmn,q}^p +$$

$$+ G_{p,q}^m Z_{kmn,l}^p + G_p^m Z_{kmn,lq}^p + \frac{1}{n-1}(R_{,lq}G_{nk} + R_{,l}G_{nk,q} + R_{,q}G_{nk,l} + RG_{nk,lq}) = 0$$

By use of (1.6) and Theorem 2, the latter relation from above becomes

$$2a_{lq}\left(G_{pk}G_n^p + G_p^m Z_{kmn}^p + \frac{R}{n-1}G_{nk}\right) + G_{pk,l}G_{n,q}^p + G_{pk,q}G_{n,l}^p + G_{p,l}^m Z_{kmn,q}^p +$$

$$+ G_{p,q}^m Z_{kmn,l}^p + \frac{1}{n-1}(R_{,l}G_{nk,q} + R_{,q}G_{nk,l}) = 0$$

From the above relation and from (2.5) there follows

THEOREM 4. *In a BZ_n-space the identity*

$$G_{pk,l}G_{n,q}^p + G_{pk,q}G_{n,l}^p + G_{p,l}^m Z_{kmn,q}^p + G_{p,q}^m Z_{kmn,l}^p + \frac{1}{n-1}(R_{,l}G_{nk,q} + R_{,q}G_{nk,l}) = 0 \qquad (2.7)$$

holds.

THEOREM 5. *A necessary and sufficient condition that a Riemannian space V_n be BG_n is that the equation*

$$W_{hijk,mn} - a_{mn}W_{hijk} = \alpha(Z_{hijk,mn} - a_{mn}Z_{hijk}) \qquad (2.8)$$

holds for some non-zero tensor field a_{mn} of V_n, where W_{hijk} is the tensor introduced by K. Yano *and* S. Sawaki [4].

Proof. Let V_n be BG_n with the birecurrent tensor a_{mn}. Then (2.8) easily follows from the expression of W_{hijk}:

$$W_{hijk} = \alpha Z_{hijk} + \frac{\beta}{n-2}(g_{hk}G_{ij} - g_{hj}G_{ik} + g_{ij}G_{hk} - g_{ik}G_{hj})$$

where α and β are constants ($n > 2$), and from (1.7).
Conversely, if (2.8) holds, transvecting it with g^{hk} we obtain

$$(\alpha + \beta)(G_{ij,mn} - a_{mn}G_{ij}) = \alpha(G_{ij,mn} - a_{mn}G_{ij})$$

or

$$\beta(G_{ij,mn} - a_{mn}G_{ij}) = 0$$

In general $\beta \neq 0$ and therefore it follows from the above relation that V_n is BG_n.

3. *The case of an Einstein space* ($G_{ij} = 0$)

The fact that V_n is an Einstein space implies that its scalar curvature R is constant. Therefore $Z_{hijk,l} = R_{hijk,l}$ and it follows that Z_{hijk} satisfies Bianchi's second identity

$$Z_{hijk,l} + Z_{hikl,j} + Z_{hilj,k} = 0 \tag{3.1}$$

If V_n is also BZ_n with a_{mn} as recurrence tensor field then

$$a_{lm}Z_{hijk} + a_{jm}Z_{hikl} + a_{km}Z_{hilj} = 0 \tag{3.2}$$

Contracting the above equation with g^{hk}, we have

$$a_{lm}G_{ij} - a_{jm}G_{il} + a_{km}Z^{k}_{ijl} = 0$$

By use of $G_{ij} = 0$ it follows

$$a_{km}Z^{k}_{ilj} = 0 \tag{3.3}$$

Transvecting (3.2) with a^{km} and using (3.3) we have

$$a^{km}a_{km}Z_{hilj} = 0$$

or

$$\theta_1 Z_{hilj} = 0 \tag{3.4}$$

where [5] $\theta_1 = a^{km}a_{km}$. Contracting now (3.2) with g^{lm} and substituting (3.3) we have $a^r_r Z_{hijk} = 0$ or

$$\theta_0 Z_{hijk} = 0 \tag{3.5}$$

where [5] $\theta_0 = a^r_r = g^{rs}a_{rs}$. If $\theta_1 = 0$ but $\theta_0 \neq 0$, we deduce from (3.4) and (3.5) that the space is of constant curvature. If $\theta_1 = 0$ but $\theta_0 = 0$ the same result holds and we may therefore state.

THEOREM 3.1. An Einstein BZ_n is a space of constant curvature.

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ASUPRA SPAȚIILOR *Z*-BIRECURENTE

(Rezumat)

Se definesc spațiile riemanniene V_n $(n > 2)$ cu tensorul curburii concirculare birecurent, se definesc proprietăți ale acestor spații și condiții necesare și suficiente pentru ca un spațiu V_n birecurent în sensul lui Roter să fie BZ_n sau BG_n. Se studiază cazul cînd un spațiu BZ_n este un spațiu Einstein.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXV, 2, 1980

ASUPRA MIŞCĂRII PLANE IROTAŢIONALE A UNUI FLUID IDEAL INCOMPRESIBIL ANIMAT DE DEPLASAREA UNUI PROFIL DEFORMABIL ÎN PREZENŢA UNUI PERETE NELIMITAT

TITUS PETRILA

1. Mişcarea fluidă plană, irotaţională, generată de deplasarea arbitrară, în masa de fluid ideal şi incompresibil, a unui profil deformabil oarecare, a fost deja studiată de R. Laporte [1], [2]. Considerînd atît cazul cînd aria profilului deformabil rămîne constantă, cît şi cazul, mai general, cînd aria profilului este variabilă în timp, în condiţiile în care nu se produc detaşări turbionare nici chiar la un bord de fugă „ascuţit" al profilului, R. Laporte reuşeşte să evalueze şi eforturile aerodinamice exercitate asupra profilului, arătînd că formulele obţinute, pentru profilul rigid, de G. Couchet, rămîn încă valabile.

Obiectul prezentei note îl constituie reluarea problemei de mai sus *în condiţiile unui perete nelimitat fix de formă arbitrară.* Plasîndu-ne în cazul profilului deformabil, de arie variabilă în timp, vom rezolva problema utilizînd o tehnică folosită deja de noi în evaluarea efectului prezenţei pereţilor relativ la problema profilului rigid nedeformabil [3], [4].

2. Să considerăm reprezentarea conformă $z = H(Z, t) = a_{-1}Z + \sum_{n\geq 1} \frac{a_n}{Z^n}$, care aplică exteriorul profilului deformabil (C_t), raportat la un sistem de axe centrat Oxy,[1] solidar legat de profil, pe exteriorul circumferinţei (C), de rază R, din planul (Z).

Fie $Z = \alpha(u)$ reprezentarea parametrică a curbei (Δ), transformata peretelui (δ) din planul fizic, la momentul considerat t; curba (Δ) o presupunem jordaniană şi regulată în sensul definiţiei din [5]. Fie de asemenea $Z = \alpha_1(u) = R^2[\bar{\alpha}(u)]^{-1}$ reprezentarea parametrică a curbei (Δ_1), inversa curbei (Δ) în raport cu circumferinţa (C).

Ponteuţialul complex[2] al mişcării fluide produse de deplasarea profilului deformabil în fluid nelimitat, presupus *a priori cunoscut* va fi de forma

$$F(Z, t) = lG^{(1)}(Z) + mG^{(2)}(Z) + \omega G^{(3)}(Z) + G^{(4)}(Z) + \frac{1}{2\pi}\left(\frac{dA}{dt} - i\Gamma\right) \operatorname{Log} Z,$$

unde $(l(t), m(t), \omega(t))$ definesc, faţă de reperul Oxy, rototranslaţia profilului (C_t), $G^{(i)}(Z)$ sînt funcţii olomorfe în exteriorul lui (C) şi satisfăcînd pe (C) con-

---

[1] Un astfel de reper se zice centrat dacă $\forall t, \int_C H(Z, t) \frac{dZ}{Z} = 0.$

[2] Mai precis transformatul său în planul (Z).

diţii Dirichlet cunoscute[3], $\frac{dA}{dt}$ este viteza de variaţie a ariei A a profilului deformabil şi care este egală cu $\int_{C_t} \text{Im}\left(\frac{\partial \bar{z}}{\partial t}\,\delta z\right)$[4], Γ este circulaţia [2].

Să introducem acum un potenţial *complementar* $F_c(Z, t)$, potenţial care caută să „corecteze" potenţialul $F(Z, t)$ în condiţiile prezenţei peretelui nelimitat (Δ). Acest potenţial complementar îl vom construi, la fel ca şi în cazul profilului nedeformabil [3], [4], plasînd o repartiţie continuă de duble surse atît pe curba (Δ) cît şi pe curba (Δ_1) astfel încît partea imaginară a acestui potenţial complementar pe cercul (C) să fie nulă şi deci el să nu „deranjeze" condiţia de alunecare de-a lungul profilului mobil.

Aplicînd teorema cercului (Milne Thomson) obţinem atunci pentru $F_c(Z, t)$ expresia

$$F_c(Z, t) = \int_{\Delta} \varphi(Q)\frac{d\alpha}{\alpha - Z} + \overline{\int_{\Delta} \varphi(Q)\frac{d\alpha}{\alpha - \frac{R^2}{Z}}} = \int_{\Delta} \varphi(Q)\frac{d\alpha}{\alpha - Z} + \int_{\Delta_1} \varphi_1(Q_1)\frac{Z\,d\alpha_1}{\alpha_1(\alpha - Z)}$$

unde densitatea φ este o funcţie reală de punctul Q de afixă $\alpha(u) \in \Delta$, a priori necunoscută, verificînd pe Δ condiţiile din [5]. S-a notat aici prin Q_1, inversul lui Q în raport cu (C), iar, pe de altă parte, s-a pus $\varphi_1(Q_1) = \varphi(R^2/Q_1)$.

Să considerăm acum $f(Z, t) = F(Z, t) + F_c(Z, t)$ transformata potenţialului complex *total* al mişcării în prezenţa peretelui nelimitat fix (δ). Impunînd condiţia de alunecare pe peretele (Δ), adică

$$\text{Im}\,[F(Z, t) + F_c(Z, t)] = 0 \quad \text{pentru } Z \to \alpha(u_0) \in \Delta, \tag{1}$$

densitatea $\varphi(Q)$ va trebui atunci să fie astfel determinată încît

$$\lim_{Z \to \alpha(u_0)} \text{Im}\left[f(Z, t) + \int_{\Delta} \varphi(Q)\frac{d\alpha}{\alpha - Z} + \int_{\Delta_1} \varphi_1(Q_1)\frac{Z\,d\alpha_1}{\alpha_1(\alpha_1 - Z)}\right] = 0, \quad \forall Q_0 \in \Delta \tag{1'}$$

Dar această ultimă condiţie împreună cu proprietăţile potenţialului de dublu strat conduc la

$$\text{Im}\left\{f(\alpha(u_0), t) - i\pi\varphi(Q_0) + i\int_{\Delta} \varphi(Q)\left(\frac{\cos\mu}{r}\right)_{Z=\alpha(u_0)} ds + \right.$$
$$+ \int_{\Delta} [\varphi(Q) - \varphi(Q_0)]\left(\frac{\sin\mu}{r}\right)_{Z=\alpha(u_0)} ds - i\int_{\Delta_1} \varphi_1(Q_1)\left(\frac{\cos\mu}{r}\right)_{Z=0} ds_1 -$$
$$- \int_{\Delta_1} \varphi_1(Q_1)\left(\frac{\sin\mu}{r}\right)_{Z=0} ds_1 + i\int_{\Delta_1} \varphi_1(Q_1)\left(\frac{\cos\mu}{r}\right)_{Z=\alpha(u_0)} ds_1 +$$
$$\left. + \int_{\Delta_1} \varphi_1(Q_1)\left(\frac{\sin\mu}{r}\right)_{Z=\alpha(u_0)} ds_1\right\} = 0,$$

[3] Forma efectivă a funcţiilor $G^{(i)}(Z)$ poate fi găsită în [2].
[4] S-a notat prin δ diferenţierea la moment t constant.

adică

$$\pi\varphi(Q_0) - \int_{\Delta} \varphi(Q) \left(\frac{\cos\mu}{r}\right)_{Z=\alpha(u_0)} ds - \int_{\Delta_1} \varphi_1(Q_1) \left[\left(\frac{\cos\mu}{r}\right)_{Z=\alpha(u_0)} - \left(\frac{\cos\mu}{r}\right)_{Z=0}\right] ds_1 =$$
$$= \mathrm{Im}[f(\alpha(u_0), t)]$$

şi care este o ecuaţie integrală de tip Fredholm, cu nucleu slab singular, a cărei rezolvare conduce la soluţionarea completă a problemei propuse.

Ecuaţiei (2) i se pot aplica, atît pentru stabilirea existenţei şi unicităţii soluţiei cît şi pentru rezolvarea ei aproximativă, metode de lucru echivalente celor date deja de noi în cadrul problemei corespunzătoare cu profil nedeformabil [3], [4].

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SUR L'ÉCOULEMENT PLAN IRROTATIONNEL D'UN FLUIDE IDÉAL INCOMPRESSIBLE PRODUIT PAR LE DÉPLACEMENT D'UN PROFIL DÉFORMABLE EN PRÉSENCE D'UNE PAROI ILLIMITÉE

(R é s u m é)

L'objet du travail présent constitue l'étude de l'influence des parois illimitées sur l'écoulement fluide idéal plan, potentiel, incompressible, produit par la rototranslation d'un profil déformable dont l'aire est variable en temps. On emploie des résultats de M.R. L a p o r t e sur le déplacement arbitraire dans la masse de fluide illimité d'un profil déformable [1], [2], aussi bien qu'une technique utilisée déjà par nous dans le cas d'un profil indéformable en présence d'une paroi infinie [3], [4].

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ON A CERTAIN VIBRATIONAL PROBLEM

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There are dynamical systems, whose parameters, included in the coefficients of the corresponding differential equation, are variable when the system is subject to external influences. This occurs for linear or nonlinear parametric vibrations of the material systems, whose parameters may have time dependence, in general a periodical dependence.

The differential equation of the linear parametric vibration of a dynamical system with only one degree of freedom has, as it is well-known, the form

$$m(t)\ddot{x} + c(t)\dot{x} + k(t)x = F(t),$$

where x is the parameter of the instantaneous position of the mass (in other words the zero-order acceleration), $\dot{x}$ the velocity (that is, the first order acceleration), and $\ddot{x}$ the second order acceleration.

In the present paper a method is given for the calculus of the accelerations higher or lower than the order of the differential equation ($m \gtrless n$), "n" being here the order of the equation, and "m" the order of the acceleration. This method allows the simultaneous deduction of two accelerations of different orders. It may also be applied when the coefficients of the differential equation are constants.

1. **Description of the method.** In the most general case, we shall consider the vectorial *differential* equation of n order, with time-dependent coefficients

$$\sum_{e=k}^{n} a_e(t)\overset{(e)}{\bar{r}} + \sum_{p=0}^{k-1} a_p(t)\overset{(p)}{\bar{r}} = \bar{A}(t), \quad [a_n(t) \neq 0], \qquad (n = 2, 3, 4, \ldots). \tag{1}$$

We denote the k and m order accelerations in the following manner

$$\overset{(k)}{\bar{r}} = \bar{\varphi}_k(t), \quad \overset{(m)}{\bar{r}} = \bar{\varphi}(t), \qquad (m = n+1, n+2, \ldots),$$

where $k = m < n$, and we calculate, using these notations, the p and e order acceleration respectively, and we obtain

$$\overset{(p)}{\bar{r}} = \int_0^t \frac{(t-s)^{k-p-1}}{(k-p-1)!}\bar{\varphi}_k(s)ds + \sum_{\sigma=0}^{k-p-1} \overset{(p+\sigma)}{\bar{r}}(0)\frac{t^\sigma}{\sigma!}, \tag{2}$$

$$(p = 0, 1, 2, \ldots, k-1),$$

$$\overset{(e)}{\bar{r}} = \int_0^t \frac{(t-s)^{m-e-1}}{(m-e-1)!}\bar{\varphi}_m(s)ds + \sum_{\sigma=0}^{m-e-1} \overset{(e+\sigma)}{\bar{r}}(0)\frac{t^\sigma}{\sigma!}, \tag{3}$$

$$(e = k, k+1, \ldots, n).$$

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Substituting (2) and (3) in (1), we have

$$\int_0^t [N_{k,m,n}(t, s)\bar{\varphi}_m(s) + N_k(t, s)\bar{\varphi}_k(s)]ds = \bar{F}_{k,m,n}(t), \tag{4}$$

where

$$N_{k,m,n}(t, s) = \sum_{e=k}^{n} a_e(t) \frac{(t-s)^{m-e-1}}{(m-e-1)!}, \tag{5}$$

$$N_k(t, s) = \sum_{p=0}^{k-1} a_p(t) \frac{(t-s)^{k-p-1}}{(k-p-1)!}, \tag{6}$$

$$\bar{F}_{k,m,n}(t) = \bar{A}(t) - \left[\sum_{e=k}^{n} \sum_{\sigma=0}^{m-e-1} a_e(t) \overset{(e+\sigma)}{\bar{r}}(0) + \sum_{p=0}^{k-1} \sum_{\sigma=0}^{k-p-1} a_p(t) \overset{(p+\sigma)}{\bar{r}}(0)\right] \frac{t^\sigma}{\sigma!}. \tag{7}$$

Next, by setting in (3) $e = k$, it results

$$\bar{\varphi}_k(t) - \int_0^t N_{k,m}(t, s)\bar{\varphi}_m(s)ds = \bar{E}_{k,m}(t), \tag{8}$$

where

$$N_{k,m}(t, s) = \frac{(t-s)^{m-k-1}}{(m-k-1)!}, \quad \bar{E}_{k,m}(t) = \sum_{\sigma=0}^{m-k-1} \overset{(k+\sigma)}{\bar{r}}(0) \frac{t^\sigma}{\sigma!}. \tag{9}$$

By inspecting these formulas we immediately see that (4) and (8) represent a system of Volterra integral equations.

If the conditions

$$N_{k,m,n}(t, t) \neq 0, \qquad \bar{F}_{k,m,n}(0) = \bar{0}, \tag{10}$$

are fulfilled, equations (4) reduce to a Volterra linear equation of the second kind.

Taking into account (7), the second of the conditions (10) becomes

$$\bar{A}(0) - \left[\sum_{e=k}^{n} a_e(0) \overset{(e)}{\bar{r}}(0) + \sum_{p=0}^{k-1} a_p(0) \overset{(p)}{\bar{r}}(0)\right] = \bar{0}. \tag{11}$$

The initial conditions $\overset{(i)}{\bar{r}}(0)$, $(i = 0, 1, 2, \ldots, n-1)$ are to be taken arbitrarily, but the initial conditions for $i > n-1$ follow from eq. (1) and its derivative.

By substituting the functions $\bar{\varphi}_k(t)$ from (8) in (4), we obtain

$$\int_0^t N_{k,m,n}(t, s)\bar{\varphi}_m(s)ds + \int_0^t N_k(t, s)ds \int_0^s N_{k,m}(s, \sigma)\bar{\varphi}_m(\sigma)d\sigma = \bar{G}_{k,m,n}(t), \tag{12}$$

where

$$\bar{G}_{k,m,n}(t) = \bar{F}_{k,m,n}(t) - \int_0^t N_k(t, s)\bar{E}_{k,m}(s)ds. \tag{13}$$

Making use in (12) of a formula given by Dirichelet, and replacing s by σ, we have

$$\int_0^t H_{k,m,n}(t, s)\bar{\varphi}_m(s)ds = \bar{G}_{k,m,n}(t), \tag{14}$$

where

$$H_{k,m,n}(t, s) = N_{k,m,n}(t, s) + \int_s^t N_k(t, \sigma)N_{k,m}(\sigma, s)d\sigma. \tag{15}$$

The Volterra linear integral equation of first kind (14) can be converted into a linear integral equation of second kind, if the following conditions are satisfied

$$H_{k,m,n}(t, t) \neq 0, \qquad \bar{G}_{k,m,n}(0) = \bar{0}. \tag{16}$$

If the first condition (16) is not verified, one deduces by the differentiating equation (14) $m - n$ times, the following Volterra integral equation of second kind

$$\bar{\varphi}_m(t) + \int_0^t Q_{k,m,n}(t, s)\bar{\varphi}_m(s)ds = \bar{\Gamma}_{k,m,n}(t), \tag{17}$$

where

$$Q_{k,m,n}(t, s) = \tilde{H}^{-1}_{k,m,n}(t, t)H^{*}_{k,m,n}(t, s), \ \bar{\Gamma}_{k,m,n}(t) = \tilde{H}^{-1}_{k,m,n}(t, t)\,\bar{G}^{*}_{k,m,n}(t), \tag{18}$$

$$\tilde{H}_{k,m,n}(t, t) = \left[\frac{\partial^{m-n-1}H_{k,m,n}(t, s)}{\partial t^{m-n-1}}\right]_{s=t} \neq 0, \ \bar{G}^{*}_{k,m,n}(t) = \bar{G}^{(m-n)}_{k,m,n}(t), \tag{19}$$

$$H^{*}_{k,m,n}(t, s) = \frac{\partial^{m-n}H_{k,m,n}(t, s)}{\partial t^{m-n}}.$$

The second condition (16) will be satisfied only if the second condition (10) is also satisfied.

Applying the method of successive approximations, the solution of (17) is obtained under the form

$$\bar{\varphi}_m(t) = \bar{\varphi}_{m,0}(t) + \bar{\varphi}_{m,1}(t) + \bar{\varphi}_{m,2}(t) + \ldots + \bar{\varphi}_{m,\nu}(t) + \ldots, \tag{20}$$

where

$$\bar{\varphi}_{m,0}(t) = \bar{\Gamma}_{k,m,n}(t),$$

$$\bar{\varphi}_{m,1}(t) = -\int_0^t Q_{k,m,n}(t, s)\bar{\varphi}_{m,0}(s)\,ds, \tag{21}$$

. . . .

$$\bar{\varphi}_{m,\nu}(t) = -\int_0^t Q_{k,m,n}(t, s)\bar{\varphi}_{m,\nu-1}(s)ds.$$

If the acceleration $\bar{\varphi}_m(t)$ is known, it follows from (8), after performing the integral, the acceleration $\bar{\varphi}_k(t)$.

The solution of equation (17) can be deduced by an interpolation method. Making use of a quadrature formula of the following form

$$\int_0^{t_i} \bar{f}(s)ds \approx \sum_{j=0}^{i} A_j \bar{f}(s_j),$$

(17) becomes

$$\bar{\varphi}_m(t_i) + \sum_{j=0}^{i} A_j Q_{k,m,n}(t_i, t_j)\bar{\varphi}_m(t_j) = \bar{\Gamma}_{k,m,n}(t_i),$$

$$(i = 0, 1, 2, \ldots, h).$$

This is a linear algebraic system of $h + 1$ equations with $h + 1$ unknowns $\bar{\varphi}_m(t_i)$.

2. **Application.** As an example we shall consider in what follows the third order differential equation

$$\dddot{\varepsilon} + \chi\ddot{\varepsilon} + \alpha\dot{\varepsilon} + \omega^2\chi\varepsilon = F \sin \nu t, \tag{22}$$

$$(\alpha, \chi, \nu, \omega, F, \text{ constants}),$$

that governs a vibrational system with internal friction and relaxation.

Our task is to calculate the second and fourth order accelerations, taking into account the initial conditions

$$\varepsilon(0) = \varepsilon_0, \ \dot{\varepsilon}(0) = -\frac{\chi}{\alpha}\omega^2\varepsilon_0, \ \ddot{\varepsilon}(0) = 0, \ \dddot{\varepsilon}(0) = 0,$$

which fulfil both the second conditions (10) and (16) as well as (22).

We have successively for $k = 2$, $m = 4$, $n = 3$

$$F_{2,3,4}(t) = F \sin \nu t + \frac{\chi^2\omega^4\varepsilon_0}{\alpha} t, \ E_{2,4}(s) = 0, \tag{23}$$

$$N_2(t, s) = \alpha + \chi\omega^2(t - s), \ N_{2,4,3}(t, s) = 1 + \chi(t - s), \tag{24}$$

$$N_2(t, \sigma) = \alpha + \chi\omega^2(t - \sigma), \qquad N_{2,4}(\sigma, s) = \sigma - s. \tag{25}$$

By virtue of (23), (24) and (25), and by making use of (13) and (15), we obtain

$$G_{2,4,3}(t) = F_{2,4,3}(t), \quad H_{2,4,3}(t, s) = 1 - \alpha ts +$$

$$+ \chi\left(1 - \frac{1}{2}\omega^2 ts\right)(t - s) + \frac{1}{2}\alpha(t^2 + s^2) + \frac{1}{6}\chi\omega^2(t^3 - s^3).$$

Performing the derivatives as indicated in (19), it results

$$\tilde{H}_{2,4,3}(t, t) = H_{2,4,3}(t, t) = 1, \quad G^{*}_{2,4,3}(t) = \nu F \cos \nu t + \frac{1}{\alpha} \chi^2\omega^4\varepsilon_0,$$

$$H^{*}_{2,4,3}(t, s) = \alpha(t - s) + \chi\left\{1 - \omega^2\left[ts - \frac{1}{2}(t^2 + s^2)\right]\right\}. \tag{26}$$

Replacing (26) in (18), we have

$$Q_{2,4,3}(t, s) = H^{*}_{2,4,3}(t, s), \Gamma_{2,4,3}(t) = G^{*}_{2,4,3}(t).$$

For $k = 2$, $m = 4$, $n = 3$, the solution of the integral equation (17), in the second approximation is

$$\varphi_4(t) \approx \varphi_{4,0}(t) + \varphi_{4,1}(t).$$

Now, taking into account (21), we obtain the fourth order acceleration

$$\ddddot{\varepsilon}(t) = \varphi_4(t) \approx \sum_{i=0}^{3} A_i t^i + B \cdot \sin \nu t + C \cdot \cos \nu t, \tag{27}$$

where

$$A_0 = -\frac{\alpha}{\nu} F + \frac{1}{\alpha}\chi^2\omega^4\varepsilon_0, \quad A_1 = -\chi\omega^2\left(\frac{F}{\nu} + \frac{1}{\alpha}\chi^2\omega^2\varepsilon_0\right),$$

$$A_2 = -\frac{1}{2}\chi^2\omega^4\varepsilon_0, \quad A_3 = -\frac{1}{6\alpha}\chi^3\omega^6\varepsilon_0,$$

$$B = \chi F\left(\frac{\omega^2}{\nu^2} - 1\right), \quad C = F\left(\nu + \frac{\alpha}{\nu}\right).$$

As

$$N_{2,4}(t, s) = t - s, \qquad E_{2,4}(t) = 0,$$

and by virtue of (27), it follows from (8), the second-order acceleration in second approximation

$$\ddot{\varepsilon}(t) = \varphi_2(t) \approx \sum_{i=0}^{3} \frac{A_i t^{i+2}}{(i + 1)(i + 2)} + D(t) + E \cdot \sin \nu t + G \cdot \cos \nu t,$$

where

$$D(t) = \frac{1}{\nu}\left(Bt + \frac{C}{\nu}\right), \quad E = -\frac{B}{\nu^2}, \quad G = -\frac{C}{\nu^2}.$$

The above developed method can be used to deduce the acceleration of any order, in the wanted approximation.

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ASUPRA UNEI PROBLEME DE VIBRAŢII

(Rezumat)

În prezenta lucrare se dă o metodă de determinare a acceleraţiilor de ordin superior, existente la vibraţiile parametrice liniare. Metoda permite determinarea simultană a două acceleraţii de ordin diferit, şi poate fi aplicată şi în cazul ecuaţiilor diferenţiale cu coeficienţi constanţi.

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SOME RADIATION EFFECTS IN THE NODAL PERIOD OF ARTIFICIAL SATELLITES

VASILE MIOC* and EUGENIA RADU*

1. **Introduction.** One of the most important disturbing factors acting upon the motion of artificial satellites with high orbits and great area-to-mass ratios is the solar radiation pressure. Out of the multiple aspects of its influence, the direct solar radiation pressure (Musen, 1960; Parkinson et al., 1960; Bryant, 1961; Kozai, 1961; Lála, 1968; Rousseau, 1970), the re-emitted solar radiation pressure (Levin, 1962; Sehnal, 1963; Baker, 1966), the Poynting-Robertson effect (Sehnal, 1963; Radziewsky, Chernikov, 1965), the shadow effect (Polyakhova, 1963; Horváth, 1968; Ferraz-Mello, 1972) were studied both quantitatively and qualitatively. Such influences on some particular satellites (or type of satellites) were studied by Shapiro and Jones (1960), Swalley (1962), Shapiro (1963), Lucas (1974) etc. It was pointed out that for balloon-satellites the solar radiation pressure effect is on the same order of magnitude as the air-drag effect, at heights of 900—1 000 km, and becomes predominant at greater heights.

The great majority of these authors deals with the influence of solar radiation pressure on the orbital elements of satellites. This one's effect on the satellite periods was studied by very few authors, as Horváth (1968) for the shadow effect on the anomalistic period, or Mioc and Radu (1977) for the direct solar radiation pressure influence on the nodal period. The present paper also deals with the nodal period variations caused by the direct solar radiation pressure, considering the Poynting-Robertson effect. Our previous qualitative analysis (Mioc, Radu, 1977) appears now as a particular case of the results obtained in this paper.

2. **Basic equations.** We will use the following system of notations:

p = parameter of the orbit;
e = eccentricity;
r = geocentric radius-vector of the satellite;
Ω = longitude of the ascending node with respect to ecliptic;
i = inclination to the ecliptic plane;
v = true anomaly;
ω = argument of perigee, defined by the new node;
u = argument of latitude, defined by the new node;
μ = product of gravitation constant and Earth mass;
L = longitude of the Sun;
K = reflectivity coefficient of the satellite surface;
$k_{\odot}$ = solar constant;
c = velocity of the light;

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$d_\odot$ = mean Sun-Earth distance;
$r_\odot$ = Sun-Earth distance (neglecting Earth-satellite distance);
A = satellite cross-section;
m = satellite mass;
$\mathbf{F}$ = disturbing acceleration;
S, T, W = radial, transversal and binormal components of $\mathbf{F}$;
$\mathbf{D}_0$ = unit vector of the direction Sun-Earth;
α, β, γ = direction angles of $\mathbf{D}_0$ in the frame S, T, W;
V = instantaneous velocity of the satellite;
$\mathbf{V}_0$ = unit vector of the instantaneous velocity;
ψ = angle between $\mathbf{D}_0$ and $\mathbf{V}_0$;
ξ = angle between $\mathbf{V}_0$ and the direction of S.

The elliptic disturbed motion of an artificial satellite with respect to a geocentric frame is described by the following system of osculating elements (Ochocimskij et al., 1959):

$$dp/du = 2r^3\eta\ T/\mu,$$
$$d\Omega/du = r^3\eta \sin u\ W/\mu p \sin i,$$
$$di/du = r^3\eta \cos u\ W/\mu p,$$
$$dq/du = r^3\eta\ k \sin u \cot i\ W/\mu p + r^2\eta[r(q + \cos u)/p + \cos u]\ T/\mu + r^2\eta \sin u\ S/\mu, \tag{1}$$
$$dk/du = -r^3\eta q \sin u \cot i\ W/\mu p + r^2\eta[r(k + \sin u)/p + \sin u]T/\mu - r^2\eta \cos u\ S/\mu,$$
$$dt/du = r^3/\sqrt{\mu p}\,[1 - (r^2 \cos i\ d\Omega/dt)/\sqrt{\mu p}\,].$$

The quantities η, q and k are defined by the relations:

$$\eta = 1/[1 - (r^2 \cos i\ d\Omega/dt)/\sqrt{\mu p}\,], \tag{2}$$
$$q = e \cos \omega; \quad k = e \sin \omega. \tag{3}$$

The disturbing acceleration of the satellite due to the direct solar radiation pressure, considering the Poynting-Robertson effect, but neglecting the shadow effect, is (Radziewsky, Chernikov, 1965):

$$\mathbf{F} = F\,\{[1 - (V/c) \cos \xi]\mathbf{D}_0 - (V/c)\mathbf{V}_0\}, \tag{4}$$

where for F different expressions were given, as for instance (Vercheval, 1974):

$$F = K(d_\odot/r_\odot)^2 k_\odot A/mc. \tag{5}$$

The components of $\mathbf{F}$ are:

$$S = F\{[1 - (V/c) \cos \xi] \cos \alpha - (V/c) \cos \psi\},$$
$$T = F\{[1 - (V/c) \cos \xi] \cos \beta - (V/c) \sin \psi\}, \tag{6}$$
$$W = F\{[1 - (V/c) \cos \xi] \cos \gamma\}.$$

For simplicity, we will use the abbreviations:

$$B = \sin u\,;\ C = \cos u, \tag{7}$$

$$A_1 = -\cos\lambda,$$
$$A_2 = -\sin\lambda\cos i, \tag{8}$$
$$A_0 = \sin\lambda\sin i,$$

where:

$$\lambda = L - \Omega. \tag{9}$$

With these notations, the angles α, β and γ will be given, according to Radziewsky and Chernikov (1965), by:

$$\cos\alpha = A_1C + A_2B,$$
$$\cos\beta = -A_1B + A_2C, \tag{10}$$
$$\cos\gamma = A_0.$$

Now, using Equations (3) and (7), the known relation:

$$r = p/(1 + e\cos v) \tag{11}$$

becomes:

$$r = p/(1 + Cq + Bk) \cong p(1 - Cq - Bk), \tag{12}$$

where the powers $\geqslant 2$ of q, k and the product qk were neglected, as in our future calculations.

With Equation (12) and also according to Radziewsky and Chernikov (1965), the products appearing in (6) are:

$$(V/c)\cos\xi = \sqrt{\mu/p}\,c^{-1}(-A_1B + A_2C + A_2q - A_1k), \tag{13}$$

$$(V/c)\cos\psi = \sqrt{\mu/p}\,c^{-1}(Bq - Ck), \tag{14}$$
$$(V/c)\sin\psi = \sqrt{\mu/p}\,c^{-1}(1 + Cq + Bk).$$

Substituting the expressions given by Equations (10), (13) and (14) in the system (6), the last one becomes:

$$S = F(A_2B + A_1C) - F\sqrt{\mu/p}\,c^{-1}\{A_1A_2(C^2 - B^2) + (A_2^2 - A_1^2)BC +$$
$$+ [(A_2^2 + 1)B + A_1A_2C]q + [-(A_1^2 + 1)C - A_1A_2B]k\},$$
$$T = F(-A_1B + A_2C) - F\sqrt{\mu/p}\,c^{-1}\{A_2^2 + 1 - (A_2^2 - A_1^2)B^2 - 2A_1A_2BC +$$
$$+ [(A_2^2 + 1)C - A_1A_2B]q + [(A_1^2 + 1)B - A_1A_2C]k\}, \tag{15}$$
$$W = FA_0 - F\sqrt{\mu/p}\,c^{-1}\{A_0(A_2C - A_1B) + A_0A_2q - A_0A_1k\}.$$

In the right part of Equations (15), the first term expresses the contribution of direct solar radiation pressure (neglecting the Poynting-Robertson effect) in the disturbing acceleration (for comparison, see e.g. L a l a, 1968; M i o c, R a d u, 1977), while the second one represents the contribution of Poynting-Robertson effect only in the disturbing acceleration.

With Equations (15), the system (1) becomes:

$$\begin{aligned}
dp/du &= K_1\{2(A_2C - A_1B) + 6(A_1B - A_2C)Cq + 6(A_1B - A_2C)Bk\} - \\
&- K_2\{3 - A_0^2 + [2A_1A_2(6C^2 - 1)B - 2(3A_1^2 - A_2^2 + 2)C - 6(A_2^2 - A_1^2)C^3]q + \\
&+ [2A_1A_2(6B^2 - 1)C + 2(A_1^2 - 3A_2^2 - 2)B + 6(A_2^2 - A_1^2)B^3]k\}, \\
d\Omega/du &= K_3\{\sqrt{1 - A_1^2}B - 3\sqrt{1 - A_1^2}BCq - 3\sqrt{1 - A_1^2}B^2k\} - \\
&- K_4\{\sqrt{1 - A_1^2}(A_2C - A_1B)B + \sqrt{1 - A_1^2}B[A_2 + 3(A_1B - A_2C)C]q + \\
&+ \sqrt{1 - A_1^2}B[- A_1 + 3(A_1B - A_2C)B]k\}, \\
di/du &= K_3\{A_0C - 3A_0C^2q - 3A_0BCk\} - \\
&- K_4\{A_0(A_2C - A_1B)C + A_0C[A_2 + 3(A_1B - A_2C)C]q + \\
&+ A_0C[- A_1 + 3(A_1B - A_2C)B]k\}, \\
dq/du &= K_3\{A_2 + (A_2C - A_1B)C + \\
&+ [A_2C - A_1B + (- 2A_2B^2 + 3A_1BC - 5A_2C^2)C]q + \\
&+ B[- A_2 + (- 2A_2B^2 + 3A_1BC - 5A_2C^2)]k\} - \\
&- K_4\{- A_1A_2(1 + 2C^2)B + (3 - A_0^2)C + (A_2^2 - A_1^2)C^3 + \\
&+ [3 - A_0^2 + A_1A_2(6C^2 - 1)BC - 4(A_1^2 + 1)C^2 - 3(A_2^2 - A_1^2)C^4]q + \\
&+ [2A_1A_2(- 1 + 5B^2 - 3B^4) - (4 + 5A_2^2)BC - 3(A_2^2 - A_1^2)BC^3]k\} \\
dk/du &= K_3\{- A_1 + (A_2C - A_1B)B + \\
&+ [A_2B + (2A_1C^2 - 3A_2BC + 5A_1B^2)C]q + \\
&+ [A_2C - A_1B + (2A_1C^2 - 3A_2BC + 5A_1B^2)B]k\} - \\
&- K_4\{- A_1A_2(1 + 2B^2)C + (3 - A_0^2)B - (A_2^2 - A_1^2)B^3 + \\
&+ [A_1A_2(1 + 2B^2 - 6B^4) - (6 - A_2^2 - 2A_0^2)BC + 3(A_2^2 - A_1^2)B^3C]q + \\
&+ [(3-A_0^2)(1-2B^2) + A_1A_2(6B^2-1)BC + 3(A_2^2 - A_1^2)B^4]k\}, \\
dt/du &= p^{3/2}\mu^{-1/2}(1 - 2Cq - 2Bk),
\end{aligned} \tag{16}$$

where:

$$K_1 = F\eta p^3\mu^{-1}; \; K_2 = F\eta p^{5/2}\mu^{-1/2}c^{-1} \tag{17}$$

and:

$$K_3 = K_1/p; \; K_4 = K_2/p. \tag{18}$$

3. **Method.** We plan to study the influence of the above mentioned disturbing factors on the nodal period of artificial satellites. This one, defined as the

time interval between two successive transits of the satellite through its ascending node, can be determined from the relation:

$$T_{\Omega} = \int_{0}^{2\pi} (dt/du)\, du. \tag{19}$$

Developing the last equation (1) in binominal series and neglecting the terms having the form $[(r^2 \cos i\, d\Omega/dt)/\sqrt{\mu p}]^n$ for $n \geqslant 2$, we obtain:

$$\begin{aligned} dt/du &\cong r^2/\sqrt{\mu p} + (r^4 \cos i\, d\Omega/dt)/\mu p = \\ &= f(p, \Omega, i, q, k, \sigma; u), \end{aligned} \tag{20}$$

where σ is a small parameter characterizing the disturbing factor.

The above considered orbital elements can be written in the form:

$$\begin{aligned} p &= p_0 + \Delta p, \\ \Omega &= \Omega_0 + \Delta\Omega, \\ i &= i_0 + \Delta i, \\ q &= q_0 + \Delta q, \\ k &= k_0 + \Delta k, \end{aligned} \tag{21}$$

where:

p, Ω, i, q, k = orbital elements as functions of u;

$p_0, \Omega_0, i_0, q_0, k_0$ = elements of the osculating orbit for the moment $t = t_0$ $(u = u_0)$;

$\Delta p, \Delta\Omega, \Delta i, \Delta q, \Delta k$ = variations of the orbital elements between the initial and current positions. These variations are:

$$\begin{aligned} \Delta p &= \int_{u_0}^{u} (dp/du)du, \\ \Delta\Omega &= \int_{u_0}^{u} (d\Omega/du)du, \\ \Delta i &= \int_{u_0}^{u} (di/du)du, \\ \Delta q &= \int_{u_0}^{u} (dq/du)du, \\ \Delta k &= \int_{u_0}^{u} (dk/du)\, du. \end{aligned} \tag{22}$$

Using Equations (1) with $\eta \cong 1$, the integrals (22) can be estimated by the successive approximations method (Chebotarev, 1965), limiting us to the first order approximations.

Developing f in Taylor series on the hypersurface:

$$H = H(p_0, \Omega_0, i_0, q_0, k_0, \sigma = 0; u), \tag{23}$$

with respect to the small quantities Δp, $\Delta\Omega$, Δi, Δq, Δk and σ, we obtain:

$$f = f_0 + (\partial f/\partial p)_0\Delta p + (\partial f/\partial \Omega)_0\Delta\Omega + (\partial f/\partial i)_0\Delta i + \\ + (\partial f/\partial q)_0\Delta q + (\partial f/\partial k)_0\Delta k + (\partial f/\partial \sigma)_0\sigma + \ldots \tag{24}$$

where:

f_0 = the value of f for the initial position u_0;

$(\partial f/\partial p)_0$, $(\partial f/\partial \Omega)_0$, $(\partial f/\partial i)_0$, $(\partial f/\partial q)_0$, $(\partial f/\partial k)_0$, $(\partial f/\partial \sigma)_0$ =

the partial derivatives of f for the initial position u_0.

Using Equation (20) and neglecting the terms of the type $\{\partial[(r^4 \cos i\, d\Omega/dt)/\mu p]/\partial p\}\, \Delta p, \ldots$ which contain the square of the small parameter σ, we obtain with Equations (7) and (12):

$$\begin{aligned} f_0 &= p_0^{3/2}\mu^{-1/2}(1 - 2Cq_0 - 2Bk_0), \\ (\partial f/\partial p)_0 &= (3/2)p_0^{1/2}\mu^{-1/2}(1 - 2Cq_0 - 2Bk_0), \\ (\partial f/\partial q)_0 &= -2p_0^{3/2}\mu^{-1/2}(1 - 3Cq_0 - 3Bk_0)C, \\ (\partial f/\partial k)_0 &= -2p_0^{3/2}\mu^{-1/2}(1 - 3Cq_0 - 3Bk_0)B, \\ (\partial f/\partial \sigma)_0 &= \partial[(r^4 \cos i\, d\Omega/dt)/\mu p]/\partial\sigma, \\ (\partial f/\partial \Omega)_0 &= 0, \\ (\partial f/\partial i)_0 &= 0. \end{aligned} \tag{25}$$

The nodal period can be expressed as follows:

$$T_\Omega = T_0 + \Delta T_\Omega, \tag{26}$$

where T_0 is the non-disturbed period corresponding to the osculating orbit (t_0 = osculation moment) and ΔT_Ω is the difference between nodal and osculating periods due to the disturbing factors. Taking into account Equations (24) and (25), they acquire the form:

$$T_0 = p_0^{3/2}\mu^{-1/2}\int_0^{2\pi}(1 - 2Cq_0 - 2Bk_0)du; \tag{27}$$

$$\Delta T_\Omega = I_1 + I_2 + I_3 + I_4, \tag{28}$$

where:

$$I_1 = (3/2)p_0^{1/2}\mu^{-1/2}\int_0^{\pi}(1 - 2Cq_0 - 2Bk_0)\Delta p\, du,$$

$$I_2 = -2p_0^{3/2}\mu^{-1/2}\int_0^{2\pi}(1 - 3Cq_0 - 3Bk_0)C\,\Delta q\, du,$$

$$I_3 = -2p_0^{3/2}\mu^{-1/2}\int_0^{2\pi}(1 - 3Cq_0 - 3Bk_0)B\,\Delta k\, du, \tag{29}$$

$$I_4 = \int_0^{2\pi}\{\partial[(r^4 \cos i\, d\Omega/dt)/\mu p]/\partial\sigma\}\,\sigma\, du.$$

4. Results. As a disturbing factor we take into account the direct sola radiation pressure, including the Poynting-Robertson effect. From Equation (16) and (22), with the notations (17) and (18), and with $\eta \cong 1$, we find:

$$\begin{aligned}
\Delta p = {} & K_{10}\{2A_{20}(B - B_0) + 2A_{10}(C - C_0) + \\
& + 3\,[A_{10}(B^2 - B_0^2) - A_{20}(BC - B_0C_0) - A_{20}(u - u_0)]q_0 + \\
& + 3\,[-A_{20}(B^2 - B_0^2) - A_{10}(BC - B_0C_0) + A_{10}(u - u_0)]k_0\} - \\
& - K_{20}\{(3 - A_{00}^2)(u - u_0) - \\
& - 2\,[(3 - A_{00}^2)(B - B_0) + A_{10}A_{20}(C - C_0) + \\
& + (A_{20}^2 - A_{10}^2)(BC^2 - B_0C_0^2) + 2A_{10}A_{20}(B^2C - B_0^2C_0)]q_0 - \\
& 2[-A_{10}A_{20}(B - B_0) - (3 - A_{00}^2)(C - C_0) + \\
& + 2A_{10}A_{20}(BC^2 - B_0C_0^2) + (A_{20}^2 - A_{10}^2)(B^2C - B_0^2C_0)]k_0\},
\end{aligned}$$

$$\begin{aligned}
\Delta q = {} & K_{30}\{[-A_{10}(B^2 - B_0^2) + A_{20}(BC - B_0C_0) + 3A_{20}(u - u_0)]/2 + \\
& + [-4A_{20}(B - B_0) + A_{10}(B^2C - B_0^2C_0) + A_{20}(B^3 - B_0^3)]q_0 + \\
& + [3A_{20}(C - C_0) + A_{10}(B^3 - B_0^3) + A_{20}(C^3 - C_0^3)]k_0\} - \\
& - K_{40}\{2(A_{20}^2 + 1)(B - B_0) + A_{10}A_{20}(C - C_0) - \\
& - (A_{20}^2 - A_{10}^2)(B^3 - B_0^3)/3 + 2A_{10}A_{20}(C^3 - C_0^3)/3 + \\
& + [-(A_{10}^2 + 15A_{20}^2 + 16)(BC - B_0C_0)/8 + A_{10}A_{20}(B^2 - B_0^2) + \\
& + 3A_{10}A_{20}(B^2C^2 - B_0^2C_0^2)/2 + 3(A_{20}^2 - A_{10}^2)(B^3C - B_0^3C_0)/4 - \\
& - (A_{20}^2 - A_{10}^2)(u - u_0)/8]q_0 + \\
& + [-5A_{10}A_{20}(BC - B_0C_0)/4 - (A_{10}^2 - 6A_{20}^2 - 4)(B^2 - B_0^2)/2 -
\end{aligned} \tag{30}$$

$$
\begin{aligned}
&- 3A_{10}A_{20}(BC^3 - B_0C_0^3)/2 + 3(A_{20}^2 - A_{10}^2)(B^4 - B_0^4)/4 + \\
&+ 3A_{10}A_{20}(u - u_0)/4]k_0\}. \\
\Delta k &= K_{30}\{[A_{20}(B^2 - B_0^2) + A_{10}(BC - B_0C_0) - 3A_{10}(u - u_0)]/2 + \\
&+ [2A_{10}(B - B_0) - A_{20}(C - C_0) + A_{10}(B^3 - B_0^3) + A_{20}(C^3 - C_0^3)]q_0 \\
&+ [- 4A_{10}(C - C_0) + A_{20}(BC^2 - B_0C_0^2) + A_{10}(C^3 - C_0^3)]k_0\} - \\
&- K_{40}\{- A_{10}A_{20}(B - B_0) - 2(A_{10}^2 + 1)(C - C_0) - \\
&- 2A_{10}A_{20}(B^3 - B_0^3)/3 - (A_{20}^2 - A_{10}^2)(C^3 - C_0^3)/3 + \\
&+ [11A_{10}A_{20}(BC - B_0C_0)/4 - (2A_{10}^2 + A_{20}^2 + 4)(B^2 - B_0^2)/2 - \\
&- 3A_{10}A_{20}(BC^3 - B_0C_0^3)/2 + 3(A_{20}^2 - A_{10}^2)(B^4 - B_0^4)/4 - \\
&- A_{10}A_{20}(u - u_0)/4]q_0 + \\
&+ [(15A_{10}^2 + A_{20}^2 + 16)(BC - B_0C_0)/8 + A_{10}A_{20}(B^2 - B_0^2) - \\
&- 3A_{10}A_{20}(B^2C^2 - B_0^2C_0^2)/2 + 3(A_{20}^2 - A_{10}^2)(BC^3 - B_0C_0^3)/4 + \\
&+ (A_{20}^2 - A_{10}^2)(u - u_0)/8]k_0\},
\end{aligned}
$$

where the supplementary index „o" fixes the respective quantities at the moment $t = t_0$.

Integrating Equations (29) with the values given by the system (30) and with $\sigma = F$, we obtain:

$$
\begin{aligned}
I_1 &= 3K_{50}\{- 2(A_{20}B_0 + A_{10}C_0) - \\
&- [A_{10}/2 - 3A_{20}(u_0 - \pi) - 3(A_{20}C_0 - A_{10}B_0)B_0]q_0 - \\
&- [7A_{20}/2 + 3A_{10}(u_0 - \pi) - 3(A_{10}C_0 + A_{20}B_0)B_0]k_0\} - \\
&- 6K_{60}\{(A_{00}^2 - 3)(u_0 - \pi)/2 + \\
&+ [2(A_{20}^2 + 1)B_0 + A_{10}A_{20}(3 - 2C_0^2)C_0 - (A_{20}^2 - A_{10}^2)B_0^3]q_0 + \\
&+ [3 - A_{00}^2 + A_{10}A_{20}(1 - 2B_0^2)B_0 - 2(A_{10}^2 + 1)C_0 - \\
&- (A_{20}^2 - A_{10}^2)C_0^3]k_0\}, \\
I_2 &= K_{50}\{[- 5A_{10}/4 - 9A_{20}(u_0 - \pi) - 3(A_{20}C_0 - A_{10}B_0)B_0]q_0 - \\
&- (45A_{20}/4)k_0\} - \\
&- K_{60}\{- 3A_{10}A_{20} + \\
&+ [- 12(A_{20}^2 + 1)B_0 + 2(A_{20}^2 - A_{10}^2)B_0^3 - 2A_{10}A_{20}(3 + 2C_0^2)C_0]q_0\},
\end{aligned}
$$

$$I_3 = K_{50}\{- 6A_{10} - (A_{10}/4)q_0 + \\ + [7A_{20}/4 + 9A_{10}(u_0 - \pi) - 3(A_{10}C_0 + A_{20}B_0)B_0]k_0\} - \\ - K_{60}\{3A_{10}A_{20} - A_{10}A_{20}q_0 + \\ + [(A_{20}^2 - A_{10}^2)/2 + 12(A_{10}^2 + 1)C_0 + 2A_{10}A_{20}(3 + 2B_0^2)B_0 + \\ + 2(A_{20}^2 - A_{10}^2)C_0^3]k_0\}, \\ I_4 = 5K_{50}A_{20}k_0 - K_{60}A_{10}A_{20}, \tag{31}$$

where:

$$K_{50} = F\pi p_0^{7/2}\mu^{-3/2}; \; K_{60} = F\pi p_0^3\mu^{-1}c^{-1}. \tag{32}$$

Finally, introducing the values given by the system (31) in Equation (28), we find:

$$\Delta T_\Omega^{(p+PR)} = 6K_{50}\{- A_{20}B_0 - A_{10}(1 + C_0) + \\ + [- A_{10}/2 - (A_{10}B_0 - A_{20}C_0)B_0]q_0 + \\ + [- 5A_{20}/2 + (A_{20}B_0 + A_{10}C_0)B_0]k_0\} - \\ - K_{60}\{3(A_{00}^2 - 3)(u_0 - \pi) + A_{10}A_{20} + \\ + [A_{10}A_{30}(- 1 + 12C_0 - 16C_0^3) - 4(A_{20}^2 - A_{10}^2)B_0^3]q_0 + \\ + [(11A_{10}^2 + 13A_{20}^2 + 24)/2 + 4A_{10}A_{20}(3 - 2B_0^2)B_0 - \\ - 4(A_{20}^2 - A_{10}^2)C_0^3]k_0\}, \tag{33}$$

where $\Delta T_\Omega^{(p+PR)}$ denotes the difference ΔT_Ω caused by the direct solar radiation pressure, including the Poynting-Robertson effect.

Equation (33) permits to separate the influences of the direct solar radiation pressure and of the Poynting-Robertson effect, as follows:

$$\Delta T_\Omega^{(p+PR)} = \Delta T_\Omega^{(p)} + \Delta T_\Omega^{(PR)}, \tag{34}$$

where $\Delta T_\Omega^{(p)}$ and $\Delta T_\Omega^{(PR)}$ denote the two significant terms appearing in the right side of Equation (33).

Our formula was deduced by considering the ecliptic plane as the reference plane of the geocentric frame. However, Equation (33) remains available for any reference plane of a geocentric frame, in the following conditions:

(i) the inclination is taken with respect to the new reference plane;

(ii) the other quantities appearing in $\Delta T_\Omega^{(p+PR)}$ are obtained from the corresponding ones of Equation (33) by a rotation in the orbital plane with an angle equal to the difference between the two arguments of latitude.

Therefore, Equation (33) is available also in the case of the classical frame (with respect to the equatorial plane), where A_1, A_2, A_0 take the form imposed by the above mentioned transformation (see e.g. Lála, 1968; Mioc, Radu, 1977).

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UNELE EFECTE RADIATIVE ÎN PERIOADA NODALĂ A SATELIŢILOR ARTIFICIALI

(Rezumat)

Lucrarea studiază influenţa presiunii radiaţiei solare directe, cu considerarea efectului Poynting-Robertson, asupra perioadei nodale a sateliţilor artificiali ai Pămintului. Formulele sint deduse pentru cazul orbitelor cvasi-circulare complet iluminate de Soare şi rămin valabile pentru orice sistem de referinţă geocentric.

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A GENERALIZATION OF THE CLASS OF CLOSE-TO-CONVEX FUNCTIONS

DORIN BLEZU and NICOLAE N. PASCU

Let S be the set of regular and univalent functions $f(z) = z + a_2z^2 + + \dots$ defined in the unit disc U. In this paper we shall denote by K – the sub-class of convex functions, C – the sub-class of close-to-convex functions and by S_β^* the sub-class of those functions which are starlike of order β; note that $S_0^* = S^*$ – starlike. Further, let $\mathfrak{P}_\gamma$ be the set of all functions $p(z) = = 1 + c_1, z + \dots$ which are analytic in the unit disc and verify:

$$|\arg p(z)| \leqslant \gamma \frac{\pi}{2} \qquad |z| < 1,\ 0 \leqslant \gamma \leqslant 1$$

The symbol $\overline{K}(\alpha)$ denotes the class of α – starlike functions. A regular function $f(z)$ belongs to $\overline{K}(\alpha)$ if $f(z) \cdot f'(z)/z \neq 0$, $|z| < 1$ and the function

$$F_\alpha(z) = (1 - \alpha)\ f(z) + \alpha z\ f'(z), \qquad |z| < 1$$

is starlike; see [9].

The aim of this paper is to present a new class of functions and to study its connections with other classes. Likewise, an integral representation of the functions of this class is given.

In the following let us denote by M a certain sub-class of S.

DEFINITION. Let $f(z)$ be an analytic function regular in the unit disc with $f(z) \cdot f'(z)/z \neq 0$, $|z| < 1$.
The function $f(z)$ belongs to the class $C(a, \gamma, M)$, Re $a \geqslant 0$ $0 \leqslant \gamma \leqslant 1$ iff there exists a function g, $g \in M$ so that

$$zf'(z) + a f(z) = (a + 1)\ zg'^{\gamma}(z)$$

In this paper the case $M = C$ is investigated.

THEOREM 1. *If* $\gamma_1 \leqslant \gamma_2$ *then* $C(a, \gamma_1, c) \cup C(a, \gamma_2, C)$

Proof. Let $f \in C(a, \gamma_1, C$ then

$$zf'(z) + af(z) = (a + 1)\ zg'^{\gamma_1}(z),\ g \in C$$

that is $zf'(z) + af(z) = (a + 1)\ zh'^{\gamma_2}(z)$ where $h'(z) = [g'(z)]^{\frac{\gamma_1}{\gamma_2}}$
We shall show that $h \in C$.
Differentiating logarithmically, we obtain

$$\frac{zh''(z)}{h'(z)} = \frac{\gamma_1}{\gamma_2} \cdot \frac{zg''(z)}{g'(z)}$$

that is

$$\frac{zh''(z)}{h'(z)} + 1 = \frac{\gamma_1}{\gamma_2}\left(\frac{zg''(z)}{g'(z)} + 1\right) + 1 - \frac{\gamma_1}{\gamma_2}.$$

Therefore

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left(\frac{zh''(z)}{h'(z)} + 1\right) d\theta = \frac{\gamma_1}{\gamma_2}\int_{\theta_1}^{\theta_2} \operatorname{Re}\left(\frac{zg''(z)}{g'(z)} + 1\right) d\theta + \left(1 - \frac{\gamma_1}{\gamma_2}\right)(\theta_2 - \theta_1) \geqslant -\frac{\gamma_1}{\gamma_2}\pi > -\pi$$

It is known that the above condition defines the close-to-convex functions. We notice that:

$C(0, 1, C) = C$

$C(a, 1, K) = \overline{K}\left(\frac{1}{a+1}\right)$

$C(\infty, \gamma, K) = S^*_{1-\gamma}$

$C(1, 1, K)$ is a class investigated in [6] by R.J. L i b e r a and A.E. L i v i n g s t o n.

LEMMA $f \in C(\infty, \gamma, C)$ *iff it may be represented as*

$$f(z) = P_1(z) \cdot G_1(z) \quad \text{where } P_1 \in \mathfrak{P}_\gamma \text{ and } G_1 \in S^*_{1-\gamma}$$

Proof. According to the definition of the class $C(a, \gamma, C)$ one has

$$f(z) = zg'^{\gamma}(z), \; g \in C$$

or

$$zg'^{\gamma}(z) = z^{1-\gamma}(zg'(z))^{\gamma}$$

But $zg'(z) = P(z) \cdot G(z)$ where $P \in \mathfrak{P}$ and $G \in S^*$,
which implies $f(z) = z^{1-\gamma} \cdot G^{\gamma}(z) \cdot P^{\gamma}(z)$, where $P^{\gamma} = P_1 \in \mathfrak{P}_\gamma$
We shall show that $G_1(z) = z^{1-\gamma} \cdot G^{\gamma}(z) \in S_{1-\gamma}$
Differentiating logarithmically we obtain

$$\frac{zG_1(z)}{G_1(z)} = (1 - \gamma) + \gamma\frac{zG'(z)}{G(z)}$$

or taking real part

$$\operatorname{Re}\frac{zG_1'(z)}{G_1(z)} = 1 - \gamma + \gamma \operatorname{Re}\frac{zG'(z)}{G(z)} > 1 - \gamma.$$

Let be now $f(z) = p(z) \cdot h(z)$, $p \in \mathfrak{P}_\gamma$ and $h \in S^*_{1-\gamma}$

and $g(z) = \int_0^z p^{\frac{1}{\gamma}}(t) \cdot \left(\frac{h(t)}{t}\right)^{\frac{1}{\gamma}} dt$

In other words $zg'^{\gamma}(z) = p(z) \cdot h(z)$.

Since $h \in S^*_{1-\gamma}$ it follows that $G(z) = z\left(\frac{h(z)}{z}\right)^{\frac{1}{\gamma}} \in S^*$

and $P(z) = p^{\frac{1}{\gamma}}(z) \in \mathfrak{P}$, $(0 \leqslant \gamma \leqslant 1)$

Hence we have $\frac{zg'(z)}{G(z)} = P(z)$; it follows that $g \in C$

There exists $g \in C$ so that $f(z) = zg'^{\gamma}(z)$

or $f \in C(\infty, \gamma, C)$

wich proves our proposition.

THEOREM 2. *If* Re $a \geqslant 0$ *and* $0 \leqslant \gamma \leqslant 1$ *then*

$$C(a, \gamma, C) \subset C(\infty, \gamma, C)$$

Proof. Let $f \in C(a, \gamma, C)$; using lemma 1 we can write

$$zf'(z) + af(z) = (a+1)\ P_1(z) \cdot G_1(z),$$

$$P_1 \in \mathfrak{P}_\gamma, \qquad G_1 \in S^*_{1-\gamma}$$

By means of the notation $f(z) = p_1(z) \cdot g_1(z)$ we have

$$z[p_1'(z) \cdot g_1(z) + p_1(z) \cdot g_1'(z)] + ap_1(z) \cdot g_1(z) = (a+1)\ P_1(z)\, G_1(z)$$

or

$$p_1(z)\ [zg_1'(z) + ag_1(z)] + zp_1'(z) \cdot g_1(z) = (a+1)\ P_1(z) \cdot G_1(z)$$

We shall find g so that

$$zg_1'(z) + ag_1(z) = (a+1)\ G_1(z)$$

that is

$$g_1(z) = \frac{a+1}{z^a}\int_0^z t^{a-1} G_1(z)\ dt$$

which is a starlike integral operator of the Libera type.

It is known that (see [4]) if $G_1 \in S^*_{1-\gamma}$ then $g_1 \in S^*_{1-\gamma}$

Therefore

$$p_1(z) + \frac{zp_1'(z) \cdot g_1(z)}{ag_1(z) + zg_1'(z)} = P_1(z),$$

or

$$p_1(z) + \frac{zp_1'(z)}{a + \frac{zg_1'(z)}{g_1(z)}} = P_1(z)$$

If $p_1(z) = p^\gamma(z)$ and $P_1(z) = P^\gamma(z)$ then as an account of the fact that P_1 is in $\mathfrak{L}_\gamma$ we conclude that $P \in \mathfrak{L}$
It follows that

$$p^\gamma(z) + \frac{z\gamma p^{\gamma-1}(z)\cdot p'(z)}{a + q(z)} = P^\gamma(z) \quad \text{where} \quad q(z) = \frac{zg_1'(z)}{g_1(z)}$$

$$p(z) + \frac{\gamma z p'(z)}{a + q(z)} = \frac{P^\gamma(z)}{p^{\gamma-1}(z)} \tag{1}$$

Let us show that $p \in \mathfrak{L}$; if we suppose that there is z_0, $|z_0| = r_0 < 1$ so that Re $p(z_0) = 0$ then, by means of Lemma A from [7] taking the real part we find

$$0 + N\cdot\gamma\cdot \operatorname{Re}\frac{\bar{a} + \bar{q}(z)}{|a + q(z)|^2} = \operatorname{Re}\frac{P^\gamma(z)}{p(z)^{\gamma-1}}$$

where $N < -\frac{1}{2}(1 + A^2)p(z_0) = A\cdot i$

But $g_1 \in S^*$ therefore Re $q(z) = \operatorname{Re}\frac{zg_1'(z)}{g_1(z)} \geqslant 0$

Now, we compute $\arg\frac{P^\gamma(z)}{P^{\gamma-1}(z)}$; from $P \in \mathfrak{L}$ and $\operatorname{Re} P(z) > 0$

we have

$$\left|\arg\frac{P^\gamma(z)}{P^{\gamma-1}(z)}\right| = |\gamma \arg P(z) - (\gamma - 1)\arg p(z)| =$$

$$= \left|\gamma \arg P(z) \pm (1-\gamma)\frac{\pi}{2}\right| < \frac{\pi}{2}$$

Therefore $\operatorname{Re}\frac{P^\gamma(z)}{p^{\gamma-1}(z)} > 0$, the left side being negative.
This contradiction completes the proof of the above theorem.

LEMMA 2. $f \in C(a, 1, C)$, Re $a \geqslant 0$ *if and only if*

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left(p(z) + \frac{zp'(z)}{a + p(z)}\right) d\theta > -\pi \quad \textit{where} \quad p(z) = \frac{zf'(z)}{f(z)}$$

Proof. Let f be in $C(a, 1, C)$; on that is

$$zf'(z) + af(z) = (a + 1)zg'(z) \text{ with } g \in C$$

Taking the logarithme derivative one finds

$$\frac{zg''(z)}{g'(z)} + 1 = p(z) + \frac{zp'(z)}{a + p(z)} \text{ where } p(z) = \frac{zf'(z)}{f(z)}$$

But g is close-to-convex iff

$$\int_{\theta_1}^{\theta_2}\left(\frac{zg''(z)}{g'(z)}+1\right)d\theta > -\pi$$

which proves our assertion.

Remark. Other generalization of the class of close-to-convex functions is given by D.V. Prohorov in [11] which used a condition of the kind:

$$\int_{\theta_1}^{\theta_2}\left[(a-1)\,p(z)+\frac{zp'(z)}{p(z)}\right]d\theta > -\gamma\pi$$

with Re $a \geqslant 0$ and $0 \leqslant \gamma \leqslant 1$.

THEOREM 3. *A function $f(z)$ belongs to the class $C(a, \gamma, C)$ if and only if there exists a function $F(z)$, $F \in C(\infty, \gamma, C)$ so that*

$$f(z)=\frac{a+1}{z^{a}}\int_{0}^{z}t^{a-1}F(t)\,dt \tag{2}$$

Proof. Let's suppose that $f(z)$ admits the representation given by (2). If $a = m + in$, $m > 0$ then according to the conclusion of Lemma 1, we have

$$F(z)=zg'^{\gamma}(z)=P_1(z)\;G_1(z) \text{ where } P_1 \in \mathfrak{P}_\gamma \text{ and } G_1 \in S^*_{1-\gamma}$$

Hence

$$f(z)=\frac{m+1+in}{z^{m+in}}\int_{0}^{z}t^{m}G_1(z)\cdot P_1(t)t^{in-1}\,dt.$$

Let us denote $G_2(z)=z\left(\frac{G_1(z)}{z}\right)^{\frac{1}{m+1}}$

We show that $G_2(z)$ is starlike. Indeed

$$\mathrm{Re}\,\frac{zG_2'(z)}{G_2(z)}=1-\frac{1}{m+1}+\frac{1}{m+1}\,\mathrm{Re}\,\frac{zG_1'(z)}{G_1(z)}>0$$

Further we define $f_1(z)$ by

$$f_1(z)=\left[(m+1+in)\int_{0}^{z}G_z^{m+1}(t)\cdot P_1(t)\cdot t^{in-1}\,dt\right]^{\frac{1}{m+1-in}}$$

I.B. B a z i l e v i č [1] has proved that if α and β are real numbers, and $g \in S^*$, $h \in \mathfrak{P}$ then the function $w(z)$,

$$w(z) = \left[(\alpha + i\beta)\int_0^z g^{\alpha}(t)\cdot h(t)\cdot t^{i\beta-1}dt\right]^{\frac{1}{\alpha+i}}$$

is regular and univalent in the unit disc.

By means of this result we conclude that $f_1(z)$ is regular and univalent. Therefore $f_1(z)/z \neq 0$, $|z| < 1$; this means that for $\left[\frac{f_1(z)}{z}\right]^{\frac{1}{a+1}}$ it is possible to select the uniform branch which takes the value one for $z = 0$ and which is regular for $|z| < 1$;
Taking into account that

$$f(z) = z\left[\frac{f_1(z)}{z}\right]^{\frac{1}{a+1}}$$

it follows that $f(z)$ given by (2) is regular in the unit disc, moreover $f(z) = 0$ iff $z = 0$ and $f'(0) = 1$.
Those shown above allow us to compute the derivative.
Noticing that $zf'(z) + af(z) = (a + 1)\ F(z)$, $F \in C(\infty, \gamma, C)$ we conclude that f is of the class $C(a, \gamma, C)$.
Further we prove that if $f \in C(a, \gamma, C)$ then $f(z)$ has the form (2). Under this assumption there exists $g(z)$, $g \in C$ so that

$$zf'(z) + af(z) = (a + 1)\ zg'^{\gamma}(z) \tag{3}$$

If $F(z) = zg'^{\gamma}(z)$ then $F \in C(\infty, \gamma, C)$, and by integrating (3) we obtain the equality (2).
From the proof of the Theorem 2 it follows:

COROLLARY 1. *If* $Re\ a \geqslant 0$, $0 \leqslant \gamma \leqslant 1$ *and* $P \in \mathfrak{P}$, $G \in S^*$ *then the function* $g(z)$ *defined by*

$$g^{\gamma}(z) = \frac{a+1}{z^a p^{\gamma}(z)}\int_0^z t^{a-1}P(t)\ G(t)\ dt$$

where $p(z)$ *is given by* (1), *is in the class* S^* *(is starlike).*

In the case in which $P(z) \equiv 1$ it follows that $p(z) \equiv 1$ and we find:

COROLLARY 2. *If* $Re\ a \geqslant 0$, $0 \leqslant \gamma \leqslant 1$, $G \in S^*$ *then the function* g *defined by*

$$g(z) = \left[\frac{a+1}{z^a}\int_0^z t^{a-1}G(t)\ dt\right]^{\frac{1}{\gamma}}$$

belongs to the class S^*

Finally, we want to carry out the following particular cases:

1. For $\gamma = 1$, the corollary 2 was proved in [4], [9], [10]
2. If $\gamma = 1$, $a = 2$ we rediscover an earlier result established by R.J. Libera [5]
3. The case $\gamma = 1$, $a = n$ was investigared by S.D. Bernardi [2]
4. If a is real then we find again some results from [3], [8].

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O GENERALIZARE A CLASEI DE FUNCȚII APROAPE CONVEXE

(Rezumat)

În această lucrare este introdusă o nouă clasă de funcții notată $C(a, \gamma, c)$ și se arată legătura acesteia cu alte clase de funcții. Este dată și reprezentarea integrală a funcțiilor din această clasă. Pentru valori particulare ale parametrilor se regăsesc rezultate importante cunoscute.

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SOME ESTIMATIONS FOR STRONGLY αm-CONVEX FUNCTIONS

I. MARUŞCIAC

1. Introduction and preliminaries. Recently we have defined [2] a new class of generalized convex functions, called α-mean convex (or αm-convex) functions, which contains as particular cases logarithmic, pseudo and quasi-convex functions. In this paper we define the class of strongly α-mean convex functions and we give some estimation theorems for such functions.

DEFINITION 1. Let $X \subseteq R^n$ be convex and $\alpha \in R \setminus \{0\}$. A function $f: X \to R_+$ is called α-mean convex (concave) (αm-convex (concave)) on X iff

$$\forall x,y \in X, \ \ \forall t \in [0,\ 1] \Rightarrow f((1-t)x+ty) \leqslant [(1-t)\ f^\alpha(x) + tf^\alpha(y)]^{\frac{1}{\alpha}} \quad (1)$$

f is stricly αm-convex (concave) on X iff for each $x,\ y \in X$, $x \neq y$ and $t \in\ ]0,\ 1[$ *inequality* (1) *is strict.*

DEFINITION 2. Let $X \subseteq R^n$ be convex and $\alpha > 0$. Function $f: X \to R_+$ is called strongly α-mean convex (strongly αm-convex) on X iff there is a constant $\rho > 0$ such that

$$\forall x,y \in X, \ \ \forall t \in [0,\ 1] \Rightarrow$$

$$f((1-t)x+ty) \leqslant [(1-t)(f^\alpha(x) + tf^\alpha(y) - t(1-t)\ \rho \| x-y \|^2]^{\frac{1}{\alpha}}$$

ρ is called strongly αm-convexity parameter.

Remark 1. From Definition 2 it follows that f is strongly αm-convex on X iff f^α is strongly convex on X in the usual sense.

Remark 2. A strongly αm-convex on X function, as it can be easily ascertained, is strictly αm-convex on X too, but the converse is not always true.

There are strongly αm-convex functions that are not convex, as we can see from the following

Example. Consider $f: R_+ \to R_+$

$$f(x) = \sqrt{x}.$$

This function is strongly $4m$-convex, on R_+ since for $x,\ y \in R_+$ we have

$$f^4((1-t)x+ty) - (1-t)f^4(x) - tf^4(y) = ((1-t)x+ty)^2 - (1-t)x^2 - ty^2 =$$
$$= (1-t)^2x^2 + 2t(1-t)xy + t^2y^2 - (1-t)x^2 - ty^2 = -t(1-t)(x-y)^2.$$

Therefore we have for $\rho = 1$:

$$f((1-t)x+ty) = ((1-t)f^4(x) + tf^4(y) - t(1-t)\,| x-y |^2)^{\frac{1}{4}}.$$

But clearly $f(x) = \sqrt{x}$ is not convex on R_+ (it is concave on R_+).

In [2] it has been shown that every local minimum of an αm-convex function is also global. Moreover if f is strictly αm-convex we have the following result:

THEOREM 1. *If $f : X \subseteq R_n \to R_+$ is strictly αm-convex on X and x^* is a global minimum of f on X, then*

$$\forall x \in X \Rightarrow f(x) > f(x^*)$$

i.e. x^* is unique.

Proof. Assume that there is $x' \in X$, $x' \neq x^*$, such that

$$\mu = f(x') = f(x^*) = \min \; \{f(x) \mid x \in X\}.$$

Then from the definition of strictly αm-convexity of f it follows

$$f((1-t)x^* + tx') < [(1-t)f^\alpha(x^*) + tf^\alpha(x')]^{\frac{1}{\alpha}} = ((1-t)\mu^\alpha + t\mu^\alpha)^{\frac{1}{\alpha}} = \mu$$

i.e. a contradiction.

2. **Estimations for strogly αm-convex functions.** In what follows we are going to enumerate some properties of the strongly αm-convex functions that are useful in determining the rate of convergence of the different numerical optimization methods for such functions.

In all this section α will be assumed to be positive.

THEOREM 2. *If $f \in C^1(X)$ $(f : X \to R_+)$ is strongly αm-convex on X, then for each $x, y \in X$ we have*

$$\rho \| x - y \|^2 \leqslant (\nabla f^\alpha(x) - \nabla f^\alpha(y))(x - y). \tag{2}$$

Proof. From the differential property of the convex functions and Definition 2, for $t = 1/2$, we have

$$\frac{1}{4}\rho \| x - y \|^2 \leqslant \frac{1}{2} f^\alpha(x) + \frac{1}{2} f^\alpha(y) - f^\alpha\left(\frac{x+y}{2}\right) =$$

$$= \frac{1}{2}\left(f^\alpha(x) - f^\alpha\left(\frac{x+y}{2}\right)\right) + \frac{1}{2}\left(f^\alpha(y) - f^\alpha\left(\frac{x+y}{2}\right)\right) \leqslant$$

$$\leqslant \frac{1}{4}(\nabla f^\alpha(x)(x-y) - \nabla f^\alpha(y)(x-y)) = \frac{1}{4}(\nabla f^\alpha(x) - \nabla f^\alpha(y)(x-y).$$

THEOREM 3. *If $f \in C^1(X)$ $(f : X \to R_+)$ is strongly αm-convex on X then*

1° *for each $x^0 \in X$ the level set*

$$X_0 = \{x \in X \mid f(x) \leqslant f(x^0)\}.$$

is bounded.

2° *There is a unique point $x^* \in X$ such that*

$$f(x^*) = \min \; \{f(x) \mid x \in X\}.$$

Proof. 1° Since

$$f^\alpha(x) - f^\alpha(y) = \int_0^1 \nabla f^\alpha(y + t(x-y))(x-y)dt =$$

$$= \nabla f^\alpha(y)(x-y) + \int_0^1 (\nabla f^\alpha(y + t(x-y)) - \nabla f^\alpha(y))(x-y)dt,$$

in view of inequality (2) we get

$$f^\alpha(x) - f^\alpha(y) \geq \nabla f^\alpha(y)(x-y) + \int_0^1 \rho || x - y ||^2 t dt =$$

$$= \nabla f^\alpha(y)(x-y) + \frac{1}{2}\rho || x - y ||^2.$$

For $y = x^0$ and $x \in X_0$ we have

$$0 \geq f^\alpha(x) - f^\alpha(x^0) \geq \nabla f^\alpha(x^0)(x - x^0) + \frac{1}{2}\rho || x - x^0 ||^2$$

and, therefore,

$$|| x - x_0 ||^2 \leq \frac{2}{\rho} \nabla f^\alpha(x^0)(x^0 - x) \leq \frac{2}{\rho} || \Delta f^\alpha(x^0) || \, || x - x^0 ||.$$

So for $x \in X_0$, it follows:

$$|| x - x^0 || \leq \frac{2}{\rho} || \nabla f^\alpha(x^0) ||.$$

2° Follows from Theorem 1, since a strongly αm-convex on X function is strictly αm-convex too.

THEOREM 4. *If $f: X \subseteq R^n \to R_+$ is strongly αm-convex on X, and let x^* be the minimum point of f, then*

$$\forall x \in X \Rightarrow || x - x^* ||^2 \leq \frac{2}{\rho} [f^\alpha(x) - f^\alpha(x^*)] \tag{3}$$

If moreover $f \in C^1(X)$ then

$$\forall x \in X \Rightarrow || x - x^* || \leq \frac{1}{\rho} || \nabla f^\alpha(x) ||. \tag{4}$$

and

$$0 \leq f^\alpha(x) - f^\alpha(x^*) \leq \frac{1}{\rho} || \nabla f^\alpha(x) ||^2. \tag{5}$$

Proof. From Definition 2 we have

$$f\left(\frac{1}{2}x + \frac{1}{2}y\right) \leq \left(\frac{1}{2}f^\alpha(x) + \frac{1}{2}f^\alpha(y) - \frac{1}{4}\rho || x - x^* ||^2\right)^{\frac{1}{\alpha}}.$$

As
$$f(x^*) \leqslant f\left(\frac{1}{2}x + \frac{1}{2}x^*\right)$$

it follows:
$$f^\alpha(x^*) \leqslant \frac{1}{2}f^\alpha(x) + \frac{1}{2}f^\alpha(x^*) - \frac{1}{4}\rho\,||\,x - x^*\,||^2,$$

whence we get (3).

Since x^* is a minimum point to f (and therefore to f^α) on X, we have
$$\forall x \in X \Rightarrow \nabla f^\alpha(x^*)(x - x^*) \geqslant 0$$

and in view of (2):
$$\rho\,||\,x - x^*\,||^2 \leqslant (\nabla f^\alpha(x) - \nabla f^\alpha(x^*))(x - x^*) = \nabla f^\alpha(x)(x - x^*) -$$
$$- \nabla f^\alpha(x^*)(x - x^*) \leqslant \nabla f^\alpha(x)(x - x^*) \leqslant ||\,\nabla f^\alpha(x)\,||\;||\,x - x^*\,||$$
i.e. we got (4).

Lastly, in view of (4), for each $x \in X$, we have
$$0 \leqslant f^\alpha(x) - f^\alpha(x^*) \leqslant \nabla f^\alpha(x)(x - x^*) \leqslant ||\,\nabla f^\alpha(x)\,||\,x - x^*\,|| \leqslant \frac{1}{\rho}\,||\,\nabla f^\alpha(x)\,||^2.$$

THEOREM 5. *If $f: X \to R_+$ is strongly αm-convex and ∇f^α is Lipschitzian, i.e. there is $L > 0$ such that*
$$\forall x, y \in X \Rightarrow ||\,\nabla f^\alpha(x) - \nabla f^\alpha(y)\,|| \leqslant L\,||\,x - y\,||,$$
then for each $x, y \in X$ for which $f(x) \leqslant f(y)$ we have the inequality:
$$||\,\nabla f^\alpha(x)\,|| \leqslant \sqrt{2 + 12\frac{L^2}{\rho^2}}\,||\,\Delta f(y)\,||.$$

The proof is similar to the proof of property 4 of [1], p. 43, since f^α is strongly convex on X.

DEFINITION 3. Let $f: X \to R_+$ be αm-convex function on X. Sequence $(x^k) \subset X$ is called relaxative for f if
$$f(x^{k+1}) \leqslant f(x^k),\ k = 0, 1, 2, \ldots$$
We shall need the following

LEMMA 1. [1, p. 133]. *If $(a_k) \subset R$ is a real sequence such that*
$$a_k - a_{k+1} \geqslant b_k a_k,\ b_k \geqslant 0,\ a_k > 0,$$
then
$$a_m \leqslant a_0 \exp\left(-\sum_{k=0}^{m-1} b_k\right),\ m = 1, 2, \ldots$$

THEOREM 6. *Let $f: R^n \to R_+$ be a strongly αm-convex and differentiable function and $(x^k) \subset R^n$ a relaxative sequence for f.*

Then

$$f^\alpha(x^m) - f^\alpha(x^*) \leqslant a_0 \exp\left(-\rho \sum_{k=0}^{m-1} \frac{f^\alpha(x^k) - f^\alpha(x^{k+1})}{\|\nabla f^\alpha(x^k)\|^2}\right), \; m = 1, 2, \ldots \tag{6}$$

and

$$\| x^m - x^* \|^2 \leqslant \frac{2}{\rho} a_0 \exp\left(-\rho \sum_{k=0}^{m-1} \frac{f^\alpha(x^k) - f^\alpha(x^{k+1})}{\|\nabla f^\alpha(x^k)\|^2}\right), \; m = 1, 2, \ldots \tag{7}$$

where ρ is parameter of strongly αm-convexity of f.

Proof. Denote

$$a_k = f^\alpha(x^k) - f^\alpha(x^*), \; k = 0, 1, \ldots$$

If f is a strongly αm-convex function, then from (5) of Theorem 4, we have

$$a_k \leqslant \frac{1}{\rho} \| \nabla f^\alpha(x^k) \|^2, \; k = 0, 1, \ldots \tag{8}$$

From (3) it follows:

$$\| x^k - x^* \|^2 \leqslant \frac{2}{\rho} (f^\alpha(x^k) - f^\alpha(x^*)). \tag{9}$$

But then:

$$a_k - a_{k+1} = f^\alpha(x^k) - f^\alpha(x^*) - f^\alpha(x^{k+1}) + f^\alpha(x^*) =$$
$$\frac{f^\alpha(x^k) - f^\alpha(x^{k+1})}{\|\nabla f^\alpha(x^k)\|^2} \| \nabla f^\alpha(x^k) \|^2 \geqslant \rho \frac{f^\alpha(x^k) - f^\alpha(x^{k+1})}{\|\nabla f^\alpha(x^k)\|^2} a_k.$$

Since $(x^k) \subset R^n$ is relaxative for f,

$$b_k = \rho \frac{f^\alpha(x^k) - f^\alpha(x^{k+1})}{\|\nabla f^\alpha(x^k)\|^2} \geqslant 0$$

and from Lemma 1 we get

$$a_m \leqslant a_0 \exp\left(-\rho \sum_{k=0}^{m-1} \frac{f^\alpha(x^k) - f^\alpha(x^{k+1})}{\|\nabla f^\alpha(x^k)\|^2}\right), \; k = 0, 1, \ldots \tag{10}$$

i.e. (6) holds.

Inequality (7) follows directly from (9) and (10).

Remark 3. From Theorem 6 it is seen that the relaxation method is convergent whenever the series

$$\sum_{k=0}^{\infty} \frac{f^\alpha(x^k) - f^\alpha(x^{k+1})}{\|\nabla f^\alpha(x^k)\|^2}$$

is divergent. For this it is sufficient that

$$\frac{f^{\alpha}(x^k) - f^{\alpha}(x^{k+1})}{||\nabla f^{\alpha}(x^k)||^2} \geqslant \frac{M}{k}, \quad k \geqslant k_0,$$

where M is a positive constant.

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UNELE ESTIMĂRI PENTRU FUNCŢII TARE αm-CONVEXE

(Rezumat)

Fie $X \subseteq R^n$ o mulţime convexă şi $\alpha > 0$. Funcţia $f: X \to R$ se numeşte tare convexă în medie de ordinul α (tare concavă în medie de ordinul α) pe X, dacă există o constantă $\rho > 0$ astfel incît $\forall x, y \in X, \forall t \in [0,1] \Rightarrow$

$$f((1-t)x + ty) \underset{(\geqslant)}{\leqslant} [(1-t)f^{\alpha}(x) + tf^{\alpha}(y) - t(1-t)\rho ||x-y||^2]^{1/\alpha}.$$

Coeficientul ρ se numeşte parametru de tare αm — convexitate (αm — concavitate) a funcţiei f pe X.

În lucrare se stabilesc teoreme de estimare privind comportarea funcţiilor tare αm — convexe în vecinătatea punctului de minim, analoge cu cele cunoscute pentru funcţiile tare convexe în sens clasic. Aceste estimări sînt utile în determinarea rapidităţii de convergenţă a diferiţilor algoritmi în teoria optimizării, în care intervin asemenea funcţii.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXV, 2, 1980

SUR L'ASSOCIATIVITÉ DE LA CONVOLUTION DES FONCTIONS PRESQUE-PÉRIODIQUES

IOAN MUNTEAN

1. **Introduction.** On dit qu'une fonction continue $x: R \to C$ est *presque-périodique* au sens de H. Bohr [1] si pour chaque $\varepsilon > 0$, il existe $l > 0$ tel que pour tout $a \in R$ on peut trouver $\tau \in [a, a+l]$ vérifiant $|x(t+\tau) - x(t)| < \varepsilon$ quelque soit $t \in R$. On note par AP l'algèbre complexe des fonctions presque-périodiques. Rappelons que la moyenne $M: AP \to C$, définie par

$$M(x) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)dt, \; x \in AP,$$

existe et elle est finie ([1], pag. 34—36). F. Rellich [3] a introduit et étudié le *produit de convolution* $x*y: R \to C$ des deux fonctions $x, y \in AP$, donné par la moyenne

$$(x*y)(s) = M(x_{s-} \cdot y) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(s-t)y(t)dt, \; s \in R. \tag{1.1}$$

On voit aussitôt que $x*y \in AP$, donc $*$ est une opération binaire dans AP, appelée *convolution* dans AP.

F. Riesz et B. Sz.—Nagy [4], pag. 252, affirment sans démonstration que la convolution dans AP est associative. Les mêmes auteurs, [4], pag. 253—257, donnent une application intéressante de cette propriété à l'analyse harmonique des fonctions presque-périodiques. Dans cette note on propose une démonstration de l'associativité de la convolution dans AP, qui s'appuie sur le produit de convolution, regardé comme limite uniforme, et sur les théorèmes de passage à la limite sous l'intégrale et d'interversion des limites.

2. **Le produit de convolution comme limite uniforme.** Notons par R_+ l'intervalle ouvert $]0, \infty[$. Soient $x, y, z \in AP$. Considérons la fonction $f: R_+ \times R \to C$, définie par

$$f(T, u) = \frac{1}{2T}\int_{-T}^{T} y(u-t)z(t)dt, \; T \in R_+, \; u \in R.$$

THÉORÈME 2.1. *Si $\varepsilon > 0$, il existe $T_0 \in R_+$ tel que pour tous $T', T'' > T_0$ et $u \in R$ on ait*

$$|f(T', u) - f(T'', u)| < \varepsilon. \tag{2.1}$$

Démonstration. Définissons les fonctions $g, h: R_+ \times R \to C$ par

$$g(T, u) = \frac{1}{T}\int_0^T y(u-t)z(t)dt \quad \text{et} \quad h(T, u) = \frac{1}{T}\int_0^T y(u+t)z(-t)dt.$$

Étant donné que $f = \frac{1}{2}(g + h)$, le théorème sera établi dès que l'on met en évidence un $T_0 \in R_+$ tel que pour tous $T', T'' > T_0$ et $u \in R$ on ait

$$|g(T', u) - g(T'', u)| < \varepsilon \text{ et } |h(T', u) - h(T'', u)| < \varepsilon. \tag{2.2}$$

Puisque les fonctions $y, z \in AP$ sont bornées, il existe $L > 0$ tel que $|y(t)| \leqslant L$ et $|z(t)| \leqslant L$, $t \in R$. L'ensemble constitué des fonctions y et z est également presque-périodique ([1], pag. 31—32), donc il existe $l > 0$ tel que pour tout $a \in R$ on peut trouver $\tau \in [a, a + l]$ vérifiant

$$|y(t+\tau) - y(t)| < \frac{\varepsilon}{8L} \text{ et } |z(t+\tau) - z(t)| < \frac{\varepsilon}{8L},\ t \in R. \tag{2.3}$$

Notons $T_0 = 16\, lL^2/\varepsilon$. Soient $T', T'' > T_0$ et $u \in R$. Si l'on écrit que la fonction $g(T, u)$ est continue dans les points $T = T'$ et $T = T''$, on peut choisir deux nombres rationnels $T_1, T_2 > T_0$ tels que

$$|g(T', u) - g(T_1, u)| < \frac{\varepsilon}{8} \text{ et } |g(T'', u) - g(T_2, u)| < \frac{\varepsilon}{8}. \tag{2.4}$$

On a $T_1/T_2 = m/n$, où m et n sont des nombres naturels. Pour tout $k \in \in \{1, \ldots, n\}$ il existe $\tau_k \in [(k-1)T_1, (k-1)T_1 + l]$ tel que les inégalités (2.3) soient vraies, donc

$$|y(u-t)z(t) - y(u-\tau_k-t)z(\tau_k+t)| \leqslant |[y(u-t) - y(u-\tau_k-t)]z(t)| +$$
$$+ |[z(t) - z(\tau_k+t)]y(u-\tau_k-t)| < \frac{\varepsilon}{8L}L + \frac{\varepsilon}{8L}L = \frac{\varepsilon}{4}$$

et, par suite,

$$|g(T_1, u) - g(nT_1, u)| = \left|\frac{1}{T_1}\int_0^{T_1} y(u-t)z(t)dt - \right.$$
$$\left. - \frac{1}{nT_1}\sum_{k=1}^{n}\int_{(k-1)T_1}^{kT_1} y(u-t)z(t)dt\right| \leqslant \frac{1}{n}\sum_{k=1}^{n}\left[\left|\frac{1}{T_1}\int_0^{T_1} y(u-t)z(t)dt - \right.\right.$$
$$\left. - \frac{1}{T_1}\int_{\tau_k}^{\tau_k+T_1} y(u-t)z(t)dt\right| + \left|\frac{1}{T_1}\int_{(k-1)T_1}^{\tau_k} y(u-t)z(t)dt\right| +$$

$$+\left|\frac{1}{T_1}\int_{kT_1}^{\tau_k+T} y(u-t)\ z(t)dt\right|\Bigg] \leqslant \frac{1}{n}\sum_{k=1}^{n}\Bigg[\frac{1}{T_1}\int_0^{T_1} |\,y(u-t)\ z(t)-$$

$$-y(u-\tau_k-t)\ z(\tau_k+t)\,|\,dt+\frac{2lL^2}{T_1}\Bigg] \leqslant \frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{T_1}\frac{\varepsilon}{4}T_1+\frac{2lL^2}{T_1}\right)=$$

$$=\frac{\varepsilon}{4}+\frac{2lL^2}{T_1}. \tag{2.5}$$

De la même manière on arrive à l'inégalité

$$|\,g(T_2,\ u)-g(mT_2,\ u)\,|<\frac{\varepsilon}{4}+\frac{2lL^2}{T_2}. \tag{2.6}$$

De (2.4), (2.5) et (2.6) on déduit la première inégalité (2.2):

$$|\,g(T',\ u)-g(T'',\ u)\,| \leqslant |\,g(T',u)-g(T_1,u)\,|+|\,g(T_1,u)-g(nT_1,u)\,|+$$

$$+|\,g(mT_2,\ u)-g(T_2,\ u)\,|+|\,g(T_2,\ u)-g(T'',\ u)\,|<\frac{\varepsilon}{8}+\frac{\varepsilon}{4}+\frac{2lL^2}{T_1}+$$

$$+\frac{\varepsilon}{4}+\frac{2lL^2}{T_2}+\frac{\varepsilon}{8}<\frac{3\varepsilon}{4}+\frac{4lL^2}{T_0}=\varepsilon.$$

On raisonne de même pour aboutir à la deuxième inégalité (2.2).

Remarque 2.2. Lorsque les fonctions y et z vérifient $y(u-t)=x(u+t)$ et $z(t)=\overline{x(t)}$, où $x\in AP$, le théorème 2.1 a été établi par H. Bohr [1], pag. 38–39.

Soit $s\in R$. Considérons maintenant la fonction $f_s: R\times R_+\to C$, définie par

$$f_s(t,\ U)=\frac{1}{2U}\int_{-U}^{U} x(s-u)\ y(u-t)du,\ t\in R,\ U\in R_+.$$

Le raisonnement de la démonstration du théorème 2.1 conduit au résultat suivant:

THÉORÈME 2.3. *Si* $\varepsilon>0$, *il existe* $U_0\in R_+$ *tel que pour tous* $U',\ U''>U_0$ *et* $t\in R$ *on ait*

$$|\,f_s(t,\ U')-f_s(t,\ U'')\,|<\varepsilon.$$

COROLLAIRE 2.4. *Si* $U\in R_+$, *alors*

$$f(T,\ u)x(s-u)\to(y*z)(u)x(s-u) \tag{2.7}$$

uniformément pour $u\in[-U,\ U]$ *lorsque* $T\to\infty$.

Démonstration. Puisque la fonction $x\in AP$ est bornée, il existe $L>0$ tel que $|\,x(t)\,|\leqslant L,\ t\in R$. Soit $\varepsilon>0$. On associe à $\frac{\varepsilon}{2L}$ le nombre T_0 du théorème 2.1. Soient $T > T_0$ et $u \in [-U, U]$. Comme la moyenne $(y*z)(u)$ existe, on peut trouver un $T_* > T$ tel que

$$|f(T_*, u) - (y*z)(u)| < \frac{\varepsilon}{2L}. \tag{2.8}$$

De (2.1) et (2.8) on déduit (2.7) :

$$|f(T,u)\,x(s-u) - (y*z)(u)x(s-u)| \leqslant |[f(T, u) - f(T_*, u)]\,x(s-u)| + $$
$$+ |[f(T_*, u) - (y*z)(u)]\,x(s-u)| < \varepsilon.$$

COROLLAIRE 2.5. *Si* $T \in R_+$, *alors*

$$f_s(t, U)z(t) \to (x*y)(s-t)z(t)$$

uniformément pour $t \in [-T, T]$ *lorsque* $U \to \infty$.

Démonstration. On répète le raisonnement de la démonstration du corollaire 2.4, en utilisant le théorème 2.3 au lieu du théorème 2.1, et on tient compte de

$$(x*y)(s-t) = \lim_{U\to\infty} \frac{1}{2U} \int_{-U}^{U} x(s-t-u)y(u)du =$$

$$= \lim_{U\to\infty} \frac{1}{2U}\left[\int_{-U+t}^{-U} + \int_{-U}^{U} + \int_{U}^{U+t}\right] x(s-u)y(u-t)du = \lim_{U\to\infty} f_s(t, U).$$

3. Théorèmes de passage à la limite sous l'intégrale et d'interversion des limites. Nous avons besoin de deux théorèmes classiques, la démonstration desquels peut être trouvé en [2], pag. 666–672.

THÉORÈME 3.1. *Soient* $Q \in R_+$, $\varphi : R_+ \times [-Q, Q] \to C$ *une fonction intégrable au sens de Riemann par rapport à* $q \in [-Q, Q]$ *pour chaque* $P \in R_+$ *et* $\varphi_0 : [-Q, Q] \to C$ *une fonction donnée. Si* $\varphi(P, q) \to \varphi_0(q)$ *uniformément pour* $q \in [-Q, Q]$ *lorsque* $P \to \infty$, *alors* φ_0 *est intégrable au sens de Riemann et*

$$\lim_{P\to\infty} \int_{-Q}^{Q} \varphi(P, q)dq = \int_{-Q}^{Q} [\lim_{P\to\infty} \varphi(P, q)]dq = \int_{-Q}^{Q} \varphi_0(q)dq.$$

THÉORÈME 3.2. *Soient* $F : R_+ \times R_+ \to C$ *et* $G, H : R_+ \to C$ *trois fonctions données. Si* $F(T, U) \to G(U)$ *uniformément pour* $U \in R_+$ *quand* $T \to \infty$ *et* $F(T, U) \to H(T)$ *pour chaque* $T \in R_+$ *quand* $U \to \infty$, *alors les limites* $\lim_{T\to\infty} \lim_{U\to\infty} F(T, U)$ *et* $\lim_{U\to\infty} \lim_{T\to\infty} F(T, U)$ *existent et elles sont égales.*

Soient $x, y, z \in AP$ et $s \in R$. Considérons la fonction $F_s : R_+ \times R_+ \to C$, définie par

$$F_s(T, U) = \frac{1}{2T}\frac{1}{2U} \int_{-T}^{T}\int_{-U}^{U} x(s-u)y(u-t)z(t)du\,dt.$$

COROLLAIRE 3.3. *Si* $T \in R_+$, *alors*

$$F_s(T, U) \to \frac{1}{2T} \int_{-T}^{T} (x * y)(s-t)z(t)dt \text{ lorsque } U \to \infty.$$

Démonstration. Le corollaire 2.5 et le théorème 3.1 permettent le passage à la limite sous l'intégrale:

$$\lim_{U\to\infty} F_s(T, U) = \frac{1}{2T} \int_{-T}^{T} \left[\lim_{U\to\infty} \frac{1}{2U} \int_{-U}^{U} x(s-u)y(u-t)du\right] z(t)dt =$$

$$= \frac{1}{2T} \int_{-T}^{T} \left[\lim_{U\to\infty} \frac{1}{2U} \int_{-U}^{U} x(s-t-u)y(u)du\right] z(t)dt = \frac{1}{2T} \int_{-T}^{T} (x * y)(s-t)z(t)dt.$$

COROLLAIRE 3.4. *On a*

$$F_s(T, U) \to \frac{1}{2U} \int_{-U}^{U} x(s-u)(y * z)(u)du \tag{3.1}$$

uniformément pour $U \in R_+$ *lorsque* $T \to \infty$.

Démonstration. Puisque la fonction $x \in AP$ est bornée, il existe $L > 0$ tel que $|x(t)| \leqslant L$, $t \in R$. Soit $\varepsilon > 0$. On associe à $\frac{\varepsilon}{2L}$ le nombre T_0 de théorème 2.1. Soient $T > T_0$ et $U \in R_+$. Le corollaire 2.4 et le théorème 3.1 permettent le passage à la limite sous l'intégrale:

$$\lim_{V\to\infty} F_s(V, U) = \frac{1}{2U} \int_{-U}^{U} x(s-u)\left[\lim_{V\to\infty} \frac{1}{2V} \int_{-V}^{V} y(u-t)z(t)dt\right] du =$$

$$= \frac{1}{2U} \int_{-U}^{U} x(s-u)(y * z)(u)du.$$

Il en résulte l'existence d'un $T_U > T_0$ tel que

$$\left| F_s(T, U) - \frac{1}{2U} \int_{-U}^{U} x(s-u)(y * z)(u)du \right| < \frac{\varepsilon}{2} \cdot \tag{3.2}$$

De (3.2) et (2.1) on obtient

$$\left|F_s(T, U) - \frac{1}{2U}\int_{-U}^{U} x(s-u)(y*z)(u)du\right| \leqslant |F_s(T, U) - F_s(T_U, U)| +$$

$$+\left|F_s(T_U, U) - \frac{1}{2U}\int_{-U}^{U} x(s-u)(y*z)(u)du\right| <$$

$$\left|\frac{1}{2U}\int_{-U}^{U} x(s-u)\left[\frac{1}{2T}\int_{-T}^{T} y(u-t)z(t)dt - \frac{1}{2T_U}\int_{-T_U}^{T_U} y(u-t)z(t)dt\right]du\right| +$$

$$\frac{\varepsilon}{2} < \frac{1}{2U}\int_{-U}^{U} L\,|f(T, u) - f(T_U, u)|\,du + \frac{\varepsilon}{2} < \varepsilon.$$

4. L'associativité de la convolution dans AP.

THÉORÈME 4.1 (F. Riesz et B. Sz.-Nagy). *Si* $x, y, z \in AP$ *et* $s \in R$, *alors* $(x*(y*z))(s) = ((x*y)*z)(s)$.

Démonstration. En utilisant successivement le corollaire 2.4, le théorème 3.1 avec $\varphi(T, u) = f(T, u)x(s-u)$, $u \in [-U, U]$, les corollaires 3.4 et 3.3, le théorème 3.2 avec $F = F_s$, le corollaire 2.5 et le théorème 3.1 avec $\varphi(t, U) = f_s(t, U)z(t)$, $t \in [-T, T]$, on a

$$(x*(y*z))(s) = \lim_{U\to\infty} \frac{1}{2U}\int_{-U}^{U} x(s-u)(y*z)(u)du =$$

$$= \lim_{U\to\infty}\frac{1}{2U}\int_{-U}^{U}[\lim_{T\to\infty} f(T, u)x(s-u)]\,du = \lim_{U\to\infty}\frac{1}{2U}\lim_{T\to\infty}\int_{-U}^{U} f(T, u)x(s-u)du =$$

$$= \lim_{U\to\infty}\lim_{T\to\infty} F_s(T, U) = \lim_{T\to\infty}\lim_{U\to\infty} F_s(T, U) =$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[\lim_{U\to\infty} f_s(t, U)z(t)]dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(x*y)(s-t)z(t)dt = ((x*y)*z)(s).$$

Remarque 4.2. Si dans la définition (1.1) du produit $x*y$ on remplace le facteur $y(t)$ par $\overline{y(t)}$, alors l'opération correspondante $*$ n'est plus associative. En effet, pour $x(t) = \overline{y(t)} = z(t) = e^{it}$ on a $(x*(y*z))(s) = e^{is}$, tandis que $((x*y)*z)(s) = 0$.

(Manuscrit reçu le 10 juin 1979)

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ASUPRA ASOCIATIVITĂŢII CONVOLUŢIEI FUNCŢIILOR APROAPE PERIODICE

(Rezumat)

Se prezintă o demonstraţie a teoremei lui F. Riesz şi B. Sz.—Nagy asupra asociativităţii convoluţiei în spaţiul funcţiilor aproape periodice. Demonstraţia utilizează produsul de convoluţie, privit ca limită uniformă, şi teoremele de trecere la limită sub integrală şi de intervertire a limitelor.

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THE ENIGMATIC VARIABLE STAR *RT PERSEI*

IOAN TODORAN

The variability of *RT Persei* was discovered in 1904 by Ceraski (see Ahnert 1974). Since that time, this star has been intensively observed in order to obtain large series of minima and to pursue the corresponding period variation.

For the first time, Dugan [2], using all available minima, found that the „observed" $O-C$ differences were beautifully represented by a light-time orbit with a long period of 37,2 years. Therefore, the observed minima up to 1937 showed a periodic variation in the length of the short period and it was explained by the hypothesis of the presence of a third component. But soon after Dugan's paper was published, the next series of observations showed systematic deviations from the predictions. That is why, in the next years the problem of the period variation of the eclipsing binary system *RT Persei* was discussed by several authors: Vasilieva [8], Todoran [7], Frieboes – Conde *and* Herczeg [3], Ahnert [1] etc. Nevertheless, the new observed minima and their $O-C$ differences make it conspicuously necessary that the problem of the period variation of *RT Persei* should be resumed (see also Mancuso et al, [6]).

Now, if G_1 and G denote the mass centers of the eclipsing binary system and of the postulated triple system respectively, in figure 1 the motion of G_1 around G in a long-periodic orbit is represented. Here the common notations for the orbital elements are used: ω = periastron longitude; r = radius vector; i = inclination of the orbital plane $\Omega B \mho$ to a plane which is tangent to the celestial sphere at the origin of coordinates (G); θ = longitude of G_1 reckoned from $\overline{GB}$; v = true anomaly, and t_0 denotes the instant when the eclipsing binary system passes through periastron. With these notations, we have at once

$$z = r \cos \theta \sin i \tag{1}$$

where

$$\theta = v + \omega - 90° \tag{2}$$

Now, if c represents the velocity of light, we may write

$$\tau = \frac{z}{c} = \frac{a \sin i}{c}\left[\frac{r}{a} \sin v \cdot \cos \omega + \frac{r}{a} \cos v \cdot \sin \omega\right], \tag{3}$$

that is, the movement of G_1 around G causes a periodic change in the length of the apparent period of an eclipsing binary system. In such a case, the corresponding $O-C$ differences will be defined by the formula

$$O-C = \tau + T_0 + N \cdot P_1 \tag{4}$$

where T_0, N and P_1 are the „constants" in the linear ephemeris

$$T_N = T_0 + N \cdot P_1$$

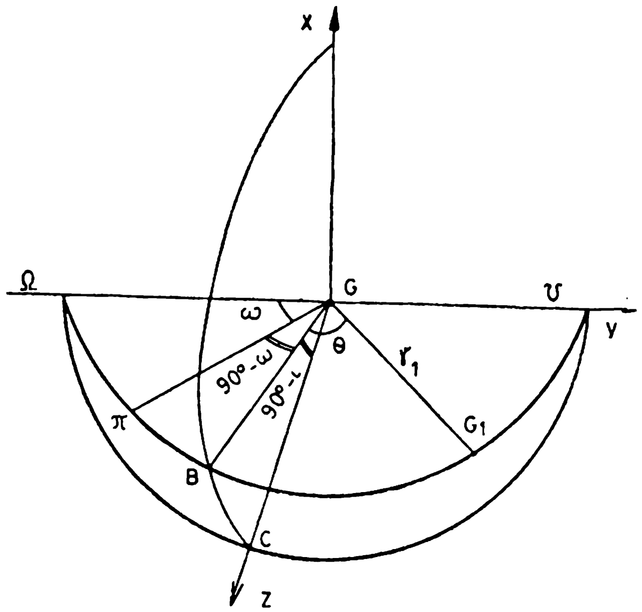

Fig. 1

It is easy to see that in the case of *RT Persei*, in the corresponding $O-C$ diagram besides the periodic variation there are some sudden changes which could be caused by some abrupt period changes. The latter may render an evaluation of the periodic variation difficult. Nevertheless, Frieboes-Conde and Herczeg [3] find a thoroughly satisfactory representation of all available minima if, in addition to the periodic variation, τ, a relatively sudden decreasing of the period P_1 is postulated. But, again, the new minima observed after 1973, showed that the above mentioned sudden change of the period is not satisfactory. That is why we consider that this problem must be resumed. With this end in view we have brought together 342 minima observed by different observers and 38 normal minima have been performed. The corresponding results are given in Table 1, where the following ephemerides are suited:

$$\text{Min.hel.} = JD2419550.251 + 0^{d}8494135 \cdot N_1, \qquad t \leqslant 1912,$$
$$\text{Min.hel.} = JD2419550.251 + 0.8494061 \cdot N_1, \qquad 1912 < t \leqslant 1956,$$
$$\text{Min.hel.} = JD2435433.296 + 0.8494033 \cdot N, \qquad t > 1956.$$

These ephemerides have been used in order to obtain the corresponding „observed" $(O-C)_0$ differences. The results are given in the fourth column of Table 1 and they are represented by full dots in figure 2, where the abscissa is the heliocentric Julian Data.

Consequently, once more, a new sudden change of the period P_1, somewhere around $JD2435433$ (1956) must be postulated, if we are going to consider that the periodic variation in the „observed" $O-C$ diagram is caused by the presence of a third component.

Table 1

Min. hel. 2400000	n	N	$(O-C)_0$	$(O-C)_c$	
16839.778	11	− 3191	$+0^d.005$	$+0^d.006$	$P=0^d.8494135$
17247.499	14	− 2711	+ .008	+ .009	
17833.600	14	− 2021	+ .014	+ .013	
18615.064	11	− 1101	+ .017	+ .017	
19031.275	2	− 611	+ .016	+ .018	
19659.841	14	+ 129	+ .017	+ .018	$P=0.8494061$
20101.530	13	+ 649	+ .014	+ .017	
20458.284	5	1069	+ .018	+ .016	
21375.637	4	2149	+ .012	+ .013	
22386.426	3	3339	+ .008	+ .007	
23524.624	8	4679	+ .002	+ .001	
24484.448	11	5809	− .003	− .004	
24798.726	13	6179	− .005	− .005	
26106.806	4	7719	− .011	− .011	
27168.563	6	8969	− .011	− .015	
28383.211	8	10399	− .014	− .017	
29597.863	6	11829	− .013	− .012	
30838.006	6	13289	− .003	− .004	
31746.878	3	14359	+. 005	.000	
33046.483	7	15889	+ .018	+ .010	
33641.061	9	16589	+ .012	+ .014	
34524.446	7	17629	+ .015	+ .017	
35433.310	15	18699	+ .014	+ .018	
35730.598	12	350	+ .011	+ .017	$P=0.8494033$
36104.337	12	790	+ .012	+ .016	
36478.071	11	1230	+ .009	+ .015	
37106.629	16	1970	+ .008	+ .013	
37565.311	11	2510	+ .013	+ .010	
37998.504	4	3020	+ .010	+ .007	
38686.518	4	3830	+ .007	+ .004	
39094.225	13	4310	+ .001	+ .002	
39561.393	7	4860	− .003	− .001	
40827.001	11	6350	− .006	− .007	
41328.145	5	6940	− .010	− .009	
41684.892	9	7360	− .012	− .010	
41982.181	7	7710	− .014	− .012	
42619.234	11	8460	− .014	− .014	
43375.202	15	9350	−0 .015	−0 .016	

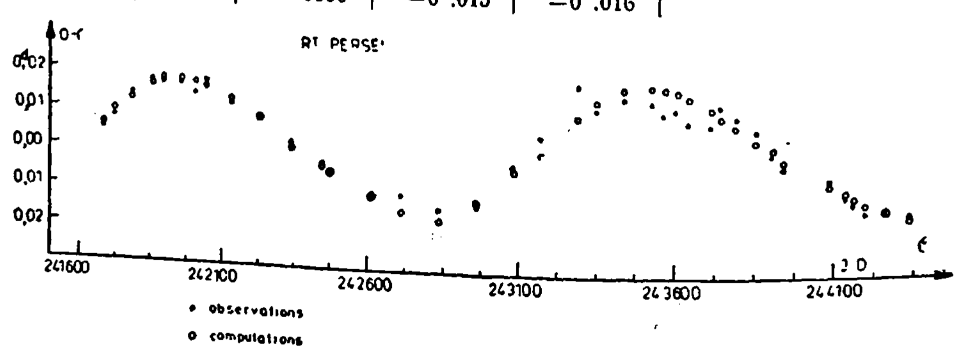

Fig. 2

It is already a well-established observational fact that there is a great number of semi-detached systems whose periods are characterized by sudden changes. That is why the two „observed" sudden variations in the orbital period of *RT Persei* could be considered as being real phenomena, and we must take them into consideration if the presence of a third component is postulated.

Of course, the „observed" sudden changes in the orbital period of an eclipsing binary system must be regarded as a disturbing phenomenon in the periodic variation of the $O-C$ differences. That is why, we must remove them before the corresponding parameters of the long-periodic orbit should be determined.

From the three above established ephemerides and the „observed" $O-C$ diagram (full dots in fig. 2), it is easy to see that the period of *RT Persei* is shortening.

Now, if we take also into consideration the well-known fact that the eclipsing binary system *RT Persei* is a semi-detached one (see Z. Kopal and B.M. Shapley [5]), that is, the secondary component is overfilling its critical Roche surface, we may consider that the corresponding period shortenings could be caused by the secondary component instability. In addition, we may postulate here the fact that some peculiar positions of the close binary system in its long-periodic orbit could be favourable circumstances for the growth of such instabilities. As an exemple we could mention here the passage of the eclipsing binary system through the periastron of the long-periodic orbit.

Now, from the available $O-C$ diagram (see Fig. 2) we have determined the following constants:

the abscissa of the first maximum $= JD2419300$,
the abscisa of the second maximum $= JD2435000$,

and then

$$P = 15700 \text{ days} \approx 43 \text{ years}$$

$$\mu = 2\pi \frac{P_1}{P} = 360^\circ \frac{0.8494}{15700} = 0^\circ.0195$$

$$\left.\begin{array}{l} e \cos\omega = 0{,}25 \\ e \sin\omega = 0{,}076 \end{array}\right\} \Rightarrow \omega = 17^\circ, \quad e = 0{,}26$$

$a \sin i = 3.04$ a.u,

$t_{02} = JD2432459$ and $t_{01} = JD2416759$.

$$q = \frac{2\pi}{P}(T_0 - t_0) = 64^\circ, \quad t \leqslant 1956,$$
$$= 87^\circ \quad t > 1956.$$

In the above presented considerations we have already mentioned the fact that t_0 represents the instant when the eclipsing binary system is passing through the long-periodic orbit periastron. Therefore, if we compare the above determined values of t_0 and the two instants of the corresponding sudden changes of the short period: $t_1 = JD2419500$ and $t_2 = JD2435433$, it is easy to see that the two „observed" abrupt variations occur somewhere around the two maxima of the „observed" $O-C$ diagram. In other words, the two sudden changes of the short period occur somewhere after the eclipsing binary system has passed through the periastron of the long-periodic orbit. That is why the above presented results may be regarded in good agreement with the hypothesis of the secondary component instability, such instability being also favoured by the corresponding position of the binary system in its long-periodic orbit.

If we take into consideration the two masses: $m_1 = 1{,}6$ and $m_2 = 0.36\ m_\odot$ (see Frieboes-Conde and Herczeg, [3]) we can write

$$f(m) = \frac{m_3^3 \sin^3 i}{(m_1 + m_2 + m_3)^2} = \frac{(a \sin i)^3}{P^2}$$

whence

$$\frac{m_3^3 \sin^3 i}{(1.96 + m_3)^2} = \frac{(3.04)^3}{43^2}.$$

Now, if we consider the inclination of the short-periodic orbital plane $i = 86^\circ\!.9$ (see Z. Kopal and B.M. Shapley [5]), and if we assume that the short- and long-periodic orbits have to lie almost in the same plane, we can write

$$60^\circ < i < 90^\circ$$

and as a consequence, it follows

$$m_3 \approx 0.5\ m_\odot$$

In order to „prove" the validity of the above determined orbital constants, the following empirical formulae have been established for the computed $(O-C)_c$ differences:

$$(O-C)_c = 0^d.017 \sin(0^\circ.0195N_1 + 81^\circ) + 0^d.0022 \sin 2\,(0^\circ.0195N_1 + 73^\circ), \quad t < 1956$$

$$(O-C)_c = 0.017 \sin(0.0195N + 85^\circ) + 0.0022 \sin 2(0.0195N + 77), \quad t > 1956,$$

and the corresponding results are listed in the last column of Table 1. In figure 2 these results are represented by open circles.

From a great number of situations in which the change in orbital period may be caused by mass ejection, we shall consider here only the *so-called slow-mode*. In this case, the general aspect of the period variation (see Huang, [4]) is given by

$$\frac{\delta P_1}{P_1} = \frac{4\delta(m_1 + m_2)}{m_1 + m_2} - 3\left(\frac{\delta m_1}{m_1} + \frac{\delta m_2}{m_2}\right) + 3\frac{\delta h_0}{h_0} + 3\frac{e\delta e}{1 - e^2} \tag{5}$$

where m_1 and m_2 are the masses of the two components, e = orbital eccentricity (for RT Persei $e = 0$) and h_0 = angular momentum per unit mass.

In the case of simple mass transfer $\delta(m_1 + m_2) = 0$, $\delta h_0 = 0$ and from (5) we have at once

$$\frac{\delta P_1}{P_1} = -3\frac{m_1 - m_2}{m_1}\frac{\delta m_2}{m_2}. \tag{6}$$

Since $m_1 > m_2$ and $-3(m_1 - m_2)/m_1 < 0$, we must have $\frac{\delta P_1}{P_1} > 0$, as being caused by a mass ejection $\delta m_2 < 0$; while for RT Persei we have $\delta P_1/P_1 < 0$. Therefore, such a case is not suitable for our problem.

When the particles ejected from the less massive component form a ring around the more massive component we may suppose that there is a change only in orbital momentum and $\delta(m_1 + m_2) = 0$ remains further valid. In such a case (see Huang, 1963) we have

$$\left.\begin{aligned} \frac{\delta P_1}{P_1} &= \alpha \frac{\delta m_2}{m_2}; \qquad \alpha = 3\left(\gamma_1 - 1 + \frac{m_2}{m_1}\right); \\ \gamma_1 &= \left[\frac{(m_1 + m_2)a_i}{m_1 a(1 - e^2)}\right]^{1/2} \end{aligned}\right\} \tag{7}$$

For $\delta P_1/P_1 < 0$ and $\delta m_2/m_2 < 0$ we must have $\alpha > 0$. That is why, α being computed for different values of a_i/a (a_i = ring radius), from Huang's Table 1 (H u a n g, [4]) we have $\alpha > 0$ for

$$a_i/a \geqslant 0{,}45 \text{ and for } m_{2/m_1} = 1/4.$$

Now, from figure 3, we can see that a_i/a cannot be greater than 0,5, which is the radius of the corresponding Roche-lobe. Therefore, if the two above considered assumptions are true, around the more massive component there will be a small ring, its radius being $0{,}45 < a_i/a < 0{,}51$, where a is the corresponding orbital radius.

From observations we have

$$\frac{\delta P_1}{P_1} = -\,0^d.0000094 \text{ and } -0^d.0000035$$

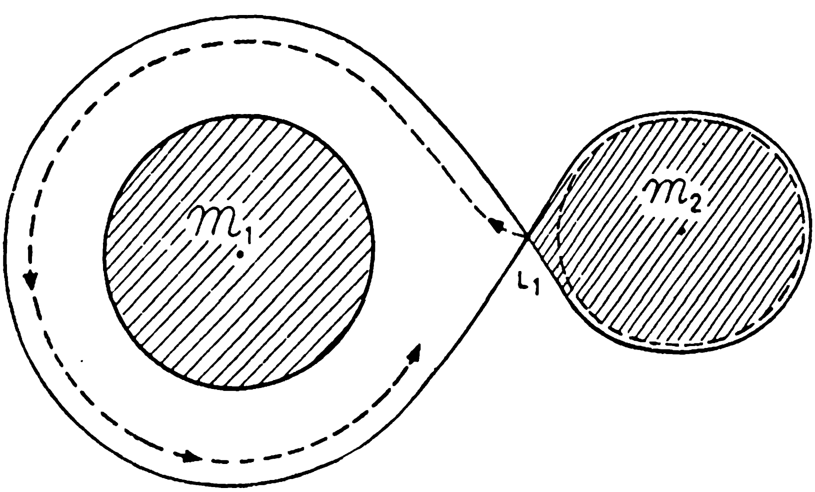

Fig. 3.

and if we adopt $\alpha = 0{,}5$, from (7) we have

$$\delta m_2 = -0.6\cdot 10^{-5} m_\odot \text{ and } \delta m_2 = -0.2\cdot 10^{-5} m_\odot$$

respectively.

Remark. In order to explain the observed $O-C$ diagram of RT Persei, two effects are simultaneously considered: the presence of a third component and the mass ejection from the secondary component. In such a way between the observed and computed ordinates (see figure 2) there is a satisfactory agreement. But, in order to prove the validity of the two assumptions, new series of photometric observations are required in order to see if the observed and the „empirical" curves will further remain in the same good agreement. Also a comparison between the $O-C$ curve and the changes of the radial velocity of the eclipsing pair could furnish, in principle, conclusive evidence for or against a light-time hypothesis. That is why, spectroscopic observations would be of a great importance for RT Persei.

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STEAUA ENIGMATICĂ *RT PERSEI*

(Rezumat)

În prezenta lucrare, luînd în considerare toate momentele minimelor observate de diferiţi autori, se studiază variaţia periodică a perioadei stelei *RT Persei*. Se ajunge la concluzia că ipoteza prezenţei unui al treilea corp poate fi susţinută numai dacă se admite existenţa a cel puţin două salturi bruşte în durata perioadei orbitale. Aceste salturi sînt atribuite instabilităţii componentei secundare a sistemului dublu fotometric *RT Persei*, instabilitate care poate fi favorizată şi de anumite poziţii speciale pe care le poate avea binara respectivă pe orbita sa lung-periodică.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXV, 2, 1980

SPAŢII 2-METRICE GENERALIZATE

D. BORŞAN

1. În lucrarea *2-metrische Räume und ihre topologische Struktur*, apărută în 1963 (Mathematische Nachrichten, t. 26, pp. 114—148), Siegfrid Gähler studiază o clasă de spaţii, *spaţiile 2-metrice*, care pot fi privite ca analogul bidimensional al spaţiilor metrice. Ideile lui S. Gähler îşi au punctul de plecare în lucrarea lui K. Menger, *Entwurf einer Theorie der n-dimensionalen Metrik*, apărută în 1928.

Prin spaţiu 2-metric, S. Gähler înţelege un cuplu (X, ρ), unde, X este o mulţime oarecare iar $\rho : X \times X \times X \to \mathbf{R}$, o aplicaţie satisfăcînd următoarelor axiome:

$$\text{(1a)} \quad x, y \in X, \ x \neq y \Rightarrow \exists\ z \in X : \rho(x, y, z) \neq 0;$$

$$\text{(1b)} \quad \rho(x, y, z) = 0 \Leftrightarrow x = y \text{ sau } x = z \text{ sau } y = z;$$

$$\text{(2)} \quad \rho(x, y, z) = \rho(x, z, y) = \rho(y, z, x); \ \forall\ x, y, z \in X$$

$$\text{(3)} \quad \rho(x, y, z) \leqslant \rho(x, y, t) + \rho(x, t, z) + \rho(t, y, z), \ \forall\ x, y, z, t \in X$$

Aplicaţia ρ cu proprietăţile (1a), (1b), (2) şi (3) se numeşte 2-metrică, iar numărul $\rho(x, y, z)$ se numeşte măsura tripletului (x, y, z).

Din axioma (2) se deduce că măsura unui triplet nu depinde de ordinea punctelor; din (1b) ,(2) şi (3) rezultă că ea este nenegativă.

Prezenta lucrare îşi propune să generalizeze noţiunea de 2-metrică, considerînd funcţii de măsură cu valori într-o structură mai puţin restrictivă decît aceea a numerelor reale.

2. Fie $(\mathfrak{M}, \leqslant)$ o mulţime parţial ordonată oarecare. Notăm cu „$\preccurlyeq$" relaţia de ordonare parţială, definită în mod natural în $\mathfrak{M} \times \mathfrak{M} \times \mathfrak{M}$, prin:

$$(a, b, c) \preccurlyeq (a_1, b_1, c_1) \underset{\text{def}}{\Leftrightarrow} a \leqslant a_1, \ b \leqslant b_1, \ c \leqslant c_1.$$

Considerăm o operaţie ternară $\varphi : \mathfrak{M} \times \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$, care satisface condiţiile:

$$(\varphi_1) \quad \varphi(a, b, c) = \varphi(a, c, b) = \varphi(b, c, a), \ \forall\ a, b, c \in \mathfrak{M};$$

$$(\varphi_2) \quad (a, b, c) \prec (a_1, b_1, c_1) \Rightarrow \varphi(a, b, c) < \varphi(a_1, b_1, c_1)$$

În sfîrşit, fie $\mathcal{S} \subseteq \mathfrak{M}$.

DEFINIŢIA (2.1). Prin 2-metrică generalizată (mai scurt g-2-metrică) pe mulţimea X, înţelegem o aplicaţie $\rho : X \times X \times X \to \mathfrak{M}$, satisfăcînd următoarele axiome:

(ρ_{1a}) $x, y \in X,\ x \neq y \Rightarrow \exists\ z \in X,\ \exists\ e \in \mathcal{E} : \rho(x, y, z) \not< e$;

(ρ_{1b}) $\forall e \in \mathcal{E} : \rho(x, y, z) < e \Leftrightarrow x = y$, sau $x = z$, sau $y = z$;

(ρ_2) $\forall x, y, z \in X : \rho(x, y, z) = \rho(x, z, y) = \rho(y, z, x)$;

(ρ_3) $\forall x, y, z, t \in X : \rho(x, y, z) \leqslant \varphi[\rho(x, y, t), \rho(x, t, z), \rho(t, y, z)]$

$\rho(x, y, z)$ se numeşte g-măsura tripletului (x, y, z); cuplul (X, ρ) se numeşte spaţiu g-2-metric.

Observaţia (2.1). Dacă putem defini o g-2-metrică $\rho : X \times X \times X \to \mathfrak{M}$, atunci $\mathcal{E}$ admite minoranţi în $\mathfrak{M}\setminus\mathcal{E}$; în caz contrar, cerinţa (ρ_{1b}) nu ar putea fi îndeplinită. Urmează că, dacă $(\mathfrak{M}, \leqslant)$ are un cel mai mic element m_0, atunci $\mathcal{E} \subseteq M\setminus\{m_0\}$.

Observaţia (2.2). Conceptul de g-2-metrică generalizează noţiunea de 2-metrică a lui S. Gähler. Într-adevăr, dacă $\mathfrak{M} = \mathbf{R}_+$ (mulţimea numerelor reale nenegative), cu ordonarea uzuală, operaţia ternară φ o interpretăm ca adunarea uzuală (adică $\varphi(a, b, c) = a + b + c$, $\forall a, b, c \in \mathbf{R}_+$), iar $\mathcal{E} = \mathbf{R}_+\setminus\{0\}$, atunci orice g-2-metrică pe X este o 2-metrică în sensul lui S. Gàhler.

Exemplu (3.1). (de g-2-metrică). Fie M o mulţime oarecare (care conţine cel puţin 2 elemente) şi $\mathfrak{M}$ o algebră de părţi ale lui M (care nu se reduce la $\{\Phi, M\}$. Definim ordonarea parţială „$\leqslant$" $\subseteq \mathfrak{M} \times \mathfrak{M}$ prin $A \leqslant B \underset{\text{def}}{\Leftrightarrow} A \subseteq B$ şi operaţia $\varphi : \mathfrak{M} \times \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$ prin $\varphi(A, B, C) = A \cup B \cup C$, pentru A, B, $C \in \mathfrak{M}$; considerăm $E = \mathfrak{M}\setminus\}\Phi\}$. Atunci φ verifică condiţiile (φ_1) şi (φ_2), iar aplicaţia $\rho : \mathfrak{M} \times \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$, definită prin

$$\rho(A, B, C) = \begin{cases} A \cup B \cup C\setminus A \cap B \cap C & \text{dacă } A \neq B \text{ şi } A \neq C \text{ şi } B \neq C \\ \Phi & \text{în caz contrar} \end{cases}$$

este o g-2-metrică pe $\mathfrak{M}$, în sensul definiţiei (2.1)

3. K. Menger nu a fost preocupat de introducerea unei topologii cu ajutorul unei n-metrici. S. Gähler defineşte topologia într-un spaţiu 2-metric (pe care o numeşte topologia naturală) de aşa manieră încît, în $\mathbf{R}^n$ ($n \geqslant 2$) metrica euclidiană şi 2-metrica euclidiană să genereze aceeaşi topologie, unde 2-metrica euclidiană $\sigma : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}_+$ se defineşte prin

$$\sigma(a, b, c) = \frac{1}{2}\left[\sum_{i<j} \begin{vmatrix} \alpha_i & \alpha_j & 1 \\ \beta_i & \beta_j & 1 \\ \gamma_i & \gamma_j & 1 \end{vmatrix}^2\right]^{\frac{1}{2}}, \quad \alpha_i,\ \beta_i,\ \gamma_i (i = \overline{1, n})$$

fiind respectiv coordonatele punctelor $a, b, c \in \mathbf{R}^n$.

(X, ρ) fiind un spaţiu g-2-metric, ne propunem în cele ce urmează să definim o topologie pe X, intim legată de g-2-metrica ρ şi care, în cazul particular cînd ρ este o 2-metrică, să coincidă cu topologia naturală definită de S. Gähler.

Facem notaţia

$$V_e(x, y) = \{z \in X \mid \rho(x, y, z) < e\}, \text{ unde } x, y \in X \text{ şi } e \in \mathcal{E}.$$

DEFINIŢIA (3.1.). Topologia care admite ca subbază familia

$$\mathcal{S} = \{V_e(x, y) \mid x, y \in X, e \in \mathcal{E}\}$$

se numeşte topologia generată de g-2-metrica ρ şi se notează cu $\mathcal{T}_\rho$.

Din definiţia (3.1) urmează că familia $\mathcal{B}$, a mulţimilor de forma $\bigcap_{i=1}^{n} V_{e_i}(x_i, y_i)$ (intersecţii a unui număr finit de elemente din $\mathcal{S}$), este o bază a topologiei $\mathcal{T}_\rho$. Prin urmare, familia tuturor mulţimilor de această formă, care conţine un punct $x \in X$, formează o bază de vecinătăţi ale punctului x. Folosind proprietăţile unei g-2-metrici, şi făcînd anumite ipoteze suplimentare asupra mulţimii $\mathcal{E}$, putem ameliora acest rezultat indicînd baze de vecinătăţi mult mai convenabile.

Introducem în prealabil următoarele notaţii. Pentru o mulţime oarecare A, notăm cu $\mathfrak{L}(A)$ familia tuturor părţilor lui A şi cu $\mathfrak{L}_0(A)$, familia părţilor finite ale lui A.

Presupunem că mulţimea E satisface următoarea condiţie:

$$(E_1) e_0 \in \mathcal{E},\ a \in \mathcal{M},\ a < e_0 \Rightarrow \exists\ e_1 \in \mathcal{E} \exists\ e_2 \in \mathcal{E} : \varphi(a, e_1, e_2) \leqslant e_0$$

Observaţii (3.1). 1. Dacă $\mathcal{M} = \mathbf{R}_+$, $\mathcal{E} = \mathbf{R}_+ \setminus \{0\}$ şi φ adunarea numerelor reale (deci ne găsim în cazul unei 2-metrici), ipoteza (E_1) este satisfăcută.

2. Dacă $\mathcal{M}$, $\mathcal{E}$ şi φ sînt cele care intervin în exemplul (2.1), condiţia (E_1) este îndeplinită. Într-adevăr, dacă $A \in \mathcal{M}$, $E_0 \in \mathcal{E}$ şi $A \subset E_0$ (incluziunea strictă), punînd $E_1 = E_2 = E_0$, avem evident $\varphi(A, E_1, E_2) = A \cup E_1 \cup E_2 = A \cup E_0 = E_0$.

TEOREMA (3.1.) *Fie* (X, ρ) *un spaţiu g-2-metric şi x un punct oarecare din X. Familia de mulţimi*

$$\mathcal{V}_x^0 = \left\{W_\Sigma(x) = \bigcap_{(y,e) \in \Sigma} V_e(x, y) \mid \Sigma \in \mathfrak{L}_0(X \times \mathcal{E})\right\}$$

este o bază de vecinătăţi ale punctului x.

Să observăm, în primul rînd că $x \in W_\Sigma(x)$ (pentru orice $\Sigma \in \mathfrak{L}_0(X \times \mathcal{E})$). Într-adevăr, avem $\rho(x, y, x) < e$ pentru orice $e \in \mathcal{E}$ şi orice $y \in X$ (în baza axiomei (ρ_{1b})), deci $x \in V_e(x, y)$, pentru orice $(y, e) \in \Sigma$, adică $x \in W_\Sigma(x)$. Cum $W_\Sigma(x) \in \mathcal{B}$ urmează că, pentru orice $\Sigma \in \mathfrak{L}_0(X \times \mathcal{E})$, mulţimea $W_\Sigma(x)$, este un element al bazei $\mathcal{B}$, care conţine pe x, deci o vecinătate a lui x.

Demonstraţie. Cum am menţionat mai sus, elementele bazei $\mathcal{B}$, care conţin punctul x, formează o bază de vecinătăţi în x; să o notăm cu $\mathcal{V}_x^*$. Va fi suficient să demonstrăm că:

$$\forall V^* \in \mathcal{V}_x^*\ \exists V^0 \in \mathcal{V}_x^0 : V^0 \subseteq V^*.$$

Fie acum $V^* = \bigcap_{i=1}^{n} V_{e_i}(x_i, y_i) \in \mathcal{V}_x^*$, deci $x \in V^* \in \mathcal{B}$.

Demonstrăm că există $\Sigma \in \mathfrak{L}_0(X \times \mathcal{E})$, astfel că $x \in W_\Sigma(x) = V^0 \subseteq \subseteq V^*$. Avem:

$$x \in \bigcap_{i=1}^{n} V_e(x_i, y_i) \Rightarrow \forall i \in \{1, 2, \ldots, n)\} : x \in V_{e_i}(x_i, y_i) \Rightarrow$$

$$\Rightarrow \forall i \in \{1, 2, \ldots, n\} : \rho(x_i, y_i, x) < e_i.$$

Notăm $\rho(x_i, y_i, x) = f_i \in \mathfrak{M}$, $i = \overline{1, n}$. Cu această notaţie avem $f_i \in \mathfrak{M}$, $e_i \in \mathcal{E}$ şi $f_i < e_i$; în baza ipotezei (E_1), urmează că există e_i' şi e_i'' în mulţimea $\mathcal{E}$, astfel ca $\varphi(f_i, e_i', e_i'') \leqslant e_i$, pentru fiecare $i \in \{1, 2, \ldots, n\}$. Considerăm acum

$$\Sigma_i = \{(e_i', x_i), (e_i'', y_i)\} \text{ şi } W_{\Sigma_i}(x) = V_{e_i'}(x, x_i) \cap V_{e_i''}(x, y_i).$$

Vom demonstra că: $W_{\Sigma_i}(x) \subseteq V_{e_i}(x_i, y_i)$. Pentru aceasta, fie $x^* \in W_{\Sigma_i}(x)$. Urmează că $\rho(x, x_i, x^*) < e_i'$ şi $\rho(x, y_i, x^*) < e_i''$. Conform axiomei (ρ_3) avem însă

$\rho(x_i, y_i, x^*) \leqslant \varphi[\rho(x_i, y_i, x), \rho(x_i, x, x^*), \rho(x, y_i, x^*)] < \varphi(f_i, e_i', e_i'') \leqslant e_i$; din $\rho(x_i, y_i, x^*)' < e_i$, deducem că $x^* \in V_{e_i}(x_i, y_i)$ şi cum x^* a fost ales arbitrar în $W_{\Sigma_i}(x)$, urmează că $W_{\Sigma_i}(x) \subseteq V_{e_i}(x_i, y_i)$.

Consideraţiile făcute rămîn valabile pentru fiecare $i \in \{1, 2, \ldots, n\}$. Să considerăm acum mulţimea:

$\Sigma = \}(x_1, e_1'), (y_1, e_1''), (x_2, e_2'), (y_2, e_2''), \ldots, (x_n, e_n'), (y_n, e_n'')\} \in \mathfrak{L}_0(X \times \mathcal{E})$. Cu notaţiile adoptate $W_\Sigma(x) = \bigcap_{i=1}^{n} W_{\Sigma_i}(x)$.

Avem aşa dar:

$x \in W_\Sigma(x) = \bigcap_{i=1}^{n} W_{\Sigma_i}(x) \subseteq \bigcap_{i=1}^{n} V_{e_i}(x_i, y_i)$. Cu aceasta teorema este demonstrată.

Impunînd noi condiţii mulţimii $\mathcal{E}$, vom putea indica baze de vecinătăţi şi mai avantajoase.

O asemenea condiţie este ipoteza $(E \aleph_0)$.

$(E \aleph_0)$ Există o mulţime $\mathcal{E}_0 \subseteq \mathcal{E}$ cu proprietăţile:

(i) $$\text{card } \mathcal{E}_0 \leqslant \aleph_0$$

(ii) $$\forall e \in \mathcal{E}\ \exists e_0 \in \mathcal{E}_0 : e_0 \leqslant e.$$

TEOREMA (3.2). *Presupunem că $\mathcal{E}$ satisface ipotezele (E_1) şi $(E \aleph_0)$. Dacă $\rho : X \times X \times X \to \mathfrak{M}$ este o g-2-metrică şi $\mathcal{E}_0 \subseteq \mathcal{E}$ este mulţimea furnizată de condiţia $(E \aleph_0)$, atunci familia*
$\mathcal{V}_x^1 = \{W_\Sigma(x) \mid \Sigma \in \mathfrak{L}_0(X \times \mathcal{E}_0)\}$ este o bază de vecinătăţi pentru punctul $x \in X$.

Demonstraţie. Evident $\mathcal{V}_x^1 \subseteq \mathcal{V}_x^0 = \{W_\Sigma(x) \mid \Sigma \in \mathfrak{L}_0(X \times \mathcal{E})\}$. Vom demonstra că oricare ar fi $\Sigma \in \mathfrak{L}_0(X \times \mathcal{E})$, există $\Sigma_0 \in \mathfrak{L}_0(X \times \mathcal{E}_0)$, astfel ca $W_{\Sigma_0}(x) \subseteq W_\Sigma(x)$.

Fie deci $W_\Sigma(x) \in \mathcal{V}_x^0$, cu $\Sigma \in \mathfrak{L}_0(X \times \mathcal{E})$. Pentru comoditate scriem $\Sigma = \{(y_i, e_i) \mid i = \overline{1, n}\}$, cu observaţia că elementele $y_i \in X$ şi elementele $e_i \in \mathcal{E}$, nu sînt neapărat distincte.

În baza ipotezei $(E\,\aleph_0)$ avem:

$\forall i \in \{1, 2, \ldots, n\}\, \exists e_i^0 \in \mathcal{E}_0 : e_i^0 \leqslant e_i$; Considerăm

$\Sigma_0 = \{(y_i, e_i^0) \mid i = \overline{1, n}\} \in \mathfrak{L}_0(X \times \mathcal{E}_0)$ şi arătăm că

$W_{\Sigma_0}(x) \subseteq W_\Sigma(x)$. Într-adevăr avem:

$$z \in W_{\Sigma_0}(x) = \bigcap_{i=1}^{n} V_{e_i^0}(x, y_i) \Rightarrow \forall i \in \{1, 2, \ldots, n\} : z \in V_{e_i^0}(x, y_i) \Rightarrow$$

$$\Rightarrow \forall i \in \{1, \ldots, n\} : \rho(x, y_i, z) < e_i^0 \leqslant e_i \Rightarrow \forall i \in \{1, \ldots, n\} :$$

$$z \in V_{e_i}(x, y_i) \Rightarrow z \in \bigcap_{i=1}^{n} V_{e_i}(x, y_i) = W_\Sigma(x).$$

Cu aceasta teorema este demonstrată.

Să introducem acum o nouă ipoteză asupra mulţimii $\mathcal{E}$, ipoteza (E_2).

(E_2) $\qquad e_1 \in \mathcal{E},\ e_2 \in \mathcal{E} \Rightarrow \exists e \in \mathcal{E} : e \leqslant e_1,\ e \leqslant e_2.$

Observaţie (3.2). Dacă $(\mathcal{E}, \leqslant)$ satisface condiţia (E_2), atunci orice parte finită a ei admite minoranţi în $\mathcal{E}$.

LEMA (3.1). *Dacă* $(\mathcal{E}, \leqslant)$ *satisface condiţiile* $(E\,\aleph_0)$ *şi* (E_2), *există un şir de elemente din* $\mathcal{E}$, $(e_n)_{n \in N}$, *cu proprietăţile:*

(i) $\qquad \forall n \in \mathrm{N} : e_{n+1} \leqslant e_n$;

(ii) $\qquad \forall e \in \mathcal{E}\ \exists n \in \mathrm{N} : e_n \leqslant e.$

Demonstraţie. Fie $\mathcal{E}_0 = \{f_n \mid n \in N\} \subseteq \mathcal{E}$, furnizată de condiţia $(E\,\aleph_0)$. Punem $e_1 = f_1$. Conform condiţiei (E_2), pentru e_1 şi f_2 există $e^1 \in \mathcal{E}$, astfel ca $e^1 \leqslant e_1$ şi $e^1 \leqslant f_2$; în baza axiomei $(E\,\aleph_0)$, există $f_{k_2} \in \mathcal{E}_0$, astfel încît $f_{k_2} \leqslant e^1 \leqslant f_2, e_1$.

Notăm $e_2 = f_{k_2}$. Avem $e_2 \leqslant e^1$ şi $e_2 \leqslant f_2$. Pentru elementele e_2 şi f_3, există, în baza ipotezei (E_2) un element $e^2 \in \mathcal{E}$, cu proprietatea $e^2 \leqslant e_2$ şi $e^2 \leqslant f_3$; ipoteza $(E\,\aleph_0)$ ne permite să considerăm $f_{k_3} \in \mathcal{E}_0$, astfel ca $f_{k_3} \leqslant e^2$; notăm $e_3 = f_{k_3}$; avem evident, $e_3 \leqslant e_2$, $e_3 \leqslant f_3$. Continuăm în acest mod. Considerînd că am obţinut $e_1 \geqslant e_2 \geqslant e_3 \geqslant \ldots \geqslant e_p$, pentru perechea e_p, f_{p+1} alegem $e^p \in \mathcal{E}$, cu $e^p \leqslant e_p$, $e^p \leqslant f_{p+1}$; în sfîrşit considerăm $f_{k_{p+1}} \in \mathcal{E}_0$ astfel că $f_{k_{p+1}} \leqslant e^p$ şi punem $e_{p+1} = f_{k_{p+1}}$. Avem $e_{p+1} \leqslant e_p$ şi $e_{p+1} \leqslant f_{p+1}$. Am arătat că din mulţimea $\mathcal{E}_0$ putem extrage un şir de elemente, necrescător.

Rămîne să arătăm că, pentru şirul $(e_n)_{n \in N}$, este satisfăcută şi cerinţa (ii) din enunţ. Pentru aceasta fie $e \in \mathcal{E}$; există conform condiţiei $(E\,\aleph_0)$, un element $f_k \in \mathcal{E}_0$ cu $f_k \leqslant e$. Avem însă $e_k \leqslant f_k \leqslant e$; prin urmare, $\forall e \in E\ \exists k \in N : e_k \leqslant e$, ceea ce demonstrează complet lema.

TEOREMA (3.3). *Fie* $\rho : X \times X \times X \to \mathfrak{M}$ *o g-2-metrică. Presupunem că mulţimea* $\mathcal{E} \subseteq \mathfrak{M}$ *(care intervine în formularea axiomelor g-2-metricii) se bucură de proprietăţile* (E_1), (E_2) *şi* $(E\,\aleph_0)$. *În aceste condiţii, există o mulţime numărabilă* $\mathcal{E}_0 \subseteq \mathcal{E}$, *astfel ca pentru* $x \in X$, *familia de mulţimi*:

$$\mathcal{V}_x^* = \{W_\Sigma(x) \mid \Sigma = M \times \{e\},\ M \in \mathfrak{L}_0(X),\ e \in \mathcal{E}_0\}$$

formează o bază de vecinătăţi pentru x, *în spaţiul* $(X, \mathcal{T}_\rho)$.

Demonstraţie. În baza lemei, există o submulţime $\mathcal{E}_0$ a lui $\mathcal{E}$, numărabilă total ordonată, relativ la ordonarea din $\mathfrak{M}$, cu proprietatea că $\forall e \in \mathcal{E}\ \exists e' \in \mathcal{E}_0 : e' \leqslant e$. Conform teoremei (3.2) familia de mulţimi $\{W_\Sigma(x) \mid E \in \mathfrak{L}_0(X \times \mathcal{E}_0)\}$ este o bază de vecinătăţi pentru x. Să observăm acum că, $\mathcal{E}_0$ fiind total ordonată, orice parte finită a ei admite un cel mai mic element. Pentru un $\Sigma_0 \in \mathfrak{L}_0(X \times \mathcal{E}_0)$ dat, să notăm cu $e_0 = \min\ \{e \in \mathcal{E}_0 \mid (y, e) \in \Sigma_0\}$.

Avem atunci $W_{\Sigma_0}(x) = \bigcap\limits_{(y,e) \in \Sigma_0} V_e(x, y) = \bigcap\limits_{y \in M} V_{e_0}(x, y)$ unde $M = \mathrm{pr}_1\ \Sigma_0 \in \mathfrak{L}_0(X)$. Prin urmare, pentru un $\Sigma_0 \in \mathfrak{L}_0(X \times \mathcal{E}_0)$, dat există un Σ' de forma $\Sigma' = M \times \{e\}$, unde $M \in \mathfrak{L}_0(x)$ iar $e \in \mathcal{E}_0$, astfel ca $W_{\Sigma_0}(x) = W_{\Sigma'}(x)$. Cu aceasta teorema este demonstrată.

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GENERALIZED 2-METRIC SPACES

(Summary)

The paper extends the concept of 2-metric introduced by S. Gähler [2]. Some properties of the topology induced by the generalized 2-metric are also discussed.

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A NOTE ON FIXED POINT THEOREM OF MAIA

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1. M.G. Maia [12] has proved the following theorem: *Let X be a nonempty set endowed in two metrics ρ, d, and let f be a mapping of X into itself. Suppose that (i) $\rho(x, y) \leqslant d(x, y)$ for all x, y in X, (ii) (X, ρ) is a complete space, (iii) f is continuous with respect to ρ, and (iv) $d(fx, fy) \leqslant k \cdot d(x, y)$ for all x, y in X, where $0 \leqslant k < 1$. Then, f has a unique fixed point.*

This result generalizes the Banach fixed point principle and is connected with Bielecki's method [2] (see also [3, ch. III], [13, ch. I]) of changing the norm in the theory of differential equations.

Many papers related to the Maia's result have been published, see: [21], [14], [6], [16], [15] and [1]. In particular, I.A. Rus [16] observed that the above theorem will remain true if (i) is replaced by the following condition: $\rho(fx, fy) \leqslant c \cdot d(x, y)$ for all x, y in X, where c is a positive constant. We present some fixed point results of Maia type with the Rus conditions, and give examples of their applications to differential-like equations.

Assumptions and notations given below, are valid throughout this paper. Suppose that:

(a) $(E, ||\cdot||)$ is a Banach space with a normal cone S(see e.g. [8], [9]), and $\preccurlyeq$ denote the partial order in E generated by S;

(b) X is a nonempty set, $\rho_E: X \times X \to S$ and $d_E: X \times X \to S$ are two generalized metrics in X (see [8, ch. II 6.3]),
$\rho_E^+(x, y) = ||x - y||$ for x, y in X, and (X, ρ_E^+) is a complete metric space;

c) L is a bounded linear operator of S into itself with the spectral radius less than one;

(d) $(I - L)^{-1}$ maps S into itself, where I is a identity operator and $(I - L)^{-1}$ denote an inverse of $I - L$ (the facts that $I - L$ is inverticable and $(I - L)^{-1}$ is bounded linear operator, are a consequence of Banach theorem [7, ch, V 2]).

2. Let f be a continuous mapping of (X, ρ_E^+) into itself such that $d_E(fx, fy) \preccurlyeq Ld_E(x, y)$ and $\rho_E(fx, fy) \preccurlyeq Cd_E(x, y)$ for all x, y in X, where C is a bounded linear operator acting in S.

Further, let $x_0 \in X$ and let us put $x_i = fx_{i-1}$ for $i = 1, 2, \dots$.

Modifying the reasoning from [1] and [8, th. 6.2, ch. II], we obtain that f has a unique fixed point u^* in X, $\rho_E^+(x_i, u^*) \to 0$ as $i \to \infty$, and $d_E(x_i, u^*) \preccurlyeq L^i d_E(x_0, u^*)$ for $i \geqslant 1$. Since $d_E(x_0, u^*) \preccurlyeq d_E(x_0, x_1) + Ld_E(x_0, u^*)$ we have from (d) that $d_E(x_0, u^*) \preccurlyeq (I - L)^{-1} d_E(x_0, x_1)$. Hence $d_E(x_i, u^*) \preccurlyeq$
$\preccurlyeq L^i(I - L)^{-1} d_E(x_0, x_1)$ for each $i \geqslant 1$. In particular, $d_E(fx_0, u^*) \preccurlyeq$
$\preccurlyeq L(I - L)^{-1} d_E(x_0, fx_0)$.

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Using the above remarks, we prove the following coincidence theorem (cf. [5]):

PROPOSITION 1. *Suppose that F and T are two mappings of X into itself such that*

(i) $F[X] \subset T[X]$ *and* $T[X]$ *is a closed set with respect to* ρ^N_+,

(ii) $d_E(Fx, Fy) \preccurlyeq Ld_E(Tx, Ty)$ *for all* x, y *in* X,

(iii) $\rho_E(Fx, Fy) \preccurlyeq Cd_E(Tx, Ty)$ *for all* x, y *in* X, *where* C *is a bounded linear operator of* S *into itself,*

(iv) $\lim_{n\to\infty} ||\rho_E(Fx_n, Fx)|| = 0$ *for every sequence* (x_n) *in* X *with* $\lim_{n\to\infty} ||\rho_E(Tx_n, Tx)|| = 0$.

Then there exists an element u *in* X *such that* $Fu = Tu$ *and, moreover,* $Tu_1 = Tu_2$ *for each* u_1, u_2 *with* $Fu_i = Tu_i (i = 1,2)$.

Proof. Let us put $fx = \{Fy : Ty = x\}$ for x in $T[X]$. If $z_i \in fx (i = 1,2)$, then $z_i = Fy_i$ and $Ty_i = x$. Hence $\theta \preccurlyeq d_E(z_1, z_2) \preccurlyeq Ld_E(Ty_1, Ty_2) = \theta$ (θ denotes the zero a space E), and therefore fx contains only one element.

It is easy to verify that the mapping f of $T[X]$ into itself is continuous with respect to ρ^+_E, $d_E(fx, fy) \preccurlyeq Ld_E(x, y)$ and $\rho_E(fx, fy) \preccurlyeq Cd_E(x, y)$ for x, y in $T(X)$. Consequently, there exists a unique x^* in $T(X)$ such that $fx^* = x^*$. Obviously, $Fu = Tu$ for all u in $\{x \in X : Tx = x^*\}$.

Let us denote by $r(L)$ the spectral radius of L. Suppose that $Fu_i = Tu_i (i = 1,2)$ and $Tu_1 \neq Tu_2$. Then $d_E(Tu_1, Tu_2) \preccurlyeq Ld_E(Tu_1, Tu_2)$, $-d_E(Tu_1, Tu_2] \not\in S$ and, by Stecenko theorem [8, th. 5.4 ch. II], we obtain $r(L) \geqslant 1$. This contradiction completes the proof.

PROPOSITION 2. *Let* $\Phi_i (i = 0, 1, \ldots)$ *be mappings of* X *into itself such that*

(i) Φ_i *are a continuous with respect to* ρ^+_E,

(ii) $d_E(\Phi_i x, \Phi_i y) \preccurlyeq Ld_E(x, y)$ *for all* x, y *in* X,

(iii) $\rho_E(\Phi_i x, \Phi_i y) \preccurlyeq C_i d_E(x, y)$ *for all* x, y *in* X, *where* C_i *is a bounded linear operator of* S *into itself.*

Denote by $u_i (i = 0, 1, \ldots)$ *a unique fixed point of* Φ_i, *and suppose that* $\lim_{n\to\infty} ||d_E(\Phi_n x, \Phi_0 x)|| = 0$ *for every* x *in* X. *Then* $||d_E(u_n, u_0)|| \to 0$ *as* $n \to \infty$.

Proof. From the above remarks $d_E(\Phi_n u_0, u_n) \preccurlyeq L(I - L)^{-1} d_E(u_0, \Phi_n u_0)$ for $n = 1,2, \ldots$. Since S is a normal cone we have

$$||d_E(u_n, u_0)|| \leqslant N||L(I - L)^{-1} d_E(u_0, \Phi_n u_0) + d_E(\Phi_n u_0, u_0)|| \leqslant$$
$$\leqslant M \; ||d_E(\Phi_n u_0, \Phi_0 u_0)||$$

for each $n \geqslant 1$, and therefore $||d_E(u_n, u_0)|| \to 0$ as $n \to \infty$.

Note that our theorem can be obtained (using the above results) also as a coincidence theorem. The details are left to the reader. Finally, let us remark that further results and its applications can be obtained if the concept of a metric space in the *Luxemburg* sense [11] (not every two points have necessarily a finite distance) will be used. See [17] — [20].

3. In this section, let us denote: by I — the closed interval $[0, a]$; by $E = \mathbf{R}^m$ — the m-dimensional Euclidean space; by S — the set of all $(t_1, t_2, \ldots, t_m)$ in $\mathbf{R}^m$ with $t_i \geqslant 0$ for $i = 1, 2, \ldots, m$ (obviously, $(u_1, u_2, \ldots, u_m) \preccurlyeq (v_1, v_2, \ldots, v_m)$ in $\mathbf{R}^m$ means $u_i \leqslant v_i$ for every $i = 1, 2, \ldots, m$); by $X = C(I, \mathbf{R}^m)$ — the set of all continuous function from I to $\mathbf{R}^m$ (in particular, $C(I) = C(I, \mathbf{R}^1)$).

We define the generalized metrics ρ, $\rho_p (p > 0)$ and d as follows:

$$\rho(x, y) = (\sup_{t \in I} |x_1(t) - y_1(t)|, \ldots, \sup_{t \in I} |x_m(t) - y_m(t)|),$$

$$\rho_p(x, y) = (\sup_{t \in I} \exp(-pt)|x_1(t) - y_1(t)|, \ldots\ldots, \sup_{t \in I} \exp(-pt)|x_m(t) - y_m(t)|),$$

$$d(x, y) = \left(\left(\int_0^a |x_1(t) - y_1(t)|^2 dt\right)^{1/2}, \ldots\ldots \left(\int_0^a |x_m(t) - y_m(t)|^2 dt\right)^{1/2}\right)$$

for $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_m)$ in $C(I, \mathbf{R}^m)$. Notice [4] that a non-negative and non-zero matrix $A = [a_{ij}]$ $(1 \leqslant i, j \leqslant m)$ has the spectral radius $r(A)$ less than one if and only if

$$\begin{vmatrix} 1 - a_{11} & -a_{12} \ldots\ldots & -a_{1i} \\ -a_{21} & 1 - a_{22} \ldots\ldots & -a_{2i} \\ \ldots & \ldots & \ldots \\ -a_{i1} & -a_{i2} \ldots\ldots & 1 - a_{ii} \end{vmatrix} > 0$$

for all $i = 1, 2, \ldots, m$. Moreover, there exists a positive constant p_0 such that $r(p \cdot A) < 1$ for every $0 < p \leqslant p_0$.

First, we give some application of Proposition 1 to integral equations system of Volterra type.

Let $\varphi_i \in C(I)$ $(i = 1,2, \ldots, m)$ be a function such that $\varphi_i(0) = 0$ and $\varphi_i(t) < 0$ for $t < 0$, and let K_i $(i = 1, 2, \ldots, m)$ be a mesurable function which is finite almost everywhere on $I \times I \times \mathrm{R}$.

Suppose that:

1° $|K_i(t, s, u_1, \ldots, u_m) - K_i(t, s, v_1, \ldots, v_m)| \leqslant$

$$\leqslant \sum_{j=1}^{m} h_{ij}(t, s)|u_j - v_j| \text{ with function } h_{ij} (j = 1, 2, \ldots, m)$$

measurable and finite almost everywhere on $I \times I$;

2° for every y in $C(I, \mathbf{R}^m)$ the functions

$$t \mapsto \varphi_i(t) \int_0^t K_i(t, s, y(s))ds \text{ are continuous on } I, \text{ and } \lim \int^t K(t, s, y(s))ds < \infty.$$

Moreover, let us put:

$$a_{ij} = \left(\int_0^a\int_0^a \frac{\varphi_i^2(t)h_{ij}(t,s)}{\varphi_j^2(s)}\,dtds\right)^{2/1},$$

$$c_{ij} = \sup_{t\in I} \varphi_i(t)\left(\int_a^0 \frac{h_{ij}^2(t,s)}{\varphi_j^2(s)}\,ds\right)^{1/2}$$

for $1 \leqslant i, j \leqslant m$. Under these conditions we have the following theorem:

Assume that $[a_{ij}]$ $(1 \leqslant i, j \leqslant m)$ *is a matrix with the spectral radius less than* 1, *and* $c_{ij} < \infty$ *for* $1 \leqslant i, j \leqslant m$. *Then there exists exactly one function* $y = (y_1, y_2, \ldots, y_m)$ *in* $C(I, \mathbf{R}^m)$ *such that*

$$y_i(t) = \int_0^t K_i(t, s, y(s))ds$$

$(i = 1, 2, \ldots, m)$ *for* $0 < t \leqslant a$.

Proof. For $z = (z_1, z_2, \ldots, z_m) \in C(I, \mathbf{R}^m)$ we define on I:

$$(T_iz)(t) = \varphi_i(t)z_i(t),$$

$$(F_iz)(t) = \varphi_i(t)\cdot\int_0^t K_i(t, s, z(s))ds$$

and

$$(Tz)(t) = ((T_1z)(t), (T_2z)(t), \ldots, (T_mz)(t)),$$
$$(Fz)(t) = ((F_1z)(t), (F_2z)(t), \ldots, (F_mz)(t)).$$

Denote by L, C linear operators generated by the matrix $[a_{ij}]$ and $[c_{ij}]$, respectively. Let $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m) \in C(I, \mathbf{R}^m)$. Evidently

$$|(F_ix)(t) - (F_iy)(t)| \leqslant \varphi_i(t)\int_t^0 \sum_{j=1}^m h_{ij}(t,s)\,|x_j(s) - y_j(s)|\,ds \leqslant$$

$$\leqslant \varphi_i(t)\sum_{j=1}^m \int_0^a \frac{h_{ij}(t,s)}{\varphi_j(s)}\cdot\varphi_j(s)\,|x_j(s) - y_j(s)|\,ds,$$

hence

$$\left(\int_0^a |(F_ix)(t) - (F_iy)(t)|^2 dt\right)^{1/2} \leqslant$$

$$\leqslant \left(\int_0^a \left(\sum_{j=1}^m \int_0^a \frac{\varphi_i(t)h_{ij}(t,s)}{\varphi_j(s)} \varphi_j(s)\,|x_j(s) - y_j(s)|\,ds\right)^2 dt\right)^{1/2} \leqslant$$

$$\leqslant \sum_{j=1}^m \left(\int_0^a \left(\int_0^a \frac{\varphi_i(t)h_{ij}(t,s)}{\varphi_j(s)} \varphi_j(s)\,|x_j(s) - y_j(s)|\,ds\right)^2 dt\right)^{1/2} \leqslant$$

$$\leqslant \sum_{j=1}^m \left(\left(\int_0^a \left(\int_0^a \frac{\varphi_i^2(t)h_{ij}^2(t,s)}{\varphi_j^2(s)} ds \cdot \int_0^a \varphi_j^2(s)\,|x_j(s) - y_j(s)|^2 ds\right) dt\right)^{1/2}\right) =$$

$$= \sum_{j=1}^m a_{ij}\left(\int_0^a \varphi_j^2(s)\,|x_j(s) - y_j(s)|^2 ds\right)^{1/2}$$

and

$$|(F_ix)(t) - (F_iy)(t)| \leqslant$$

$$\leqslant \varphi(t) \cdot \sum_{j=1}^m \left(\int_0^a \frac{h_{ij}^2(t,s)}{\varphi_j^2(s)} ds\right)^{1/2} \cdot \left(\int_0^a \varphi_j^2(s)\,|\,x_j(s) - y_j(s)|^2 ds\right)^{1/2} \leqslant$$

$$\leqslant \sum_{j=1}^m c_{ij}\left(\int_a^0 \varphi_j^2(s)\,|x_j(s) - y_j(s)|^2 ds\right)^{1/2}$$

for $0 \leqslant t \leqslant a$ and $i = 1, 2, \ldots, m$. Consequently, $d_E(Fx, Fy) \preccurlyeq L d_E(Tx, Ty)$ and $\rho_E(Fx, Fy) \preccurlyeq Cd\,(Tx, Ty)$ for all x, y in $C(I, \mathbf{R}^m)$.

It is easy to verify that conditions (i), (iv) of Proposition 1 are satisfied. Therefore, applying Proposition 1, we can conclude the proof.

Now, suppose that $l_{ij}(1 \leqslant i, j \leqslant m)$ are non-negative constants. Denote by $\mathfrak{F}$ the set of continuous functions $F = (f_1, f_2, \ldots, f_m)$ from $I \times \mathbf{R}^m$ into $\mathbf{R}^m$ such that $|f_i(t, u) - f_i(t, v)| \leqslant \sum_{j=1}^m l_{ij}|u_j - v_j|$ $(i = 1, 2, \ldots, m)$ for every $t \in I$ and $u = (u_1, \ldots, u_m)$, $v = (v_1, \ldots, v_m)$ in $\mathbf{R}^m$.

Let us put

$$\Phi(F\,\eta, x)(t) = \eta + \int_0^t F(s, x(s))ds$$

for $F \in \mathfrak{F}$, $\eta \in \mathbf{R}^m$ and $x \in C(U, \mathbf{R}^m)$. The set $\mathfrak{F}$ will be considered as $\mathfrak{F}^*$ — space (see e.g. [10, ch. II §20]), and $C(I, \mathbf{R}^m)$ with the usual supremum metric. Assume that for every x in $C(I, \mathbf{R}^m)$ the transformation $(F, \eta) \mapsto (F, \eta, x)$ maps continuously $\mathfrak{F}^*$ — product $\mathfrak{F} \times \mathbf{R}^m$ into $C(I, \mathbf{R})$, (F endowed with almost uniform convergence is $\mathfrak{L}^*$ —space satisfying the above condition. If set $\mathfrak{F}$ is uniformly bounded and endowed with pointwise convergence, then using the Lebesgue convergence theorem we obtain that our condition holds.

We shall prove the following theorem:

For an arbitrary $F \in \mathfrak{F}$ and $\eta \in \mathbf{R}^m$ there exists a unique function $x_{(F,\eta)}$ in $C(I, \mathbf{R}^m)$ such that $x(0) = \eta$ and $x'(t) = F(t, x(t))$ for $t \in I$). Moreover, $(F, \eta) \mapsto$ $\mapsto x_{(F,\eta)}$ maps continuously $\mathfrak{F} \times \mathbf{R}^m$ into $C(I, \mathbf{R}^m)$.

Proof. Let $k = 0, 1, \ldots$. Assume that $F_k = (f_1^{(k)}, \ldots, f_m^{(k)}) \in \mathfrak{F}$ and $\eta_{ik} \in \mathbf{R}^m$ are such that $\lim_{n\to\infty} (F_n, \eta_n) = (F_0, \eta_0)$. Using the method of Bielecki, we obtain

$$\exp(-pt)\left|\int_0^t (f_i^{(k)}(s, x(s)) - f_i^{(k)}(s, y(s)))ds\right| \leqslant$$

$$\leqslant p^{p-1} \cdot \sum_{j=1}^{m} l_{ij} exp(-pt)\,|x_j(t) - y_j(t)|$$

for $t \in |$ and $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$ in $C(I, \mathbf{R}^m)$. Hence $\rho_p(\Phi(F_k, \eta_{ik}, x), \Phi(F_k, \eta_k, y)) \leqslant L\rho_p(x, y)$, and $\rho(\Phi(F_k, \eta_k, x), \Phi(F_k, \eta_k, y)) \leqslant c \cdot \rho_p(x, y)$ for each x, y in $C(I, \mathbf{R}^m)$, wherec $\geqslant 0$ is some constant anf L denotes the operator generated by matrix $[p^{-1} \cdot l_{ij}(1 \leqslant i, j \leqslant m)$. Therefore, applying Proposition 2, we obtain our result.

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O NOTĂ DESPRE TEOREMA DE PUNCT FIX A LUI MAIA

(Rezumat)

În lucrare se stabilesc teoreme de punct fix de tip Maia în spaţii metrice generalizate cu metrica luînd valori într-o mulţime ordonată. Rezultatele obţinute se aplică la stabilirea unor teoreme de existenţă şi unicitate pentru sisteme de ecuaţii integrale de forma

$$y_i(t) = \int_0^t K_i(t, s, y, \ldots, y_m)ds, \quad i = \overline{1, m}, \quad 0 < t \leqslant a$$

şi pentru problema lui Cauchy

$$x(t) = F(t, x(t)), \quad t \in I,$$
$$x(0) = \eta.$$

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REZOLVABILITATEA PROBLEMEI LUI DIRICHLET PENTRU O ECUAŢIE ELIPTICĂ CU SINGULARITĂŢI

P. SZILÁGYI

1. **Introducere.** În această lucrare se studiază rezolvabilitatea problemei lui Dirichlet pentru ecuaţia eliptică

$$Lw = \frac{\partial^2 w}{\partial \bar{z} \partial \bar{z}_1} + \frac{\alpha}{\bar{z}} \frac{\partial w}{\partial \bar{z}_1} + \frac{\beta}{\bar{z}_1} \frac{\partial w}{\partial \bar{z}} + \frac{\alpha\beta}{\bar{z}\bar{z}_1} w = f \tag{1}$$

în domenii $\Omega \subset \mathbf{R}^2$ care conţin punctul $0(0,0)$. Aici

$$z = x + iy; \; z_1 = x + iky, \; k > 0, \; k \neq 1, \; w = u + iv, \tag{2}$$

$$\frac{\partial}{\partial \bar{z}} = \frac{k}{k-1} \frac{\partial}{\partial x} - i \frac{1}{k-1} \frac{\partial}{\partial y}, \; \frac{\partial}{\partial \bar{z}_1} = - \frac{1}{k-1} \frac{\partial}{\partial x} + i \frac{1}{k-1} \frac{\partial}{\partial y}. \tag{3}$$

Deoarece 0 este un punct singular al ecuaţiei (1), soluţiile se vor căuta în $C^2(\bar{\Omega})\setminus 0$. Polinomul caracteristic al ecuaţiei (1), $P(\lambda, 1) = \frac{1}{(k-1)^2} [-k^2\lambda^2 + 1 + i(1+k)\lambda]$ se anulează în două puncte din semiplanul complex $\operatorname{Im} \lambda > 0$, de aceea nu putem aştepta ca problema lui Dirichlet pentru ecuaţia (1) să fie de tip Fredholm.

2. **Rezolvarea problemei omogene.** Considerăm problema omogenă

$$Lw = 0 \text{ în } \Omega\setminus 0, \; w|_{\partial\Omega} = 0. \tag{4}$$

Se vede uşor că Lw poate fi descompus în felul următor

$$Lw = \left(\frac{\partial}{\partial \bar{z}} + \frac{\alpha}{\bar{z}}\right)\left(\frac{\partial}{\partial \bar{z}_1} + \frac{\beta}{\bar{z}_1}\right) w.$$

Folosind această descompunere, ecuaţia $Lw = 0$ poate fi rezolvată astfel: Notînd

$$W = \frac{\partial w}{\partial \bar{z}_1} + \frac{\beta}{\bar{z}_1} w$$

ecuaţia omogenă $Lw = 0$ este echivalentă cu sistemul linear

$$\frac{\partial w}{\partial \bar{z}_1} + \frac{\beta}{\bar{z}_1} w = W, \quad \frac{\partial W}{\partial \bar{z}} + \frac{\alpha}{\bar{z}} W = 0,$$

care se rezolvă prin metode elementare şi se obţine că mulţimea soluţiilor ecuaţiei omogene este familia

$$w(x, y) = \bar{z}_1^{-\beta}\varphi(\bar{z}) + \bar{z}^{-\alpha}\psi(\bar{z}_1), \tag{5}$$

unde φ şi ψ sînt funcţii olomorfe arbitrare.

Fie în continuare Ω discul unitate, adică

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}.$$

Funcţia w din (5) satisface condiţia la limită $w|_{\partial\Omega} = 0$, atunci şi numai atunci dacă

$$\bar{z}_1^{-\beta}\varphi(\bar{z}) + \bar{z}^{-\alpha}\psi(\bar{z}_1) = 0 \quad \forall z \in \partial\Omega, \tag{6}$$

sau

$$\bar{z}^{\alpha}\varphi(\bar{z}) = -\bar{z}_1^{\beta}\psi(\bar{z}_1) \qquad \forall z \in \partial\Omega. \tag{6'}$$

Fie în continuare α şi β numere întregi pozitive. Atunci funcţiile $\bar{z} \to \bar{z}^{\alpha}\varphi(\bar{z})$ şi $\bar{z}_1 \to \bar{z}_1^{\beta}\psi(z_1)$ sînt olomorfe în raport cu $\bar{z}$ resp. $\bar{z}_1$ în Ω; însă pentru ca $\bar{z}^{\alpha}\varphi(\bar{z})$ să fie valoarea unei funcţii olomorfe pe frontiera $\partial\Omega$ după $\bar{z}_1$ este necesar şi suficient ca

$$\int_{|z|=1} \bar{z}^{\alpha}\varphi(\bar{z})\ \bar{z}_1^{m}d\bar{z}_1 = 0 \quad m = 0,\ 1,\ 2,\ \ldots,\left(\varphi(\bar{z}) = \sum_{j=1}^{\infty} a_j\bar{z}^j\right) \tag{7}$$

sau

$$\int_{|z|=1} z^{\alpha}\bar{\varphi}(z)\bar{z}_1^{m}d\bar{z}_1 = 0 \quad m = 0,\ 1,\ 2,\ \ldots,\left(\bar{\varphi}(z) = \sum_{j=1}^{\infty} \bar{a}_j z^j\right). \tag{7'}$$

Pe cercul $|z| = 1$ avem

$$\bar{z}_1 = \frac{1-k}{2}z + \frac{k+1}{2}\frac{1}{z} \quad \text{şi} \quad d\bar{z}_1 = \frac{1-k}{2}dz - \frac{k+1}{2}\frac{1}{z^2}dz.$$

Cazul $\alpha = 1$. Condiţia (7') pentru $m = 0$ dă

$$\int_{|z|=1} z\ \bar{\varphi}(z)\left[\frac{1-k}{2}dz - \frac{k+1}{2}\frac{1}{z^2}dz\right] = 0$$

sau

$$\int_{|z|=1} z^{-1}[(1-k)z^2 - (k+1)]\sum_{j=0}^{\infty}\bar{a}_j z^j\, dz = 0$$

de unde pe baza teoremei reziduurilor obţinem $a_0 = 0$. În mod analog pentru $m = 1$ se găseşte $a_1 = 0$; $m = 2$, $a_2 = 0$ ş.a.m.d., prin urmare $\varphi = 0$.

Cazul $\alpha > 1$. Condiţia (7') cu $m = 0, 1, 2, \ldots, \alpha - 1$ se îndeplineşte pentru orice funcţie olomorfă, pentru $m = \alpha$, $\alpha + 1, \ldots$ obţinem succesiv $a_0 = 0$, $a_1 = 0$, $a_2 = 0, \ldots$, adică şi în acest caz $\varphi = 0$. Condiţia (6) şi $\varphi = 0$ atrage după sine şi $\psi = 0$, astfel am ajuns la următoarea teoremă.

TEOREMA 1. *Dacă Ω este discul unitate, atunci problema omogenă la limită (4) are numai soluţia $w = 0$.*

Studiem mai departe problema (4) dacă Ω este elipsa $\Omega = \{(x, y) \in \in \mathbf{R}^2 \mid x^2 + ky^2 < 1\}$. Ca şi în cazul cercului condiţia la limită $w|_{\partial\Omega} = 0$ este satisfăcută atunci şi numai atunci dacă pentru funcţiile olomorfe φ şi ψ au loc egalităţile (6) şi (6′). Condiţia (6′) este echivalentă cu condiţia

$$\int_{\partial\Omega} \bar{z}^{\alpha}\varphi(\bar{z})z_1^m dz_1 = 0 \quad m = 0, 1, 2, \ldots \tag{8}$$

În continuare determinăm funcţiile olomorfe φ pentru care condiţia (8) este satisfăcută. În acest scop introducem variabilia ζ prin

$$z = \frac{\sqrt{k}+1}{2}\zeta + \frac{\sqrt{k}-1}{2}\frac{1}{\zeta}, \tag{9}$$

care transformă elipsa $x^2 + ky^2 - 1 < 0$ în discul $|\zeta| < 1$. Pentru $|\zeta| = 1$ avem $\bar{\zeta} = \frac{1}{\zeta}$, prin urmare

$$z_1 = \frac{1-\sqrt{k}}{2}\frac{1}{\zeta} + \frac{1+\sqrt{k}}{2}\zeta \text{ şi } dz_1 = \left[\frac{1+\sqrt{k}}{2} - \frac{1-\sqrt{k}}{2}\frac{1}{\zeta^2}\right]d\zeta$$

Fie $\varphi_1(\bar{z}) = \bar{z}^{\alpha}\ \varphi(\bar{z})$ şi Φ_1 imaginea lui φ_1 după efectuarea transformării (9) Avem

$$\Phi_1(\zeta) = \sum_{n=-\infty}^{\infty} a_n\zeta^n \tag{10}$$

Condiţia (8) înseamnă că

$$\int_{|\zeta|=1} \sum_{n=-\infty}^{\infty} a_n\zeta^n\left[\frac{1-\sqrt{k}}{2}\frac{1}{\zeta} + \frac{1+\sqrt{k}}{2}\zeta\right]^m\left[-\frac{1-\sqrt{k}}{2}\frac{1}{\zeta^2} + \frac{1+\sqrt{k}}{2}\right]d\zeta = 0 \quad m = 0, 1, \ldots \tag{11}$$

Folosind teorema reziduurilor obţinem egalităţile

$$a_{-n} = \left(\frac{1-\sqrt{k}}{1+\sqrt{k}}\right)^n a_n \qquad n = 1, 2, \ldots \tag{12}$$

Ca şi în lucrarea [2] putem arăta că

$$a_{-n} = (-1)^n \rho_0^{2n} a_n \text{ dacă } k < 1 \text{ şi } a_{-n} = \rho_0^{2n} a_n \text{ dacă } k > 1, \tag{13}$$

unde $\rho_0^2 = \frac{|1-\sqrt{k}|}{1+\sqrt{k}}$. Egalităţile (12) şi (13) ne arată că $a_{2l+1} = 0$ $l = 0, 1, 2, \ldots,$ $a_{2l} =$ arbitrar, astfel avem o infinitate de posibilităţi de a alege pe φ şi ψ.

TEOREMA 2. *In cazul elipsei* $\Omega = \{(x, y) \in \mathbf{R}^2 \mid x^2 + ky^2 - 1 < 0\}$ *problema omogenă (4) are o infinitate de soluții linear independente.*

Observație. Teorema 2 rămîne valabilă şi pentru elipsele $x^2 + ky^2 - a^2 < 0$, însă în orice altă elipsă de forma $b^2x^2 + a^2y^2 - a^2b^2 < 0$ cu $\frac{a^2}{b^2} \neq k$ problema omogenă Dirichlet are numai soluția nulă.

3. **Rezolvarea problemei neomogene.** Considerăm problema neomogenă

$$Lw = f \text{ în } \Omega\setminus 0, \quad w|_{\partial\Omega} = 0, \tag{14}$$

unde $f \in C(\overline{\Omega})$ este o funcție dată.

Prima dată vom determina „soluția generală" a ecuației neomogene. Pentru aceasta folosim „metoda variației funcțiilor olomorfe", adică vom căuta o soluție a ecuației neomogene sub forma soluției generale a ecuației omogene, însă în locul funcțiilor olomorfe punem două funcții nedeterminate f_1 şi f_2. Fie deci

$$w(x, y) = \bar{z}_1^{-\beta} f_1(x, y) + \bar{z}^{-\alpha} f_2(x, y) \tag{15}$$

Avem

$$\frac{\partial w}{\partial \bar{z}} = \bar{z}_1^{-\beta} \frac{\partial f_1}{\partial \bar{z}} - \alpha\bar{z}^{-\alpha-1} f_2 + \bar{z}^{-\alpha} \frac{\partial f_2}{\partial \bar{z}}, \quad \frac{\partial w}{\partial \bar{z}_1} = -\beta\bar{z}_1^{-\beta-1} f_1 + \bar{z}_1^{-\beta} \frac{\partial f_1}{\partial \bar{z}_1} + \bar{z}^{-\alpha} \frac{\partial f_2}{\partial \bar{z}}.$$

Impunem condiția

$$\bar{z}_1^{-\beta} \frac{\partial f_1}{\partial \bar{z}} + \bar{z}^{-\alpha} \frac{\partial f_2}{\partial \bar{z}} = 0 \tag{16}$$

atunci

$$\frac{\partial^2 w}{\partial \bar{z}_1 \partial \bar{z}} = -\alpha\bar{z}^{-\alpha-1} \frac{\partial f_2}{\partial \bar{z}_1},$$

iar ecuația $Lw = f$ ne dă

$$\alpha\bar{z}_1^{-\beta}\bar{z}^{-1} \frac{\partial f_1}{\partial \bar{z}_1} = f. \tag{17}$$

În continuare trebuie să găsim o soluție (f_1, f_2) a sistemului (16)—(17). Pentru aceasta demonstrăm următoarea lemă:

LEMA 1. *Dacă* Ω *este un domeniu plan simplu conex şi* $u \in C^1(\overline{\Omega})$, *atunci au loc următoarele două reprezentări integrale*

$$u(\zeta) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{u(x, y)}{\bar{z} - \zeta} d\bar{z} + \frac{1-k}{2\pi} \iint_{\Omega} \frac{\frac{\partial u}{\partial \bar{z}_1}(x, y)}{\bar{z} - \xi} dx\, dy, \tag{18}$$

$$u(\zeta) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{u(x, y)}{\bar{z}_1 - \zeta} d\bar{z}_1 + \frac{k-1}{2\pi} \iint_{\Omega} \frac{\frac{\partial u}{\partial \bar{z}}(x, y)}{\bar{z}_1 - \zeta} dx\, dy. \tag{19}$$

Demonstrație. Dacă $v \in C^1(\overline{\Omega})$, atunci pe baza formulei lui Green

$$\int_{\partial\Omega} v(x, y)\, d\bar{z}_1 = -\, i(k-1) \iint_{\Omega} \frac{\partial v}{\partial \bar{z}}\, dx\, dy \tag{20}$$

și

$$\int_{\partial\Omega} v(x, y)\, d\bar{z} = i(k-1) \iint_{\Omega} \frac{\partial v}{\partial \bar{z}_1}\, dx\, dy. \tag{21}$$

Fie acum $u \in C^1(\overline{\Omega})$, $\zeta \in \Omega$, $0 < \varepsilon < d(\zeta, C\Omega)$, $\Omega_\varepsilon = \{z \in \Omega \mid |\bar{z}_1 - \zeta| > \varepsilon\}$. Astfel funcția $(x, y) \to v(x, y) = \frac{u(x, y)}{\bar{z}_1 - \zeta}$ aparține lui $C(\overline{\Omega}_\varepsilon)$ și pe baza formulei (20)

$$\iint_{\Omega_\varepsilon} \frac{\partial u}{\partial \bar{z}}(x, y)(\bar{z}_1 - \zeta)^{-1} dx\, dy =$$

$$= \frac{i}{k-1} \int_{\partial\Omega_\varepsilon} \frac{u(x, y)}{\bar{z}_1 - \zeta}\, d\bar{z}_1 = \frac{i}{k-1} \int_{\partial\Omega} \frac{u(x, y)}{\bar{z}_1 - \zeta}\, d\bar{z}_1 + \frac{1}{k-1} \int_0^{2\pi} u(\zeta + \varepsilon e^{i\theta})\, d\theta.$$

Trecînd la limită în această egalitate, pentru $\varepsilon \to 0$ obținem reprezentarea (18).

În mod asemănător se obține și (19).

Cu ajutorul reprezentărilor (18)—(19) putem demonstra următoarea teoremă.

TEOREMA 3. *Dacă $f \in C(\overline{\Omega})$, atunci funcțiile*

$$u(\zeta) = \frac{1-k}{2\pi} \iint_{\Omega} \frac{f(x, y)}{\bar{z} - \zeta}\, dx\, dy, \quad v(\zeta) = \frac{k-1}{2\pi} \iint_{\Omega} \frac{f(x, y)}{\bar{z}_1 - \zeta}\, dx\, dy \tag{22}$$

aparțin lui $C^1(\overline{\Omega})$ și au loc egalitățile

$$\frac{\partial u}{\partial \bar{z}_1} = f, \qquad \frac{\partial v}{\partial \bar{z}} = f. \tag{23}$$

Folosind această teoremă putem rezolva sistemul (16)—(17). Din (17) și din teorema 3 rezultă că

$$f_1(x, y) = \frac{1-k}{2\pi\alpha} \iint_{\Omega} \frac{f(\xi, \eta)\overline{\zeta}\,\overline{\zeta}_1^{\beta}}{\overline{\zeta} - z}\, d\xi d\eta \quad (\zeta = \xi + i\eta,\ \zeta_1 = \xi + ik\eta) \tag{24}$$

este o soluţie a ecuaţiei (19) şi

$$\frac{\partial f_1}{\partial \bar z}(x, y) = \frac{k+1}{2\pi\alpha} \iint\limits_{\Omega} \frac{f(\xi, \eta)}{(\bar\zeta - z)^2} \bar\zeta \bar\zeta_1^{\beta} d\xi d\eta,$$

iar de aici şi din (16) obţinem

$$\frac{\partial f_2}{\partial \bar z_1}(x, y) = -\bar z^{\alpha} \bar z_1^{-\beta} \frac{\partial f_1}{\partial \bar z} = -\frac{k+1}{2\pi\alpha} \bar z^{\alpha} \bar z_1^{-\beta} \iint\limits_{\Omega} \frac{f(\xi, \eta)\bar\zeta\bar\zeta_1^{\beta}}{(\bar\zeta - z)^2} d\xi d\eta.$$

De aici şi din teorema 3 avem

$$f_2(x, y) = \frac{1-k}{x\pi} \iint\limits_{\Omega} \frac{1}{\bar\zeta - z}\left\{-\frac{k+1}{2\pi\alpha} \bar\zeta^{\alpha} \bar\zeta_1^{-\beta} \iint\limits_{\Omega} \frac{f(\tau_1, \tau_2)}{(\bar\tau - \zeta)^2} d\tau_1 d\tau_2\right\} d\xi d\eta, \ (\tau = \tau_1 + i\tau_2). \tag{25}$$

Astfel (f_1, f_2) construit este o soluţie a sistemului (16)—(17).

TEOREMA 4. *Dacă $f \in C^1(\bar\Omega)$, atunci mulţimea soluţiilor ecuaţiei neomogene $Lw = f$ este familia de funcţii*

$$w(x, y) = \bar z_1^{-\beta}\varphi(\bar z) + \bar z^{-\alpha}\psi(\bar z_1) + \bar z_1^{-\beta} f_1(x, y) + \bar z^{-\alpha} f_2(x, y), \tag{26}$$

unde φ şi ψ sînt funcţii olomorfe arbitrare, iar f_1 şi f_2 sînt funcţiile determinate de (24)—(25).

Dacă $\Omega = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1\}$ atunci w dat de (26) va fi o soluţie a problemei (14) atunci şi numai atunci dacă $w|_{\partial\Omega} = 0$, adică

$$[\bar z_1^{-\beta}\varphi(\bar z) + {}^{-}\bar z^{\alpha}\psi(\bar z_1) + \bar z_1^{-\beta} f_1(x, y) + \bar z^{-\alpha} f_2(x, y)]\,|_{|z|=1} = 0.$$

Aceasta înseamnă

$$\bar z_1^{\beta}\psi(\bar z_1) = -\bar z^{\alpha}\varphi(\bar z) - \bar z^{\alpha} f_1(x, y) - \bar z_1^{\beta} f_2(x, y) \text{ pt. } |z| = 1,$$

care este posibil atunci şi numai atunci dacă

$$\int\limits_{|z|=1} [z^{\alpha}\bar\varphi(z) - z^{\alpha}\overline{f_1(x, y)} - z_1^{\beta}\overline{f_2(x, y)}] z_1^{m} dz_1 = 0 \text{ pt. } m = 0, 1, 2, \dots \tag{27}$$

Dacă $\alpha > 1$ avem

$$\int\limits_{|z|=1} z^{\alpha}\bar\varphi(z)\ z_1^{m} dz_1 = 0 \text{ pt. } m = 0, 1, \dots, \alpha - 1,$$

astfel condiţia (27) nu va fi verificată pentru orice $f \in C^1(\bar\Omega)$, adică problema lui Dirichlet nu are soluţie pentru orice $f \in C^1(\bar\Omega)$, deşi problema omogenă corespunzătoare are numai soluţia banală.

(Intrat în redacţie la 27 decembrie 1979)

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ON THE SOLVABILITY OF THE DIRICHLET PROBLEM FOR AN ELLIPTICAL EQUATION WITH SINGULARITIES

(Summary)

In this paper the solvability of the Dirichlet Problem for equation (1) is studied. It is shown that in the case of the domain $\Omega = \{(x, y) \in R^2 \mid x^2 + y^2 < 1\}$ the homogeneous problem (4) has only the solution $u = 0$; but the inhomogeneous problem (14) is not solvable for any $f \in C^1(\bar{\Omega})$. It is also proved that in $\Omega = \{(x, y) \in R^2 \mid x^2 + hy^2 - 1 < 0\}$ the problem (4) has infinitely many solutions.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXV, 2, 1980

IN MEMORIAM

DR. PROFIRA SANDOVICI

(1927 – 1980)

În ziua de 3 aprilie 1980 s-a stins din viaţă, în plină activitate creatoare, lector univ. dr. Profira Sandovici, distins cadru didactic al Facultăţii de matematică. Întreaga sa viaţă s-a caracterizat printr-o muncă plină de dăruire depusă în slujba Universităţii clujene, în folosul învăţămîntului şi al cercetării matematice româneşti.

S-a născut la 8 ianuarie 1927, în comuna Tărnăuca, jud. Dorohoi. A urmat cursurile secundare la Dorohoi şi la liceul „Domniţa Ileana" din Bucureşti, ultimele clase absolvindu-le la Cluj. S-a înscris apoi la Facultatea de matematică a Universităţii din Cluj, pe care a absolvit-o în anul 1950, numărîndu-se printre studenţii eminenţi ai acestei Universităţi. La terminarea studiilor a fost numită asistentă la disciplina de geometrie analitică în cadrul Catedrei de geometrie a Facultăţii de matematică, unde după cîţiva ani, în 1956, a fost avansată ca lector.

În lunga sa activitate de peste 30 de ani, depusă în cadrul facultăţii noastre, colega Profira Sandovici a ţinut numeroase cursuri şi seminarii de geometrie analitică, geometrie proiectivă, geometrie diferenţială şi bazele geometriei. Lecţiile ţinute de dînsa s-au caracterizat printr-un înalt nivel ştiinţific şi o rigurozitate desăvîrşită, contribuind astfel la formarea a numeroase generaţii de profesori şi cercetători în domeniul matematicii. A fost întotdeauna deosebit de apropiată faţă de studenţii pe care i-a îndrumat, dovedind multă înţelegere pentru problemele lor. Cursul de geometrie diferenţială redactat pentru studenţii anilor III şi IV descifrează cu multă migală tainele acestei discipline, căutînd să o prezinte într-un mod cît mai accesibil şi mai actual.

Întreaga activitate ştiinţifică a regretatei noastre colege s-a desfăşurat în cadrul Colectivului de geometrie diferenţială de la facultate.

Dintre preocupările sale ştiinţifice menţionăm studiul varietăţilor scufundate într-un spaţiu euclidian cu n dimensiuni, unele probleme privind grupul de mişcări al spaţiilor riemanniene şi spaţii cu tensor recurent. Generalizarea unor proprietăţi de recurenţă pentru fibrate vectoriale, a constituit subiectul tezei de doctorat pe care a susţinut-o la Universitatea din Iaşi.

Toate lucrările sale ştiinţifice atestă o mare fineţe de raţionament şi o deosebită capacitate de aprofundare şi generalizare, reprezentînd valoroase contribuţii la tezaurul geometriei.

Conştiinciozitatea fără margini în toate acţiunile a reprezentat o constantă a vieţii sale, care oferă un exemplu demn de urmat pentru colegi şi studenţi.

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