STUDIA
UNIVERSITATIS BABEȘ-BOLYAI

MATHEMATICA

1

1978

CLUJ-NAPOCA

ANUL XXIII 1978

STUDIA
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA

1

Redacţia: 3400 CLUJ-NAPOCA, str. M. Kogălniceanu, 1 ● Telefon 1 34 50

SUMAR — CONTENTS — SOMMAIRE — INHALT

STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, 1, 1978

MAGNETOHYDRODYNAMIC AND FREE CONVECTION EFFECTS ON THE OSCILLATORY FLOW PAST A VERTICAL PLATE

S. NARASIMHA MURTHY*, M. R. KRISHNA MURTHY** and H. M. THIMMARAYAPPA**

Introduction. The experimental and theoretical studies on oscillatory flows are important from the technological point, for they have many practical applications. Stuart (1955) extended the study of steady two-dimensional flow of an incompressible viscous fluid which has been earlier studied by Lighthill (1954) in relation to a steady two-dimensional flow past an infinite porous plate when the free stream oscillates in time about a constant mean.

Recently Soudalgekar (1973) considered effects of free convection on the oscillatory flow past an infinite, vertical porous plate with constant suction. The free convection currents are caused by maintaining the difference between the plate temperature T'_w and the free stream temperature T'_∞ namely $T'_w - T'_\infty$ as fairly large. The free stream velocity oscillates about a constant mean in the direction of the flow. He (1973) observed that with constant temperature along the plate, the mean flow, though not affected by the frequency of the oscillatory flow, is affected by the free convection currents, Eckert number E and Prandtl number P. He further observed that there is a reverse flow with small Prandtl number in the boundary layer.

The object of the present investigation is to study the effects of transverse magnetic field and free convection on the oscillatory flow past a vertical plate. The fluid is considered to be electrically conducting. Such a type of study is not only of theoretical interest but also of aerodynamic interest in regard to heat transfer characteristics in the aerofoil. Gupta (1960, 1962), also studied the free convection and transverse magnetic field effects on the flow past a vertical plate when the flow is not oscillatory by using similarity variables. He found that the similarity solutions are valid only if the magnetic field varies inversely as the fourth power of the distance along the plate. The present analysis shows no such constraint on the magnetic field. We present here only the mean flow. The oscillatory flow phenomenon will be discussed separately.

Mathematical formulation. We consider the motion of a two-dimensional, unsteady, viscous, incompressible, electrically conducting flow

---

* Department of Mathematics, College of Science, University of Mosul, Mosul, IRAQ (To whom communications may be addressed)

** Department of Mathematics, College of Basic Sciences and Humanities University of Agricultural Sciences, Bangalore-24, INDIA.

past an infinite, porous vertical plate with constant suction. The x'-axis is taken along the plate in the upward direction, being the direction of the flow and y'-axis normal to the plate. The applied magnetic field is in the $y°$-direction. We also assume the well known Boussinesq approximation which implies that density variation with temperature is considered only in the body force term. The magnetic Reynolds number is also assumed small, which is true in most of the laboratory conducting fluids, so that the applied magnetic field is hardly affected by the induced magnetic field. The physical variables are functions of y' and t'. Thus the governing equations are: Momentum equations:

$$\rho'\left(\frac{\partial u'}{\partial t'} + v'\frac{\partial u'}{\partial y'}\right) = -\frac{\partial p'}{\partial x'} - p' g_x + \mu\frac{\partial^2 u'}{\partial y'^2} - \sigma B_0^2 u', \tag{1}$$

$$\frac{\partial v'}{\partial t'} = -\frac{1}{\rho'}\frac{\partial p'}{\partial y'}. \tag{2}$$

Continuity equation:

$$\frac{\partial v'}{\partial y'} = 0. \tag{3}$$

Energy equation:

$$\rho' c_p\left(\frac{\partial T'}{\partial x} + v'\frac{\partial y'}{\partial T'}\right) = K\frac{\partial^2 T'}{\partial y'^2} + \mu\left(\frac{\partial u'}{\partial y'}\right)^2, \tag{4}$$

where u', v' are the velocity components in x' and y' direction, g_x is the acceleration due to gravity, σ is the electrical conductivity, K is the thermal conductivity, B_0 is the strength of the magnetic field. The other quantities have usual meaning. The heat due to ohmic dissipation is neglected in comparison with the viscous dissipation in the energy equation.

The boundary conditions are

$$\left.\begin{array}{lll} u' = 0, & T' = T'_w & \text{at } y' = 0, \\ u' = U'(t'), & T' = T'_\infty & \text{as } y' \to \infty. \end{array}\right] \tag{5}$$

Equation (1) in the free stream is

$$\rho'\frac{\partial u'}{\partial t'} + \rho'_\infty g_x + \sigma B_0^2 U' = -\frac{\partial p'}{\partial x'}. \tag{6}$$

Further from the equation of state

$$g_x(\rho'_\infty - \rho') = g_x\beta\rho'(T' - T'_\infty). \tag{7}$$

Using (6) and (7) in (1) we get

$$\frac{\partial u'}{\partial t'} + v'\frac{\partial u'}{\partial y'} = \frac{\partial u'}{\partial t'} + g_x\beta(T' - T'_\infty) + \nu\frac{\partial^2 u'}{\partial y'^2} + \sigma\frac{\sigma B_0^2}{\rho'}(U' - u') \tag{8}$$

where $\nu = \frac{\mu}{\rho'}$, the kinematic viscosity.

As there is a constant suction velocity v_0 at the plate, equation (3) yields

$$v_0' = -v_0. \tag{9}$$

In view of (9) equations (8) and (4) now reduce to

$$\frac{\partial u'}{\partial t'} - v_0 \frac{\partial u'}{\partial y'} = \frac{\partial U'}{\partial t'} + g_x\beta(T' - T'_\infty) + \nu \frac{\partial^2 u'}{\partial y'^2} + \sigma \frac{\sigma B_0^2}{\rho'}(U' - u') \tag{10}$$

$$\frac{\partial T}{\partial t'} - v_0 \frac{\partial u'}{\partial y'} = \frac{\partial u'}{\partial t'} + g_x\beta(T' - T'_\infty) + \nu \frac{\partial^2 u'}{\partial y'^2}. \tag{11}$$

Introducing now the following non-dimensional quantities:

$$\begin{gathered} y = \frac{y'v_0}{\nu}, \quad t = \frac{t'v_0^2}{4\nu}, \quad \omega = \frac{4\nu\omega'}{v_0^2} \\ u = \frac{u'}{u_0}, \quad U = \frac{U_0}{U'}, \quad \theta = \frac{T' - T'_\infty}{T'_w - T'_\infty} \\ G = \frac{\nu g_g \beta(T'_w - T'_\infty}{U_0^2 v_0^2}, \quad P = \frac{\mu c_p}{k}, \\ E = \frac{U_0^2}{c_p(T'_w - T'_\infty)}, \; M = \frac{\sigma B_0^2 \nu}{v_0^2 \rho'}, \end{gathered} \tag{12}$$

where G is the Grashof number, P is the Prandtl number, E is the Eckert number and M is the magnetic parameter, in equations (10) and (11) gives

$$\frac{1}{4}\frac{\partial u}{\partial t} - \frac{\partial u}{\partial y} = \frac{1}{4}\frac{\partial v}{\partial t} + G\theta + \frac{\partial^2 u}{\partial y^2} + M(U - u), \tag{13}$$

$$\frac{P}{4}\frac{\partial \theta}{\partial t} - P\frac{\partial \theta}{\partial y^2} = \frac{\partial^2 \theta}{\partial y^2} + PE\left(\frac{\partial u}{\partial y}\right)^2. \tag{14}$$

The boundary conditions are:

$$\left.\begin{array}{lll} u = 0, & \Theta = 1m & \text{at } y = 0, \\ u = U(t), & \Theta = 0 & \text{as } y \to \infty \end{array}\right| \tag{15}$$

In the neighbourhood of the plate we assume

$$\begin{aligned} U(y, t) &= u_0(y) + \in e^{i\omega t} u_1(y), \\ \Theta(y, t) &= \Theta_0(y) + \in e^{i\omega t}\Theta_1(y) \end{aligned} \tag{16}$$

and the free stream

$$U(y, t) = 1 + \in e^{i\omega t},$$

where $\in << 1$.

Substituting (16) in (14) and (15), equating harmonic terms, neglecting coefficients of ϵ^2, we get

$$u_0'' + u_0' + M(1 - u_0) = - G\Theta_0, \tag{17}$$

$$u_1'' + u_1' - \frac{i\omega}{4} u_1 + M(1 - u_1) = - \frac{i\omega}{4} - G\Theta_1, \tag{18}$$

$$\Theta_0'' + P\Theta_0' = - PEu_0'^2, \tag{19}$$

$$\Theta_1'' + P\Theta_1' - \frac{i\omega}{4} P\Theta_1 = - 2EPu_0'u_1'., \tag{20}$$

where the primes denote differentiation with respect to y.

The corresponding boundary conditions are:

$$\begin{aligned} &u_0 = 0,\ u_1 = 0,\ \Theta_0 = 1,\ \Theta_1 = 0 \text{ at } y = 0, \\ &u_0 = 1,\ u_1 \neq 0,\ \Theta_1 = 0,\ \Theta_1 = 0 \text{ as } y \to \infty. \end{aligned} \tag{21}$$

In order to solve the system of equations (17) to (20), following Soundalegekar (1973) we assume that the heat due to viscous dissipation is superimposed on the motion. This is achieved mathematically by expanding the velocity and temperature in powers of Eckert number. Thus we assume.

$$\left.\begin{aligned} u_0(y) &= u_{01}(y) + Eu_{02}(y) + 0(E^2), \\ u_1(y) &= u_{11}(y) + Eu_{12}(y) + 0(E^2), \\ \Theta_1(y) &= \Theta_{01}(y) + E\Theta_{02}(y) + 0(E^2), \\ \Theta_1(y) &= \Theta_{11}(y) + E\Theta_{12}(y) + 0(E^2). \end{aligned}\right] \tag{22}$$

Using (22) in (17) to (21), equating to zero the coefficient of various powers of E, we obtain the following set of equations (for the mean flow only)

$$u_{01}' + u_{01}' + M(1 - u_{01}) \neq - G\Theta_{01}, \tag{23}$$

$$u_{02}'' + u_{02}' - Mu_{02} = - G\Theta_{02}, \tag{24}$$

with boundary conditions:

$$\begin{aligned} &u_{01} = 0,\ u_{02} = 0 \text{ at } y = 0, \\ &u_{01} = 1,\ u_{02} = 0 \text{ as } y \to \infty. \end{aligned} \tag{25}$$

$$\Theta_{01}'' + P\Theta_{01}' = 0, \tag{26}$$

$$\Theta_{02}'' + P\Theta_{02}' = - Pu_{01}'^2, \tag{27}$$

with boundary conditions:

$$\begin{aligned} &\Theta_{01} = 1,\ \Theta_{02} = 0 \text{ at } y = 0, \\ &\Theta_{01} = 0,\ \Theta_{02} = 0 \text{ as } y \to \infty. \end{aligned} \tag{28}$$

The solutions for mean flow are

$$u_0(y) = 1 - \left[\frac{G}{P(1-P)+M} + 1\right] e^{-\beta y} - \frac{G e^{-Py}}{P^2 - P - M}$$

$$+ EGP\left[\left(\frac{G}{P^2 - P - M} - 1\right)^2 \frac{\beta}{P(P-2\beta)\,(4\beta^2 - 2\beta - M)(P^2 - P - M)}\right.$$

$$\{e^{-\beta y}(P^2 - P - 4\beta^2 + 2\beta) - (e^{-2\beta y}\,\overline{P^2 - P - M} - e^{-Py}\overline{4\beta^2 - 2\beta - M})\}$$

$$+ \frac{2PS\left(\frac{G}{P^2 - P - M}\right)^{-1}}{(P^2 - P - M)^2(\beta + P)(\beta + P)^2 - (\beta + P) - M)} \times \tag{36}$$

$$\times \{e^{-\beta y}(\beta - 2\beta P - \beta^2) - (e^{-(\beta+P)y}\,\overline{P^2 - P - M} -$$

$$- e^{-Py}\,\overline{(\beta + P)2 - (\beta + P) - M})\} +$$

$$+ \frac{G^2}{2(P^2 - P - M)^2(4P^2 - 2P - M)} \times$$

$$\left.\times \{e^{-\beta y}P(3P - 1) - (e^{-Py}\,\overline{4P^2 - 2P - M} - e^{-2Py}\,\overline{P^2 - P - M}\}\right].$$

$$\Theta_0(y) = e^{-Py} + E\left[\left(\frac{G}{P^2 - P - M} - 1\right)^2 \frac{\beta P}{2(P - 2\beta)}(e^{-2\beta y} - e^{-Py}) +\right.$$

$$+ \frac{2P^2G}{(P^2 - P - M)\,(\beta + P)}\left(\frac{G}{(P^2 - P - M)} - 1\right)(e^{-(\beta+P)y} - e^{-Py}) +$$

$$\left.+ \frac{PG^2}{2(P^2 - P - M)}(e^{-Py} - e^{-2Py})\right]. \tag{37}$$

By letting $M = 0$ in the above expressions we get back Soundalgekar's (1973) result.

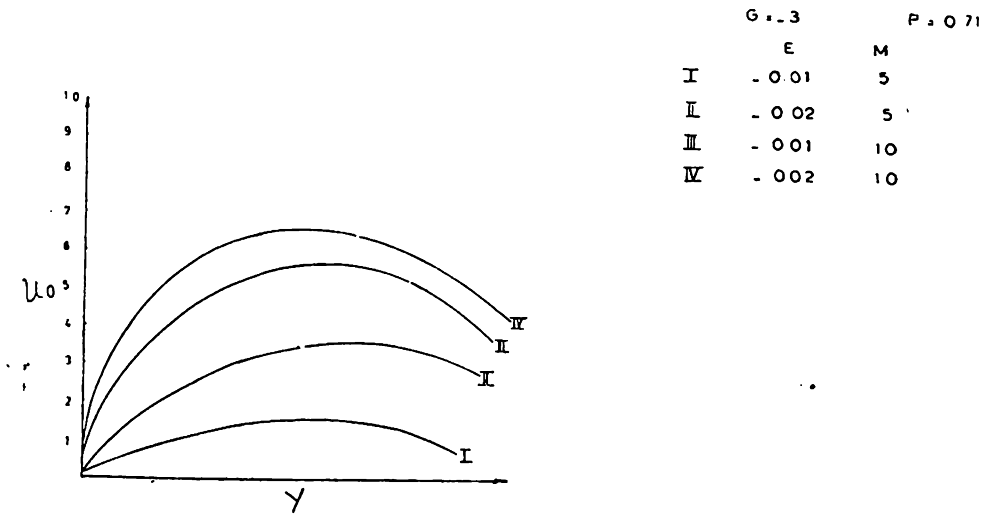

Fig. 1. Mean velocity profiles

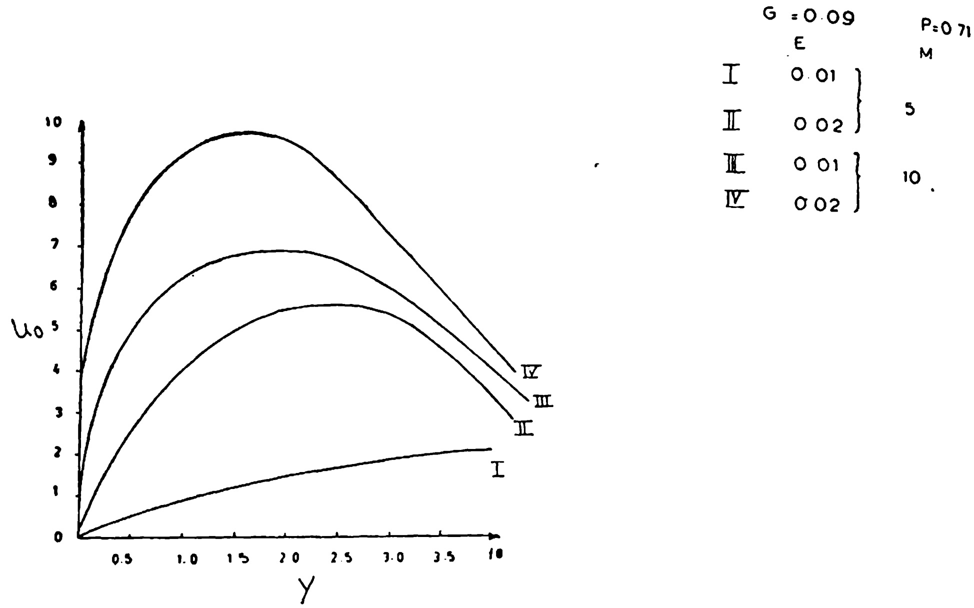

Fig. 2. Mean velocity profiles

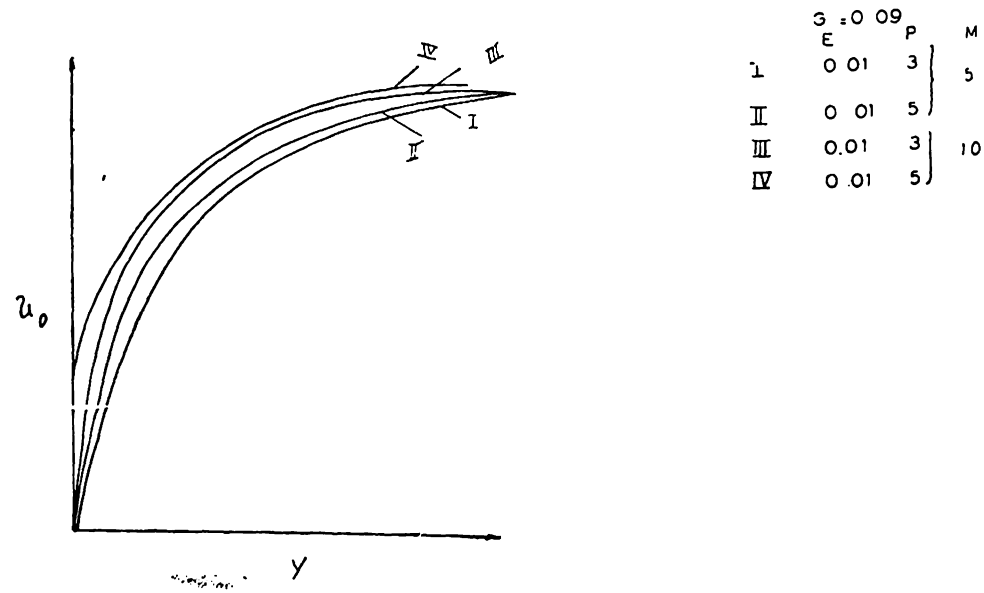

Fig. 3. Mean velocity profiles

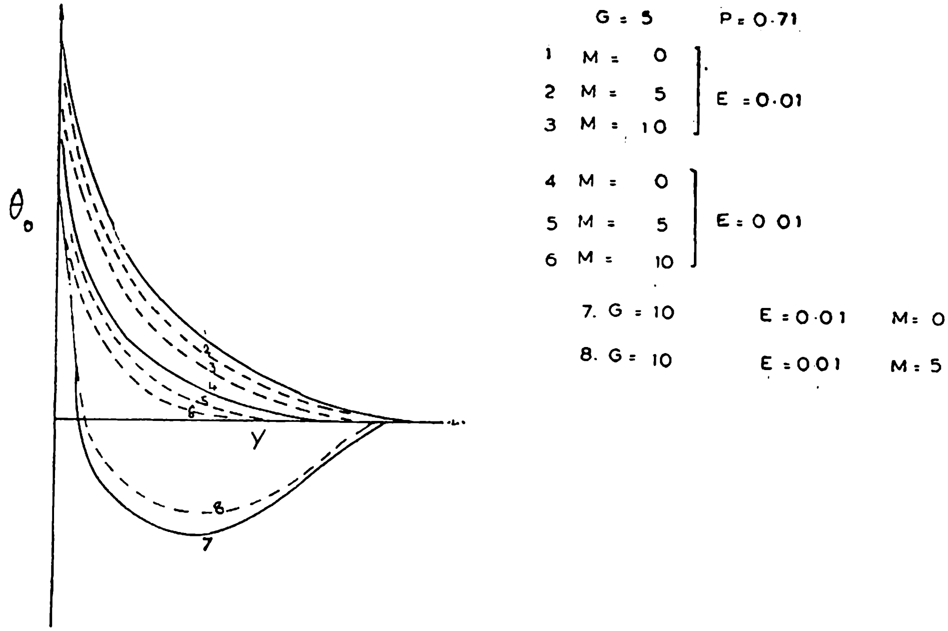

Fig. 4. Mean temperature profiles

Discussion. The mean velocity $u_0(y)$ and the mean temperature $\Theta_0(y)$ are plotted in figures 1—4 for different values of M, G, E and P. Physically $G < 0$ corresponds to an externally heated plate as the free convective currents are carried towards the plate. $G > 0$ corresponds to an externally cooled plate and $G = 0$ corresponds to the absence of the free convection currents. Figure 1, shows that the mean velocity increases with increase in magnetic parameter when $G < 0$. This is due to the fact that when the plate is being heated by free convection currents, the magnitude of the magnetic force is so large in the x-direction that the velocity increases. Even in the process of cooling of the plate ($G > 0$), Figures 2—3 the mean velocity still increases with increase in magnetic parameter. Thus we conclude that both in the process of cooling of the plate, and heating of the plate, the tendency of the magnetic force is to increase the mean velocity. The same behaviour is observed for other arbitrary values of P such as $P = 3$ and 5 and for other stes of values of G and M, E (Figure 3). Thus we see that there is no reverse flow both in the case of high or low Prandtl numbers and when $G \lessgtr 0$. In other words, the phenomenon of separation is removed with the application of magnetic field. This result is in agreement with the general behaviour of effect of magnetic field on conducting flows.

The mean temperature profiles are drawn in figure 4. We notice that because of the greater cooling of the plate, the mean temperature in the boundary layer decreases with increase in magnetic parameter. In other words, the heat carried away from the plate by free convection currents is not affected by the mechanical force which is magnetic force. For large Prandtl numbers the decrease in mean temperature is not significant. The fact that the presence of magnetic field has no effect on mean temperature may be due to neglect of ohmic heating or ohmic dissipation of heat. The fluctuating flow will be described elsewhere.

(Received June 20, 1976)

REFERENCES

1. Stuart, J. T., Proc. Roy. Soc. Lond, A **231**, 1955, 116.
2. Lighthill, M. J., Proc. Roy. Soc. Lond, A **224**, 1954, 1.
3. Soundalgekar, V. M., Proc. Roy. Soc. Lond, A **333**, 1973, 25.
4. Gupta, A. S., Applied Sci. Res., **9**, 1960, 315.
5. Gupta, A. S., Z. Angew. Math. Phys., **13**, 1962, 324.

EFECTE MAGNETOHIDRODINAMICE ŞI DE CONVECŢIE LIBERĂ ASUPRA SCURGERII OSCILATORII PESTE O PLACĂ VERTICALĂ

(Rezumat)

S-au obţinut soluţii pentru scurgerea bidimensională a unui fluid viscos incompresibil peste o placă verticală, infinită, poroasă, in diferite ipoteze menţionate in lucrare. Se arată că viteza medie creşte cu numărul lui Grashof şi cu parametrul magnetic, in timp ce temperatura medie descreşte.

SUBSPAŢII ÎNTR-UN SPAŢIU K_m^* (II)

Subspaţii total ombilicale într-un spaţiu K_m^* recurent

P. ENGHIŞ

Un subspaţiu V_n a unui spaţiu riemannian V_m este total ombilical [1], dacă în orice punct

$$\Omega_{p|ij} = \rho_p \, g_{ij} \tag{2,1}$$

unde ρ_p sînt funcţii scalare, iar $\Omega_{p|ij}$ componentele celui de al doilea tensor fundamental date de (1,9) [5].

Pentru un subspaţiu V_n total ombilical a unui spaţiu K_m^* recurent cu vector de recurenţă neortogonal la subspaţiu, relaţiile (1,13), [5] ţinînd cont de (2,1) şi (1,10) devin:

$$\begin{aligned} R_{ijkh,r} - \varphi_r \; R_{ijkh} = & - \varphi_r \sum_p e_p \rho_p^2 (g_{ik}g_{jh} - g_{ih}g_{jk}) + \\ & + \sum_p e_p \rho_p [2\rho_{p,r}(g_{ik}g_{jh} - g_{ih}g_{jk}) + \rho_{p,h}(g_{ik}g_{jr} - g_{jk}g_{ir}) + \\ & + \rho_{p,k}(g_{jh}g_{ir} - g_{ih}g_{jr}) + \rho_{p,j}(g_{ik}g_{hr} - g_{hi}g_{kr}) + \rho_{p,i}(g_{hj}g_{kr} - g_{kj}g_{hr}). \end{aligned} \tag{2,2}$$

Din (2,2) prin înmulţire cu g^{ik} şi însumare se obţine

$$\begin{aligned} R_{jh,r} - \varphi_r R_{jh} = & - (n-1)\varphi_r g_{jh} \sum_p e_p \rho_p^2 + \sum_p e_p \rho_p [2n g_{jh} \rho_{p,r} + \\ & + (n-2)\rho_{p,h} g_{jr} + (n-2)\rho_{p,j} g_{hr}]. \end{aligned} \tag{2,3}$$

După o nouă înmulţire cu g^{jh} şi însumare se obţine:

$$R_{,r} - \varphi_r R = - n(n-1)\varphi_r \sum_p e_p \rho_p^2 + 2(n^2 + n - 2) \sum_p e_p \rho_p \rho_{p,r} \tag{2,4}$$

PROPOZIŢIA 2.1. *Tensorul de curbură, tensorul lui Ricci şi curbura scalară a unui subspaţiu total ombilical a unui spaţiu K_m^* recurent cu vector de recurenţă neortogonal la subspaţiu, verifică* (2,2), (2,3) şi (2,4).

Transcriind relaţiile (1,16), (1,17), (1,18), [5] ţinînd seama de (2,1) se obţine

$$\begin{aligned} \varphi_r R_{ijkh} + \varphi_k R_{ijhr} + \varphi_h R_{ijhr} = & \sum_p e_p \rho_p^2 [\varphi_r (g_{ik}g_{jh} - g_{ih}g_{jk}) + \\ & + \varphi_k g_{ih} g_{jr} - g_{ir}g_{jh}) + \varphi_h (g_{ir}g_{jk} - g_{ik}g_{jr})] \end{aligned} \tag{2,5}$$

$$A^i_{jkh}\varphi_i = (2-n)(\varphi_k g_{jh} - \varphi_h g_{jk}) \sum_p e_p \rho_p^2 \tag{2,6}$$

$$(R\,\delta^i_h - 2\;R^i_h)\;\varphi_i = (n-2)(n-1)\;\varphi_h \sum_p e_p \rho_p^2 \tag{2,7}$$

Relaţii de forma (2,6) şi (2,7) au fost obţinute pentru prima dată de M. Prîvarovih [8] pe o altă cale. Din (2,6) şi (2,7) rezultă:

PROPOZIŢIA 2.2. *Tensorul A^i_{jkh} a unui subspaţiu V_n total ombilical dintr-un spaţiu K^*_m recurent cu vector de recurenţă neortogonal la subspaţiu verifică relaţiile:*

$$A^i_{jkh}\varphi_i = \frac{2}{n-1}(R^i_k g_{jh} - R^i_h g_{jk})\varphi_i + \frac{R}{n-1}(\varphi_h g_{jk} - \varphi_k g_{jh}) \tag{2,8}$$

Dacă subspaţiul V_n total ombilical este şi el un spaţiu K^*_n, ţinînd seama că în acest caz $A^i_{jkh}\varphi_i = 0$ [3], din (2,8) rezultă.

PROPOZIŢIA 2.3. *Vectorul φ_i dintr-un subspaţiu K^*_n total ombilical, al unui spaţiu K^*_m recurent cu vector de recurenţă neortogonal la subspaţiu este soluţie a sistemului*

$$[2(R^i_k g_{jh} - R^i_h g_{jk}) + R(g_{jk}\delta^i_h - g_{jh}\delta^i_k)]\varphi_i = 0.$$

Derivînd covariant (2,4) în raport cu g_{ij}, schimbînd indicii r şi s între ei şi scăzînd relaţiile obţinute, ţinînd seama de (2,4) şi de faptul că φ_r este gradient se obţine:

$$\left(\sum_p e_p\rho_p^2\right)_{,r}\varphi_s = \left(\sum_p e_p\rho_p^2\right)_{,s}\varphi_r$$

din care rezultă relaţia lui T. Myazawa [7]:

$$\left(\sum_p e_p\rho_p^2\right)_{,r} = \alpha\,\varphi_r \tag{2,9}$$

unde α este o funcţie scalară.

Din (2,2) şi (2,9) se obţine acum teorema lui T. Myazawa [7], potrivit căreia condiţia necesară şi suficientă ca un subspaţiu V_n total ombilical a unui spaţiu K^*_m recurent cu vector de recurenţă neortogonal la subspaţiu să fie recurent este ca $\sum_p e_p\rho_p^2 = 0$.

Observaţie. În teorema lui Myazawa este conţinut şi cazul $\rho_p = 0$ cînd subspaţiul este total geodezic.

Dacă subspaţiul V_n este o hipersuprafaţă, deci $n = m - 1$, atunci din (2,9) şi teorema lui Myazawa rezultă $\rho = 0$ drept condiţie necesară şi suficientă ca hipersuprafaţa să fie recurentă. Dar în acest caz, potrivit observaţiei precedente, hipersuprafaţa este total geodezică. Deci

PROPOZIŢIA 2.4. *Singurele hipersuprafeţe total ombilicale recurente ale unui spaţiu K^*_m recurent cu vector de recurenţă neortogonal la hipersuprafaţă sînt cele total geodezice.*

Dacă se consideră în (2,9), $\sum_p e_p\rho_p^2 = \text{const} \neq 0$ se obţine din (2,2) şi (2,3), $W_{ijkh,r} = \varphi_r W_{ijkh}$ şi reciproc, de unde rezultă teorema lui Myazawa [7] potrivit căreia condiţia necesară şi suficientă ca un subspaţiu total ombilical a unui spaţiu K_m^* recurent, cu vector de recurenţă neortogonal la subspaţiu, să fie proiectiv recurent, este ca $\sum_p e_p \rho_p^2 = \text{const.}$

Observaţie. Cum orice spaţiu recurent este şi proiectiv recurent [2], dar reciproc nu, din rezultatul lui Myazawa se deduce că dacă $\sum_p e_p \rho_p^2$ const $\neq 0$ subspaţiul este proiectiv recurent fără să fie şi recurent, constituind astfel un nou exemplu de spaţii proiectiv recurente, care nu sînt şi recurente.

Pe subspaţiul V_n a spaţiului K_m^* considerînd tensorul

$$Z_{ijkh} = R_{ijkh} + \frac{1}{n-2}(R_{ih}g_{jk} - R_{jh}g_{ik} + R_{jk}g_{ih} - R_{ik}g_{jh})$$

numit tensorul coharmonic [6], T. Myazawa [7] stabileşte că dacă funcţia α în (2,9) este

$$\alpha = \frac{n}{n+2}\sum_p e_p \rho_p^2 \tag{2,10}$$

atunci $Z_{ijkh,r} = \varphi_r Z_{ijkh}$ şi de aci condiţia necesară şi suficientă ca subspaţiul V_n total ombilical al spaţiului K_m^* recurent cu vector de recurenţă neortogonal la subspaţiu să fie coharmonic recurent, este ca $\sum_p e_p \rho_p^2 = C e^{\frac{n}{n+2}f}$ unde C este o constantă iar f o funcţie scalară ce satisface condiţia $\frac{\partial f}{\partial x^r} = \varphi_r$.

Dacă acum pentru subspaţiul V_n total ombilical relaţia (2,10) are loc, din (2,4) rezultă $R_{,r} = \varphi_r R$ şi subspaţiul este cu curbură scalară recurentă. Reciproc, dacă curbura scalară a lui V_n este recurentă de vector φ_r, sau nulă, din (2,4) rezultă că în (2,9) funcţia α este dată de (2,10) şi deci afirmaţia lui T. Myazawa poate fi reformulată astfel:

PROPOZIŢIA 2.5. *Condiţia necesară şi suficientă pentru ca un subspaţiu V_n, total ombilical, a unui spaţiu K_m^* recurent cu vector de recurenţă neortogonal la subspaţiu să fie coharmonic recurent este ca subspaţiul să fie de curbură scalară recurentă sau nulă.*

Presupunînd subspaţiul V_n Ricci-recurent, de vector φ_r, atunci el este şi de curbură scalară recurentă, deci coharmonic recurent şi din (2,3) şi (2,4) rezultă

$$\begin{gathered} -(n-1)\varphi_r g_{jh}\sum_p e_p\rho_p^2 + \sum_p e_p\rho_p\,[2n\, g_{jh}\rho_{p,r} + \\ +(n-2)\, g_{jr}\rho_{p,h} + (n-2)\, g_{hr}\rho_{p,j}] = 0 \end{gathered} \tag{2,11}$$

$$-n(n-1)\,\varphi_r\sum_p e_p\rho_p^2 + 2(n^2+n-2)\sum_p e_p\rho_p\rho_{p,r} = 0.$$

Dacă în prima din aceste relații se ține seama de a doua rezultă

$$\sum_p e_p \rho_p (2\rho_{p,r} g_{jk} - n\ \rho_{p,k} g_{jr} - n\ \rho_{p,j} g_{kr}) = 0 \qquad (2,12)$$

care dacă se înmulțește contractat cu g^{kr} se obține $\left(\sum_p e_p \rho_p^2\right)_{,j} = 0$ și din teorema lui Myazawa rezultă

PROPOZIȚIA 2.6. *Subspațiile total ombilicale Ricci-recurente de vector φ_r, ale unui spațiu K_m^* recurent cu vector de recurență φ_α neortogonal la subspațiu sînt de curbură proiectivă recurentă, cu vector de proiectiv recurență φ_r.*

Într-o lucrare anterioară [4] s-a arătat că o condiție necesară și suficientă pentru ca un spațiu V_n Ricci-recurent să fie de curbură recurentă este ca spațiul să fie proiectiv recurent de același vector de recurență.

Din această proprietate și din propoziția 2.6 rezultă

PROPOZIȚIA 2.7. *Subspațiile total ombilicale Ricci-recurente ale unui spațiu K_m^* recurent cu vector de recurență neortogonal la subspațiu sînt recurente.*

Dacă spațiul V_n este coharmonic recurent, atunci relația a doua din (2,11) are loc. Presupunînd în plus că și (2,12) are loc, atunci înlocuind în (2,12) din a doua relație (2,11) se obține $\sum_p e_p \rho_p^2 = 0$ dacă $\varphi_i \neq 0$ și din teorema lui Myazawa rezultă:

PROPOZIȚIA 2.8. *Subspațiile total ombilicale coharmonic recurente ale unui spațiu K_m^* recurent cu vector de recurență neortogonal la subspațiu sînt recurente dacă relația (2,12) are loc.*

Dacă în (2,9) funcția α este arbitrară din (2,2), (2,3) și (2,4) se obține rezultatul lui M. Prîvanovih [8] $C_{ijkh,r} = \varphi_r C_{ijkh}$ unde C_{ijkh} este tensorul de curbură conformă.

Dacă subspațiul V_n total ombilical este simetric Cartan, din (2,2) și (2,9) rezultă:

$$\begin{aligned}\varphi_r R_{ijkh} = \varphi_r\Big(\sum_p e_p \rho_p^2 - \alpha\Big)(g_{ik} g_{jh} - g_{ih} g_{jk}) - \\ - \frac{1}{2}\alpha\varphi_h(g_{ik} g_{jr} - g_{jk} g_{ir}) - \frac{1}{2}\alpha\varphi_k(g_{jh} g_{ir} - g_{ih} g_{jr}) - \\ - \frac{1}{2}\alpha\varphi_j(g_{ik} g_{hr} - g_{ki} g_{hr}) - \frac{1}{2}\alpha\varphi_i(g_{hj} g_{kr} - g_{kj} g_{hr})\end{aligned} \qquad (2,13)$$

Dacă vectorul de recurență φ_α este presupus neortogonal la subspațiu din (2,13) rezultă că reducînd vectorul φ_r la o formă canonică, (2,13) se poate scrie de forma:

$$R_{ijkh} = \Big(\sum_p e_p \rho_p^2 - \alpha\Big)(g_{ik} g_{jh} - g_{ih} g_{jk})$$

şi spaţiul este de curbură constantă. Deci

PROPOZIŢIA 2.9. *Subspaţiile total ombilicale ale unui spaţiu K_m^* recurent cu vector de recurenţă neortogonal la subspaţiu sînt simetrice Cartan dacă şi numai dacă sînt de curbură constantă.*

Considerînd cazul în care vectorul de recurenţă φ_α al spaţiului K_m^* este ortogonal la subspaţiul V_n total ombilical din (1,25), [5], rezultă

$$R_{ijkh} = \sum e_p \rho_p^2 (g_{ik}g_{jh} - g_{ih}g_{jk})$$

şi subspaţiul este de curbură constantă, regăsindu-se astfel rezultatul lui T. Myazawa [7].

(Intrat în redacţie la 20 octombrie 1976)

BIBLIOGRAFIE

1. L. P. Eisenhart, *Riemannian geometry*, Princeton, 1949.
2. P. Enghiş, *Sur les espaces V_n récurrents et Riccirécurrents*, Studia Univ. Babeş-Bolyai, Math-Mech., f. 1 (1972), 3–6.
3. P. Enghiş, *Sur les espaces K_n^* de Walker*, Studia Univ. Babeş-Bolyai, Math., 1 (1977), 14–16.
4. P. Enghiş, *Quelques remarques sur les espaces symétriques et récurrents*, Mathematica, t. 17 (40), 1 (1975), 67–69.
5. P. Enghiş, *Subspaţii într-un spaţiu K_m^** (I), Studia Univ. Babeş-Bolyai, Math., 2 (1977), 24–30.
6. I. Ishii, *On conharmonic transformations*, Tensor, N.S., 7 (1957), 70–80.
7. T. Myazawa and Goro Chuman, *On certain subspaces of Riemannian recurrent spaces*, Tensor N.S., 2 (1972), 253–260.
8. M. Prîvanovih, *Neke teoreme o podirastorima sa neodrejeniam liniama Krivine recurentnîh Riemannovîh prostora*, Mat. Vesnic, 1 (16) (1964), 81–87.

DES SOUS-ESPACES DANS UN ESPACE K_m^* (II)

Des sous-espaces totalement ombilicaux dans un espace K_m^* récurrent

(Résumé)

On donne premièrement dans ce travail les relations qui sont vérifiées dans un sous-espace totalement ombilical d'un espace K_m^* récurrent.

On donne ensuite les conditions nécessaires et suffisantes pour que le sous-espace totalement ombilical soit récurrent, coharmoniquement récurrent, projectivement récurrent, Ricci-récurrent et symétriquement Cartan, en étudiant la liaison réciproque entre les divers cas. Pour le cas où le vecteur de récurrence est orthogonal au sous-espace, on retrouve le résultat de T. Myazawa.

STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, 1, 1978

LEGI DE DERIVARE DE CURBURĂ RECURENTĂ ÎNTR-UN FIBRAT VECTORIAL

PROFIRA SANDOVICI

1. Fie $\Pi : E \to M$ un fibrat vectorial diferenţiabil C^∞ şi local trivial cu fibra tip R^m. Vom nota prin $\Sigma(\Pi)$ modulul secţiunilor fibratului Π şi cu ∇^Π o lege de derivare în $\Sigma(\Pi)$. Curbura derivării ∇^Π pe care o notăm prin R^Π este o secţiune a fibratului vectorial A_2 $(T(M)\,;\, L(E\,;E))$, [1]. Legea de derivare ∇^Π induce în fibratul $L(\Pi,\Pi) : L(E\,;E) \to M$ o lege de derivare definită pentru o secţiune oarecare ω a lui $L(\Pi\,;\Pi)$ prin

$$\nabla^L_X \omega = [\nabla^\Pi_X, \omega] \ \forall \ X \in \mathfrak{X}(M) \tag{1}$$

unde

$$[\nabla^\Pi_X, \omega] = \nabla^\Pi_X \omega - \omega \nabla^\Pi_X$$

PROPOZIŢIA 1. *Pentru* $X,Y,Z \in \mathfrak{X}(M)$ *arbitrari,* R^Π *verifică următoarea identitate a lui Bianchi:*

$$\underset{X,Y,Z}{S} \{\nabla^L_X(R^\Pi(Y,Z)) - R^\Pi([X,Y], Z)\} = 0 \tag{2}$$

unde $\underset{X,Y,Z}{S}$ *indică suma după permutările circulare ale lui* X,Y,Z.

Demonstraţie. Se explicitează $\nabla^L_X(R^\Pi(Y,Z)) = [\nabla^\Pi_X, R^\Pi(Y, Z)]$ ţinînd seama de definiţia lui $R^\Pi(Y,Z)$.

Identitatea (2) exprimă anularea diferenţialei exterioare a formei de curbură R^Π în raport cu legea de derivare ∇^L, adică $dR^\Pi = 0$ [1].

Vom nota în continuare cu ∇^τ o lege de derivare în fibratul tangent $\tau : T(M) \to M$. Atunci perechea $\nabla^* = (\nabla^\tau, \nabla^\Pi)$ defineşte o lege de derivare ∇^* în fibratul $A_2(T(M)\,;\, L(E\,;\, E))$ prin formula

$$(\nabla^*_X \sigma)(Y,Z) = \nabla^L_X(\sigma(Y,Z)) - \sigma(\nabla^\tau_X Y,Z) - \sigma(Y, \nabla^\tau_X Z) \tag{3}$$

unde σ este o secţiune a acestui fibrat iar $X,Y,Z \in \mathfrak{X}(M)$ arbitrari.

PROPOZIŢIA 2. R^Π *verifică în raport cu derivarea* $\nabla^* = (\nabla^\tau, \nabla^\Pi)$ *următoarea identitate a lui Bianchi:*

$$\underset{X,Y,Z}{S} \{(\nabla^*_X R^\Pi)(Y,Z) + R^\Pi(T^\tau(X,Y),Z)\} = 0 \tag{4}$$

unde T^τ *este torsiunea lui* ∇^τ.

Demonstraţie. Vom ţine seama de identitatea (2), de formula (3) şi de definiţia lui T^τ.

Să notăm prin R^{τ}, R^{Π}, R^{L}, R^{*} curburile derivărilor ∇^{τ}, ∇^{Π}, ∇^{L}, ∇^{*} respectiv. Atunci avem pentru o secțiune arbitrară ω a fibratului $L(\Pi\,;\Pi)$ şi pentru $X,Y,Z,V \in \mathfrak{X}\ (M)$ oarecari

$$R^{L}(X, Y) \cdot \omega = [R^{\Pi}(X, Y),\omega] \tag{5}$$

$$(R^{*}(X, Y) \cdot R^{\Pi})(Z, V) = R^{L}(X, Y) \cdot R^{\Pi}(Z, V) - R^{\Pi}(R^{\tau}(X, Y)Z, V) - \\ - R^{\Pi}(Z, R^{\tau}(X, Y)V) \tag{6}$$

PROPOZIȚIA 3. *Secțiunile următoare ale fibratului* $L(\Pi,\Pi)$ *:*

$$R^{\Pi}(Y, Z),\ \nabla^{L}_{X_1}(R^{\Pi}(Y, Z)),\ \ldots,\ \nabla^{L}_{X_K} \ldots \nabla^{L}_{X_1}(\,R^{\Pi}(Y, Z)),\ \ldots \tag{7}$$

pentru $Y,Z,X_1, \ldots, X_k \in \mathfrak{X}(M)$ *arbitrari şi* $k = 1, 2, \ldots$ *generează în fiecare punct* $x \in M$ *o algebră Lie* $L_x(\nabla^{\Pi})$, *subalgebră a algebrei endomorfismelor fibrei* E_x *a fibratului* Π.

Demonstrație. Ținînd seama de (1) şi de identitatea lui Jacobi, obținem

$$[\nabla^{L}_{X}(R^{\Pi}(Y, Z)),\ \omega] = -\,[R^{\Pi}(Y,Z),\ \nabla^{L}_{X}\omega] + \nabla^{L}_{X}[R^{\Pi}(Y, Z),\ \omega] \tag{8}$$

şi în general

$$[\nabla^{L}_{X_k} \ldots \nabla^{L}_{X_1}(R^{\Pi}(Y, Z)),\ \omega] = -\,[\nabla^{L}_{X_{k-1}} \ldots \nabla^{L}_{X_1}(R^{\Pi}(Y, Z)),\ \nabla^{L}_{X_k}\omega] + \\ + \nabla^{L}_{X_k}[\nabla^{L}_{X_{k-1}} \ldots \nabla^{L}_{X_1}(R^{\Pi}(Y, Z)),\omega]. \tag{9}$$

Notăm prin A_0, $\mathcal{F}\ (M)$ — modulul generat de $R^{\Pi}(Y,Z)$ pentru Y, Z arbitrari şi, în general, notăm prin A^{k}, $\mathcal{F}\ (M)$ — modulul generat de

$$\nabla^{L}_{X_k} \ldots \nabla^{L}_{X_1}(R^{\Pi}(Y, Z))\ \forall\, Y, Z, X_1, \ldots, X_k \in \mathfrak{X}(M)$$

Avem din (1), pentru o secțiune arbitrară ω a fibratului $L(\Pi\,;\Pi)$,

$$[R^{\Pi}(Y, Z),\ \omega] = \nabla^{L}_{Y}\nabla^{L}_{Z}\,\omega - \nabla^{L}_{Z}\nabla^{L}_{Y}\,\omega - \nabla^{L}_{[Y,Z]}\,\omega$$

de unde deducem că

$$[A^{0},\ A^{k}] \subset A^{k} + A^{k+1} + A^{k+2},\ k = 0,\ 1, \ldots \tag{10}$$

Din (8) şi (10) avem

$$[A^{1},\ A^{k}] \subset A^{k} + A^{k+1} + A^{k+2} + A^{k+3},\ k = 1,\ 2, \ldots$$

Continuînd în acest mod, din (9) se obține în general

$$[A^{h},\ A^{k}] \subset A^{k} + A^{k+1} + \ldots + A^{k+h+2},\ h \leqslant k$$

PROPOZIŢIA 4. *Algebra Lie $L_x(\nabla^\Pi)$ este generată de valorile în punctul x ale următoarelor secţiuni ale fibratului $L(\Pi, \Pi)$:*

$$R^\Pi(Y, Z),\ (\nabla^*_{X_1} R^\Pi)(Y,Z),\ \ldots,\ (\nabla^*_{X_k} \ldots \nabla^*_{X_1} R^\Pi)(Y, Z),\ \ldots$$

considerate pentru $Y,Z,X_1 \ldots, X_k \in \mathfrak{X}(M)$ arbitrari.

Demonstraţie. Din (3) deducem prin inducţie completă

$$(\nabla^*_{X_k} \ldots \nabla^*_{X_1} R^\Pi)(Y, Z) = \nabla^L_{X_k} \ldots \nabla^L_{X_1}(R^\Pi(Y, Z)) + a_{k-1}$$

unde $a_{k-1} \in \sum_{i=1}^{k-1} A^i$.

Observaţie. Fie $\xi = (P, GL(F), M, \Pi_p)$ fibratul principal al reperelor fibratului vectorial $\Pi : E \to M$ de fibră tip $F = R^m$ şi Φ'_z grupul de olonomie infinitezimală al lui ξ în raport cu conexiunea care induce legea de derivare ∇^Π. Atunci, algebra $L_x(\nabla^\Pi)$, într-un punct $x \in M$ astfel că $x = \Pi_p(z)$, este izomorfă cu algebra Lie a grupului Φ'_z.

2. DEFINIŢIA 1. O secţiune nenulă $u \in \Sigma(\Pi)$ este recurentă în raport cu o lege de derivare ∇^Π (sau ∇^Π recurentă) dacă există o 1-formă φ pe M astfel încît

$$\nabla^\Pi_X u = \varphi(x) u \quad \forall X \in \mathfrak{X}(M) \tag{11}$$

Secţiunea $u \in \Sigma(\Pi)$ este ∇^Π — paralelă dacă $\nabla^\Pi_X u = 0$.

PROPOZIŢIA 5. *Dacă secţiunea $u \in \Sigma(\Pi)$ este ∇^Π — recurentă, cu 1-forma de recurentă φ, atunci avem*

$$\begin{gathered} R^\Pi(X,Y)u = (d\varphi)(X,Y)u \\ \{\nabla^L_{X_k} \ldots \nabla^L_{X_1}(R^\Pi(Y, Z))\}u = X_k \ldots X_1\{(d\varphi)(Y, Z)\}u \end{gathered} \tag{12}$$

pentru $Y,Z,X, \ldots, X_k \in \mathfrak{X}(M)$ arbitrari şi $k = 1, 2, \ldots$.

Demonstraţie. Relaţiile (12) se obţin ţinînd seama de (1), de definiţia lui R^Π şi de (11).

COROLAR. *Dacă o secţiune $u \in \Sigma(\Pi)$ este ∇^Π-recurentă, atunci algebra $L_x(\nabla^\Pi)$ invariază subspaţiul fibrei E_x subîntins de $u(x)$.*

Considerînd fibratul vectorial $L(\Pi,\Pi)$ şi legea de derivare ∇^L definită de (1), propoziţia 5 are următoarea formulare:

PROPOZIŢIA 5'. *Dacă o secţiune ω a fibratului $L(\Pi,\Pi)$ este ∇^L-recurentă. cu 1-forma de recurenţă φ, atunci avem*

$$[R^\Pi(Y, Z), \omega] = (d\varphi)(Y, Z)\omega$$

$$[\nabla^L_{X_k} \ldots \nabla^L_{X_1}(R_\Pi(Y, Z)), \omega] = X_k \ldots X_1\{(d\varphi)(Y, Z)\}\omega$$

$$\forall Y, Z, X_1, \ldots, X_k \in \mathfrak{X}(M),\ k = 1, 2, \ldots$$

COROLAR. Fie V_x subspaţiul lui $L(E_x, E_x)$ subîntins de $\omega(x)$. *Dacă secţiunea* ω *este* ∇^L*-recurentă, atunci avem*

$$[L_x(\nabla^{\Pi}), \omega(x)] \subset V_x \qquad \forall x \in M.$$

În particular, dacă 1-forma de recurenţă φ este închisă, atunci

$$[L_x(\nabla^{\Pi}), \omega(x)] = 0 \qquad \forall x \in M.$$

Considerăm în încheiere o lege de derivare $\nabla^* = (\nabla^{\tau}, \nabla^{\Pi})$ astfel încît R^{Π} să fie recurent, deci să avem

$$\nabla^*_X R^{\Pi} = \varphi(X) R^{\Pi} \qquad \forall X \in \mathfrak{X}(M) \tag{13}$$

PROPOZIŢIA 6. *Fiecare din relaţiile următoare este o condiţie necesară pentru ca* R^{Π} *să fie* ∇^**-recurent*

$$\varphi(X)R^{\Pi}(Y, Z) + \varphi(Y)R^{\Pi}(Z, X) + \varphi(Z)R^{\Pi}(X, Y) + R^{\Pi}(T^{\tau}(X, Y), Z) + $$
$$+ R^{\Pi}(T^{\tau}(Y, Z), X) + R^{\Pi}(T^{\tau}(Z, X), Y) = 0 \tag{14}$$

$$(R^{\Pi}(X, Y), R^{\Pi}(Z, V)] = R^{\Pi}(R^{\tau}(X, Y)Z, V) +$$
$$+ R^{\Pi}(Z, R^{\tau}(X, Y)V) + (d\varphi)(X, Y)R^{\Pi}(Z, V) \tag{15}$$

Demonstraţie. Condiţia (14) rezultă din identitatea (4) unde se ţine seama de (13). Aplicînd lui R^{Π} propoziţia 5, avem $R^*(X, Y)R^{\Pi} = (d\varphi)(X, Y)R^{\Pi}$. Condiţia (15) se obţine acum din (6) unde $R^L(X, Y)R^{\Pi}(Z, V)$ se substituie prin expresia sa dată de (5).

Urma lui $R^{\Pi}(X, Y)$ pentru X, Y arbitrari determină pe M o 2-formă ρ cu valori scalare

$$\rho : (X, Y) \to T_r . R^{\Pi}]X, Y)$$

ρ este o 2-formă închisă.

Vom presupune în continuare ρ nenul. Atunci, în ipoteza 13, din (3) pentru $\sigma = R^{\Pi}$ deducem

$$(\nabla^{\tau}_X \rho)(Y, Z) = \varphi(X)\rho(Y, Z) \qquad \forall X,Y,Z \in \mathfrak{X}(M)$$

şi prin urmare ρ este ∇^{τ}-recurent cu aceeaşi 1-formă de recurenţă ca şi R^{Π}. Din (14) şi (15) obţinem

$$(\varphi \wedge \rho)(X, Y, Z) + \underset{X,Y,Z}{S}\{\rho(T^{\tau}(X, Y), Z)\} = 0 \tag{14'}$$

$$(d\varphi)(X, Y)\rho(Z, V) + \rho(R^{\tau}(X, Y)Z, V) + \rho(Z, R^{\tau}(X,Y)V)\} = 0 \tag{15'}$$

Din (14') pentru $T^{\tau} = 0$, rezultă $\varphi \wedge \rho = 0$. Avem astfel:

PROPOZIŢIA 7. *O condiţie necesară pentru ca* R^{Π} *cu* $\rho \neq 0$ *să fie recurent în raport cu legea de derivare* $\nabla^* = (\nabla^{\tau}, \nabla^{\Pi})$ *unde* ∇^{τ} *este de torsiune nulă* ($T^{\tau} = 0$) *este ca 2-forma* ρ *să fie divizibilă prin 1-forma de recurenţă* φ.

COROLAR. *Dacă* $T^{\tau} = 0$ *şi* $\rho \neq 0$, *pentru ca* R^{Π} *să fie* ∇^**-recurent, este necesar ca rang* $\rho = 2$.

Considerînd propoziția 4 în ipoteza (13) avem:

PROPOZIȚIA 8. *Dacă* R^{Π} *este* ∇^**-recurent, atunci algebra* $L_x(\nabla^{\Pi})$ *este generată de* $R_x^{\Pi}(X_x, Y_x)$ *unde* X_x, Y_x *sînt vectori arbitrari ai spațiului tangent* $T_x(M)$.

COROLAR. *Dacă* R^{Π} *este* ∇^**-recurent avem* $\dim L_x(\nabla^{\Pi}) \leqslant n(n-1)/2$ *unde* $n = \dim M$.

(Intrat în redacție la 14 martie 1977)

BIBLIOGRAFIE

1. Gheorghiev, Gh., Oproiu, V. *Varietăți diferențiabile finit și infinit dimensionale*, vol. I, Ed. Acad., București, 1976.
2. Sandovici, P. *Secțiuni recurente ale unui fibrat vectorial în raport cu o lege de derivare*, Studia Univ. Babeș-Bolyai, Math.: (I) 1975, 21–25; (II) 1976, 19–22.

LOIS DE DÉRIVATION À COURBURE RÉCURRENTE DANS UN FIBRÉ VECTORIEL

(Résumé)

Dans cette note on considère des sections récurrentes d'un fibré vectoriel doué d'une loi de dérivation. On établit des conditions nécessaires afin que la 2-forme de courbure d'une telle loi de dérivation soit récurrente.

STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, 1, 1978

THE SECONDARY EFFECTS OF XZ CYGNI

VASILE POP*

1. **Introduction.** The main pulsation of a star, shown by the light curves and radial velocities periodicity and by a periodic variation of some physical and photometric parameters too, is accompanied — in the case of RR Lyrae stars — by secondary phenomena such as:

— the variation of the main period of pulsation;

— the periodic variation in the shape of light and radial velocity curves;

— simultaneous excitation of several pulsation modes;

— multiple periodicities in the light curve variation.

Taking into account the pulsation theory and the horizontal branch evolutionary tracings, we may assume that the RR Lyrae stars' periods increase or decrease during instability strip crossing. Pointing out the real pulsation period variations, we obtain valuable information in the evolution theory for stars with masses smaller than the solar mass. Moreover, it seems that this is the single observational evidence for the evolution in the horizontal branch phase.

The study about the stability of pulsation in the fundamental mode and in the first harmonic shows that there is a domain in the instability strip where the two oscillation modes are stable. In this common stability domain, the star will oscillate in the first harmonic or in the fundamental mode, according to its evolution in the instability strip: from high to low or from low to high temperatures, respectively.

These problems have not exact and complete solutions, neither theoretically, nor observationally. Therefore, K u k a r k i n [5] pointed out that it is important and necessary to perform and analyse annual series of observations about several RR Lyrae stars from the galactic field and about some globular clusters with different metal abundance, but rich in stars of this type. We consider that XZ Cygni-with a very rapid decrease (after 1965) of its main pulsation period (P o p, [9])-is suitable for such research.

2. **Secular Variation of the Main Pulsation Period and Osculating Photometric Elements of XZ Cygni.** Using the observations until 1953 of XZ Cygni, K l e p i k o v a [4] studied the secular variation of its main pulsation period, determining the following parabolic elements:

$$\text{Max.Hel.} = \text{JD } 2417201 \cdot 241 + 0^d \cdot 4665878\, E - 0.000107 \times 10^{-6} E^2, \quad (1)$$

where E is the main pulsation cycles number, counted from JD 2417201. The observations performed after 1965 verify neither elements (1), nor linear elements given by G.C.V.S. (1969):

$$\text{Max.Hel.} = \text{JD } 2436933.981 + 0^d \cdot 466579\, E, \quad (2)$$

* Astronomical Observatory, Cluj-Napoca.

Between 16 April 1972 – 29 October 1975, we performed 3500 BV photoelectric observations of XZ Cygni with the 50 cm reflector of the Cluj-Napoca Astronomical Observatory. From them, we determined 47 maxima and, with the least squares method (in FORTRAN IV for the computer FELIX C–256), we found the following osculating photometric elements:

$$\text{Max.Hel.} = \text{JD } 2441453{\cdot}3856 + 0^d{\cdot}4664731\ E. \qquad (3)$$

XZ Cygni is a RR_{ab} pulsating star. It will be pulsating in the fundamental mode, with a main period $P = 0 \cdot 4664731$ days. The maximum moments and the O–C differences calculated with (3) are listed in Table 1.

Table 1

Max. Hel. 2400000+	O–C	E	Max. Hel. 2400000+	O–C	E
41424.4425	–0.0218	–62	41967.4457	+0.0067	1102
453.3980	+ .0124	0	974.4486	+ .0126	1117
461.3390:	+ .0234	17	982.3767	+ .0106	1134
467.3851	+ .0053	30	988.4330	+ .0028	1147
468.3187	+ .0060	32	41997.2900:	– .0032	1166
507.5088	+ .0124	116	42047.2115	+ .0056	1273
515.4295	+ .0030	133	161.5074	+ .0156	1518
522.4295	+ .0059	148	162.4356	+ .0109	1520
544.3180	– .0298	195	245.4596	+ .0027	1698
622.2557	+ .0068	362	280.4547	+ .0123	1717
834.4689	– .0252	817	288.3879	+ .0155	1790
835.4036	– .0235	819	308.4504	+ .0196	1833
849.4293	+ .0080	849	331.3051	+ .0171	1882
856.4236	+ .0052	864	338.3027	+ .0176	1897
862.4851	+ .0026	877	.538.4000	– .0020	2326
869.4707	– .0089	892	543.5350	+ .0018	2337
910.5357	+ .0065	980	573.4026	+ .0151	2401
913.3325	+ .0044	986	579.4637	+ .0120	2414
932.4564	+ .0029	1027	614.4458	+ .0087	2489
933.3921	+ .0057	1029	634.5004	+ .0049	2532
939.4489	– .0017	1042	684.4146	+ .0065	2639
947.3668	– .0142	1059	685.3517	+ .0106	2641
960.4363	– .0056	1087	686.2820	+ .0080	2643
41962.3079	+0.0002	1091	42712.3834	–0.0131	2699

Using the maxima published by Kle̓pikova [4], Lange and Guseev [7], Kunchev [6], Baldwin [1] and those of Table 1, we formed 59 normal maxima, listed in Table 2, first column. The second column contains the number of individual maxima used for each normal maximum, the third and fourth columns contain E and $(O-C)_2$ differences calculated with (2). As Figure 1 shows, after $E = 4\,000$ (year 1965), there is a real and very rapid decrease of the main pulsation period.

The elements given by (2) and (3) are in accordance with the observations performed in a short time interval before JD 2438576 and around JD 2442686, respectively. Assuming that the pulsation period of XZ Cygni

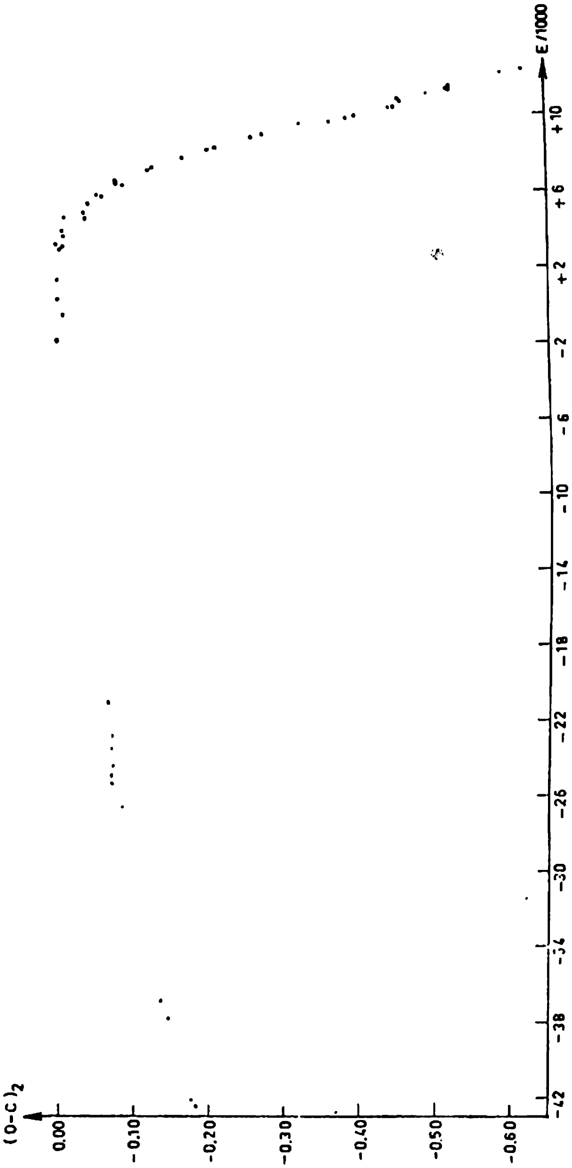
(O–C)2
0.00
-0.10
-0.20
-0.30
-0.40
-0.50
-0.60
E/1000
-42
-38
-34
-30
-26
-22
-18
-14
-10
-6
-2
+2
+6
+10

Fig. 1.

Table 2

Max. Hel. JD 2400000+	Num. max.	E	$(O-C)_2$	$(O-C)_3$	Max. Hel. JD 2400000+	Num. max.	E	$(O-C)_2$	$(O-C)_3$
17074.793	15	−42563	−0.181	−0.003	392.375	3	+5269	−0.011	+0.004
232.041	9	42226	− .175	+ .005	471.667	3	5439	− .037	− .016
19248.158	33	37905	− .146	− .002	718.481	3	5968	− .043	+ .002
627.497	8	37092	− .136	+ .002	826.706	3	6200	− .065	− .007
20758.499	11	34668	− .121	+ .001	890.171	1	6336	− .055	+ .011
22932.319	10	30009	− .093	− .004	40084.699	3	6753	− .090	− .001
24410.994	15	26842	− .073	− .003	40154.695	2	6903	− .081	+ .018
963.373	12	25656	− .057	+ .007	40211.616	1	7025	− .082	+ .024
25146.739	26	25263	− .057	+ .005	467.726	6	7574	− .124	+ .019
418.288	10	24681	− .057	+ .002	529.772	5	7707	− .133	+ .020
860.138	15	23734	− .057	− .002	742.494	1	8163	− .171	+ .017
26181.613	9	23045	− .055	− .003	885.702	3	8470	− .203	+ .010
982.271	3	21329	− .047	− .003	949.610	2	8607	− .216	+ .009
27713.406	5	19762	− .041	− .004	41177.253	13	9095	− .264	+ .006
32759.494	23	8947	− .005	+ .002	241.628	8	9233	− .277	+ .005
33531.682	48	7292	− .005	− .001	461.339	8	9704	− .325	+ .004
856.418	6	6596	− .008	− .005	530.351	20	9852	− .366	− .023
34194.224	12	5872	− .005	− .003	586.321	10	9972	− .386	− .029
634.680	23	4928	+ .000	+ .001	604.504	10	10011	− .399	− .039
36322.764	1	1310	+ .002	+ .001	683.357	2	10180	− .398	− .019
872.387	1	− 132	− .005	− .005	849.414	6	10537	− .444	− .024
37372.566	1	+ 940	+ .001	+ .001	908.662	16	10663	− .451	− .017
820.481	11	1900	+ .000	+ .001	979.574	18	10815	− .459	− .008
38576.333	2	3520	− .006	− .004	42047.230	2	10960	− .457	+ .011
609.458	1	3591	− .008	− .006	162.436	4	11207	− .496	+ .002
702.312	4	3790	− .003	− .001	228.659	6	11349	− .527	− .012
935.594	3	4290	− .009	− .008	280.453	3	11460	− .523	+ .007
39027.980	6	4488	− .008	− .004	331.307	3	11569	− .526	+ .017
324.692	5	5124	−0.040	−0.031	573.396	5	12088	− .592	+ .019
					42686.277	4	+12330	−0.623	+0.022

decreases linearly between these epochs, there results a decrease of 0.0001059 days in 8810 main pulsation cycles, corresponding to :

$$\alpha = (1/P)(dP/dt) \cong -20 \cdot 156 \text{ in } 10 \text{ years.}$$

3. **The Blazhko Effect.** At some RR Lyrae variables, over the main phenomenon of pulsation, a periodic variation of the light curve shape (the so-called Blazhko effect) is superposed. In the case of XZ Cygni, both the light curve and the phase of maxima are changing periodically. Such a secondary pulsation cycle is specific to the normal Blazhko effect.

Using the maxima published by B o g d a n o v [2], K u n c h e v [6] and those from Table 1, we respresented the O−C differences calculated with (3) as functions of E (Figures 2, 3, 4). A periodic variation of O−C versus time is pointed out. The upper part of Figure 3 (maxima heights ΔV_{max} versus E) also shows their periodicity. This periodic oscillation of O−C and ΔV_{max} versus E is the secondary pulsation (second pulsation cycle) or the Blazhko effect of XZ Cygni.

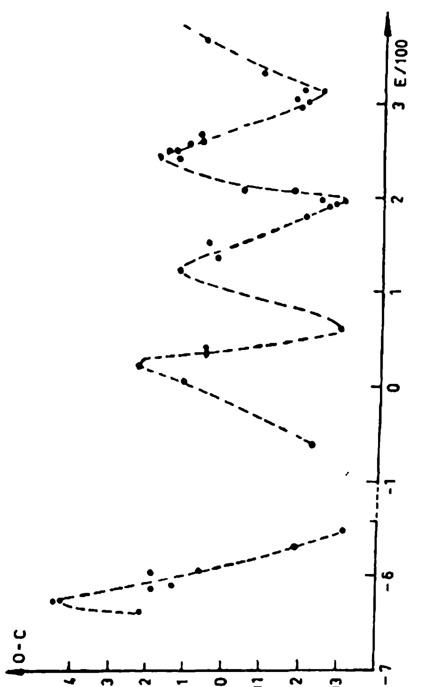
O-C
+0.04
+0.03
+0.02
+0.01
0.00
-0.01
-0.02
-0.03
-7
-6
-1
0
1
2
3
E/100

Fig. 2.

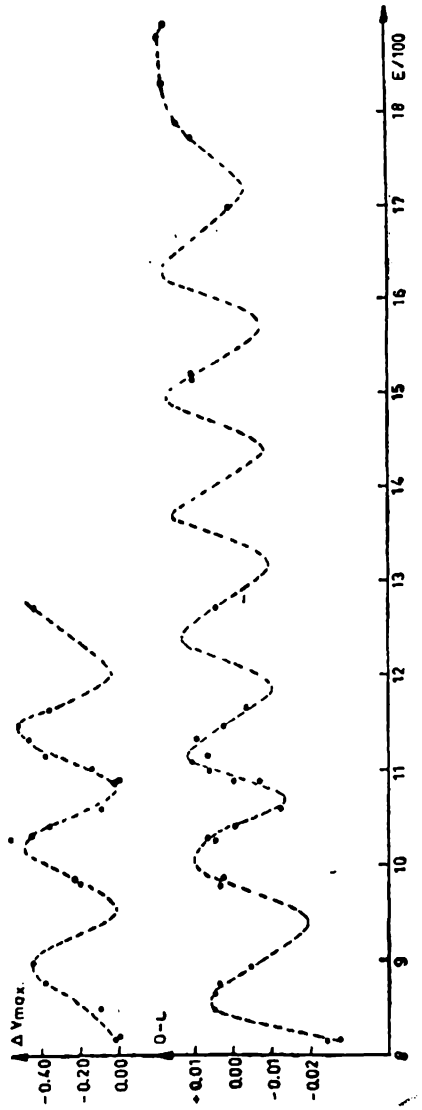
Δ Vmax.
-0.40
-0.20
0.00
O-L
+0.01
0.00
-0.01
-0.02
8
9
10
11
12
13
14
15
16
17
18
E/100

Fig. 3.

We determined the moments of 11 secondary pulsation maxima, finding the following photometric elements:

$$\text{Max.Hel. Sec.} = \text{JD } 2441977{\cdot}886 + 58{\cdot}532\ N, \qquad (4)$$
$$\pm 1{\cdot}540 \quad \pm 0.202$$

where N is the number of secondary pulsation cycles, counted from JD 2441977 and 58.532 days (P_b) is the period of Blazhko effect. Table 3 contains the observed Blazhko effect maxima with the corresponding secondary cycles and the (O—C)" differences calculated with (4).

Analysing the diagrams 2, 3 and 4, we find out that:

(a) The secondary pulsation cycle is not exactly reproducing; although there is a mean period of 58.532 days.

(b) The secondary pulsation period increases from $P_b = 57.38$ days (K l e p i k o v a, [4]) to 58.532 days, corresponding to the increase of the main pulsation period.

(c) There is a phase-shift of 0.19 secondary cycles in the ΔV curve as reported to the O—C curve (Figure 3).

Till now there is no satisfactory theoretical interpretation of the Blazhko effect observed at some RR Lyrae pulsating stars. From several explanatory hypotheses (P o p, [8]), the one closest to reality seems to be the one which considers the Blazhko effect as a result of a beat phenomenon produced by the interference of two pulsations with near periods.

If P is the main pulsation period and P_1 a pulsation period close to P, with the above-mentioned hypothesis, we may write:

$$nP = (n + 1)P_1 = P_b, \qquad (5)$$

where n is a whole number and P_b is the beat period. Out of (5) results that:

$$1/nP = 1/P_b, \qquad (6)$$

$$nP = nP_1 + P_1$$

Table 3

Max.Hel. JD 2400000+	N	(O—C)" days
41160.441	−14	+2.004
460.850	− 9	+9.753
507.497	− 8	−2.132
564.407	− 7	−3.755
852.687	− 2	−8.135
923.591	− 1	+4.237
41973.970	0	−3.916
42031.813	+ 1	−4.605
569.190	+10	+5.982
623.301	+11	+1.561
42679.277	+12	−0.994

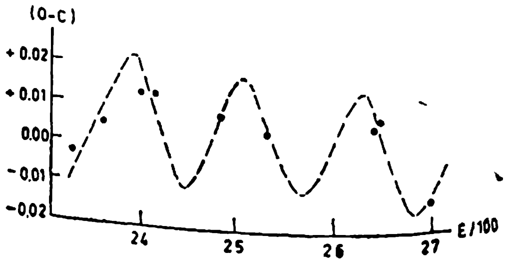

Fig. 4.

and

$$n = P_1/(P - P_1). \tag{7}$$

Replacing (7) in (6), we obtain:

$$1/P_1 - 1/P = 1/P_b. \tag{8}$$

In the case of XZ Cygni, we determined: $P = 0.4664731$ days and $P_b = 58.532$ days. Substituting these values in (8), we find $P_1 = 0.4627849$ days. Considering P_1 a constant in (8), we get the following relation between the variations of P and P_b:

$$\Delta(1/P) + \Delta(1/P_b) = 0, \tag{9}$$

whereby: if P increases, P_b decreases and vice-versa. Because before 1965, $P = 0.466579$ days and $P_b = 57.38$ days and now $P = 0.4664731$ days, it would result from (9) that for the time being $P_b = 59.028$ days. This value corresponds quite well to the observed value ($P_b = 58.532$ days).

4. **The Long-Period Variation.** The ephemerides of maxima observed before JD 2439027 may be calculated with (1). For the calculation of maxima moments after the above-mentioned epoch, we determined:

$$\text{Max.Hel.} = \text{JD } 2441453.4046 + 0^d\cdot4664772\, E - \\ - 0.699 \times 10^{-8}E^2. \tag{10}$$

The last column of Table 2 gives $(O-C)_3$ calculated with (1) up to JD 2439027 and with (10) after this epoch.

The $(O-C)_3$ variation in time (measured in main pulsation cycles number E) up to JD 2439027 is shown in Figure 5. The variation is periodic, with an amplitude of $\cong 0^d.01$ and a mean period $\Pi = 9.33$ years. As Figure 6 — where the $(O-C)_3$ calculated with (10) are represented versus E-shows, the variation amplitude is $\cong 0^d.05$. Because only an oscillation cycle is covered, we estimated the period value at $\Pi = 6.46$ years. The rapid decrease of the main pulsation period is accompanied by a substantial decrease of the long-period oscillation, from 9.33 to 6.46. years.

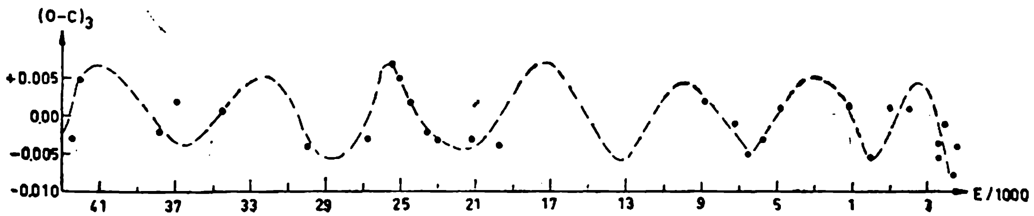

Fig. 5.

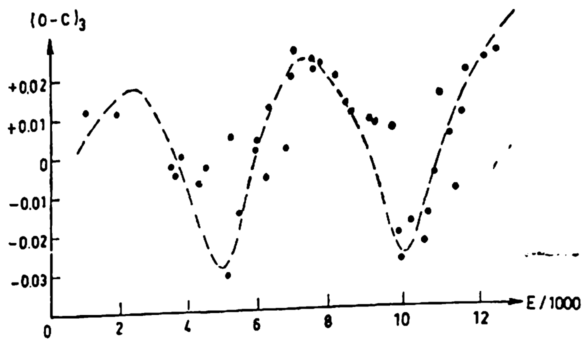

Fig. 6.

5. The Evolutionary Stage of XZ Cygni. The pulsation is only a transitory phase in the life of a star, as the evolution theory shows. The RR Lyrae stars are in a rather late phase of the general stellar evolution. They left the main sequence with a luminosity close to that of the Sun, they crossed the red giants region and the helium flash and are now in the horizontal branch phase where the helium is in the nucleus and the hydrogen in a shell burning. The pulsating stars of the horizontal branch are situated in the HR instability strip. During its evolution, a RR Lyrae star may cross — from the right or from the left, once or several times — the instability strip.

Using theoretical models, Iben and Rood [3] showed that for any value of Z (heavy elements abundance) there is a critical value of helium abundance $Y_1(Z)$ so that if for a model $Y < Y_1(Z)$, it will evolve towards the red and if $Y > Y_1(Z)$, the model will evolve towards the blue in the HR diagram, during a quite large interval of time from the phase of horizontal branch. The more Y increases with respect to $Y_1(Z)$, the longer is the evolution time of the model towards the blue. To $Z = 10^{-3}$ corresponds $Y_1(Z) = 0.25$. In the case of XZ Cygni, we find that $Z = 0.002$ and $Y = 0.32$ (Pop, [10]) i.e. this star may evolve a long time towards the blue.

It is known that between the period P, mass M and radius R of a star exists the relation:

$$P \propto M^{-1/2} R^{3/2}. \tag{11}$$

Differentiating logarithmically the relation (11), we obtain for M constant:

$$\alpha = (1/P)(dP/dt) = (3/2)(1/R)(dR/dt), \tag{12}$$

whereby the secular variation of the main pulsation period is caused by the evolutionary change of its radius. It was stated theoretically that, at the end of the evolutionary tracing towards the blue, the decreasing rate of the period is $0.03 < \alpha < 0.00$ in 10^6 years, and at the end of the evolutionary tracing towards the red, $0.00 < \alpha < +0.3$ in 10^6 years. Comparing the observed rate of the pulsation period variation at RR

Lyrae stars in M 3 cluster with the theoretical one, Iben and Rood [3] found out that the observed values are with an order of magnitude higher than the theoretical ones. They explain this disagreement by a great contribution of the „noise" to the observed value of dP/dt.

At XZ Cygni we pointed out the very rapid decrease of the main pulsation period, whereby this star is evolving towards the blue in the instability strip. In Section 2 we showed that the decreasing rate of the period is $\alpha = -20.156$ in 10^6 years, this value being with 3 orders of magnitude higher than the theoretical one. Although the "noise" has here a great contribution, being amplified in the Blazhko effect and long-period variation presence. We believe that the observed α is really greater than the theoretical one.

Because XZ Cygni is pulsating in the fundamental mode, it passed not beyond the blue edge of the transition region of the instability strip. Its position is not far from the red edge of the instability strip and is evolving towards the blue. For a more precise determination of the position and evolution of XZ Cygni in the instability strip, it is necessary to determine their physical parameters.

(Received March 30, 1977)

REFERENCES

1. Baldwin, M. E., Private communication (1975).
2. Bogdanov, M. B., The Variable Stars Suppl., 1, 5 (1972), 309.
3. Iben Jr. I., Rood, R. T., Astrophys. J., **161** (1970), 587.
4. Klepikova, L. A., The Variable Stars, **12** (1959), 164.
5. Kukarkin, B. V., *RR Lyrae and W Virginis Type Stars*, Preprint, Symposium IAU No. 67, Moscow, 1974.
6. Kunchev, P., I.B.V.S., **927** (1974).
7. Lange, G. A., Guseev, P. P., The Variable Stars Suppl., 1, 5 (1972), 333.
8. Pop, V., *Anuarul Observatorului Astronomic din București*, (1974), p. 239.
9. Pop, V., I.B.V.S., **990** (1975).
10. Pop, V., *Determinarea parametrilor fizici ai stelelor pulsante de tip RR Lyrae. Aplicație la pulsanta XZ Cygni*, Communication at the Colloquium of Astronomy and Space Research, 13—14 Nov. (1976), Cluj-Napoca (in press).

EFECTELE SECUNDARE ALE STELEI XZ CYGNI

(Rezumat)

Sînt date 47 maxime de lumină ale pulsantei XZ Cygni, pe baza cărora au fost determinate elementele fotometrice osculatoare. Se pune în evidență o descreștere foarte rapidă a perioadei principale de pulsație după anul 1965 și se determină elementele fotometrice parabolice valabile după această dată. Din 11 maxime ale ciclului secundar de pulsație am stabilit că perioada efectului Blažko este de 58.532 zile, interpretînd-o ca rezultat al interacțiunii pulsațiilor de perioadă 0.4664731 zile și 0.4627849 zile. Este relevată o variație lung-periodică cu perioada de 6.46 ani. Ipoteza că XZ Cygni este situată în banda de instabilitate nu departe de marginea roșie și evoluează către albastru este justificată de descreșterea perioadei fundamentale de pulsație cu o rată $\alpha = (1/P)(dP/dt) \simeq -20.156$ în 10^6 ani.

STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, 1, 1978

ASUPRA FUNCŢIILOR SPLINE HERMITE DE DOUĂ VARIABILE

PETRU P. BLAGA

În lucrarea lui D. W. Arthur [2] se dă o metodă de construcţie a funcţiilor spline naturale de două variabile, folosind noţiunea de nucleu reproduizant al unui spaţiu Hilbert de funcţii de două variabile şi se dau cîteva proprietăţi ale acestora. Folosind această metodă se construiesc funcţiile spline Hermite de două variabile şi se extind unele proprietăţi ale funcţiilor spline Hermite de o variabilă la funcţii spline Hermite de două variabile.

Fie domeniul

$$D = \{(x, y) \in \mathbf{R}^2,\ a \leq x \leq b,\ c \leq y \leq d\}$$

şi o diviziune Δ_0 a acestui domeniu, realizată prin

$$\Delta_x^\circ : a \leq x_0 < x_1 < \ldots < x_p \leq b \text{ şi}$$

$$\Delta_y^\circ : c \leq y_0 < y_1 < \ldots < y_q \leq d,$$

unde nodurile x_i, $i = \overline{0, p}$ au respectiv ordinul de multiplicitate r_i, $i = \overline{0, p}$, iar nodurile y_j, $j = \overline{0, q}$ au respectiv ordinul de multiplicitate s_j, $j = \overline{0, q}$. Notăm $m = \sum_{i=0}^{p} r_i$ şi $n = \sum_{j=0}^{q} s_j$.

Considerăm spaţiul de funcţii

$\mathcal{H}_{\Delta_0}^{(m,n)}(D) = \{f : D \to \mathbf{R}$, cu proprietăţile (i) — (iv)$\}$

(i) $f^{(i,j)}(.,.) \in C(D)$ pentru $i < m$ şi $j < n$.

(ii) $f^{(m-1,l)}(., y_j)$ este absolut continuă pe $[a, b]$ şi $f^{(m,l)}(., y_j) \in L^2[a, b]$, pentru $j = \overline{0, q}$ şi $l = \overline{0, s_j - 1}$.

(iii) $f^{(k,n-1)}(x_i, .)$, este absolut continuă pe $[c, d]$ şi $f^{(k,n)}(x_i, .) \in L^2[c, d]$, pentru $i = \overline{0, p}$ şi $k = \overline{0, r_i - 1}$.

(iv) $f^{(m-1,n-1)}(.,.)$ este absolut continuă pe D şi $f^{(m,n)}(.,.) \in L^2(D)$.

TEOREMA 1. *Spaţiul de funcţii $\mathcal{H}_{\Delta_0}^{(m,n)}(D)$ este un spaţiu Hilbert în raport cu produsul scalar definit prin*

$$(f, g) = \sum_{i,j=0}^{p,q} \sum_{k,l=0}^{r_i-1, s_j-1} f^{(k,l)}(x_i, y_j) g^{(k,l)}(x_i, y_j) +$$

$$+ \sum_{i=0}^{p} \sum_{k=0}^{r_i-1} \int_c^d f^{(k,n)}(x_i, y) g^{(k,n)}(x_i, y) dy +$$

$$+\sum_{j=0}^{q}\sum_{l=0}^{s_j-1}\int_a^b f^{(m,l)}(x, y_j)g^{(m,l)}(x, y_j)dx + \iint_D f^{(m,n)}(x, y)g^{(m,n)}(x, y)dxdy,$$

pentru orice $f, g \in \mathcal{H}_{\Delta_0}^{(m,n)}(D)$.

Demonstrație. Afirmația este imediată dacă ținem seamă de faptul că în spațiul de funcții considerat putem scrie formula de interpolare a lui Hermite relativă la diviziunea Δ_0,

$$f(x, y) = \sum_{i,j=0}^{p,q}\sum_{k,l=0}^{r_i-1,s_j-1} c_{ik}(x)d_{jl}(y)f^{(k,l)}(x_i, y_j) +$$

$$+\sum_{j=0}^{q}\sum_{l=0}^{s_j-1} d_{jl}(y)\int_a^b g(x, s)f^{(m,l)}(s, y_j)\,ds +$$

$$+\sum_{i=0}^{p}\sum_{k=0}^{r_i-1} c_{ik}(x)\int_c^d h(y, t)f^{(k,n)}(x_i, t)dt + \iint_D g(x, s)h(y, t)f^{(m,n)}(s,t)dsdt,$$

unde c_{ik}, $i = \overline{0, p}$, $k = \overline{0, r_i - 1}$ și d_{jl}, $j = \overline{0, q}$, $l = \overline{0, s_j - 1}$ sînt polinoamele fundamentale de interpolare ale lui Hermite de o singură variabilă relative la diviziunile Δ_x^0 și Δ_y^0 respectiv, iar funcțiile de două variabile g și h sînt date de formulele

$$g(x, s) = \frac{(x-s)_+^{m-1}}{(m-1)!} - \sum_{i=0}^{p}\sum_{k=0}^{r_i-1}\frac{(x_i-s)_+^{m-k-1}}{(m-k-1)!}c_{ik}(x) \text{ și}$$

$$h(y, t) = \frac{(y-t)_+^{n-1}}{(n-1)!} - \sum_{j=0}^{q}\sum_{l=0}^{s_j-1}\frac{(y_j-t)_+^{n-l-1}}{(n-l-1)!}d_{jl}(y).$$

DEFINIȚIA 1. [1]. Numim nucleu reproduizant al spațiului Hilbert de funcții $H = \{f: \Omega \to \mathbf{R},\ \Omega \subset \mathbf{R}^N\}$ o funcție K definită pe produsul cartezian $\Omega \times \Omega$ cu valori reale și care verifică proprietățile:

1° $K(\cdot, y) \in H$, pentru orice $y \in \Omega$ fixat,

2° $(K(x, \cdot), f(x)) = f$, pentru orice $f \in H$.

Observație. Nu orice spațiu Hilbert admite nucleu reproduizant, dar dacă admite acesta este unic [1].

TEOREMA 2. *Spațiul Hilbert* $\mathcal{H}_{\Delta_0}^{(m,n)}(D)$ *posedă nucleu reproduizant.*

Demonstrație. Procedînd ca în [6] se obține nucleul reproduizant al spațiului Hilbert considerat, ca fiind

$$K(X, Y) = K_1(x, s)K_2(y, t), \text{ unde } X = (x, y),\ Y = (s, t),$$

$x, s \in [a, b], y, t \in [c, d]$, iar pentru K_1 şi K_2 se obţin expresiile

$$K_1(x, s) = \sum_{i=0}^{p} \sum_{k=0}^{r_i-1} c_{ik}(x)c_{ik}(s) + \int_a^b g(x, t)g(s, t)dt \text{ şi}$$

$$K_2(y, t) = \sum_{j=0}^{q} \sum_{l=0}^{s_j-1} d_{jl}(y)d_{jl}(t) + \int_c^d h(y, s)h(t, s)ds$$

folosind metoda dată de C. de Boor şi R. E. Lynch [5]. Se calculează integralele din membrul drept al fiecărei formule şi obţinem

$$K_1(x, s) = \sum_{i=0}^{p} \sum_{k=0}^{r_i-1} c_{ik}(x)c_{ik}(s) +$$

$$+(-1)^m \left\{\frac{(x-s)_+^{2m-1}}{(2m-1)!} - \sum_{i=0}^{p} \sum_{k=0}^{r_i-1} \left[\frac{(x_i-s)_+^{2m-k-1}}{(2m-k-1)!} c_{ik}(x) + (-1)^k \frac{(x-x_i)_+^{2m-k-1}}{(2m-k-1)!} c_{ik}(s)\right] + \right.$$

$$\left. + \sum_{i,j=0}^{p} \sum_{k,l=0}^{r_i-1, r_j-1} (-1)^l \frac{(x_i - x_j)_+^{2m-k-l-1}}{(2m-k-l-1)!} c_{ik}(x)c_{jl}(s)\right\}$$

şi o expresie analogă pentru K_2.

Prin calcul direct se verifică cele două proprietăţi din definiţia nucleului reproduizant.

Considerăm o nouă diviziune Δ a domeniului D, care se obţine prin

$$\Delta_x : a \le a_0 < a_1 < \ldots < a_P \le b, \quad P \ge p \text{ şi}$$

$$\Delta_y : b \le b_0 < b_1 < \ldots < b_Q \le d, \quad Q \ge q,$$

unde a_i, $i = \overline{0, P}$ au respectiv ordinul de multiplicitate u_i, $i = \overline{0, P}$, iar b_j, $j = \overline{0, Q}$ au respectiv ordinul de multiplicitate v_j, $j = \overline{0, Q}$ şi în plus Δ_0 este o subdiviziune a diviziunii Δ, adică

$$\{x_0, x_1, \ldots, x_p\} \subset \{a_0, a_1, \ldots, a_P\} \quad \text{şi}$$

$$\{y_0, y_1, \ldots, y_q\} \subset \{b_0, b_1, \ldots, b_Q\},$$

iar dacă $x_i = a_j$ atunci $r_i \le u_j$ şi dacă $y_i = b_j$ atunci $s_i \le v_j$.

Notăm $M = \sum_{i=0}^{P} u_i$ şi $N = \sum_{j=0}^{Q} v_j$.

Considerăm operatorul T definit în felul următor

$$T : \mathcal{H}_{\Delta_0}^{(m,n)}(D) \to Z = L^2(D)X(L^2[a, b])^n X(L^2[c, d])^m$$

pentru orice $f \in \mathcal{H}_{\Delta_0}^{(m,n)}(D)$ avem

$$Tf = (f^{(m,n)}(.,.)\,;\, f^{(m,l)}(., y_j),\ j = \overline{0, q},\ l = \overline{0, s_j - 1}\,;$$
$$f^{(k,n)}(x_i, .),\ i = \overline{0, p},\ k = \overline{0, r_i - 1}).$$

Spaţiul Z este un spaţiu Hilbert în raport cu produsul scalar definit prin

$$(f, g)_Z = \iint_D f_0(x, y) g_0(x, y) dx dy + \sum_{i=1}^{n} \int_a^b f_i(x) g_i(x) dx +$$
$$+ \sum_{j=1}^{m} \int_c^d f_{n+j}(y) g_{n+j}(y) dy,$$

pentru orice $f = (f_0; f_1, \ldots, f_n; f_{n+1}, \ldots, f_{n+m}) \in Z$ şi orice $g = (g_0; g_1, \ldots, g_n; g_{n+1}, \ldots, g_{n+m}) \in Z$.

Nucleul operatorului T este mulţimea polinoamelor de două variabile de grad $m - 1$ şi $n - 1$ respectiv în raport cu prima şi a doua variabilă, deci

$$\dim(\operatorname{Ker} T) = mn \leq MN.$$

Fie vectorul $r \in R^{N+M}$ şi mulţimea de funcţii

$$U(r) = \{f \in \mathcal{H}_{\Delta_0}^{(m,n)}(D),\ f^{(k,l)}(a_i, b_j) = r_{ij}^{kl},\ i = \overline{0, P},$$
$$k = \overline{0, u_i - 1},\ j = \overline{0, Q},\ l = \overline{0, v_j - 1}\}$$

DEFINIŢIA 2. [3]. Numim funcţie spline Hermite de interpolare relativ la diviziunea Δ şi vectorul r, orice element $s \in U(r)$ pentru care avem

$$||Ts||_Z = \min_{f \in U(r)} ||Tf||_Z$$

Existenţa funcţiei spline Hermite de interpolare astfel definită este asigurată de faptul că nucleul lui T are dimensiune finită, iar unicitatea rezultă din $U(0) \cap \operatorname{Ker} T = \{0\}$.

Să considerăm funcţia de reprezentare e_{ij}^{kl} a funcţionalei Hermite definită prin $\delta(f_{ij}^{kl}) = f^{(k,l)}(a_i, b_j)$ şi care se obţine cu ajutorul nucleului reproduizant după formula

$$e_{ij}^{kl} = \frac{\partial^{k+l}}{\partial s^k \partial t^l} [K(., s\,;\, ., t)]_{\substack{s=a_i \\ t=b_j}} = \frac{\partial^k}{\partial s^k} [K_1(., s)]_{s=a_i} \cdot \frac{\partial^l}{\partial t^l} [K_2(., t)]_{t=b_j}$$

pentru $i = \overline{0, P}$, $k = \overline{0, u_i - 1}$, $j = \overline{0, Q}$, $l = \overline{0, v_j - 1}$.

Considerăm subspaţiul generat de aceste funcţii

$$S = \operatorname{span}\ (\{e_{ij}^{kl},\ i = \overline{0, P},\ k = \overline{0, u_i - 1},\ j = \overline{0, Q},\ l = \overline{0, v_j - 1}\})$$

şi definim operatorul de proiecţie

$$P_S : \mathcal{H}_{\Delta_0}^{(m,n)}(D) \ni f \mapsto P_S f \in S$$

unde $P_S f$ este elementul de cea mai bună aproximare al lui f prin elemente din S.

PROPRIETATEA 1. $P_S f$ este funcţia spline Hermite de interpolare relativă la diviziunea Δ pentru funcţia $f \in \mathcal{H}_{\Delta_0}^{(m,n)}(D)$.

Demonstraţie. Pentru a demonstra această proprietate ne folosim de următoarele relaţii

$$(P_S f,\ g) = (P_S f,\ P_S g) = (f,\ P_S g),\ \forall f,\ g \in \mathcal{H}_{\Delta_0}^{(m,n)}(D) \tag{2}$$

$$||f||^2 = ||f - P_S f||^2 + ||P_S f||^2,\ \forall f \in \mathcal{H}_{\Delta_0}^{(m,n)}(D). \tag{3}$$

Notăm

$$U(f) = \{h \in \mathcal{H}_{\Delta_0}^{(m,n)}(D),\ h^{(k,l)}(a_i,\ b_j) = f^{(k,l)}(a_i,\ b_j),\ i = \overline{0,\ P},$$
$$k = \overline{0,\ u_i - 1},\ j = \overline{0,\ Q},\ l = \overline{0,\ v_j - 1}\}$$

Scriem relaţia (2) pentru $g = e_{ij}^{kl}$

$$(P_S\, f,\ e_{ij}^{kl}) = (f,\ e_{ij}^{kl}) \text{ pentru } i = \overline{0,\ P},\ k = \overline{0,\ u_i - 1},\ j = \overline{0,\ Q},$$
$$l = \overline{0,\ v_j - 1}.$$

Ţinînd seama de modul de obţinere a funcţiilor de reprezentare e_{ij}^{kl} cu ajutorul nucleului reproduizant obţinem

$$\frac{\partial^{k+l}}{\partial x^k \partial y^l}[P_S f(x,y)]_{\substack{x=a_i\\ y=b_j}} = \frac{\partial^{k+l}}{\partial x^k \partial y^l}[f(x,\ y)]_{\substack{x=a_i\\ y=b_j}}$$

pentru $i = \overline{0,\ P}$, $k = \overline{0,\ u_i - 1}$, $j = \overline{0,\ Q}$, $l = \overline{0,\ v_j - 1}$, deci $P_S f \in U(f)$. Mai rămîne de arătat că

$$||TP_S f||_Z = \min_{h \in U(f)} ||Th||_Z, \tag{4}$$

pentru aceasta scriem relaţia (3) pentru $h \in U(f)$,

$$||h||^2 = ||h - P_S h||^2 + ||P_S h||^2$$

şi deoarece $P_S h = P_S f$ obţinem

$$||P_S f|| \leq ||h||, \quad h \in U(f) \tag{5}$$

cu egalitate dacă şi numai dacă $h = P_S f$. Ţinînd seama de faptul că $P_S f$ şi h sînt din $U(f)$ din (5) rezultă (4).

PROPRIETATEA 2. Fie $f \in \mathcal{H}_{\Delta_*}^{(m,n)}(D)$ şi $s = P_S f$, atunci

$$\|T(f - s)\|_Z \leq \|T(f - \tilde{s})\|_Z, \qquad \forall s \in S,$$

cu egalitate dacă şi numai dacă $\tilde{s} \in s + \mathfrak{L}_{m-1,n-1}$.

Proprietatea este un corolar al definiţiei funcţiei spline Hermite de interpolare cu ajutorul operatorului T.

PROPRIETATEA 3. Dacă F este o funcţională din clasa

$$\mathfrak{L}^{(m,n)} = \Big\{F,\ Ff = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\iint_D f^{(i,j)}(x, y)d\mu_{ij}(x, y), \text{ unde } \mu_{ij} \text{ sînt funcţii cu variaţie mărginită pe } D\Big\}$$

atunci aproximanta optimală în sensul lui A. Sard [9]

$$F^*f = \sum_{i,j=0}^{P,Q}\ \sum_{k,l=0}^{u_i-1,v_j-1} A_{ij}^{kl} f^{(k,l)}(a_i, b_j)$$

se obţine prin aplicarea funcţionalei F asupra lui $P_S f$.

Demonstraţia este analogă cu cea dată de D. W. Arthur [2], iar restul $R = F - F^*$ are reprezentarea

$$Rf = \sum_{i=0}^{p}\sum_{k=0}^{r_i-1}\int_c^d V_i^k(t) f^{(k,n)}(x_i, t)dt +$$

$$+\sum_{j=0}^{q}\sum_{l=0}^{s_j-1}\int_a^b U_j^l(s) f^{(m,l)}(s, y_j)ds + \iint_D W(s, t) f^{(m,n)}(s, t)dsdt$$

unde

$$U_j^l(s) = R\,[d_{jl}(y)g(x, s)] = R\left[d_{jl}(y)\,\frac{(x-s)_+^{m-1}}{(m-1)!}\right],\ j = \overline{0, q},\ l = \overline{0, s_j - 1}$$

$$V_i^k(t) = R[c_{ik}(x)h(y, t)] = R\left[c_{ik}(x)\,\frac{(y-t)_+^{n-1}}{(n-1)!}\right],\ i = \overline{0, p},\ k = \overline{0, r_i - 1}$$

$$W(s, t) = R\left[\frac{(x-s)_+^{m-1}}{(m-1)!}\ \frac{(y-t)_+^{n-1}}{(n-1)!}\right] -$$

$$-\sum_{j=0}^{q}\sum_{l=0}^{s_j-1}\frac{(y_j - t)_+^{n-l-1}}{(n-l-1)!}U_j^l(s) - \sum_{i=0}^{p}\sum_{k=0}^{r_i-1}\frac{(x_i - s)_+^{m-k-1}}{(m-k-1)!}V_i^k(t).$$

Metoda de calcul a funcţiilor spline Hermite de interpolare este cea din [2], cu modificarea că apar în diferenţele divizate considerate noduri cu un anumit ordin de multiplicitate.

Cazuri particulare. 1°. Dacă diviziunile Δ_0 şi Δ sînt identice obţinem spaţiul S ca fiind mulţimea polinoamelor de două variabile, de grad $m-1$ şi $n-1$ respectiv în raport cu prima variabilă şi a doua variabilă, iar funcţia spline Hermite de interpolare se reduce la polinomul de interpolare al lui Hermite de două variabile.

2°. Dacă $r_i = s_j = 1$ pentru $i = \overline{0, p}$, $j = \overline{0, q}$ se obţin funcţiile spline naturale de două variabile, construite în acest mod de către D. W. Arthur [2].

3° Dacă $p = 0$, $q = 0$, $r_0 = m$, $s_0 = n$ se obţin funcţiile spline de două variabile relative la spaţiul de funcţii $T^{(m,n)}(\alpha, \beta)$ considerat de L. E. Mansfield [7].

4°. Dacă $p = q = 1$, $x_0 = a$, $x_1 = b$, $y_0 = c$, $y_1 = d$ se obţine un spaţiu Hilbert de funcţii pentru care condiţiile (ii) şi (iii) sînt relative numai la frontiera domeniului D, ceea ce simplifică mult forma nucleului reproduizant precum şi calculul efectiv al funcţiilor spline Hermite de interpolare.

(Intrat în redacţie la 10 mai 1977)

BIBLIOGRAFIE

1. N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc., **68** (1950), 337–404.
2. D. W. Arthur, *Multivariate spline functions I. Construction, properties, and computation*, J. Approx. Theory, 12 (1974), 396–411.
3. M. Atteia, *Généralisation de la définition et des propriétés des „spline functions"*, C. R. Acad. Sci., 260 (1965), 3550–3553.
4. G. Birkhoff, M. H. Schultz, R. S. Varga, *Piecewise Hermite interpolation in one and two variables with applications to partial differential equations*, Numer. Math., 11 (1968), 232–256.
5. C. de Boor, R. E. Lynch, *On splines and their minimum properties*, J. Math. Mech., 15 (1966), 953–969.
6. L. E. Mansfield, *On optimal approximation of linear functionals in spaces of bivariate functions*, SIAM J. Numer. Anal., 8 (1971), 115–126.
7. L. E. Masfield, *Optimal approximation and error bounds in spaces of bivariate functions*, J. Approx. Theory., 5 (1972), 77–96.
8. L. E. Mansfield, *On the variational characterization and convergence of bivariate splines*, Numer. Math., 20 (1972), 99–114.
9. A. Sard, *Linear Approximation*, Math. Surveys, No. 9, American Mathematical Society, Providence, R I, 1963.

SUR LES FONCTIONS „SPLINE" DE HERMITE DE DEUX VARIABLES

(Résumé)

Dans cet article on définit les fonctions „spline" de Hermite de deux variables utilisant la méthode de D. W. Arthur [2] et on étudie quelques propriétés minimales de ces fonctions. Pour obtenir les fonctions „spline" de Hermite on utilise la notion du noyau reproduisant d'un espace de Hilbert des fonctions réelles.

STUDIA UNIV, BABEŞ-BOLYAI, MATHEMATICA, 1, 1978

L'INFORMATION ASSOCIÉE A UNE RÉGION DE CONFIANCE

E. OANCEA et M. RĂDULESCU

Le problème posé, c'est la détermination d'un indicateur pour l'information comportée d'une région de confiance.

On considère une suite d'éxperiences (E_t), $t = 1, \ldots, n$ indépendantes, pour la même caractéristique X. Soient $x_2^{(t)}, x_2^{(t)}, \ldots, x_{n_t}^{(t)}$, $t = 1, \ldots, n$ les observations correspondantes à l'experience E et on suppose qu'elles ont la même loi de probabilité pour $t = 1, \ldots, n$, la loi normale. La valeur moyenne x , $t = 1, \ldots, n$ détermine une distribution statistique qui peut être ajustée par la méthode des plus petites carrés. La courbe d'ajustage théorique correspondante aux variables $x_t = y$, $t = x$, est d'équation

$$y = a + bx \tag{1}$$

et la forme estimative obtenue par la méthode des plus petites carrés:

$$\hat{y} = \alpha + \beta x$$

où

$$\alpha = m + \beta\bar{x}, \quad m = \frac{1}{n}\sum_{1}^{n} y_i = \bar{y}, \quad \bar{x} = \frac{1}{n}\sum_{1}^{n} x_i, \quad \beta = \frac{S_{xy}}{S_{xx}},$$

$$S_{xx} = \sum_{1}^{n} (x_i - \bar{x})^2, \quad S_{xy} = \sum_{1}^{n} (x_i - \bar{x})(y_i - \bar{y}).$$

Si on considère aussi le facteur correcteur η qui représente la déviation à la vraie valeur de y donné par (1), on a

$$y = \alpha + \beta x + \eta = \hat{y} + \eta.$$

où η est une variable aléatoire normale $N(0, \sigma_\eta)$. On observe que μ est aussi normale $N(a + bx, \sigma_\eta)$. D'autre part, de la forme théorique (1) de la variable, il résulte qu'elle est normale de paramètres μ et σ_y ($\mu = a + bx$).

Pour le calcul de la variation de y, en tenant compte que (1) on peut écrire

$$y = \bar{y} + b(x - \bar{x}) \tag{2}$$

d'où il résulte

$$\sigma_y^2 = \sigma_{\bar{y}}^2 + \sigma_b^2(x - \bar{x})^2$$

Parce que

$$\sigma_{\bar{y}}^2 = \sigma_\eta^2 \frac{1}{n}, \qquad \sigma_b^2 = \sigma_\eta^2 \frac{1}{S_{xx}}$$

on obtient

$$\sigma_y^2 = \sigma_\eta^2 \left[\frac{1}{n} + \frac{(x - \bar{x})^2}{S_{xx}}\right].$$

La variable aléatoire

$$\frac{y-\mu}{\sigma_y}$$

est évidemment $N(0, 1)$. Pour estimer le paramètre μ on peut considérer la statistique

$$T_{n-2} = \frac{y-\mu}{s_\eta \sqrt{\frac{1}{n} + \frac{(x-\bar{x})^2}{S_{xx}}}}, \tag{3}$$

où

$$s_\eta^2 = \frac{1}{n-2}\left[S_{yy} - \frac{S_{xy}^2}{S_{yy}}\right], \quad S_{yy} = \sum_1^n (y_i - \bar{y})^2$$

qui est de type Student avec $n-2$ degrés de liberté. En choisissant la probabilité de risque q, on détermine dans le plan du (x, y) pour le paramètre μ une région R_q nommée „région de confiance", donnée par:

$$P[T_{n-2} \in (-t_q, t_q)] = 1 - q.$$

La frontière de cette région s'obtient de la relation

$$\pm t_q = \frac{y-\mu}{s_\eta \sqrt{\frac{1}{n} + \frac{(x-\bar{x})^2}{S_{xx}}}} \tag{4}$$

qui représente une famille d'hyperboles de paramètre μ. (le paramètre μ est variable avec x).

L'équation (4) peut être rationnalisée et amenée à la forme

$$\frac{(x-\bar{x})^2}{\gamma^2} + \frac{(y-a-b\bar{x})^2}{\delta^2} - 1 = 0 \tag{5}$$

où

$$\delta^2 = \frac{b^2 + 1 - \frac{t_q^2 s_\eta^2}{S_{xx}} + \sqrt{\left(b^2 + 1 - \frac{t_q^2 s_\eta^2}{S_{xx}}\right)^2 + 4\frac{t_q^2 s_\eta^2}{S_{xx}}}}{2\frac{t_q^2 s_\eta^2}{n}}$$

$$\gamma^2 = \frac{b^2 + 1 - \frac{t_q^2 s_\eta^2}{S_{xx}} - \sqrt{\left(b^2 + 1 - \frac{t_q^2 s_\eta^2}{S_{xx}}\right)^2 + 4\frac{t_q^2 s_\eta^2}{S_{xx}}}}{2\frac{t_q^2 s_\eta^2}{n}}.$$

Le lieu géométrique des sommets correspondants à l'axe nontransversale détermine la frontière de la région R_q. (On observe que cette frontière est semblable à l'hyperbole, mais ses „axes" ne sont pas orthogonales; c'est pourquoi cette région est connue dans la littérature de spécialité comme une région „hyperbolique".)

De l'équation (4) on voit que R_p contient la droite $y = a + bx$ et les points de la frontière R_q sont également éloignés sur une direction parallèle à Oy, au point correspondant de la droite (1).

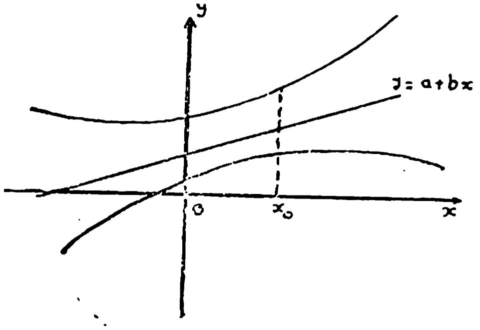

Fig. 1.

On observe que pour un x_0 fixé, la droite $x = x_0$ intersectée avec la frontière de R_q, détermine un intervalle de confiance I_{qx_0} qui en fait est un intervalle de confiance pour μ.

Pour déterminer l'information correspondante, apportée par la région de confiance R_q, on observe de la fig. 1, que dans le cas où les hyperboles de la famille d'équation (4) tendent vers les droites

$$y - \mu = \pm t_q \frac{s_\eta}{\sqrt{n}}$$

alors la frontière de R_q est formée de ces droites et l'information apportée de région R_q pour chaque $x = x_0$, $x_0 \in R$, tend à donner la valeur maximale.

Dans le cas où les hyperboles s'approchent à la droite

$$x = \bar{x}$$

l'intervalle de confiance I_{qx_0} pour chaque $x_0 \neq \bar{x}$, devient infini et donc l'information de la région R_q est minimale.

Il résulte d'ici que les situations limites avec lesquelles on compare l'information contenue dans une région de confiance quelconque R_q sont :

$$\text{I} \quad y = \mu \qquad (R_q \text{ max}) \tag{6}$$

qui représente la situation optimale et

$$\text{II} \quad x = \bar{x} \qquad (R_q \text{ min}) \tag{7}$$

qui représente la situation avec l'information minimale.

Pour établir un indicateur de la mesure d'information correspondant à une région de confiance R_q, on considère le vecteur

$$V_q\left(\frac{\delta}{\sqrt{\delta^2 + \gamma^2}}, \frac{\gamma}{\sqrt{\delta^2 + \gamma^2}}\right) \tag{8}$$

dénommé *le vecteur d'information de la région de confiance* R_q avec le niveau de signification q.

On observe de l'expression de δ^2 et γ^2 qu'elles ne s'annulent pas simultanément et de l'analyse graphique de la région de confiance il résulte la signification informationelle donnée par le vecteur V_q relatif à la quantité d'information associée à une région de confiance, c'est à dire :

Si

$$\gamma \to 0, \quad V_q \to (1,0) \tag{1}$$

les hyperboles (5) tendent vers la droite (7) qui correspond à la situation d'information minimale.

Si

$$\delta \to 0, \quad V_q \to (0, 1) \tag{2}$$

les hyperboles (5) tendent vers la droite (6) qui détermine la situation d'information maximale.

Pratiquement, le vecteur V_q sera différent à ces situations extrêmes et donc le chercheur fixera un niveau de signification $\varepsilon > 0$, suffisamment petit, pour accepter avec le risque q la conclusion que l'information de R_q est minimale, si

$$\gamma < \varepsilon,$$

ou l'information est maximale si

$$\gamma > 1 - \varepsilon.$$

Dans ces intermédiaires on apprécie la grandeur d'information d'après les valeurs des composantes de vecteur V_q.

Remarque : Pour une région de confiance quelconque R_q, on observe que l'intervalle de confiance I_{qx_0} avec l'information maximale correspond à $x_0 = \bar{x}$.

(Manuscrit reçu le 12 mai 1977)

BIBLIOGRAPHIE

1. F. S. Action, *Analysis of Straight – Line Data*, London, 1959.
2. P. D. Lark, B. R. Craven, *The Handling of Chemical Data*, London, 1968–1969.

INFORMAȚIA ASOCIATĂ UNEI REGIUNI DE ÎNCREDERE
(Rezumat)

În lucrare, unei regiuni de încredere R_q corespunzătoare unei drepte de regresie i se asociază indicatorul informațional (8). Sînt analizate cazurile extreme cînd informația este minimă, respectiv maximă. În cazurile concrete intermediare acestor două cazuri extreme, coeficientul informațional propus în lucrare poate da indicații cercetătorului asupra informației aduse de R_q.

STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, 1, 1978

FUZZY CORRELATION

D. DUMITRESCU

The fuzzy sets theory (Zadeh [2]) was introduced in order to give an instrument to describe those classes of problems exhibiting an imprecision which is not of aleatory nature. Such problems appear in modelling of so-called soft sciences (biology, psychology, sociology), natural languages and in general systems theory. Many mathematical branches, such as topology, probability, measure theory, formal languages and logic have been enlarged in the context of fuzzy sets theory. There are many works dedicated to fuzzy sets and their applications in pattern recognition, artificial intelligence, medical diagnosis, switching circuits, control systems, management information systems (for a complete bibliography up to 1976 see Kandel and Davis [6]).

In this note we consider properties defined on finite classes of objects, such that every object can enjoy to a different degree a property, and we propose a measure for the correlation of two such properties.

A fuzzy set f (Goguen [1]) on a set X is a function $f: X \rightarrow L$, where L is a partially ordered set. X is called the carrier of f and L the truth set of f. Generally one assumes that L is a completely distributive lattice. In this paper the case $L = [0, 1]$ will be considered.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a class of objects and P a characteristic of these objects. We assume that objects of X can enjoy to a different degree the characteristic P. The measure of the degree to which x_i enjoys P is given by $f_P(x_i)$ where f_P is the fuzzy set $f_P: X \rightarrow [0, 1]$. In this way we can associate to every characteristic P a fuzzy set f_P. In the following no distinction is made between the characteristic P and the corresponding fuzzy set f_P.

Let us consider a finite class of objects $X = \{x_1, x_2, \ldots, x_n\}$ and two characteristics f, g defined on it.

DEFINITION 1. The correlation of characteristics f and g is defined by:

$$C(f, g) = \sum_{i=1}^{n} [f(x_i)g(x_i) + \bar{f}(x_i)\bar{g}(x_i)], \tag{1}$$

where $\bar{f}$ is the complement of f, $\bar{f}(x) = 1 - f(x)$ for all x of X. In the following we shall denote $f(x_i) = f_i$.

From (1) we have $C(f, f) = E(f)$ where $E(f)$ is the energy of fuzzy set f (Dumitrescu [3]). $E(f)$ can be interpreted as a measure of non-fuzziness.

DEFINITION 2. The correlation coefficient of characteristics f and g is defined by

$$\alpha(f, g) = \frac{C(f, g)}{\left[\left(\sum_{i=1}^{n} (f_i^2 + \bar{f}_i^2)\right)\left(\sum_{i=1}^{n} (g_i^2 + \bar{g}_i^2)\right)\right]^{1/2}} = \frac{C(f, g)}{[E(f)E(g)]^{1/2}}. \quad (2)$$

We have evidently $\alpha(f, f) = 1$ and $C(f, g) = C(\bar{f}, \bar{g})$.

PROPOSITION 1. *The correlation coefficient satisfies the inequalities:*

$$0 \le \alpha(f, g) \le 1.$$

Proof. Using the well-known inequality

$$\sum_{i=1}^{n} f_i g_i \le \left[\left(\sum_{i=1}^{n} f_i^2\right)\left(\sum_{i=1}^{n} g_i^2\right)\right]^{1/2},$$

and the notations

$$\sum_{i=1}^{n} f_i^2 = a, \quad \sum_{i=1}^{n} g_i^2 = b, \quad \sum_{i=1}^{n} \bar{f}_i^2 = c, \quad \sum_{i=1}^{n} \bar{g}_i^2 = d,$$

we obtain

$$\alpha(f, g) = \frac{\sqrt{ab} + \sqrt{cd}}{(a + c)(b + d)},$$

from which we have:

$$\alpha^2(f, g) - 1 \le -\frac{(\sqrt{bc} + \sqrt{ad})^2}{(a + c)(b + d)} \le 0.$$

PROPOSITION 2. $C(f, g) = 0$ *if and only if we have* $f_i = 0$, $g_i = 1$ *for* $i \in I$ *and* $f_j = 1$, $g_j = 0$, *for* $j \in J$, *where* $I \cap J = \emptyset$ *and* $I \cup J = \{1, 2, \ldots, n\}$.

If $C(f, g) = 0$ *we say that* f *and* g *are uncorrelated characteristics.*

Let us consider now X as a time evoluting system the states of system being $x_1, x_2, \ldots, x_n$.

DEFINITION 3. The retarded correlation of characteristics f and g is by definition

$$C^{\tau}(f, g) = \sum_{i=1}^{n} [f(x_i)g(x_{i+\tau}) + \bar{f}(x)\bar{g}(x_{i+\tau})], \quad (4)$$

where $\tau \in \{0, 1, \ldots, n\}$ and $f(x_{i+\tau}) = 0$ for $i + \tau > n$.

The retarded correlation coefficient is

$$\alpha^{\tau}(f, g) = \frac{C(f, g)}{[C^{\tau}(f, f)C^{\tau}(g, g)]^{1/2}}. \quad (5)$$

We have $C^{0}(f, f) = C(f, f) = E(f)$.

DEFINITION 4. The total correlation of characteristics f and g is

$$\mathcal{C}(f,\ g) = \sum_{\substack{i,j=1\\ i\neq j}}^{n} (f_i g_j + \bar{f}_i \bar{g}_j), \tag{6}$$

where we assume that $f_i \leq f_{i+1}$

PROPOSITION 3. *The total correlation, the retarded correlation and the energy of a fuzzy set f are connected by:*

$$\mathcal{C}(f,\ f) + E(f) = \sum_{\tau=0}^{n} C^{\tau}(f,\ f). \tag{7}$$

Proof. From

$$\mathcal{C}(f,\ g) = \sum_{\tau=1}^{n} C^{\tau}(f,\ g) = \sum_{\tau=0}^{n} C^{\tau}(f,\ g) - C^{\circ}(f,\ g)$$

and $C^{\circ}(f,\ f) = E(f)$ we obtain (7).

DEFINITION 5. The total correlation coefficient is defined by

$$\alpha_t(f,\ g) = \frac{\mathcal{C}(f,\ g)}{[\mathcal{C}(f,\ f)\mathcal{C}(g,\ g)]^{1/2}}. \tag{8}$$

DEFINITION 6. The correlation of the three characteristics $f,\ g,\ h$ is

$$C(f,\ g,\ h) = \sum_{i=1}^{n} (f_i g_i h_i + \bar{f}_i \bar{g}_i \bar{h}_i), \tag{9}$$

and the correlation coefficient is given by:

$$\alpha(f,\ g,\ h) = \frac{C(f,\ g,\ h)}{\left[\left(\sum_{i=1}^{n} (f_i^3 + \bar{f}_i^3)\right) \sum_{i=1}^{n} (g_i^3 + \bar{g}_i^3) \left(\sum_{i=1}^{n} (h_i^3 + \bar{h}_i^3)\right)\right]^{1/3}}. \tag{10}$$

The generalisation for the case of n characteristics is obvious.

Let us now consider an experiment in which for every element x_i of X the characteristics f and g appear with probabilities p_i and q_i respectively$\left(p_i \geqslant 0,\ q_i \geqslant 0, \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1\right)$. The correlation of characteristics f and g is defined in this case by:

$$C^*(f,\ g) = \sum_{i=1}^{n} p_i q_i (f_i g_i + \bar{f}_i \bar{g}_i). \tag{11}$$

When $f_i,\ g_i \in \{0,\ 1,\}$, (11) reduce to Onicescu's informational correlation (Onicescu [4]).

The form of the retarded and the total correlation in this probabilistic case is evident.

(Received May 15, 1977)

REFERENCES

1. J. A. Goguen, *L-fuzzy sets*, J. Math. Anal. Appl., **18** (1967), 145–171.
2. L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (3) (1965), 338–353.
3. D. Dumitrescu, *A definition of an informational energy in fuzzy sets theory*, Studia Univ. Babeş-Bolyai, Math., 2 (1977), 57–59.
4. O. Onicescu, *Corelaţia informaţională*, Revista de statistică, **6** (1972), 3–13.
5. D. Zorilescu, *Covariograma informaţională*, Analele Univ. Bucureşti, ser. Mat. - Mec., 2 (1971), 149–157.
6. A. Kandel, H. A. Davis, *The first fuzzy decade (bibliography on fuzzy sets and their applications)*, Dept. of Computer Science, C S R 140, New Mexico Institute of Mining and Technology, Socorro, 1976.

CORELAŢIA FUZZY

(Rezumat)

Se consideră caracteristici imprecise (fuzzy) ale unei clase de obiecte şi se propun cîteva măsuri ale gradului de corelaţie între astfel de caracteristici.

STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, 1, 1978

O GENERALIZARE A INVARIANTULUI LUI H. LEPTIN

CONSTANŢA MOCANU

Fie G un grup local compact şi μ o măsură pozitivă netrivială relativ invariantă la stînga pe G de modul Δ^μ. Se ştie că în acest caz are loc formula

$$\int_G f(x)d\mu(x) = \int_G f(x)\Delta^\mu(x)dx, \ \forall f \in \mathcal{K}(G), \tag{1}$$

unde $\mathcal{K}(G)$ este spaţiul funcţiilor continue pe G cu suport compact, iar dx este măsura Haar invariantă la stînga pe G. Dacă X este o submulţime a lui G măsurabilă în raport cu μ atunci măsura sa este dată de

$$\mu(X) = \int_G \chi_G(x)d\mu(x) = \int_G \chi_G(x)\Delta^\mu(x)dx,$$

unde χ_G este funcţia caracteristică a lui X.

Avem, evident, $\mu(sX) = \Delta^\mu(s)\mu(X)$, $\forall s \in G$.

Notînd cu $\mathcal{C}(G)$ familia submulţimilor compacte din G, vom defini

$$I_\mu(G) = \sup_{C\in\mathcal{C}(G)} \inf_{\substack{U\in\mathcal{C}(G)\\ \mu(U)>0}} \frac{\mu(CU)}{\mu(U)} \tag{2}$$

În cazul cînd μ este măsura lui Haar, $I_\mu(G)$ este chiar invariantul $I(G)$ introdus de L e p t i n [1].

Fie G şi K două grupuri local compacte şi $\varphi : G \to K$ un epimorfism continuu deschis. Notăm $H = \ker(\varphi)$. Generalizînd o inegalitate a lui Leptin în [2], am arătat că

$$I(G) \leqslant I(K) \cdot I(H) \tag{3}$$

În cele ce urmează, se va da o generalizare a inegalităţii (3), în cazul cînd invariantul lui Leptin se înlocuieşte cu I_μ.

TEOREMA. *Fie G şi K două grupuri local compacte şi $\varphi : G \to K$ un epimorfism continuu deschis, iar $H = ker(\varphi)$. Fie μ şi ν două măsuri relativ invariante pe G, respectiv K, de module Δ^μ, respectiv Δ^ν. Atunci are loc inegalitatea*

$$I_\mu(G) \leqslant M \cdot I_\nu(K) \cdot I_\lambda(H), \tag{4}$$

unde λ este o măsură relativ invariantă pe H de modul Δ^λ definit de formula

$$\Delta^\lambda(s) = \Delta^\mu(s)/\Delta^\nu(\varphi(s)), \quad s \in G,$$

iar

$$M = \sup_{s\in G} \Delta^\lambda(s).$$

Pentru demonstrarea teoremei se utilizează două leme analoge cu cele din lucrarea [2]. Notăm cu x, y, z elementele generice din G, H, K. Fie dy şi dz măsurile Haar invariante la stînga pe H, respectiv K. Se ştie că pe H se poate defini măsura $\lambda = \mu/\nu$, care este relativ invariantă pe H de modul Δ^{λ} definit de egalitatea

$$\Delta^{\mu}(s) = \Delta^{\lambda}(s) \cdot \Delta^{\nu}(\varphi(s)), \ \forall s \in G, \tag{5}$$

Măsura λ este bine definită, astfel încît măsura μ este produsul omomorf al măsurilor λ şi ν, adică $\mu = \lambda \otimes \nu$.

Fiecărei funcţii $f \in \mathfrak{L}^1(G)$ îi corespunde o funcţie unică $\bar{f} \in \mathfrak{L}^1(K)$ cu proprietatea

$$\forall x \in G, \ \bar{f}[\varphi(x)] = \int_H f(xy)d\lambda(y) = \int_H f(xy)\Delta^{\lambda}(xy)\,dy, \tag{6}$$

iar

$$\int_G f(x)d\mu(x) = \int_K \bar{f}(z)d\nu(z)$$

sau

$$\int_G f(x)\Delta^{\mu}(x)dx = \int_K \bar{f}(z)\Delta^{\nu}(z)dz \tag{7}$$

LEMA 1. *Fie* $C \in \mathcal{C}(G)$, $W \in \mathcal{C}(H)$, $Q = C^{-1}C \cap H \in \mathcal{C}(H)$. *Atunci,*

$$\mu(CW) \leqslant M_C \nu[\varphi(C)]\lambda(QW),$$

unde

$$M_C = \max_{x \in C} \Delta^{\lambda}(x).$$

Demonstraţie. Fie $z \notin \varphi(C)$ şi $x \in \varphi^{-1}(z)$. Dacă $y \in H$, atunci $xy \notin CW$ [în adevăr, dacă $xy \in CW$, atunci $\varphi(xy) = \varphi(x)\varphi(y) = \varphi(x) = z \in \varphi(CW) \in \varphi(C)$]. Deci, $\chi(xy) = 0$ şi din (6) rezultă

$$\bar{\chi}_{CW}(z) = \bar{\chi}_{CW}(\varphi(x)) = \int_H \chi_{CW}(xy)d\lambda(y) = 0.$$

Prin urmare,

$$z \notin \varphi(C) \Rightarrow \bar{\chi}_{CW}(z) = 0. \tag{8}$$

Fie $x \in C$. Atunci, $x^{-1}CW \cap H \subset C^{-1}CW \cap H = (C^{-1}C \cap H)W = QW$, de unde

$$\chi_{CW}(xy) = \chi_{x^{-1}CW}(y) \leqslant \chi_{QW}(y).$$

Deci,

$$\bar{\chi}_{CW}(z) = \bar{\chi}_{CW}(\varphi(x)) = \int_H \chi_{CW}(xy)\Delta^{\lambda}(xy)\,dy \leqslant \Delta^{\lambda}(x) \int_H \chi_{QW}(y)\Delta^{\nu}(y)dy =$$
$$= \Delta^{\lambda}(x) \cdot \lambda(QW) \leqslant M_C\lambda(QW). \tag{9}$$

Din (7), (8) şi (9) rezultă

$$\mu(CW) = \int_G \chi_{CW}(x)\Delta^{\mu}(x)dx = \int_K \overline{\chi}_{CW}(z)\Delta^{\nu}(z)dz =$$

$$= \int_{\varphi(C)} \overline{\chi}_{CW}(z)\Delta^{\lambda}(z)dz \leq M_C\lambda(QW) \int_{\varphi(C)} \Delta^{\lambda}(z)dz = M_C\lambda(QW)\nu(\varphi(C)).$$

LEMA 2. *Pentru* $C \in \mathcal{C}(G)$ *avem*

$$m_C\nu[\varphi(C)] \leqslant \inf_{W\in\mathcal{C}(H)} \frac{\mu(CW)}{\lambda(W)} \leqslant M_C\nu[\varphi(C)]I_{\lambda}(H),$$

unde

$$M_C = \max_{x\in C} \Delta^{\lambda}(x) \text{ şi } m_C = \min_{x\in C} \Delta^{\lambda}(x).$$

Demonstraţie. Din (1) deducem

$$I_{\lambda}(H) \geqslant \inf_{\substack{W\in\mathcal{C}(H)\\ \lambda(W)>0}} \frac{\lambda(QW)}{\lambda(W)}.$$

Ţinînd seama de Lema 1, obţinem

$$\nu[\varphi(C)]I_{\lambda}(H) \geqslant \nu[\varphi(C)] \inf_{\substack{W\in\mathcal{C}(H)\\ \lambda(W)>0}} \frac{\lambda(QW)}{\lambda(W)} =$$

$$= \inf_{\substack{W\in\mathcal{C}(H)\\ \lambda(W)>0}} \frac{\nu(\varphi(C)) \cdot \lambda(QW)}{\lambda(W)} \geqslant \frac{1}{M_C} \inf_{\substack{W\in\mathcal{C}(H)\\ \lambda(W)>0}} \frac{\mu(CW)}{\lambda(W)},$$

de unde rezultă partea dreaptă a inegalităţii.

Pentru a demonstra partea stîngă a inegalităţii, fie $x \in C$ şi $W \in \mathcal{C}(H)$. Pentru $y \in H$ avem

$$\chi_{CW}(xy) = \chi_{x^{-1}CW}(y) \geqslant \chi_W(y)$$

deci

$$\overline{\chi}_{CW}[\varphi(x)] = \int_H \chi_{CW}(xy)\Delta^{\lambda}(y)dy \geqslant \int_H \chi_W(y)\Delta^{\lambda}(xy)dy =$$

$$= \Delta^{\lambda}(x) \int_H \chi_W(y)\Delta^{\lambda}(y)dy = \Delta^{\lambda}(x)\lambda(W) \geqslant m_C\lambda(W),$$

adică

$$\overline{\chi}_{CW}(z) \geqslant m_C\lambda(W) \text{ pentru orice } z \in \varphi(C).$$

De aici rezultă

$$\mu(CW) = \int_G \chi_{CW}(x)\Delta^{\lambda}(x)dx = \int_K \chi_{CW}(z)\Delta^{\nu}(z)dz =$$

$$= \int_{\varphi(C)} \bar{\chi}_{CW}(z)\Delta^{\nu}(z) \geqslant m_C\lambda(W) \int_{\varphi(C)} \Delta^{\nu}(z)dz = m_C\lambda(W)\nu[\varphi(C)].$$

Deci,

$$m_C\nu[\varphi(C)] \leqslant \frac{\mu(CW)}{\lambda(W)} \text{ oricare ar fi } W \in \mathcal{C}(H),$$

de unde, inegalitatea din partea stîngă.

Vom trece acum la demonstrarea teoremei.

Fie C şi U din $\mathcal{C}(G)$ cu $\mu(G) > 0$. Din Lema 2 rezultă că oricare ar fi $\varepsilon > 0$, există $W \in \mathcal{C}(H)$ şi $\lambda(H) > 0$, astfel încît

$$m_{CU}\nu[\varphi(CU)] = m_{CU}\nu[\varphi(C)\varphi(U)] \leqslant \frac{\mu(CUW)}{\lambda(W)} \leqslant M_{CU}\nu[\varphi(C)\varphi(U)](I_{\lambda}(H) + \varepsilon),$$

şi

$$m_U\nu[\varphi(U)] \leqslant \frac{\mu(UW)}{\lambda(W)}$$

sau

$$\frac{\lambda(W)}{\mu(UW)} \leqslant \frac{1}{m_U} \cdot \frac{1}{\nu[\varphi(U)]}.$$

De aici rezultă

$$\inf_{\substack{V \in \mathcal{C}(G)\\ \mu(V)>0}} \frac{\mu(CV)}{\mu(V)} \leqslant \frac{\mu(CUW)}{\mu(UW)} = \frac{\mu(CUW)}{\lambda(W)} \cdot \frac{\lambda(W)}{\mu(UW)} \leqslant \frac{M_{CU}}{m_U} \cdot \frac{\nu[\varphi(C)\cdot\varphi(U)]}{\nu(\varphi(U))} \cdot [I_{\lambda}(H) + \varepsilon].$$

Dar, $M_{CU} = M_C M_U$ (deoarece $\max\limits_{x \in CU} \Delta^{\lambda}(x) = \max\limits_{x_1 \in C, x_2 \in U} \Delta^{\lambda}(x_1 x_2) =$

$= \max\limits_{x_1 \in C} \Delta^{\lambda}(x_1) \max\limits_{x_2 \in U} \Delta^{\lambda}(x_2)$, şi inegalitatea de mai sus se poate scrie

$$\inf_{\substack{V \in \mathcal{C}(G)\\ \mu(V)>0}} \frac{\mu(CV)}{\mu(V)} \leqslant M_C \frac{M_U}{m_U} \cdot \frac{\nu[\varphi(C)\cdot\varphi(U)]}{\nu(\varphi(U))} \cdot [I_{\lambda}(H) + \varepsilon].$$

Deoarece φ este un epimorfism continuu deschis, rezultă că

$$\mathcal{C}(K) = \{\varphi(U) : U \in \mathcal{C}(G)\}.$$

Luînd în (10) infimul după $U \in \mathcal{C}(G)$, adică $\varphi(U) \in \mathcal{C}(K)$, şi apoi supremul după $C \in \mathcal{C}(G)$ şi ţinînd seama că $\inf \frac{M_U}{m_U} = 1$, deducem

$$I_{\mu}(G) \leqslant MI_{\nu}(K)[I_{\lambda}(H) + \varepsilon,],$$

unde $M = \sup\limits_{C \in \mathcal{C}(G)} M_C = \sup\limits_{S \in G} \Delta^{\lambda}(x)$.

Deoarece ε este arbitrar, deducem inegalitatea (4).

(Intrat în redacţie la 2 iunie 1977)

BIBLIOGRAFIE

1. H. Leptin, *On certain invariant of locally compact group*, Bull. Amer. Math. Soc. **72** (1966), 870–874.
2. C. Mocanu, *Asupra unui invariant al lui H. Leptin*, Studia Univ. Babeş-Bolyai, Math., **1** (1977), 22–25.

A GENERALIZATION OF AN INVARIANT OF H. LEPTIN

(Summary)

Let G be a locally compact group and μ a (left) relatively invariant measure on G with modulus Δ^{μ}. We define $I_{\mu}(G)$ by the formula (2), where $\mathcal{C}\,(G)$ denotes the family of compact subsets of G. If μ is the Haar measure on G then $I_{\mu}(G)$ reduces to the invariant $I(G)$ of H. Leptin [1]. Let G and K be two locally compact groups, $\varphi : G \to K$, an open continuous epimorphism and $H = \ker(\varphi)$. Let μ and ν be two relatively invariant measures on G, with moduli Δ^{μ} and Δ^{ν} respectively. Then the inequality (4) holds, where λ is a relatively invariant measure on H with modulus Δ^{λ} defined by (5) and $M = \sup\limits_{s \in G} \Delta^{\lambda}(s)$. This result is a generalization of an inequality due to H. Leptin [1].

STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, 1, 1978

REMARQUES CONCERNANT LE RAPPORT ENTRE LES PROBLÈMES DE PROGRAMMATION QUADRATIQUE INDÉFINIE ET LES PROBLÈMES DE PROGRAMMATION HYPERBOLIQUE

LIANA LUPŞA

Dans ce travail on étudie le problème de programmation nonlinéaire suivant : déterminer le maximum de la fonction f, définie sur l'espace R^n par

$$f(x) = (c^Tx + \alpha)(d^Tx + \beta) + e^Tx + \gamma \text{ pour tout } x \in R^n, \tag{1}$$

pour les valeurs de $x \in R^n$, qui vérifient le système

$$Ax \leqq b. \tag{2}$$

Dans l'énoncé de ce problème c, d, e sont des éléments donnés de l'espace R^n, α, β, γ sont des nombres réels donnés, A est une matrice de nombres réels de type (m, n) et b est un élément de l'espace R^m.

Le problème ainsi formulé, que nous allons désigner, par la suite, pour simplifier l'écriture, par (P), représente une généralisation immédiate des problèmes où l'on demande la maximisation du produit de deux fonctions linéaires, sur l'ensemble des solutions du système (2) et qui ont été étudiés dans [4—8]. De plus, ce problème est étroitement lié aux problèmes de programmation hyperbolique étudiés dans [3, 9].

Dans ce travail on met en évidence le rapport qui existe entre les points de maximum local (respectivement maximum) de la fonction (1) sur l'ensemble (2) d'une part et les points de maximum local (respectivement maximum) d'une fonction hyperbolique sur l'ensemble (2) d'autre part. Ensuite on relève la possibilité de l'utilisation d'un algorithme de détermination du maximum de la fonction (1) sur l'ensemble (2).

Nous introduisons les ensembles suivants :

$$\Omega = \{x \in R^n \mid Ax \leqq b\},$$
$$E = \{x \in R^n \mid d^Tx + \beta > 0\}.$$

Nous allons supposer que $\Omega \subset E$.

Pour caractériser les points de maximum de la fonction f dans Ω, nous définissons la fonction $g : E \times R \to R$, par

$$g(x, t) = \frac{f(x) - t}{d^Tx + \beta} \text{ pour tout } (x, t) \in E \times R. \tag{3}$$

On vérifie que la fonction g peut être exprimée aussi sous la forme

$$g(x, t) = c^Tx + \alpha + \frac{e^Tx + \gamma - t}{d^Tx + \beta} \text{ pour tout } (x, t) \in E \times R. \tag{3'}$$

THÉORÈME 1. *Soit* $x^0 \in \Omega$ *et* $t_0 = f(x^0)$. *Pour que* x^0 *soit un point de maximum local de la fonction* f *en* Ω *il est nécessaire et suffisant qu'il soit un point de maximum local de la fonction* $g(\cdot, t_0)$ *dans* Ω.

Démonstration. Nécessité. Soit x^0 un point de maximum local de la fonction f dans Ω. Il existe un voisinage V du point x^0 tel que

$$f(x) \leqq t_0 \quad \text{pour tout } x \in V \cap \Omega.$$

En tenant compte de la définition de la fonction $g(\cdot, t_0)$ il résulte que

$$g(x, t_0) \leqq 0 \quad \text{pour tout} \quad x \in V \cap \Omega.$$

D'autre part nous avons $g(x^0, t_0) = 0$. Donc $g(x, t_0) \leqq g(x^0, t_0)$ pour tout $x \in V \cap \Omega$.

Suffisance. Nous supposons que x^0 est un point de maximum local de la fonction $g(\cdot, t_0)$ dans Ω. Dans ce cas il existe un voisinage V du point x^0 tel que

$$g(x, t_0) \leqq g(x^0, t_0) \quad \text{pour tout } x \in V \cap \Omega.$$

Parce que $g(x^0, t_0) = 0$, nous avons $g(x, t_0) \leqq 0$ pour tout $x \in V \cap \Omega$. En tenant compte du fait que $x^0 \in \Omega$ et que $\Omega \subset E$ il résulte

$$f(x) \leqq t_0 = f(x^0) \quad \text{quel que soit } x \in V \cap \Omega,$$

ce qui démontre que x^0 est un point de maximum de la fonction f dans Ω.

COROLLAIRE. *Si* x^0 *est un point de maximum de la fonction* f *sur* Ω *alors* $\max\limits_{x \in \Omega} g(x, f(x^0)) = 0$.

Démonstration. Soit x^0 un point de maximum de la fonction f sur Ω. Alors, du théorème 1, il résulte que x^0 est un point de maximum de la fonction $g(\cdot, f(x^0))$ sur Ω. Mais $g(x^0, f(x^0)) = 0$. Donc $\max\limits_{x \in \Omega} g(x, f(x^0)) = 0$.

THÉORÈME 2. *Soit* $x^0 \in \Omega$ *et* $t_0 = f(x^0)$. *Une condition suffisante pour qu'un point de maximum* y^0 *de la fonction* $g(\cdot, t_0)$ *dans* Ω *soit un point de maximum de la fonction* f *sur* Ω *est que* $g(y^0, t_0) = 0$.

Démonstration. Soit y^0 un point de maximum de la fonction $g(\cdot, t_0)$ sur Ω et $g(y^0, t_0) = 0$. Puisque y^0 est un point de maximum de la fonction $g(\cdot, t_0)$ dans Ω nous avons

$$g(x, t_0) \leqq 0 \quad \text{pour tout} \quad x \in \Omega.$$

Cette relation implique

$$f(x) \leqq t_0 \quad \text{pour tout} \quad x \in \Omega.$$

D'autre part, de $g(y^0, t_0) = 0$ il résulte que $t_0 = f(y^0)$. Donc y^0 est un point de maximum de la fonction f dans Ω.

Il est bien connu que le produit de deux fonctions linéaires positives est une fonction quasiconcave. Il se pose la question : quelle allure aura la fonction qui s'obtient en ajoutant à ce produit une fonction linéaire? La réponse à cette question est donnée par le théorème suivant.

THÉORÈME 3. *Si la fonction $g(\cdot, f(x))$ est quasiconvexe sur Ω, pour tout $x \in \Omega$, alors la fonction f est quasiconvexe sur Ω.*

Démonstration. Soient x^1, x^2 deux points de Ω et

$$X = \{x \in R^n \mid x = tx^1 + (1-t)x^2,\ t \in [0, 1]\}.$$

Il faut montrer que

$$f(x) \leqq \max \{f(x^1), f(x^2)\} \text{ pour tout } x \in X. \tag{4}$$

Supposons, par exemple, que $\max \{f(x^1), f(x^2)\} = f(x^1)$. Puisque X est un sousensemble convexe de Ω, et $g(\cdot, f(x^1))$ est quasiconvexe sur Ω, il résulte que la restriction de la fonction $g(\cdot, f(x^1))$ à X peut atteindre son maximum seulement en x^1 ou en x^2. Si

$$\max_{x \in X} g(x, f(x^1)) = g(x^1, f(x^1)),$$

alors du théorème 1 il résulte que

$$\max_{x \in X} f(x) = f(x^1).$$

Par conséquent la relation (4) a lieu. Si

$$\max_{x \in X} g(x, f(x^1)) = g(x^2, f(x^1)),$$

alors nous avons

$$\frac{f(x) - f(x^1)}{d^T x + \beta} \leqq \frac{f(x^2) - f(x^1)}{d^T x^2 + \beta} \text{ pour tout } x \in X. \tag{5}$$

Compte tenu que $f(x^1) \geqslant f(x^2)$, l'inégalité (5) implique

$$\frac{f(x) - f(x^1)}{d^T x + \beta} \leqq 0 \quad \text{pour tout } x \in X$$

donc $f(x) \leqq f(x^1)$ quel que soit $x \in X$. Donc, dans ce cas aussi nous avons (4).

Les points x^1, x^2 ayant été choisis arbitrairement, il résulte que f est quasiconvexe sur Ω.

En nous appuyant sur les résultats contenus dans les théorèmes 1 et 2 nous pouvons présenter maintenant l'algorithme suivant pour la détermination des points de maximum de la fonction f dans Ω.

Le pas 0. On prend $i = 0$.

On détermine un point $x^0 \in \Omega$ (en utilisant par exemple la méthode simplex pour la résolution d'un système linéaire d'inéquations).

On continue par le pas 1.

Le pas 1. On calcule $t_i = f(x^i)$.

On détermine un point x^{i+1} de maximum de la fonction $g(\cdot,\ t_i)$ sur Ω (en utilisant par exemple l'algorithme de Maruşciac, I. [3].

On continue par le pas 2.

Le pas 2. Si $g(x^{i+1},\ t_i) \neq 0$, i augmente d'une unité et on revient au pas 1. Selon le théorème 2, si $g(x^{i+1},\ t_i) = 0$, alors x^{i+1} est un point de maximum de la fonction f dans Ω, donc x^{i+1} est une solution optimal du problème (P).

THÉORÈME 4. *Si* Ω *est un ensemble compact et* $\max\limits_{y\in\Omega} g(y,\ f(x)) = \max\limits_{y\in\ddot{\Omega}} g(y,\ f(x))$ *quel que soit* $x \in \ddot{\Omega}$, *où* $\ddot{\Omega}$ *est le profil de* Ω, *alors en appliquant l'algorithme ci-dessus on obtient après un nombre fini de pas une solution optimale du problème* (P).

Démonstration. Parce que $\max\limits_{y\in\Omega} g(y,\ f(x)) = \max\limits_{y\in\ddot{\Omega}} g(y,\ f(x))$ quel que soit $x \in \ddot{\Omega}$, nous pouvons supposer que les points x^i, $i \geqslant 1$ obtenus au pas 1 sont des sommets de Ω. Mais le nombre de sommets de Ω est fini, donc pour montrer que le nombre de pas est fini, il suffit de démontrer que nous ne pouvons pas revenir au même sommet. Pour le faire, supposons que nous avons trouvé le sommet x^{i+1}. Il peut se présenter deux cas :

1° $g(x^{i+1},\ t_i) = 0$ et alors nous nous arrêtons.

2° $g(x^{i+1}, t_i) \neq 0$ ce qui implique $f(x^{i+1}) - t^i > 0$, c'est-à-dire $f(x^{i+1}) > f(x^i)$ et alors nous continuous par le pas 1.

Il en résulte que nous passons dans l'algorithme à la détermination d'un nouveau point x^{i+1}, si et seulement $f(x^{i+1}) < f(x^i)$, donc la séquence $(f(x^k))_{k=0}^{i+1}$ est une séquence de nombres strictement croissante, ce qui implique que le séquence $(x^k)_{k=0}^{i+1}$ est une séquence d'éléments de R^n distincts.

Exemple. On considère le problème suivant : déterminer le maximum de la fonction f, définie sur l'espace R^2 par

$$f(x_1,\ x_2) = x_1 x_2 - 1 \quad \text{pour tout } (x_1,\ x_2) \in R^2$$

pour le valeurs de $(x_1,\ x_2) \in \Omega$, où

$$\Omega = \{(x_1,\ x_2) \in R^2 \mid x_1 \geqslant x_2,\ x_1 \leqq 2,\ x_2 \geqslant 1\}.$$

Première itération.

Le pas 0. Nous prenons $i = 0$ et $x^0 = (1,\ 1)$.

Le pas 1. Nous calculons la valeur de la fonction f sur le point x^0 et nous prenons $t_0 = f(x^0) = 0$; nous construisons la fonction

$$g(\cdot,\ 0) : \Omega \to R,\ g(x_1,\ x_2,\ 0) = x_1 - \frac{1}{x_2} \text{ pour tout } (x_1,\ x_2) \in \Omega.$$

Le point de maximum de la fonction $g(\cdot,\ 0)$, dans Ω obtenant à l'aide de l'algorithme de Maruşciac I. [3] est $x^1 = (2,\ 2)$. Parce que $g(2,\ 2) = \frac{3}{2} \neq 0$, nous augmentons i d'une unité et nous revenons au pas 1.

Deuxième itération.

Le pas 1. Nous calculons la valeur de la fonction f sur le point x^1 et nous prenons $t = f(x^1) = 3$. Nous construisons la fonction $g(\cdot, 3) : \Omega \to R$, $g(x_1, x_2, 3) = x_1 - \frac{4}{x_2}$ pour tout $(x_1, x_2) \in \Omega$.

Le point de maximum de la fonction $g(\cdot, 3)$ dans Ω est $x^2 = (2, 2)$. Parce que $g(2, 2, 3) = 0$, $x = x^2 = (2, 2)$ sera une solution optimale de la fonction f dans Ω.

Le principe de l'algorithme décrit ci-dessus est semblable à celui utilisé par Florian M. et Robillard P. [2] pour réduire la résolution d'un problème de programmation fractionnaire en variables bivalentes par l'introduction d'un paramètre à la résolution de plusieurs problèmes de programmation linéaire en variabiles bivalentes.

(Manuscrit reçu la 27 juin 1977)

BIBLIOGRAPHIE

1. Charnes, A., Cooper, W. W., *Programming with Linear Fractional Functions*, Naval Res. Logist. Quart., **9**, 3–4, (1962), 181–187.
2. Florian, M., Robillard, P., *Programmation Hyperbolique en Variables Bivalentes*, Revue Française A.I.R.O., **5**, V–1, (1971), 3–9.
3. Maruşciac, I., *Asupra unei programări hiperbolice*, Studii Cerc. Mat., **26** (1974), 419–430.
4. Maruşciac, I., Rădulescu, M., *Programarea unui produs de funcţii liniare*, Studii Cerc. Mat., **25** (1973), 833–845.
5. Orden, A., *Stationary points of quadratic functions under linear contraints* Computer Journal, **7**, 3 (1964).
6. Swarup Kanti., *Programming with indefinite quadratic function with linear constraints*, Cahiers de Centre d'Études de Rech. Op., **8**, 2 (1966), 132–136.
7. Swarup Kanti., *Quadratic programming*, Cahiers de Centre d'Études de Rech. Op., **8**, 4 (1966), 223–234.
8. Schaible, S., *Maximization of quasi-concave quotients and products of finitely many functionals*, Cahiers de Centre d'Études de Rech. Op., **16**, 1 (1974), 45–53.
9. Teterev, A., G., *Ob odnom obobscenii lineinovo i drobno lineinovo programmirovania*, Ekonomika i matem. metodî, 5 (1969), 440–447.

OBSERVAŢII PRIVIND LEGĂTURA DINTRE PROBLEMELE DE PROGRAMARE PĂTRATICĂ NEDEFINITĂ ŞI PROBLEMELE DE PROGRAMARE HIPERBOLICĂ

(Rezumat)

În lucrare, în teoremele 1 şi 2, se pune în evidenţă legătura ce există între punctele de maxim local respectiv de maxim ale funcţiei (1) pe mulţimea (2) pe de o parte şi punctele de maxim local respectiv de maxim ale unei funcţii hiperbolice pe mulţimea (2), pe de altă parte. Se indică apoi posibilitatea folosirii acestei legături pentru elaborarea unui algoritm de determinare a maximului funcţiei (1) pe mulţimea (2).

STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, 1, 1978

SUR LE PROLONGEMENT DES FONCTIONNELLES MAJORÉES PAR UNE FONCTIONNELLE SOUS-ADDITIVE

DUMITRU BARAC

1. H. Nakano [1] a prouvé le théorème suivant qui montre que, pour les fonctionnelles linéaires majorées par certaines fonctionnelles sous-additives, un théorème de prolongement du type du théorème de Hahn-Banach reste vrai :

Soient X un espace vectoriel réel, Y un sous-espace vectoriel de X, $p : X \to R$ une fonctionnelle sous-additive et $f_0 : Y \to R$ une fontionnelle linéaire telles que les conditions suivantes soient vérifiées :

(i) $0 \leqq p(x)$ *pour tout* $x \in X$;

(ii) $\lim\limits_{t \searrow 0} p(tx) = 0$ *pour tout* $x \in X$;

(iii) $f_0(y) \leqq p(y)$ *pour tout* $y \in Y$.

Alors il existe une fonctionnelle linéaire $f: X \to R$ *qui possède les propriétés suivantes :*

$f(y) = f_0(y)$ *pour tout* $y \in Y$.

$f(x) \leqq p(x)$ *pour tout* $x \in X$.

Dans cette note on démontrera une généralisation de ce résultat, généralisation qui s'obtient, d'une part, en affaiblissant les conditions (i) et (ii) imposées à la fonctionnelle p, et, d'autre part, en permettant que f_0 soit aussi une fonctionnelle affine ou convexe.

2. Dans ce qui suit, nous désignons par X un espace vectoriel réel. Une fonctionnelle $p : X \to R$ est dite

a) *sous-additive,* si

$$p(x + y) \leqq p(x) + p(y) \text{ pour tous } x, y \in X;$$

b) *sur-additive,* si

$$p(x) + p(y) \leqq p(x + y) \text{ pour tous } x, y \in X;$$

c) *radialement supérieurement semi-continue,* si

$$\overline{\lim_{t \to 1}}\, p(tx) \leqq p(x) \text{ pour tout } x \in X;$$

d) *radialement inférieurement semi-continue,* si

$$p(x) \leqq \underline{\lim_{t \to 1}}\, p(tx) \text{ pour tout } x \in X.$$

PROPOSITION 1. *Soient $p: X \to R$ une fonctionnelle sous-additive et c un nombre réel. La fonctionnelle $x \in X \mapsto p(x) - c$ est aussi sous-additive si et seulement si*

$$c \leqq \inf \{p(x) + p(y) - p(x + y) : x, y \in X\}.$$

La démonstration est immédiate.

PROPOSITION 2. *Si la fonctionnelle sous-additive* $p: X \to R$ *satisfait aux conditions suivantes:*

$$p(0) = \inf \{p(x) + p(y) - p(x+y) : x,\ y \in X\}, \tag{1}$$

$$\overline{\lim_{t \to 1}}\, p(tx) \leqq p(0) \text{ pour tout } x \in X, \tag{2}$$

alors elle est aussi radialement supérieurement semi-continue.

Démonstration. La fonctionnelle $q: X \to R$ définie par

$$q(x) = p(x) - p(0) \text{ pour tout } x \in X,$$

est sous-additive en vertu de (1) et de la proposition 1. Il en résulte que pour tout $x \in X$ on a

$$q(tx) \leqq q(x) + q((t-1)x) \text{ quel que soit } t \in R,$$

d'où, conformément à la condition (2), on déduit

$$\overline{\lim_{t \to 1}}\, q(tx) \leqq q(x) + \overline{\lim_{t \to 1}}\, q((t-1)x) \leqq q(x),$$

c'est-à-dire q et p sont radialement supérieurement semi-continues.

PROPOSITION 3. *Si la fonctionnelle* $p: X \to R$ *est sous-additive, alors, pour tout* $x \in X$, *le nombre*

$$\inf \{n^{-1}p(nx) : n \in N\} \tag{3}$$

est fini et la fonctionnelle $\hat{p}: X \to R$, *définie par*

$$\hat{p}(x) = \inf \{n^{-1}p(nx) : n \in N\} \text{ pour tout } x \in X, \tag{4}$$

est sous-additive. De plus, si p est radialement supérieurement semi-continue, alors $\hat{p}$ est sous-linéaire.

Démonstration. La sous-additivité de p implique, pour $n \in N$ et $x \in X$

$$0 \leqq p(0) \leqq p(nx) + p(-nx) \leqq p(nx) + np(-x)$$

d'où on déduit $-p(-x) \leqq n^{-1}p(nx)$. Par conséquent le nombre (3) est fini.

Prouvons maintenant la sous-additivité de $\hat{p}$. Soient $x,\ y \in X$. Si $m,\ n \in N$, alors on peut écrire

$$\hat{p}(x+y) \leqq (mn)^{-1}p(mn(x+y)) \leqq (mn)^{-1}[p(mnx) + p(mny)] \leqq$$
$$\leqq (mn)^{-1}[np(mx) + mp(ny)] = m^{-1}p(mx) + n^{-1}p(ny).$$

d'où on obtient $\hat{p}(x+y) \leqq \hat{p}(x) + \hat{p}(y)$.

On remarque que pour tout nombre rationnel $t = mn^{-1}$, avec $m, n \in N$, on a

$$\hat{p}(x) \leqq m^{-1}p(mx) \leqq nm^{-1}p(mn^{-1}x) = t^{-1}p(tx) \text{ pour tout } x \in X.$$

Considérons maintenant $t_0 > 0$ un nombre réel positif quelconque. Il existe une suite $(t_n)_{n \in N}$ décroissante de nombres rationnels qui converge vers t_0. Puisque $\hat{p}(tx) \leqq t_n^{-1} p(t_n x)$ et $\lim_{n \to \infty} t_n t_0^{-1} = 1$, de la semi-continuité supérieure radiale de p on déduit

$$\hat{p}(x) \leqq \overline{\lim_{n \to \infty}}\, t_n^{-1} p(t_n x) = t_0^{-1} \overline{\lim_{n \to \infty}}\, (t_n^{-1} t_0) p((t_n t_0^{-1}) t_0 x) \leqq t_0^{-1} p(t_0 x)$$

pour tout $x \in X$. De cette dernière relation on déduit que $\hat{p}$ est sous-linéaire.

Corollaire 4. *Si la fonctionnelle $q: X \to R$ est sur-additive, alors pour tout $x \in X$, le nombre*

$$\sup \{n^{-1}q(nx) : n \in N\} \tag{5}$$

est fini et la fonctionnelle $\hat{q}: X \to R$, définie par

$$\hat{q}(x) = \sup \{n^{-1}q(nx) : n \in N\} \text{ pour tout } x \in X, \tag{6}$$

est sur-additive. De plus, si q est radialement inférieurement semi-continue, alors $\hat{q}$ est sur-linéaire.

Théorème 5. *Soient Y un sous-espace vectoriel de X, $p: X \to R$ une fonctionnelle sous-additive radialement supérieurement semi-continue et $f_0: Y \to R$ une fonctionnelle linéaire (respectivement affine, convexe), telles que les conditions suivantes soient vérifiées :*

(j) $f_0(0) \leqq \inf \{p(x) + p(y) - p(x+y) : x, y \in X\}$;

(jj) $f_0(y) \leqq p(y)$ *pour tout* $y \in Y$.

Alors il existe une fonctionnelle linéaire (respectivement affine, convexe) $f: X \to R$ qui possède les propriétés suivantes :

$f(y) = f_0(y)$ *pour tout* $y \in Y$,

$f(x) \leqq p(x)$ *pour tout* $x \in X$.

Démonstration. Supposons d'abord $f_0(0) \leqq 0$. La convexité de f_0 et (jj) entraînent, pour $y \in Y$ et $n \in N$ quelconques,

$$f_0(y) \leqq f_0(y) - (1 - n^{-1})f_0(0) \leqq n^{-1}f_0(ny) + (1 - n^{-1})f_0(0) - $$
$$-(1 - n^{-1})f_0(0) = n^{-1}f_0(ny) \leqq n^{-1}p(ny),$$

donc

$$f_0(y) \leqq \hat{p}(y) \text{ pour tout } y \in Y,$$

où $\hat{p}: X \to R$ est la fonctionnelle définie par (4). Vu la sous-linéarité de $\hat{p}$ donnée par la proposition 3, on applique le théorème pour le prolongement des fonctionnelles linéaires (respectivement affines, convexes) majorées par une fonctionnelle convexe (voir, par exemple, F. A. Valentine [2, p.29]) et on obtient l'existence d'un prolongement f de f_0 de même type, defini sur X et majoré par $\hat{p}$. En employant (4), il résulte

$$f(x) \leqq p(x) \leqq \hat{p}(x) \text{ pour tout } x \in X.$$

Si $0 < f_0(0)$, alors $g_0(y) = f_0(y) - f_0(0)$ est affine (respectivement convexe), s'annule dans l'origine et est majorée par $q(x) = p(x) - f_0(0)$ qui est sous-additive (proposition 1). L'application du cas précédent conduit à un prolongement affine (respectivement convexe) $g : X \to R$ de g_0 majoré par q. Pour $f(x) = g(x) + f_0(0)$ l'énoncé est vérifié.

Remarques. 1) La condition pour p d'être radialement supérieurement semi-continue ne peut pas être omise. Pour le montrer, considérons d'abord R comme un espace vectoriel sur le corps Q des rationnels. Si nous désignons $b_1 = 1$ et b_2 un nombre irrationel sur-unitaire, alors b_1 et b_2 seront linéairement indépendants. Soit B une base de R avec $\{b_1, b_2\} \subseteq B$. Maintenant nous définissons une fonction $h : R \to R$ de la manière suivante : chaque $x \in R$ a une représentation unique

$$x = \sum_{i=1}^{n} a_i b_i, \text{ où } b_i \in B,\ a_i \in Q,\ i = 1, \ldots, n ;$$

nous posons

$$h(x) = \sum_{i=1}^{n} a_i.$$

On voit aisément que h est additive. Choisissons $X = R^2$, définissons $f_0 : R \otimes \{0\} \to R$ par $f_0(x^1, 0) = 0$ pour tout $x^1 \in R$ et $p : X \to R$ par $p(x^1, x^2) = h(x^2)$ pour tout $(x^1, x^2) \in R^2$. Il est clair que p est (sous) additive, f_0 est linéaire et $f_0(x^1, 0) = 0 = p(x^1, 0)$ pour tout $x^1 \in R$. Mais on ne pourra pas prolonger f_0. En effet, supposant le contraire, on trouverait $a, b \in R, f(x^1, x^2) = ax^1 + bx^2$ pour tout $(x^1, x^2) \in R^2$; de $f_{R\otimes\{0\}} = f_0$ on obtiendrait $a = 0$; de $f(0, x^2) \leqq p(0, x^2)$ on obtiendrait $bx^2 \leqq h(x^2)$ pour tout $x^2 \in R$, par conséquent, aussi pour $x^2 = b_1$, $x^2 = -b_1$, ce qui exige $b = 1$; enfin, en remplaçant $x^2 = b_2$, on obtiendrait $b_2 \leqq 1$ ce qui est une contradiction.

2) Dans le théorème 5 on ne peut pas renoncer à la condition (j) qui, visiblement, est remplie si f_0 est linéaire. La fonctionnelle $p : R^2 \to R$, définie par

$$p(x^1, x^2) = 1 + (1 + |x^2|)^{-1} \text{ pour tout } (x^1, x^2) \in R^2 ;$$

est sous-additive et radialement supérieurement semi-continue, mais il n'y a aucun prolongement affine pour la fonctionnelle $f_0 : R \otimes \{0\} \to R$ donnée par $f_0(x^1, 0) = 3/2$ pour tout $x^1 \in R$.

3) Le théorème de H. Nakano, rappelé au début de la note, est un cas particulier du théorème 5. En effet, si on pose dans (ii) $x = 0$,

on obtient $p(0) = 0$. Grâce à la proposition 1 on obtient alors $\inf \{p(x) + p(y) - p(x+y) : x, y \in X\} = 0$, donc la condition (j) est vérifiée. De la proposition 2 on déduit la semi-continuité supérieure radiale de p.

THÉORÈME 6. *Soient Y, Z sous-espaces vectoriels de X, $p : X \to R$ une fonctionnelle sous-additive radialement supérieurement semi-continue, $q : Z \to R$ une fonctionnelle sur-additive radialement inférieurement semi-continue et $f_0 : Y \to R$ une fonctionnelle linéaire. Alors les affirmations suivantes sont équivalentes :*

1° $f_0(y) + q(z) \leqq p(y + z)$ *pour tout* $(y, z) \in Y \otimes Z$.

2° *Il existe une fonctionnelle linéaire $f : X \to R$ telle que*

$f(y) = f_0(y)$ *pour tout* $y \in Y$,

$f(x) + q(z) \leqq p(x + z)$ *pour tout* $(x, z) \in X \otimes Z$.

3° *Il existe une fonctionnelle linéaire $f : X \to R$ telle que*

$f(y) = f_0(y)$ *pour tout* $y \in Y$,

$q(z) \leqq f(z)$ *pour tout* $z \in Z$,

$f(x) \leqq p(x)$ *pour tout* $x \in X$.

Demonstration. Supposons 1° remplie. Pour tout $x \in X$ et tout $z \in Z$ on a :

$$-p(-x) \leqq -p(-x) + p(z) - q(z) \leqq p(x + z) - q(z).$$

Par conséquent, on peut définir la fonctionnelle $\tilde{p} : X \to R$ par

$$\tilde{p}(x) = \inf \{p(x + z) - q(z) : z \in Z\} \text{ pour tout } x \in X.$$

Pour tous $x, y \in X$ et tous $z, z' \in Z$ on a

$$\tilde{p}(x + y) \leqq p(x + y + z + z') - q(z + z') \leqq$$
$$\leqq p(x + z) - q(z) + p(y + z') - q(z') ;$$

il en résulte la sous-additivité de $\tilde{p}$. Prouvons maintenant la semi-continuité supérieure radiale de $\tilde{p}$. Remarquons d'abord l'égalité

$$\tilde{p}(x) = \inf \{p(x) + sz) - q(sz) : s \in R, z \in Z\} \text{ pour tout } x \in X ;$$

pour tout $t \in R$ et tout $x \in X$ on a

$$\tilde{p}(tx) \leqq \inf \{p(tx + tz) - q(tz) : z \in Z\},$$

par conséquent,

$$\overline{\lim_{t \to 1}}\, p(tx) \leqq \inf \{\overline{\lim_{t \to 1}}\, [p(tx + tz) - q(tz)] : z \in Z\} \leqq$$
$$\leqq \inf \{p(x + z) - q(z) : z \in Z\} = \tilde{p}(x),$$

puisque p et $-q$ sont radialement supérieurement semi-continues.

Vu que

$$f_0(y) \leqq \tilde{p}(y) \text{ pour tout } y \in Y,$$

d'après le théorème 5 on obtient $f: X \to R$ qui vérifie 2°.

Supposons maintenant 2° satisfaite. Pour $m, n \in N$, $(x, z) \in X \otimes Z$ on a

$$f(x) + m^{-1}q(mz) = (mn)^{-1}[f(mnx) + nq(mz)] \leqq$$
$$\leqq (mn)^{-1}[f(mnx) + q(mnz)] \leqq (mn)^{-1}p(mn(x + z)) \leqq$$
$$\leqq n^{-1}p(n(x + z)),$$

donc

$$f(x) + q(z) \leqq f(x) + \hat{q}(z) \leqq \hat{p}(x + z) \leqq p(x + z) \text{ pour tout } (x, z) \in X \otimes$$

$\otimes Z$, où $\hat{p}$ et $\hat{q}$ sont données par (4) et (6). En remplaçant $x = -z$ et $z = 0$, on obtient 3°.

Le fait que 3° implique 1° est immédiat.

(Manuscrit reçu le 16 juillet 1977)

BIBLIOGRAPHIE

1. H. Nakano., *On the Hahn-Banach theorem*, Bull. Acad. Pol. Sci. Ser. Sc. Math., Astr. Phys., **19** (1971), 743–745.
2. F. A. Valentine., *Convex Sets*, New-York, Mc. Graw-Hill, 1964.

ASUPRA PRELUNGIRII FUNCŢIONALELOR MAJORATE DE O FUNCŢIONALĂ SUBADITIVĂ

(Rezumat)

În nota de față este demonstrat următorul rezultat, care constituie o generalizare a unei teoreme a lui H. Nakano [1]:

Fie X un spațiu vectorial real, Y un subspațiu vectorial al lui X, $p: X \to R$ o funcțională subaditivă și $f_0: Y \to R$ o funcțională liniară (respectiv afină, convexă) astfel încît următoarele condiții să fie îndeplinite:

(i) $f_0(0) \leq \inf \{p(x) + p(y) - p(x + y) : x, y \in X\}$;

(ii) $\varlimsup_{t \to 1} p(tx) \leq p(x)$ *pentru orice* $x \in X$;

(iii) $f_0(y) \leq p(y)$ *pentru orice* $y \in Y$.

Atunci există o funcțională liniară (respectiv afină, convexă) $f: X \to R$ cu proprietățile

$f(y) = f_0(y)$ *pentru orice* $y \in Y$,

$f(x) \leq p(x)$ *pentru orice* $x \in X$.

STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, 1, 1978

CONSTRAINT QUALIFICATIONS IN NONLINEAR PROGRAMMING IN COMPLEX SPACE

DOREL DUCA

1. Introduction. Let us consider the problem:

$$\min \operatorname{Re} f(z, \bar{z}) \tag{1}$$

subject to

$$g(z, \bar{z}) \in S \tag{2}$$

where: $f: C^{2n} \to C$, $g: C^{2n} \to C^m$ and S is a polyhedral cone in C^m.

Considering that the vector function g satisfies the Kuhn-Tucker's complex constraint qualification on the local minimum point $(z^0, \bar{z}^0)$, Abrams, R. A. and Ben-Israel, A. [2] proved a Kuhn-Tucker type necessary condition for the (1)—(2) problem.

Considering int $S \neq \emptyset$, Craven, B. D. and Mond, B. [5] have established a necessary condition of the Fritz John type.

In this paper, some other complex constraint qualification conditions are given for which a necessary condition of the Kuhn-Tucker type are then derived.

2. Notation and preliminaries. Let $C^n(R^n)$ denote n-dimensional complex (real) vector space, $R^n_+ = \{x = (x_j) \in R^n / x_j \geq 0, j = 1, \ldots, n\}$ the non-negative orthant of R^n and $C^{m x n}$ the set of mxn complex matrices. For $A = (a_{kj}) \in C^{mxn}$, $\bar{A} = (\bar{a}_{kj})$, $A^T = (a_{jk})$ and $A^H = \bar{A}^T$ denote conjugate, transpose and conjugate transpose of A respectively.

For $z = (z_j)$, $u \in C^n$: $< z,u > = u^H z$ — denote the inner product of z and u.

For a vector function $g: C^n \times C^n \to C^m$ analytic in the $2n$ variables (u^1, u^2) at $(z^0, \bar{z}^0) \in C^n \times C^n$:

$$\nabla_z g(z^0, \bar{z}^0) = \left(\frac{\partial g_k}{\partial u_j^1}(z^0, \bar{z}^0)\right), \; k = 1, \ldots, m; \; j = 1, \ldots, n$$

and

$$\nabla_{\bar{z}} g(z^0, \bar{z}^0) = \left(\frac{\partial g_k}{\partial u_j^2}(z^0, \bar{z}^0)\right), \; k = 1, \ldots, m; \; j = 1, \ldots, n.$$

DEFINITION 1. A nonempty set S in C^m is a:

(a) convex cone if $S + S \subset S$ and $\alpha S \subset S$ for $\alpha \geq 0$,

(b) pointed convex cone if it satisfies (a) and $S \cup (-S) = \{0\}$,

(c) polyhedral cone if, for some positive integer k and $A \in C^{n\times k}$,

$$S = AR_+^k = \{Ax/x \in R_+^k\}$$

For any nonempty set S in C^m let

$$S^* = \{w \in C^m/v \in S \Rightarrow \text{Re}\,\langle w,v\rangle \geq 0\}$$

be the polar of S, and

$$\text{int}\, S^* = \{w \in C^m/0 \neq v \in S \Rightarrow \text{Re}\,\langle w,v\rangle > 0\},$$

the interior of S^*.

It is known that S^* is a closed convex cone [3]. Since $S = S^{**}$ if and only if S is a closed convex cone [3], it follows that for a closed convex cone S, int S can be expressed by

$$\text{int}\, S = \{v \in C^m/0 \neq w \in S^* \Rightarrow \text{Re}\,\langle w, v\rangle > 0\}$$

and is nonempty if and only if S^* is pointed convex cone [4].

We define the manifold:

$$Q = \{(u^1,u^2) \in C^{2n}/u^2 = \bar{u}^1\}.$$

DEFINITION 2. The vector function $g : C^{2n} \to C^m$ is concave with respect to the polyhedral cone $S \subset C^m$ on the manifold Q if for any z^1 and z^2 from C^n and $0 < \lambda < 1$,

$$g[\lambda z^1 + (1-\lambda)z^2, \lambda\bar{z}^1 + (1-\lambda)\bar{z}^2] - \\ - \lambda g(z^1, \bar{z}^1) - (1-\lambda)g(z^2, \bar{z}^2) \in S. \tag{3}$$

When g is analytic, a condition equivalent to (3) is:

$$[\nabla_z g(z^2,\bar{z}^2)]^T(z^1 - z^2) + [\nabla_{\bar{z}} g(z^2, \bar{z}^2)]^T(\bar{z}^1 - \bar{z}^2) + g(z^2, \bar{z}^2) - g(z^1, \bar{z}^1) \in S.$$

3. Complex constraint qualification.

DEFINITION 3. The vector function $g : C^{2n} \to C^m$ is said to satisfy Slater's complex constraint qualification with respect to the polyhedral cone $S \subset C^m$ on Q, if there exists a point $(z^1,\bar{z}^1) \in Q$ such that:

$$g(z^1, \bar{z}^1) \in \text{int}\ S.$$

DEFINITION 4. The vector function $g : C^{2n} \to C^m$ is said to satisfy the strict complex constraint qualification with respect to the polyhedral cone $S \subset C^m$ on Q, if there exist two points $(z^1, \bar{z}^1)$, $(z^2, \bar{z}^2) \in Q$, $(z^1, \bar{z}^1) \neq (z^2, \bar{z}^2)$ with $g(z^1, \bar{z}^1), g(z^2, \bar{z}^2) \in S$ and if there exists a $0 < \lambda < 1$ such that:

$$g[\lambda z^1 + (1-\lambda)z^2, \lambda\bar{z}^1 + (1-\lambda)\bar{z}^2] - \lambda g(z^1, \bar{z}^1) - (1-\lambda)g(z^2, \bar{z}^2) \in \text{int}\ S.$$

Remark 1. If g is strictly concave with respect to S on Q [7], then the strict complex constraint qualification with respect to S on Q is satisfied.

THEOREM 1. *The strict complex constraint qualification with respect to the polyhedral cone S on the manifold Q implies Slater's complex constraint qualification with respect to S on Q.*

Proof. If the strict complex constraint qualification with respect to S on Q is satisfied then there exist two points $(z^1, \bar{z}^1)$, $(z^2, \bar{z}^2) \in Q$, $(z^1, \bar{z}^1) \neq \neq (z^2, \bar{z}^2)$ with $g(z^1, \bar{z}^1)$, $g(z^2, \bar{z}^2) \in S$ and there exists a $0 < \lambda < 1$ such that: $0 \neq w \in S^*$ imply $\operatorname{Re} < w, g[\lambda z^1 + (1-\lambda)z^2, \lambda\bar{z}^1 + (1-\lambda)\bar{z}^2] > > \lambda \operatorname{Re} < w, g(z^1, \bar{z}^1) > + (1-\lambda) \operatorname{Re} < w, g(z^2, \bar{z}^2) >$.

Since $g(z^1, \bar{z}^1)$, $g(z^2, \bar{z}^2) \in S$, it follows that

$0 \neq w \in S^*$ imply $\operatorname{Re} < w, g[\lambda z^1 + (1-\lambda)z^2, \lambda\bar{z}^1 + (1-\lambda)\bar{z}^2] > > 0$, hence the point $(z, \bar{z}) = (\lambda z^1 + (1-\lambda)z^2, \lambda\bar{z}^1 + (1-\lambda)\bar{z}^2) \in Q$ has the property that $g(z, \bar{z}) \in \operatorname{int} S$, consequently Slater's complex constraint qualification with respect to S on Q is satisfied.

DEFINITION 5. The analytic vector function $g: C^{2n} \to C^m$ is said to satisfy the weak complex constraint qualification at $(z^0, \bar{z}^0) \in Q$ with respect to the polyhedral cone $S \subset C^m$ on the manifold Q, if

$$\begin{pmatrix} \overline{\nabla_z g(z^0, \bar{z}^0)} w + \nabla_{\bar{z}} g(z^0, \bar{z}^0)\bar{w} = 0 \\ \operatorname{Re} < w, g(z^0, \bar{z}^0) > = 0 \\ w \in S^* \end{pmatrix} \text{imply } w = 0.$$

THEOREM 2. *If the analytic vector function $g: C^{2n} \to C^m$ is concave with respect to the polyhedral cone $S \subset C^m$ on the manifold Q, then Slater's complex constraint qualification with respect to S on Q implies the weak complex constraint qualification at all point $(z^0, \bar{z}^0) \in Q$ for which $g(z^0, \bar{z}^0) \in S$ with respect to S on Q.*

Proof. Let $(z^1, \bar{z}^1) \in Q$ be such that $g(z^1, \bar{z}^1) \in \operatorname{int} S$, or, equivalently,

$$0 \neq w \in S^* \text{ imply } \operatorname{Re} < w, g(z^1, \bar{z}^1) > > 0. \tag{4}$$

If the weak complex constraint qualification is not satisfied at the point $(z^0, \bar{z}^0) \in Q$ for which $g(z^0, \bar{z}^0) \in S$, then there exists a $w^1 \in S^*$, $w^1 \neq 0$ such that:

$$\begin{cases} \overline{\nabla_z g(z^0, \bar{z}^0)} w^1 + \nabla_{\bar{z}} g(z^0, \bar{z}^0)\bar{w}^1 = 0 \\ \operatorname{Re} < w^1, g(z^0, \bar{z}^0) > = 0. \end{cases} \tag{5}$$

On the other hand g being concave with respect to S on Q, for any $w \in S^*$ we have

$$\operatorname{Re} < w, g(z^1, \bar{z}^1) > \leq \operatorname{Re} < w, g(z^0, \bar{z}^0) > + \\ + \operatorname{Re} < w, [\nabla_z g(z^0, \bar{z}^0)]^T (z^1 - z^0) > + \operatorname{Re} < w, [\nabla_{\bar{z}} g(z^0, \bar{z}^0)]^T (\bar{z}^1 - \bar{z}^0) >. \tag{6}$$

Taking in (6) $w = w^1 \in S^*$, from (5) we have

$$\operatorname{Re} < w^1, g(z^1, \bar{z}^1) > \ \leq 0.$$

This inequality contradicts (4) for $w = w^1 \neq 0$.

4. Necessary conditions of Kuhn-Tucker type. We shall use the following results of C r a v e n, B. D. and M o n d, B. [5]:

THEOREM 3. *Let $f: C^{2n} \to C$ and $g: C^{2n} \to C^m$ be analytic functions and let $S \subset C^m$ be a polyhedral cone with nonempty interior.*

A necessary condition for $(z^0, \bar{z}^0)$ to be a local minimum of the (1)-(2) problem is that there exist $\tau \in R_+$ and $v \in S^*$, not both zero, such that:

$$\tau\overline{\nabla_z f(z^0, \bar{z}^0)} + \tau\nabla_{\bar{z}} f(z^0, \bar{z}^0) -$$

$$- \overline{\nabla_z g(z^0, \bar{z}^0)} v + \nabla_{\bar{z}} g(z^0, \bar{z}^0)\bar{v} = 0 \qquad (7)$$

$$\operatorname{Re} < v, g(z^0, \bar{z}^0) > \ = 0. \qquad (8)$$

THEOREM 4. *Let $f: C^{2n} \to C$ and $g: C^{2n} \to C^m$ be analytic functions, and let $S \subset C^m$ be a polyhedral cone with nonempty interior. Let $(z^0, \bar{z}^0) \in C^{2n}$ a local minimum of the (1) – (2) problem.*

If the vector function g is concave with respect to S on Q and satisfies one of the following three conditions:

(i) *the strict complex constraint qualification with respect to S on Q, or*

(ii) *Slater's complex constraint qualification with respect to S on Q, or*

(iii) *the weak complex constraint qualification at $(z^0, \bar{z}^0)$ with respect to S on Q,*

then there exists a $w \in S^*$ such that:

$$\overline{\nabla_z f(z^0, \bar{z}^0)} + \nabla_{\bar{z}} f(z^0, \bar{z}^0) - \overline{\nabla_z g(z^0, \bar{z}^0)} w - \nabla_{\bar{z}} g(z^0, \bar{z}^0)\bar{w} = 0 \qquad (9)$$

$$\operatorname{Re} < w, g(z^0, \bar{z}^0) > \ = 0. \qquad (10)$$

Proof. In view of Theorems 1 and 2 we need only to prove the theorem under assumption (iii).

Since the conditions of Theorem 3 are satisfied, there exists a couple $0 \neq (\tau, v) \in R_+ \times S^*$, such that (7), (8) are satisfied.

If $\tau = 0$, we have that $v = 0$ (since g satisfies (iii)), which is a contradiction with $(\tau, v) \neq 0$. Consequently $\tau > 0$.

Dividing (7) and (8) by $\tau > 0$ and setting $w = (1/\tau)v \in S^*$, we get (9) and (10).

Remark 2. If the real part of f is pseudoconvex with respect to R_+ on Q [6] and the vector function g is concave with respect to S on Q, then the conditions (9) and (10) are also sufficient for $(z^0, \bar{z}^0) \in C^{2n}$ to be the solution to (1)–(2).

5. Example. Consider the problem

$$\min \ \operatorname{Re} z^2$$

subject to

$$g(z, \bar{z}) = \begin{pmatrix} z \\ z + i \end{pmatrix} \in S,$$

where

$$S = \left\{ v = (v_1, v_2) \in C^2 / |\arg v_1| \leqslant \frac{\pi}{6}, \ |\arg v_2| \leqslant \frac{\pi}{4} \right\}$$

which has the optimal solution $z^0 = (\sqrt{3} - i)/(1 + \sqrt{3})$.

The vector function g satisfies Slater's complex constraint qualification with respect to S on Q, because

$$g(5{,}5) = (5{,}5 + i)^T \in \text{int } S.$$

It is easy to show that the vector

$$w = 2/(1 + \sqrt{3})^2 (\sqrt{3} - 1 + (3 - \sqrt{3})i, \ 4 - 4i) \in S^* =$$
$$= \left\{ v \in C^2 / |\arg v_1| \leqslant \frac{\pi}{3}, \ |\arg v_2| \leqslant \frac{\pi}{4} \right\}.$$

fulfils the equations (9) and (10).

(Received july 26, 1977)

REFERENCES

1. Abrams, Robert A., *Nonlinear programming in complex space: sufficient conditions and duality*, J. Math. Anal. Appl. 38 (1972), 619–632.
2. Abrams, Robert A. and Ben-Israel, A., *Nonlinear programming in complex space: necessary conditions*, SIAM J. Control, 9 (1971), 606–620.
3. Ben-Israel, A., *Linear equations and inequalities on finite dimensional, real or complex, vector spaces: a unified theory*, J. Math. Anal. Appl., 27 (1969), 367–389.
4. Ben-Israel, A., *Theorems of the alternative for complex linear inequalities*, Israel J. Math., 7 (1969), 129–136.
5. Craven, B. D. and Mond, B., *A Fritz John theorem in complex space*, Bull. Austral. Math. Soc., 8 (1973), 215–220.
6. Craven, B. D. and Mond, B., *Real and complex Fritz John theorems*, J. Math. Anal. Appl., 44 (1973), 773–778.
7. Gulati, T. R., *A Fritz John type sufficient optimality theorem in complex space*, Bull. Austral. Math. Soc., 11 (1974), 219–224.

CONDIŢII DE CALIFICARE ÎN PROGRAMAREA NELINIARĂ DIN DOMENIUL COMPLEX

(Rezumat)

În prezenta lucrare se dau trei noi condiţii complexe de calificare (definiţiile 3, 4, 5) pentru problema (1)–(2). Dacă una din aceste condiţii complexe de calificare este îndeplinită, S are interior nevid, funcţiile f şi g sînt analitice, iar g este concavă în raport cu conul S; se stabilesc condiţii ca un punct $(z^0, \bar{z}^0) \in C^{2n}$ să fie optim.

STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, 1, 1978

AN ALGORITHM FOR THE BEST L_p APPROXIMATION IN THE COMPLEX PLANE

I. MARUŞCIAC

1. **Introduction.** In this paper we describe a new algorithm for the best L_p approximation to a function on a discrete set of the complex plane. It is an adaptation of Newton-Raphson's method. In the real case a similar algorithm has been given recently by Kahng, S. W. [1]. In the same case another algorithm for finding the Tchebycheff approximation on a finite real point set was constructed in 1961 by Lawson, C. L. [2] and then extended (in 1968) by Rice, J. R. and Usow, K. H. [6]. The algorithm proposed here is based on a special explicit form of the L_2 approximation with a weight $\mu_\nu \geqslant 0$, given by the author in [4]. Our method can be used as well for finding the L_p approximation to a continuous function on a rectificable curve in the complex plane.

2. **Best L_p approximation.** Let f be a continuous function on a compact point set $K \subset C$ and let $\varphi = (\varphi_j)_1^n$ be a system of functions, continuous and linearly independent on K. Let $\mathfrak{L}(\varphi)$ be the set of generalized polynomials of the form:

$$P(a; z) = \sum_{j=1}^{n} a_j \varphi_j(z), \tag{1}$$

where $a = (a_1, a_2, \ldots, a_n) \in C^n$.

We consider a finite subset $Z_m = \{z_\nu\}_1^m \subset K$ such that $\varphi = (\varphi_j)$ is a Tchebycheff system on Z_m.

Let $p \in]2, +\infty[$ and let $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ be a weight, i.e. $\mu_\nu \geqslant 0$, $\Sigma\, \mu_\nu = 1$.

DEFINITION 1. *The polynomial $P(a^*; z) \in \mathfrak{L}(\varphi)$ is said to be the best $L_{p,\mu}$ (or L_p) approximation to the function f on Z_m, if*

$$\sum_{\nu=1}^{m} \mu_\nu |f(z_\nu) - P(a^*; z_\nu)|^p \leqslant \sum_{\nu=1}^{m} \mu_\nu |f(z_\nu) - P(a; z_\nu)|^p \tag{2}$$

for all $a \in C^n$.

Let us denote

$$F(a) = \sum_{\nu=1}^{m} |f(z_\nu) - P(a; z_\nu)|^p.$$

To minimize the function F it is necessary to find the solution of the system

$$\frac{\partial F}{\partial a_i} = 0, \quad i = 1, 2, \ldots, n. \tag{3}$$

But we have

$$\frac{\partial F}{\partial a_i} = -\frac{p}{2}\sum_{\nu=1}^{m} (f(z_\nu) - P(a;z_\nu))^{\frac{p}{2}-1}\,\overline{(f(z_\nu) - P(a;z_\nu))}^{\frac{p}{2}}\,\varphi_i(z_\nu)$$

$$\frac{\partial F}{\partial \bar{a}_i} = -\frac{p}{2}\sum_{\nu=1}^{m} (f(z_\nu) - P(a;z_\nu))^{\frac{p}{2}}\,\overline{(f(z_\nu) - P(a;z_\nu))}^{\frac{p}{2}-1}\,\overline{\varphi_i(z_\nu)} \tag{4}$$

and therefore we see that the system (3) and

$$\frac{\partial F}{\partial \bar{a}_i} = 0,\ i = 1,\ 2, \dots, n. \tag{4'}$$

are equivalent.

We will solve the system (4′) by the Newton-Raphson's method. So the system (4′) will be solved iteratively by finding the solutions of the system

$$\frac{\partial F(a^{k-1})}{\partial \bar{a}_i} + \sum_{j=1}^{n}\frac{\partial^2 F(a^{k-1})}{\partial a_j \partial \bar{a}_i}\Delta a_j^k = 0,\ i = 1,\ 2,\ \dots,\ n, \tag{5}$$

where $\Delta a^k = a^k - a^{k-1}$.

From (4) we find that

$$\frac{\partial^2 F(a)}{\partial a_j \partial \bar{a}_i} = \frac{p^2}{4}\sum_{\nu=1}^{m} (f(z_\nu) - P(a;\ z_\nu))^{\frac{p}{2}-1}\,\overline{(f(z_\nu) - P(a;\ z_\nu))}^{\frac{p}{2}-1}\,\varphi_j(z_\nu)\overline{\varphi_i(z_\nu)} =$$

$$= \frac{p^2}{4}\sum_{\nu=1}^{m} |f(z_\nu) - P(a;\ z_\nu)|^{p-2}\varphi_j(z_\nu)\overline{\varphi_j(z_\nu)}. \tag{6}$$

If we put

$$w_k(z) = |f(z) - P(a^{k-1};\ z)|^{p-2},$$

where $P(a^{k-1}; z) \in \mathfrak{L}(\varphi)$ is the best $L_{p,w_{k-1}}$ approximation to the function f on Z_m with the weight

$$w_{k-1} = (w_{k-1}(z_1),\ w_{k-1}(z_2),\ \dots,\ w_{k-1}(z_m)),$$

then we have

$$\frac{\partial^2 F(a^{k-1})}{\partial a_j \partial \bar{a}_i} = \frac{p^2}{4}\sum_{\nu=1}^{m} w_k(z_j)\varphi_j(z_\nu)\bar{\varphi}_i(z_\nu)$$

$$\frac{\partial F(a^{k-1})}{\partial \bar{a}_i} = -\frac{p}{2}\sum_{\nu=1}^{m} w_k(z_\nu)(f(z_\nu) - P(a^{k-1};\ z_\nu))\ \overline{\varphi_i(z_\nu)}. \tag{7}$$

Substituting (7) in (5) we obtain

$$\frac{p}{2}\sum_{\nu=1}^{m} w_k(z_\nu)\overline{\varphi_i(z_\nu)}\ \sum_{j=1}^{n}\varphi_j(z_\nu)\Delta a_j^k =$$

$$= \sum_{\nu=1}^{m} w_k(z_\nu)(f(z_\nu) - P(a^{k-1};\ z_\nu))\ \overline{\varphi_i(z_i)},\ i = 1,\ 2,\ \dots,\ n. \tag{8}$$

If we denote

$$H_k = \left(\sum_{\nu=1}^{m} w_k(z_\nu)\overline{\varphi_i(z_\nu)}\varphi_j(z_\nu)\right),\ i, j = 1, 2, \ldots, n, \tag{9}$$

$$D_k = \left(\sum_{\nu=1}^{m} w_k(z_\nu)f(z_\nu)\ \overline{\varphi_1(z_\nu)}, \ldots, \sum_{\nu=1}^{m} w_k(z_\nu)f(z_\nu)\overline{\varphi_n(z_\nu)}\right)^T,$$

then the system (8) can be written under the form

$$\frac{p}{2} H_k \Delta a^k = D_k - H_k a^{k-1}. \tag{10}$$

Now, if $P(b^k ;z) \in \mathfrak{L}(\varphi)$ is the polynomial of the best L_{2,w_k} approximation to the function f on Z , then it follows that

$$\frac{\partial F(b^k)}{\partial \bar{a}_i} = -\sum_{\nu=1}^{m} w_k(z_\nu)(f(z_\nu) - P(b^k ; z_\nu))\overline{\varphi_i(z_\nu)} = 0,\ i = 1, 2, \ldots, n$$

hence

$$D_k = H_k b^k. \tag{11}$$

Substituting D_k from (11) in (10) we obtain

$$\frac{p}{2} H_k \Delta a^k = H_k(b^k - a^{k-1}). \tag{12}$$

If the matrix H_k is nonsingular (as we will show), from (12) we obtain

$$\Delta a^k = \frac{2}{p}(b^k - a^k)$$

or

$$a^k = a^{k-1} + \Delta a^k = \frac{1}{p}((p-2)a^{k-1} + 2b^k). \tag{13}$$

From above it follows therefore the following algorithm for the best L_p approximation to a function on a complex finite set:

Start from the initial coefficient $a_0 \in C^n$.

Step 1. Set $w_k = (w_k(z_1), w_k(z_2), \ldots, w_k(z_m))$, where

$$w_k(z) = |f(z) - P(a^{k-1}; z)|^{p-2}.$$

Step 2. Find the least squares approximation $P(b^k ; z) \in P(\varphi)$ to the function f on Z_m with the weight w_k(i.e. the best L_{2,w_k} approximation to f on Z_m)

Step 3. Set $a^k = \frac{1}{p}((p-2)a^{k-1} + 2b^k)$.

3. Convergence of the algorithm. First we will show that the matrix H_k is nonsingular. For this we will use one of our results contained in [4], that

$$\det H_k = \frac{1}{n!}\Sigma\ w_k(z_{\nu_1}) \ldots w_k(z_{\nu_n})\ |\ D(z_{\nu_1}, \ldots, z_{\nu_n};\ \varphi)|^2, \tag{14}$$

where

$$D(x_1, x_2, \ldots, x_n; \varphi) = \begin{vmatrix} \varphi_1(x_1) & \ldots & \varphi_1(x_n) \\ \varphi_2(x_1) & \ldots & \varphi_2(x_n) \\ \cdot & \cdot & \cdot \\ \varphi_n(x_1) & \ldots & \varphi_n(x_n) \end{vmatrix}.$$

LEMMA 1. *If* det $H_0 > 0$, *then* det $H_k > 0$ *for all* $k \in N$.

Proof. We denote by

$$W^k = \{z \in Z_m \mid w_k(z) > 0\}.$$

Then the proof is by induction. Assume that det $H_k > 0$. Then from (14) it follows that W_k contains at least n points. If $W^{k+1} = W^k$ then no polynomial $p \in \mathfrak{L}(\varphi)$ agrees with f on the set $W^{k+1} = W^k$ and therefore W^{k+1} contains at least $n+1$ points. Because φ is a Tchebycheff system on Z_m (and hence on W^{k+1} too), from (14) it follows that det $H_{k+1} > 0$.

If $W^k \setminus W^{k+1} \neq \Phi$, then it is seen that $P(b^k; z) \in \mathfrak{L}(\varphi)$ is the best L_{2,w_k} approximation to f on W^{k+1} as well as on W^k, because $w_k(z) = 0$ for $z \in W^{k+1} \setminus W^k$. This shows again that no polynomial $P \in \mathfrak{L}(\varphi)$ agrees with the functions f on W^{k+1}. Therefore W^{k+1} contains at least $n+1$ points, and from (14) it follows that det $H_{k+1} > 0$. This concludes the proof.

Remark 1. This lemma together with the assumption that φ is a Tchebycheff system on Z_m, implies that

$$\bigcap_{k=1}^{\infty} W^k$$

contains at least $n+1$ points.

LEMMA 2 *If* $f \notin \mathfrak{L}(\varphi)$ *on* Z_m, *then the function*

$$J(a) = \sum_{\nu=1}^{m} \mu_\nu |f(z_\nu) - P(a; z_\nu)|^p$$

is strictly convex with respect to a, *for every* $p > 1$, *and* $\mu_\nu \geqslant 0$.

Proof. If $a', a'' \in C^n$ and $t \in]0,1[$, then from Minkowski's inequality we have

$$(J((1-t)a' + ta''))^{1/p} = \left(\sum_{\nu=1}^{m} \mu_\nu |f(z_\nu) - (1-t)P(a'; z_\nu) - tP(a''; z_\nu)|^p\right)^{1/p} =$$

$$= \left(\sum_{\nu=1}^{m} \mu_\nu |(1-t)[f(z_\nu) - P(a'; z_\nu)] + t[f(z_\nu) - P(a''; z_\nu)]|^p\right)^{1/p} \leqslant$$

$$\leqslant \left(\sum_{\nu=1}^{m} \mu_\nu (1-t)^p |f(z_\nu) - P(a'; z_\nu)|^p\right)^{1/p} + \left(\sum_{\nu=1}^{m} \mu_\nu t^p |f(z_\nu) - P(a''; z_\nu)|^p\right)^{1/p} =$$

$$= (1-t)[J(a')]^{1/p} + t[J(a'')]^{1/p},$$

hence $[J(a)]^{1/p}$ is convex. Therefore $J(a)$ is convex. It is strictly convex because Minkowski's inequality becomes equality if and only if there exists a constant $\lambda > 0$ such that

$$(1-t)[f(z_\nu) - P(a ; z_\nu)] = \lambda t[f(z_\nu) - P(a'' ; z_\nu)], \quad \nu = 1, 2, \ldots, m$$

or

$$P((1-t)a' + \lambda t a'' ; z_\nu) = (1 - t - \lambda t) f(z_\nu), \quad \nu = 1, 2, \ldots, m$$

which contradicts the assumption that $f \notin \mathfrak{L}(\varphi)$ on Z_m.

Lemmas 1 and 2 show that the convergence conditions of Newton-Raphson method are satisfied (see for instance [5], pp. 501–506), so we have.

THEOREM. *If the matrix H_0 is nonsingular, then the algorithm described at 2. is always convergent and the convergence of the iterations is quadratic.*

Obviously that in the case when $f \in \mathfrak{L}(\varphi)$ or $m \leqslant n$, the algorithm is finite. In this case there is a polynomial from $\mathfrak{L}(\varphi)$ which agrees with the function f on Z_m.

Remark 2. We note that this algorithm is valid for $p > 2$, so when p tends to infinity we obtain Lawson's like algorithm, which gives us the best uniform approximation (L_∞ approximation) to a complex function on a discrete set Z_m.

Remark 3. This algorithm can be extended to the case when instead of a finite set Z_m we take a rectifiable curve Γ, replacing the operator „summation over a discrete set" by the operator „integration on Γ" (see [1]).

4. **An explicit form of the best L_p approximation.** In our paper [3, p. 330] has been given an explicit form of the best $L_{p,\mu}$ approximation on a finite point set of the complex plane. So, if $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \geqslant 0$, $\Sigma\mu_\nu = 1$, then the polynomial $P(b ; z) \in \mathfrak{L}(\varphi)$ of the best $L_{2,\mu}$ approximation to the function f on Z_m is given by the formula

$$P(b ; z) = \frac{\Sigma \mu_{\nu_1} \cdots \mu_{\nu_n} |D(z_{\nu_1}, \ldots, z_{\nu_n} ; \varphi)|^2 L(z_{\nu_1}, \ldots, z_{\nu_n} ; \varphi ; f | z)}{\Sigma \mu_{\nu_1} \cdots \mu_{\nu_n} |D(z_{\nu_1}, \ldots, z_{\nu_n} ; \varphi)|^2} \tag{15}$$

where $L(x_1, x_2, \ldots, x_n ; \varphi ; f|z)$ is the generalized interpolatory polynomial from $\mathfrak{L}(\varphi)$ to the function f on the knots $x_1, x_2, \ldots, x_n$. The summ Σ is taken for all $\nu_1, \nu_2, \ldots, \nu_n$ from 1 to m.

Now combining the algorithm described at 2. with (15) we obtain the following variant of the algorithm:

Start from an arbitrary polynomial $\mathfrak{L}(a^0 ; z) \in P(\varphi)$.

Step 1. Set $W_k = (w_k(z_1), w_k(z_2), \ldots, w_k(z_m))$, where

$$W_k(z_\nu) = |f(z_\nu) - P(a^{k-1} ; z_\nu)|^{p-2} \Big/ \sum_{\nu=1}^{m} |f(z_\nu) - P(a^{k-1} ; z_\nu)|^{p-2}. \tag{16}$$

Step 2. Write the best L_{2,w_k} approximation to f on Z_m under the form (15).

In this variant of the algorithm at each iteration it is necessary to calculate only the weight (16). The polynomial of the best L_{2,w_k} approximation to f on Z_m is written explicitly. We note that in (15) the determinants $D(z_{\nu_1}, \ldots, z_{\nu_n}; \varphi)$ and the interpolatory polynomials $L(z_{\nu_1}, \ldots, z_{\nu_n}; \varphi; f|z)$ are unchanged at each iteration. So they are calculated only once, at the beginning of the algorithm. At each iteration changes only the weight w_k.

Remark 4. Obviously the convergence of the algorithm depends on the starting point a^0. To accelerate the algorithm we may take for the starting polynomial $P(a^0; z) \in \mathfrak{L}(\varphi)$, the best $L_{2,\mu}$ approximation to f on Z_m with the arbitrary but given weight μ.

(Received September 1, 1977)

REFERENCES

1. Kahng, S. W., *Best L_p Approximation*, Math. Comp., **28**, 118 (1972), 505–508.
2. Lawson, C. L., *Contribution to the Theory of Linear Least Maximum Approximation*, Ph. D. Thesis, UCLA, 1961, 55–61.
3. Maruşciac, I., *Asupra polinoamelor de cea mai bună aproximaţie a unei funcţii pe o mulţime finită de puncte din planul complex*, Studii Cerc. Matem. (Cluj), **11**, 2 (1960), 325–335.
4. Maruşciac, I., *Une forme explicite du polynome de meilleure approximation d'une fonction dans le domaine complex*, Mathematica (Cluj), **6** (29), 2 (1964), 257–263.
5. Ortega, J. M., Rheinbold, W. C., *Iterative solution of Nonlinear Equations in Several Variables*, Academic Press, New York, 1970.
6. Rice J. R., Usow, K. H., The Lawson algorithm and extentions, Math. Comp. **22** (1968), 118–127.

UN ALGORITM PENTRU CEA MAI BUNĂ L_p – APROXIMARE ÎN PLANUL COMPLEX

(Rezumat)

Se dă un algoritm pentru calculul celei mai bune L_p-aproximări a unei funcţii pe o mulţime discretă din planul complex. Algoritmul propus este o adaptare a metodei lui Newton Raphson şi se bazează pe o formă explicită a polinomului de grad cel mult n de cea mai bună L_2-aproximare a unei funcţii pe o mulţime finită de puncte din planul complex, dată de autor în lucrarea [4]. Metoda propusă poate fi utilizată de asemenea şi pentru calculul celei mai bune L_p-aproximări a unei funcţii pe un arc de curbă rectificabilă din planul complex.

STUDIA UNIV. „BABEŞ-BOLYAI, MATHEMATICA, 1, 1978

ALGÈBRES PARTIELLES ET STRUCTURES RELATIONNELLES

MICHELINE FRODA-SCHECHTER

Les homomorphismes relationnels se présentent aussi bien dans l'étude des structures ordonnées (groupes ordonnés) que dans la théorie des modèles. Deux particularisations principales on été mises en évidence dans ces mêmes domaines sous des appellations très diverses[1]. Dans des ouvrages antérieurs ([2], [3] et [4]) j'ai essayé de faire une étude systématique de ces homomorphismes et de leurs particularisations, la factorisation (terminologie empruntée à Cohn [1]) et la factorisation forte, „dans" ou „sur" la structure[2] ainsi que de la congruence relationnelle (si φ est une factorisation forte, ker φ est une congruence relationnelle).

Si on tient compte du fait qu'on peut faire correspondre à toute opération n-aire ω_A sur A, éventuellement partielle, $\omega_A : S \to A$, où $S^n \subseteq A$, une relation $(n+1)$-aire sur A, ρ_A, à savoir

$$(a_1, a_2, \ldots, a_n, a_0) \in \rho_A \Leftrightarrow \omega_A(a_1, a_2, \ldots, a_n) = a_0$$

on peut se demander quelles liaisons existent entre certains homomorphismes des algèbres universelles partielles considérés par Grätzer [5] et tous ces homomorphismes relationnels par rapport à une relation quand elle est une opération. La même question se pose pour les congruences. Un résultat de ce type est connu, c'est la partie 1°) du th. *I* dans le cas plus particulier d'une opération sur A (non partielle, c'est-à-dire $S = A^n$) : Malcev [6].

DÉFINITIONS I ([5] p. 81) Soient ω_A et ω_B des opérations partielles sur A respectivement sur B, $\varphi : A \to B$ et $a_1, a_2, \ldots, a_n, a_0$ des éléments quelconques de A,

(i) φ est *ω-homomorphisme* si du fait que $\omega_A(a_1, a_2, \ldots, a_n)$ existe[3] il résulte que $\omega_B(\varphi a_1, \varphi a_2, \ldots, \varphi a_n)$ existe et l'égalité suivante est vérifiée:

$$\varphi(\omega_A(a_1, a_2, \ldots, a_n)) = \omega_B(\varphi a_1, \varphi a_2, \ldots, \varphi a_n). \quad (1)$$

(ii) φ est un *ω-homomorphisme plein* si φ est un ω-homomorphisme tel que $\omega_B(\varphi a_1, \varphi a_2, \ldots, \varphi a_n) = \varphi a_0$ implique l'existence d'éléments $x_1, x_2, \ldots, x_n, x_0 \in A$ tels que $\varphi x_i = \varphi x_i (i = 1, 2, \ldots, n)$ et $x_0 = \omega_A(x_1, x_2, \ldots, x_n)$;

(iii) φ est un *ω-homomorphisme fort* si φ est un ω-homomorphisme tel que $\omega_A(a_1, a_2, \ldots, a_n)$ existe si et seulement si $\omega_B(\varphi a_1, \varphi a_2, \ldots, \varphi a_n)$ existe.

---

[1] Une bibliographie d'ouvrages utilisant ces homomophismes se trouve dans [2], pour les congruences relationnelles voy. la bibliographie dans [4].

[2] Cette classification des factorisations (fortes) introduite dans [2] est naturelle si celles-ci ne sont pas des surjections. Dans l'ouvrage présent je n'utilise que la factorisation dans la structure (déf. II (ii)) et la factorisation forte sur la structure (déf. II(iii)).

[3] On dit que $\omega_A(a_1, a_2, \ldots, a_n)$ existe si $(a_1, a_2, \ldots, a_n) \in S$ et on écrit $\omega_A(a_1, a_2, \ldots, a_n) \in A$.

DÉFINITIONS II (cf. [2]). Soient ρ_A et ρ_B des relations $(n+1)$-aires sur A respectivement B, $\varphi : A \to B$, et $a_1, a_2, \ldots, a_n, a_0$ des éléments quelconques de A,

(i) φ est un ρ-*homomorphisme* si l'on a

$$(a_1, a_2, \ldots, a_n, a_0) \in \rho_A \Rightarrow (\varphi a_1, \varphi a_2, \ldots, \varphi a_n, \varphi a_0) \in \rho_B \tag{2}$$

(ii) φ est une ρ-*factorisation dans* la structure relationnelle $< \varphi A, \{\rho\} >$ si φ est un ρ-homomorphisme et $(\varphi a_1, \varphi a_2, \ldots, \varphi a_n, \varphi a_0) \in \rho_B$ implique l'existence d'éléments $x_1, x_2, \ldots, x_n, x_0 \in A$ tels que $\varphi x_i = \varphi a_i$ $(i = 0, 1, 2, \ldots, n)$ et $(x_1, x_2, \ldots, x_n, x_0) \in \rho_A$;

(iii) φ est une ρ-*factorisation forte dans la structure* $< B, \{\rho\} >$ si φ est un ρ-homomorphsime tel que $\rho_B \subseteq (\varphi A)^{n+1}$ et $(b_1, b_2, \ldots, b^n, b_0) \in \rho_B$ et $\varphi a_i = b_i$ $(i = 0, 1, 2, \ldots, n)$ implique $(a_1, a_2, \ldots, a_n, a_0) \in \rho_A$.

THÉORÈME 1. *Si ω_A et ω_B sont deux opérations partielles sur A respectivement B et ρ_A et ρ_B sont les relations correspondantes et $\varphi : A \to B$ une application alors*

1°) *φ est un ω-homomorphisme si et seulement si φ est un ρ-homomorphisme,*

2°) *φ est un ω-homomorphisme plein si et seulement si φ est une ρ-factorisation dans $<\varphi A, \{\rho\}>$,*

3°) *Si φ est une ρ-factorisation forte dans $< B, \{\rho\} >$ alors φ est un ω-homomorphisme fort.*

Démonstration : 1° Soit φ un ω-homomorphisme et supposons que $(a_1, a_2, \ldots, a_n, a_0) \in \rho_A$; il résulte que $a_0 = \omega_A(a_1, a_2, \ldots, a_n)$ et donc $\omega_A(a_1, a_2, \ldots, a_n)$ existe[3]. Par l'hypothèse $\omega_B(\varphi a_1, \varphi a_2, \ldots, \varphi a_n)$ existe aussi et l'on a (1), c'est-à-dire $(\varphi a_1, \varphi a_2, \ldots, \varphi a_n, \varphi(\omega_A(a_1, a_2, \ldots, a_n))) \in \rho_B$ donc $(\varphi a_1, \varphi a_2, \ldots, \varphi a_n, \varphi a_0) \in \rho_B$

Réciproquement, supposons que φ est un ρ-homomorphisme et que $\omega_A(a_1, a_2, \ldots, a_n) \in A$, donc qu'il existe un élément $a_0 \in A$ tel que $\omega_A(a_1, a_2, \ldots, a_n) = a_0$. On a par conséquent $(a_1, a_2, \ldots, a_n, a_0) \in \rho_A$ et, par l'hypothèse, aussi $(\varphi a_1, \varphi a_2, \ldots, \varphi a_n, \varphi a_0) \in \rho_B$. Mais selon la définition de ρ_B ceci équivaut à $\varphi a_0 = \omega_B(\varphi a_1, \varphi a_2, \ldots, \varphi a_n)$, ce qui prouve que φ est un ω-homomorphisme.

2°) Soit φ un ω-homomorphisme plein ; par définition φ est un ω-homomorphisme donc, selon 1°), un ρ-homomorphisme. Considérons maintenant $(\varphi a_1, \varphi a_2, \ldots, \varphi a_n, \varphi a_0) \in \rho_B$, donc $\varphi a_0 = \omega_B(\varphi a_1, \varphi a_2, \ldots, \varphi a_n)$. Mais, par l'hypothèse, des éléments $x_1, x_2, \ldots, x_n, x_0 \in A$ existent tels que $\varphi x_i = \varphi a_i$ pour $i = 1, 2, \ldots, n$ et $x_0 = \omega_A(x_1, x_2, \ldots, x_n)$, donc $(x_1, x_2, \ldots, x_n, x_0) \in \rho_A$, c'est-à-dire φ vérifie la définition II (ii).

Réciproquement soit φ une ρ-factorisation, elle sera donc un ρ-homomorphisme et par conséquent, selon 1°) un ω-homomorphisme. Supposons que $\omega_B(\varphi a_1, \varphi a_2, \ldots, \varphi a_n) = \varphi a_0$, c'est-à-dire $(\varphi a_1, \varphi a_2, \ldots, \varphi a_n, \varphi a_0) \in \rho_B$, d'où par la définition II (ii), il résulte qu'il existe $x_1, x_2, \ldots, x_n, x_0 \in A$

tels que $\varphi a_i = \varphi x_i$ $(i = 0, 1, 2, \ldots, n)$ et $(x_1, x_2, \ldots, x_n, x_0) \in \rho_A$. Ceci peut encore s'écrire $x_0 = \omega_A(x_1, x_2, \ldots, x_n)$, donc φ est un ω-homomorphisme plein.

3°) Si φ est une ρ-factorisation forte il en résulte selon 1°) que φ est un ω-homomorphisme. Il reste seulement à prouver que si $\omega_B(\varphi a_1, \varphi a_2, \ldots, \varphi a_n)$ existe alors $\omega_A(a_1, a_2, \ldots, a_n)$ existe aussi. L'hypothèse signifie qu'il existe un élément $b_0 \in B$ tel que $(\varphi a_1, \varphi a_2, \ldots, \varphi a_n, b_0) \in \rho_B$. Mais $\rho_B \subseteq (\varphi A)^{n+1}$ implique l'existence d'au moins un élément $a_0 \in A$ tel que $\varphi a_0 = b_0$ et l'on a $(a_1, a_2, \ldots, a_n, a_0) \in \rho_A$ c'est-à-dire $\omega_A(a_1, a_2, \ldots, a_n)$ existe.

Remarque: L'implication contraire de 3°) n'est pas vraie; en effet considérons l'exemple suivant: $A = \{a_1, a_2, a_0, a_0'\}$, $B = \{b_1, b_2, b_0\}$, $\varphi : A \to B$, $\varphi a_i = b_i$ $(i = 0, 1, 2)$, $\varphi(a_0') = b_0$, $\omega_A(a_1, a_2) = a_0$, $\omega_B(b_1, b_2) = b_0$ où φ est un ω-homomorphisme fort mais n'est pas une ρ-factorisation forte car $\varphi a_0 = \varphi a_0' = b_0$, $(b_1, b_2, b_0) \in \rho_B$ mais $(a_1, a_2, a_0) \notin \rho_A$.

On définit pour les algèbres (partielles), les congruences de la manière suivante (cf. [5], p. 35, 82):

Définitions III. Soit γ une équivalence sur A et $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ des éléments quelconques de A:

(i) Si ω_A est une opération n-aire sur A, γ s'appelle *ω_A-congruence* si

$$x_i = y_i \ (\gamma) \text{ pour } i = 1, 2, \ldots, n \tag{3}$$

implique

$$\omega_A(x_1, x_2, \ldots, x_n) = \omega_A(y_1, y_2, \ldots, y_n) \ (\gamma) \tag{4}$$

(ii) Si ω_A est une opération n-aire partielle sur A, γ s'appelle *ω_A-congruence* si (4) résulte de (3) et du fait que $\omega_A(x_1, x_2, \ldots, x_n)$ et $\omega_A(y_1, y_2, \ldots, y_n)$ sont définis.

(iii) Si ω_A est une opération n-aire partielle sur A, une ω-congruence est *forte* si du fait que $\omega_A(x_1, x_2, \ldots, x_n)$ existe et que (3) est vérifiée il résulte que $\omega_A(y_1, y_2, \ldots, y_n)$ existe aussi.

J'ai nommé dans [3] et [4] une équivalence γ sur A une *ρ-congruence (congruence relationnelle)* si γ satisfait la condition suivante pour tout $x_1, x_2, \ldots, x_n, x_0 \in A$:

$$(x_1, x_2, \ldots, x_n, x_0) \in \rho_A \quad \text{et} \quad x_i \equiv y_i(\gamma) \quad \text{pour } i = 0, 1, 2, \ldots, n \Rightarrow (y_1, y_2, \ldots, y_n, y_0) \in \rho_A$$

THÉORÈME 2. *Si ω_A est une opération partielle n-aire sur A et ρ_A la relation correspondante, toute ρ_A-congruence est une ω_A-congruence forte. En particulier si ω_A est une opération sur A toute ρ-congruence est une ω_A-congruence.*

Démonstration. Soit γ une ρ_A-congruence et supposons que l'on a (3) et que $\omega_A(x_1, x_2, \ldots, x_n)$ existe, c'est-à-dire qu'il existe $x_0 \in A$ tel que $\omega\ (x_1, x_2, \ldots, x_n) = x_0$ donc $(x_1, x_2, \ldots, x_n, x_0) \in \rho_A$. Mais $x_0 \equiv x_0$ (γ) donc $(y_1, y_2, \ldots, y_n, x_0) \in \rho_A$ ce qui montre que $\omega_A(y_1, y_2, \ldots, y_n) = = x_0$ et donc, à fortiori, (4) est vérifiée.

Remarque : La réciproque n'est pas valable, d'ailleurs la congruence algébrique qu'on obtient à partir d'une congruence relationnelle est très particulière. On peut aussi donner un contre-exemple simple, comme suit : Soit γ la congruence modulo 3 sur l'ensemble Z des entiers, par rapport à l'addition. Mais γ n'est pas une congruence pour la relation ternaire ρ_z correspondant à l'addition car l'on a : $2 + 3 = 5 \Leftrightarrow (2, 3, 5) \in \rho_Z$, $2 \equiv 8 \pmod 3$, $3 \equiv 0 \pmod 3$, $5 \equiv 11 \pmod 3$, mais $(8, 0, 11) \notin \rho_Z$ car $8 + 0 \neq 11$.

(Manuscrit reçu la 10 septembre 1977)

BIBLIOGRAPHIE

1. Cohn, P. M., *Universal Algebra*, Harper and Row, New York, 1965.
2. Froda—Schechter, M., *Sur les homomorphismes des structures relationnelles* (I), Studia Univ. Babeş-Bolyai, ser. Math.—Mech., 2 (1974), 20—25.
3. Froda—Schechter, M., *Sur les homomorphismes des structures relationnelles* (II), Studia Univ. Babeş-Bolyai, Math. (1975), 11—15.
4. Froda—Schechter, M., *Propriétés des congruences relationnelles*, Publ. du Centre de recherches en Math. pures, Neuchâtel, série I, **11** (1976), 1—5.
5. Grätzer, G., *Universal Algebra*, Van Nostrand Comp. Toronto, 1968.
6. Malcev, A. I., *The Metamatemathematics of Algebraic Systems*, Collected papers 1936— —1967 (Studies in Logic), North—Holland Publ. Comp., Amsterdam, 1971.

ALGEBRE PARȚIALE ȘI STRUCTURI RELAȚIONALE

(Rezumat)

Se consideră operații n-are (parțiale) care se privesc și ca relații $(n + 1)$-are arătindu-se legătura dintre diferite tipuri de omomorfisme față de aceste operații și unele omomorfisme relaționale studiate de autor în lucrări anterioare. Același tip de problemă se rezolvă și pentru congruențe.

STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, 1, 1978

A FIXED POINT THEOREM OF MAIA-PEROV TYPE

MONICA ALBU

In the present paper a fixed point theorem is established, by which we can get a new existence and uniqueness theorem relating to a system of integral equations of Fredholm type.

THEOREM 1. *Let X be a set endowed with the generalized metrics d, $\delta : X \times X \to \mathbf{R}^n$. If the conditions are fulfilled:*

(1) $d(x,y) \leq \delta(x,y), \quad \forall\ x,y \in X$

(2) (X,d) *is complete*

(3) the application $f: (X,d) \to (X,d)$ *is continuous*

(4) there exists a matrix convergent toward 0

$$A \in M_{nn}\ (\mathbf{R}) \text{ such that}$$

$$\delta(f(x),\ f(y) \leq A\,\delta(x,\ y),\ \forall\ x,\ y \in X$$

then the application f has a unique fixed point x^*, which can be obtained by the succesive approximations' method starting from any element $x_0 \in X$. Moreover, the following estimation takes place:

$$\delta(x_m,\ x^*)) \leq A^m(I - A)^{-1}\delta(x_0,\ x_1)$$

Proof. Let $x_0 \in X$. By plotting the sequence of successive approximations $x_1 = f(x_0)$, $x_2 = f^2(x_0)$, ..., $x_m = f^m(x_0)$, ..., we get

$$\delta\,(f^{m+p}\ (x_0),\ f^m(x_0)) \leq \delta(f^{m+p-1}(x_0),\ f^{m+p}(x_0)) +$$
$$+\ \delta(f^{m+p-2}(x_0),\ f^{m+p-1}(x_0)) + \ldots + \delta(f^m(x_0),\ f^{m+1}(x_0)) \leq$$
$$\leq A^{m+p-1}\delta(f(x_0),\ x_0) + A^{m+p-2}\delta(f(x_0),\ x_0) + \ldots + A^m\delta(f(x_0),\ x_0) =$$
$$= A^m[\mathrm{I} + A + \ldots + A^{p-1}]\ \delta(f(x_0),\ x_0)$$

As $A^m \underset{m\to\infty}{\longrightarrow} 0$ it follows that $\delta(f^{m+p}(x_0),\ f^m(x_0)) \underset{m\to\infty}{\longrightarrow} 0$

The sequence $(f^m(x_0))_{m\in N}$ is fundamental relative to the metric δ

But $d(x,y) \leq \delta(x,y)$, $\forall\ x,y \in X$. It follows that the sequence $(f^m(x_0))_{m\in N}$ is fundamental relative to the metric d.

The metric space (X,d) being complete, the sequence $(f^m(x_0))_{m\in N}$ is convergent.

Let $x^* = \lim\limits_{m\to\infty} f^m(x_0) = f(\lim\limits_{m\to\infty} f^{m-1}(x_0)) = f(x^*)$

Accordingly x^* is a fixed point for f. We suppose that x^{**} is another fixed point of the mapping f, $x^* \neq x^{**}$. Then $\delta(x^*, x^{**}) = \delta(f(x^*), f(x^{**})) \leq A\delta(x^*, x^{**}) \leq A^2\delta(x^*, x^{**}) \leq \ldots \leq A^m \delta(x^*, x^{**})$.

It follows that $\delta(x^*, x^{**}) = 0$. Therefore $x^* = x^{**}$, the fixed point is unique.

$$\delta(x_m, x^*) = \delta(f^m(x_0), x^*) \leq A\delta(f^{m-1}(x_0), x^*) \leq$$
$$\leq A^2\delta(f^{m-2}(x_0), x^*) \leq \ldots \leq A^m\delta(x_0, x^*)$$
$$\delta(x_0, x^*) \leq \delta(x_0, x_1) + \delta(x_1, x^*) \leq \delta(x_0, x_1) + A\delta(x_0, x^*)$$
$$\Rightarrow \delta(x_0, x^*) \leq (I - A)^{-1}\, \delta(x_0, x_1)$$

Then $\delta(x_m, x^*) \leq A^m(I - A)^{-1}\, \delta(x_0, x_1)$

Remark. The conclusion of theorem 1 does not change if condition (1) is replaced by the condition (see [4])

$$\exists C > 0 : d(f(x), f(y)) \leq C\delta(x,y), \quad \forall\, x,y \in X$$

From theorem 1 we obtain the following two known results:

THEOREM 2. (Maia [1]). *Let X be a metric space with the following metrics $d : X \times X \to \mathbf{R}_+$ and $\delta : X \times X \to \mathbf{R}_+$. If the following conditions are fulfilled:*

(1) $d(x, y) \leq \delta(x, y) \quad \forall\, x,y \in X$

(2) (X, d) is complete

(3) the mapping $f : (X, d) \to (X, d)$ is continuous

(4) $f : (X, \delta) \to (X, \delta)$ is contraction

then the mapping f has a unique fixed point in X.

Proof. $f : (X, \delta) \to (X, \delta)$ contraction means:

$$\exists \alpha \in\,]0,1[\, : \delta(f(x), f(y)) \leq \alpha\delta(x,y) \quad \forall\, x,y \in X$$
$$\alpha \in\,]0,1[\, \Rightarrow \alpha^m \underset{m\to\infty}{\longrightarrow} 0$$

Theorem 1 is applied using $n = 1$ and $A = \alpha$

THEOREM 3 (Perov [2]). *Let (X, d) be a generalized complete metric space with the metric $d : X \times X \to \mathbf{R}^n$ and $f : X \to X$ an mapping such that there exists a convergent matrix toward 0, $A \in M_{nn}(\mathbf{R})$, having the property of $d(f(x), f(y)) \leq Ad(x,y)$, $\forall\, x,y \in X$.*

Under these conditions the mapping f has a unique fixed point x^* which can be obtained by the successive approximations' method starting from any element $x_0 \in X$. Moreover, the estimation $d(x_m, x^*) \leq A^m(I - A)^{-1} \cdot d(x_0, x_1)$ takes place.

Proof. We consider a sequence $(x_m)_{m\in N} \subset (X,d)$ convergent toward an element $x \in X$. It means that $d(x_m, x) \underset{m\to\infty}{\longrightarrow} 0$

As $d(f(x_m), f(x)) \leqslant Ad(x_m, x)$ it follows that $f(x_m) \underset{x_m \to x}{\longrightarrow} f(x)$, that means that $f: (X,d) \to (X,d)$ is continuous.

The conditions of theorem 1 are checked for $d = \delta$

Using theorem 1 we get an existence and uniqueness theorem for the integral equations system of Fredholm type.

$$\varphi(x) = \lambda \int_{\Omega} K(x, y, \varphi(y))dy + f(x) \qquad (*)$$

where $\lambda \in \mathbf{R}$, $K \in C(\bar{\Omega} \times \bar{\Omega} \times \mathbf{R}^m, \mathbf{R}^m)$, $f \in C(\bar{\Omega}, \mathbf{R}^m)$ and $\Omega \in \mathbf{R}^n$ bounded domaine.

Let $A : C(\bar{\Omega}, \mathbf{R}^m) \to C(\bar{\Omega}, \mathbf{R}^m)$ be the operator defined by:

$$A(\varphi)(x) = \lambda \int_{\Omega} K(x, y, \varphi(y))\, dy + f(x)$$

We consider the generalized metric space

$X = C(\bar{\Omega}, \mathbf{R}^m)$ containing the metrics

$$d(\varphi, \psi) = ||\varphi - \psi||_{C(\bar{\Omega},\mathbf{R}^m)} = (||\varphi_1 - \psi_1||_{C(\bar{\Omega})}, \ldots, ||\varphi_m - \psi_m||_{C(\bar{\Omega})})$$

$$\delta(\varphi, \psi) = ||\varphi - \psi||_{L^2(\Omega,\mathbf{R}^m)} = (||\varphi_1 - \psi_1||_{L^2(\Omega)}, \ldots, ||\varphi_m - \psi_m||_{L^2(\Omega)})$$

where $||u_i||_{C(\bar{\Omega})} = \max \{|u_i(t)| : t \in \bar{\Omega}\}$

$$||u_i||_{L^2(\Omega)} = \left(\int_{\Omega} |u_i(t)|^2 dt\right)^{\frac{1}{2}}$$

We get the following existence and uniqueness theorem of the solution:

THEOREM 4. *If the following conditions are fulfilled:*

(1) $K \in C(\bar{\Omega} \times \bar{\Omega} \times \mathbf{R}^m, \mathbf{R}^m)$, $f \in C(\bar{\Omega}, \mathbf{R}^m)$, $\lambda \in \mathbf{R}$

(2) $\exists L : \Omega \times \Omega \to M_{mm}(\mathbf{R})$ *nonnegative,*

and $$\sup_{x \in \Omega} \left(\int_{\Omega} |L_{ij}(x, y)|^2\, dy\right)^{1/2} < \infty, \ \forall\, i,j = \overline{1, m}$$

so that $|K(x,y,u) - K(x,y,v)| \leqslant L(x,y)|u - v|$, $\forall\ x, y \in \Omega$, $\forall\ u,v \in \mathbf{R}^m$ *and* $|K(x,y,0)| \leqslant r(x,y)$ *where* $r \in C(\bar{\Omega} \times \bar{\Omega}, \mathbf{R}^m)$ *nonnegative.*

(3) there exists a convergent matrix toward 0

$S \in M_{mm}(\mathbf{R})$ *such that* $\left(|\lambda| \left\{ \int_{\Omega \times \Omega} |L_{ij}(x,y)|^2\, dx\, dy \right\}^{\frac{1}{2}} \right)_m^m \leqslant S$ *then the system of equations* (*) *has in* $C(\bar{\Omega}, \mathbf{R}^m)$ *one and only one solution which can be obtained by the successive approximations' method starting from any element from* $C(\bar{\Omega}, \mathbf{R}^m)$. *Moreover, if* φ *is the solution the following estimation takes place:*

$$||\varphi||_{L^2(\Omega,\mathbf{R}^m)} \leqslant (I - S)^{-1} \left(|\lambda| \left\| \int_{\Omega} r(\cdot, y)\, dy \right\|_{L^2(\Omega,\mathbf{R}^m)} + ||f||_{L^2(\Omega,\mathbf{R}^m)} \right).$$

(Received October 10, 1977)

REFERENCES

1. M. G. Maia, *Un'Osservazione sulle contrazioni metriche*, Rend. Sem. Mat. Univ. Padova, **40** (1968), 139—143.
2. A. I. Perov, A. V. Kibenco, *Ob adnom obscem metode isledovania kraevîh zadaci*, Isv. Akad. Nauk SSSR, Ser. Mat., **30** (1966), 249—264.
3. I. A. Rus, *Teoria punctului fix*, II, Cluj, 1973.
4. I. A. Rus, *On a fixed point theorem of Maia*, Studia Univ. Babeş-Bolyai, Math., **1** (1977) ,40—42.

O TEOREMĂ DE PUNCT FIX DE TIP MAIA-PEROV

(Rezumat)

Se dă o generalizare a teoremelor de punct fix stabilite de Maia şi de Perov şi o aplicaţie a acestora la rezolvarea sistemelor de ecuaţii integrale de tip Fredholm.

RECENZII

I. Păvăloiu, **Introduceere în teoria aproximării soluţiilor ecuaţiilor** (Introduction in the theory of approximation of the solutions of equations), Ed. Dacia, Cluj-Napoca, 1976, p. 207.

For the sake of a general presentation of the numerical solving of equations, the author develops at first the fundamentals in the theory of linear normed spaces, which permits he to present the approximation of solutions of the linear operational equations of solutions of the linear operatorial equations in the sense of L. V. Kantorovic. There are given then some applications to the solutions of infinite systems of linear equations, integral equations, many-point boundary value problems for differential equations, etc.

The investigations of the author about the inverse interpolation method in solving of equations in normed spaces have an important role in the monography. Thus a unitary approach is given in the treatment of the methods of Chebyshev and Steffensen type, and those obtained from the Lagrange's inverse interpolation formula. The convergence problem of these methods are considered by using a general convergence criterion given by the author, from that as particular cases are obtained convergence criteria for the methods of Newton, Cebyshev, Steffensen, etc.

The combined methods. are presented in a general setting Here are studied the iterative methods obtained by combination of interpolatory type methods and methods in which the inverse of linear operator is determined.

The author studies finally the stability problems of the iterative methods, the problems of order of convergence and the convergence of the Gauss-Seidel method for solving operatorial equations.

There are given various applications of the considered iterative methods in solving algebraic and differential equations, systems of linear and nonlinear equations, etc.

A. B. NÉMETH

Beiträge zur Numerischen Matematik 4, VEB Deutscher Verlag der Wissenschaften, Berlin, 1975.

Dieser Band, der Herrn Prof. Dr. – Ing. habil. Dr. tech. h.c. Helmut Heinrich zum 70 Geburtstag gewidmet ist, enthält eine grosse Menge Arbeiten aus verschiedenen Gebieten der Numerischen Mathematik, wie zum Beispiel: Optimierungsaufgaben und Rand-Kontrollprobleme (S. Ulm, W. Krabs, H. Kleinmichel), Eigenwertprobleme (L. Boubelikova, I. Marek, I. Neuman, W. Dück, G. Hämmerlin, W. R. Richert, F. Kuhnert, T. Riedrich, T. Stoer), Numerische Verfahren für Differential – und Partialdifferentialgleichungen (S. Dietze, R. Ansorge, H. Kreth, E. Lauckan, T. W. Schmidt, A. N. Tichonov, W. Törnig) Programierungsaufgaben (W. Terke, W. Schiebel, T. Terno, Ç. Unger) Aproximationsverfahren für algebraische Gleichungen (K. H. Bachmann, S. Filippi, W. C. Rheinboldt, Ch. K. Mesztenyi, P. H. Müller, G. Richter) Numerische Analysis (G. Braunss, L. Elsner, G. Merz, A. A. Samarskii, I. V. Freazinov, H. Schwetlick).

G. MICULA

I. P. Cluj, Municipiul Cluj-Napoca cd. 695/1978

În cel de al XXIII-lea an (1978) *Studia Universitatis Babeş—Bolyai* apare semestrial în specialităţile:

matematică

fizică

chimie

geologie—geografie

biologie

filozofie

ştiinţe economice

ştiinţe juridice

istorie

filologie

На XXIII году издания (1978) *Studia Universitatis Babeş—Bolyai* выходит два раза в году со следующими специальностями:

математика

физика

химия

геология—география

биология

философия

экономические науки

юридические науки

история

филология

Dans sa XXIII-e année (1978) *Studia Universitatis Babeş—Bolyai* paraît semestriellement dans les spécialités:

mathématiques

physique

chimie

géologie—géographie

biologie

philosophie

sciences économiques

sciences juridiques

histoire

philologie

43 875

Abonamentele se fac la oficiile poştale, prin factorii poştali şi prin difuzorii de presă, iar pentru străinătate prin ILEXIM Departamentul Export—Import Presă, P. O. Box 136—137, telex 11226, Bucureşti, str. 13 Decembrie nr. 3.

Lei 10