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MECHANICAL CONSIDERATIONS UPON THE DISTRIBUTION OF THE TEMPERATURE IN INCOMPRESSIBLE SEMILIMITED VISCOUS JETS

ŞTEFAN MAKSAY

1. Introduction and formulation of the problem. The dynamic problem of the spreading of the semilimited incompressible jet on one of the sides of a semiinfinite plane plate having a punctual source in the attack border is studied in [1].

In [2] the distribution of temperature is determined for the same kind of movement, in the case of the plate with a temperature equal to that of the external fluid and also in the case of the thermally isolated plate.

This work is dedicated to the mechanical considerations on the distribution of the temperature in incompressible semilimited viscous jets in the two cases studied in [2].

We consider a semifinite plate, placed on the positive side of the Ox axis and we suppose the surrounding space occupied by an incompressible viscid fluid at rest.

On the top of the plate ($x = 0$) there is a punctual source from which a fluid of the same nature as that from the surrounding space spouts, moving stationary and having the shape of a jet, on the $y > 0$ side of the plate.

We shall admit that the movement of the jet satisfies the equations of the temperature boundary layer that, in comparison with Mises's variables [3], have the following form [4]

$$\frac{\partial u}{\partial x} = \nu \frac{\partial}{\partial \psi}\left(u \frac{\partial u}{\partial \psi}\right), \tag{1}$$

$$\frac{1}{\nu} \frac{\partial i}{\partial x} = u\left(\frac{\partial u}{\partial \psi}\right)^2 + \frac{1}{\sigma} \frac{\partial}{\partial \psi}\left(u \frac{\partial i}{\partial \psi}\right), \tag{2}$$

$$\left(u = \frac{\partial \psi}{\partial y},\ v = -\frac{\partial \psi}{\partial x},\ \nu = \frac{\mu}{\rho} = \text{const.}\right).$$

2. The study of the temperature distribution in the case of the plate having the same temperature as that of the external fluid ($\sigma = 1$). In this case the equation of energy (2) to which the limit conditions have been attached

$$T(x, 0) = T_w, \qquad T(x, \psi_\infty) = T_\infty, \qquad T_w = T_\infty, \tag{3}$$

x, ψ — Mises' variables in the boundary layer,
μ, ν, ρ, c_p — coefficients of dynamic and kinematic viscosity, density and specific heat,
λ — coefficient of thermal conductibility,
$\sigma = \mu c_p/\lambda$ — Prandtl's number,
T — absolute temperature in the fluid,
$i = c_p T$ — the enthalpy of the mass unit of the fluid,
E_0, I, I_1 — given constants resulting from integral conservation conditions,

supposing that the movement is undissipative (which is admitted in the incompressible fluid) has the similarity solution [2]

$$T(x, \eta) = T_w - \frac{I\eta_\infty^2}{6c_p}\sqrt{\frac{\nu}{E_0 x}}\left(\sqrt{\frac{\eta}{\eta_\infty}} - \frac{\eta^2}{\eta_\infty^2}\right), \tag{4}$$

where

$$\eta = \psi(E\ \nu x)^{-1/4}, \qquad \eta_\infty = 2{,}515. \tag{5}$$

At the level of the self-modelling variables (x, η) the distribution of the temperature in a $x = x_0$ section, normal for the plate, is represented in figure 1.

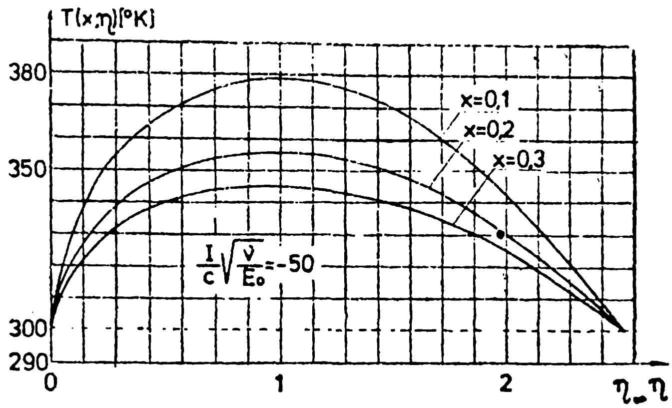

Fig. 1.

The extreme value of the temperature in the jet, corresponding to the ordinate

$$\eta = 0{,}397\eta_\infty \tag{6}$$

is

$$T_{ext}(x) = T_w - 0{,}498\frac{I}{c_p}\sqrt{\frac{\nu}{E_0 x}}, \tag{7}$$

wherefrom it results that in the same time with the removing from the source of the jet, the temperature of the fluid in the jet becomes uniform coming close to the value $T_\infty = T_w$.

The local thermic flux going through the surface unit situated on $\eta =$ constant is expressed

$$q(x, \eta) = \frac{5\lambda I}{9c_p\eta_\infty}(E_0\ \nu x^5)^{-1/4}\left[1 - 5\left(\frac{\eta}{\eta_\infty}\right)^{3/2} + 4\left(\frac{\eta}{\eta_\infty}\right)^3\right],$$

and the diagram of its variation in a $x = x_0$ section is represented in figure 2.

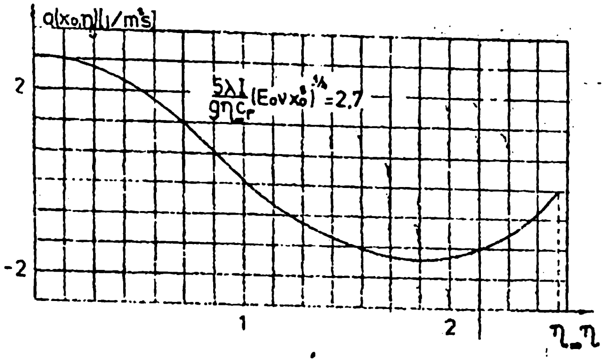

Fig. 2.

3. The study of the temperature distribution in the case of the thermal solated plate. In this case the equation of energy (2) to which the limit conditions have

$$\frac{\partial T}{\partial y}(x,\ 0) = 0, \qquad T(x,\ \infty) = T_\infty. \tag{9}$$

has the solution [2]

$$T(x,\ \eta\,;\ \sigma) = \frac{I_1(E_0 \nu x)^{-1/4}}{\rho c_p A(\sigma)\eta_\infty}\left[1 - \left(\frac{\eta}{\eta_\infty}\right)^{3/2}\right]^\sigma + T_\infty, \tag{10}$$

where

$$A(\sigma) = \int (1 - t^{3/2})^\sigma\, dt, \qquad (t = \eta/\eta_\infty). \tag{11}$$

The temperature T of the plate is obtained from (10) for $\eta = 0$.

The expression of the local thermic flux is

$$q(x,\ \eta\,;\ \sigma) = \frac{I_1}{4A(\sigma)x}\ \frac{\eta}{\eta_\infty}\left[1 - \left(\frac{\eta}{\eta_\infty}\right)^{3/2}\right]^\sigma \tag{12}$$

wherefrom we can notice the fact that the sense of the heat transfer is the same in all the mass of the fluid in the jet depending only on the sign of the conservation thermic constant, namely, when it is positive, heat is conveyed towards the fluid in the infinite, and when it is negative towards the fluid in the neighbourhood of the plate.

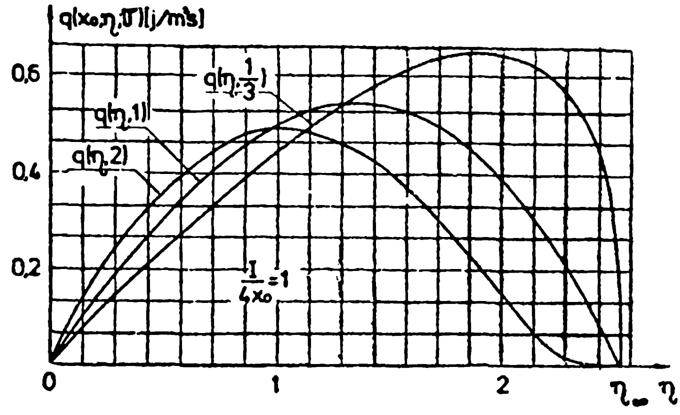

Fig. 3.

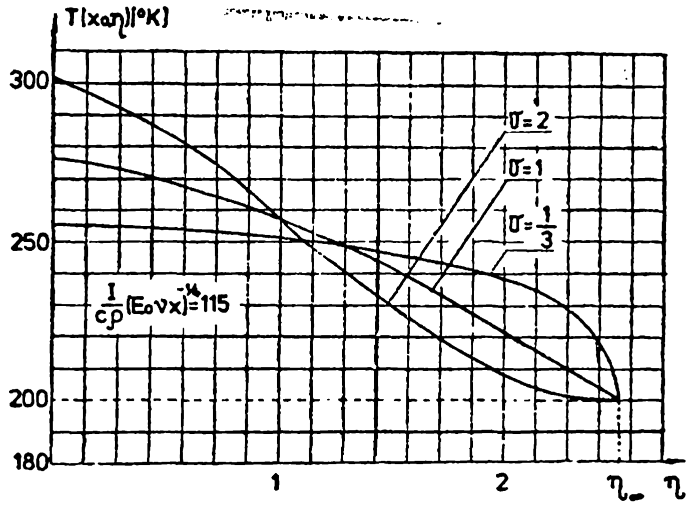

Fig. 4.

In figures 3, 4 there is the study of the local termic flux variation respectively the distribution of the temperature in the jet in a normal section at a plate.

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REFERENCES

1. Akatnov, N. I., *Primenenie peremennîh mizesa k zadăce o rasprostranenii laminarnoj strui vdol' stenki*, Prikl. Mat. i Meh., **XXIV**, 1, 1960, 154.
2. Brădeanu, P., Maksay, Şt., *Asupra determinării temperaturii în jeturi vîscoase semimărginite*, Studia Univ. Babeş-Bolyai, ser. Math.-Mech., f. 2, 1972, 79–86.

3. I a c o b, C a i u s, *Introduction mathématique à la méchanique des fluides*, Ed. Academiei, București și Gauthier-Villars, Paris, 1959.
4. G h e o r g h i ț ă, Șt. I., *Teoria stratului limită și turbulență*. Centrul de multiplicare al Universității din București, 1973.

CONSIDERAȚII MECANICE ASUPRA DISTRIBUȚIEI TEMPERATURII ÎN JETURILE VÎSCOASE NELIMITATE INCOMPRESIBILE

(R e z u m a t)

Această lucrare este dedicată unor considerații mecanice asupra distribuției temperaturii în jeturile vîscoase nelimitate incompresibile în cazul unei plăci de temperatură egale cu aceea a fluidului exterior și în cazul plăcii izolate termic. În aceste două situații s-a determinat soluția automodelată a energiei, s-a studiat într-o secțiune normală a plăcii distribuția temperaturii și fluxul termic local.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXIV, 2, 1979

CONTROLLABILITY OF LINEAR PARABOLIC SYSTEMS WITH DISTRIBUTED PARAMETERS AND BOUNDARY CONDITIONS

VIOREL HIRIŞ*

1. Introduction. The classical aspects of the controllability theory in finite-dimensional spaces are extended to linear infinite dimensional systems in which the control is exerted by means of parameters distributed all over the domain G and by means of the conditions on the boundary S of the domain G. Such a study was initiated in the papers of Egorov Iu. V., Galc'uk L.I., Russel D.L., Fattorini H.O., Sakawa Y., Glashoff K., and other authors, who used the theory of non-harmonic Fourier series, moments problem, fundamental solutions a.s.o. In the sequel we use the idea of H.O. Fattorini [1] to replace the boundary controls by distributed controls, that have the same effect on the system.

2. The parabolic system. We shall consider the following abstract model of distributed and boundary control linear parabolic system:

$$(\mathfrak{L}) \begin{cases} u'(t) = \sigma u(t) + B_G f_G(t) \\ \tau u(t) = B_S f_S(t), \ (t \geq 0), \end{cases}$$

where notations have the next meaning:

$u(t) \in E$, E any complex reflexive Banach space
$\sigma : D(\sigma)(\subset E) \to E$, a closed linear operator
$\tau : D(\tau)(\subset E) \to E_1$, a linear operator ($E_1$-complex Banach space)
$D(\sigma) \subset D(\tau)$
$f_G(.) \in C^1(R_+, F_G)$-the space $\mathfrak{F}_G$ of distributed control
$f_S(.) \in C^1(R_+, F_S)$-the space $\mathfrak{F}_S$ of boundary control
F_G, F_S, complex Banach space
$B_G \in \mathfrak{L}(F_G, E)$
$B_S \in \mathfrak{L}(F_S, E_1)$.

Let A be an operator with the domain $D(A)$ and values defined by

$$D(A) = \{u/u \in D(\sigma) \quad \& \ \tau u = 0\}, \qquad Au = \sigma u,$$

which satisfies the following assumptions:

(A_i) $\overline{D(A)} = E$;
(A_{ii}) $\rho(A) \neq \emptyset$;
(A_{iii}) the Cauchy problem for the equation $u'(t) = Au(t)$ is uniformly well-posed in R_+.

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In the assumptions $(A_I) - (A_{III})$, the unique solution of the inhomogeneous Cauchy problem

$$\begin{cases} u'(t) = Au(t) + g(t) \\ u(0) = u \in D(A) \end{cases} \tag{2.1}$$

$(t \geqslant 0,\ g \in C^1(R_+, E))$ is given by

$$u(t) = T(t)u + \int_0^t T(t-s)\,g(s)ds \tag{2.2}$$

where $\{T(t),\ t \geqslant 0\}$ is a semigroup in $\mathfrak{L}(E)$, strongly continuous on E.

In addition, A is the infinitesimal generator of $T(t), t \geqslant 0$. Now, if $g \in C_0(R_+, E)$ and $u(.)$ is the solution for the problem (2.1), then it is uniquely defined from u, by the same representation (2.2) (see. T. Kato [2]).
For constructing the solutions of the original problem, we must impose additional requirements on B_S. Namely, we suppose the existence of a bounded linear operator $B: F_S \to E$ such that

$$Bf_S \in D(\sigma) \quad \text{and} \quad \tau(B_S f) = B_S f_S, \text{ for every } f_S \in F_S. \tag{2.3}$$

Finally, we shall call

$$\mathfrak{S} = \{u / u \in D(\sigma) \quad \& \ \sigma u \in B_S F_S\},$$

the set of possible initial data for the solutions of $(\mathfrak{L})$, and let $\mathfrak{F}_S(u)$ be

$$F_S(u) = \{f_S(.) \in \mathfrak{F}_S / B_S f_S(0) = \tau u,\ u \in \mathfrak{S}\}.$$

3. **Fattorini's result.** Here we consider the results of H. O. Fattorini in [1], some of them being used in sequel.

Before, we recall the next fundamental lemma, through which, as we show in Introduction, we can replace the boundary controls by distributed controls.

3.1. Lemma. *Let u and $f(.)$ be, where $u \in \mathfrak{S}$, $f_S(.) \in \mathfrak{F}_S(u)$. Then the unique solution $u(.)$ of $(\mathfrak{L})$ with $u(0) = u$ is given by*

$$u(t) = v(t) + Bf_s(t), \tag{3.1}$$

where $v(.)$ is the solution of the Cauchy problem

$$\begin{cases} v'(t) = Av(t) - Bf'_S(t) + (\sigma B)f_S(t) + B_G f_G(t) \\ v(0) = u - Bf_S(0) \qquad (t \geqslant 0). \end{cases} \tag{3.2}$$

3.2. Definition. The system $(\mathfrak{L})$ is called *t-controllable* if, for any $u \in \mathfrak{S}$ and $\varepsilon > 0$ there exist $f_G(\cdot) \in \mathfrak{F}_G$, $f_S(\cdot) \in \mathfrak{F}_S(0)$, such that the adequate solution $u(\cdot)$ of $(\mathfrak{L})$ with initial data $u(0) = 0$ satisfies $\|u(t) - u\| < \varepsilon$. Let K_t be the subspace of E consisting of the values at the moment t of all solutions of $(\mathfrak{L})$ with u(0) = 0,

$$f_G(.) \in \mathfrak{F}_G,\ f_S(.) \in \mathfrak{F}_S(0) \text{ and } K = \bigcup_{t > 0} K_t.$$

Then ($\mathfrak{L}$) is t-controllable if and only if $\overline{K}_t = E$, the closure being taken in E-topology.

3.3. Definition. If $\overline{K} = E$, then the system (L) is said to be *controllable.*//.
This Definition comes back to consider in Definition 3.2. the time t depending on u, ε.

In fact, here we define null-controllability, (at the time t); however, we have that if ($\mathfrak{L}$) is null-controllable at the time t, then it is t-controllable with respect to every initial data $u(0) = u$.

If $\{T(t),\ t > 0\}$ is an analytic semigroup, then $\overline{K}_t = \overline{K}$ for any $t > 0$. We are interested in this case, in order to apply our results to equations of parabolic type.

That is why we consider only t-controllability.

Because E is reflexive, the semigroup $\{T^*(t),\ t \geqslant 0\}$
generated by A^*, as smooth as $\{T(t),\ t \geqslant 0\}$,
verifies $T^*(t) = T(t)^*,\ t \geqslant 0$.

If E is not reflexive, the results can be formulated in a "weak" form with trivial modifications.

In examples we shall consider that $E = H$ is a Hilbert space and A is self-adjoint. Then $\overline{D(A)} = H$ and $\rho(A) \neq \emptyset$ are automatically satisfied and Assumption (A_{iii}) is equivalent to the requirement that A is semi-upper bounded, i.e., the spectrum $\sigma(A)$ in the real line is upper-bounded.

It follows from Lemma 3.1. the next lemma, which is to be used later.

3.4. Lemma. $u^* \in K_t^{\perp}$ if and *only if the conditions are satisfied:*

$$\left(B^*(T(s)^* - I) - (\sigma B)^* \int_0^s T(r)\, dr\right) u^* = 0 \tag{3.3}$$

$$B_G^* T(s)^*\ u = 0, \tag{3.4}$$

for $0 \leqslant s \leqslant t$.

4. Controllability of parabolic systems. We introduce the bounded linear operator

$$\mathcal{G}(s) = T(s) B_G : F_G \to E \tag{4.1}$$

$$\mathcal{S}(s) = (T(s) - I) B - \left(\int_0^s T(r) dr\right)(\sigma B) : F_S \to E \tag{4.2}$$

for any $s \geqslant 0$.

With these notations, we have that $u^* \in K_t^{\perp}$ if and only if $u^* \in \bigcap_{s \in [0,t]} \{Ker\ \mathcal{G}(s)^* \cap \mathrm{Ker}\ \mathcal{S}(s)^*\}$ and, from Lemma 3.2. it results the following theorem:

4.1. THEOREM. *The system* ($\mathfrak{L}$) *is* t-*controllable if and only if:*

$$\bigcap_{s \in [0,t]} \{\text{Ker } \mathcal{G}(s)^* \cap \mathcal{S}(s)^*\} = \{0\} \tag{4.3}$$

4.2. LEMMA. *Let* F_G, F_S *be Hilbert spaces and consider the following linear operators:*

$$W_G(t) = \int_0^t \mathcal{G}(s)\, \mathcal{G}(s)^* ds : E^* \to E \tag{4.4}$$

$$W_S(t) = \int_0^t \mathcal{S}(s)\, \mathcal{S}(s)^* ds : E^* \to E. \tag{4.5}$$

Then $u^* \in K_t^{\perp}$ *if and only if*

$$u^* \in Ker\ W_G(t) \cap Ker\ W_S(t). \tag{4.6}$$

Proof. If $u^* \in K_t^{\perp}$, from Lemma 3.2. it follows that

$u^* \in \bigcap_{s \in [0,t]} \{\text{Ker } \mathcal{G}(s)^* \cup \text{Ker } \mathcal{S}(s)^*\}$ and hence, for every $v^* \in E^*$,

we have: $0 = \int_0^t < \mathcal{G}(s)^* v^*, \mathcal{G}(s)^* u^* > ds = \left\langle \int_0^t \mathcal{G}(s)\, \mathcal{G}(s)^* v^* ds,\ u^* \right\rangle$,

$$0 = \int_0^t < \mathcal{S}(s)^* v^*, \mathcal{S}(s)^* u^* > ds = \left\langle \int_0^t \mathcal{S}(s) \mathcal{S}(s)^* v^* ds,\ u^* \right\rangle.$$

Therefore

$$u^* \in \{\overline{W_G(t)\, E^*}\}^{\perp} \cap \{\overline{W_S(t)\, E^*}\}^{\perp} = \text{Ker}_G W\ (t) \cap \text{Ker } W_S(t).$$

Conversely, if $u^* \in \text{Ker } W_G(t) \cap \text{Ker } W_S(t)$, it follows that

$$< W_G(t) u^*, u^* > = \text{and} < W_S(t) u^*, u^* > = 0$$

Replacing the operators by their integral definitions (4.4) and (4.5), we can write:

$$\int_0^t \| \mathcal{G}(s)^* u^* \|^2 ds = 0 \ \text{ and } \int_0^t \| \mathcal{S}(s)^* u^* \|^2 ds = 0.$$

From the continuity of the functions $\mathcal{G}(\cdot)^* u^*$ and $\mathcal{S}(.) u^*$, it results that:

$$\mathcal{G}(s)^* u^* = 0 \text{ and } \mathcal{S}(s)^* u^* = 0 \text{ for every } s \in [0,t].$$

Consequently, we obtain $u^* \in \bigcap_{s \in [0,t]} \{\text{Ker} \mathcal{G}(s)^* \cap \text{Ker } \mathcal{S}(s)^*\}$ and, from Lemma 3.2., that $u^* \in K_t^{\perp}$.

4.3. THEOREM. *The system* ($\mathfrak{L}$) *is t-controllable if and only if*

$$Ker\, W_G(t) \cap Ker\, W_S(t) = \{0\} \tag{1.7}$$

4.4. COROLLARY. *The „purely-distributed" control system:*

$$\begin{cases} u'(t) = \sigma u(t) + B_G f_G(t) \\ \tau u(t) = 0 \qquad\qquad (t \geqslant 0) \end{cases} \tag{4.8}$$

is t-controllable if and only if the operator $W_G(t)$ is injective and hence, if and only if the distributed-parameters system (A, B_G) is t-controllable.

For the last assertion see, for example [3], since, if $B_S = 0$ we can take $B = 0$ and thus $W_S(t) = 0$, i.e., Ker $W_S(t) = E$.

4.5. COROLLARY. *The boundary-control system:*

$$\begin{cases} u'(t) = \sigma u(t) \\ \tau u(t) = B_S f_S(t) \qquad (t) \geqslant 0) \end{cases} \tag{4.9}$$

is t-controllable if and only if the operator $W_S(t)$ is injective.

4.6. LEMMA. $u^* \in K_t^{\perp}$ *if and only if*

$$u^* \in \{\bigvee_{s\in[0,t]} \mathcal{G}(s)F_G\}^{\perp} \cap \{\bigvee_{s\in[0,t]} \mathcal{S}(s)F_S\}^{\perp}.^* \tag{4.10}$$

Proof. Taking $u^* \in \{\bigvee_{s\in[0,t]} \mathcal{G}(s)F_G\}^{\perp} \cap \{\bigvee_{s\in[0,t]} \mathcal{S}(s)F_S\}^{\perp}$ we see that vanishing of $\langle \mathcal{G}(s)f_G, u^*\rangle = 0$ and $\langle \mathcal{S}(s)f_S, u^*\rangle = 0$, for all vectors $f_G \in F_G$ and $f_S \in F_S$, it implies that $u^* \in \bigcap_{s\in[0,t]} \{\mathrm{Ker}\, \mathcal{G}(s)^*\} \cap \mathrm{Ker}\, \mathcal{S}(s)^*\}$. Then by Lemma 3.4. it results that $u^* \in K_t^{\perp}$. Conversely, if $u^* \in K_t^{\perp}$, then by analogous considerations we obtain

$$u^* \in \{\bigvee_{s\in[0,t]} \mathcal{G}(s)F_G\}^{\perp} \cap \{\bigvee_{s\in[0,t]} \mathcal{S}(s)F_S\}^{\perp}.$$

4.7. THEOREM. *The system* ($\mathfrak{L}$) *is t-controllable if and only if*

$$\{\bigvee_{s\in[0,t]} \mathcal{G}(s)F_G\}^{\perp} \cap \{\bigvee_{s\in[0,t]} \mathcal{S}(s)F_S\}^{\perp} = \{0\}. \tag{4.11}$$

4.8. *Remark.* Because $A^{\perp} \cap B^{\perp} = (A \cup B)^{\perp}$, the condition (4.11) is equivalent to

$$\{\bigvee_{s\in[0,t]} \mathcal{G}(s)F_G\} \cup \{\bigvee_{s\in[0,t]} \mathcal{S}(s)F_S\} = E \tag{4.12}$$

4.9. *Remark.* The previous propositions extend the methods of R. E. Kalman [4] for the analysis of the finite dimensional systems.

The extensions of rank conditions for controllability to distributed-parameter control system have been obtained by R. Triggiani [5].

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* $\bigvee_{\gamma\in\Gamma} M_\gamma$ denotes the span of the sets $M_\gamma, \gamma \in \Gamma$.

Namely, we consider

$$D_\infty(A) = \bigcap_{n=1}^{\infty} D(A^n) \qquad (\overline{D_\infty(A)} = E)$$

and

$$F_\infty^G = \{f_G \in F_G / B_G f_G \in D_\infty(A)\},$$

$$F_S^\infty = \{f_S \in F_S / Bf_s \in D_\infty(A)\}.$$

4. 10. THEOREM. (i) If

$$\bigvee_{n=0}^{\infty} A^n B_G F_G^\infty = E, \tag{4.13}$$

then the system ($\mathfrak{L}$) *is t-controllable for every* $t > 0$ (ii) *Conversely, if:*

(a) ($\mathfrak{L}$) *is controllable;*

(b) *A generates an analytic semigroup* $\{T(t),\ t > 0\}$;

(c) $\overline{BF_S^\infty} = BF_S$ and $\overline{BF_G^\infty} = BF_G$,

then, for any arbitrary time $t > 0$, *we have*

$$\bigvee_{n=0}^{\infty} T(t) A^n B_G F_G^\infty = E \tag{4.14}$$

Proof. First of all, we need the following facts.

Let $f_S^\infty \in F_S^\infty$. Then, the function

$$\Phi(s) = \left\langle \left((T(s) - I)B - \left(\int_0^s T(r)dr \right) (\sigma B) \right) f_S^\infty,\ u^* \right\rangle$$

$u^* \in E^*$, $u^* \neq 0$, $s \in [0, t]$) can be differentiated and we have:

$$\Phi'(s) = \left\langle T(s) Bf_S^\infty - BF_S^\infty - \int_0^s T(r)(\sigma B) f_S^\infty\, dr,\ u^* \right\rangle =$$

$$= \langle T'(s) Bf_S^\infty - T(s)(\sigma B) f_S^\infty,\ u^* \rangle =$$

$$= \langle T(s) A Bf_S^\infty - T(s)(\sigma B) f_S^\infty,\ u^* \rangle$$

$$= \langle T(s)(A(Bf_S^\infty) - \sigma(Bf_S)),\ u^* \rangle = \langle 0,\ u^* \rangle = 0,$$

since $Bf_S^\infty \in D(A)$.

Hence

$$\Phi(s) = \Phi(0) = 0,$$

and this

$$\left\langle \left((T(s) - I)B - \left(\int_0^s T(r)dr \right)(\sigma B) \right) F_S^{\infty}, u^* \right\rangle = 0, \quad \forall s \in [0, t]$$

(i) Suppose that there exists $t_0 > 0$ such that (£) is not t_0 — controllable. We can choose $u^* \neq 0$ for which

$$B^*(T(s)^* - I) - (\sigma B)^* \int_0^s T(r)^* dr) u^* = 0$$

and

$$B_G^* T(s)^* u^* = 0$$

$(0 \leqslant s \leqslant t)$.

Particularly, we obtain

$$< T(s) B_G F_G^{\infty}, u^* > = 0 \qquad (0 \leqslant s \leqslant t).$$

If we successively differentiate in the last identities, using the equalities

$$\frac{d^n T(s) u}{ds^n} = T(s) A^n u, \quad n = 0, 1, 2, \ldots, u \in D_{\infty}(A),$$

it follows for $s = 0$:

$$< A^n B_G F_G^{\infty}, u^* > = 0, \quad n = 0, 1, 2, \ldots$$

Hence

$$u^* \in \left\{ \bigvee_{n=0}^{\infty} A^n B_G F_G^{\infty} \right\}^{\perp}$$

Now, by hypotheses, $u^* = 0$, which is a contradiction.

(ii) We suppose that there exists $u^* = 0$, such that for one $t_0 > 0$ is true

$$u^* \in \left\{ \bigvee_{n=0}^{\infty} T(t_0) A^n B_G F_G^{\infty} \right\}^{\perp},$$

that is

$$< T(t_0) A^n B_G F_G^{\infty}, u^* > = 0, \quad n = 0, 1, 2, \ldots$$

If we differentiate in the same manner, it follows that

$$< T(s) B_G F_G^{\infty}, u^* > = 0,$$

in a neighbourhood of t_0 and hence for all $t \geqslant 0$.

Using now the previous facts, we obtain

$$\left\langle \left((T(s) - I)\ B - \left(\int_0^s T(r) dr \right)(\sigma B) \right) F_S^{\infty},\ u^* \right\rangle = 0$$

Since $\overline{BF_S^{\infty}} = BF_S$ and $\overline{B_G F_G^{\infty}} = B_G F_S$, it follows, by continuity, that for every $t > 0$ we have

$$\left\langle \left((T(s) - I)\ B - \left(\int_0^s T(r)^*\ dr \right)(\sigma B) \right) F_S,\ u^* \right\rangle = 0$$

$$< T(s)\ B_G F_G,\ u^* > = 0 \qquad (0 \leqslant s \leqslant t),$$

and this contradicts the controllability hypotheses.

4.11. COROLLARY. *Let b, $c \in D_{\infty}(A)$ be and the system*

$$(\mathfrak{L}_{b,c}) \quad \begin{cases} u'(t) = \sigma u(t) + b f_G(t) \\ \tau u(t) = c f_S(t). \end{cases}$$

We have

(i) *If $\bigvee_{n=0}^{\infty} A^n b = \chi$, then the system $(\mathfrak{L}_{b,c})$ is t-controllable for every $t > 0$*

(ii) *If $(\mathfrak{L}_{b,c})$ is controllable and A generates an analytic semigroup, then for any arbitrary time $t > 0$ we have*

$$\bigvee_{n=0}^{\infty} T(t) A^n b = E$$

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REFERENCES

1. Fattorini, H. O., *Boundary Control systems*, SIAM J Control, **4**, 1966, 349–385.
2. Kato, T., *Integrations of the Equations of Evolution in a Banach Space*, Math. Soc, Japan, **5**, 1953, 208–234.
3. Megan, M., Hiriş, V., *On the Space of Linear Controllable Systems in Hilbert Spaces*, Glasnik Matematiki, **10**, 1975, 161–167.
4. Kalman, R. E., *On the general Theory of Control Systems*, Proc. 1-st IFAC Congress 1960, Moscow, Butherworths, London.
5. Triggiani, R., *Extension of Rank Conditions for Controllability and Observability to Banach Spaces and Unbounded Operators*, Report, 28, 1973.

CONTROLABILITATEA SISTEMELOR LINIARE PARABOLICE CU PARAMETRI SEPARAŢI ŞI CONDIŢII DE MĂRGINIRE

(Rezumat)

În lucrare se studiază controlabilitatea sistemului parabolic liniar

$$(\mathfrak{L}) \quad u'(t) = \sigma u(t) + B_G f_G(t), \ \tau u(t) = B_S f_S(t), \ t \geq 0$$

unde $u(t) \in E$; $\sigma : D(\sigma) \to E$ este un operator liniar închis;

$\tau : D(\tau) \to E_1$ este un operator liniar;

$$D(\sigma) \subset D(\tau) \subset E; \quad B_G \in \mathfrak{L}(F_G, E); \quad B_S \in \mathfrak{L}(F_S, E_1);$$

$$f_G \in C^1(R_+, F_G) = \mathcal{F}_G : f_S \in C^1(R_+, F_S) = \mathcal{F}_S; \quad E, E_1, F_G \text{ şi } F_S$$

fiind spaţii Banach.

Cînd E este reflexiv se stabilesc cîteva caracterizări ale t-controlabilităţii lui ($\mathfrak{L}$) de tipul celor finit-dimensionale ale lui R. E. Kalman, precum şi de tipul celor infinit-dimensionale ale lui R. Triggiani, M. Megan şi V. Hiriş.

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UNE MÉTHODE D'AJUSTATION STATISTIQUE POLYNOMIALE D'ORDRE SUPÉRIEUR

E. OANCEA, M. RĂDULESCU

Dans cette note on généralise une méthode d'ajustation proposée dans [1] pour le cas où la dépendance entre deux caractéristiques statistiques X et Y est polynomiale d'ordre $N > 2$, c'est a dire:

$$y = a_0x^N + a_1x^{N-1} + \ldots + a_N \tag{1}$$

Pour simplification on expose en détail le cas $N = 3$.

Soit la courbe de dépendance d'équation:

$$y = ax^3 + bx^2 + cx + d \tag{2}$$

En utilisant les données d'observation x_i, $\bar{y}_i$, $i = \overline{1, n}$ où

$$\bar{y}_i = \frac{1}{n}\sum_{j=1}^{n_i} y_{ij},$$

on a

$$m_i = \frac{\bar{y}_{i+1} - \bar{y}_i}{x_{i+1} - x_i} = \frac{\bar{y}_{i+1} - \bar{y}_i}{h}, \; i = \overline{1, n-1} \tag{3}$$

où

$$h = x_{i+1} - x_i, \; i = \overline{1, n-1}.$$

D'autre côté, les valeurs m_i dans l'hypothèse que h soit suffisamment petit, sont approximativement:

$$m_i = 3ax_i^2 + 2bx_i + c, \; i = \overline{1, n-1} \tag{4}$$

et

$$\frac{m_{i+1} - m_i}{h} = 6ax_i + 2b, \; i = \overline{1, n-1}. \tag{5}$$

Après la sommation de (4) et (5) on obtient

$$\sum_{i=1}^{n-1} m_i = 3a\sum_{i=1}^{n-1} x_i^2 + 2b\sum_{i=1}^{n-1} x_i + (n-1)c \tag{6}$$

$$\frac{m_n - m_1}{h} = 6a\sum_{i=1}^{n-1} x_i + 2b(n-1)$$

De ces rélations on peut calculer les paramètres b et c comme fonctions de a:

$$2b = \frac{1}{n-1}\left[\frac{m_n - m_1}{h} - 6a\sum_{i=1}^{n-1} x_i\right] \tag{7}$$

$$c = \frac{1}{n-1}\left[\sum_{i=1}^{n-1} m_i - 3a\sum_{i=1}^{n-1} x_i^2 - \frac{1}{n-1}\cdot\frac{m_n - m_1}{h}\sum_{i=1}^{n-1} x_i + \frac{6a}{n-1}\left(\sum_{i=1}^{n-1} x_i\right)^2\right] \tag{8}$$

Alors (4) devient :

$$m_i = 3ax_i^2 + \frac{x_i}{n-1}\left[\frac{m_n - m_1}{h} - 6a \sum_{i=1}^{n-1} x_i\right] +$$
$$+ \frac{1}{n-1}\left[\sum_{i=1}^{n-1} m_i - 3a \sum_{i=1}^{n-1} x_i^2 - \frac{1}{n-1}\cdot\frac{m_n - m_1}{h}\sum_{i=1}^{n-1} x_i + \frac{6a}{n-1}\left(\sum_{i=1}^{n-1} x_i\right)^2\right] \tag{9}$$

d'où

$$3a = \frac{m_i - \frac{m_n - m_1}{h(n-1)}\left(x_i - \frac{1}{n-1}\sum_{i=1}^{n-1} x_i\right) - \frac{1}{n-1}\sum_{i=1}^{n-1} m_i}{x_i^2 - 2x_i\frac{1}{n-1}\sum_{i=1}^{n-1} x_i - \frac{1}{n-1}\sum_{i=1}^{n-1} x_i + \frac{2}{(n-1)^2}\left(\sum_{i=1}^{n-1} x_i\right)^2} \tag{10}$$

On observe que l'expression de $3a$ dépend de la valeur de X d'indice i et à cause de ce motif on note par $3a_i$ l'expression :

$$3a_i = \frac{m_i - \frac{m_n - m_1}{h(n-1)}(x_i - \bar{x}) - \bar{m}}{x_i^2 - 2x_i\bar{x} - \bar{M}_2 + 2\bar{x}^2}, \quad i = \overline{1, n-1} \tag{11}$$

où

$$\bar{x} = \frac{1}{n-1}\sum_{i=1}^{n-1} x_i, \quad M_2 = \frac{1}{n-1}\sum_{i=1}^{n-1} x_i^2, \quad \bar{m} = \frac{1}{n-1}\sum_{i=1}^{n-1} m_i$$

Dans le cas où l'hypothèse de dépendance de type (2) este vraie, il résulte que $3a_i$ donné par (11) est approximativement constante :

$$3a_i \simeq \text{const.} \quad , \quad i = \overline{1, n-1}$$

et on attribue à $3a$ la valeur

$$3a = \frac{1}{n-1}\sum_{i=1}^{n-1} 3a_i. \tag{12}$$

Pour le calcul du coefficient d on utilise la formule :

$$d = \frac{1}{n}\sum_{i=1}^{n}(\bar{y}_i - ax_i^3 - bx_i^2 - cx_i).$$

où $a, b, c,$ sont donnès par (12), (7), (8).

Remarques. 1. Pour le cas $N > 3$ on procède analoguement, en considérant au lieu du système (6) un système de $N - 1$ équations obtenues dans le même mode, comme celles du (6), d'où on détermine les coefficients $a_1, \ldots, a_{N-1}$ en fonction de a_0, puis on détermine la valeur correspondante de a_{0i}, $i = \overline{1, n-1}$. Si

$$a_{0i} \simeq \text{const.}, \; i = \overline{1, n-1}$$

l'hypothèse de dépendance polynomiale d'ordre N est vraie et on attribue à a_0 la valeur

$$a_0 = \frac{1}{n-1} \sum_{i=1}^{n-1} a_{0i}.$$

2. Il est possible que la dépendance entre X et Y soit polynomiale d'ordre $k > N$. Mais parce que le nombre des extrèmes résultant de l'allure du nuage statistique est $N - 1$, pratiquement on prend comme courbe d'ajustation de la dépendance un polynôme d'ordre N. Une telle situation peut être rencontrée quand le polynôme dérivé de la courbe de dépendance a $N - 1$ racines réelles, distinctes, où quand le pas h n'est pas suffisamment petit et les données d'observation ne sont pas si précises.

Soit les variables aléatoires indépendantes :

$$Z_i = \frac{(n-1)Y_{i+1} - Y_i) - (Y_n - Y_1) - (Y_n - Y_{n-1} - Y_2 + Y_1)k_i}{(k_i^2 - \overline{M}_2 - \bar{x}^2)h(n-1)} \tag{13}$$

où $k_i = x_i - x$, $i = \overline{1, n-1}$ dont les valeurs sont $3a_i$, $i = \overline{1, n-1}$. Alors la statistique

$$Z = \frac{\bar{Z} - M}{\frac{\sigma}{\sqrt{n-1}}} \tag{14}$$

où

$$\bar{Z} = \frac{1}{n-1} \sum_{i=1}^{n-1} Z_i, \quad M = M\bar{Z}, \quad \frac{\sigma^2}{n-1} = D^2\bar{Z}$$

et σ^2 est la variance des variables aléatoires Z_i, $i = \overline{1, n-1}$, dans l'hypothèse que Z_i, $i = \overline{1, n-1}$ vérifient les conditions du théorème central du calcul des probabilités, est normale $N(0, 1)$. Par conséquent pour déterminer un intervalle de confiance relatif à M, dont la valeur est en fait $3a$, on choisit une probabilité de risque q et de la probabilité

$$P(|Z| < z_q) = 1 - q$$

on obtient

$$\bar{z} - z_q \frac{\sigma}{\sqrt{n-1}} < M < \bar{z} + z_q \frac{\sigma}{\sqrt{n-1}} \tag{15}$$

où $\bar{z}$ est la valeur moyenne d'échantillon calculée aves les données d'observation.

On observe que (14) peut être utilisé aussi pour vérifier une hypothèse de forme :

$$3a = a_0.$$

Pour déterminer un intervalle de confiance pour la variance Z (c'est à dire la variance de $3a$), dans l'hypothèse que les variables aléatoires Y_i, $i = \overline{1, n-1}$ sont normales $N(m, \sigma)$, les variables

$$Y_i - \bar{Y}, \quad \bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i$$

sont normales $N(0, \sigma)$, on considère la statistique

$$\chi^2 = \frac{s^2}{\sigma^2} (n - 1)$$

où

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2$$

qui est de type χ^2 avec $n - 1$ degrés de liberté et de paramètre 1. En choisissant une probabilité de risque q, on obtient un intervalle de confiance pour σ:

$$\sqrt{\frac{1}{b_q} \sum_{i=1}^{n} (y_i - \bar{y})^2} < \sigma < \sqrt{\frac{1}{a_q} \sum_{i=}^{n} (y_i - \bar{y})^2}$$

où a_q, b_q sont donnés par :

$$P(\chi^2 \in (a_q, b_q)) = 1 - q.$$

Parce que

$$D^2 Z_i = 2\sigma^2 \frac{(n-1)^2 + 1 + 2k_i^2}{h^2(n-1)^2(k_i^2 - \overline{M}_2 - \bar{x}^2)^2}, \quad i = \overline{1, n-1} \tag{17}$$

il résulte

$$D^2 \bar{Z} = \frac{2\sigma^2}{h^2(n-1)^4} \sum_{i=1}^{n-1} \frac{(n-1)^2 + 1 + 2k_i^2}{(k_i^2 - \overline{M}_2 - \bar{x}^2)^2} = \sigma^2 L. \tag{18}$$

L'intervalle de confiance pour $D^2 Z$ est :

$$\frac{\sum_{i=1}^{n} (y_i - \bar{y})^2}{b_q} L < D^2 \bar{Z} < \frac{\sum_{i=1}^{n} (y_i - \bar{y})^2}{a_q} L \tag{19}$$

qui donne une indication relativement à la dispersion des valeurs $3a_i$, $i = \overline{1, n-1}$.

(Manuscrit reçu le 15 avril 1978)

BIBLIOGRAPHIE

1. E. Oancea, M. Rădulescu, *Metodă de ajustare statistică*, Rev. de statistică, Bucureşti, XXVI, 1, 1977, 29–33.

O METODĂ DE AJUSTARE STATISTICĂ POLINOMIALĂ DE ORDIN SUPERIOR

(Rezumat)

Lucrarea prezintă o metodă de ajustare statistică pentru o dependenţă polinomială de ordin $N > 2$ dintre două caracteristici statistice. Metoda utilizează anumite elemente geometrice care furnizează un sistem de ecuaţii din care se pot determina, pe de o parte coeficienţii ecuaţiei de dependenţă, pe de altă parte verificarea dependenţei de tipul considerat. Se determină de asemenea un interval de încredere pentru un anumit coeficient al curbei de dependenţă precum şi pentru dispersia acestuia.

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A METHOD FOR FINDING THE STRONG COMPONENTS OF A DIAGRAPH

DĂNUȚ MARCU*

In [2], [3], and [4] there are three algorithms for computing the strongly connected components of a digraph.

Our paper suggests a new method in this direction, a method for which we can write a very simple and efficient computer program.

Let $D = \langle \mathfrak{N}, \mathfrak{A} \rangle$ be a *digraph* [1] with $\mathfrak{N} = \{n_1, n_2, \ldots, n_N\}$ the set of *nodes*, and $\mathfrak{A}$ the set of *arcs*. If certain members of $\mathfrak{A}$ can be placed in a sequence of the form $\langle n_{i_1}, n_{i_2}\rangle, \langle n_{i_2}, n_{i_3}\rangle, \ldots, \langle n_{i_{s-1}}, n_{i_s}\rangle$, then the set $\{\langle n_{i_1}, n_{i_2}\rangle, \langle n_{i_2}, n_{i_3}\rangle, \ldots, \ldots, \langle n_{i_{s-1}}, n_{i_s}\rangle\}$ is a *path* from n_{i_1} to n_{i_s} in D, a path denoted by $\gamma[n_{i_1}, n_{i_s}]$. For a digraph we consider the *path matrix* $P = (P_{ij})_{\substack{i=1,2,\ldots,N,\\ j=1,2,\ldots,N}}$

where:

$$P_{ij} = \begin{cases} 1, & \text{if there exists a path } \gamma[n_i, n_j], \\ 0, & \text{otherwise.} \end{cases}$$

In [5] we suggested a very simple algorithm for computing the path matrix of a digraph from his *adjacency matrix*

$A = (a_{ij})_{\substack{i=1,2,\ldots,N,\\ j=1,2,\ldots,N}}$ defined as follows:

$$a_{ij} = \begin{cases} 1, & \text{if } \langle n_i, n_j\rangle \in \mathfrak{A}, \\ 0, & \text{if } \langle n_i, n_j\rangle \notin \mathfrak{A}. \end{cases}$$

Two nodes n and m are *mutually strongly connected* $(n \sim m)$ if and only if there exist the paths $\gamma[n, m]$ and $\gamma(m, n]$. Evidently, „$\sim$" is an equivalence and induces a partition of $\mathfrak{N}$ in the classes: $C_1, C_2, \ldots, C_K$, called *strongly connected components*.

Our problem is to find an algorithm for the calculus of $C_1, C_2, \ldots, C_K$. Having this purpose we consider the matrix $P^* = (p^*_{ij})_{\substack{i=1,2,\ldots,N,\\ j=1,2,\ldots,N}}$

where:

$$P^*_{ij} = \begin{cases} P_{ij}, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

THEOREM 1. *If C_0 is the strongly connected component that contains the node n_i, then* $[(P^*)^2]_{ii} = |C_0|$.

Proof. Suppose that C_0 is $C_0 = \{n_i, n_{k_1}, n_{k_2}, \ldots, n_{k_t}\}$.

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Because the set C_0 is a strongly connected component of D, then, there exist the paths $\gamma[n_i, n_{k_\alpha}]$ and $\gamma[n_{k_\alpha}, n_i]$, for all $\alpha = 1, 2, \ldots, t$, fact that implies:

$$p^*_{i_{k_\alpha}} = p^*_{k_\alpha i} = 1, \text{ for all } \alpha = 1, 2, \ldots, t \quad (1)$$

From (1) we obtain: $[(P^*)^2]_{ii} = \sum_{r=1}^{N} P^*_{ir} \cdot P^*_{ri} = \sum_{\beta \in \{i, k_1, k_2, \ldots, k_t\}} P^*_{i\beta} \cdot P^*_{\beta i} + $
$+ \sum_{\sigma \in \{1, 2, \ldots, N\} \setminus \{i, k_1, k_2, \ldots, k_t\}} \cdot P^*_{i\sigma} \cdot P^*_{\sigma i} = \underbrace{1 + 1 + \ldots + 1}_{t+1} + \underbrace{0 + 0 + \ldots + 0}_{N-t-1} =$
$t + 1 = |C_0| \cdot$ (Q.E.D.)

THEOREM 2. *Let C_0 be the strongly connected component that contains the node n_i. The node n_j belongs to C_0 if and only if*

$$[(P^*)^2]_{ij} = [(P^*)^2]_{ii}.$$

Proof. Suppose that n_j belongs to C_0 and let $C_0 = \{n_i, n_{k_1}, n_{k_2}, \ldots, n_{k_t}\}$. Because C_0 is a strongly connected component, there exist the paths $\gamma[n_i, n_\beta]$ and $\gamma[n_\beta, n_j]$, for all $\beta \in \{i, j, k_1, k_2, \ldots, k_t\}$, fact that implies:

$$P^*_{i\beta} = P^*_{\beta j} = 1, \text{ for all } \beta \in \{i, j, k_1, k_2, \ldots, k_t\}. \quad (2)$$

From (2) and the theorem 1 we obtain: $[(P^*)^2]_{ij} = \sum_{r=1}^{N} P^*_{ir} \cdot P^*_{rj} =$

$= \sum_{\beta \in \{i, j, k_1, k_2, \ldots, k_t\}} P^*_{i\beta} \cdot P^*_{\beta j} + \sum_{\sigma \in \{1, 2, \ldots, N\} \setminus \{i, j, k_1, k_2, \ldots, t_t\}} \cdot P^*_{\sigma i} \cdot P^*_{\sigma j} = \underbrace{1 + 1 + \ldots + 1}_{t+2} +$
$+ \underbrace{0 + 0 + \ldots + 0}_{N-t-2} = t + 2 = |C_0| = \; = [(P^*)^2]_{ii} \cdot$ (Q.E.D.)

Now we suppose $[(P^*)^2]_{ij} = [(P^*)^2]_{ii} = |C_0|$ and we show that n_j belongs to C_0.

Let $I = \{i, j, k_1, k_2, \cdots, k_t\} \subseteq \{1, 2, \ldots, N\}$ be a set of indices so that: $P^*_{i\beta} \cdot P^*_{\beta j} = 1$, for all $\beta \in I$ and $P^*_{i\sigma} \cdot P^*_{\sigma j} = 0$, for all $\sigma \in \{1, 2, \ldots, N\} \setminus I$.

Evidently we have:

$$[(P^*)^2]_{ij} = |\mathfrak{M}|, \quad (3)$$

where $\mathfrak{M} = \{n_i, n_j, n_{k_1}, n_{k_2}, \ldots, n_{k_t}\}^2{}_{k_t} \subseteq \mathfrak{N}$

Let n_ω be an arbitrary node belonging to C_0. Because C_0 is a strongly connected component of D, then there exist the paths:

$$\gamma[n_i, n_\omega] \text{ and } \gamma[n_\omega, n_i]. \quad (4)$$

Because $P^*_{i\beta} \cdot P^*_{\beta j} = 1$, for all $\beta \in I$, there exists at least a path $\gamma[n_i, n_j]$. Hence, having in view the relation (4), there exists a path $\gamma[n_i, n_j] = [n, \cdots, n_\omega, \ldots, n_j]$, fact that implies $P^*_{i\omega} \cdot P^*_{\omega j} = 1$. Hence, the node n_ω belongs to $\mathfrak{M}$ and we have:

$$C_0 \subseteq \mathfrak{M}. \quad (5)$$

From (3) and (5) results, having in view the equality $[(P^*)^2]_{ij} = [(P^*)^2]_{ii} = |C_0|$, that $C_0 = \mathfrak{M}$, and evidently $n_j \in C_0$. (Q.E.D.)

The theorem 2 suggests, for computing the strongly connected components of D, the following very simple

ALGORITHM

1. Set $v_i = 1$ for all $i = 1, 2, \ldots, N$, and $k = 0$.
2. Set $i = 1$.
3. If $v_i = 0$, go to 9.
4. Set $k = k + 1$, constructs the set $C_k = \{n_i\}$, and set $v_i = 0$.
5. Set $j = 1$.
6. If $v_j = 0$, go to 8.
7. If $[(P^*)^2]_{ij} = [(P^*)^2]_{ii}$, then set $C_k = C_k \cup \{n_j\}$ and $v_j = 0$.
8. Set $j = j + 1$. If $j \leqslant N$, go to 6.
9. Set $i = i + 1$. If $i \leqslant N$, go to 3; else STOP.

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REFERENCES

1. Berztiss, A. T., *Data Structures*, New York, 1971.
2. Foulkes, J. D., *Directed Graphs and Assembly Schedules*, 10th Symp. in Applied Math., Amer. Soc. Math., 1960.
3. Muller, Y., *Applications Pratiques des Graphes à la Recherche d'un Optimum*, Paris, 1972.
4. Marcu, D., *Un algoritm de determinare a componentelor tari conexe ale unui graf orientat finit.* Studii şi Cercetări Matematice, **28**, 1, 1976.
5. Marcu, D., *Determinarea arcelor esenţiale într-un graf orientat finit*, Studii şi Cercetări de Calcul Economic şi Cibernetică Economică, **2**, 1975.

O METODĂ PENTRU AFLAREA COMPONENTELOR TARI ALE UNUI GRAF

(Rezumat)

În lucrare se prezintă o metodă pentru determinarea componentelor conexe în sens tare ale unui graf orientat. Metoda este dată sub forma unui algoritm.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXIV, 2, 1979

ON AN ENRICHED THEORY OF MODULES (I)

GRIGORE CĂLUGĂREANU

0. Introduction. This paper uses entirely the terminology from the Eilenberg-Kelly's ample monography [3]. Various results from this monography will also be used. Therefore, in order to simplify the references, we shall denote by [III, 2.4] (square brackets) the theorem (lemma, proposition or corollary) 2.4. from the chapter III of [3].

In the third chapter of the well-known Mac Lane's paper *Categorical Algebra* [4], the author is bounded by a symmetric monoidal category $\underline{V}$ (a category with a multiplication) adding some restrictively enough conditions: V_0 is an abelian category and the bifunctor $\otimes : V_0 \times V_0 \to V_0$ is additive and right exact in each argument separately. In such a category (called tensored) he defines the notion of $\underline{V}$-algebra (in the present paper this notion will be called a monoidal monoid over $\underline{V}$) in a natural way (such as this is done by Bénabou [1]), showing that $\underline{V}$-algebras also form a symmetric monoidal category, the tensor product of two $\underline{V}$-algebras being essentially determined by the „middle four interchange" isomorphism.

Further on, for a fixed $\underline{V}$-algebra R, he defines the notion of left R-module and the corresponding notion of morphism. The results obtained are the following: if $\underline{V}$ is a tensored category, so is the category of modules over a commutative $\underline{V}$-algebra R, and, if $\underline{V}$ is a tensored category and R a $\underline{V}$-algebra, the functor $F : \underline{V}_0 \to {}_R M V$ with $F(\overline{A}) = R \otimes A$ is an adjoint to the forgetfull functor $G : {}_R M V \to V_0$; both functors are additive, G is exact and F is right exact.

Stimulated by the excellent monograph [3], the author of the present paper works out the closed and monoidal closed part of the theory of modules over a fixed monoid, theory, which for MacLane only worked out the monoidal part. Secondly, the author elaborates this theory in considerably weaker assumptions, assumptions present in almost all concrete categories (the abelianity condition evidently being not of this kind).

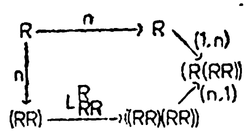
In the first section, the basic notions: closed and monoidal monoids and the corresponding morphisms, left and right R-modules over a fixed closed (monoidal) monoid R, are defined, and the basic situations: in a monoidal closed category, closed and monoidal monoids may be identified, etc., are studied.

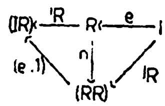
In the second section we successively prove the "enriched" versions of the following classical results: every ring can be viewed as a category with a single object, the category of modules can be identified with a suitable category of functors from the category with a single object mentioned above, the category of modules inherits properties, such as completeness, from the category Ab. Assuming that the basic category V_0 has equalizers, the monoid of the biendomorphisms is constructed and the classical results about it, proved.

In the third and final section we prove several results that are leading us to the following result: if $\underline{V}$ is a symmetric monoidal closed category with equalizers and the functor of subjacency V preserves equalizers, then the category ${}_R M V$ of the left modules over a commutative monoid R, is closed.

After recovering , in much more weaker conditions, the results of MacLane concerning the monoidal structure of ${}_RMV$, we prove our principal result : *if $\underline{V}$ is a symmetric monoidal closed category, V_0 has equalizers and coequalizers, R is a commutative monoid over $\underline{V}$, V preserves equalizers and $R \otimes -$ preserves coequalizers, then ${}_RMV$ is a symmetric monoidal closed category.*

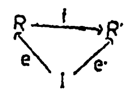
1. Basic notions and results. Definition 1.1. Let $\underline{V}$ be a closed category. A *closed monoid* (R, e, n) *over* $\underline{V}$ consists of the following data : an object R in V_0 and two morphisms $e : I \to R$, $n : R \to (RR)$ in V_0. These data are to satisfy the following axioms :

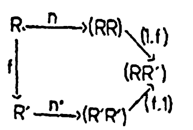
CM 1 

CM 2 

are commutative diagrams.

Definition 1.2. If (R, e, n) and (R', e', n') are two closed monoids over $\underline{V}$, a *morphism* $f : (R, e, n) \to (R', e', n')$ *of closed monoids* is a morphism $f : R \to R'$ in V_0 which satisfies the following axioms :

MCM 1 

MCM 2 

are commutative diagrams.

Proposition 1.1. *For each object A from V_0, $((AA), j_A, L^A_{AA})$ is a closed monoid.*

Proof. Straightforward from axioms [CC1−3] for the closed structure of $\underline{V}$.

We shall denote by *MonV* the category of the closed monoids over $\underline{V}$ and of the morphisms of closed monoids.

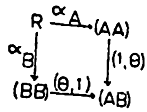
Remark. A closed structure on *MonV* seems to be difficult to be found. For this purpose one could expect to use ends.

— Let (R, e, n) be a closed monoid over $\underline{V}$.

Definition 1.3. An object A in V_0 together with a morphism $\alpha_A : (R, e, n) \to ((AA), j_A, L^A_{AA})$ of closed monoids is called a *closed left R-module over* $\underline{V}$.

DEFINITION 1.4. If (A, α_A) and (B, α_B) are two left R-modules over $\underline{V}$, $\theta : (A, \alpha_A) \to (B, \alpha_B)$ is a *morphism of closed left R-modules* if $\theta : A \to B$ is a morphism in V_0 which satisfies the axiom

$CR1$



We shall denote by ${}^{c}_{R}MV$ the category of the closed left R-modules with the morphisms of closed left R-modules.

PROPOSITION 1.2. *R admits a canonic structure of closed left R-module ovcr $\underline{V}$.*

Proof. Obviously, $n : (R, e, n) \to ((RR), j_R, L^R_{RR})$ is a morphism of closed monoids.

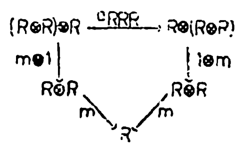
PROPOSITION 1.3. *Each object A in V_0 has a canonic structure of (AA)-module, over the closed monoid defined in proposition 1.1.*

Proof. The identity of (AA) gives the required structure.

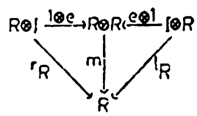
— Let $\underline{V}$ be a monoidal category.

DEFINITION 1.5. (B é n a b o u) A *monoidal monoid* (R, e, m) *over* $\underline{V}$ consists of the following data: an object R in V_0 and two morphisms $e : I \to R$, $m : R \otimes R \to R$ in V_0. These data are to satisfy the following axioms

$MM1$



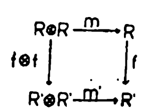
$MM2$



are commutative diagrams.

DEFINITION 1.6. If (R, e, m) and (R', e', m') are two monoidal monoids over $\underline{V}$, a *morphism* $f : (R, e, m) \to (R', e', m')$ *of monoidal monoids* is a morphism $f : R \to R'$ in V_0 which satisfies the axiom $MMM1 = MCM1$ and the axiom

$MMM2$



PROPOSITION 1.4. *$((AA), j_A, M^A_{AA})$ is a monoidal monoid over $\underline{V}$, for each object A of V_0.*

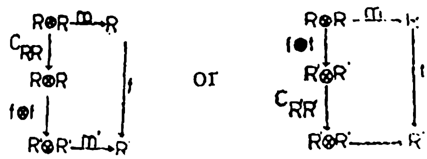
Proof. Obvious by axioms $[VC1']$, $[VC2']$ and $[VC3']$ for the special case $\underline{A} = \underline{V}$.

We shall denote by ${}^m MonV$ the category of the monoidal monoids over $\underline{V}$ and of the morphisms of monoidal monoids.

PROPOSITION 1.5 (MacLane) — *If $\underline{V}$ is a symmetric monoidal category then so is ${}^m MonV$.*

We refer to [4] for the proof of this result.

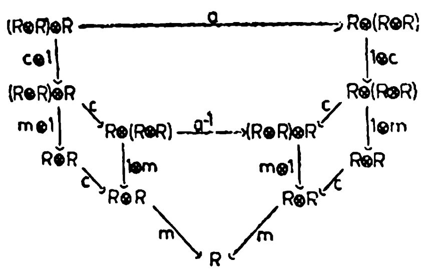
DEFINITION 1.7. Let $\underline{V}$ be a symmetric monoidal category. An *antimorphism $f: (R, e, m) \to (R', e', m')$ of monoidal monoids* is a morphism $f: R \to R'$ in V_0 which satisfies the axiom $AMM1 = MMM1 = MCM1$ and the axiom

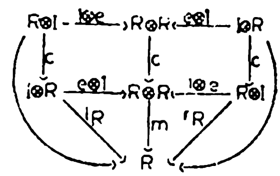
$AMM2$ 

these conditions being equivalent by the naturality of the isomorphisms c

PROPOSITION 1.6. *If (R, e, m) is a monoidal monoid over $\underline{V}$, then $(R, e, m . c_{RR})$ is also a monoidal monoid over $\underline{V}$.*

Proof. The commutativity of the following diagrams





follows by coherence, naturality of the isomorphisms c and by axioms $MM1$ and $MM2$ for the monoidal monoid (R, e, m).

DEFINITION 1.8. If (R, e, m) is a monoidal monoid over $\underline{V}$, the monoidal monoid defined above will be called *the opposite monoid of* (R, e, m).

Obviously we then have

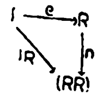
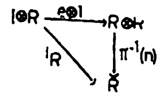
COROLLARY 1.7. *$f: (R, e, m) \to (R', e', m')$ is an antimorphism of monoidal monoids iff f is a morphism from the opposite monoid of (R, e, m) to (R', e', m') or iff f is a morphism of monoidal monoids from (R, e, m) to the opposite monoid of (R', e', m').*

As for the following, we shall consider a (symmetric) monoidal closed category $\underline{V}$. We shall make full use of the bijections

$$\pi_{ABC} : V_0(A \otimes B, C) \to V_0(A(BC)).$$

PROPOSITION 1.8. *If (R, e, n) is a closed monoid over $\underline{V}$ then $(R, e, \pi^{-1}_{RRR}(n))$ is a monoidal monoid over $\underline{V}$. Conversely, if (R, e, m) is a monoidal monoid over $\underline{V}$, then $(R, e, \pi_{RRR}(m))$ is a closed monoid over $\underline{V}$.*

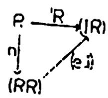
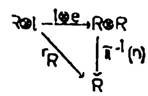
Proof. We shall show that

(a)  implies 

π_{IRR} being a bijection, the commutativity of the right triangle is equivalent to $\pi_{IRR}(l_R) = \pi_{IRR}(\pi^{-1}(n) \cdot e \otimes 1)$. In proving the last equality we shall use equalities from [chap. II]. Indeed

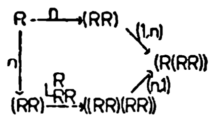
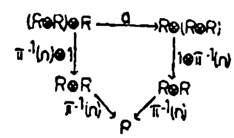
$$\pi(\pi^{-1}(n) \cdot e \otimes 1)^{[(3.1)]} = \pi(\pi^{-1}(n) \cdot e) = n \cdot e = j_R^{[(3.15)]} = \pi(l_R).$$

Next, we have to show that

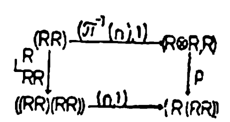
(b)  implies 

Indeed, $\pi(\pi^{-1}(n) \cdot 1 \otimes e)^{[(3.1)]} = (e, 1) \cdot \pi(\pi^{-1}(n)) = (e, 1) \cdot n = i_R^{[(3.17)]} = \pi(r_R)$.

Similarly,

(c)  implies 

In this case, we shall prove that $\pi_{R,R,(RR)}\pi_{R \otimes R,R,R}$ applied to the right diagram is commutative. We shall use [(3.22)] in the following form



Indeed, $\pi(\pi(\pi^{-1}(n) \cdot 1 \otimes \pi^{-1}(n) \cdot a))_{[(3.19)]} = p \cdot \pi(\pi^{-1}(n) \cdot 1 \otimes \pi^{-1}(n))_{[(3.1)]} =$

$$= p \cdot (\pi^{-1}(n), 1) \cdot n^{[(3.22)]} = (n, 1) \cdot L_{RR}^{R} \cdot n = (1, n) \cdot n^{[(3.1)]} =$$

$$= \pi(n \cdot \pi^{-1}(n))^{[(3.1)]} = \pi(\pi(\pi^{-1}(n) \cdot \pi^{-1}(n))^{[(3.1)]} = \pi(\pi(\pi^{-1}(n) \cdot \pi^{-1}(n) \otimes 1)).$$

Similarly, for the second part of the proposition, we only have to use the following equalities

(a′) $\pi(m) \cdot e^{[(3.1)]} = \pi(m \cdot e \otimes 1) = \pi(l_R)^{[(3.15)]} = j_R$

(b′) $(e, 1) \cdot \pi(m)^{[(3.1)]} = \pi(m \cdot e \otimes 1) = \pi(r_R)^{[(3.17)]} = i_R$

(c′) $(1, \pi(m)) \cdot \pi(m)^{[(3.1)]} = \pi(\pi(m) \cdot m)^{[(3.1)]} = \pi\pi(m \cdot m \otimes 1) =$

$$= \pi\pi(m \cdot 1 \otimes m \cdot a)^{[(3.19)]} = p \cdot \pi(m \cdot 1 \otimes m)^{[(3.1)]} = p \cdot (m, 1) \cdot \pi(m)^{[(3.22)]} =$$

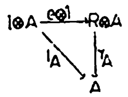
$$= (\pi(m), 1) \cdot L^R_{RR} \cdot \pi(m).$$

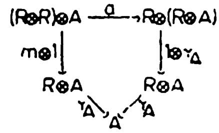
By a simple application of [(3.1)] we obtain

PROPOSITION 1.9. *If $f: (R, e, n) \to (R', e', n')$ is a morphism of closed monoids over $\underline{V}$ then $f: (R, e, \pi^{-1}(n)) \to (R', e', \pi^{-1}(n'))$ is a morphism of monoidal monoids over $\underline{V}$ and conversely.*

According to these two last propositions, in a (symmetric) monoidal closed category $\underline{V}$, we shall identify closed and monoidal monoids, speaking roughly of *monoids over* $\underline{V}$. In doing so, (R, e, n, m) will denote a monoid over $\underline{V}$ such that $\pi(m) = n$. Similarly, we shall use morphisms and antimorphisms of monoids.

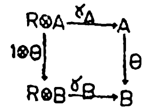
DEFINITION 1.9. (M a c L a n e) — An object A in V_0 together with a morphism $\gamma_A: R \otimes A \to A$ in V_0 is called a *monoidal left R-module over* $\underline{V}$ if the following diagrams are commutative





Remark. A monoidal left R-module may also be defined as an object A in V_0 together with a morphism $\varepsilon_A : (R, e, m) \to ((AA), j_A, M^A_{AA})$ of monoidal monoids (using proposition 1.4). One simply gets the above definition applying π^{-1}.

DEFINITION 1.10 (M a c L a n e) — A morphism $\theta: (A, \gamma_A) \to (B, \gamma_B)$ is called a *morphism of monoidal left R-modules* if $f: A \to B$ is a morphism in V_0 and

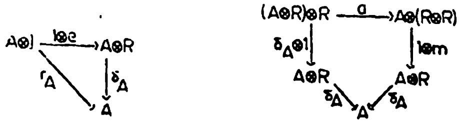


is a commutative diagram.

One can easily show (in a similar way to the last two propositions), that monoidal left R-modules over a symmetric monoidal closed category $\underline{V}$ can be identified with the closed ones. According to this, in what follows, we shall use only the notion of left R-module and morphism of left R-modules. As for monoids, by (A, α_A, γ_A) we shall denote a left R-module with $\alpha_A = \pi(\gamma_A)$; we shall denote by ${}_R M \underline{V}$ the corresponding category.

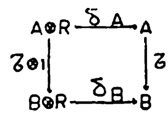
Let $\underline{V}$ be a symmetric monoidal closed category and (R, e, n, m) a fixed monoid over $\underline{V}$.

Definition 1.11. (MacLane) An object A in V_0 together with a morphism $\delta_A : A \otimes R \to A$ in V_0 is called a *right R-module over* $\underline{V}$ if the following diagrams are commutative

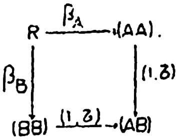


Remark. A right R-module over $\underline{V}$ may also be defined as an object A in V_0 together with an antimorphism of monoids $\beta_A : (R, e, n, m) \to ((AA), j_A, L^A_{AA}, M^A_{AA})$. In proving that these two definitions are equivalent one has to make use of the following natural isomorphisms $s_{RAA} : (R(AA)) \to (A(RA))$ which exist, the V-functor $L^A : \underline{V} \to \underline{V}$ being a self $\underline{V}$-adjoint to the left.

Definition 1.12 (MacLane) — A morphism $\tau : (A, \delta_A) \to (B, \delta_B)$ is called a *morphism of right R-modules* if $\tau : A \to B$ is a morphism in V_0 making commutative the following diagram



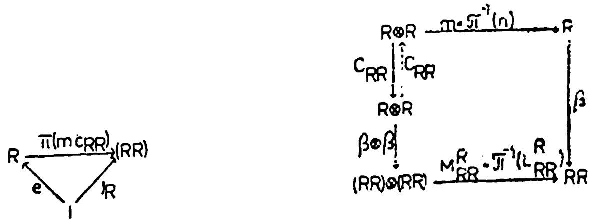
In a similar manner, $\tau : (A, \beta_A) \to (B, \beta_B)$ is a morphism of right R-modules if the following diagram commutes



We shall denote this category with MV_R.

Proposition 1.10. — *R admits a canonic structure of right R-module over* $\underline{V}$.

Proof. We have to verify that $\beta_R = \pi(\pi^{-1}(n) \cdot c_{RR}) : (R, e, n, m) \to ((RR), j_R, L^R_{RR}, M^R_{RR})$ actually is an antimorphism of monoids, i.e. the following diagrams are commutative



Using π^{-1}, the first is equivalent to $l_R = m \cdot c_{RR} \cdot c \otimes 1 = m \cdot 1 \otimes e \cdot c_{IR}$ which is $MM2$ for (R, e, m) by c's naturality and coherence. As for the second, using $[MC6]$ we have $c_{RR} \cdot c_{RR} = 1$ and so we have to check $\beta \cdot \pi^{-1}(n) \cdot c_{RR} = \pi^{-1}(L^R_{RR}) \cdot \beta \otimes \beta$ or $\beta \cdot \pi^{-1}(\beta) = \pi^{-1}(L^R_{RR}) \cdot \beta \otimes \beta$; using [(3.1)] in the application of π we get $CM1$ for β. From proposition 1.8 this is equivalent to $MM1$ for $\pi^{-1}(\beta)$. But this is true by proposition 1.6.

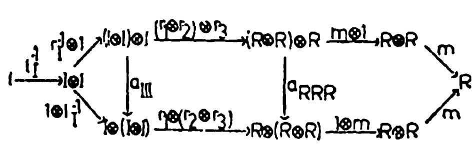
— Let (R', e', n', m') be a monoid over $\underline{V}$ and $f : R' \to R$ a morphism of monoids over $\underline{V}$. If (A, α_A) is a left R-module then using $\alpha_A \cdot f$, f induces on A a structure of left R'-module. $(A, \alpha_A \cdot f)$ is called *the R'-ification of A*. In this way, a morphism $f : R' \to R$ of monoids over $\underline{V}$ induces (details are straightforward) a covariant faithful functor $f : {}_R M V \to {}_{R'} M V$ (identical on morphisms). Now, if $f : R' \to R$ is an antimorphism of monoids over $\underline{V}$, it induces a covariant faithful functor ${}_R M V \to M V_R$. The identity $1_R : (R, e, m) \to (R, e, m \cdot c_{RR})$ is a canonic antimorphism from the monoid (R, e, m) to its opposite. This antimorphism induces two category equivalences ${}_{R^{op}} M V \to M V_R$ and ${}_R M V \to M V_{R^{op}}$, equivalences which permit, in a symmetric monoidal closed category $\underline{V}$, the well-known identifications.

One can now define and recover the basic situations for bimodules. Details are now straightforward.

2. **"Enriched" versions of classical results.** Let $\underline{V}$ be a monoidal closed category and (R, e, n, m) a fixed monoid over $\underline{V}$.

PROPOSITION 2.1. *The following data form a category $[R]$ with a single object R:*
(i) $[R](R, R) = V_0(I, R)$; (ii) *for each $r_1, r_2 \in [R](R, R)$ the "composition" is $r_1 * r_2 = m \cdot r_1 \otimes r_2 \cdot l_I^{-1} = m \cdot r_1 \otimes r_2 \cdot r_I^{-1} : I \to R$;* (iii) *the morphism $e : I \to R$ is "the identity" in $[R](R, R)$.*

Proof. The associativity of the composition $(r_1 * r_2) * r_3 = r_1 * (r_2 * r_3)$ can be deduced from the following diagram



where regions commute by $[MC2, MC5]$, naturality of a and $MM1$ for the monoid (R, e, m). Using $r_I = l_I$ and the naturality of the isomorphisms r and l, one easily checks that e actually is an identity.

Using this result and the well-known representation of the subjacency functor $V : \underline{V} \to Ens$ given by [I, 2.1] one gets

COROLLARY 2.2. *The following data form a category $[R]'$ with a single object* R:

(i) $[R]'(R, R) = V(R)$; (ii) *for each $r_1, r_2 \in V(R)$ "the composition" is $r_1 \square r_2 = Vi_R^{-1}(Vi_R^{-1}(r_1) * Vi_R^{-1}(r_2))$;* (iii) *the element $Vi_R^{-1}(e)$ is "the identity" in $[R]'(R, R)$. The categories $[R]$ and $[R]'$ are isomorphic.*

PROPOSITION 2.3. *The following data form a $\underline{V}$-category $\{R\}$: (i) obj $\{R\}$: only R; (ii) $\{R\}(R, R) = R$, object in V_0; (iii) a morphism $j_R : I \to \{R\}(R, R)$, namely $e : I \to R$; (iv) a morphism $M^R_{RR} : \{R\}(R, R) \otimes \{R\}(R, R) \to \{R\}(R, R)$ namely $m : R \otimes R \to R$.*

Proof. Axioms of $\underline{V}$-category are easily seen to coincide with the monoid axioms.

More important for what follows is now

THEOREM 2.4. *There is a canonic identification between the category ${}_RMV$ of left R-modules and morphisms of left R-modules and the category of the $\underline{V}$-functors and the $\underline{V}$-natural transformations from the $\underline{V}$-category $\{R\}$ to $\underline{V}$ (as $\underline{V}$-category over itself).*

Proof. A $\underline{V}$-functor $T : \{R\} \to \underline{V}$ consists of a function obj $\{R\} \to$ obj$\underline{V}$, that is, an object $T(R) = A$ in V_0 and a morphism $T_{RR} : \{R\}(R, R) \to (AA)$, that is, $T_{RR} = \alpha_A : R \to (AA)$ a morphism in V_0, such that

are commutative. But this proves that $\alpha_A : (R, e, m) \to ((AA), j_A, M^A_{AA})$ is a morphism of monoids. The rest of this proof goes similarly.

COROLLARY 2.5. *There is a canonic identification between the category MV_R of the right R-modules and the morphisms of right R-modules and the category of the $\underline{V}$-functors and the $\underline{V}$-natural transformations from the $\underline{V}$-category $\{R^{op}\}$ to $\underline{V}$.*

— Before giving the next result, we shall introduce here an obvious functor of subjacency $W : {}_RMV \to V_0$ defined by $W(A, \alpha_A) = A$, $W(f) = f$. W is obviously a faithful functor.

PROPOSITION 2.6. *If V_0 is complete, so is ${}_RMV$.*

Proof. This result can be derived from a more general one from [2].

We also derive

COROLLARY 2.7. *The functor W reflects monomorphisms, epimorphisms and limits.*

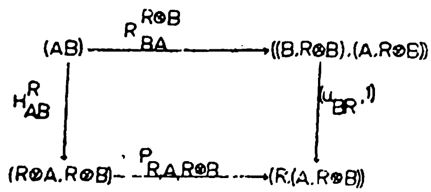
Remark. At this point two directions could also be followed: conditions on $\underline{V}$ in which some properties of V_0 are inherited by ${}_RMV$; the second, the construction of a left adjoint for the functor W, or, the construction of the "free" left R-module over an arbitrary object of V_0. Satisfactory results can be found in [4].

In what follows, we suppose that $\underline{V}$ is a symmetric monoidal closed category and that V_0 *has equalizers.*

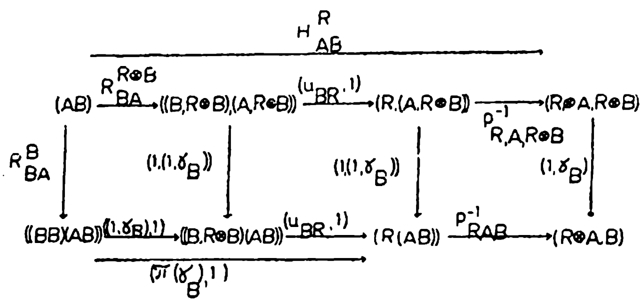
LEMMA 2.8. *Let* (A, α_A, γ_A) *and* (B, α_B, γ_B) *be two left R-modules. We have*

$$\mathrm{equ}\,((1_{R\otimes A}, \gamma_B)\cdot H^R_{AB}, (\gamma_A, 1_B)) \cong \mathrm{equ}((\alpha_B, 1_{(AB)})\cdot R^B_{BA}, (\alpha_A, 1_{(AB)})\cdot L^A_{AB})$$

Proof. First, since $\alpha_A = \pi(\gamma_A)$, an easy application of [II, (3.22)] shows that $(\gamma_A, 1_B) = p^{-1}_{RAB}\cdot(\alpha_A, 1_{(AB)})\cdot L^A_{AB}$. We show that $p^{-1}_{RAB}\cdot(\alpha_B, 1_{(AB)})\cdot R^B_{BA} = (1_{R\otimes A}, \gamma_B)\cdot H^R_{AB}$ and this will prove our lemma modulo the isomorphism p_{RAB}.
From [p. 545] we already have the following commutative diagram



Further, the following diagram commutes by the naturality of R and p^{-1}



using for the bottom triangle [II, (3.4)]. But the exterior is just the required equality.

We shall denote by $\{AB\}$ and call *the object of the morphisms of R-modules between* (A, α_A, γ_A) *and* (B, α_B, γ_B), the object defined by the previous lemma.

Remark. If V preserves equalizers, and $f \in V(\{AB\})$, one has for the first member in lemma 3.1

$$V_0(\gamma_A, 1_A)(f) = V_0(1_{R\otimes A}, \gamma_B)(V(H^R_{AB})(f)) = V_0(1_{R\otimes A}, \gamma_B)(1_R \otimes f)$$

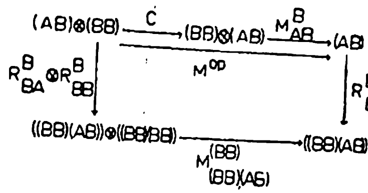
which is the definition of the morphism of monoidal left R-modules. Similarly, using the second member, one finds the definition of the morphism of closed left R-modules.

Hence, if V preserves equalizers, $V(\{AB\}) = {}_R MV((A, \alpha_A), (B, \alpha_B))$.

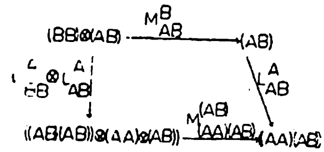
— We then have a subobject (in V_0) $\{AB\}$ of (AB). The corresponding monomorphism will be denoted by $\mathrm{equ}_{AB}: \{AB\} \to (AB)$.

LEMMA 2.9. *We have* $(\alpha_B, 1_{(AB)})\cdot R^B_{BA}\cdot M^B_{AB}\cdot \mathrm{equ}_{BB}\otimes \mathrm{equ}_{AB} = (\alpha_A, 1_{(AB)})\cdot L^A_{AB}\cdot M^B_{AB}\cdot \mathrm{equ}_{BB}\otimes \mathrm{equ}_{AB}$, *i.e.,* $M^B_{AB}\cdot \mathrm{equ}_{BB}\otimes \mathrm{equ}_{AB}$ *factors through* $\{AB\}$.

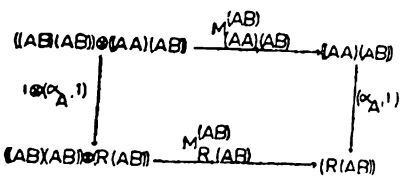
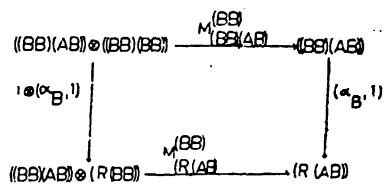
Proof. We shall use, among others, the following five facts: the $\underline{V}$-functoriality of $R^B : \underline{V}^{op} \to \underline{V}$, that is, the commutativity of



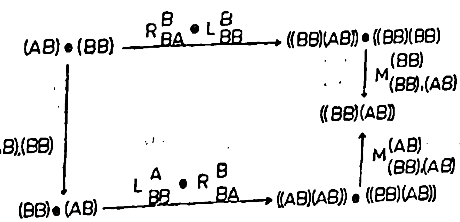
the $\underline{V}$-functoriality of $L^A : \underline{V} \to \underline{V}$, that is, the commutativity of



the naturality of M, that is, the commutativity of the next diagrams

and the commutativity of the following diagram, derived from [III, (4.4)]



Using all these, we have $(\alpha_B, 1) \cdot R^B_{BA} \cdot M^B_{AB} \cdot \mathrm{equ}_{BB} \otimes \mathrm{equ}_{AB} =$

$= (\alpha_B, 1) \cdot M^{(BB)}_{(BB),(AB)} \cdot R^B_{BA} \otimes R^B_{BB} \cdot c_{(BB),(AB)} \cdot \mathrm{equ}_{BB} \otimes \mathrm{equ}_{AB} =$

$= M^{(RR)}_{R,(AB)} \cdot 1 \otimes (\alpha_B, 1) \cdot R^B_{AB} \otimes R^B_{BB} \cdot \mathrm{equ}_{AB} \otimes \mathrm{equ}_{BB} \cdot c_{(AB),(BB)} =$

$= (\alpha_B, 1) \cdot M^{(BB)}_{(BB),(AB)} \cdot R^B_{BA} \otimes L^B_{BB} \cdot \mathrm{equ}_{AB} \otimes \mathrm{equ}_{BB} \cdot c_{(AB),(BB)} =$

$= (\alpha_B, 1) \cdot M^{(AB)}_{(BB),(AB)} \cdot L^A_{BB} \otimes R^B_{BA} \cdot c_{(AB),(BB)} \cdot \mathrm{equ}_{AB} \otimes \mathrm{equ}_{BB} \cdot c_{(AB),(BB)} =$

$= (\alpha_B, 1) \cdot M^{(AB)}_{(BB),(AB)} \cdot L^A_{BB} \otimes R^B_{BA} \cdot \mathrm{equ}_{BB} \otimes \mathrm{equ}_{AB} =$

$= M^{(AB)}_{R,(AB)} \cdot 1 \otimes (\alpha_B, 1) \cdot L^A_{BB} \otimes R^B_{BA} \cdot \mathrm{equ}_{BB} \otimes \mathrm{equ}_{AB} =$

$= M^{(AB)}_{R,(AB)} \cdot 1 \otimes (\alpha_A, 1) \cdot L^A_{BB} \otimes L^A_{AB} \cdot \mathrm{equ}_{BB} \otimes \mathrm{equ}_{AB} =$

$= (\alpha_A, 1) \cdot M^{(AB)}_{(AA),(AB)} \cdot L^A_{BB} \otimes L^A_{AB} \cdot \mathrm{equ}_{BB} \otimes \mathrm{equ}_{AB} =$

$= (\alpha_A, 1) \cdot L^A_{AB} \cdot M^B_{AB} \cdot \mathrm{equ}_{BB} \otimes \mathrm{equ}_{AB}.$

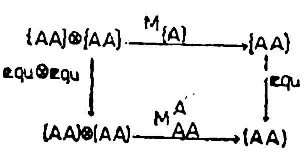
We now prove our main theorem

THEOREM 2.10. *Let (R, c, n, m) be a monoid over $\underline{V}$ and (A, α_A, γ_A) be a left R-module. $\{AA\}$ admits a structure of monoid such that equ_{AA} is a morphism of monoids over $\underline{V}$.*

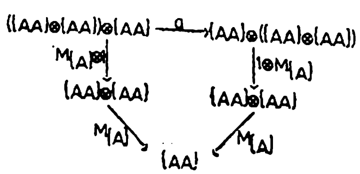
Proof. The components of this monoid are denoted by $j_{\{A\}} : I \to \{AA\}$ and $M_{\{A\}} : \{AA\} \otimes \{AA\} \to \{AA\}$.

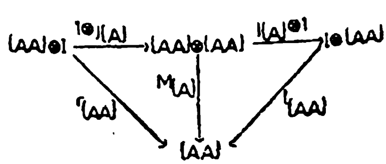
We shall prove that $(\gamma_A, 1) \cdot j_A = (1_{R\otimes A}, \gamma_A) \cdot H^R_{AA} \cdot j_A$ and derive from here a factorization of $j_A : I \to (AA)$ through $\{AA\}$ which yields our $j_{\{A\}}$. Indeed, one reads from the diagram of [p. 483] that $K^R_{AA} \cdot j_A = j_{A\otimes R}$ and hence $H^R_{AA} \cdot j_A = j_{R\otimes A}$ using the naturality of j. The stated equality now follows using again the naturality of j. Thus, we have $j_A = \mathrm{equ}_{AA} \cdot j_{\{A\}}$.

Next, using the previous lemma for $A = B$, we derive the existence of $M_{\{A\}}$ on the commutative diagram



So, if we show that $(\{AA\}, j_{\{A\}}, M_{\{A\}})$ is a monoid over $\underline{V}$, the above equality and commutative diagram will show that $\mathrm{equ}_{AA} : (\{AA\}, j_{\{A\}}, M_{\{A\}}) \to ((AA), j_A, M^A_{AA})$ actually is a morphism of monoids. We must check $MM1$ and $MM2$, that is, the commutativity of the following diagrams





As for the first, the equality to be checked is equivalent to the one obtained by composition to the left with equ_{AA}. We then have to verify $M^A_{AA} \cdot (M^A_{AA} \cdot \mathrm{equ} \otimes \mathrm{equ}) \otimes \mathrm{equ} = M^A_{AA} \cdot \mathrm{equ} \otimes (M^A_{AA} \cdot \mathrm{equ} \otimes \mathrm{equ}) \cdot a$. Applying $\pi\pi$ to both members, we successively have (using the naturality of L, [CC3], [II, (3.19) and (3.22)])

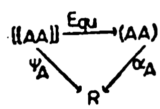
$$\begin{aligned}\pi\pi(M^A_{AA}\cdot(M^A_{AA}\cdot \mathrm{equ}\otimes \mathrm{equ})\otimes \mathrm{equ}) &= \pi((\mathrm{equ},1)\cdot L^A_{AA}\cdot M^A_{AA}\cdot \mathrm{equ}\otimes \mathrm{equ}) =\\ &= (\mathrm{equ},(\mathrm{equ},1)\cdot L^A_{AA})\cdot L^A_{AA}\cdot \mathrm{equ} = (\mathrm{equ},(\mathrm{equ},1))\cdot(1,L^A_{AA})\cdot L^A_{AA}\cdot \mathrm{equ} =\\ &= (\mathrm{equ},(\mathrm{equ},1))\cdot(L^A_{AA},1)\cdot L^{(AA)}_{(AA),(AA)}\cdot L^A_{AA}\cdot \mathrm{equ} =\\ &= (\mathrm{equ},1)\cdot(L^A_{AA},1)\cdot(1,(\mathrm{equ},1))\cdot L^{(AA)}_{(AA),(AA)}\cdot L^A_{AA}\cdot \mathrm{equ} =\\ &= (\mathrm{equ},1)\cdot(L^A_{AA},1)\cdot((\mathrm{equ},1),1)\cdot L^{\{AA\}}_{(AA),(AA)}\cdot L^A_{AA}\cdot \mathrm{equ} =\\ &= (\pi(M^A_{AA}\cdot \mathrm{equ}\otimes \mathrm{equ}),1)\cdot L^{\{AA\}}_{(AA),(AA)}\cdot L^A_{AA}\cdot \mathrm{equ} =\\ &= p\cdot(M^A_{AA}\cdot \mathrm{equ}\otimes \mathrm{equ},1)\cdot L^A_{AA}\cdot \mathrm{equ} = p\cdot\pi(M^A_{AA}\cdot \mathrm{equ}\otimes(M\cdot \mathrm{equ}\otimes \mathrm{equ})) =\\ &= \pi\pi(M^A_{AA}\cdot \mathrm{equ}\otimes(M^A_{AA}\cdot \mathrm{equ}\otimes \mathrm{equ})\cdot a).\end{aligned}$$

Finally, the last two equalities are equivalent (by left composition with equ_{AA}) to $\mathrm{equ}\cdot r_{\{AA\}} = M^A_{AA}\cdot \mathrm{equ}\otimes j_A$, $\mathrm{equ}\cdot l_{\{AA\}} = M^A_{AA}\cdot j_A\otimes \mathrm{equ}$. Applying π to these, [II, (3.1), (3.15), (3.17) and (6.2)] and also axioms [$CC1$] and [$CC2$], one easily gets $(1,\mathrm{equ})\cdot i_{\{AA\}} = i_{(AA)}\cdot \mathrm{equ}$, $(1,\mathrm{equ})\cdot j_{\{AA\}} = (\mathrm{equ},1)\cdot j_{(AA)}$, true by naturality of i and j.

COROLLARY 2.11. *Each left R-module (A,α_A) has a canonic structure of left $\{AA\}$-module, namely (A,equ_{AA}).*

DEFINITION 2.1. The monoid $(\{AA\}, j_{\{A\}}, M_{\{A\}})$ is called the *monoid of the R-endomorphisms of* (A,α_A). One can now iterate this construction getting the *monoid of the biendomorphisms of the left R-module* (A,α_A). More exactly, this monoid is $\{\{AA\}\} = \mathrm{Equ}\,((\mathrm{equ},1)\cdot R^A_{AA}, (\mathrm{equ},1)\cdot L^A_{AA})$ where $\mathrm{Equ}:\{\{AA\}\}\to(AA)$ is a morphism of monoids over $\underline{V}$.

THEOREM 2.12. *If (A,α_A) is a left R-module, then there exists a canonic morphism of monoids over $\underline{V}$, $\psi_A: R\to\{\{AA\}\}$, such that the diagram*



is commutative.

Proof. By the equ_{AA}'s definition we have $(\alpha_A,1)\cdot R^A_{AA}\cdot \mathrm{equ} = (\alpha_A,1)\cdot L^A_{AA}\cdot \mathrm{equ}$. In fact, we must prove that α_A factors through Equ, i.e., $(\mathrm{equ},1)\cdot L^A_{AA}\cdot\alpha_A = (\mathrm{equ},1)\cdot R^A_{AA}\cdot\alpha_A$.

Using the following analogous of [II, (3.1)]: $\pi^{-1}((g,h)xf) = h\cdot\pi^{-1}(x)\cdot f\otimes g$ and applying π^{-1} to the equalities above we get the following equivalent ones

(1) $\qquad M^A_{AA}\cdot c_{(AA),(AA)}\cdot \mathrm{equ}\otimes\alpha_A = M^A_{AA}\cdot \mathrm{equ}\otimes\alpha_A$, respectively

(2) $M^A_{AA}\cdot\alpha_A\otimes \mathrm{equ} = M^A_{AA}\cdot c_{(AA),(AA)}\cdot\alpha_A\otimes \mathrm{equ}$. Using $c^{-1}_{(AA),(AA)} = c_{(AA),(AA)}$ and the naturality of c, one obtains (2) by a right composition of (1) with $c_{R,\{AA\}}$.

Finally, we show that $\psi_A : R \to \{\{AA\}\}$ is a morphism of monoids over $\underline{V}$. First, $\psi_A \cdot e = j_{\{\{A\}\}}$ is easily seen to be equivalent to $\alpha_A \cdot e = j_A$ by a simple left composition with Equ. Next, $\psi_A \cdot m = M_{\{\{A\}\}} \cdot \psi_A \otimes \psi_A$ combined with Equ $\cdot M_{\{\{A\}\}} = M^A_{AA} \cdot$ Equ $\otimes$ Equ (true, Equ being morphism of monoids over $\underline{V}$) is seen to be equivalent with $M^A_{AA} \cdot \alpha_A \otimes \alpha_A = \alpha_A \cdot m$ which is true, α_A being a morphism of monoids over $\underline{V}$.

— We shall call $\psi_A : R \to \{\{AA\}\}$ the *canonic morphism* from R to the monoid of the biendomorphisms of (A, α_A).

Remark. Using ψ_A one can define faithful and balanced left R-modules.

We conclude this section with the enriched version of a wellknown result:

PROPOSITION 2.13. $\{\{\{AA\}\}\} = \{AA\}$.

Proof. We have $\{\{\{AA\}\}\} = \mathcal{E}$qu $((\text{Equ}, 1) \cdot R^A_{AA}, \ (\text{Equ}, 1) \cdot L^A_{AA})$ and $\{AA\}$=equ $((\alpha_A, 1) \cdot R^A_{AA}, \ (\alpha_A, 1) \cdot L^A_{AA})$, these being subobjects of (AA).

We only have to check the following two equalities $(\alpha_A, 1) \cdot R^A_{AA} \cdot \mathcal{E}$qu $= (\alpha_A, 1) \cdot L^A_{AA} \cdot \mathcal{E}$qu, $(\text{Equ}, 1) \cdot R^A_{AA} \cdot$ equ $= (\text{Equ}, 1) \cdot L^A_{AA} \cdot$ equ. Using $\alpha_A =$ Equ $\cdot \psi_A$, the first one obviously follows from the definition of $\mathcal{E}$qu. The second can be deduced from $(\text{equ}, 1) \cdot R^A_{AA} \cdot$ Equ $= (\text{equ}, 1) \cdot L^A_{AA} \cdot$ Equ (which is true by Equ's definition) in a very analogous manner to the first part of the proof of theorem 2.12.

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REFERENCES

1. Bénabou, J., *Algèbre élémentaire dans les catégories avec multiplication*, C.R. Acad. Sci., Paris, **258**, 1964, 771–774.
2. Bunge, M., *Relative functor categories and categories of algebras*, J. of Algebra, **11**, 1969, 64–101.
3. Eilenberg, S., G. M. Kelly, *Closed categories*, Proc. Conf. on Categorical Algebra (La Jolla, 1965), Springer Verlag.
4. MacLane, S., *Categorical Algebra*, Bull. Amer. Math. Soc. **71**, 1965, 40–106.

ASUPRA UNEI TEORII ÎMBOGĂȚITE A MODULELOR (I)

(Rezumat)

Stimulat de excelenta monografie despre categoriile închise, monoidale și monoidal închise a lui Eilenberg și Kelly [3], autorul elaborează partea închisă și monoidal închisă a teoriei modulelor peste un monoid fixat, teorie pentru care MacLane a elaborat în [4] partea monoidală în condiții mai restrictive.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXIV, 2, 1979

A COMMON FIXED POINT THEOREM FOR A SEQUENCE OF MULTIFUNCTIONS

VALERIU POPA*

The purpose of this note is to prove a common fixed point theorem for a sequence of multifunction on a nonempty complete metric space (X, d), which generalizes results of Rus [1], Avram [2], Ray [3], [4], Iséki [5] and Wong [6]. The method used is a combination of methods used in [7] and [8].

We denote by $CB(X)$ the set of all nonempty closed bounded subsets of X, and by H the Hausdorff-Pompeiu metric on $CB(X)$

$$H(A, B) = \max [\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)]$$

where $A, B \in CB(X)$ and

$$d(x, A) = \inf_{y\in A} d(x, y).$$

Let $A, B \in CB(X)$ and $k > 1$. In what follows, the following well-known fact will be used: For each $a \in A$, there is a, $b \in B$ such that

$$d(a, b) \leqslant k \cdot H(A, B).$$

Let $T : X \to X$ a multifunction. Denote $F(T) = \{x \in X;\ x \in T(x)\}$.

LEMMA. *Let $T_0, T : (X, d) \to CB(X)$ be two multifunctions. If*

$$H(T_0x, Ty) \leqslant a_1d(x, y) + a_2d(x, T_0x) + a_3d(y, Ty) + a_4d(x, Ty) + + a_5d(y, T_0x) \tag{1}$$

holds for non-negative a_i, $i = 1, \ldots, 5$ with

$$\sum_{i=1}^{5} a_i < 1 \tag{2}$$

for all $x, y \in X$ and $F(T_0) \neq \emptyset$, then $F(T) = F(T_0)$.

Proof. If $u \in F(T_0)$, then by (1)

$$d(u, Tu) \leqslant H(T_0u, Tu) \leqslant (a_3 + a_4)\, d(u, Tu)$$

which implies $d(u, Tu) = 0$. Since Tu is closed, this shows that $u \in Tu$, which implies $F(T_0) \subset F(T)$. Analogously, $F(T) \subset F(T_0)$.

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THEOREM. *Let (X,d) be a nonempty complete metric space and $\{T_n\}_{n\in N}$ a sequence of multifunctions of X into $CB(X)$. Suppose there are nonnegative real uumbers a_i, $i = 1, \dots, 5$, such that*

$$H(T_1x, T_ny) \leqslant a_1d(x, y) + a_2d(x, T_1x) + a_3d(y, T_ny) + a_4d(x, T_ny) + a_5d(y, T_1x) \quad (3)$$

for all $x, y \in X$ and $n \geqslant 2$

$$\sum_{i=1}^{5} a_i < 1 \quad (2)$$

$$a_2 = a_3 \text{ or } a_4 = a_5 \quad (4)$$

Then $\{T_n\}_{N\in N}$ has common fixed points and $F(T_1) = F(T_n)$.

Proof. Choose a real number k with

$$1 < k < \min\left[\frac{1}{a_3 + a_4};\ \frac{1}{a_2 + a_5};\ \frac{1}{\sqrt{\sum_{i=1}^{5}}}\right]. \quad (5)$$

Let $x_0 \in X$ and $x_1 \in T_1x_0$. Then there is an $x_2 \in T_2x_1$ so that $d(x_1, x_2) \leqslant$ $\leqslant kH(T_1x_0, T_2x_1)$. Suppose $x_3, \dots, x_{2n-1}, x_{2n}, \dots$ could be chosen so that

$$x_{2n-1} \in T_1x_{2n-2};\ x_{2n} \in T_2x_{2n-1} \quad \text{and}$$

$$d(x_{2n-1}, x_{2n}) \leqslant kH(T_1x_{2n-2}, T_2x_{2n-1});$$

$$d(x_{2n-2}, x_{2n-1}) \leqslant kH(T_1x_{2n-2}, T_2x_{2n-3})$$

By (3) for $x = x_{2n-2}$, $y = x_{2n-1}$ we have

$$d(x_{2n-1}, x_{2n}) \leqslant kH(T_1x_{2n-2}, T_2x_{2n-1}) \leqslant k[a_1d(x_{2n-2}, x_{2n-1}) + a_2d(x_{2n-2}, T_1x_{2n-2}) + a_3d(x_{2n-1}, T_2x_{2n-1}) + a_4d(x_{2n-2}, T_2x_{2n-1}) + a_5d(x_{2n-1}, T_1x_{2n-2})] \leqslant k[a_1d(x_{2n-2}, x_{2n-1}) + a_2d(x_{2n-2}, x_{2n-1}) + a_3d(x_{2n-1}, x_{2n}) + a_4d(x_{2n-1}, x_{2n}) + a_5d(x_{2n-1}, x_{2n-1})]$$

which implies

$$d(x_{2n-1}, x_{2n}) \leqslant rd(x_{2n-2}, x_{2n-1})$$

where

$$r = \frac{k(a_1 + a_2 + a_4)}{1 - k(a_3 + a_4)}.$$

Analogously

$$d(x_{2n-2}, x_{2n-1}) \leqslant sd(x_{2n-2}, x_{2n-3})$$

where

$$s = \frac{k(a_1 + a_3 + a_5)}{1 - k(a_2 + a_5)}$$

and by (5) $r, s \in R_+$.

Repeating the above argument, we obtain

$$d(x_{2n-1}, x_{2n}) \leqslant (rs)^n d(x_0, x).$$

By (2), (4) and (5) $rs < 1$.

Then by routine calculation we can show that x_n is Cauchy sequence and, since X is complete, we have $\lim_{n\to\infty} x_n = u$ for some $u \in X$.

Now if $n \in N$, (3) implies

$$d(x_{2n}, T_1u) \leqslant H(T_2x_{2n-1}, T_1u) \leqslant a_1d(x_{2n-1}, u) + a_2d(x_{2n-1}, T_2x_{2n-1}) + \\ + a_3d(u, T_1u) + a_4d(x_{2n-1}, T_1u) + a_5d(u, T_2x_{2n-1}) \leqslant a_1d(x_{2n-1}, u) + \\ + a_2d(x_{2n-1}, x_{2n}) + a_3d(u, T_1u) + a_4d(x_{2n-1}, T_1u) + a_5d(u, x_{2n}).$$

Hence by letting $n \to \infty$, we obtain

$$d(u, T_1u) \leqslant (a_3 + a_4)d(u, T_1u)$$

which implies $d(u, T_1u) = 0$. Since T_1u is closed, this shows that $u \in T_1u$. By (2), (3) and lemma $u \in T_nu$ and $F(T_1) = F(T_n)$. This completes the proof of the theorem.

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REFERENCES

1. Rus, A. I., *Fixed points theorems for multivalued mappings in complete metric spaces*, Math. Japonica, **25**, 1975, 21–24.
2. Avram, M., *Points fixes communs pour les applications multivoques dans les espaces métriques*, Mathematica, **17** (40), 1975, 153–156.
3. Ray, K. B., *A note on multi-valued contraction mappings*, Atti Acad. Naz. Lincei, Rend. Classe fis., mat. natur., serie VIII, LVI, fasc. 4, 1974, 500–503.
4. Ray, K. B., *On Ciric's fixed point theorem*, Fundamenta Math., XCIV, 1977, 221–229.
5. Iséki, K., *Multi-valued contraction mappings in complete metric* space, Rend. Sem. Mat. Univ. Padova, **53**, 1975, 15–19.
6. Wong, Song Chi, *Common fixed points of two mappings*, Pacific J. Math., **48**, 1, 1973, 299–312.
7. Kita, Toru, *A common fixed point theorem for multivalued mappings*, Math. Japonica, **22**, 1977, 113–116.
8. Achari, J., *On a family of maps and fixed points*, Comptes rendus de l'Académie bulgare des sciences, **30**, 2, 1977, 171–174.

O TEOREMĂ DE PUNCT FIX PENTRU UN ŞIR DE MULTIFUNCŢII

(Rezumat)

Se demonstrează următoarea

TEOREMĂ. Fie (X, d) un spaţiu metric complet şi $\{T_n\}_{n \in N}$ un şir de multifuncţii definite pe X cu valori în $CB(X)$. Presupunem că există numerele $a_1, \ldots, a_4 \in R_+$ satisfăcînd relaţiile (2) (3) şi (4). Atunci şirul $\{T_n\}_{n \in N}$ are puncte fixe comune şi $F(T_1) = F(T)_n$.

Teorema generalizează rezultate date de Rus [1], Avram [2], Ray [3], [4], Iséki, [5] şi Wong [6].

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ON ALMOST SEQUENTIAL LOCALLY CONVEX SPACES (II)

CSABA NÉMETHI

We shall make free use of the terminology and notations of the first part [3] of our study concerning *almost sequential* locally convex spaces. These spaces are also called in the literature *convex-sequential*, or *C-sequential*, and have been considered by R. M. Dudley [1], A. Wilansky [7], J. H. Webb [6], R. F. Snipes [4] and J. Mikusiński [2]. Unfortunately, the last two articles were not known to the author when he was preparing the first part. Theorem 1.5 and Corollary 1.8 of [3] can be found in Snipes' paper [4] (we mention that his proofs are quite different from ours). Theorem 2.3 established by the author in [3] yields several examples of non-bornological (in fact even non-*sequentially bornological* [4]) almost sequential spaces, and Snipes' Example 3 in [4] is just a particular case of this result. We also mention that Mikusiński [2] uses sequentially continuous semi-norms instead of sequential neighbourhoods, but the two methods are equivalent. For a good survey on sequential convergence see R. F. Snipes [5].

In this second part of our study it is shown that the category ASLCS of almost sequential spaces is coreflectively singly generated, and further characterizations of the almost sequential spaces are given.

1. **The Category** $\mathcal{S}$. Besides the category LCS of locally convex spaces it will be convenient to consider the category $\mathcal{S}$, whose objects are the couples $(X, \mathcal{C}_0)$, where X is a vector space and C_0 is a set of sequences in X, and the morphisms from $(X, \mathcal{C}_0)$ to $(Y, \mathcal{C}_0')$ are the $(\mathcal{C}_0, \mathcal{C}_0')$ — continuous linear maps of X into Y.

If $\{(X^i, \mathcal{C}_0^i) : i \in I\}$ is a family of objects in $\mathcal{S}$, Y is a vector space, and for each $i \in I$ $f^i : X^i \to Y$ is a linear map, then there exists a unique set $\mathcal{C}_0$ of sequences in Y with the following property: given an object $(Z, \mathcal{C}_0')$ in $\mathcal{S}$ and a linear map $g : Y \to Z$, g is $(\mathcal{C}_0, \mathcal{C}_0')$-continuous iff each $g \circ f^i : X^i \to Z$ is $(\mathcal{C}_0^i, \mathcal{C}_0')$-continuous. The set $\mathcal{C}_0$ consists of the sequences (y_n) in Y for which there exist an $i \in I$ and a sequence $(x_n) \in \mathcal{C}_0^i$ such that $y_n = f(x_n)$ for each $n \in N$. We shall say that $\mathcal{C}_0$ is *inductively generated* by the family $\{(\mathcal{C}_0^i, f^i) : i \in I\}$.

It follows from these considerations that colimits in $\mathcal{S}$ can easily be constructed, using colimits in the category of linear spaces and endowing them with the corresponding inductively generated sets of sequences.

THEOREM 1.1. *Consider the functors* $F : \text{LCS} \to \mathcal{S}$ *and* $G : \mathcal{S} \to \text{LCS}$, *where* $F((X, \mathcal{T})) = (X, c_0(\mathcal{T}))$, $G((X, \mathcal{C}_0)) = (X, \tau(\mathcal{C}_0))$, *and on morphisms* F *and* G *are defined naturally. Then* G *is a left adjoint of* F.

Moreover, given $(X, \mathcal{T}) \in \text{Ob(LCS)}$, *the couple*

$$((X, c_0(\mathcal{T})),\ 1_X : (X, \tau c_0(\mathcal{T}))) \to (X, \mathcal{T}))$$

is a G-couniversal morphism for $(X, \mathcal{T})$.

Proof. From Proposition 1.1 in [3] it follows that for each $(X, \mathcal{T}) \in \mathrm{Ob}(LCS)$ the identity map 1_X is a morphism in LCS from $(G \circ F)((X, \mathcal{T}))$ to $(X, \mathcal{T})$, and for each $(X, \mathcal{C}_0) \in \mathrm{Ob}(\mathcal{S})$ the identity map 1_X is a morphism in $\mathcal{S}$ from $(X, \mathcal{C}_0)$ to $(F \circ G)((X, \mathcal{C}_0))$. Hence the maps 1_X are the components of the desired natural transformations $\Phi : G \circ F \to 1_{LCS}$ and $\psi : 1_{\mathcal{S}} \to F \circ G$.

The second affirmation follows by well-known categorical arguments, and we remark that it is equivalent to Proposition 1.4. c) in [3].

COROLLARY 1.2. *For each category $\mathfrak{I}$, the functor G preserves $\mathfrak{I}$-colimits.*

More generally, we have

COROLLARY 1.3. *The functor G preserves inductive generations, in the following sense: Let $\{(X^i, \mathcal{C}_0^i) : i \in I\}$ be a family of objects in $\mathcal{S}$, let Y be a vector space, and for each $i \in I$ let $f^i : X^i \to Y$ be a linear map. If $\mathcal{C}_0$ is the set of sequences in Y inductively generated by the family $\{(\mathcal{C}_0^i, f^i) : i \in I\}$, then $\tau(\mathcal{C}_0)$ is the locally convex topology on Y inductively generated by the family $\{(\tau(\mathcal{C}_0^i), f^i) : i \in I\}$.*

Proof. Let $(Z, \mathcal{T}')$ be a locally convex space and $g : Y \to Z$ a linear map. Our aim is to show that g is $(\tau(\mathcal{C}_0), \mathcal{T}')$-continuous iff each $g \circ f^i : X^i \to Z$ is $(\tau(\mathcal{C}_0^i), \mathcal{T}')$-continuous.

Let $\alpha = ((Z, \mathcal{C}_0'), 1_Z : (Z, \tau(\mathcal{C}_0')) \to (Z, \mathcal{T}'))$ be the G-couniversal morphism for $(Z, \mathcal{T}')$. By the couniversality of α, g is $(\tau(\mathcal{C}_0), \mathcal{T}')$-continuous iff it is $(\mathcal{C}_0, \mathcal{C}_0')$-continuous. Because of the definition of $\mathcal{C}_0$, this is the case iff each $g \circ f^i$ is $(\mathcal{C}_0^i, c_0(\mathcal{T}'))$-continuous. Using again the couniversality of α, the last property is equivalent to the $(\tau(\mathcal{C}_0^i), \mathcal{T}')$-continuity of each $g \circ f^i$.

It is easy to see that Corollaries 1.7 and 1.8 in [3] can be reobtained from the above Corollaries 1.2 and 1.3 respectively.

THEOREM 1.4. *For a locally convex space $(X, \mathcal{T})$ the following three statements are equivalent:*

(a) *$(X, \mathcal{T})$ is almost sequential;*

(b) *for every locally convex space $(Y, \mathcal{T}')$ and every $(\mathcal{T}', \mathcal{T})$-continuous linear map $f : Y \to X$ for which $\tau(c_0(\mathcal{T}))$ is inductively generated by $\{(\tau(c_0(\mathcal{T}')), f)\}$, $\mathcal{T}$ is inductively generated by $\{(\mathcal{T}', f)\}$;*

(c) *for every locally convex space $(Y, \mathcal{T}')$ and every $(\mathcal{T}', \mathcal{T})$-continuous linear map $f : Y \to X$ for which $c_0(\mathcal{T})$ is inductively generated by $\{(c_0(\mathcal{T}'), f)\}$, $\mathcal{T}$ is inductively generated by $\{(\mathcal{T}', f)\}$.*

Proof. (a) $\Rightarrow$ (b). Assuming that $(X, \mathcal{T})$ is almost sequential, let $(Y, \mathcal{T}')$ and f satisfy the hypotheses of (b). Then $\tau(c_0(\mathcal{T})) = \mathcal{T}$ is inductively generated by $\{(\tau(c_0(\mathcal{T}')), f)\}$. Taking into account the $(\mathcal{T}', \mathcal{T})$-continuity of f and the relation $\mathcal{T}' \subseteq \tau(c_0(\mathcal{T}'))$, one can easily deduce that $\{(\mathcal{T}', f)\}$ inductively generates $\mathcal{T}$.

(b) $\Rightarrow$ (c). It is sufficient to apply Corollary 1.3.

(c) $\Rightarrow$ (a). Consider the space $(X, \tau(c_0(\mathcal{T})) = \mathcal{T}')$ and the map 1_X, which is linear and $(\mathcal{T}', \mathcal{T})$-continuous. In addition, $c_0(\mathcal{T})$ is inductively generated by $\{(c_0(\mathcal{T}'), 1_X)\}$, because, by the dual of Corollary 1.2 in [3], $c_0(\mathcal{T}') = c_0(\tau(c_0(\mathcal{T}))) =$

$= c_0(\mathscr{T})$. By the hypothesis $\{(\mathscr{T}', 1_X)\}$ inductively generates $\mathscr{T}$, i.e. $\mathscr{T} = \mathscr{T}'$, hence $\mathscr{T}$ is almost sequential.

2. **The Structure of Almost Sequential Spaces.** We have already seen ([3], Theorem 1.6) that the category ASLCS of almost sequential spaces is coreflective in LCS, hence it is the coreflective hull of some class of locally convex spaces. Next we shall find such a class (other than ASLCS itself).

Consider the vector space l^0 of the sequences $x = (x^j)$, where $x^j \in K$, $j \in N$, and $x^j = 0$ for all but a finite number of values of j. The sequences $e_n = (\delta^j_n)$, $n \in N$, form an algebraic base of l^0. Taking $\mathscr{C}^*_0 = \{(e_n)\}$, let $\mathscr{T}^* = \tau(\mathscr{C}^*_0)$, i.e. $\mathscr{T}^*$ is the finest locally convex topology on l^0 for which (e_n) converges to 0. Hence $\mathscr{T}^*$ is almost sequential. It is also separated, being finer than the topology of componentwise convergence.

PROPOSITION 2.1. *Let $(X, \mathscr{T})$ be a locally convex space, $S = (x_n)$ a sequence in X, and let $f_S : l^0 \to X$ be the linear map for which $f_S(e_n) = x_n$, $n \in N$. We have $(x_n) \in c_0(\mathscr{T})$ iff f_S is $(\mathscr{T}^*, \mathscr{T})$-continuous.*

Proof. Obviously, $(x_n) \in c_0(\mathscr{T})$ iff f_S is $(\mathscr{C}^*_0, c_0(\mathscr{T}))$-continuous. But (see Proposition 1.4.c) in [3]) this means precisely that f_S is $(\tau(\mathscr{C}^*_0) = \mathscr{T}^*, \mathscr{T})$-continuous.

PROPOSITION 2.2. *Let $(X, \mathscr{T})$ be a locally convex space. The almost sequential modification of $\mathscr{T}$ is inductively generated in* LCS *by the family $\{(\mathscr{T}^*, f_S) : S \in c_0(\mathscr{T})\}$, and $X = \bigcup_{S \in c_0(\mathscr{T})} f_S(l^0)$.*

Proof. It is clear that $c_0(\mathscr{T})$ is inductively generated in $\mathscr{S}$ by the family $\{(\mathscr{C}^*_0, f_S) : S \in c_0(\mathscr{T})\}$. Then by Corollary 1.3, $\tau(c_0(\mathscr{T}))$ is inductively generated in LCS by the family $\{(\tau(\mathscr{C}^*_0) = \mathscr{T}^*, f_S) : S \in c_0(\mathscr{T})\}$.

The second affirmation follows from the fact that for every $x \in X$, taking $S = \left(\frac{1}{n} x\right)$, we have $S \in c_0(\mathscr{T})$ and $x = f_S(e_1)$.

THEOREM 2.3. *The category* ASLCS *is the coreflective hull in* LCS *of the space $(l^0, \mathscr{T}^*)$.*

Proof. Let $\mathscr{A}$ be the coreflective hull of $(l_0, \mathscr{T}^*)$. The inclusion $\mathrm{Ob}(\mathscr{A}) \subseteq \mathrm{Ob}(\mathrm{ASLCS})$ follows from the fact that $(l^0, \mathscr{T}^*) \in \mathrm{Ob}(\mathrm{ASLCS})$.

Conversely, let $(X, \mathscr{T}) \in \mathrm{Ob}(\mathrm{ASLCS})$. Then by Proposition 2.2, $\mathscr{T}$ is inductively generated by the family $\{(\mathscr{T}^*, f_S) : S \in c_0(\mathscr{T})\}$, and $X = \bigcup_{S \in c_0(\mathscr{T})} f_S(l^0)$. This implies that the space $(X, \mathscr{T})$ is an extremal quotient object of the locally convex direct sum $(Y, \mathscr{T}') = \bigoplus_{S \in c_0(\mathscr{T})} (l^0, \mathscr{T}^*)$, hence $(X, \mathscr{T}) \in \mathrm{Ob}(\mathscr{A})$. Indeed, if $i_S : l^0 \to Y$, $S \in c_0(\mathscr{T})$ are the canonical injections and $g : Y \to X$ is the unique linear map which satisfies $g \circ i_S = f_S$ for each $S \in c_0(\mathscr{T})$, then g is onto and, together with $\mathscr{T}'$, inductively generates $\mathscr{T}$.

COROLLARY 2.4. *The almost sequential locally convex space $(l^0, \mathscr{T}^*)$ fails to be bornological.*

Proof. If $(l^0, \mathscr{T}^*)$ were bornological, then so would be every almost sequential space, contradicting the fact that there exist non-bornological almost sequential spaces.

THEOREM 2.5. *Let* $(X, \mathcal{T})$ *be an almost sequential locally convex space. For each* $S = (x_n) \in c_0(\mathcal{T})$ *let* $[S]$ *denote the linear subspace of* X *spanned by the set* $\{x_n : n \in N\}$, *let* $\mathcal{T}_S$ *be the subspace topology on* $[S]$ *and* $i_S : [S] \to X$ *the inclusion map. Then the topology* $\mathcal{T}$ *is inductively generated in* LCS *by the family* $\{(\mathcal{T}_S, i_S) : S \in c_0(\mathcal{T})\}$, *and* $X = \bigcup_{S \in c_0(\mathcal{T})} [S]$.

Proof. Clearly, $c_0(\mathcal{T})$ is inductively generated in $\mathcal{S}$ by the family $\{([S], i_S) : S \in c_0(\mathcal{T})\}$. Then by Corollary 1.3, $\tau(c_0(\mathcal{T})) = \mathcal{T}$ is inductively generated in LCS by the family $\{(\tau(\{S\}), i_S) : S \in c_0(\mathcal{T})\}$. Since each i_S is $(\mathcal{T}_S, \mathcal{T})$-continuous and $\mathcal{T}_S \subseteq \tau(\{S\})$, one can easily infer that $\mathcal{T}$ is also inductively generated by the family $\{(\mathcal{T}_S, i_S) : S \in c_0(\mathcal{T})\}$.

For the second affirmation it is sufficient to observe that for each $x \in X$, taking $S = \left(\frac{1}{n} x\right)$, we have $S \in c_0(\mathcal{T})$ and $x \in [S]$.

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REFERENCES

1. Dudley, R. M., *On sequential convergence*, Trans. Amer. Math. Soc., **112**, 1964, 483–507.
2. Mikusiński, J., *Convergence and locally convex spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., **23**, 1975, 1171–1173.
3. Némethi, Cs., *On almost sequential locally convex spaces* (I), Studia Univ. Babeş-Bolyai Math., **23**, 2, 1978, 54–58.
4. Snipes, R. F., *C-sequential and S-bornological topological vector spaces*, Math. Ann., **202**, 1973, 273–283.
5. Snipes, R. F., *Convergence and continuity: the adequacy of filter-bases, the inadequacy of sequences, and spaces in which sequences suffice*, Nieuw Arch. voor Wiskunde, (3) **23**, 1975, 8–31.
6. Webb, J. H., *Sequential convergence in locally convex spaces*, Proc. Cambridge Philos. Soc., **64**, 1968, 341–364.
7. Wilański, A., *Topics in functional analysis*, Lecture Notes in Mathematics, vol. 45, Springer, Berlin-Heidelberg-New York, 1967.

DESPRE SPAŢII LOCAL CONVEXE APROAPE SECVENŢIALE (II)

(Rezumat)

Se stabilesc cîteva teoreme de caracterizare şi de reprezentare a spaţiilor local convexe aproape secvenţiale.

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NUMERISCHE BEHANDLUNG DER VOLTERRA-INTEGRALGLEICHUNGEN MIT SPLINES

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1. **Einführung.** In den letzten Jahren ershien eine grosse Anzahl von Veröffentlichungen, in denen Spline-Approximationen zur Lösung von Differential- und Integralgleichungen verwendet wurden. Diese Arbeiten beschäftigen sich mit sehr verschiedenen Problemen; z. B: Anfangswertproblemen Randwertproblemen, Integral- und Integrodifferentialgleichungs problemen, u.s.w.

Im Zuge der schnellen Entwicklung der Computer-Technik sind praktische numerische Methoden zur Lösung von solchen Problemen wünschenswert.

Von besonderem Interesse sind die Splineapproximationsmethoden, weil sie einige Vorteile gegenüber klassischen numerischen Methoden besitzen.

In dieser Arbeit werden einige Beitrege zur numerischen Lösung der Volterra-Integralgleichungen mit Hilfe von Spline-Funktionen.

Einige Definitionen und einleitende Betrachtungen sind dazu erforderlich.

DEFINITION 1. Die Funktion $s: [a, b] \to R$ wird als Polynon Spline-Funktion für Δ vom Grade m und dem „Defekt" q, $(q \leqslant m)$ bezeichnet, falls die folgenden Beziehungen gelten:

(i) $s \in P_m,\ x \in \,]x_i, x_{i+1}[,\quad 1 \leqslant i \leqslant N$

(ii) $D^k s(x_i -) = D^k s(x_i +)$, für jedes $k,\ 0 \leqslant k \leqslant m - q,\ 1 \leqslant i \leqslant N$

Hierbei ist P_m der lineare Raum der reellen Polynome auf $[a, b]$ deren Grad hochstens m ist, und Δ eine Zerlegung des Intervalls $[a, b]$.

Es sei (S_m, C^k) die Klasse der polynomialen Spline-Funktionen in bezug auf die Zerlegung Δ, die stückweise aus Polynomen vom Grad m bestehen und deren ersten k-Ableitungen $(k \leqslant m)$ in den Knotenpunkten stetig sind.

Loscalzo, Talbot, Schoenberg von 1967 ab haben die Klasse (S_m, C^{m-1}) gebraucht, um die Lösung des Aufangswertproblemes $y' = f(x, y)$, $y(x_0) = y_0$, approximieren zu konnen. Danach erschienen eine gauze Reihe Verallgemeinerungen und Anwendungen der Spline-Methoden für verschiedene Probleme.

In weiteren Verlauf werden die Spline-Funktionen der Klasse (S_m, C^k) zur Approximation der Losung von nicht linearen Volterra-Integralgleichungen verwendet.

2. **Problemstellung.** Betrachtet wird die nichtlineare Volterra Integralgleichung:

$$x(s) = y(s) + \int_a^s K(s, t, x(t))dt, \quad s \geqslant a, \quad (a \in I \subset R) \tag{1}$$

wobei die Funktion x unbekannt ist, und die Funktionen y und K gegebene reelle Funktionen sind.

Es werden folgende Annahmen getroffen:

(i) $y: I \to R$ sei eine genügend glatte Funktion

(ii) $K: I^2 \times R \to R$ sei eine genügend glatte Funktion bezüglich aller Variablen und genüge bezüglich der Variablen x der folgenden Lipschitz-Bedingung:

$$|K(x, t, x_1) - K(s, t, x_2)| \leqslant L|x_1 - x_2|, \; (s, t, x_1), \; (s, t, x_2) \in I^2 \times R$$

Diese Bedingungen garantieren die Existenz genau einer Lösung

$$x: [a, b] \to R \text{ zu } (1)$$

Bekanntlich können die Approximationsmethoden zur Lösung von (1) folgendermassen eingeteilt werden:

a) Schritt für Schritt-Methoden („Step by step") die den Mehrschrittverfahren für gewöhnliche Differentialgleichungen entsprechen (B. Noble, 1964)

b) Methoden vom Runge-Kutta Typ: (Pouzet, P., 1962; Laudet, M-Oulés, H., 1960; Beltjukov, B. A., 1965: F. de Hoog-R. Weiss, 1975)

c) Block-Verfahren (Young, A., 1954)

d) Methoden unter Verwendung von Spline-Funktionen (H. Brunner, von 1971 ab; El Tom, M. E. A., 1971; Micula, G. 1972).

Kürzlich hat M. E. A. El Tom (1974) eine Vollstandige Untersuchung der Stabilitätseigenschaften der Splineapproximationsmethoden bezüglich Gleichung (1) gegeben. Daraus folgt, dass die Näherungssplineslösungen von Gleichung (1) von höheren Stetigkeitsordnung im allgemeinen keine Stäbilitatseingenschaften besitzen.

Darum ist die Weglassung der Stetigkeitsforderungen der Splinenäherungslösung erforderllich, um Verfahren von höherer Ordnung angeben zu können.

Wir beschäftigen uns mit der numerischen Approximation an die Lösung von (1) durch eine Spline-Funktion $S: [a, b] \to R$ vom Grade m und der Klasse C^k $[a, b]$. $(k < m)$

Die approximierende Spline-Funktion wird durch die Kollokationsmethode bestimmt. Die Konvergenz und Stabilitätseigenschaften werden in Abhängigkeit von k angegeben.

3. **Das Konstructionsverfahren.** Wir teilen das Interval $[a, b]$ in N gleiche Teile, deren Lange Th ist, wobei T eine feste, ganze, pozitive Zahl ist. Auf dem p-ten Teilinterval (i.e. $Tph \leqslant s \leqslant T(p+1)h$), $p = 0(1)\; N-1$ wird die Spline-Funktion S definiert durch:

$$S(s) := \sum_{r=0}^{m-T} \frac{(s - Tph)^r}{r!} S_{Tp}^{(r)} + \sum_{r=1}^{T} (s - Tp)^{m-T+r} a_r^{(p)} \tag{2}$$

$$S_0^{(r)} = x_0^{(r)}, \; r = 0(1)m - T, \quad 1 \leqslant T \leqslant m,$$

wobei die Koeffizienten $a_r^{(p)}$ so bestimmt werden, dass die Integralgleichung:

$$S(s) = y(s) + \int_a^s K(s, t, S(t))dt \tag{3}$$

in den Knotenpunkten

$$s = (Tp + q)h, \quad q = 1(1)T \tag{4}$$

erfüllt ist.

$S^{(r)}$ bezeichnet die r-te Ableitung von S.

Dieses Verfahren ergibt eine Spline-Funktion S vom Grade m, deren Defekt $(T - 1)$ ist, mit dem Abstand T.h zwischen Knotenpunkten. Offensichtlich gehort die Funktion S zu $C^{m-T}[a, b]$.

Wir erwähnen auch, dass der Fall $T = 1$ einer Spline-Funktion mit maximalen Stetigkeiteigenschaften entspricht. (spline-function with full continuity).

Der Fall $T = m$ führt uns zu einer Spline-Funktion vom Grade m und Defekt $m - 1$ (d.h. $S \in C\,[a, b]$), die für $mph \leqslant s \leqslant m(p + 1)h$, $p = 0(1)\,N-1$ definiert ist durch:

$$S(s) = S_{mp} + \sum_{r=1}^{m} (s - mph)^r a_r^{(p)}, \quad S_0 = x_0, \tag{5}$$

wobei die Koeffizienten $a_r^{(p)}$ so gewählt werden, dass die Integralgleichung (3) in den Knotenpunkten

$$s = (mp + q)h, \quad q = 1(1)m \tag{6}$$

erfüllt ist.

In (5) bezeichnen wir mit S_k und x_k die Werte $S(kh)$, bzw $x(kh)$.

Dass die obige Konstruktion wohl-definiert ist, und genau eine Spline-Funktion existiert, wenn h genügend klein ist, kann man durch einen Fixpunksatz, genauso wie bei El Tom [5–6], Micula [16] beweisen

4. **Das Konvergenz-und numerische Stabilitätsverhalten.** Der Einfachheit halber werden wir die Konvergenz nur für $T = m$ untersuchen.

Es sei

$$e(s) = S(s) - x(s), \quad s \in [a, b], \tag{7}$$

wobei S die durch (5) gegebene Spline-Funktion ist und x die exakte Lösung von (1) ist.

Für $p = 0(1)\; N - 1$ schreiben wir:

$$a_r^{(p)} = \frac{x_{mp}^{(r)}}{r!} + h^{m+1-r} b_r^{(p)}, \quad r = 1(1)m, \tag{8}$$

wobei $b_r^{(p)} = b_r^{(p)}(h)$. Wenn $x \in C^{m+1}\,[a, b]$ gilt, so folgt aus (8) und (5) für $mph \leqslant s \leqslant m(p + 1)h$:

$$e(s) = e_{mp} + \sum_{r=1}^{m} h^{m+1-r}(s - mph)^r b_r^{(p)} - \frac{(s - mph)^{m+1}}{(m + 1)!} x^{(m+1)}(\xi_p), \tag{9}$$

$$mph \leqslant \xi_p \leqslant s.$$

LEMMA 1. *Es sei $x \in C^{m+1}[a, b]$. Dann sind die Koeffizienten $b_r^{(p)}$, die durch* (8) *definiert sind, gleichmässig beschränkt.*

Der Beweis ist derselbe wie bei [6, El Tom].
Aus Lemma 1 folgt:

LEMMA 1. *Wenn $x \in C^{m+1}[a, b]$, dann gilt:*

$$e_{m(p+1)} = 0(h^{m+1}),\ p = 0(1)\ N - 1$$

Da $e_0 = 0$, führen Lemma 2 und Gleichung (9) zu.

THEOREM 1. *Wenn $x \in C^{m+1}[a, b]$, dann genügt der Fehler* (7) *der Methoden* (5)—(3) *für beliebige $s \in [a, b]$ der folgenden Abschätzung:*

$$e(x) = 0(h^{m+1}) \tag{10}$$

Theorem 1 bleibt auch für die allgemeine Methode (2)—(3)—(4) gültig.
Zur Vereinfachung der Schreibweise werden wir eine m-Spline-Methode mit dem $(T - 1)$ Defekt einfach eine *(m, T)-Methode nennen.*

Nun soll das Verhalten der Methode untersucht werden für die lineare Integralgleichung

$$s(x) = 1 + \lambda \int_0^s x(t)dt, \tag{11}$$

wobei λ eine Konstante mit negativen Realteil ist.

DEFINITION 2. Man nennt eine (m, T)-Methode stabil wenn alle Lösungen S_{Tr} beschränkt bleiben für $r \to \infty$, $h \to 0$ während $s = Trh$ fest bleibt.

DEFINITION 3. Eine (m, T)-Methode wird A-stabil gennant, wenn alle Lösungen S_{Tr} gegen Null streben, für $r \to \infty$, falls die. Methode mit einem festen h auf eine Gleichungen der form (11) angewandt wird.

Die Hauptresultate des Stabilitätsverhaltens sind in dem nächsten Theorem enthalten.

THEOREM 2. (El Tom [7]).
Die $(m, 1)$-Methode divergiert für $m \geqslant 3$.
Die $(m, 2)$-Methode divergiert für $m \geqslant 4$.
Die $(m, 3)$-Methode divergiert für $m \geqslant 5$.
Die (m, m)-Methode ist stabil für beliebiges m. Ausserdem ist sie A-stabil für $m \geqslant 3$.
Die $(m, m - 1)$-Methode ist stabil für $m > 1$.

Das obige Theorem zeigt uns, dass eine (m, T)-Methode
(i) A-stabil für $T = m$, (m willkürlich) ist und
(ii) divergiert für $T \leqslant m - 2$, $m > 2$.
Ausserdem hat eine (m, T)-Methode gegensätzliche Eigenschaften für $T = 1$ im Vergleich zu $T = m$.

Auf der einen Seite ist eine $(m, m - 1)$-Methode Stabil, auf der anderen Seite divergiert die $(m, 1)$-Methode für $m \geqslant 3$.

Beweis. Um das algebraische System der Methode (2)—(3)—(4) auflösen zu können, betrachtem wir die folgenden Quadraturformeln:

$$\int_0^1 u(s)ds = \sum_{i=1}^{n} w_i u(s_i), \ 0 \leqslant s_i \leqslant 1, \text{ für jedes } i \text{ deren}$$

Genauigkeitsgrad $\geqslant m$ ist. Zur Abkürzung bezeichnen wir weiter mit $R_p(u)$:

$$R_p(u) = h \sum_{i=1}^{n} \sum_{j=0}^{p-1} w_i u(jk + hs_i)$$

$R_p(u)$ ergibt sich durch wiederholte Anwendung der Quatraturformeln.

Zur Vereinfachung der Schreibweise wollen wir bezeichnen:

$$a_r = (h^{m-T+1}a_1^{(r)}, h^{m-T+2}a_2^{(r)}, \ldots, h^m a_T^{(r)})$$
$$r = 0(1)\ N - 1 \tag{12}$$

$$S_r = \left(S_{Tr}, \frac{h}{1!} S_{Tr}^{(1)}, \ldots, \frac{h^{m-T}}{(m+k)!} S_{Tr}^{(m-T)}\right)$$

Mit B bezeichnen wir die $T \times T$-Matrix, deren (i, j)-Element

$$b_{ij} = \left(1 - \frac{h\lambda i}{(m - T + j + 1)}\right) i^{m-T+j-1}, \qquad i, j = 1(1)T \text{ ist.} \tag{13}$$

Mit C bezeichnen wir die $T \times (m - T + 1)$-Matrix, deren (i, j)-Element ist:

$$C_{ij} = \begin{cases} \lambda h, \ j = 1, \ i = 1(1)T \\ \left(\frac{\lambda hi}{j} - 1\right) i^{j-1}; \qquad j = 2(1)\ m - T + 1, \ i = 1(1)T \end{cases} \tag{14}$$

Nun können wir das algebraische System der Spline-Methode kurz schreiben:

$$Ba_r = CS_r, \qquad r = 0, 1, \ldots, N \tag{15}$$

Durch Ableitung der (2) erhält man:

$$S_{r+1} = DS_r + Ea_r, \tag{16}$$

wobei D eine $(m - T \times 1) \times (m - T + k)$-Matrix ist, deren (i, j)-Element ist:

$$d_{ij} = \begin{cases} \binom{i}{j} T^{j-1}, & i = 0, 1, \ldots, m - T, \ j \geqslant i \\ 0 & j < i \end{cases} \tag{17}$$

und E eine $(m - T + 1) \times T$-Matrix ist deren (i, j)-Element

$$e_{ij} = \binom{m - T + j}{i} T^{m-T+j-1}, \qquad i = 0, 1, \ldots, m - T, \ j = 1, 2, \ldots T \text{ ist} \tag{18}$$

Durch Eliminierung von a_r, erhalten wir aus (15) und (16):

$$S_{r+1} = AS_r, \tag{19}$$

wobei

$$A = D + EB^{-1}C \text{ ist} \tag{20}$$

Wir bezeichnen mit A_0 die Matrix A, falls $h = 0$ gilt, und mit μ_0 und μ die Eingenwerte von A_0, bzw A.

Jetzt kommen wir zurück zum Beweis des Theorems.

$\underline{T = 1}$. Es ist klar, dass

$$\mu = \mu_0 + 0(h) \text{ gilt}$$

Für $m \geqslant 3$ hat mindestens ein μ_0 die Eigenschaft $|\mu_0| > 1$, weil

$$tr\,(A_0) = m + 2 - 2$$

und

$\mu_0^{(1)} + \ldots + \mu_0^{(m)} = tr\,(A_0)$ gelten und weil $\mu = 1$ ein Eigenwert von A_0 für jedes m ist.

So ist die $(m, 1)$-Methode devergent für $m \geqslant 3$.

$\underline{T = 2}$. Die Diagonalelement a_{ij} von A_0 sind:

$$a_{11} = 1$$

$$a_{ii} = 1 + \binom{m-1}{i-2}\left(2^{m-i+1} - \frac{m+i-1}{i-1}\right), \quad i = 2(1)\, m - 1$$

Also gilt:

$$t_r(A_0) = 3^{m-1} - 3 \cdot 2^{m-1} + m + 2$$

und es folgt, dass die $(m, 3)$-Methode divergiert für $m \geqslant 4$.

$\underline{T = 3}$. Ähnlich folgt

$$t_r(A_0) = m - 4^{m-2} + 8(5/2)^{m-2} - 5 \cdot 2^{m-1} + 2.$$

$\underline{T = m}$. In diesem Fall ist die Spline-Function $S \in C\,[a, b]$. Dabei müssen wir die Bezeichnungen ein wenig ändern, und zwar:

$$S_{r+1} = (S_{mr+1}, S_{mr+2}, \ldots, S_{m(r+1)\, r+1}),\ r = 0(1)\, N - 1$$

und (16) lautet dann

$$S_{r+1} = (1, 1, \ldots, 1)S_{mr} + \hat{E}a_r,$$

wobei die Element von $\hat{E}$

$$e_{ij} = i^j, \quad i, j = 1(1)\, m, \text{ sind.}$$

Wir haben weiter:

$$\hat{B}a_r = \hat{C}S_{m},$$
$$S_{r+1} = \hat{C}S_{mr},$$
$$\hat{A} = (1, 1, \ldots, 1) + \hat{E}\hat{B}^{-1}\hat{C},$$

wobei

$$\hat{b}_{ij} = ib_{ij}, \qquad i, j = 1(1)\, m$$
$$\hat{C}_{ij} = iC_{ij}, \qquad i = 1(1)m\,;\ j = 1 \text{ ist.}$$

In diesem Fall ist C eine $m \times 1$-Matrix.

Man sieht also, dass

$$\hat{B} = \hat{E} + 0(h)$$
$$\hat{C} = 0(h)$$

und
$$\hat{A} = (1, 1, \ldots, 1) + \hat{C} + 0(h^2)$$

gelten. Schliesslich kann man schreiben:

$$S_{mr+t} = (1 + t\lambda h + 0(\lambda^2 h^2))S_{mr}, \quad t = 1(1)m.$$

Beispielsweise haben wir für $m = 1, 2, 3$;

$$S_{m(r+1)} = \begin{cases} \dfrac{1 + \lambda h/2}{1 - \lambda h/2}\, S_r & m = 1 \\[2ex] \dfrac{1 + \lambda h + \lambda^2 h^2/3}{1 - \lambda h + \lambda^2 h^2/3}\, S_{2r}, & m = 2 \\[2ex] \dfrac{1 + 3\lambda h/2 + 11\lambda^2 h^2/12 + \lambda^3 h^3/4}{1 - 3\lambda h/2 + 11\lambda^2 h^2/12 - \lambda^3 h^3 4}\, S_{3r}, & m = 3 \end{cases}$$

Die rationalen Funktionen sind regulär in der linken Seite der Ebene und ihr Betrag (absolute Werte) ist kleiner als 1.

5. **Numerische Ergebnisse.** *Beispiel* 1. Man betracht die Integralgleichung:

$$x(s) = e^{-s}(1 + s + s^2) - s + \int_0^s st\, x(t)dt, \ 0 \leqslant s \leqslant 1$$

deren exakte Lösung $x(s) = e^{-s}$ ist.

Für $m = 3$, $T = 2$, haben wir die folgenden Ergebnisse:

N	5	10	15
Maximalfehler	3.34×10^{-4}	1.4×10^{-4}	6.31×10^{-3}

Beispiel 2. Man betrachtet die Integralgleichung

$$x(s) = e^s + \int_0^s e^{s-t}\, x(t)dt, \qquad 0 \leqslant s \leqslant 4$$

deren genaue Lösung $x(s) = 1$ ist.

Die Maximalfehler für zwei verschiedene Methoden sind in der folgenden Tabelle enthalten:

h	0.8	0.4	0.2
(2,2)-Methode	5×10^{-3}	3.9×10^{-4}	6.9×10^{-3}
(4,3)-Methode	5.9×10^{-5}	8×10^{-4}	0.9×10^{-8}

6. Bemerkungen. (a) Das dargestellte Verfahren hat einige Vorteile:

(i) Man bracht keine zusätzlichen Startwerte.

(ii) Es liefert eine globale Approximation der Lösung.

(iii) Die Schrittweite h kann notfalls von Schritt zi Schritt geändert werden und gleichfalls der Grad m der Spline-Approximation

(b) Das Verfahren ist numerisch besonders einfach zu handhaben durch die Benutzung einer geeigneten Quadraturformel auf jedem p-Intervall.

(c) Bei der Auwendung der Methode muss ein nichtlineares Gleichungssystem gelöst werden. Dafür können Iterationsmethoden, wie Newton-Raphson-Methode u.s.w., angewendet werden.

(Eingegangen an 6 Oktober 1978)

LITERATUR

1. Antes, H., *Splinefunctionen bei der Lösung von Integral gleichungen*, Numer. Math., **19**, 1972, 116–126.
2. Beltjukov, B. A., *An analogue of Runge-Kutta method for the solution of a nonlinear integral equation of the Volterra type*, Differential Equations, **I**, 1965, 417–433.
3. Brunner, H., *The solution of nonlinear Volterra integral equations by piecewise polynomials*, in *Proc. Manitoba Conf. Numerical Mathematics* (R. S. Thomas and H. C. Williams, eds.), Univ. of Manitoba Winnipeg, 1971, 65–68.
4. Brunner, H., *On the numerical solution of nonlinear Volterra integro-differential equtions*, BIT **13**, 1973, 381–390.
5. El Tom, M. E. A., *Numerical solution of Volterra integral equations by spline functions*, BIT **13**, 1973, 1–7.
6. El Tom, M. E. A., *Application of spline functions to Volterra integral equations*, J. Inst. Maths. Applics, **8**, 1971, 354–357.
7. El Tom, M. E. A., *On the numerical stability of spline function approximations to solutions of Volterra integral equations of the second kind*, BIT **14**, 1974, 136–43.
8. El Tom, M. E. A., *Spline function approximation to the solution of singular. Volterra integral equations of the second kind*, J. Inst. Maths. Applics., **14**, 1974, 303–309.
9. Guzek, J. A. and Kemper, G. A., *A new error analysis for a cubic spline approximate solution of a class of Volterra integro-differential equations*. Math. Comp., **27**, 1973, 563–570.
10. De Hoog, F. and Weiss, R., *Implicit Runge-Kutta Methods for second kind Volterra Integral Equations*, Numer. Math., **23**, 1975, 199–213.

11. H u n g, H. S., *The numerical solution of differential and integral equations by spline functions*, in *M.R.C. Tch. Summary Rept*, 1053, Madison, 1970.
12. L a u d e t, M. et O u l é s, H., *Sur l'integration numérique des équations integrales du type de Volterra*, in *Symposium on the Numerical Treatment of Ordinary Diff. Eqs. Integral and Integro-differential Eqs*, Birkhauser Verlag, Basel, 1960, 117–121.
13. L i n z, P., *The numerical solution of Volterra integral equations by finite methods*, in *MRC Tech. Summary Rept*, 825, Madison, 1967.
14. M a l i n a, L u b o r, *A note on convergence for the numerical scheme from the Miculas paper*, Mathematica, Cluj, **17**, 1975, 187–190.
15. M i c u l a, M., M i c u l a, G., *Sur la résolution numérique des équations intégrales du type de Volterra de second espèce à l'aide des fonctions splines*, Studia Univ. Babeş-Bolyai, ser. Math. Mech., f. 2, 1973, 64–68.
16. M i c u l a, G., *The numerical solution of Volterra integro-differential equations by spline functions*, Revue Roumaine de Math. pures et appliquées, **20**, 1975, 341–358.
17. N e t r a v a l i, A. N., *Spline approximation to the solution of the Volterra integral equation of the Second Kind*, Math. Comp., **27**, 1973, 99–106.
18. N o b l e, B., *The numerical solution on nonlinear integral equations and related topics*, in *Non-linear Integral Equations*, ed. by P. M. A n s e l o n e, Univ. of Wisconsin Press, Madison, 1964, 215–318.
19. P o u z e t, P., *Étude, en vue de leur traitment numérique d'équations intégrales et integro-differentielles du type de Volterra pour des problémes de conditions initiales*, Thesis, Univ. of Strassbourg, 1962.
20. Y o u n g, A., *The application of approximate product integration to the numerical solution of integral equations*, Proc. Roy. Soc. London (A), **224**, 1954, 561–573.

TRATAREA NUMERICĂ A ECUAŢIILOR INTEGRALE CU AJUTORUL FUNCŢIILOR SPLINE

(R e z u m a t)

În această lucrare se aduc contribuţii la rezolvarea numerică a ecuaţiilor integrale de tip Volterra de speţa a doua cu ajutorul funcţiilor spline polinomiale.

Folosind procedeul colocaţiei se construieşte o funcţie spline de grad m şi de clasă C^k pentru aproximarea soluţiei ecuaţiei date.

Se studiază convergenţa şi stabilitatea metodei în funcţie de valorile parametrilor m şi k.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXIV, 2, 1979

ASUPRA PUNCTELOR FIXE ALE APLICAŢIILOR DEFINITE PE UN PRODUS CARTEZIAN (III)

IOAN A. RUS

1. Fie (X, d) un spaţiu metric complet şi (Y, τ) un spaţiu topologic separat, cu proprietatea de punct fix. În prezenta notă ne propunem să stabilim teoreme de punct fix pentru aplicaţii

$$f: X \times Y \to X \times Y, \ (x, y) \to (f_1(x, y), f_2(x, y)),$$

folosind condiţii asupra lui f_1 şi f_2. În cazul în care X şi Y sînt mulţimi, sau mulţimi ordonate, sau spaţii metrice, sau spaţii topologice, asemenea probleme au fost studiate în [1], [2] şi [3] (a se vedea de asemenea indicaţiile bibliografice din [5]).

2. Rezultatele din prezenta lucrare se bazează pe următoarele leme:

LEMA 1. *Fie (X, d) un spaţiu metric şi (Y, τ) un spaţiu topologic separat. Presupunem că*

(i) *$f_1(\cdot, y)$ are un punct fix unic $x^*(y)$ şi că aplicaţia $P: Y \to X$, $y \mapsto x^*(y)$ este continuă;*

(ii) *spaţiul topologic Y are proprietatea de punct fix.*

In aceste condiţii aplicaţia f are cel puţin un punct fix.

LEMA 2. *Fie (X, d) un spaţiu metric, $(Y, ||\cdot||)$ un spaţiu Banach şi $Z \subset Y$ o submulţime convexă şi închisă. Fie $f: X \times Z \to X \times Z$, continuă, astfel încît*

(i) *$f_1(\cdot, y)$ are un punct fix unic, ce depinde continuu de y*

(ii) *$f_2(X, Z) \subset Z$ este compactă în Z.*

In aceste condiţii, f are cel puţin un punct fix.

Avînd în vedere aceste leme şi anumite teoreme de dependenţă continuă a punctului fix de parametru ([4]) obţinem:

TEOREMA 1. *Fie (X, d) un spaţiu metric complet, Y un spaţiu topologic $f: X \times Y \to X \times Y$ continuă şi $\varphi: R_+ \to R_+$, continuă, crescătoare, cu proprietatea că $r - \varphi(r) \to +\infty$ cînd $r \to +\infty$ şi $\varphi(r) < r$, $\forall r > 0$. Presupunim că*

(i) *$d(f_1(x_1, y), f(x_2, y)) \leqslant \varphi(d(x_1, x_2))$, $\forall x_1, x_2 \in X$, $y \in Y$*

(ii) *spaţiul topologic Y are proprietatea de punct fix. Atunci f are cel puţin un punct fix*

Demonstraţie. Considerăm aplicaţia $f_1: X \times Y \to X$. Pe baza teoremei 5 din [4] rezultă că $f_1(\cdot, y)$ are un punct fix unic $x^*(y)$, ce depinde continuu de y. Teorema 1 rezultă din lema 1.

TEOREMA 2. *Fie (X, d) un spaţiu metric complet, Y un spaţiu Banach $Z \subset Y$ o submulţime convexă şi închisă, $f: X \times Z \to X \times Z$ continuă şi φ: $R_+ \to R^+$ cu proprietăţile din teorema 1. Presupunem că*

(i) *$d(f_1(x_1, y), f(x_2, y)) \leqslant \varphi(d(x_1, x_2)$, $\forall x_1, x_2 \in X, y \in Y$*

(ii) *$f_2(X, Z) \subset Z$ ste compactă în Z*

În aceste condiţii, f are cel puţin un punct fix.

Demonstraţie. Se aplică teorema 5 din [4] şi lema 2

3. Observaţii. Din cele de mai sus rezultă că oricărei teoreme de dependenţă continuă a punctului fix de parametri îi corespund teoreme de existenţă a punctului fix pentru aplicaţii de forma $f: X \times Y \to X \times Y$.

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BIBLIOGRAFIE

1. R. F. Brown, *On some old problems of fixed point theory* The Rocky Mountain Journal of Mathematics, 4, 1974, 3–14.
2. H. Cohen, *A fixed point problem for products of metric spaces*, Nieuw Archief voor Wiskunde, 21, 1973, 59–63.
3. I. A. Rus, *Asupra punctelor fixe ale aplicaţiilor definite pe un produs cartezian*, II, Stud. şi Cerc. Mat., 24, 1972, 897–904.
4. I. A. Rus, *Resultats and problems in the metrical fixed point theory*, The first Romanian American seminar on operator theory and applications, Iaşi, 20–24 martie, 1978.
5. I. A. Rus., *Principii şi aplicaţii ale teoriei punctului fix*, Ed. Dacia, Cluj-Napoca, 1979.

ON THE FIXED POINTS OF THE MAPPINGS DEFINED ON A CARTESIAN PRODUCT (III)

(Summary)

Let (X, d) be a complete metric space and (Y, τ) a Hausdorff space. Fixed point theorems for the mappings $f: X \times Y \to X \times Y$, $(x, y) \mapsto (f_1(x, y), f_2(x, y))$ are given.

STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXIV, 2, 1979

A FIFTH ORDER FAMILY RUNGE-KUTTA TYPE METHODS

I. COROIAN*

1. The purpose of this paper is to derive a family of Runge-Kutta type methods of order 5 with 6 stages ($R - K_{5,6}$) for numerical integration of the initial values problem

$$y'(x) = f(x, y(x)), \ y(x_0) = y_0, \tag{1.1}$$

on the set of points $x_{n+1} = x_n + h = x_0 + nh, \ h > 0, \ n = 0, 1, \ldots, N - 1$.

We assume that (1.1) has a unique solution, and the function f is smooth enough in a domain $\mathfrak{D} \subset \mathbf{R}^2$, which is a neighbourhood of the point (x_0, y_0).

DEFINITION 1.1. An explicit Runge-Kutta type method of order 5 with 6 stages, for the numerical solution of (1.1), on the set of points $x_{n+1} = x_0 + nh$, $n = 0, 1, \ldots, N - 1$ is defined by the formulas

$$\begin{gathered} y_{n+1} = y_n + h\sum_{i=1}^{6} c_i k_i, \ n = 0, 1, 2, \ldots, N-1 \\ k_1 = f(x_n, y_n), \\ k_i = f\left(x_n + a_i h, \ y_n + h\sum_{j=1}^{i-1} b_{ij} k_i\right), \ i = 2, 3, 4, 5, 6 \end{gathered} \tag{1.2}$$

where y_{n+1} is the approximate value of $Y_n(x_{n+1})$. $Y_n(x)$ denotes the solution of the "local" Cauchy problem

$$Y_n'(x) = f(x, Y_n(x)), \ \ Y_n(x_n) = y_n, \tag{1.3}$$

and a_i, b_{ij}, c_i; $i = 2, 3, 4, 5, 6$; $j = 1, 2, \ldots, i - 1$, are real parameters satisfying a system of algebraic equations which ensures the order five, of the method, that is

$$Y_n(x_{n+1}) - y_{n+1} = 0(h^6). \tag{1.4}$$

This means that the Taylor series of the powers of h of $Y_n(x_n + h)$ is identical with similar development of y_{n+1}, for the terms including the term in h^5.

2. To shorten our paper, we omit the derivation of Taylor development for $Y_n(x_{n+1})$ and y_{n+1}, and the derivation of the algebraic equations for a_i, b_{ij}, c_i. We can find these equations, for instance, in H. A. Luther and H. P. Konen, [8].

If we claim the same simplifying conditions, like in [8]

$$c_2 = 0, \tag{2.1}$$

$$a_i^2 = 2\sum_{j=2}^{i-1} b_{ij} \ a_j, \ i = \overline{2,6}, \tag{2.2}$$

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system of algebraic equations satisfied by a_i, b_{ij}, c_i, is:

$$c_1 + c_3 + c_4 + c_5 + c_6 = 1, \tag{2.3}$$

$$c_3a_3 + c_4a_4 + c_5a_5 + c_6a_6 = 1/2, \tag{2.4}$$

$$c_3a_3^2 + c_4a_4^2 + c_5a_5^2 + c_6a_6^2 = 1/3 \tag{2.5}$$

$$c_3a_3^3 + c_4a_4^3 + c_5a_5^3 + c_6a_6^3 = 1/4, \tag{2.6}$$

$$c_3a_4^3 + c_4a_4^4 + c_5a_5^4 + c_6a_6^4 = 1/5, \tag{2.7}$$

$$\sum_{i=4}^{6} c_i \sum_{j=3}^{i-1} b_{ij}a_j = 1/6, \tag{2.8}$$

$$\sum_{i=4}^{6} c_i \sum_{j=3}^{i-1} b_{ij}a_j^2 = 1/12, \tag{2.9}$$

$$\sum_{i=4}^{6} c_i \sum_{j=3}^{i-1} b_{ij}a_j^3 = 1/20 \tag{2.10}$$

$$\sum_{i=4}^{6} c_ia_i \sum_{j=3}^{i-1} b_{ij}a_j = 1/8, \tag{2.11}$$

$$\sum_{i=4}^{6} c_ia_i \sum_{j=3}^{i-1} b_{ij}a_j^2 = 1/15, \tag{2.12}$$

$$\sum_{i=5}^{6} c_i \sum_{j=4}^{i-1} b_{ij} \sum_{k=3}^{j-1} b_{jk}a_k = 1/24, \tag{2.13}$$

$$\sum_{i=5}^{6} c_i \sum_{j=4}^{i-1} b_{ij} \sum_{k=3}^{j-1} b_{jk}a_k^2 = 1/60, \tag{2.14}$$

and we must add

$$\sum_{j=1}^{i-1} b_{ij} = a_i\,;\; i = \overline{2,6}. \tag{2.15}$$

First, a $R-K_{5,6}$ method was given in 1901 by W. Kutta [6], then by E. J. Nyström [10], in 1925, and more recently the solutions of the equations (2.3)–(2.15) have been discussed by H. A. Luther [9], H. A. Luther and H. P. Konen [8], C. R. Cassity [3].

3. We shall present a three parameters family solution of the system (2.3)–(2.15) in the assumptions (2.1), (2.2), and requiring

$$c_5 = 0. \tag{3.1}$$

From the equations (2.4)—(2.7) we get

$$c_3 = \frac{\frac{1}{2} a_4 a_6 - \frac{1}{3}(a_4 + a_6) + \frac{1}{4}}{a_3(a_4 - a_3)(a_6 - a_3)},$$

$$c_4 = \frac{\frac{1}{2} a_3 a_6 - \frac{1}{3}(a_3 + a_6) + \frac{1}{4}}{a_4(a_3 - a_4)(a_6 - a_4)}, \qquad (3.2)$$

$$c_6 = \frac{\frac{1}{2} a_3 a_4 - \frac{1}{3}(a_3 + a_4) + \frac{1}{4}}{a_6(a_3 - a_6)(a_4 - a_6)},$$

if $a_i \neq 0$; $a_i \neq a_j$, $i \neq j$, $i, j = 3, 4, 6$ and

$$\frac{1}{5} - \frac{1}{4}(a_3 + a_4 + a_6) + \frac{1}{3}(a_3 a_4 + a_3 a_6 + a_4 a_6) - \frac{1}{2} a_3 a_4 a_6 = 0 \qquad (3.3)$$

From (2.8)—(2.14) we can obtain

$$\frac{1}{6} a_3 - \frac{1}{12} = c_6[a_4(a_3 - a_4)b_{64} + a_5(a_3 - a_5)b_{65}], \qquad (3.4)$$

$$\frac{1}{8} a_3 - \frac{1}{15} = c_6 a_6[a_4(a_3 - a_4)b_{64} + a_5(a_3 - a_5)b_{65}], \qquad (3.5)$$

$$\frac{1}{12} a_3 - \frac{1}{20} = c_6[a_4^2(a_3 - a_4)b_{64} + a_5^2(a_3 - a_5)b_{65}]. \qquad (3.6)$$

By division of (3.4) to (3.5) give

$$a_6 = \frac{15a_3 - 8}{10(2a_3 - 1)}, \qquad (3.7)$$

and then, from (3.3) we get

$$a_4 = \frac{a_3}{2(5a_3^2 - 4a_3 + 1)}, \qquad (3.8)$$

Because $c \neq 0$, in opposite case our method (1.2) would be a $R - K_{5,5}$ method, and this is a contradiction (I. C. Butcher, [1]).

Now, (2.4)—(2.7) give

$$\frac{1}{5} - \frac{1}{4}(a_3 + a_4 + a_5) + \frac{1}{3}(a_3 a_4 + a_3 a_5 + a_4 a_5) - \frac{1}{2} a_3 a_4 a_5 =$$
$$= c_6 a_6 (a_6 - a_3)(a_6 - a_4)(a_6 - a_5) \neq 0 \qquad (3.9)$$

Taking into account (3.7), it follows that

$$a_5 \neq \frac{150a_3^3 - 260a_3^2 + 141a_3 - 24}{10(20a_3^3 - 34a_3^2 + 18a_3 - 3)}. \qquad (3.10)$$

The free parameters will be now a_2, a_3, a_5 and because $a_3 \neq a_6$, $a_3 \neq a_4$, $a_4 \neq a_6$, a_3, a_4, $a_6 \neq 0$, it follows that

$$a_2 \neq \frac{2}{5}, \quad a_3 \neq \frac{2}{3}, \quad a_3 \neq \frac{4 \pm \sqrt{6}}{10} \tag{3.11}$$

The parameter a_5 cannot be 1, because then (3.10) is not satisfied, i.e.

$$a_5 \neq 1 \tag{3.12}$$

From (3.4), (3.6), it results

$$b_{64} = \frac{10a_3a_5 - 5(a_3 + a_5) + 3}{60c_6a_4(a_5 - a_4)(a_3 - a_4)}, \quad b_{65} = \frac{10a_3a_4 - 5(a_3 + a_4) + 3}{60c_6 a_5(a_3 - a_5)(a_4 - a_5)} \tag{3.13}$$

and from (2.13), (2.14) we obtain

$$b_{54} = \frac{5a_3 - 2}{120c_6a_4b_{65}(a_3 - a_4)}. \tag{3.14}$$

In order to obtain b_{i3}, $i = 4, 5, 6$, first we derive from (2.13), (2.14), (2.8), the equations:

$$\frac{1}{24}a_4 - \frac{1}{60} = c_6a_3(a_4 - a_3)(b_{43}b_{64} + b_{53}b_{65}),$$

$$\frac{1}{6}a_5 - \frac{1}{8} - c_6(a_5 - a_6)(a_4b_{64} + a_5b_{65}) = c_4a_3b_{43}(a_5 - a_4) +$$
$$+ c_6a_3b_{63}(a_5 - a_6),$$

$$c_4a_3b_{43} + c_6a_3b_{63} = \frac{1}{6} - c_6(b_{64}a_4 + b_{65}a_5).$$

The last equations give

$$\begin{aligned} b_{43} &= \frac{4a_6 - 3}{24c_4a_3(a_6 - a_4)}, \\ b_{53} &= \frac{1}{b_{65}}\left[\frac{5a_4 - 2}{120c_6a_3(a_4 - a_3)} + \frac{b_{64}(4a_6 - 3)}{24c_4a_3(a_4 - a_6)}\right], \\ b_{63} &= \frac{4a_4 - 3}{24c_6a_3(a_4 - a_6)} - \frac{1}{a_3}(b_{64}a_4 + b_{65}a_6). \end{aligned} \tag{3.15}$$

From (2.2), follows

$$b_{i2} = \frac{1}{a_2}\left(\frac{1}{2}a_i^2 - b_{i3}a_3 - \ldots - b_{i,i-1}\, a_{i-1}\right), \quad i = \overline{3,6}, \tag{3.16}$$

and (2.15) gives

$$b_{i1} = a_i - \sum_{j=2}^{i-1} b_{ij}, \quad i = \overline{2,6}. \tag{3.17}$$

Thus we have

THEOREM 3.1. *If a_2, a_3, a_5 have real values satisfying* (3.10), (3.11), (3.12) $a_i \neq 0$, $a_i \neq a_j$, $i \neq j$, $i, j = 2, 3, 5$, *then the formulas* (2.1), (3.1), (3.2), (3.7), (3.8), (3.13), (3.14), (3.15), (3.16) *and* (3.17) *furnishes three parameters explicit* $R - K_{5,6}$ *methods.*

The choice

$$a_2 = \frac{9}{10}, \quad a_3 = \frac{3}{5}, \quad a_5 = -\frac{3}{20},$$

gives the following explicit $R - K_{5,6}$ method

$$\begin{aligned}
y_{n+1} &= y_n + h\left(\frac{7}{54}k_1 - \frac{125}{54}k_3 + \frac{32}{27}k_4 + 2k_6\right),\\
k_1 &= f(x_n, y_n),\\
k_2 &= f\left(x_n + \frac{9}{10}h, y_n + \frac{9}{10}hk_1\right),\\
k_3 &= f\left(x_n + \frac{3}{5}h, \ y_n + \frac{h}{5}(2k_1 + k_2)\right),\\
k_4 &= f\left(x_n + \frac{3}{4}h, \ y_n + \frac{h}{64}(23k_1 + 10k_2 + 15k_3)\right),\\
k_5 &= f\left(x_n - \frac{3}{20}h, \ y_n + h\left(\frac{61}{320}k_1 + \frac{67}{160}k_2 - \frac{87}{64}k_3 + \frac{3}{5}k_4\right)\right),\\
k_6 &= f\left(x_n + \frac{h}{2}, \ y_n + h\left(\frac{47}{108}k_1 + \frac{5}{36}k_2 - \frac{1}{81}k_3 - \frac{5}{81}k_5\right)\right).
\end{aligned} \tag{3.18}$$

For a comparison of our method (3.15) to the $R - K_{5,6}$ methods of Kutta, [6], and of Nytröm, [10], we proceed like A. H. Luther and H. P. Konen, [8].

For our method (3.18), $\sum_{j=1}^{i-1} |b_{ij}|$, $i = 4, 5, 6$, has the values 0.75; 0.56; 0.61, while for the method of Kutta [6] these values are 11; 3.16; 2.34, respectively, and for Nytröm method, [10], the correspondent values are 7; 1.9; 0.8.

The above-mentioned three methods: (3.18), the method of Kutta and the method of Nyström, have the same number of multiplications, namely 18, like $\frac{7}{54}k_1, \frac{125}{54}k_3$, etc.

Thus, our method would be preferable to the method of Kutta, and as A. Coţiu, [4], asserts the method of Nyström would be preferable to the method of Kutta.

Table 3.1. illustrates the above comparison between our $R - K_{5,6}$ method and the $R - K_{5,6}$ method of Kutta for the problem

$$y' = -2xy^2, \qquad y(0) = 1,$$

with exact solution $y(x) = (1 + x^2)^{-1}$

Table 3.1

x	Exact solution	Method (3.18)	Kutta's method
0.1	0.9900990	0.9901004	0.9900903
0.2	0.9615384	0.9615435	0.9615096
0.3	0.9174311	0.9174402	0.9173789
0.4	0.8620689	0.8620807	0.8619990
0.5	0.8000000	0.8000122	0.7999239

The computation was carried out by a FELIX C-256 computer.

4. The last paragraph is devoted to obtain two bounds for the principal truncation error term, $\psi_5 h^6$, of the $R - K_{5,6}$ method (3.18).

THEOREM 4.1. *The $R - K_{5,6}$ method* (3.18), *has for the principal truncation error term, with the assumptions of L. Bieberbach* [2], *the bound*

$$|\psi_5 h^6| < 0{,}1862\ ML\frac{L^5 - 1}{L - 1}h^6, \tag{4.1}$$

and with the assumptions of M. *Lotkin,* [7], *the bound*

$$|\psi_5 h^6| < 0{,}2827\ ML^6 h^6, \tag{4.2}$$

where M, L are known positive constants.

The proof requires a tedious computation, which we omit here. Briefly, with the coefficients of the method (3.18) one can obtain for the principal error function ψ_5.

$$\begin{aligned}\psi_5 = {} & \frac{7}{288000} D^5 f + \frac{1}{4608} f_1 D^4 f + \frac{11}{14400} f_1^2 D^3 f - \frac{113}{28800} f_1^3 D^2 f + \\ & + \frac{13}{2880} f_1^4 Df + \frac{1}{3600} Df_1 D^3 f + \frac{119}{6400} f_1 Df_1 D^2 f + \\ & + \frac{41}{3200} f_1^2 Df_1 Df + \frac{7}{28800} D^3 f_1 Df + \frac{5}{2304} f_1 D^2 f_1 Df - \\ & - \frac{31}{57600} D^3 f_1 D^2 f - \frac{31}{57600} f_2 D^2 f Df + \frac{7}{4608} f_1 f_2 (Df)^2 + \\ & + \frac{1}{19200} Df_2 (Df)^2 + \frac{1}{1200} (Df_1)^2 Df,\end{aligned} \tag{4.3}$$

where $f_1 = \dfrac{\partial f(x_n, y_n)}{\partial x}$, $f_2 = \dfrac{\partial^2 f(x_n, y_n)}{\partial y^2}$, and

$$D^k f = \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)^{(k)} f = \sum_{i=0}^{k} \binom{k}{i} \frac{\partial^k f(x_n, y_n)}{\partial x^{k-i} \partial y^i} f^i(x_n, y_n), \quad k = \overline{1,5} \tag{4.4}$$

From Bieberbach [2] assumptions it follows that

$$|f_1| \leqslant L,\ |f_2| \leqslant LM^{-1},\ |D^k f| \leqslant 2^k LM,$$
$$|D^k f_i| \leqslant 2^k LM^{1-i},\ i = 1, 2;\ K = \overline{1,5},$$

and from Lotkin's [7], assumptions one obtains

$$|f_1| \leqslant L,\ |f_2| \leqslant L^2 M^{-1},\ |D^k f| \leqslant 2^k L^k M,$$
$$|D^k f_i| \leqslant 2^k L^{k+i} M^{1-i},\ K = \overline{1,5};\ i = 1,2. \tag{4.6}$$

Now, using (4.5) and (4.6), in (4.3) one obtains (4.1) and (4.2).

Remark 4.1. For the $R - K_{5,6}$, method of Nyström, [10], the similar bounds to (4.1) and (4.2) were obtained by V. Jukl [5].

$$|\psi_5 h^6| < 0{,}36\, ML \frac{L^6 - 1}{L - 1} h^6, \tag{4.7}$$

$$|\psi_5 h^6| < 0{,}48\, ML^5 h^6. \tag{4.8}$$

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REFERENCES

1. Butcher, J. C., *On Runge-Kutta processes of higher order*, J. Austral Math. Soc. **4**, 1964, 179–194.
2. Bieberbach, L., *On the remainder of the Runge-Kutta formulae in the theory of ordinary diff. eqs.*, ZAMP, **2**, 1951, 233–248.
3. Cassity, C. R., *Solution of the fifth order Runge-Kutta equations*, SIAM J. Numer. Anal., **3**, 4, 1960, 598–606.
4. Coţiu, A., *Studiu comparativ al delimitărilor erorilor unor formule de integrare numerică a ecuaţiilor diferenţiale*, Lucrări şt. Inst. Politehnic Cluj, 1960, 31–42.
5. Jukl, V., *Fehlerabschätzung der Nyström schem Formel*, Acta Fac. Rerum Nat. Univ. Comenianae, Math. **24**, 1970, 81–100.
6. Kutta, W., *Beitrag zur nöherungsweissen Integration totaler Differentialgleichungen*, Zeitschr., f. Math. und Physic, **46**, 1901, 435–453.
7. Lotkin, M., *On the accuracy of Runge-Kutta's method*, Math. Tables and Other Aids to Computation, **5**, 1951, 128–133.
8. Luther, H. A., Konen, H. P., *Some fifth order classical. Runge-Kutta formulas*, SIAM REVIEW, **7**, 1965, 551–558.
9. Luther, H. A., *Further explicit fifth – order Runge-Kutta formulas*, SIAM REVIEW, **8**, 1966, 374–380.
10. Nyström, E. J., *Über die numerische Integration von Diff.*, Acta Soc. Sci. Fennicae, Helsingfors, **50**, 1925, 13.

O FAMILIE DE METODE DE TIP RUNGE-KUTTA DE ORDINUL CINCI

(Rezumat)

În lucrare se deduce o familie de metode de tip Runge-Kutta cu ordinul de exactitate 5 şi cu 6 substituţii de forma (1.2) pentru problema (1.1).

Se alege o metodă (3.18), din această familie care, comparată cu metoda de ordinul 5 al lui Kutta [6], şi cu cea a lui Nystrom [10], are anumite avantaje.

În ultimul paragraf se deduc 2 margini superioare, (4.1) şi (4.3), pentru termenul principal al erorii de trunchiere.

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FORMULE DE CUADRATURĂ DE TIP PRODUS CU GRAD MARE DE EXACTITATE

PETRU P. BLAGA

1. Fie $f, g \in C[a, b]$ şi fie două diviziuni ale intervalului $[a, b]$:

$$\Delta_1 : a \leqslant x_0 < x_1 < \ldots < x_m \leqslant b \quad \text{şi}$$
$$\Delta_2 : a \leqslant y_0 < y_1 < \ldots < y_n \leqslant b,$$

care în general sînt distincte.

Definiţia 1.1. Se numeşte formulă de cuadratură de tip produs formula

$$\int_a^b f(x)g(x)dx = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} f(x_i) g(y_j) + R(f, g), \tag{1.1}$$

unde $a_{ij} \in \mathbf{R}$, $i = \overline{0, m}$, $j = \overline{0, n}$, iar $R(f, g)$ se numeşte restul formulei.

Definiţia 1.2. Formula de cuadratură de tip produs (1.1) este de tip interpolatoriu dacă

$$a_{ij} = \int_a^b l_i^{(1)}(x) l_j^{(2)}(x)dx, \; i = \overline{0, m}, \; j = \overline{0, n}, \tag{1.2}$$

unde $l_i^{(1)}$, $i = \overline{0, m}$ şi $l_j^{(2)}$, $j = \overline{0, n}$ sînt respectiv polinoamele fundamentale ale lui Lagrange relative la diviziunile Δ_1 şi Δ_2.

Definiţia 1.3. Formula de cuadratură de tip produs (1.1) are gradul de exactitate (p, q) dacă $R(f, g) = 0$, pentru orice $(f, g) \in \mathfrak{P}_p \times \mathfrak{P}_q$, unde $\mathfrak{P}_s$ notează mulţimea polinomelor de grad cel mult s.

Teorema 1.1. *Dacă* (1.1) *este o formulă de cuadratură de tip produs interpolatorie, atunci are gradul de exactitate* (m, n) *şi restul se poate exprima astfel:*

$$R(f, g) = \int_a^b u(x)[x_0, x_1, \ldots, x_m, x; f]g(x)dx +$$
$$+ \int_a^b v(x)[y_0, y_1, \ldots, y_n, x; g]f(x)dx -$$
$$- \int_a^b u(x)v(x)[x_0, x_1, \ldots, x_m, x; f][y_0, y_1, \ldots, y_n, x; g]dx$$

Demonstraţia se poate găsi în [4].

COROLAR 1.1. *Dacă* (1.1) *este formulă de cuadratură de tip produs interpolatorie atunci*

$$R(f, g) = \int_a^b v(x)[y_0, y_1, \ldots, y_n, x; g]f(x)dx, \quad \forall f \in \mathfrak{P}_m \text{ şi}$$

$$R(f, g) = \int_a^b u(x)[x_0, x_1, \ldots, x_m, x; f]g(x)dx, \quad \forall g \in \mathfrak{P}_n.$$

TEOREMA 1.2 *Dacă formula de cuadratură de tip produs* (1.1) *are gradul de exactitate* (p, q) şi $f \in C^{p+1}[a, b]$, $g \in C^{q+1}[a, b]$, *atunci restul se poate exprima astfel:*

$$R(f, g) = \sum_{k=0}^{p} f^{(k)}(a) \int_a^b M_k(t) g^{(q+1)}(t)dt + \tag{1.4}$$

$$+ \sum_{l=0}^{q} g^{(l)}(a) \int_a^b L_l(s) f(s)^{(p+1)}ds + \int_a^b \int_a^b N(s, t) f^{(p+1)}(s) g^{(q+1)}(t)dt\, ds,$$

unde

$$L_l(s) = R_{(x)}\left[\frac{(x-s)_+^p}{p!}, \frac{(x-a)^l}{l!}\right], \quad l = \overline{0, q},$$

$$M_k(t) = R_{(x)}\left[\frac{(x-a)^k}{k!}, \frac{(x-t)_+^q}{q!}\right], \quad k = \overline{0, p}, \tag{1.5}$$

$$N(s, t) = R_{(x)}\left[\frac{(x-s)_+^p}{p!}, \frac{(x-t)_+^q}{q!}\right].$$

Demonstraţie. Se scrie formula lui Taylor relativă la funcţia f şi respectiv funcţia g;

$$f(x) = \sum_{k=0}^{p} \frac{(x-a)^k}{k!} f^{(k)}(a) + \int_a^b \frac{(x-s)_+^p}{p!} f^{(p+1)}(s)ds,$$

$$g(x) = \sum_{l=0}^{q} \frac{(x-a)^l}{l!} g^{(l)}(a) + \int_a^b \frac{(x-t)_+^q}{q!} g^{(q+1)}(t)\, dt.$$

Ţinînd seama de biliniaritatea lui R se obţine

$$R(f, g) = \sum_{k=0}^{p} \sum_{l=0}^{q} f^{(k)}(a) g^{(l)}(a) R\left[\frac{(x-a)^k}{k!}, \frac{(x-a)^l}{l!}\right] +$$

$$+ \sum_{k=0}^{p} f^{(k)}(a) R\left[\frac{(x-a)^k}{k!}, \int_a^b \frac{(x-t)_+^q}{q!} g^{(q+1)}(t) dt\right] +$$

$$+ \sum_{l=0}^{q} g^{(l)}(a) R\left[\int_a^b \frac{(x-s)_+^p}{p!} f^{(p+1)}(s) ds, \frac{(x-a)^l}{l!}\right] +$$

$$+ R\left[\int_a^b \frac{(x-s)_+^p}{p!} f^{(p+1)}(s) ds, \int_a^b \frac{(x-t)_+^q}{q!} g^{(q+1)}(t) dt\right].$$

Dar $R\left[\frac{(x-a)^k}{k!}, \frac{(x-a)^l}{l!}\right] = 0$, pentru $k = \overline{0, p}$, $l = \overline{0, q}$, deoarece formula de cuadratură de tip produs considerată are gradul de exactitate (p, q) şi introducînd pe R sub integrală se obţine reprezentarea

$$R(f, g) = \sum_{k=0}^{p} f^{(k)}(a) \int_a^b R_{(x)}\left[\frac{(x-a)^k}{k!}, \frac{(x-t)_+^q}{q!}\right] g^{(q+1)}(t)\, dt +$$

$$+ \sum_{l=0}^{q} g^{(l)}(a) \int_a^b R_{(x)}\left[\frac{(x-s)_+^p}{p!}, \frac{(x-a)^l}{l!}\right] f^{(p+1)}(s)\, ds +$$

$$+ \int_a^b \int_a^b R_{(x)}\left[\frac{(x-s)_+^p}{p!}, \frac{(x-t)_+^q}{q!}\right] f^{(p+1)}(s) g^{(q+1)}(t)\, dt\, ds$$

adică (1.4), dacă se ţine seama de (1.5)

COROLAR 1.2. *Dacă formula de cuadratură de tip produs* (1.1) *are gradul de exactitate* (p, q), *atunci*

$$R(f, g) = \sum_{l=0}^{q} g^{(l)}(a) \int_a^b L_l(s) f^{(p+1)}(s) ds, \quad \forall (f, g) \in C^{p+1}[a, b] \times \mathfrak{P}_q$$

$$R(f, g) = \sum_{k=0}^{p} f^{(k)}(a) \int_a^b M_k(t) g^{(q+1)}(t) dt, \quad \forall (f, g) \in \mathfrak{P}_p \times C^{q+1}[a, b].$$

COROLAR 1.3. *Dacă formula de cuadratură de tip produs* (1.1) *are gradul de exactitate* (p, q), atunci

$$R(f, g) = \int_a^b \int_a^b N(s, t) f^{(p+1)}(s) g^{(q+1)}(t)\, dt\, ds$$

pentru orice $(f, g) \in F_0^{(p)} \times F_0^{(q)}$, *unde s-a notat prin* $F_0^{(h)}$ *spațiul de funcții*

$$F_0^{(h)}[a, b] = \{f \mid f \in C^{h+1}[a, b],\ f^{(k)}(a) = 0,\ k = \overline{0, h}\}.$$

2. În [4] se găsesc demonstrate următoarele două leme.

LEMA 2.1. *Dacă formula de cuadratură de tip produs* (1.1) *are gradul de exactitate* (m, n), *atunci este de tip interpolatoriu.*

LEMA 2.2. *Formula de cuadratură de tip produs* (1.1) *nu poate avea gradul de exactitate* $(m + 1, n + 1)$.

DEFINIȚIA 2.1. Formula de cuadratură de tip produs (1.1) se numește formulă de cuadratură de tip produs cu grad mare de exactitate, dacă are gradul de exactitate $(m + k, n)$ sau $(m, n + k)$, $k \geqslant 1$.

TEOREMA 2.1. *Dacă* $n \geqslant m + 2$, *există o diviziune* $\Delta_2^{(1)}$ *pentru care gradul de exactitate al formulei* (1.1) *este* $(m, 2n - m - 1)$, *oricare ar fi diviziunea* Δ_1.

Demonstrație. Din Lema 2.1 rezultă că în acest caz formula (1.1) este de tip interpolatoriu. Pe baza formulei restului dată prin (1.3), formula (1.1) va avea gradul de exactitate $(m, 2n - m - 1)$ dacă

$$R(f, g) = \int_a^b f(x) v(x)\, [y_0, y_1, \ldots, y_n, x; g]\, dx = 0 \qquad (2.1)$$

pentru orice $(f, g) \in \mathfrak{L}_m \times \mathfrak{L}_{2n-m-1}$. Această scriere a restului cu un singur termen rezultă din Corolarul 1.1. Condiția (2.1) este echivalentă cu condiția

$$\int_a^b x^k v(x) dx = 0, \qquad k = \overline{0, n - 2}, \qquad (2.2)$$

deoarece polinomul $f(x)\, [y_0, y_1, \ldots, y_n, x; g]$ are gradul cel mult $n - 2$, oricare ar fi $(f, g) \in \mathfrak{L}_m \times \mathfrak{L}_{2n-m-1}$.

Se scrie în continuare v sub forma

$$v(x) = (x - a)(x - b) P_{n-1}(x),$$

unde $P_{n-1} \in \mathfrak{L}_{n-1}$ și are coeficientul lui x^{n-1} egal cu 1. În acest fel condiția (2.2) devine

$$\int_a^b (x - a)(b - x) x^k P_{n-1}(x) dx = 0, \quad k = \overline{0, n - 2}$$

şi care determină în mod unic polinomul P_{n-1}, acesta fiind polinomul ortogonal de grad $n-1$ relativ la intervalul $[a, b]$ şi ponderea $p(x) = (x-a)(b-x)$. Astfel polinomul P_{n-1} are toate rădăcinile reale, distincte şi situate în intervalul (a, b). Se va alege

$$\Delta_2^{(1)}: a = y_0^{(1)} < y_1^{(1)} < \ldots < y_n^{(1)} = b,$$

unde $y_j^{(1)}$, $j = \overline{1, n-1}$, sînt rădăcinile polinomului P_{n-1}.

TEOREMA 2.2 *Dacă $n \geq m + 1$, există două diviziuni $\overline{\Delta}_2^{(2)}$ şi $\widetilde{\Delta}_2^{(2)}$ pentru care gradul de exactitate al formulei* (1.1) *este* $(m, 2n-m)$, *oricare ar fi diviziunea* Δ_1.

Demonstraţie. Se urmează acelaşi raţionament ca şi la teorema precedentă, cu deosebirea că în acest caz se alege ca funcţie pondere

$$p(x) = \begin{cases} x - a, & \text{pentru } \overline{\Delta}_2^{(2)} \\ b - x, & \text{pentru } \widetilde{\Delta}_2^{(2)} \end{cases},$$

iar diviziunile care se obţin sînt respectv

$$\overline{\Delta}_2^{(2)}: a = \overline{y}_0^{(2)} < \overline{y}_1^{(2)} < \ldots < \overline{y}_n^{(2)} < b \text{ şi}$$

$$\widetilde{\Delta}_2^{(2)}: a < \widetilde{y}_0^{(2)} < \widetilde{y}_1^{(2)} < \ldots < \widetilde{y}_n^{(2)} = b,$$

unde $\overline{y}_j^{(2)}$, $j = \overline{1, n}$ sînt rădăcinile polinomului ortogonal de grad n relativ la intervalul $[a, b]$ şi ponderea $p(x) = x - a$, iar $\widetilde{y}_j^{(2)}$, $j = \overline{0, n-1}$, sînt rădăcinile polinomului ortogonal de grad n relativ la intervalul $[a, b]$ şi ponderea $p(x) = b - x$.

TEOREMA 2.3. *Dacă $n \geq m$, există o diviziune $\Delta_2^{(3)}$ pentru care gradul de exactitate al formulei* (1.1) *este* $(m, 2n - m + 1)$, *oricare ar fi diviziunea* Δ_1.

Demonstraţie. Funcţia pondere care se consideră în acest caz este $p(x) = 1$, iar diviziunea care trebuie aleasă este

$$\Delta_2^{(3)}: a < y_0^{(3)} < y_1^{(3)} < \ldots < y_n^{(3)} < b,$$

unde $y_j^{(3)}$, $j = \overline{0, n}$, sînt rădăcinile polionmului Legendre de grad $n + 1$ relativ la intervalul $[a, b]$,.

3. Se consideră următorul exemplu: $[a, b] \equiv [0, 1]$, $m = 1$, $n = 2$, Δ_1: $x_0 = 0$, $x_1 = 1$.

Dacă se alege $\Delta_2: y_0 = 0, \quad y_1 = \frac{1}{2}, \; y_2 = 1$, se obţine formula de cuadratură de tip produs trapez-Simpson, care a fost dată în [4] şi care se scrie sub formă matricială astfel:

$$\int_0^1 f(x)g(x)dx = \frac{1}{6}\,(f(0)\; f(1)) \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} g(0) \\ g\left(\frac{1}{2}\right) \\ g(1) \end{pmatrix} + R(f, g)$$

cu gradul de exactitate (1, 2).

Folosind reprezentarea restului dată de Teorema 1.1 se poate obține o margine superioară pentru rest, şi anume:

$$|R(f, g)| \leqslant \frac{1}{12} M_f^2 M_g^0 + \frac{1}{192} M_f^0 M_g^3 + \frac{1}{2304} M_f^2 M_g^3, \ \forall (f, g) \in C^2 [0, 1] \times C^3 [0, 1],$$

$$|R(f, g)| \leqslant \frac{1}{192} M_f^0 M_g^3, \qquad \forall (f, g) \in \mathfrak{P}_1 \times C^3[0, 1],$$

$$|R(f, g)| \leqslant \frac{1}{12} M_f^2 M_g^0, \qquad \forall (f, g) \in C^2 [0, 1] \times \mathfrak{P}_2,$$

unde s-a notat

$$M_h^r = \max_{x \in [0,1]} |h^{(r)}(x)|.$$

Conform Teoremei 1.2 şi Corolarului 1.3 pentru rest se obțin reprezentările:

$$R(f, g) = f(0) \int_0^1 M_0(t) g'''(t) dt + f'(0) \int_0^1 M_1(t) g'''(t) dt +$$

$$+ g(0) \int_0^1 L_0(s) f''(s) ds + g'(0) \int_0^1 L_1(s) f''(s) ds + g''(0) \int_0^1 L_2(s) f''(s) ds +$$

$$+ \int_0^1 \int_0^1 N(s, t) f''(s) g'''(t) dt\, ds, \quad \forall (f, g) \in C^2[0, 1] \times C^3[0, 1]$$

şi respectiv

$$R(f, g) = \int_0^1 \int_0^1 N(s, t) f''(s) g'''(t) dt\, ds, \quad \forall (f, g) \in F_0^{(1)} \times F_0^{(2)}$$

unde

$$L_0(s) = -\frac{s(1-s)}{2},$$

$$L_1(s) = -\frac{s(1-s^2)}{6},$$

$$L_2(s) = -\frac{s(1-s^3)}{24},$$

$$M_0(t) = \frac{(1-2t)(1-t)^2}{12} - \frac{1}{3}\left(\frac{1}{2} - t\right)_+^2,$$

$$M_1(t) = \frac{(1-t)^2(1-2t-t^2)}{24} - \frac{1}{6}\left(\frac{1}{2} - t\right)_+^2,$$

$$N(s, t) = \frac{(s-t)_+^4}{24} - \frac{(1-t)^4}{24} + \frac{(1-s)(1-t)^3}{6} - \frac{(1-s)(1-t)^2}{12} + \frac{1-s}{6}\left(\frac{1}{2} - t\right)_+^2.$$

Pe baza Teoremei 2.2 se obţin respectiv formulele de cuadratură de tip produs cu gradul de exactitate (1, 3):

$$\int_0^1 f(x)g(x)dx = \frac{1}{36}f(0)\ f(1))\begin{pmatrix} 4 & 7+2\sqrt{6} & 7-2\sqrt{6} \\ 0 & 9-\sqrt{6} & 9+\sqrt{6} \end{pmatrix}\begin{pmatrix} g(0) \\ g\left(\frac{6-\sqrt{6}}{10}\right) \\ g\left(\frac{6+\sqrt{6}}{10}\right) \end{pmatrix} + \bar{R}^{(2)}(f, g)$$

$$\int_0^1 f(x)g(x)dx = \frac{1}{36}(f(0)\ f(1))\begin{pmatrix} 9+\sqrt{6} & 9-\sqrt{6} & 0 \\ 7-2\sqrt{6} & 7+2\sqrt{6} & 4 \end{pmatrix}\begin{pmatrix} g\left(\frac{4-\sqrt{6}}{10}\right) \\ g\left(\frac{4+\sqrt{6}}{10}\right) \\ g(1) \end{pmatrix} + \tilde{R}^{(2)}(f, g).$$

Folosind Teorema 1.1 se obţine aceeaşi margine superioară pentru cele două resturi:

$$|\bar{R}^{(2)}(f, g)| \leqslant \frac{1}{12}M_f^2M_g^0 + \frac{\sqrt{6}}{625}M_f^0M_g^2 + \frac{0.40\,32\sqrt{6} - 0\cdot 3125}{2826}M_f^2M_g^2 \quad \text{şi}$$

$$|\tilde{R}^{(2)}(f, g)| \leqslant \frac{1}{12}M^2M^0 + \frac{\sqrt{6}}{625}M^0M^2 + \frac{0.4032\sqrt{6} - 0.3125}{2826}M^2M^3,$$

$$\forall (f, g) \in C^2[0, 1] \times C^3[0, 1].$$

Dacă se aplică Teorema 2.3 se obţine formula de cuadratură de tip produs cu gradul de exactitate (1, 4)

$$\int_0^1 f(x)g(x)dx = \frac{1}{36}(f(0)\ f(1))\begin{pmatrix} 5+\sqrt{15} & 8 & 5-\sqrt{15} \\ 5-\sqrt{15} & 8 & 5+\sqrt{15} \end{pmatrix}\begin{pmatrix} g\left(\frac{5-\sqrt{15}}{10}\right) \\ g\left(\frac{1}{2}\right) \\ g\left(\frac{5+\sqrt{15}}{10}\right) \end{pmatrix} + R^{(3)}(f, g)$$

avînd următoarea margine superioară pentru rest

$$|R^{(3)}(f, g)| \leqslant \frac{1}{12}M_f^2M_g^0 + \frac{13}{4800}M_f^0M_g^3 + \frac{7}{18\,000}M_f^2M_g^3,\ \forall (f, g) \in C^2[0, 1] \times C^3[0, 1].$$

Ca şi în cazul formulei de cuadratură de tip produs trapez-Simpson se pot da, în condiţiile Teoremei 1.2, reprezentări ale restului de forma (1.4).

În continuare se folosesc cele patru formule de cuadratură de tip produs, pentru calculul aproximativ al integralei definite $\int_0^1 \frac{2x}{1+x^2}dx = \ln 2 = 0.693147\ldots$, pentru care se obţin respectiv valorile:

$$\frac{7}{10} = 0.7,\ \frac{134}{193} = 0.6943\ldots,\ \frac{301}{435} = 0.69195\ldots,\ \frac{1882}{2715} = 0.693186\ldots,$$

dacă se consideră $f(x) = 2x$ şi $g(x) = \frac{1}{1+x^2}$.

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BIBLIOGRAFIE

1. Boland, W. R., *The convergence of product-type quadrature formulas*, SIAM J. Numer. Analysis, **9**, 1972, 6–13.
2. Boland, W. R., *Properties of product-type quadrature formulas*, BIT, **13**, 1973, 287–291.
3. Boland, W. R., *The numerical solution of Fredholm integral equations using product-type quadrature formulas*, BIT, **12**, 1972, 5–16.
4. Boland, W. R., Duris, C. S., *Product-type quadrature formulas*, BIT, **11**, 1971, 139–158.

PRODUCT-TYPE QUADRATURE FORMULAS WITH GREAT DEGREE OF EXACTNESS

(Summary)

Using Taylor's formula, the integral form for the remainder of the product-type quadrature formulas is given. The choice of the knots, for getting the product-type quadrature formulas with great degree of exactness is presented. Some numerical examples are given.

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O METODĂ DE CALCUL A SOLUŢIEI MINIME L_∞ A SISTEMELOR DE ECUAŢII LINIARE NEDETERMINATE

MONICA ALBU

Se consideră sistemul de ecuaţii liniare $Ax = b$, unde $A \in M_{nm}(\mathbf{R})$, $n < m$, $b \in M_{n1}(\mathbf{R})$, $b \neq 0$. Se cere să se determine $x^* \in \mathbf{R}^m$ astfel încît $||x^*||_\infty = \inf\{||x||_\infty : Ax = b\}$.

Această problemă a fost studiată de Cadzow [2, 3], care dă şi algoritmi de rezolvare a problemei.

Recent, Abdelmalek [1] reduce problema dată la problema de programare liniară

$$(P)\begin{cases} h \to \min \\ x + he \geqslant 0 \\ -x + he \geqslant 0 \\ Ax = b \\ h \geqslant 0 \end{cases} \qquad \text{cu } h = ||x||_\infty = \max\{|x_1|, |x_2|, \ldots, |x_m|\}$$

$$\text{şi } e = \begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix} \in M_{m1}$$

pe care o rezolvă cu ajutorul algoritmului simplex al lui Dantzig.

În prezenta lucrare vom descrie un nou algoritm pentru găsirea soluţiei x^* utilizînd o altă variantă a algoritmului simplex [4], care nu necesită introducerea unor variabile auxiliare.

Vom rezolva duala problemei (P):

$$(D)\begin{cases} b^T v \to \max \\ [E \; -E \;\; A^T]\begin{bmatrix} u \\ v \end{bmatrix} = 0 \\ [e^T \quad e^T \;\; 0]\begin{bmatrix} u \\ v \end{bmatrix} \leqslant 1 \\ u \geqslant 0 \end{cases} \qquad \text{unde } E \in M_{m}(\mathbf{R}), \quad u \in \mathbf{R}^{2m},\ v \in \mathbf{R}^n$$

Întroducem variabila $w \geqslant 0$ astfel încît să avem

$$\begin{bmatrix} E & -E & A^T & 0 \\ e^T & e^T & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e_{m+1}$$

LEMA 1. *Dacă w intră în baza optimă atunci valoarea optimă a funcției de scop este $b^T v^* = 0$.*

Dar $b^T v^* = 0$ implică $h^* = ||x^*||_\infty = 0$ și

$$b = Ax^* = 0,$$

ceea ce contrazice ipoteza.

CONSECINȚA 1. *Componenta w^* nu poate intra în baza optimă.*

Atunci vom putea presupune de la început $w = 0$, iar problema (D) se va transcrie

$$\begin{cases} b^T v \to \max \\ \begin{bmatrix} E & -E & A^T \\ e^T & e^T & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = e_{m+1} \\ u \geqslant 0 \end{cases}$$

Coustruim tabelul simplex pentru această problemă.

	D_1	D_2	...	D_m	D_{m+1}	...	D_{2m}	D_{2m+1}	...	D_{2m+n}	
	$-u_1$	$-u_2$	...	$-u_m$	$-u_{m+1}$	...	$-u_{2m}$	$-v_1$	...	$-v_n$	1
$0 = y_1$											0
$0 = y_2$			E			$-E$			A^T		0
...											...
$0 = y_m$											0
$0 = y_{m+1}$	1	1		1	1		1	0		0	1
	0	0		0	0		0	$-b_1$		$-b_n$	0

În prima etapă eliminăm variabilele v_i, acestea nefiind supuse restricției de nenegativitate. Coloanele $y_j = 0$, care trec în locul variabilelor v_i eliminate, pot fi omise din tabel. Obținem un tabel de forma:

	$\tilde{D}_1$	...	$\tilde{D}_m$	$\tilde{D}_{m+1}$	...	$\tilde{D}_{2m}$	$\tilde{D}_{2m+i_{k+1}}$	...	$\tilde{D}_{2m+i_n}$	
	$-u_1$	...	$-u_m$	$-u_{m+1}$	...	$-u_{2m}$	$-v_{i_{k+1}}$	...	$-v_{i_n}$	
$v_{i_1} =$										0
$v_{i_2} =$										
...										...
$v_{i_k} =$										
$0 = y_{j_{k+1}}$								0		0
...										
$0 = y_{j_m}$										
$0 = y_{m+1}$	1	...	1	1	...	1	0	...	0	1
	q_1	..	q_m	q_{m+1}	...	q_{2m}	$\tilde{q}_{i_{k+1}}$	...	$\tilde{q}_{i_n}$	

Liniile $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ pot fi omise din tabel. Coloanele $\tilde{D}_{2m+i_{k+1}}, \ldots, \tilde{D}_{2m+i_n}$ nu au putut fi eliminate dacă au avut zero în dreptul liniilor $y_{j_{k+1}}, \ldots, y_{j_m}$. Lucrînd cu acest tabel simplex coloanele variabilelor $v_{i_{k+1}}, \ldots, v_{i_n}$ vor rămîne neschimbate (inclusiv $\tilde{q}_{i_{k+1}}, \ldots, \tilde{q}_{i_n}$).

Înseamnă că aceste coloane le putem omite din noul tabel simplex. Vom lucra în continuare cu tabelul

	C_1	C_2	$\ldots$	C_m	C_{m+1}	$\ldots$	C_{2m}	
	$-u_1$	$-u_2$	$\ldots$	$-u_m$	$-u_{m+1}$	$\ldots$	$-u_{2m}$	1
$0 = y_{j_{k+1}}$								0
$\ldots$								
$0 = y_{j_m}$								0
$0 = y_{m+1}$	1	1	$\ldots$	1	1	$\ldots$	1	1
	q_1	q_2	$\ldots$	q_m	q_{m+1}	$\ldots$	q_{2m}	0

(1)

Coloanele C_j, $j = \overline{1, 2m}$ conţin ultimele $m - k + 1$ componenete ale coloanelor $\tilde{D}_j$, $j = \overline{1, 2m}$

LEMA 2. *Au loc relaţiile* $\begin{cases} C_j + C_{m+j} = 2e_{m+1-k} & j = \overline{1, m} \\ q_j + q_{m+j} = 0 & j = \overline{1, m} \end{cases}$

Demonstraţie. Coloanele $\tilde{D}_j$, $j = \overline{1, 2m}$ se obţin din coloanele D_j, $j = \overline{1, 2m}$ prin efectuarea unor paşi Gauss—Jordan modificaţi. Iar pentru coloanele D_j aveau loc relaţiile

$$\begin{cases} D_j + D_{m+j} = 2e_{m+1} & j = \overline{1, m} \\ q_j + q_{m+j} = 0 & j = \overline{1, m} \end{cases}$$

Variabilele u_j şi u_{j+m} le vom numi variabile corespondente, iar coloanele lor din tabelul simplex, coloane corespondente.

Eliminăm acum liniile nule din ultimul tabel simplex şi construim o bază.

Dacă variabilele $u_{i_1}, u_{i_2}, \ldots, u_{i_{m+1-k}}$ se găsesc în bază şi notăm cu B matricea formată din coloanele $C_{i_1}, C_{i_2}, \ldots, C_{i_{m+1-k}}$, atunci coloanele noului tabel simplex notate cu Y, verifică relaţiile:

$$\begin{bmatrix} Y_j \\ q_j' \end{bmatrix} + \begin{bmatrix} Y_{m+j} \\ q_{m+j}' \end{bmatrix} = \begin{bmatrix} B & 0 \\ q^T & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} C_j \\ q_j \end{bmatrix} + \begin{bmatrix} C_{m+j} \\ q_{m+j} \end{bmatrix} \right) = 2 \begin{bmatrix} B^{-1} e_{m+1-k} \\ -q^T B^{-1} e_{m+1-k} \end{bmatrix} = 2 \begin{bmatrix} b_B \\ z \end{bmatrix}$$

unde $q^T = [q_{i_1}, q_{i_2}, \ldots, q_{i_{m+1-k}}]$, $j \not\in \{i_1, \ldots, i_{m+1-k}\}$ Dacă C_j este coloana variabilei corespondente celei care a intrat în bază pe linia r atunci $\begin{bmatrix} Y_j \\ q_j' \end{bmatrix} = 2 \begin{bmatrix} b_B \\ z \end{bmatrix} - \begin{bmatrix} e_r \\ 0 \end{bmatrix}$

LEMA 3. *Fie b_B o soluţie de bază. Atunci există două baze B_1 şi B_2 cărora le corespunde $b_B^=$. Valoarea funcţiei de scop pentru cele două baze este egală în valoarea absolută dar de semn opus.*

Demonstraţie. Fie $B_1 = (u_{i_1}, u_{i_2}, \ldots, u_{i_{m+1-k}})$ o bază şi soluţia de bază corespunzătoare ei b_B. Atunci $B_2 = (u_{j_1}, u_{j_2}, \ldots, u_{j_{m+1-k}})$, u_{i_s} şi u_{j_s} corespondente este baza căutată.

LEMA 4. *Considerăm cele două baze B_1 şi B_2 definite ca în lema 3. Fie T_1 şi T_2 tabelele simplex corespunzătoare celor două baze. Atunci, dacă $|i - j| = m$, avem*

$$\begin{bmatrix} Y_i \\ q_i' \end{bmatrix} \text{ în } T_1 = \begin{bmatrix} Y_j \\ -q_j' \end{bmatrix} \text{ în } T_2$$

Observăm că dacă se cunosc coloanele $Y_1, Y_2, \ldots, Y_m$ şi $q_1', q_2', \ldots, q_m'$ atunci sînt cunoscute şi $Y_{m+1}, Y_{m+2}, \ldots, Y_{2m}, q_{m+1}', q_{m+2}', \ldots, q_{2m}'$. Înseamnă că vom putea lucra de la început cu un tabel simplex condensat, conţinînd doar cîte o singură coloană din fiecare pereche de coloane corespondente.

Dacă valoarea funcţiei de scop corespunzătoare primei baze admisibile construite este negativă atunci, conform lemei 3, se înlocuieşte baza cu baza corespondentă ei. Soluţia de bază b_B rămîne aceeaşi, dar fiecare coloană din tabelul simplex se înlocuieşte cu coloana corespondentă conform lemei 4. Pozitivăm linia funcţiei de scop. La fiecare pas calculăm q_j', $j = \overline{1, 2m}$. Dacă pentru un j avem $q_j' < 0$ şi coloana $\begin{bmatrix} Y_j \\ q_j' \end{bmatrix}$ nu este în tabelul simplex atunci se înlocuieşte coloana corespondentă ei din tabelul simplex cu aceasta. Dacă o variabilă u_j este în bază atunci $q_j' = 0$ iar în coloana corespondentă ei $q_i' = 2z \geq 0$.

Presupunem că am ajuns la ultimul tabel simplex

	$-u_{j_{m+2-k}} \quad \cdots \quad -u_{j_m}$	
u_{j_1} $\cdots$ $u_{j_{m+1-k}}$		$\tilde{b}_{B^*}$
	$q^*_{j_{m+2-k}} \quad \cdots \quad q^*_{j_m}$	z^*

Din acest tabel citim şi soluţia optimă a problemei (P).

LEMA 5

$$j = \overline{1, m} \quad x_j^* = \begin{cases} -z^* & \text{dacă } u_j \text{ este în baza optimă} \\ z^* & \text{dacă } u_{j+m} \text{ este în baza optimă} \\ q_j^* - z^* & \text{dacă } u_j \text{ nu este în baza optimă dar apare în ultimul tabel simplex} \\ z^* - q_{j+m}^* & \text{dacă } u_{j+m} \text{ nu este în baza optimă dar apare în ultimul tabel simplex} \end{cases}$$

Observaţie. Se constată că baza optimă nu poate conţine variabile corespondente.

Descrierea algoritmului

Pasul 1. Construim tabelul simplex corespunzător problemei

$$\begin{cases} b^T v \to \max \\ \begin{bmatrix} E & A^T \\ e^T & 0 \end{bmatrix} \begin{bmatrix} \tilde{u} \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \tilde{u} \geqslant 0 \qquad \text{unde } \tilde{u} = (u_1, u_2, \ldots, u_m) \end{cases}$$

Pasul 2. Se elimină variabilele v_i, $i = \overline{1, n}$ şi se construieşte o bază admisibilă în modul următor: presupunem că s-au eliminat liniile nule $i_1, i_2, \ldots, i_k$. Ne fixăm asupra unui indice i_s, $s \in \{1, 2, \ldots, k\}$ şi notăm cu

$$C_{i_s} = (c_{j_1 i_s}, c_{j_2 i_s}, \ldots, c_{j_{m-k} i_s}, 1)^T$$

$j_r \in \{1, 2, \ldots, m\} \setminus \{i_1, i_2, \ldots, i_k\}$, $r = \overline{1, m-k}$, coloana variabilei u_{i_s} din ultimul tabel simplex. Variabila u_{i_s} împreună cu variabilele u_j, $j \in \{j_1, j_2, \ldots, j_{m-k}\}$ pentru care $c_{ji_s} \leqslant 0$ şi corespondentele variabilelor u_j, $j \in \{j_1, j_2, \ldots, j_{m-k}\}$ pentru care $c_{ji_s} > 0$ formează o bază admisibilă. Dacă valoarea funcţiei de scop corespunzătoare acestei baze este negativă, se înlocuieşte baza cu corespondenta sa. Soluţia de bază nu se schimbă, dar coloanele tabelului simplex se înlocuiesc cu coloanele corespondente lor.

Pasul 3. Se trece la pozitivarea liniei funcţiei de scop, calculînd la fiecare pas q_j pentru variabilele nebazice şi pentru corespondentele acestora. Dacă există $q_{j_0} < 0$ şi variabila u_{j_0} nu apare în tabelul simplex, atunci se înlocuieşte corespondenta acesteia din tabelul simplex cu u_{j_0}. Dacă toţi q_j, $j = \overline{1, 2m}$ sînt nenegativi, atunci se calculează soluţia optimă a problemei (P) conform lemei 5.

Dacă problema (D) are optim infinit atunci sistemul $Ax = b$ este incompatibil.

Exemplu. Să se obţină soluţia minimă L_∞ pentru sistemul de ecuaţii

$$\begin{cases} 7x_1 - 4x_2 + 5x_3 + 3x_4 + x_5 = -30 \\ -2x_1 + x_2 + 5x_3 + 4x_4 + x_5 = 15 \\ 5x_1 - 3x_2 + 10x_3 + 7x_4 + 2x_5 = -15 \end{cases}$$

Pasul 1. Construim tabelul simplex

	$-u_1$	$-u_2$	$-u_3$	$-u_4$	$-u_5$	$-v_1$	$-v_2$	$-v_3$	1
$0 = y_1$	1	0	0	0	0	7	-2	5	0
$0 = y_2$	0	1	0	0	0	-4	1	-3	0
$0 = y_3$	0	0	1	0	0	5	5	10	0
$0 = y_4$	0	0	0	1	0	3	4	7	0
$0 = y_5$	0	0	0	0	1	1	1	2	0
$0 = y_6$	1	1	1	1	1	0	0	0	1
f	0	0	0	0	0	30	-15	15	0

Pasul 2. Eliminăm variabilele v_1, v_2, v_3 şi liniile nule. Obţinem tabelul

$-u_1$	$-u_2$	$-u_3$	$-u_4$	$-u_5$		1
0 =	25	45	1	0	0	0
0 =	19	34	0	1	0	0
0 =	5	9	0	0	1	0
0 =	1	1	1	1	1	1
	-30	-45	0	0	0	0

Vom construi o bază admisibilă cu variabilele u_8, u_9, u_{10}, u_1. Rezultă tabelul

	$-u_2$	1
u_8 =	$-1/2$	1/2
u_9 =	$-9/50$	19/50
u_{10} =	$-1/10$	1/10
u_1 =	89/50	1/50
	42/5	3/5

Deoarece $q_2 = 2 \cdot 3/5 - 42/5 < 0$ se înlocuieşte variabila u_2 cu corespondenta sa:

	$-u_7$	1
u_8 =	3/2	1/2
u_9 =	47/50	19/50
u_{10} =	3/10	1/10
u_1 =	$-87/50$	1/50
	$-36/5$	3/5

Efectuăm un pas Gauss-Jordan modificat cu elementul pivot $a_{8,7} = 3/2$ şi ajungem la ultimul tabel simplex

	$-u_8$	1
u_7 =	2/3	1/3
u_9 =	$-47/75$	2/30
u_{10} =	1/5	0
u_1 =	29/25	3/5
	24/5	3

Soluţia optimă este $x^* = (-3,\ 3,\ -9/5,\ 3,\ 3)$.

BIBLIOGRAFIE

1. Abdelmalek, N., *Minimum L_∞ solution of underdetermined systems of linear equations*, Journal of Approximation Theory, **20**, 1977, 57–69.
2. Cadzow, J., *A finite algorithm for the minimum l_∞ solution to a system of consistent linear equations*, SIAM J. Numer. Anal., **10**, 4, 1973, 607–617.

3. Cadzow, J., *An efficient algorithmic procedure for obtaining a minimum l_∞-norm solution to a system of consistent linear equations*, SIAM J. Numer. Anal., **11**, 6, 1974, 1151–1165.
4. Maruşciac, I., *Metode de rezolvare a problemelor de programare neliniară*, Ed. Dacia, Cluj, 1973.

A COMPUTATION METHOD FOR THE MINIMAL L_∞ SOLUTION OF THE UNDETERMINED LINEAR EQUATIONS

(Summary)

In this paper an algorithm for finding the minimal L_∞ solution of the und etermined linear equation system $Ax = b$ is described using a variant of the simplex algorithm which does not require the introduction of any auxiliary variables.

RECENZII

Ioan Maruşceac, **Programare geometrică şi aplicaţii** (Geometric programming and applications), Ed. Dacia, Cluj-Napoca, 1978, 348 pp. + 30 pp. programs.

This book provides a systematic and easy readable, but yet a rather penetrating presentation of both the theory and application of geometric programming. Following the publication in 1967 of the book by R. J. Duffin, E. L. Peterson and C. Zener, geometric programming has developed to an important branch of mathematical programming with various fine applications.

Since only minimal prerequisites are supposed, the first chapter gives an account of the main theoretical backgrounds of mathematical programming such as convex functions, theorem of Helly, Farkas-Minkowski, Fritz John, Kuhn-Tucker, duality in linear programming etc.

Chapter II is devoted to the foundation of geometric programming. The geometric inequality and duality in geometric programming are treated in full details, including a recent result by A. C. Williams and M. Avriel.

Chapter III covers a great deal of material, focusing attention on a series of key theorems. The paragraph headings of this chapter are: classification, generalized geometric program, rational geometric programming, polynomial geometric programming. Mention should be made of the author's elegant way to approach the complementary geometriç program.

Chapter IV deals with numerical methods to solve geometric programs. Among others, the reader will be able to handle the cutting-plane, the condensation and the interior penalty (or SUMT) methods.

In chapter V, devoted entirely to applications, we find vectorial programming, applications in approximation theory, approximation of positive solutions of some nonlinear equations, chemical equilibrium problems, electrical networks, realiability achieved by redundancy, optimization of sea water desalting technology, nonlinear assignment problems, all treated with geometric programming. The book ends with three FORTRAN programs for the SUMT, linearizing and condensing algorithms.

The book under review will certainly ease the way to learn and handle the tools of geometric programming for anyone who wants to learn it.

F. RADO

Németi Ladislau, **Programarea în timp a fabricaţiei** (La programation temporelle de la fabrication), Ed. Dacia, Cluj-Napoca, 1975, 197 pag.

Le travail traite le bienconnu problème de l'ordonnancement de la fabrication, constituant un problème important dans le lancement et dans la programmation de la fabrication d'une entreprise.

L'auteur, spécialiste reconnu de ce domaine, a synthétisé et réuni dans ce livre ses principales recherches effectuées dans la période 1963—1975, ainsi que plusieurs résultats importants des autres auteurs concernant les problèmes de l'ordonnancement de la fabrication.

Le livre est constitué de trois parties: ,,Fondements mathématiques", ,,Modèles mathématiques de l'ordonnancement" et ,,Méthodes approximatives". Dans la première partie ,,Fondements mathématiques" on introduit des notions de la théorie des graphes et on donne les formulations des quelques problèmes mathématiques utilisés dans la résolution des problèmes de l'ordonnancement de la fabrication. Ainsi, dans le premier chapitre on présente le problème de la programmation mathématique à contraintes disjonctives et un algorithme pour la résolution de celui-ci. Le deuxième chapitre contient quelques notions de la théorie des graphes et une étude du bien connu problème des potentiels.

La deuxième partie du livre ,,Modèles mathématiques de l'ordonnancement", comprend 4 chapitres où l'on donne des modèles mathématiques pour les divers cas particuliers du problème de l'ordonnancement de la fabrication. On étudie les cas suivants: la fabrication des pièces uniques dans le cas où il y a seulement un exemplaire de chaque type de machine-outil (chapitre 3), ou plusieurs exemplaires du même type de machine-outil (chapitre 4), la fabrication en série (chapitre 5) et le cas où on travaille par plusieurs relèves (chapitre 6).

Dans la troisième partie intitulée ,,Méthodes approximatives" (chapitres 7—9) on présente des méthodes approximatives pour la résolution des problèmes de l'ordonnancement de la fabrication, à savoir: les méthodes euristiques de type global (chapitre 9) ou de type local (chapitre 8) et des méthodes d'approximation ayant à la base des modèles mathématiques.

Certains chapitres possèdent des exemples résolus à l'aide des algorithmes présentés. A la fin du travail, l'auteur fait une analyse critique des possibilités d'application des modèles et des méthodes étudiées.

ŞTEFAN ŢIGAN

S. Grossmann, **Mathematischer Einführungskurs für die Physik**, Teubner Studienbücher, B. G. Teubner, Stuttgart, 1976.

This book is intended as a first course for students in Physics. The subjects treated cover the necessities of the basic courses of experimental and of clasical theoretical physics. The author justifies through various examples from physics the necessity of the introduced mathematical notions. They are introduced rigorously and in certain parts equivalent formulations are given. These allow a better understanding of the notion and they show the students the possibility of using these notions in physics. A positive feature of the book is the emphasy on the applications of the theory in practice, and many methods of computing and solving problems. Each chapter contains a lot of examples, many of them solved.

The content of the book is as follows:

1. Vectors, transformation of coordinates, matrices, determinants.
2. Vector-functions, differentiation of vector-functions, space curves.
3. Fields, partial derivatives, gradient, divergence, rotation, Δ-operator.
4. Integration, improper integral, integrals with respect to parameters, δ-function.
5. Integral of vector functions, line integral, surface integral, volume integral.
6. Integral theorems, representations of the Δ-operator. Gauss theorem, partial integration, Stokes' theorem.
7. Curvilinear coordinates, differential operators in curvilinear-orthogonal coordinates.

P. SZILAGYI

R. D. Grigorieff, **Numerik gewöhnlicher Differentialgleichungen**, Band 2 Mehrschrittverfahren, Teubner Studienbücher, B. G. Teubner, Stuttgart, 1977.

The second volume of R. D. Grigorieff's book is devoted to the study of multistep methods for ordinary differential equations.

The chapters of the book are as follows: Multistep methods for first order systems; Asymptotic properties of multistep methods; Domains of stability of multistep methods; Predictor-corrector methods; Some special methods and Methods for higher order systems.

The way in which the material of the book was arranged makes it useful for the specialist interested only in the algorithmical side of most efficient methods, as well as for the mathematician interested in the most recent development of the theory. The latter will find a thorough study of topics like: stiff systems, variable step-sizes, asymptotic developments, optimal error estimates, etc.

The references contain more than 300 entries.

E. SCHECHTER

I. P. Cluj, Municipiul Cluj-Napoca — Cd. 3062

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