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SOME PROPERTIES OF TOPOLOGICAL Ω-GROUPS (I)

M. SZILÁGYI

As the notion *Ω-group* (a group with multioperators) introduced by P. Higgins [10] is a natural generalization of the notions of group and ring, the notion of the topological Ω- *group* is in the same manner a natural generalization of the notions of topological group and topological ring.

The topological groups appear at first referring to the abelian groups in the works of H. Prüfer [23] and W. Krull [13]. H. Shöneborn [25] studies this problem with more details. In the works mentioned above the topology compatible with the structure of an abelian group is introduced by a family of subgroups serving as the base of the filter of the neighborhood of the zero element of the group. For arbitrary groups the same problem was studied by D. van Dantzig [6], M. Hall jr. [9] H. Schöneborn [26], F. D. Bolker [2] and W. Krull [14]. In the works [26] and [14], a complete system of neighborhoods of the zero element is introduced as a family of normal subgroups.

The general notion of a topological ring appears explicitly for the first time in the works of D. van Dantzig [7] and [8]. These works of van Dantzig were preceded by some studies referring to special classes of topological rings (see e.g. [27]) A particular and very important case is the class of those topological rings in which a system of complete neighborhoods is presented by a family of special subrings (bilateral ideals, left ideals, etc.) The same cases were studied by Zelinsky [28], [29], [30], and Ballier [1]. The topologies which are compatible with a structure of group with operators — in which the topology is given by a family of permitted normal subgroups — are studied in the works [16], [17]. The importance of this case results from the fact that these topologies-in a particular case- are the so called m-adic topologies of the local rings studied in the works [21], [20], [24], etc. A short description of the topological groups with operators can be found in work [3].

The idea of topologization of the algebraic structures with multioperators is to be found in the work [15] by A.I. Malcev. The study of topological universal algebras is treated by Malcev with the help of translations of universal algebra. That is Malcev draws conclusions referring to the topological universal algebra from the structure of the translations. The notions introduced by Malcev are very comprehensive and some of his ideas are used in this work too.

In this work will be studied some of the properties of the topological Ω-*groups:* In the first part are given the definition of the topological Ω-*group* and some examples. We shall reformulate in this part some of the properties of topological groups and rings, which are preserved in a natural way in the case of the topological Ω-*groups*. Though the notion of topological Ω-*groups* is a more particular notion than the notion of topological universal algebra, nevertheless for the unity of notations and presentation we shall give the detailed definition of the topological Ω-*group*. It is worth remarking that the notion of Ω-*group* was introduced by Higgins in 1956, whereas the definition of the topological universal algebra was given by Malcev in 1953. In the §2 we shall state some of the elementary properties referring to the notion of the topological Ω-*group*. The product topology in topological Ω-*groups* is the subject of §3. In §4 we shall study the quotient topology within the topological Ω-groups. The topology of the Ω-*groups* is interesting when we have a complete system of neighborhoods of the zero element containing only ideals. This case is described in §5. §6 deals with the theorems of izomorphism referring to topological Ω-*groups*.

Let G be a set having the algebraic structure of an Ω-*group* and a topological structure τ. The group operation will be denoted by "+" — generally operation "+" is not commutative — and let:

$$\Omega(n) = \{\omega \in \Omega \mid a(\omega) = n\}^1.$$

G^n as a topological space whenever appears will mean the cartesian product $\Sigma\{G_k|(G_k = G) \wedge (k \in \{1, 2, \ldots, n\})\}$ supplied with the product topology whereas $(G^n, +, \Omega)$ as an Ω-*group* is considered to be the discrete direct product. If A and B are topological spaces and $f: A \to B$ is a continuous mapping, this quality will be denoted by $C_A^B(f)$ or, if there is no danger of confusion, by $C(f)$. We shall introduce some very useful mappings in the following. So the "addition mapping" $s: G \times G \to G$ defined by $s(x, y) = x + y$; "the opposite element mapping" $i: G \to G$ defined by $i(x) = -x$; the "translation mappings" $\varphi_a, {}_a\varphi, {}_a\varphi_b: G \to G$ defined by $\varphi_a(x) = x + a$, ${}_a\varphi(x) = a + x$, ${}_a\varphi_b(x) = a + x + b$ for each $x \in G$ where $a, b \in G$; "the subtraction mapping" defined by $s_1: G \times G \to G$, $s_1(x, y) = x - y$; "the Higgins mapping" attached to the operator $\omega \in \Omega(n)$ is $h_\omega^n: G^n \to G$, defined by $h_\omega^n(x_1, x_2, \ldots, x_n) = \omega x_1 x_2 \ldots x_n$.

§ 1. Definition of the Topological Ω-group. Example. Let $(G, +, \Omega)$ be an Ω-*group* and let (G, τ) be a topological space.

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[1] In the following the symbols $n, n_1, n_2, \ldots,$ will represent the elements of the set $\mathbf{N} = \{1, 2, 3, \ldots, n, \ldots\}$; the symbol $\mathbf{N}$ will always represent this set; the symbol ω whenever it appears, will represent an element of Ω; the mapping $a: \Omega \to \mathbf{N}$ is defined as $a(\omega) = n$ whenever ω is an n — ary operator [5].

DEFINITION 1.1. $(G, +, \Omega, \tau)$ is called τ — topological Ω-group[2], if G is supplied with an Ω-group structure and with a topological structure τ, and these two structures are compatible in the following sense:

$$1.\ C_{G^2}^{G}(\mathrm{s}) \qquad 2.\ C_{G}^{G}(\mathrm{i}) \qquad 3.\ \forall\omega\,(a(\omega) = \mathrm{n} \Rightarrow C_{G^n}^{G}(\mathrm{h}_{\omega}^{n}))$$

Observation 1. Each Ω-group may be considered as a topological Ω-group with respect to the discrete topology or to the indiscrete topology.

Observation 2. If $\Omega = \Phi$, the notion of topological Ω-group is identic with the notion of topological group; if Ω contains only one binar operator to which it corresponds an operation which is distributive with respect to the group operation and $(G,+)$ is an abelian group, then the notion of topological Ω-group is identical with the notion of topological ring.

Observation 3. If $(G,+,\Omega,\tau)$ is a τ — topological Ω-group, then $(G,+)$ as a group is, of course, a τ-topological group. The same thing will be sometimes formulated as: $(G,+,\tau)$ *is the additive topological group of the topological Ω-group*. The properties of the topological group $(G,+,\tau)$ which do not depend on Ω are true to $(G,+,\Omega,\tau)$ too. So we have ([3], [11], =12]):

a) In the definition of the topological Ω-*group* the axioms 1. and 2. may be replaced with the following axiom:

1. $C_{G^3}^{G}(\mathrm{s}_1)$

b) $\forall a\forall b((a \in G) \wedge (b \in G) \Rightarrow C_{G}^{G}(\varphi_a) \wedge C_{G}^{G}({}_a\varphi) \wedge C_{G}^{G}({}_a\varphi_b))$.

c) $\forall x\forall A((x \in G) \wedge (A \in \tau) \Rightarrow (x + A \in \tau) \wedge (A + x \in \tau) \wedge (-A \in \tau))$.

d) If we denote by $\mathcal{V}_0$ the neighborhood filter of the zero element and by $\mathcal{V}_A$ the neighborhoord filter of the subset $A \subseteq G$ we have:

$$\forall V\forall A((V \in \mathcal{V}_0) \wedge (A \subseteq G) \Rightarrow A + V \in \mathcal{V}_A).^{3}$$

e) If $\mathcal{B}$ is a base of the neighborhood filters of the zero element $0 \in G$, then the family $\mathcal{B} + a = \{X \subseteq G \mid \exists\, B((B \in \mathcal{B}) \wedge (X = B + a))\}$ as well as the family $a + \mathcal{B}$ (defined in a similar manner) is a complete system of neighborhoods of the arbitrary element $a \in G$ moreover:

$$\mathcal{B} = \mathcal{V}_0 \Rightarrow (\mathcal{B} + a = \mathcal{V}_a) \wedge (a + \mathcal{B} = \mathcal{V}_a).$$

f) The topological Ω-*group* $(G, +, \Omega, \tau)$ is a Hausdorff space (separated space) if and only if: $\bigcap \{V \mid V \in \mathcal{B}\} = \{0\}$, whichever be $\mathcal{B}$ a base of $\mathcal{V}_0$.

---

[2] When there is no danger of confusion instead of τ-topological Ω-*group* we shall say only topological Ω-*group*.

[3] If $A = \{a\}$, instead of $\mathcal{V}_{\{a\}}$ we write $\mathcal{V}_a$. We also remark that the symbols V_a, V_a^i $(i = 1, 2, 3, \ldots)$ always represent neighborhoods of the element $a \in G$. As in the above mentioned instead of $V_{\{a\}}$ we write only V_a.

g) $\forall H((H, +, \Omega) < (H, +, \Omega) \Rightarrow (H \in \tau \Rightarrow \overline{H} = H)$,[4] and

$$\forall H((H. +, \Omega) < (G, +, \Omega) \Rightarrow (H \in \tau \Leftrightarrow$$
$$\Leftrightarrow \exists a \exists V_a((a \in H) \wedge (V_a \in \mathcal{V}_a) \wedge (V_a \subseteq H)))).$$

h) Let $(G, +, \Omega, \tau), (G_1, +, \Omega, \tau_1) \in \text{Top } \mathcal{G}_\omega$[5] be and $\varphi : G \to G_1$ an omomorphism. Then we have: $C_G^{G_1}(\varphi) \Leftrightarrow \varphi$ is continuous in an arbitrary point.

l) We have: $(G, +, \Omega, \tau) \in Top\ \mathcal{G}_\omega^{T_2} \Leftrightarrow (G, +, \Omega, \tau) \in Top\ \mathcal{G}_\omega^{T_0}$.

Observation 4. If $(G_1, +, \Omega, \tau_1), (G_2, +, \Omega_1 \tau_2) \in Top\ \mathcal{G}_\omega$, then the set of all continuous omomorphisms $\varphi : G_1 \to G_2$ will be denoted by *Hom* $((G_1, +, \Omega, \tau_1), (G_2, +, \Omega_1 \tau_2))$ or, if there is no danger of confusion, by *Hom* (G_1, G_2). If $\alpha \in Hom\ (G_1, G_2)$ and $\beta \in Hom\ (G_2, G_3)$ referring to the usual composition of mappings we have $\beta \circ \alpha \in Hom\ (G_1, G_3)$. Moreover for each $(G, +, \Omega, \tau) \in Top\ \mathcal{G}_\omega$ identic mapping $1_G : G \to G$ is an element of *Hom* (G, G). So *Top* $\mathcal{G}_\omega$ together with the class of all continuous omomorphisms *Hom* $(\mathcal{G}_\omega)$ of the topological Ω — *groups* belonging to a primitive class form a category that will be called the *category of topological* Ω-*groups.*

Let *Top* $\mathcal{G}_\omega^{T_2}$ be the class of the topological Ω-*groups* belonging to a certain primitive class and supplying the axiom of Hausdorff for separability.

We consider the complete subcategory supplied by *Top* $\mathcal{G}_\omega^{T_2}$ in the category of the topological Ω-*groups*. This category will be called the *category of Hausdorff* Ω-*groups.*

The observations 1. and 2. have the result that each Ω-*group* (especially group, ring, group with operators etc.) serves as an example for a topological Ω-*group*. In the same way each topological group, topological ring, etc., is an example for the topological Ω-*group*.

Another example for the topological Ω-*group* can be obtained in the following way: let **R** be the set of real numbers with usual topology. Of course, **R** referring to the addition of real numbers is a topological group. For each $n \in \mathbf{N} = \{1, 2, 3, \ldots\}$ we consider the set of any continuous mappings of $\mathbf{R}_n$ in **R** "passing through the origin":

$$\Omega(n) = \{f : \mathbf{R}^n \to \mathbf{R} \mid C_{\mathbf{R}^n}^{\mathbf{R}}(f) \wedge f(0, 0, \ldots, 0) = 0\}. \text{ Let } \Omega = \bigcup\{\Omega(n) \mid n \in \mathbf{N}\}$$

be. So we obtain a topological Ω-*group*.

§ 2. Elementary Properties of the Topological Ω-groups. After the definition of the topological Ω-*group* we gave a list of some evident properties linked by this notion. These properties were independent by the third

[4] If $A \subseteq G$, $\overline{A}$ means its topological closure. If $(G, +, \Omega)$ is an arbitrary Ω-group, the symbol: $(H, +, \Omega) < (G, +, \Omega)$ means that $(H, +, \Omega)$ is an Ω-subgroup of $(G, +, \Omega)$, and $(H, +, \Omega)\, \Delta\, (G, +, \Omega)$ means that $(H, +, \Omega)$ is the ideal of $(G, +, \Omega)$.

[5] In the following by $\mathcal{G}_\omega$ we denote — the class of Ω-groups belonging to a certain primitive class. Top $\mathcal{G}_\omega$ will represent the class of the topological Ω-*groups* belonging to a primitive class and *Top* $\mathcal{G}_\omega^{T_i}$ $(i = 0, 1, 2, 3, 4)$ represents the class of topological Ω-groups satisfying the axiom T_i $(i = 0, 1, 2, 3\ 4)$ and belonging to a primitive class.

axiom of the topological Ω-*group*, thus they are trivially true. In the following we transpose for the topological Ω-*group* the most important properties of the topological groups and topological rings, that do not trivially result from the definition 1.1. and which will be necessary and useful for the presentation of the results included in the other parts of the work.

LEMMA 2.1. *Let* $(G, +, \Omega, \tau) \in Top\, \mathcal{G}_\omega$ *and* $(G_1, +, \Omega) < (G, +, \Omega)$. *Then the relative topology* τ_{G_1} *for* G_1 *is compatible with the structure of* Ω-*group for* G_1.

Proof: Because $(G_1, +, \Omega) < (G, +, \Omega)$ we have $(G_1, +, \Omega) < (G, +)$ and thus ([3], [11], [12], [19].) topology τ_{G_1} is compatible with the structure of additive group for G_1. We also have to show the validity of the axiom 3. from the definition of the topological Ω-*group* for the τ_{G_1} topology. But this axiom is true for the topology τ; moreover:

$$\forall G_1 \forall \omega((a(\omega) = n) \wedge (G_1 \subseteq G) \wedge (h^n_\omega(G_1) \subseteq G_1) \wedge C^G_{G^n}(h^n_\omega) \Rightarrow C^{G_1}_{G_1^n}(h^n_{\omega|G_1})),$$

where $h^n_{\omega|G_1}$ is the restriction of the mapping $h^n_\omega : G^n \to G$, for the subset G_1. Because $(G_1, +, \Omega) < (G, +, \Omega)$ we have $\forall \omega(a(\omega) = n \Rightarrow h^n_\omega(G_1) \subseteq G_1)$ from where relying on the above mentioned $\forall \omega(a(\omega) = n \Rightarrow C^{G_1}_{G_1^n}(h^n_\omega|_{G_1}))$ that is exactly what we wanted to prove.

DEFINITION 2.1. *Let* $(G, +, \Omega, \tau) \in Top\, \mathcal{G}_\omega$ *and* $(G_1, +, \Omega) < (G, +, \Omega)$ *be. Then we say that the topological* Ω-*group* $(G_1, +, \Omega, \tau_{G_1})$ *is a topological* Ω-*subgroup of the topological* Ω-*group* $(G, +, \Omega, \tau)$ *and we write:* $(G_1, +, \Omega, \tau_{G_1}) < (G, +, \Omega, \tau)$.

THEOREM 2.2. A) *Let* $(G, +, \Omega) \in \mathcal{G}_\omega$ *be and* $\mathfrak{F}$ *a filter for the set* G *satisfying the following properties:*

1° $\forall \mathrm{U} \exists \mathrm{V}(\mathrm{U} \in \mathfrak{F} \Rightarrow (\mathrm{V} \in \mathfrak{F}) \wedge (\mathrm{V} + \mathrm{V} \subseteq \mathrm{U}))$

2° $\forall \mathrm{U}(\mathrm{U} \in \mathfrak{F} \Rightarrow -\mathrm{U} \in \mathfrak{F})$;

3° $\forall a \forall \mathrm{U}((a \in G) \wedge (\mathrm{U} \in \mathfrak{F}) \Rightarrow a + \mathrm{U} - a \in \mathfrak{F})$

4° $\forall \omega \forall \mathrm{U} \exists \mathrm{V}((a(\omega) = n) \wedge (\mathrm{U} \in \mathfrak{F}) \Rightarrow (\mathrm{V} \in \mathfrak{F}) \wedge (\omega \mathrm{V}\mathrm{V} \ldots \mathrm{V} \subseteq \mathrm{U}))$

5° *The filter* $\mathfrak{F}$ *a base of filter containing only ideals of* $(G, +, \Omega)$ *that is:* $\forall \mathrm{V} \exists \mathrm{U}((\mathrm{V} \in \mathfrak{F}) \Rightarrow (\mathrm{U} \in \mathfrak{F}) \wedge (\mathrm{U} \subseteq \mathrm{V}) \wedge ((\mathrm{U}, +, \Omega) \Delta (G, +, \Omega))$.

Then there exists only one single topology τ *for the set* G *compatible with the structure of the* Ω-*group for* G, *in which* $\mathfrak{F}$ *is the filter of the neighborhoods of the neuter element* $0 \in G$. *For this topology the filter of the neighborhoods of an arbitrary point* $a \in G$ *is each of the filters* $a + \mathfrak{F}$ *or* $\mathfrak{F} + a$.

B) *If* $(G, +, \Omega, \tau) \in \mathrm{Top}\ \mathcal{G}_\omega$ *then the filter* $\mathfrak{F}$ *of the neighborhood of the neuter element* $0 \in G$ *satisfies the conditions* 1°—4°.

Proof. A) We observe first that the conditions 1°—3° asure the existance of a topology τ determined uniquely for the set G which is compatible with the structure of the group for G, and in this topology the filter of the neighborhoords of the neuter element $0 \in G$ is $\mathcal{F}$ and for an arbitrary element $a \in G$ is any of the filters $a + \mathcal{F}$ and $\mathcal{F} + a$ ([3], [12]). We have also to show, that the axiom 3. is also true. For this reason let us consider an arbitrary element $\omega \in \Omega$. Let $a(\omega) = n$ be and $a_1, a_2, \ldots, \ldots, a_n \in G$ be an arbitrary system of elements. In accordance with the above mentioned

$$\forall V \exists U'(V \in \mathcal{V}_{\omega a_1 a_2 \ldots a_n} \Rightarrow (U' \in \mathcal{F}) \wedge (V = \omega a_1 a_2 \ldots a_n + U') \qquad (1)$$

In accordance with the condition 5° results

$$\forall U' \exists U(U' \in \mathcal{F} \Rightarrow (U \in \mathcal{F}) \wedge (U \subseteq U') \wedge ((U, +, \Omega) \Delta (G, +, \Omega))) \quad (2)$$

From (1) and (2) results:

$$\forall V \exists U(V \in \mathcal{V}_{\omega a_1 a_2 \ldots a_n} \Rightarrow (U \in \mathcal{F}) \wedge ((U, +, \Omega) \Delta$$
$$\Delta (G, +, \Omega) \wedge (\omega\, a_1 a_2 \ldots a_n + U \subseteq V)). \qquad (3)$$

From the formula (3) we obtain that:

$$\forall V \exists U(V \in \mathcal{V}_{\omega a_1 a_2 \ldots a_n} \Rightarrow (U \in \mathcal{F}) \wedge ((V \supseteq \omega a_1 a_2 \ldots a_n + U =$$
$$= \omega(a_1 + U)(a_2 + U) \ldots (a_n + U))). \qquad (4)$$

We have that $a_1 + U = V_{a_1} \in \mathcal{V}_{a_1}$, $a_2 + U = V_{a_2}, \ldots, a_n + U = V_{a_n} \in \mathcal{V}_{a_n}$. Consequently from (3) and (4) results:

$$\forall V \exists_1^n V_{a_i}(V \in \mathcal{V}_{\omega a_1 a_2 \ldots a_n} \Rightarrow (V_{a_1} \in \mathcal{V}_{a_1}) \wedge (V_{a_2} \in \mathcal{V}_{a_2}) \wedge \ldots$$
$$(V_{a_n} \in \mathcal{V}_{a_n}) \wedge (\omega V_{a_1} V_{a_2} \ldots V_{a_n} \subseteq V))^{6)}.$$

that is the mapping $h^n_\omega : G^n \to G$ is continued.

B) The conditions 1°—3°. are satisfied by the filter of the neighborhoods $\mathcal{V}_0$ of the neuter element $0 \in G$ in each topological group, especially in each topological Ω-group. In the following, because $\forall\omega\, (a(\omega) = = n \Rightarrow \omega 00 \ldots 0 = 0)$ using axiom 3. from the definition of the topological *Ω-group* for the point $(0, 0, \ldots, 0) \in G^n$ in the case of the Higgins mapping $h^n_\omega : G^n \to G$ we obtain:

$$\forall\omega \forall U \exists_1^n V_i((a(\omega) = n) \wedge (U \in \mathcal{V}_0) \Rightarrow (V_1, V_2, \ldots, V_n \in \mathcal{V}_0) \wedge$$
$$\wedge (\omega V_1 V_2 \ldots V_n \subseteq U).$$

Replacing in this last formula

$$V = \bigcap \{V_i \mid (V_i \in \mathcal{V}_0) \wedge (i \in \{1, 2, \ldots, n\})\} \in \mathcal{V}_0$$

[6] In the place of $\exists V_1 \exists V_2 \ldots \exists V_n$ we write $\exists_i^n V_i$; in the place of $\forall x_1 \forall x_2 \ldots \forall x_n$ we write $\forall_i^n x_i$; and in the place of $(x_1 \in A) \wedge (x_2 \in A) \wedge \ldots \wedge (x_n \in A)$ only $x_1, x_2, \ldots, x_n$.

we obtain:

$$\forall\omega\forall\, \mathrm{U}\, \exists \mathrm{V}((a(\omega) = \mathrm{n}) \wedge (\mathrm{U} \in \mathcal{V}_0) \Rightarrow (\mathrm{V} \in \mathcal{V}_0) \wedge (\omega\mathrm{VV} \ldots \mathrm{V} \subseteq \mathrm{U}),$$

which is identical to the condition 4°.

THEOREM 2.3. *Let* $(G, +, \Omega, \tau) \in Top\, \mathcal{G}_\omega$ *be. Then we have:*

$$\forall \mathrm{H}\, ((\mathrm{H}, +, \Omega) < (\mathrm{G}, + \ \Omega) \Rightarrow (\overline{\mathrm{H}} \ + \ \Omega) < (\mathrm{G} \ + \ \Omega)) \ \textit{and}$$

$$\forall \mathrm{H}((\mathrm{H} \ +, \Omega)\, \Delta\, (\mathrm{G}, +, \Omega) \Rightarrow (\overline{\mathrm{H}}, +, \Omega)\, \Delta\, (\mathrm{G}, +, \Omega)).$$

Proof. Because $(H, +) < (G, +)$ respectively $(H, +)\,\Delta\,(G, +)$ we have ([3], [12], [19]) $(\overline{H}, +) < (G, +)$ respectively $(\overline{H}, +)\,\Delta\,(G, +)$.

In accordance with the axiom 3. of the definition of the topological Ω-group we have

$$\forall\omega\forall\, \mathrm{U}\, \forall_1^n a_i \exists_1^n \mathrm{V}_i((a(\omega) = \mathrm{n}) \wedge (\mathrm{U} \in \mathcal{V}_{\omega a_1 a_2 \ldots a_n}) < (a_1, a_2, \ldots, a_n \in \mathrm{H}) \Rightarrow$$
$$\Rightarrow (\mathrm{V}_{a_1} \in \mathcal{V}_{a_1}) \wedge (\mathrm{V}_{a_2} \in \mathcal{V}_{a_2}) \wedge \ldots \wedge (\mathrm{V}_{a_n} \in \mathcal{V}_{a_n}) \wedge (\omega \mathrm{V}_{a_1}\mathrm{V}_{a_2} \ldots \mathrm{V}_{a_n} \subseteq \mathrm{U}). \tag{5}$$

On the other hand:

$$\forall_1^n a_i \forall_1^n \mathrm{V}_{a_i} \exists_1^n g_i((a_1, a_2, \ldots, a_n \in \overline{H}) \wedge (V_{a_1} \in \mathcal{V}_{a_1}) \wedge (V_{a_2} \in \mathcal{V}_{a_2}) \wedge$$
$$\wedge \ldots \wedge (V_{a_n} \in \mathcal{V}_{a_n}) \Rightarrow (g_1 \in V_{a_1} \cap H) \wedge (g_2 \in V_{a_2} \cap H) \wedge \tag{6}$$
$$\wedge \ldots \wedge (g_n \in V_{a_n} \cap H)),$$

consequently $\omega g_1 g_2 \ldots g_n \in H$, and in accordance with the relation (5) $\omega g_1 g_2 \ldots g_n \in U$, that shows us that $\omega g_1 g_2 \ldots g_n \in H \cap U \neq \Phi$. Thus each neighborhood U of the element $\omega a_1 a_2 \ldots a_n$ intersects with H, which proves that $\omega a_1 a_2 \ldots a_n \in \overline{H}$. Consequently $(\overline{H}, +, \Omega) < (G, +, \Omega)$.

Let $(H, +, \Omega)\,\Delta\,(G, +, \Omega)$ be. Because $(H, +)\,\Delta\,(G, +)$ we have $(\overline{H}, +)\,\Delta\,(G, +)$ and in acordance with the above mentioned $(\overline{H}, +, \Omega) <$ $< (G, +, \Omega)$. Thus we also have to prove:

$$\forall\omega\forall a \forall_1^n g_j \forall i((a(\omega) = n) \wedge (a \in \overline{H}) \wedge (g_1, g_2, \ldots, g_n \in G) \wedge \tag{7}$$
$$\wedge (1 \leqslant i \leqslant n) \Rightarrow (-\omega g_1 g_2 \ldots g_n + \omega g_1 g_2 \ldots g_{i-1}(a + g_i)\, g_{i+1} \ldots g_n \in \overline{H})).$$

Due to the property a) from the observation 3) that followed the definition of the topolological Ω-*group*, we have:

$$\forall\omega\forall a \forall_1^n g_j \forall\, \mathrm{U}\, \exists_1^2 V_k((a(\omega) = n) \wedge (a \in \overline{H}) \wedge (g_1, g_2, \ldots, g_n \in G) \wedge$$
$$\wedge (U \in \mathcal{V}_{-\omega g_1 g_2 \ldots g_n + \omega g_1 g_2 \ldots g_{i-1}(a+g_i)\, g_{i+1} \ldots g_n}) \Rightarrow \tag{8}$$
$$\Rightarrow (V_1 \in \mathcal{V}_{\omega g_1 g_2 \ldots g_n}) \wedge (V_2 \in \mathcal{V}_{\omega g_1 g_2 \ldots g_{i-1}(a+g_i)\, g_{i+1} \ldots g_n}) \wedge (-V_1 + V_2 \subseteq U).$$

From the axiom 3. of the definition of the topological Ω-*group* results the existance of the neighborhoods V_{g_j} $(j = 1, 2, \ldots, i-1, \; i+1, \ldots n)$, V_{a+g_i} for the indicated elements given as indexes, so that we have:

$$\omega V_{g_1} V_{g_2} \ldots V_{g_{i-1}} V_{a+g_i} V_{g_{i+1}} \ldots V_{g_n} \subseteq V_2, \tag{9}$$

where V_2 is the neighborhood which appears in the realtion (8). Because $(G, +, \tau)$ is a topological group we have for the neighborhood V_{a+g_i} from the relation (9):

$$\exists V_a \exists V_{g_i}((V_{a+g} \in \mathcal{V}_{a+g_i}) \Rightarrow (V_a + V_{g_i} \subseteq V_{a+g_i})). \tag{10}$$

From the relations (9) and (10) results:

$$\omega V_{g_1} V_{g_2} \ldots V_{g_{i-1}} (V_a + V_{g_i}) V_{g_{i+1}} \ldots V_{g_n} \subseteq V_2. \tag{11}$$

On the other hand we have: $\forall a \forall V_a \exists a_1((a \in \overline{H}) \wedge (V_a \in \mathcal{V}_a) \Rightarrow a_1 \in H \cap V_a)$, from where in accordance with the relation (11) results:

$$\omega g_1 g_2 \ldots g_{i-1}(a_1 + g_i) g_{i+1} \ldots g_n \in V_2.$$

From (8), using the last relation, results: $\forall \omega \forall a \forall_1^n g_i \forall \; \mathsf{U} \; \exists a_1((a(\omega) = n) \wedge$ $\wedge (a \in \overline{H}) \wedge (g_1, g_2, \ldots, g_n \in G) \wedge (\mathsf{U} \in \mathcal{V}_{-\omega g_1 g_2 \ldots g_n + \omega g_1 g_2 \ldots g_{i-1}}(a + g_i) g_{i+1} \ldots g_n) \Rightarrow$ $\Rightarrow (a_1 \in H) \wedge (-\omega g_1 g_2 \ldots g_n + \omega g_1 g_2 \ldots g_{i-1}(a_1 + g_i) g_{i+1} \ldots g_n \in \mathsf{U}))$. Because $(H, +, \Omega) \, \Delta \, (G, +, \Omega)$ and $a_1 \in H$ we have: $-\omega g_1 g_2 \ldots g_n +$ $+ \omega g_1 g_2 \ldots g_{i-1}(a_1 + g_i) g_{i+1} \ldots g_n \in H$. Thus $U \cap H \neq \Phi$, which proves that each neighborhood of the element

$$\omega g_1 g_2 \ldots g_n + \omega g_1 g_2 \ldots g_{i-1}(a + g_i) g_{i+1} \ldots g_n \in G,$$

intersects with H, so:

$$\forall \omega \forall a \forall_1^n g_j \forall i((a(\omega) = n) \wedge (a \in \overline{H}) \wedge (g_1, g_2, \ldots, g_n \in G) \wedge (l \leqslant i \leqslant n) \Rightarrow$$
$$\Rightarrow -\omega g_1 g_2 \ldots g_n + \omega g_1 g_2 \ldots g_{i-1}(a + g_i) g_{i+1} \ldots g_n \in \overline{H}).$$

that is $(\overline{H}, +, \Omega) \, \Delta \, (G, +, \Omega)$.

COROLLARY. *Let* $(G, +, \Omega, \tau) \in Top \, \mathfrak{L}_\omega$ *be and* $(\{0\}, +, \Omega)$ *be ideals zero of the* Ω-*group. Then we have*: $(\overline{\{0\}}, +, \Omega) \, \Delta \, (G, +, \Omega)$.

§ 3. Product Topology of the Ω-groups. Let $\{(G_\nu, +, \Omega, \tau_\nu \mid \nu \in \Delta\}$ be a family of topological Ω-*groups* where Δ is an arbitrary set of indexes. For the cartesian product $\Sigma \{G_\nu \mid \nu \in \Delta\}$ we consider the structure of the Ω-*group* and the structure of the product topology. We shall denote this product topology by τ. We have.

THEOREM 3.1. $(\Sigma\{G_\nu \mid \nu \in \Delta\}, +, \Omega, \tau) \in Top \, \mathcal{G}_\omega$.

Proof. We know that $(\Sigma\{G_\nu \mid \nu \in \Delta\}, +, \tau)$ is a topological group. We also have to show that for all $\omega \in \Omega(n)$ Higgins mapping $\overline{h_\omega^n}: (\Sigma G_\nu)^n \rightarrow$

$\to \Sigma G_\nu$ is continuous. In accordance with [12] for this is necessary and sufficient to have: $\forall \mu(\mu \in \Delta \Rightarrow C^{G_\mu}_{(\Sigma G_\nu)^n}(P_\mu \circ \overline{h^n_\omega})$, where $P_\mu : \Sigma G_\nu \to G_o$ is the canonical mapping.

It is sufficient to show that the diagram:

$$\begin{array}{ccc} (\Sigma G_\nu)^n & \xrightarrow{\overline{h^n_\omega}} & \Sigma G_\nu \\ & & \Big\downarrow P_\mu \\ (G_\mu)^n & \xrightarrow{h^n_\omega} & G_\mu \end{array}$$

is closed commutatively by a continuons mapping: $\overline{P}_\mu : (\Sigma G_\nu)^n \to (G_\mu)^n$. Let $((g_\nu^{(1)})_{\nu\in\Delta}, (g_\nu^{(n)})_{\nu\in\Delta}, \ldots, (g_\nu^{(n)})_{\nu\in\Delta})$ be an arbitrary element of $(\Sigma G_\nu)^n$. Thorougly the definition: $\overline{P}_\mu((g_\nu^{(1)})_{\nu\in\Delta}(g_\nu^{(2)})_{\nu\in\Delta}, \ldots, (g_\nu^{(n)})_{\nu\in\Delta})) = (g_\mu^{(1)}, g_\mu^{(2)}, , \ldots, g_\mu^{(n)})$, for each $\mu \in \Delta$. We prove that this mapping closes commutatively the above mentioned diagram. Indeed:

$$(P_\mu \circ \overline{h^n_\omega})((g_\nu^{(1)})_{\nu\in\Delta}, (g_\nu^{(2)})_{\nu\in\Delta}, \ldots, (g_\nu^{(n)})_{\nu\in\Delta}) =$$

$$= P_\mu(\overline{h^n_\omega}((g_\nu^{(1)})_{\nu\in\Delta}, (g_\nu^{(2)})_{\nu\in\Delta}, \ldots, (g_\nu^{(n)})_{\nu\in\Delta}) =$$

$$= P_\mu((\omega g_\nu^{(1)} g_\nu^{(2)} \ldots g_\nu^{(n)})_{\nu\in\Delta}) = \omega g_\mu^{(1)} g_\mu^{(2)} \ldots g_\mu^{(n)} =$$

$$= h^n_\omega((g_\mu^{(1)}, g_\mu^{(2)}, \ldots, g_\mu^{n})) = h^n_\omega(\overline{P}_\mu((g_\nu^{(1)})_{\nu\in\Delta}, (g_\nu^{(2)})_{\nu\in\Delta}, \ldots (g_\nu^{(n)})_{\nu\in\Delta}) =$$

$$= (h^n_\omega \circ \overline{P}_\mu)((g_\nu^{(1)})_{\nu\in\Delta}, (g_\nu^{(2)})_{\nu\in\Delta}, \ldots, (g_\nu^{(n)})_{\nu\in\Delta});$$

We also have to show that the mapping $\overline{P}_\mu$ is continuons for all $\mu \in \Delta$. In accordance with [12] is sufficient to prove that the mapping $P \circ \overline{P}_\mu$: $(\Sigma G_\nu)^n \to G_\mu$ is continuous, where $P : (G_\mu)^n \to G_\mu$ is the projection mapping, for instance after the first component. For this reason let $V \in \tau_\mu$ be. Then we have:

$$(P \circ \overline{P}_\mu)^{-1}(V) = ((\overline{P}_\mu)^{-1} \circ)(V) = (\overline{P}_\mu)^{-1}(P^{-1}(V)) =$$

$$= (\overline{P}_\mu)^{-1}(\{g_\mu^{(1)}, g_\mu^{(2)}, \ldots, g_\mu^{(n)} | (g_\mu^{(1)} \in V) \wedge (g_\mu^{(2)}, g_\mu^{(3)}, \ldots, g_\mu^{(n)} \in G_\mu)\}) =$$

$$= \{((g_\nu^{(1)})_{\nu\in\Delta}, (g_\nu^{(2)})_{\nu\in\Delta}, \ldots, (g_\nu^{(n)})_{\nu\in\Delta}) \mid (g_\mu^{(1)} \in V) \wedge (g_\nu^{(1)}) \in G_\nu (\nu \neq \mu)) \wedge$$

$$\wedge (g_\nu^{(2)}, g_\nu^{(3)}, \ldots, g_\nu^{(n)} \in G_\nu)\}.$$

But in accordance with [12] this set is open in the product topology for $\Sigma\{G_\nu \mid \nu \in \Delta\}$.

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UNELE PROPRIETĂŢI ALE Ω-GRUPURILOR TOPOLOGICE (I–II)

(Rezumat)

Noţiunea de Ω-*grup* topologic este o generalizare firească a noţiunilor de grup topologic şi de inel topologic. Lucrarea de faţă studiază unele proprietăţi legate de Ω-*grupurile* topologice: în prima parte dăm definiţia Ω-*grupului* topologic şi cîteva exemple pentru această noţiune. Tot aici transcriem cîteva proprietăţi ale grupurilor şi inelelor topologice, care se transpun în mod firesc pentru cazul Ω-*grupurilor* topologice. În § 2. stabilim cîteva proprietăţi elementare legate de noţiunea de Ω-*grup* topologic. Topologia produs în cazul Ω-*grupurilor* topologice este subiectul § 3. În § 4. studiem topologia Ω-*grupurilor* cît în cadrul Ω-*grupurilor* topologice. Este interesantă topologia Ω-*grupurilor* cînd avem un sistem complet de vecinătăţi al elementului zero, care conţine numai ideale. Acest caz este descris în § 5. § 6. conţine descrierea teoremelor de izomorfism pentru cazul Ω-*grupurilor* topologice.

НЕКОТОРЫЕ СВОЙСТВА ТОПОЛОГИЧЕСКИХ Ω-ГРУПП (I–II)

(Резюме)

Понятие топологический Ω-*группы* является естественным обобщением понятий топологической группы и топологического кольца. В статье изучаются некоторые свойства, связанные с топологическими Ω-*группами*; в первой части статьи автор даёт определение топологической Ω-*группы* и несколько примеров для этого понятия. Переписаны также некоторые свойства топологических групп и колец, которые перестанавливаются естественным образом для случая топологических Ω-*групп*. В § 2 устанавливаются некоторые элементарные свойства, связанные с понятием топологической Ω-*группы*. Топология-произведение в случае топологических Ω-*групп* составляет предмет § 3. В § 4 изучается топология Ω-*фактор-групп* в рамках топологических Ω-*групп*. Интересна топология Ω-*групп* тогда, когда имеем полную систему окрестностей нулевого элемента, которая содержит только идеалы. Этот случай описан в § 5. В § 6 описаны теоремы изоморфизма для случая топологических Ω-*групп*.

SUR LES ESPACES À CONNEXION AFFINE AVEC TORSION RÉCURRENTE

P. ENGHIŞ

Soit A_n un espace à connexion affine non-symétrique, Γ^i_{jk} les composants de sa connexion affine et $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$ le tenseur de torsion. En contractant le tenseur de torsion T^i_{jk} en i et j on obtient le vecteur de torsion $T_k = T^i_{ik}$. Du tenseur de torsion, par produit tensoriel et produit tensoriel contracté, on déduit [1] les tenseurs

$$T^i_{jh} \cdot T^k_{lr}\,;\ T^i_{jk} \cdot T^r_{il}\,;\ T^i_{jk} \cdot T^s_{si}\,;\ T_{kl} = T^i_{jk} \cdot T^j_{il} \tag{1}$$

S'il existe dans l'espace A_n un vecteur covariant φ_r tel que

$$T^i_{jk,\,r} = \varphi_r \cdot T^i_{jk} \tag{2}$$

où la virgule désigne la dérivée covariante par raport à Γ^i_{jk}, on dit que l'espace A_n est avec torsion récurrente.

De la relation (2) par contraction en i et j on obtient :

$$T_{k,\,r} = \varphi_r \cdot T_k \tag{3}$$

donc :

1. Dans un espace A_n avec torsion récurrente le vecteur de torsion est récurrent avec le même vecteur de récurrence.

En prenant la dérivée covariante des tenseurs donnés par la relation (1) on déduit :

2. Les tenseurs obtenus du tenseur de torsion par produit tensoriel et produit tensoriel contracté, dans un espace A_n avec torsion récurrente de vecteur φ_r, sont récurrents, de vecteur $2\varphi_r$.

En tenant compte de ce résultat et du fait que le dernier des tenseurs de la relation (1) est aussi symétrique, on obtient :

3. Les espaces A_n avec torsion récurrente sont des espaces de Weyl généralisés [3].

Dans un travail antérieur [4] j'ai introduit les espaces $\widetilde{A}_n$ qui sont les espaces avec connexion affine A_n pour lesquels la différentielle extérieure de la forme de *Pfaff* σ, associée au vecteur de torsion T_k, $\sigma = T_k dx^k$ satisfait à la relation :

$$D\sigma = \frac{1}{2} T_i \cdot T^i_{hk}[dx^h dx^k] \tag{4}$$

Supposons maintenant que les espaces $\widetilde{A}_n$ sont avec torsion récurrente. De la relation

$$T_{i,j} - T_{j,i} = 0 \tag{5}$$

qui caractérise les espaces $\widetilde{A}_n$ [4 théorème 1], si on tient compte de (3) on déduit :

$$\varphi_j \cdot T_i = \varphi_i \cdot T_j \tag{6}$$

de laquelle il résulte :

$$\varphi_i = \alpha T_i \tag{7}$$

où $\alpha = \alpha(x^i)$; donc :

4. Dans un espace $\widetilde{A}_n$ avec torsion récurrente le vecteur de torsion est proportionnel au vecteur de récurrence.

De (6) on déduit que le tenseur

$$\Phi_{ij} = \varphi_i \cdot T_j \tag{8}$$

est symétrique. En prenant la dérivée covariante de (8) on obtient :

$$\Phi_{ij,r} = \left(\frac{d(ln\alpha)}{dx^r} + 2\varphi_r\right) \Phi_{ij} \tag{9}$$

donc :

5. Dans un espace $\widetilde{A}_n$ avec torsion récurrente le tenseur $\Phi_{ij} = \varphi_i \cdot T_j$ est symétrique et récurrent de vecteur de récurrence

$$\omega_r = \frac{d(ln\alpha)}{dx^r} + 2\varphi_r$$

Dans le travail cité plus haut [4] j'ai montré que le tenseur du second membre de la relation (4) de définition des espaces $\widetilde{A}_n$ est donné par

$$T_i \cdot T^i_{jk} = \Omega_{kj} \tag{10}$$

où Ω_{kj} est le tenseur contracté du tenseur Ω^i_{hkj} qui intervient dans l'expression du tenseur de courbure Γ^i_{hkj} de la connexion Γ^i_{jk}, [1], [2].

$$\Gamma^i_{hkj} = S^i_{hkj} + \frac{1}{2} \Omega^i_{hkj} \tag{11}$$

S^i_{hkj} étant le tenseur de courbure de la connexion symétrique $S^i_{jk} = \frac{1}{2} (\Gamma^i_{jk} + \Gamma^i_{kj})$ associée à la connexion Γ^i_{jk}.

En tenant compte de (2) on obtient:

6. Dans un espace $\widetilde{A}_n$ avec torsion récurrente le tenseur Ω_{kj} est récurrent de vecteur de récurrence $2\ \varphi_r$.

De la relation (2), (7) et (10) on déduit que:

7. Dans un espace $\widetilde{A}_n$ avec torsion récurrente nous avons

$$\Omega_{kj} = \frac{1}{\alpha} \cdot T^s_{jk,\,s} \tag{12}$$

Si dans la relation connue [1], qui existe entre le tenseur de courbure et le tenseur de torsion

$$\Gamma^i_{jkh} + \Gamma^i_{khj} + \Gamma^i_{hjk} = T^i_{jh,\,k} + T^i_{hk,\,j} + T^i_{kj,\,h} + T^s_{jh}T^i_{ks} + T^s_{hk}T^i_{js} + T^s_{kj}T^i_{hs} \tag{13}$$

dans un espace A_n, on tient compte de (2) et (7), on obtient:

8. Dans un espace $\widetilde{A}_n$ avec torsion récurrente, entre le tenseur de courbure et le tenseur de torsion il existe la relation

$$\begin{aligned} \Gamma^i_{jkh} + \Gamma^i_{khj} + \Gamma^i_{hjk} = \left(\alpha T_k \delta^s_h + T^s_{hk}\right) T^i_{js} + \left(\alpha T_h \delta^s_j + T^s_{jh}\right) T^i_{ks} + \\ + \left(\alpha T_j \delta^s_k + T^s_{kj}\right) T^i_{hs} \end{aligned} \tag{14}$$

De la relation (14) en contractant en i et k on déduit

$$\Gamma_{jh} - \Gamma_{hj} + R_{hj} = (\alpha + 1)\Omega_{hj} \tag{15}$$

où $\Gamma_{jh} = \Gamma^i_{jih}$ et $R_{hj} = \Gamma^i_{ihj}$, relation qui existe entre les tenseurs contractés du tenseur de courbure de l'espace $\widetilde{A}_n$ avec torsion récurrente.

Remarque. Si l'espace $\widetilde{A}_n$ est un espace riemannien doué d'une connexion asymétrique, de la relation (12) et de la relation (1,14) du travail antérieurement cité [4] il résulte que α est une constante égale à deux.

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ASUPRA SPAŢIILOR CU CONEXIUNE AFINĂ CU TORSIUNE RECURENTĂ

(R e z u m a t)

În lucrare se consideră spaţiile cu conexiune afină cu torsiune recurentă. Se dau cîteva proprietăţi generale (teoremele 1, 2, 3), apoi se consideră spaţiile $\widetilde{A}_n$ definite de (4) pentru care se dau de asemenea cîteva proprietăţi în cazul cînd torsiunea este recurentă (teoremele 4, 5, 6, 7 şi 8).

О ПРОСТРАНСТВАХ С АФФИННОЙ СВЯЗНОСТЬЮ С РЕКУРРЕНТНЫМ КРУЧЕНИЕМ

(Р е з ю м е)

Рассматриваются пространства с аффинной связностью с рекуррентным кручением. Даются некоторые общие свойства (теоремы 1, 2, 3), затем рассматриваются пространства $\widetilde{A}_n$, определённые при помощи (4), для которых даются также некоторые свойства в том случае, когда кручение является рекуррентным (теоремы 4, 5, 6, 7 и 8).

PROPRIÉTÉS DE L'INTÉGRALE M_* DANS DES ÉSPACES LINÉAIRES

MARCEL RĂDULESCU

1. On considère un espace linéaire X ayant comme corps de scalaires le corps $\mathbf{R}$ des nombres réels. Soient $u, v \in X$; on note par $[u, v]$, le segment d'extrémités u, v c'est à dire l'ensemble des points $z \in X : z = h(t) = tu + (1 - t)v$, $t \in [0, 1]$. Nous notons pour un ensemble $A \subset [u, v]$, par $t_A \subset [0, 1]$, l'ensemble des valeurs du paramètre $t \in [0, 1]$, de sorte que $h(t) \in A$, $t \in t_A$, et par t_z la valeur du paramètre t pour laquelle $h(t_z) = z$. L'ensemble A est mesurable si t_A est mesurable et $m(A) = m(t_A)$. Une base $B \subset [u, v]$ est un ensemble pour lequel $u, v \in B$ et $m(B) = 1$. Un système de points $D(z_1, z_2, \ldots, z_k)$ de sorte que $z_1 = u$, $z_k = v$, $z_i \in B$, $i \in \{1, 2, \ldots, k\}$ et $t_{z_i} < t_{z_{i+1}}$, $i \in \{1, 2, \ldots, k - 1\}$, sera nommé une division de la base B. La classe des divisions d'une base B sera notée par $\mathcal{K}_B$. Si on écrit $t_i = t_{z_i}$, $z_i \in D$, $i \in \{1, 2, \ldots, k\}$, le nombre

$$\delta = \delta(D) = \max_{1 \leqslant i \leqslant k-1} (t_{i+1} - t_i)$$

sera nommé norme de la division, D. Soient $D_1, D_2 \in \mathcal{K}_B$, et $\delta(D_1) > \delta(D_2)$, on dira que D_1 est moins fine en norme que D_2, et on écrira $D_1 <_\delta D_2$. Si $\delta(D_1) \geqslant \delta(D_2)$, on écrira $D_1 \leqslant_\delta D_2$.

Un système de paires de points $d : \langle x_i, y_i \rangle$, $i \in \{1, 2, \ldots, k\}$ de sorte que $x_i, y_i \in B$, $t_{x_i} < t_{y_i}$ et

$$(t_{x_i}, t_{y_i}) \cap (t_{x_j}, t_{y_j}) = \emptyset, \; i \neq j \tag{1,1}$$

sera nommé une sous-division de la base B. Si k est un nombre fini la sous-division d est finie. Les nombres

$$\delta = \delta(d) = \sup_{1 \leqslant i \leqslant k} (t_{y_i} - t_{x_i})$$

$$\Delta = \Delta(d) = \sum_{i=1}^{k} (t_{y_i} - t_{x_i})$$

seront nommés la norme de la sous-division et la norme absolue de la sous-division. La classe des sous-divisions d'une base B sera notée $\mathcal{H}_B^*$. Soient $d_1, d_2 \subset \mathcal{H}_B^*$; si de $\langle x_i^1, y_i^1\rangle \in d_1$ il résulte que $\langle x_i^1, y^1\rangle \in d_2$, alors $d_1 \subset d_2$. Dans le cas où $d_1 \subset d_2$ et $d_2 \subset d_1$, on a $d_1 = d_2$. On considère deux sous-divisions $d_1, d_2 \in \mathcal{H}_B^*$; si $\Delta(d_1) > \Delta(d_2)$, on dit que d_1 est moins fine que d_2 en norme absolue, et on écrit $d_1 <_\Delta d_2$. Si $\Delta(d_1) \geqslant \Delta(d_2)$ on écrira $d_1 \leqslant_\Delta d_2$. À une sous-division d: $\langle x_i, y_i\rangle$, $i \in \{1, 2, \ldots, k\}$, il correspond un ensemble ouvert

$$e_d = \bigcup_{i=1}^{k} (t_{x_i}, t_{y_i}) \tag{1, 2}$$

et aussi

$$C_d = \bigcup_{i=1}^{k} \{t_{x_i}, t_{y_i}\}$$

Réciproquement, à un ensemble ouvert du type (1, 2) pour lequel les relations (1,1) sont satisfaites, il correspond une sous-division d de la base B. Soient $d_1, d_2 \in \mathcal{H}_B^*$, alors la sous-division $d = d_1 \cup d_2$ pour laquelle $e_d = e_{d_1} \cup e_{d_2} - C_{d_1} - C_{d_2}$, sera nommée la réunion des sous-divisions d_1, d_2, et la sous-division $d = d_1 \cap d_2$ pour laquelle $e_d = e_{d_1} \cap e_{d_2}$, sera nommée l'intersection des sous-divisions d_1, d_2. Si $e_{d_1} \cap e_{d_2} = \emptyset$, on dit que d_1 d_2 sont disjointes et on écrit $d_1 \cap d_2 = \emptyset$.

Deux couples $\langle x_i, y_i\rangle$, $\langle x_j, y_j\rangle \in d$ sont liés si $x_i = y_j$ ou $y_i = x_j$. Une sous-division $\overline{d} \subset d$ est une composante de la sous-division d si tout couple $\langle x_j, y_j\rangle \in \overline{d}$, est lié à un couple au moins de la sous-division $\overline{d}$ et n'importe quel couple $\langle x_i, y_i\rangle \in d$, $\langle x_i, y_i\rangle \notin \overline{d}$ n'est lié à aucun couple de la sous-division $\overline{d}$. Les composantes d'une sous-division seront, dans tous les cas finis, tels que les couples d'une composante $\overline{d}$ peuvent être nummérotés de sorte que $y_j = x_{j+1}$, $j \in \{1, 2, \ldots, h-1\}$. Soient $\overline{d}_i : \langle x_j^i, y_j^i\rangle$, $j \in \{1, 2, \ldots, h_i\}$, $i \in \{1, 2, \ldots, k_r\}$ les composantes de la sous-division d. On notera par d_r la sous-division réduite $d_r : \langle x_1^i, y_{h_i}^i\rangle$, $i \in \{1, 2, \ldots, k_r\}$. On démontre (voir [3], [4]) que la relation

$$(d_1 \cup d_2)_r = [(d_1)_r \cup (d_2)_r]_r \tag{1, 3}$$

a lieu. Etant données $d_1, d_2 \in \mathcal{H}_B^*$, s'il existe une sous-division $d_3 \neq \emptyset$ de sorte que $d_3 = \bigcup_{i=1}^{m} d_i^3$, $d_2 = d_1 \cup d_3$ et $(d_i^3)_r \subset d_1$, $i \in \{1, 2, \ldots, m\}$, on dit que d_1 est moins fine que d_2 et on écrit $d_1 \leqslant d_2$. En utilisant la relation (1,3), il résulte de $d_1 \leqslant d_2$ que $(d_1)_r = (d_2)_r$. On peut considérer, en particulier, la relation de finesse entre les divisions $D_1 \leqslant D_2$.

Soit B une base du segment $[u, v]$; alors le couple $[\mathcal{H}_B^*, \leqslant_\delta]$ constitue un ensemble dirigé (voir [1], [2]). Les conditions de réflexivité et de transitivité de la relation $\leqslant_\delta$ sont évidentes. Et pour $D_1, D_2 \in \mathcal{H}_B$, ont lieu les relations: $D_1 \cup D_2 \in \mathcal{H}_B$, $D_1 \leqslant_\delta D_1 \cup D_2$, $D_2 \leqslant_\delta D_1 \cup D_2$. On remarque de même que le couple $[\mathcal{H}_B^*, \leqslant_\Delta]$ est un ensemble dirigé. Dans ce cas, $d \leqslant_\Delta d$, et il résulte de $d_1 \leqslant_\Delta d_2$, $d_2 \leqslant_\Delta d_3$ que $d_1 \leqslant_\Delta d_3$. Nous avons, pour $d_1, d_2 \in \mathcal{H}_B^*$, $d_1 \cap d_2 \in \mathcal{H}_B^*$ et $d_1 \leqslant_\Delta d_1 \cap d_2$, $d_2 \leqslant_\Delta d_1 \cap d_2$.

Considérons un ensemble dirigé $[Q, \prec]$ et une application $\varphi : Q \to \mathcal{H}_B$. On peut considérer la suite généralisée $[\delta[\varphi(q)], q \in Q, \leqslant]$. Si cette suite est convergente et à la limite zéro,

$$\lim_q \delta[\varphi(q)] = 0, \tag{1, 4}$$

on dit que la suite généralisée $[\varphi(q), q \in Q, \prec]$ est admise.

2. On considère un espace linéaire topologique T, ayant comme corps de scalaires le corps $\mathbf{R}$ des nombres réels. Soit B une base du segment $[u, v]$, de l'espace linéaire X. Pour une application $f : B \to T$ et $D(z_1, z_2, \ldots, z_k)$ une division de la base B, la somme

$$\sigma(f, D) = \sum_{i=1}^{k-1} \frac{f(z_i) + f(z_{i+1})}{2} (t_{i+1} - t_i)$$

est notée par $\sigma(f, D)$. De même, pour $d : \langle x_i, y_i \rangle$, $i \in \{1, 2, \ldots, k\}$, une sous-division de la base B s'écrit

$$\sigma(f, d) = \sum_{i=1}^{k} \frac{f(x_i) + f(y_i)}{2} (t_{y_i} - t_{x_i}).$$

Dans ce cas, $\sigma(f, D)$ est une application $\sigma : \mathcal{H}_B \to T$ et $\sigma(f, d)$ l'application $\sigma : \mathcal{H}_B^* \to T$. Parce que $[\mathcal{H}_B, \leqslant_\delta]$ est un ensemble dirigé, on peut considérer la suite généralisée $[\sigma, \mathcal{H}_B, \leqslant_\delta] = [\sigma(f, D), D \in \mathcal{H}_B, \leqslant_\delta]$. Dans ce qui suit on va noter par $[\tau(q), q \in Q, \prec] = (\tau, Q, \prec)$, $\tau : Q \to T$ une sous-suite généralisée de la suite généralisée $[\sigma, \mathcal{H}_B, \leqslant_\delta]$. Dans ce cas (voir [2]) il existe une application $\varphi : Q \to \mathcal{H}_B$ de sorte que $\tau = \sigma \circ \varphi$ et pour tout $D \in \mathcal{H}_B$ il existe $q_D \in Q$ tel que si $q_D \prec q$, alors

$$D \leqslant_\delta \varphi(q). \tag{2,1}$$

Définition 2.1. Si la suite généralisée $[\sigma, \mathcal{H}_B, \leqslant_\delta]$, est convergente on dit que l'application $f : B \to T$ est intégrable M_* sur B. Dans ce cas, la limite de cette suite généralisée est nommée intégrale M_* et on écrira

$$\lim_D \sigma(f, D) = (M_*) \int_B f(z)\, dz.$$

Il résulte des propriétés de la suite généralisée convergente que le théorème suivant est vérifié.

Théorème 2.1. *La condition nécessaire et suffisante pour que l'application $f : B \to T$ soit intégrable M_* est l'existence du point $I_B \in T$ tel que pour* *chaque voisinage* $V(I_B)$ *du point* I_B, *il existe* $\eta \in \mathbf{R}^+$ *tel que pour chaque* $D \in \mathcal{H}_B$, $\delta(D) \leqslant \eta$ *a lieu l'appartenance* $\sigma(f, D) \in V(I_B)$.

THÉORÈME 2.2. *Soient* B_1 *et* B_2 *deux bases du segment* $[u, v]$ *et l'application* $f: B \to T$, *où* $B = B_1 \cup B_2$. *Si les restrictions* $f_{B_1}: B_1 \to T$, *et* $f_{B_2}: B_2 \to T$ *sont intégrables* M_*, *alors*

$$(M_*) \int_{B_1} f(z)dz = M_*) \int_{B_2} f(z)dz.$$

Démonstration. La suite généralisée $[\sigma, \mathcal{H}_B, \leqslant_\delta]$, où $B = B_1 \cap B_2$ est une sous-suite généralisée de la suite $[\sigma, \mathcal{H}_{B_1}, \leqslant_\delta]$ et de la suite généralisée $[\sigma, \mathcal{H}_{B_2}, \leqslant_\delta]$. Dans les deux cas, l'application qui détermine les sous-suites généralisées est l'identité $\varphi(D) = D$, $D \in \mathcal{H}_B$. Alors, par hypothèse

$$(M_*) \int_{B_1} f(z)dz = \lim_D \sigma(f, D) = (M_*) \int_{B_2} f(z)dz.$$

THÉORÈME 2.3. *La condition nécessaire et suffisante pour qu'une suite généralisée* $(\tau, Q, \prec)$, $\tau : Q \to T$ *soit une sous-suite généralisée de la suite généralisée* $[\sigma, \mathcal{H}_B, \leqslant_\delta]$ *est qu'il existe une application* $\varphi : Q \to \mathcal{H}_B$, *de sorte que la suite généralisée* $[\varphi, Q \prec]$ *soit admise.*

Démonstration. Soit $(\tau, Q, \prec)$, $\tau : Q \to T$ une sous-suite généralisée de la suite généralisée $[\sigma, \mathcal{H}_B, \leqslant_\delta]$. À un $\varepsilon > 0$ il correspond une division $D_\varepsilon \in \mathcal{H}_B$ de sorte que $\delta(D_\varepsilon) < \varepsilon$. Conformément à (2,1) il existe $q_\varepsilon \in Q$ de sorte que pour n'importe quel $q \in Q$, $q_\varepsilon \prec q$, $D_\varepsilon \leqslant_\delta \varphi(q)$; autrement dit: $\delta[\varphi(q)] \leqslant \delta(D_\varepsilon)) < \varepsilon$. Il résulte que la suite généralisée $[\delta[\varphi(q)]$, $q \in Q$, $\leqslant]$, se trouve éventuellement dans le voisinage V_ε de l'origine et la relation (1, 4) est donc vérifiée.

Réciproquement, étant donné $(\tau, Q, \prec)$, $\tau : Q \to T$, supposons qu'il existe l'application $\varphi : Q \to \mathcal{H}_B$ de sorte que $[\varphi, q \in Q \prec]$ soit admise. Soit $D \in \mathcal{H}_B$, et $\varepsilon = \delta(D)$; la suite généralisée $[\delta[\varphi(q)], q \in Q, \leqslant]$ se trouve éventuellement dans le voisinage V_ε de l'origine. Il existe donc q_ε, de sorte que pour n'importe quel $q \in Q$, $q_\varepsilon \prec q$, $\delta[\varphi(q)] \in V_\varepsilon$, et par conséquent $\delta(\varphi(q)] \leqslant \varepsilon = \delta(D)$. Il résulte que, pour $q \in Q$, $q_\varepsilon \prec q$, nous avons $D \leqslant_\delta \varphi(q)$.

Si $f: B \to T$ est une application intégrable M_* alors, l'application $Cf: B \to T$, où C est un nombre réel, est intégrable M_* et l'égalité

$$(M_*) \int_B Cf(z)dz = C(M_*) \int_B f(z)dz \tag{2,2}$$

a lieu. Cette propriété est une conséquence de la relation $\sigma(Cf, D) = C\sigma(f, D)$, $D \in \mathcal{H}_B$ et des propriétés des suites généralisées convergentes.

Si $f_1 : B_1 \to T$, $f_2 : B_2 \to T$, où B_1, B_2 sont des bases du segment $[u, v]$ sont des applications intégrables M_*, alors l'application $f_1 + f_2 : B \to T$, $B = B_1 \cap B_2$ est intégrable M_* et l'égalité

$$(M_*) \int_B [f_1(z) + f_2(z)]dz = (M_*) \int_{B_1} f_1(z)dz + (M_*) \int_{B_2} f_2(z)dz \tag{2,3}$$

a lieu. En effet

$$\sigma(f_1 + f_2, D) = \sigma(f_1, D) + \sigma(f_2, D), \ D \in \mathcal{H}_B$$

et

$$\lim_D \sigma(f_1 + f_2, D) = \lim_D \sigma(f_1, D) + \lim_D \sigma(f_2, D).$$

On considère $w - h_1$, w, $w + h_2 \in B$, $t_{w-h_1} < t_w < t_{w+h_2}$. Soit d_w: $: \langle w - h_1, w \rangle$; $\langle w, w + h_2 \rangle$. On notera par $\mathcal{H}^*_{Bw}$ la classe des sous-divisions de la forme précédente. Alors, le couple $[\mathcal{H}^*_{Bw}, \leqslant_\Delta]$ est un ensemble dirigé. La vérification des propriétés de réflexivité et de transitivité est immédiate. Pour vérifier la troisième propriété, on remarque que de $d'_w, d''_w \in \mathcal{H}^*_{Bw}$, il résulte $d'_w \cap d''_w \in \mathcal{H}^*_{Bw}$ et $d'_w \leqslant_\Delta d'_w \cap d''_w$, $d''_w \leqslant_\Delta d'_w \cap d''_w$. Soit $f : B \to T$; nous allons noter par $\rho_w : \mathcal{H}^*_{Bw} \to T$ l'application $\rho_w(f, d_w) = \sigma(f, d_w) - \sigma [f, (d_w)_r]$. Enfin, on notera par $\theta = \theta_T$ l'élément nul de l'espace T.

Définition 2.2. On dit que l'application $f : B \to T$ vérifie la propriété S_* dans le voisinage du point $w \in B$, $w \neq u, v$, sur B, si la suite généralisée $[\rho_w, \mathcal{H}^*_{Bw}, \leqslant_\Delta] = [\rho_w(f, d_w), d_w \in \mathcal{H}^*_{Bw}, \leqslant_\Delta]$ est convergente et à la limite θ, $\lim\limits_{d_w} \rho_w(f, d_w) = \theta$.

Théorème 2.4. *On considère deux segments de l'éspace X, $[u, v]$, $[v, w]$, de sorte que $t_u < t_v < t_w$, et les bases $B_1 \subset [u, v]$, $B_2 \subset [v, w]$. Soit $B = B_1 \cup B_2$ et l'application $f : B \to T$. Si les restrictions de l'application f, $f_1 : B_1 \to T$, $f_2 : B_2 \to T$ sont intégrables M_* et si f vérifie la propriété S_* dans le voisinage du point v sur B, alors f est intégrable M_* sur B et l'égalité*

$$(M_*) \int_B f(z)dz = (M_*) \int_{B_1} f(z)dz + (M_*) \int_B f(z)dz \tag{2,4}$$

a lieu.

Démonstration. Soit $\mathcal{H}'_B$ la classe des divisions de la base B, de sorte que si $D' \in \mathcal{H}'_B$ alors $v \in D'$. En ce cas, pour $D' \in \mathcal{H}'_B$, il existe $D_1 \in \mathcal{H}_{B_1}$, $D_2 \in \mathcal{H}_{B_2}$ de sorte que $\sigma(f, D') = \sigma(f_1, D_1) + \sigma(f_2, D_2)$. Parce que, par hypothèse,

$$\lim_{D_1} \sigma(f_1, D_1) = (M_*) \int_{B_1} f(z)dz, \ \lim_{D_2} \sigma(f_2, D_2) = (M_*) \int_{B_2} f(z)dz$$

il résulte que la suite généralisée $[\sigma(f, D'), D'' \in \mathcal{H}'_B, \leqslant_\delta]$ est convergente, et, pour $D' \in \mathcal{H}'_B$ la relation

$$\lim_{D'} \sigma(f, D') = (M_*) \int_{B_1} f(z) dz + (M_*) \int_{B} f(z) dz \tag{2,5}$$

a lieu.

Si $D \in \mathcal{H}_B$, $D \notin \mathcal{H}'_B$ il existe $d_v \in \mathcal{H}'_{Bv}$ de sorte que pour la division $D' = D \cup d_v$ a lieu la relation $D' \in \mathcal{H}'_B$. Dans ce cas $\sigma(f, D') - \sigma(f, D) = \sigma(f, d_v) - \sigma[f, (d_v)_r] = \rho_v(f, d_v)$ et donc $\sigma(f, D) = \sigma(f, D') - \rho_v(f, d_v)$. On remarque que $\Delta(d_v) \leqslant \delta(D)$. De (2,5) et du fait que f vérifie la propriété S_* dans le voisinage du point v sur B, il résulte que la suite généralisée $[\sigma, \mathcal{H}_B, \leqslant \delta]$ est convergente, et l'égalité (2,4) a lieu.

DÉFINITION 2.3. On dit que l'application $f: B \to T$ vérifie la propriété S_* sur B,si la suite généralisée $[\rho(f, d), d \in \mathcal{H}^*_B \leqslant_\Delta] = [\rho, \mathcal{H}^*_B, \leqslant_\Delta]$ est convergente et a la limite $\theta: \lim\limits_{d} \rho(f, d) = \theta$, *où* l'application $\rho: \mathcal{H}^*_B \to T$ est donnée par la relation $\rho(f, d) = \sigma(f, d) - \sigma(f, d_r)$.

On remarque que pour n'importe quel $w \in B$, $w \neq u, v$, la suite généralisée $[\rho_w, \mathcal{H}^*_{Bw}, \leqslant_\Delta]$ est une sous-suite généralisée de $[\rho, \mathcal{H}^*_B, \leqslant_\Delta]$. Il résulte que si l'application $f: B \to T$ vérifie la propriété S_* sur B, alors elle vérifie aussi la propriété S_* sur B dans le voisinage de n'importe quel point $w \in \mathcal{H}$, $w \neq u, v$.

Étant donnée une suite généralisée $[\psi, P, \prec]$, $\psi: P \to \mathcal{H}^*_B$, on peut considérer la suite généralisée $[\Delta(\psi(p))], p \in P, \leqslant]$. Si la suite généralisée $[\Delta[\psi(o)], p \in P, \leqslant]$ est convergente, et à la limite zéro

$$\lim_{p} \Delta[\psi(p)] = 0, \tag{2,6}$$

on dit que la suite généralisée $[\psi, P, \prec]$ tend vèrs zéro en norme absolue. Ces notions étant données, la propriété suivante a lieu:

La condition nécessaire et suffisante pour que la suite généralisée $[\pi, P, \prec]$, $\pi: P \to T$ soit une sous-suite généralisée de la suite généralisée $[\rho, \mathcal{H}^*_B, \leqslant_\Delta]$ est qu'il existe une application $\psi: P \to \mathcal{H}^*_B$, de sorte que la suite généralisée $[\psi, P, \prec]$ tende vers zéro en norme absolue. La démonstration de cette propriété est analogue à celle du Théorème 2.3; celle-ci peut être suivie pas à pas, avec les changements correspondents.

THÉORÈME 2.5. *Si l'application $f: B \to T$ est intégrable M_* sur B, elle vérifie la propriété S_* sur B.*

Démonstration. On considère une sous-suite généralisée $[\pi, P, \prec]$, $\pi: P \to T$ de la suite généralisée $[\rho, \mathcal{H}^*_B \leqslant_\Delta]$. Alors l'application $\psi: P \to H^*_B$ existe, de sorte que la suite généralisée $[\psi, P, \prec]$ tend vers zéro en norme absolue. Soit $p \in P$ et $d_p \in \mathcal{H}^*_B$ de sorte que $\psi(p) = d_p$. La division $D_p \in \mathcal{H}_B$ existe de sorte que $(d_p)_r \subset D_p$ et

$$\delta(D_p) \leqslant \Delta(d_p). \tag{2,6}$$

Nous notons par D'_p la division $D'_p = D_p \cup d_p$. Il résulte alors de (2,5) et (2,6) que les sous-suites généralisées $[\varphi, P, \prec]$, $\varphi : P \to \mathcal{H}_B$, $\varphi(p) = D_p$ et $[\varphi', P, \prec]$, $\varphi' : P \to \mathcal{H}_B$, $\varphi'(p) = D'_p$ sont admises, et donc qu'elles sont des sous-suites généralisées de $[\sigma, \mathcal{H}_B, \leqslant_\delta]$. Parce que $\sigma(f, \varphi'(p)] - \sigma[f, \varphi(p)] = \sigma(f, d_p] - \sigma[f, (d_p)_r] = \rho(f, d_p]$, $p \in P$, il résulte de

$$\lim_p \sigma[f, \varphi'(p)] = \lim_p \sigma[f, \varphi(p)] = (M_*) \int_B f(z) dz$$

que $\lim_p \rho(f, d_p) = \theta$.

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PROPRIETĂŢI ALE INTEGRALEI M_* ÎN SPAŢII LINIARE

(Rezumat)

Se defineşte integrala M_* a unei funcţii $f: X \to T$ definite pe un spaţiu liniar şi cu valori într-un spaţiu liniar topologic. Se dau mai multe proprietăţi ale acestei integrale între care o condiţie necesară ca $f: X \to T$ să fie integrabila M_* pe baza B.

НЕКОТОРЫЕ СВОЙСТВА ИНТЕГРАЛА M_* В ЛИНЕЙНЫХ ПРОСТРАНСТВАХ

(Резюме)

Определяется интеграл M_* одной функции $f: X \to T$, определённой на линейном пространстве и со значениями в линейном топологическом пространстве. Автор даёт несколько свойств этого интеграла, среди которых и одно необходимое условие для того, чтобы $f: X \to T$ была интегрируемой M_* на основании B.

ASUPRA METODEI F_n ÎN CAZUL NELINIAR

GAROFIŢA PAVEL

1. În [1] se prezintă o metodă pentru a obţine soluţii aproximative pentru problema lui Cauchy

$$x' = f(t, x), \ x(a) = x_0. \tag{1}$$

Soluţia aproximativă se obţine pe nodurile

$$T_M = \{t_i | \ i = 0, 1, \ldots, M; \ t_0 = a, \ t_i < t_{i+1}\} \subset I$$

şi se notează cu X_i valoarea soluţiei aproximative pe nodul t_i $(i = 0, 1, \ldots, M)$.

Pentru determinarea vectorilor X_i se ataşează un sistem de ecuaţii

$$F_n(X_n, X_{n+1}, \ldots, X_{n+k}) = 0. \tag{2}$$

În lucrarea [1] se dau diferite metode de construcţie a sistemului (2).

Una din metode conduce la studiul următorului sistem neliniar:

$$X_{n+k} = h_{n+k}\beta_{nk}f(t_{n+k}, X_{n+k}) + Z \tag{3}$$

unde h_{n+k}, β_{nk} sînt numere, iar z un vector constant cu semnificaţii bine precizate.

Pentru studiul acestei probleme autorul foloseşte teorema de punct fix a lui Banach.

În nota de faţă ne propunem studierea rezolvabilităţii sistemului neliniar (3) în condiţii mai puţin restrictive, folosind o teoremă de punct fix în spaţii cu metrică vectorială.

Fie $\mathbf{R}^Q$ spaţiul liniar real Q — dimensional.

Definim o normă în $\mathbf{R}^Q$ prin

$$||X|| = |X^1| + |X^2| + \ldots + |X^S| \ \text{dacă} \ X = (X^1, X^2, \ldots, X^Q).$$

Presupunem că funcţia f satisface următoarea condiţie Lipschitz generalizată

$$||f(t, x) - f(t, y)|| \leqslant L||x - y||, \tag{4}$$

unde L este o matrice pătratică Q — *dimensională* negativă.

Considerăm aplicaţia

$$G : \mathbf{R}^S \to \mathbf{R}^Q$$

definită de

$$X \mapsto h_{n+k}\beta_{nk}f(t_{n+k}, X) + Z.$$

Ţinînd cont de condiţiile (4) obţinem

$$||G(X) - G(Y)|| \leqslant h_{n+k}|\beta_{nk}|\ ||f(t_{n+k}, X) - f(t_{n+k}, Y)|| \leqslant$$
$$\leqslant h_{n+k}|\beta_{nk}|\ L\ ||X - Y\ ||.$$

Dacă $h_{n+k}|\beta_{nk}|L$ este o matrice Hadamard atunci sîntem în condiţiile teoremei de punct fix a lui Perov [2] şi, prin urmare, ecuaţia (3) are o soluţie unică ce poate fi obţinută prin metoda aproximaţiilor succesive, pornind de la orice element din $\mathbf{R}^Q$.

Pentru a ilustra avantajul acestei metode dăm următorul exemplu:

2. *Exemplu*. Considerăm sistemul

$$\begin{aligned} x_1' &= t + \frac{1}{2\lambda}\sin x_1 + 2x_2 \\ & \qquad\qquad\qquad\qquad \text{cu } X_0 = (1, 0), \\ x_2' &= 100\ x_1 + \frac{1}{\lambda}\cos x_2 \end{aligned} \tag{5}$$

alegem $k = 1$, $F = \{0\}$, $D = \{1\}$ şi sistemul de funcţii $g_1 = 1$, $g_2 = t$.

În acest caz sistemul F — *metodei* [1] este

$$X^j_{n+k} + \sum_{\nu\in F} \alpha_{\nu n}X^j_{n+\nu} = h_{n+k}\sum_{\nu\in D} \beta_{\nu n}f^j_{n+k}$$

unde

$$f^j_{n+k} = f^j(t_{n+k}, X_{n+k}), \quad \begin{array}{l} j = 1, 2 \\ n = 0, 1, \ldots, M - k \end{array}$$

$$\alpha_{\nu n} = -g_n^T p_{n\nu}\mu_\nu,\ \nu \in F$$

$$\beta_{\nu n} = g_n^T p_{n\nu}\mu_\nu,\ \nu \in D$$

$$g_n = (g_1(h_{n+k}), g_2(h_{n+k}))^T$$

$p_n\nu$ este coloana a $\nu - a$ a matricei P_n^{-1},

$$P_n = \begin{pmatrix} g_1(t_{n+\nu} - t_{n+k-1}), & g_2(t_{n+\nu} - t_{n+k-1}) \\ h_{n+k}g_1'(t_{n+\nu} - t_{n+k-1}), & h_{n+k}g_2'(t_{n+\nu}t_{n+k-1}) \end{pmatrix}^{\nu\in F}_{\nu\in D}.$$

În cazul nostru

$$P_n^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{1}{h_{n+1}} \end{pmatrix}, \quad p_{n_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad p_{n_1} = \begin{pmatrix} 0 \\ \dfrac{1}{h_{n+1}} \end{pmatrix}$$

$$\alpha_{0n} = -g_n^T p_{n_0}\mu_0 = (-1, h_{n+1}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu_0 = -\mu_0$$

$$\beta_{1n} = g_n^T p_{n_1}\mu_1 = (1, h_{n+1}) \begin{pmatrix} 0 \\ \dfrac{1}{h_{n+1}} \end{pmatrix} u_1' = \mu_1'.$$

Sistemul va fi deci

$$(1) \begin{cases} X_{n+1}' - \mu_0 X_n' = h_{n+1}\left(t_{n+1} + \dfrac{1}{2\lambda}\sin X_{n+1}' + 2X_{n+1}^2\right)\mu_1' \\ X_{n+1}^2 - \mu_0 X_n^2 = h_{n+1}\left(5X_{n+1}' + \dfrac{1}{\lambda}\cos X_{n+1}^2\right)\mu_1', \end{cases}$$

$$n = 0, 1, \ldots, M-1.$$

Să vedem în ce condiții se poate aplica teorema lui Perov. Sistemul (1) se poate scrie

$$X_{n+1}' = G^1(X_{n+1}', X_{n+1}^2)$$

$$X_{n+1}' = G^2(X_{n+1}', X_{n+1}^2)$$

și obținem o aplicație $G: \mathbf{R}^2 \to \mathbf{R}^2$, definită de $(x, y) \mapsto (G^1(x, y), G^2(x, y))$.

Considerăm $\lambda = 100$

$$\mu_1' = \frac{1}{5}$$

$$\mu_0 = 1.$$

Sistemul, lucrînd în [0, 1], cu nodurile $t_0 = 0$, $t_1 = \dfrac{1}{4}$, $t_2 = \dfrac{1}{3}$, $t_3 = \dfrac{1}{2}$, $M = 3$, devine

$$\begin{cases} X_{n+1}' - X_n' = h_{n+1}\left(t_{n+1} + \dfrac{1}{200}\sin X_{n+1}' + 2X_{n+1}^2\right)\dfrac{1}{5} \\ X_{n+1}^3 - X_n^2 = h_{n+1}\left(100X_{n+1}' + \dfrac{1}{100}\cos X_{n+1}^2\right)\dfrac{1}{5} \end{cases}$$

$$X_0 = (1, 0);\ n = 0, 1, 2.$$

Pentru calculul lui X_1, X_2, X_3 se aplică metoda aproximaţiilor succesive. Calculele au fost efectuate la calculatorul FELIX C-256, după următorul program scris în FORTRAN:

```
      DIMENSION X(4), Y(4), H(3), T(3), U(50), V(50)
      READ (105,1) X(1), Y(1)
 1    FORMAT (2Flo. 7)
      READ (105,2) (H(N), N = 1, 3), (T(N), N = 1,3)
 2    FORMAT (6Flo. 7)
      DO 10 N = 1, 3
      M = 1
      U(M) = X(N)
      V(M) = Y(N)
 11   MPU = M + 1
      U(MPU)=X(N) + H(N)*(T(N)+ SIN(U(M))/200 +2*V(M))/5
      V(MPU) = Y(N) + H(N)*(100*U(M) + COS(V(M))/100)/5
      M = MPU-1
      IF(ABS(U(MPU) - U(M)).GE.O.OOOO1.OR.ABS(V(MPU)-
     A-V(M)).GE.O.OOO1) GOTO11
      NPU = N + 1
      X(NPU) = U(MPU)
 10   Y(NPU) = V(MPU)
      WRITE(108,3)
 3    FORMAT (1H1,5x, 47HSOLUTIA APROXIMATIVA PE
     BPUNCTELE T(1), T(2), T(3)/ 5X, 50(1H*)
      WRITE (108,4) (X(N), N = 1,4), (Y(N), N = 1,4)
 4    FORMAT (1H1, 5X, 4 (Flo. 7, 2X)/6X, 4(Flo. 7, 2X))
      STOP
      END
```

```
          Soluţia aproximativă pe punctele T(1), T(2), T(3)
**************************************************************************
   1.0000000   1.0127106   2.7358065   4.2111712
    .0000000   5.0005007  21.8794861  30.9985046
```

Prin urmare soluţia problemei (5) calculată aproximativ pe nodurile $0, \frac{1}{4}, \frac{1}{3}$ şi $\frac{1}{2}$ este:

$x_1(0) = 1.0000000,\quad x_1\left(\frac{1}{4}\right) = 1.0127106,\quad x_1\left(\frac{1}{3}\right) = 2.7358065$

$x_2(0) = .0000000,\quad x_2\left(\frac{1}{4}\right) = 5.0005007,\quad x_2\left(\frac{1}{3}\right) = 21.8794861$

$x_1\left(\frac{1}{2}\right) = 4.2111712,$

$x_2\left(\frac{1}{2}\right) = 30.9985056$

(Intrat în redacţie la 25 ianuarie 1972)

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О МЕТОДЕ F_n В НЕЛИНЕЙНОМ СЛУЧАЕ

(Резюме)

Применяя метод F_n [1], а также теорему с неподвижной точкой Перова, автор даёт метод для числового расчёта решения задачи Коши. Даётся также один пример. Расчёты были выполнены на электронной вычислительной машине FELIX C-256.

ON THE F_n METHOD IN THE NON LINEAR CASE

(Summary)

A method for the numerical calculation of the solution of Cauchy's problem is given by applying the F_n [1] method and a Perov's fixed point theorem. There is also an example for which the calculations were carried out on the computer Felix C-256.

ON COMMON FIXED POINTS

IOAN A. RUS

Let (X, d) be a complete metric space and two mapping $f, g : X \to X$. The purpose of the present paper is to investigate under what conditions f and g have a unique common fixed point, *i. e*, $x_0 \in X : f(x_0) = g(x_0) = x_0$.

TEOREM. *Let (X, d) be a complete metric space and $f, g : X \to X$, two mappings for which there exist numbers $\alpha, \beta, \gamma \in \mathbf{R}_+$, $\alpha + 2\beta + 2\gamma < 1$, such that*

$$d(f(x), g(y)) \leqslant \alpha d(x, y) + \beta[d(x, f(x)) + d(y, g(y))] + \gamma[d(x, g(y)) + d(y, f(x))] \tag{1}$$

for all $x, y \in X$.

Then f and g have a unique common fixed point.

Proof. Let x be any element of X, and we consider the sequence

$$x_1 = f(x), \ x_2 = g(x_1), \ \ldots, \ x_{2n} = g(x_{2n-1}), \ x_{2n+1} = f(x_{2n}), \ \ldots \tag{2}$$

We have

$$d(x_1, x_2) = d(f(x, g(x_1)) \leqslant \alpha d(x, x_1) + \beta[d(x, x_1) + d(x_1, x_2)] +$$
$$+ \gamma[d(x, x_2) + d(x_1, x_1)] \leqslant \alpha d(x, x_1) +$$
$$+ \beta[d(x, x_1) + d(x_1, x_2)] + \gamma[d(x, x_1) + d(x_1, x_2)].$$

Hence

$$d(x_1, s_2) \leqslant \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} d(x, x_1).$$

Similarly we have

$$d(x_2, x_3) \leqslant \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} d(x_1, x_2).$$

And by induction

$$d(x_n, x_{n+1}) \leqslant \left[\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\right]^n (d(x, ,x_1),$$

which implies that the sequence (2), is fundamental. Let x_0 be the limit of this sequence. We have

$$d(x_0, f(x_0) \leqslant d(x_0, x_{2n}) + d(x_{2n}, d(x_0)) \leqslant d(x_0, x_{2n}) +$$
$$+ \beta[d(x_0, f(x_0)) + d(x_{2n-1}, x_{2n})] + \gamma[d(x_0, x_{2n}) + d(x_{2n-1}, f(x_0)].$$

and for $n \to \infty$, we have, $d(x_0, f(x_0) = 0$, *i.e*, $f(x_0) = x_0$.

In a similar way we can prove that $g(x_0) = x_0$.

Let us prove that x_0 is a unique common fixed point for f and g. If there exist another common fixed point, x_0', then

$$d(x_0, x') = d(f(x_0), g(x_0')) \leqslant \alpha\, d(x_0, x_0') + 2\gamma d(x_0, x'),$$

which implies that $x_0 = x_0'$.

COROLLARY 1 (Kannan [4]). *Let* (X, d) be a *complete metric space and* $f, g: X \to X$, *two mappings for which there exists a number* $\beta \in \mathbf{R}_+$, $\beta < \frac{1}{2}$, *such that*

$$d(f(x), g(y) \leqslant \beta\, [d(x, f(x)) + d(y, g(y))]$$

for all $x, y \in X$.

Then f and g have a unique common fixed point.

COROLLARY 2 (Chatterjea [2]). *Let* (X, d) *be a complete metric space and* $f, g: X \to X$ *two mappings for which there exists a number* $\gamma \in \mathbf{R}_+$, $\gamma < \frac{1}{2}$, such that

$$d(f(x), g(y)) \leqslant \gamma\, [d(x, g(y)) + d(y, f(x))]$$

for all $x, y \in X$.

Then f and g have a unique a common fixed point.

COROLLARY 3 (Ciric [3]) ; see also *Avramescu* [1], Reich [5], Rus [6], [7], Zamfirescu [8]). *Let* (X, d) *be a complete metric space and* $f: X \to X$, *a mapping for which there exist numbers* $\alpha, \beta, \gamma \in \mathbf{R}_+$, $\alpha + 2\beta + 2\gamma < 1$, such that

$$d(f(x), f(y)) \leqslant \alpha d(x, y) + \beta[d(x, f(x)) + d(y, f(y))] +$$
$$+ \gamma\, [d(x, f(y)) + d(y, f(x))]$$

for all $x, y \in X$.

Then f has a unique fixed point.

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ASUPRA PUNCTELOR FIXE COMUNE

(Rezumat)

Fie (X, d) un spaţiu metric şi $f, g: X \to X$. Se dau condiţii ce asigură existenţa în mod unic a unui punct fix comun pentru aplicaţiile f şi g. Rezultatul principal al lucrării este conţinut în

Teorema 1. Fie (X, d) un spaţiu metric complet şi $f, g: X \to X$, două aplicaţii astfel încît există $\alpha, \beta, \gamma \in \mathbf{R}_+$, $\alpha + 2\beta + 2\gamma < 1$ şi

$$d(f(x), g(y)) \leqslant \alpha d(x, y) + \beta[d(x, f(x)) + d(y, g(y))] + \gamma[d(x, g(y)) + d(y, f(x))].$$

În aceste condiţii aplicaţiile f şi g au un singur punct fix comun.

ОБ ОБЩИХ НЕПОДВИЖНЫХ ТОЧКАХ

(Резюме)

Пусть (X, d) — метрическое пространство и $f, g: X \to X$. Даются условия, обеспечивающие существование одной общей единственной неподвижной точки для отображений f и g. Основной результат работы содержится в

Теореме 1. Пусть (X, d) — полное метрическое пространство и $f, g: X \to X$ — два отображения, такие, чтобы существовали

$$\alpha, \beta, \gamma \in \mathbf{R}_+, \ \alpha + 2\beta + 2\gamma < 1 \ \text{и} \ d(f(x), g(y)) \leqslant \alpha d(x, y) + \beta[d(x, f(x)) + d(y, g(y))] + \gamma[d(x, g(y)) + d(y, f(x))].$$

В этих условиях отображения f и g имеют одну общую неподвижную точку.

CONVERGENCE THEOREMS FOR SEQUENCES OF LINEAR TRANSFORMATIONS

LUCIANA LUPAŞ

1. Let $C_0[a, b]$ be the space of all real functions which are defined and continuous on $[a, b]$, $a \leqslant 0 \leqslant b$, vanishing at the point $x = 0$. This space is normed by means of the uniform norm. We define the functions $e_k \in C_0[a, b]$, $k = 1, 2, \ldots$, by

$$e_k(t) = t^k,\ t \in [a, b],\ k = 1, 2, \ldots .$$

DEFINITION 1. *A sequence of functionals* $\varphi_n : C_0[a, b] \to \mathbf{R}$, $n = 1, 2, \ldots$, of the form

$$\varphi_n(f) = \int_a^b f(t)\ dw_n(t),\quad f \in C_0[a, b],\ n = 1, 2, \ldots, \tag{1}$$

where w_n, $n = 1, 2, \ldots$, are signed measures, belongs to the *class* $\mathcal{F}$ if the following properties are verified:

(i) $\varphi_n(e_1) = 0,\ n = 1, 2, \ldots;$

(ii) $\displaystyle\int_a^u t\ dw_n(t) \leqslant 0$ for every $u \in [a, 0[$ and $n=1, 2, \ldots$

$\displaystyle\int_v^b t\ dw_n(t) \geqslant 0$ for every $v \in]0, b]$ and $n = 1, 2, \ldots .$

DEFINITION 2. *A sequence of operators* $L_n : C_0\ [a, b] \to C_0[a, b]$, $n = 1, 2, \ldots$, defined by

$$(L_n f)(x) = \int_a^b f(t)\ dw_n(t, x),\quad f \in C_0[a, b],\ x \in [a, b],\ n = 1, 2, \ldots, . \tag{2}$$

belongs to the class $\mathfrak{L}$ if:

(i) $(L_n e_1)(x) = 0, \quad x \in [a, b], \ n = 1, 2, \dots;$

(ii) $\int_a^u t \, dw_n(t, x) \leqslant 0$ for every $u \in [a, 0[, \ x \in [a, b], n = 1, 2, \dots$

and

$$\int_v^b t \, dw_n(t, x) \geqslant 0 \text{ for every } v \in]0, b], \ x \in [a, b], \ n = 1,2,\dots.$$

Here $w_n(t,.) \in C_0[a, b]$ for any $t \in [a, b]$ and $w_n(.,x)$ is a signed measure for any $x \in [a, b]$.

We note that the transformations which belong to $\mathfrak{F}$ or $\mathfrak{L}$ are linear but generally they are not positive.

2. Our aim is to find a subset H in $C_0[a, b]$ such that the pointwise convergence on H, of a sequence from $\mathfrak{F}$ (or from $\mathfrak{L}$) to the zero functional (resp. to the zero operator), to imply the pointwise convergence on the whole space $C_0[a, b]$.

DEFINITION 3. A function $f \in C_0[a, b]$ is called *starshaped* on $[a, b]$ if for any $x_1, x_2 \in [a, b]$, $x_1 \neq 0 \neq x_2$, $x_1 \neq x_2$ the following inequality is valid

$$[0, x_1, x_2; f] \geqslant 0, \tag{3}$$

where the symbol $[t_1, t_2, t_3; f]$ denotes the divided difference of the second order.

LEMMA 1. *Let* $(\varphi_n)_{n=1}^{\infty} \in \mathfrak{F}$ *and* $f \in C_0[a, b]$. *If there exist two real numbers* m_f, M_f *such that*

$$m_f \leqslant [0, x_1, x_2; f] \leqslant M_f,$$

for any distinct points $x_1, x_2, \in [a, b]$, $x_1 \neq 0 \neq x_2$, *then for* $n = 1, 2, \dots$

$$m_f \varphi_n(e_2) \leqslant \varphi_n(f) \leqslant M_f \varphi_n(e_2). \tag{4}$$

LEMMA 2. *Let* $(L_n)_{n=1}^{\infty} \in \mathfrak{L}$ *and* $f \in C_0[a, b]$. *If there exist two real numbers* m_f, M_f *such that*

$$m_f \leqslant [0, x_1, x_2; f] \leqslant M_f,$$

for any distinct points $x_1, x_2 \in [a, b]$, $x_1 \neq 0 \neq x_2$, *then for* $x \in [a, b]$ *and* $n = 1, 2, \dots.$

$$m_f(L_n e_2)(x) \leqslant (L_n f)(x) \leqslant M_f(L_n e_2)(x). \tag{5}$$

Proof. Let $m_f \leqslant [0, x_1, x_2; f] \leqslant M_f$ for $x_1, x_2 \in [a, b]$, $x_1 \neq x_2$, $x_1 \neq 0 \neq x_2$. We define the functions $g_k \in C_0[a, b]$, $k = 1, 2$, by

$$g_1(t) = f(t) - m_f t^2$$
$$g_2(t) = M_f t^2 - f(t).$$

Taking into account that $[0, x_1, x_2; g_k] \geqslant 0$ for any distinct points from $[a, b]$, not zero, we observe that g_k, $k = 1, 2$, are starshaped on $[a, b]$. By using the theorem 2.6 from [1], we have for $n = 1, 2, \ldots$

$$(L_n g_k)(x) \geqslant 0, \qquad k = 1, 2, ; \; x \in [\mathrm{a}, \mathrm{b}].$$

But these inequalities are equivalent with

$$L_n f \geqslant m_f \cdot L_n e_2$$
$$M_f \cdot L_n e_2 \geqslant L_n f$$

which completes the proof.

With the similar arguments the first lemma may be proved.

THEOREM I. *Let* $(\varphi_n)_{n=1}^{\infty}$ *be a sequence of the class* $\mathfrak{F}$ *such that*

(i) *there exist a positive number* M *for which*

$$||\varphi_n|| \leqslant M, \qquad n = 1, 2, \ldots;$$

(ii) $$\lim_{n\to\infty} \varphi_n(e_2) = 0.$$

Then

$$\lim_{n\to\infty} \varphi_n(f) = 0 \quad \textit{for every} \quad f \in C_0[a, b].$$

THEOREM II. *Let* $(L_n)_{n=1}^{\infty} \in \mathfrak{L}$ *with the properties*:

(i) *there exist a positive number M such that*

$$||L_n|| \leqslant M, \qquad n=1, 2, \ldots;$$

(ii) $$\lim_{n\to\infty} ||L_n e_2|| = 0.$$

Then

$$\lim_{n\to\infty} ||L_n f|| = 0 \quad \textit{for every} \quad f \in C_0[a, b].$$

Proof. For $k = 0, 1, \ldots$ we find the real numbers m_k, M_k such that

$$m_k \leqslant [0, x_1, x_2; e_k] \leqslant M_k,$$

for any distinct points from $[a, b]$ with $x_1 \neq 0 \neq x_2$.
According to the lemma 2 on the interval $[a, b]$ the following inequalities are true

$$m_k \cdot L_n e_2 \leqslant L_n e_k \leqslant \mathrm{M}_k \cdot L_n e_2, \qquad n = 1, 2, \ldots; \; k = 0, 1, \ldots.$$

Because e_2 is starshaped on $[a, b]$ it follows $L_n e_2 \geqslant 0$ on $[a, b]$. If we put

$$c_k = \max\{|m_k|, |M_k|\}, \; k = 1, 2, \ldots,$$

then

$$|L_n e_k| \leqslant c_k \cdot L_n e_2 \leqslant c_k \cdot ||L_n e_2||$$

that is

$$||L_n e_k|| \leqslant c_k \cdot ||L_n e_2||.$$

Therefore

$$\lim_{n\to\infty} ||L_n e_k|| = 0 \text{ for } k = 0, 1, 2, \ldots.$$

On account of theorem 15 (see [2]) and by using the fact that the sequence $(L_n)_{n=1}^{\infty}$ is uniform bounded, we conclude that the sequence of operators converges pointwise on the space $C_0[a, b]$ to the zero operator.

The first theorem has a similar proof.

3. In conclusion we observe that the following assertion is valid:

THEOREM III. *Let $(L_n)_{n=1}^{\infty}$ be a sequence of linear operators which map $C_0[a, b]$ into a linear space formed with bounded functions on $[a, b]$. If*

(i) $\lim_{n\to\infty} ||L_n e_2|| = 0$;

(*ii*) $L_n g \geqslant 0$ *for every* $g \in C_0\,[a, b]$ *which is starshaped on* $[a, b]$;

(ii) *there is a positive number* M *such that*

$$||L_n|| \leqslant M, \quad n = 1, 2, \ldots,$$

then

$$\lim_{n\to\infty} ||L_n f|| = 0 \quad \textit{for every} \quad f \in C_0\,[a, b].$$

Proof. The condition (*ii*) of this theorem enables us to prove an inequality of the form (5) for the considered class of operators. Then, the conclusion is proved in a manner similar to that of the second theorem.

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TEOREME DE CONVERGENȚĂ PENTRU APLICAȚII LINIARE

(R e z u m a t)

În lucrare se dau teoreme de convergență punctuală a unor șiruri de funcționale de forma (1), respectiv a unor șiruri de operatori de forma (2) către funcționala nulă, respectiv operatorul nul. Aplicațiile considerate sînt liniare dar, în general, nu sînt pozitive.

În încheiere se enunță o teoremă analoagă pentru o clasă mai generală de operatori.

ТЕОРЕМЫ СХОДИМОСТИ ДЛЯ ЛИНЕЙНЫХ ОТОБРАЖЕНИЙ

(Р е з ю м е)

В работе даются теоремы точечной сходимости последовательностей функционалов вида (1) и некоторых последовательностей операторов вида (2), соответственно, к нулевому функционалу и к нулевому оператору, соответственно. Рассматриваемые отображения являются линейными, но в общем не являются положительными.

В заключение формулируется аналогичная теорема для более общего класса операторов.

TWO-DIMENSIONAL MONOSPLINES AND OPTIMAL CUBATURE FORMULAE

GH. COMAN

Let $\Omega \subset \mathbf{R}^2$ and H be the class of Lebesgue integrable functions on Ω.

One considers the cubature formula

$$I(f) = Q(f) + R_Q(f), \tag{1}$$

where

$$I(f) = \iint_{\Omega} f(x, y)\,dxdy$$

$$Q(f) = \sum_{i=1}^{N} \sum_{p,q=0}^{\mu,\nu} A_i^{pq} f^{(p,q)}(M_i), \tag{2}$$

and

$$R_Q(f) = I(f) - Q(f).$$

DEFINITION 1. One says that the cubature formula (1) has the degree of exectitude (r, s), if $R_Q(f) = 0$, $\forall f \in \Pi_{rs}$ and there exist $g \in \Pi_{r+1,s}$ and $h \in \Pi_{r,s+1}$ such that $R_Q(g) \neq 0$ and $R_Q(h) \neq 0$, where Π_{mn} denotes the set of all real polynomials of the form

$$P_{mn}(x, y) = \sum_{i,j=0}^{m,n} \rho_{ij} x^i y^j.$$

DEFINITION 2. Let $\mathcal{F}$ be the set of the cubature formulae of the form (1). A formula $F \in \mathcal{F}$ is called optimal in the class of functions H, if

$$\inf_{F \in \mathcal{F}} \sup_{f \in H} |R_Q(f)| = R_{\bar{Q}}(f),$$

where $\bar{Q}$ is the functional (2) corresponding to $\bar{F}$.

We consider the following problem: Given the natural numbers N, r and s, determine a cubature formula of the form (1) which is optimal in a given class of functions H.

This paper is concerned with the above problem in the case when Ω is $D = \{(x, y) \in \mathbf{R}^2 | 0 \leqslant x, y \leqslant 1\}$, the knots are $\{(x_i, y_j) | 0 \leqslant x_0 < x_1 < \ldots < x_m \leqslant 1; 0 \leqslant y_0 < y_1 < \ldots < y_n \leqslant 1\}$, and H is $W^{(r,s)} L_2(D) = \{f \in C^{rs}(D) | \; \|f^{(r,s)}\|_{L_2(D)} \leqslant 1, \|f^{(k,s)}(0,.)\|_{L_2(0,1)} \leqslant 1, \|f^{(r,l)}(.,0)\|_{L_2(0,1)} \leqslant 1; k = 0, \ldots, r-1; l = 0, \ldots, s\}$.

In the first part is defined the notion of two-dimensional monospline function and is determined the monospline which, on D, is of least deviation from zero in the square mean. In the second part is established a one-to-one correspondence between these monosplines and the cubature formulae (1). Using this correspondence the optimal cubature formula is then constructed in the class of function $W^{(r,s)} L_2(D)$.

1. Two-Dimensional Monosplines. Let $\Delta x : 0 \leqslant x_0 < x_1 < \ldots < x_m \leqslant 1$ and $\Delta y : 0 \leqslant y_0 < y_1 < \ldots < y_n \leqslant 1$ be partitions of $[0,1]$ and $\Delta xy = \{(x_i, y_j) | x_i \in \Delta x, y_j \in \Delta y\}$.

One considers the one-dimensional spline functions

$$s_q(x) = \sum_{i=1}^{m-1} \sum_{p=0}^{\mu} \lambda_i^{pq} \frac{(x - x_i)_+^{r-p-1}}{(r-p-1)!}, \quad (q = 0, \ldots, s-1)$$

and

$$t_p(y) = \sum_{j=1}^{n-1} \sum_{q=0}^{\nu} \delta_j^{pq} \frac{(y - y_j)_+^{s-q-1}}{(s-q-1)!}, \quad (p = 0, \ldots, r-1)$$

with the knots $\{x_i\}_0^m$ respectively $\{y_j\}_0^n$, and with deficiency of order μ respectively ν.

Furthermore, let

$$s(x, y) = \sum_{i,j=1}^{m-1, n-1} \sum_{p,q=0}^{\mu, \nu} \rho_{ij}^{pq} \frac{(x - x_i)_+^{r-p-1}}{(r-p-1)!} \frac{(y - y_j)_+^{s-q-1}}{(s-q-1)!}.$$

The function (see [5])

$$S(x,y) = s(x,y) + \sum_{q=0}^{s-1} y^q s_q(x) + \sum_{p=0}^{r-1} x^p t_p(y) + P_{r-1,s-1}(x,y), \tag{1.1}$$

where $P_{r-1,s-1} \in \Pi_{r-1,s-1}$ is called a spline function of degree $(r-1, s-1)$ with deficiency of order (μ, ν) and with the knots (x_i, y_j), $(i = 0, \ldots, m; j = 0, \ldots, n)$.

One observes that

1. $S \in C^{r-\mu-2, s-\nu-2}(\mathrm{R})$
2. $S \in \Pi_{r-1,s-1}$ $(x,y) \in D_{ij}$, $(i = -\infty, 1, \ldots, m, +\infty; j = -\infty, 1, \ldots, n, +\infty)$

where

$$D_{ij} = \{(x,y) \in \mathbf{R}^2 | x_{i-1} \leqslant x \leqslant x_i ; y_{j-1} \leqslant y \leqslant y_j\}.$$

DEFINITION 3. The function

$$M(x, y) = \frac{x^r y^s}{r!s!} + S(x,y), \tag{(1.2)}$$

where S is a two-dimensional spline function, is called a two-dimensional monospline of degree (r,s), with deficiency of order (μ, ν) and with the knots (x_i, y_j), $(i = 0,\ldots, m ; j = 0, \ldots, n)$.

We denote by $\mathfrak{M}_{rs}$ (m, n, μ, ν) the set of all monosplines of the form (1.2).

One considers the following problem: *Given the natural numbers $m,n,$ r,s,μ and ν, determine the monospline $\overline{M} \in \mathfrak{M}_{rs}(m, n, \mu, \nu)$ which, on D, is of least deviation from zero in square mean, i.e.*

$$||\overline{\mathrm{M}}||_{L_2(D)} = \min_{M \in \mathfrak{M}_{rs}} ||\mathrm{M}||_{L_2(D)}.$$

We shall study this problem in the case $x_0 = y_0 = 0$, $x_m = y_n = 1$ and $\mu = r - 1$, $\nu = s - 1$.

THEOREM 1. *Let $m,n,r,$ and s be given natural numbers. Then $\exists_u\ \overline{M} \in$ $\in \mathfrak{M}_{rs}$ $(m, n, r - 1, s - 1)$ which, on D, is of least deviation from zero in square mean, namely*

$$\overline{M}(x,y) = \frac{x^r x^s}{r!s!} - \sum_{i,j=0}^{m-1,\, n-1} \sum_{p,\, q=0}^{r-1,\, s-1} \alpha_i^p \beta_j^q \frac{\left(x - \frac{i}{m}\right)_+^p}{p!} \frac{\left(y - \frac{j}{n}\right)_+^q}{p!},$$

where

$$\alpha_0^p = \frac{(-1)^{r-p}}{r!} h^{r-p} X_r^{(p)}(1)$$

$$\alpha_i^p = \frac{1 - (-1)^{r-p}}{r!} h^{r-p} X_r^{(p)}(1), \quad \begin{pmatrix} i = 1, \ldots, m - 1 \\ \mathrm{p} = 0, \ldots, \mathrm{r} - 1 \end{pmatrix}$$

$$\beta_0^q = \frac{(-1)^{s-q}}{s!} g^{s-q} X_s^{(q)}(1)$$

$$\beta_j^q = \frac{1 - (-1)^{s-q}}{s!} g^{s-q} X_s^{(q)}(1), \quad \begin{pmatrix} \mathrm{j} = 1, \ldots, \mathrm{n} - 1 \\ q = 0, \ldots, s - 1 \end{pmatrix}$$

X_r being the Legendre polynomial of the degree r having the coefficient of x^r equal with 1 and

$$||\overline{\mathrm{M}}||_{L_2(D)} = \left[\frac{\lambda^2}{(2s + 1)\,(s!)^2} + \frac{\delta^2}{(2r + 1)\,(r!)^2} - \lambda^2 \delta^2\right]^{1/2},$$

where

$$\lambda = \frac{r!}{(2r)!m^r\sqrt{2r+1}}, \quad \delta = \frac{s!}{(2s)!n^s\sqrt{2s+1}}.$$

Proof. From (1.2) it follows that in the case $\mu = r - 1$, $\nu = s - 1$ we have

$$M(x,y) = \frac{x^r y^s}{r!s!} + \sum_{i,j=0}^{m-1,n-1} \sum_{p,q=0}^{r-1,s-1} \omega_{ij}^{pq} \frac{(x-x_i)_+^p}{p!} \frac{(y-y_j)_+^q}{q!}.$$

Now we determine the parameters x_i, y_j, ω_{ij}^{pq} for which the integral

$$J = \int_0^1\int_0^1 M^2(x,y)dxdy$$

is minimal. But we have

$$J = \sum_{i,j=1}^{m,n} \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} M_{ij}^2(x,y)\,dx\,dy,$$

where

$$M_{ij}(x,y) = \frac{x^r y^s}{r!s!} + \sum_{k,l=0}^{i-1,j-1} \sum_{p,q=0}^{r-1,s-1} \omega_{kl}^{pq} \frac{(x-x_k)^p}{p!} \frac{(y-y_l)^q}{q!}. \tag{1.3}$$

It follows that the integral J is minimal if and only if all the integrals

$$I_{ij} = \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} M_{ij}^2(x,y)\,dx\,dy, \quad \begin{pmatrix} i - 1, \ldots & m \\ j = 1, \ldots, & n \end{pmatrix}$$

are minimal.

Using the notations

$$\frac{1}{p!q!}\omega_{ij}^{pq} = \frac{1}{r!s!}\theta_{ij}^{pq}, \quad \begin{pmatrix} i = 0, \ldots, m-1, & p = 0, \ldots, r-1 \\ j = 0, \ldots, n-1, & q = 0, \ldots, s-1 \end{pmatrix}, \tag{1.4}$$

we obtain

$$M_{ij}(x,y) = \frac{1}{r!s!} N_{ij}(x,y),$$

where

$$N_{ij}(x, y) = x^r y^s + \sum_{k, l=0}^{i-1, j-1} \sum_{p, q=0}^{r-1, s-1} \theta_{kl}^{pq} (x - x_k)^p (y - y_l)^q \tag{1.5}$$

and

$$J = \frac{1}{(r!)^2 (s!)^2} \sum_{i, j=1}^{m, n} \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} N_{ij}^2 (x, y) \, dx \, dy. \tag{1.6}$$

It is known (see [7]) that in the set all polynomials of the form

$$P_{rs}(x, y) = x^r y^s + \sum_{p, q=0}^{r-1, s-1} a_{pq} x^p y^q$$

the unique polynomial for which the integral

$$\int_{a-h}^{a+h} \int_{b-g}^{b+g} P_{rs}^2 (x, y) \, dx \, dy$$

takes its minimum value, is

$$R_{rs}(x, y) = x^r g^s X_s \left(\frac{y-b}{g}\right) + y^s h^r X_r \left(\frac{x-a}{h}\right) - h^r g^s X_r \left(\frac{x-a}{h}\right) X_s \left(\frac{y-b}{g}\right). \tag{1.7}$$

From (1.5) — (1.7) it follows that J takes the minimum value with regard to parameters θ_{ij}^{pq}, x_i, y_j, if and only if

$$N_{ij} (x, y) = R_{rs}^{ij}(x, y), \quad (x, y) \in D_{ij}, \begin{pmatrix} i - 1, \ldots, m \\ j = 1, \ldots, n \end{pmatrix},$$

where

$$R_{rs}^{ij}(x, y) = x^r g_j^s X_s \left(\frac{y-b_j}{g}\right) + y^s h_i^r X_r \left(\frac{x-a_i}{h}\right) - h_i^r g_j^s X_r \left(\frac{x-a_i}{h_i}\right) X_s \left(\frac{y-b_j}{g_j}\right)$$

and

$$h_i = \frac{x_i - x_{i-1}}{2}, \quad a_i = \frac{x_i + x_{i-1}}{2}, \quad g_j = \frac{y_j - y_{j-1}}{2}, \quad b_j = \frac{y_j + y_{j-1}}{2}.$$

Assuming that the knots (x_i, y_j) are fixed and taking into account that

$$\int_{a-h}^{a+h} X_n^2(x) \, dx = \frac{(n!) \, 2^{2n+1}}{(2n+1) \, [(2n)!]^2} h^{2n+1},$$

we obtain

$$\bar{J} = \frac{1}{(r!)^2(s!)^2(2r+1)(2s+1)}\left[\frac{(r!)^4}{[(2r)!]^2}\sum_{i=1}^{m}(x_i - x_{i-1})^{2r+1} + \right.$$

$$\left. + \frac{(s!)^4}{[(2s)!]^2}\sum_{j=1}^{n}(y_j - y_{j-1})^{2s+1} - \frac{(r!)^4(s!)^4}{[(2r)!]^2[(2s)!]^2}\sum_{i,j=1}^{m,n}(x_i - x_{i-1})^{2r+1}(y_j - y_{j-1})^{2s+1}\right]$$

which is a function of x_i, y_j, $(i = 1; j = 1, \ldots, n-1)$.

Now, we shall minimize the function $\bar{J}$ with regard to the parameters x_i, y_j. It is easy to see that J takes its minimum value only when

$$\begin{aligned} x_i &= \bar{x}_i = \frac{i}{m}, \quad (i - 1, \ldots, m-1) \\ y_j &= \bar{y}_j = \frac{j}{n}, \quad (j - 1, \ldots, n-1), \end{aligned} \tag{1.9}$$

and

$$\min_{x_i, y_j} \bar{J} = \frac{1}{(2r+1)(2s+1)}\left[\frac{(r!)^2}{[(2r)!]^2(s!)^2 m^{2r}} + \frac{(s!)^2}{[(2s)!]^2(r!)^2 n^{2s}} - \frac{(r!)^2(s!)^2}{[(2r)!]^2[(2s)!]^2 m^{2r} n^{2s}}\right].$$

We shall denote by N^*_{ij} the polynomials N_{ij} for $x_i = \bar{x}_i$, $\bar{y}_j = y_j$. Now we have to determine the parameters θ^{pq}_{ij} such that

$$N^*_{ij} = R^{ij}_{rs}, \qquad (i - 1, \ldots m-1, \ldots \ldots, n-1). \tag{1.11}$$

One observes that

$$N^*_{i+1,j+1}(x, y) - N^*_{i+1,j}(x, y) - N^*_{i,j+1}(x, y) + N^*_{ij}(x, y) = \sum_{p,q=0}^{r-1,s-1}\theta^{pq}_{ij}(x - \bar{x}_i)^p (y - \bar{y}_j)^q.$$

and

$$R^{i+1,j+1}_{rs}(x, y) - R^{i+1,j}_{rs_i}(x, y) - R^{i,j+1}_{rs}(x, y) + R^{ij}_{rs}(x, y) =$$

$$= -\sum_{p=1}^{r-1}\frac{1-(-1)^{r-p}}{p!}h^{r-p}X^{(p)}_r(1)(x - \bar{x}_i)^p \sum_{q=1}^{s-1}\frac{1-(-1)^{s-q}}{q!}g^{s-q}X^{(q)}(1)(y - \bar{y}_j)^q$$

where $h = \frac{1}{2m}$, $g = \frac{1}{2n}$. Form (1.11), it follows that

$$\theta^{pq}_{ij} = -\left[\frac{1-(-1)^{r-p}}{p!}h^{r-p}X^{(p)}_r(1)\right]\left[\frac{1-(-1)^{s-q}}{q!}g^{s-q}X^{(q)}_s(1)\right],$$

$$\begin{pmatrix} i = 1, \ldots, m-1 \\ j = 1, \ldots, n-1 \end{pmatrix}$$

Using the identities $N_{1j} \equiv R_{rs}^{1j}$, $(j = 1, \ldots, n-1)$ and $N_{i1} \equiv R_{rs}^{i1}$, $(i = 1, \ldots, m-1)$, we obtain

$$\theta_{i0}^{pq} = -\left[\frac{1-(-1)^{r-p}}{p!} h^{r-p} X_r^{(p)}(1)\right]\left[\frac{(-1)^{s-q}}{q!} g^{s-q} X_s^{(q)}(1)\right](i = 1, \ldots, m-1)$$

$$\theta_{0j}^{pq} = -\left[\frac{(-1)^{r-p}}{p!} h^{r-p} X_r^{(p)}(1)\right]\left[\frac{1-(-1)^{s-q}}{q!} g^{s-q} X_s^{(q)}(1)\right], (j = 1, \ldots, n-1) \tag{1.13}$$

$$\theta_{00}^{pq} = -\left[\frac{(-1)^{r-p}}{p!} h^{r-p} X_r^{(p)}(1)\right]\left[\frac{(-1)^{s-q}}{q!} g^{s-q} X_s^{(q)}(1)\right], \begin{pmatrix} p = 0, \ldots, r-1 \\ q = 0, \ldots, s-1 \end{pmatrix}$$

From (1.4), (1.10), (1.11), (1.12) and (1.13), it follows that theorem 1 is proved.

2. Optimal Cubature Formulae. One considers a cubature formula of the form (1) in which

$$I = \iint_{00}^{11} (f(x, y)\, dx\, dy \tag{2.1}$$

$$Q(f) = \sum_{p,q=0}^{r-1,s-1} [A_{00}^{pq} f^{(p,q)}(0, 0) + A_{0n}^{pq} f^{(p,q)}(0, 1) + A_{m0}^{pq} f^{(p,q)}(1, 0) +$$

$$+ A_{mn}^{pq} f^{(p,q)}(1,1)] + \sum_{i=1}^{m-1} \sum_{p,q=0}^{\mu,s-1} [A_{i0}^{pq} f^{(p,q)}(x_i, 0) + A_{in}^{pq} f^{(p,q)}(x_i, 1)] + \tag{2.2}$$

$$\sum_{j=1}^{n-1} \sum_{p,q=0}^{r-1,\nu} [A_{0j}^{pq} f^{(p,q)}(0, y_j) + A_{mj}^{pq} f^{(p,q)}(1, y_j)] + \sum_{i,j=1}^{m-1,n-1} \sum_{p,q=0}^{\mu,\nu} A_{ij}^{pq} f^{(p,q)}(x_i, y_j),$$

and

$$R_Q(f) = (-1)^{r+s} \iint_{00}^{11} M(x, y) f^{(r,s)}(x, y)\, dx\, dy +$$

$$+ \sum_{q=0}^{s-1} \frac{(-1)^{r+l}}{(l+1)!} \int_0^1 N_l(x) f^{(r,l)}(x, 1)\, dx + \sum_{k=0}^{r-1} \frac{(-1)^{k+s}}{(k+1)!} \int_0^1 L_k(y) f^{(k,s)}(1, y)\, dy, \tag{2.3}$$

where M is a two-dimensional monospline of the form (1.2), N_l and L_k are one-dimensional monosplines of the form

$$N_l(x) = \frac{x^r}{r!} + \sum_{i=1}^{m-1} \sum_{p=0}^{\mu} \lambda_i^{pl} \frac{(x-x_i)_+^{r-p-1}}{(r-p-1)!} + P_{r-1}^l(x), \quad (l = 0, \ldots, s-1)$$

$$L_k(y) = \frac{y^s}{s!} + \sum_{j=1}^{n-1} \sum_{q=0}^{\nu} \delta_j^{kq} \frac{(y-y_j)_+^{s-q-1}}{(s-q-1)!} + P_{s-1}^k(y), \quad (k = 0, \ldots, r-1) \tag{2.4}$$

where $P_{n-1}^i \in \Pi_{n-1}$, and

$$A_{oo}^{pq} = (-1)^{p+q+1} M^{(r-p-1, s-q-1)}(0+0,\ 0+0),$$

$$A_{on}^{pq} = (-1)^{p+q}\left[M^{(r-p-1, s-q-1)}(0+0,\ 1-0) - \frac{1}{(q+1)!} N_q^{(r-p-1)}(0+0)\right],$$

$$A_{mo}^{pq} = (-1)^{p+q}\left[M^{(r-p-1, s-q-1)}(1-0,\ 0+0) - \frac{1}{(p+1)!} L_p^{(s-q-1)}(0+0)\right],$$

$$A_{mn}^{pq} = (-1)^{p+q+1}\left[M^{(r-p-1, s-q-1)}(1-0,\ 1-0) - \right.$$
$$\left. - \frac{1}{(q+1)!} N_q^{(r-p-1)}(1-0) - \frac{1}{(p+1)!} L_p^{(s-q-1)}(1-0), \begin{pmatrix} p=0, \ldots, r-1 \\ q=0, \ldots, s-1 \end{pmatrix}\right.$$

$$A_{io}^{pq} = (-1)^{p+q}[M^{(r-p-1, s-q-1)}(x_i-0,\ 0+0) -$$
$$- M^{(r-p-1, s-q-1)}(x_i+0,\ 0+0)], \tag{2.5}$$

$$A_{in}^{pq} = (-1)^{(p+q+1} \{M^{(r-p-1, s-q-1)}(x_i-0,\ 1-0) -$$
$$- M^{(r-p-1, s-p-1)}(x_i+0, 1-0) - \frac{1}{(q+1)!}[N_q^{(r-p-1)}(x_i-0) -$$
$$- N_q^{(r-p-1)}(x_i+0)]\}, \begin{pmatrix} \mathrm{p}=0, \ldots, \mu \\ q=0, \ldots, s-1 \end{pmatrix}$$

$$A_{oj}^{pq} = (-1)^{p+q}[M^{(r-p-1, s-q-1)}(0+0, y_j-0) - M^{(r-p-1, s-q-1)}(0+0, y_j+0)],$$

$$A_{mj}^{pq} = (-1)^{p+q+1}\{M^{(r-p-1, s-q-1)}(1-0, y_j-0) -$$
$$- M^{(r-p-1, s-q-1)}(1-0, y_j+0) - \frac{1}{(p+1)!}[L_p^{(s-q-1)}(y_j-0) -$$
$$- L_p^{(s-q-1)}(y_j+0)]\}, \begin{pmatrix} p=0, \ldots, r-1 \\ q=0, \ldots, \nu \end{pmatrix}$$

$$A_{ij}^{pq} = (-1)^{p+q+1}[M^{(r-p-1, s-q-1)}(x_i-0,\ y_j-0) -$$
$$- M^{(r-p-1, s-q-1)}(x_i+0, y_j-0) - M^{(r-p-1, s-q-1)}(x_i-0,\ y_j+0) +$$
$$+ M^{(r-p-1, s-q-1)}(x_i+0,\ y_j+0)], \begin{pmatrix} p=0, \ldots, \mu \\ q=0, \ldots, \nu \end{pmatrix}$$

$(i = 1, \ldots, m-1;\ j = 1, \ldots, n-1)$.

From (2.3) it follows that such a cubature formula has the degree of exactitude $(r-1,\ s-1)$.

We shall denote the set of these cubature formulae with $\mathcal{F}_{rs}(m, n, \mu, \nu)$.

Let

$$\mathcal{K}_{rs} = \mathcal{M}_{rs} \times \mathcal{N}_r^s \times \mathcal{L}_s^r, \tag{2.6}$$

where $\mathcal{M}_{rs}$ is the set of two-dimensional monosplines of the form (1.2) and $\mathcal{N}_r$, $\mathcal{L}_s$ are the sets of one-dimensional monosplines from (2.4).

LEMMA. *There is a one-to-one correspondence between the set of cubature formulae* $\mathcal{F}_{rs}$ *and the set* $\mathcal{K}_{rs}$.

Proof. Let be $K \in \mathcal{K}_{rs}$. From (2.3), applying the general formula for integration by parts and taking into account that $M^{(r,s)} = 1$,

$$M^{(r,q)}(0, y) = \frac{y^{s-q}}{(s-q)!}, \; M^{(p,s)}(x, 0) = \frac{x^{r-p}}{(r-p)!}, \; N_l^{(r)} = 1, \; L_k^{(s)} = 1,$$
$$(l = 0, \ldots, s-1; \; k = 0, \ldots, r-1),$$

we obtain

$$R_Q(f) = \sum_{p,q=0}^{r-1,s-1} (-1)^{p+q} \Big\{ M^{(r-p-1,s-q-1)}(0+0, 0+0) f^{(p,q)}(0,0) -$$
$$- \Big[M^{(r-p-1,s-q-1)}(0+0, 1-0) - \frac{1}{(q+1)!} N_q^{(r-p-1)}(0, +0) \Big] f^{(p,q)}(0,1) -$$
$$- \Big[M^{(r-p-1,s-q-1)}(1-0, 0+0) - \frac{1}{(p+1)!} L_p^{(s-q-1)}(0+0) \Big] f^{(p,q)}(1,0) -$$
$$- \Big[M^{(r-p-1,s-q-1)}(1-0, 1-0) - \frac{1}{(q+1)!} N_q^{(r-p-1)}(1-0) -$$
$$- \frac{1}{(p+1)!} L_p^{(s-q-1)}(1-1) \Big] f^{(p,q)}(1, 1) \Big\} +$$
$$+ \sum_{i=1}^{m-1} \sum_{p,q=0}^{\mu, s-1} (-1)^{p+q-1} \Big\{ [M^{(r-p-1,s-q-1)}(x_i - 0, 0+0) -$$
$$- M^{(r-p-1,s-q-1)}(x_i + 0, 0+0)] f^{(p,q)}(x_i, 0) -$$
$$- \Big[M^{(r-p-1,s-q-1)}(x_i - 0, 1-0) - M^{(r-p-1,s-q-1)}(x_i + 0, 1-0) -$$
$$- \frac{1}{(q+1)!} (N_q^{(r-p-1)}(x_i - 0) - N_q^{(r-p-1)}(x_i + 0)) \Big] f^{(p,q)}(x_i, 1) \Big\} +$$
$$+ \sum_{j=1}^{n-1} \sum_{p,q=0}^{r-1,\nu} (-1)^{(p+q-1)} \Big\{ [M^{(r-p-1,s-q-1)}(0+0, y_j - 0) -$$
$$- M^{(r-p-1,s-q-1)}(0+0, y_j + 0)] f^{(p,q)}(0, y_j) -$$
$$- [M^{(r-p-1,s-q-1)}(1-0, y_j - 0) - M^{(r-p-1,s-q-1)}(1-0, y_j + 0) -$$
$$- \frac{1}{(p+1)!} (L_p^{(s-q-1)}(y_j - 0) - L_p^{(s-q-1)}(y_j + 0)) \Big] f^{(p,q)}(1, y_j) \Big\} +$$
$$+ \sum_{i,j=1}^{m-1,n-1} \sum_{p,q=0}^{\mu,\nu} (-1)^{p+q} [M(x_i - 0, y_j - 0) -$$
$$- M^{(r-p-1,s-q-1)}(x_i + 0, y_j - 0) - M^{(r-p-1,s-q-1)}(x_i - 0, y_j + 0) +$$
$$+ M^{(r-p-1,s-q-1)}(x_i + 0, y_j + 0)] f^{(p,q)}(x_i, y_j) + \int\!\!\int_{00}^{11} f(x, y)\, dx\, dy,$$

which is a cubature formula of the form (2.1) — (2.3).

Conversely, let us assume that $F \in \mathcal{F}_{rs}$. If $f \in C^{rs}(D)$, one considers the Taylor formula for the function f (see [6]), *i.e.*

$$f(x,y) = P_{r-1,s-1}(x,y) + r(x,y),$$

where $P_{r-1,s-1}$ is the Taylor polynomial of degree $(r-1, s-1)$, and

$$r(x,y) = \int_0^1\int_0^1 \frac{(x-t)_+^{r-1}}{(r-1)!}\frac{(y-\tau)_+^{s-1}}{(s-1)!} f^{(r,s)}(t,\tau)dtd\tau + \sum_{l=0}^{s-1}\frac{y^l}{l!}\int_0^1 \frac{(x-t)_+^{r-1}}{(r-1)!} f^{(r,l)}(t,0)dt +$$

$$+ \sum_{k=0}^{r-1}\frac{x^k}{k!}\int_0^1 \frac{(y-\tau)_+^{s-1}}{(s-1)!} f^{(k,s)}(0,\tau)d\tau.$$

Taking into account that F has the degree of exactitude $(r-1, s-1)$ it follows that

$$R_Q(f) = I(r) - Q(r) = R_Q(r),$$

or

$$R_Q(f) = (-1)^{r+s}\int_0^1\int_0^1 M(t,\tau)f^{(r,s)}(t,\tau)dtd\tau +$$

$$+ \sum_{l=0}^{s-1}\frac{(-1)^{r+l}}{(l+1)!}\int_0^1 N_l(x)f^{(r,l)}(t,0)\,dt + \sum_{k=0}^{r-1}\frac{(-1)^{k+s}}{(k+1)!}\int_0^1 L_k(y)f^{(k,s)}(0,\tau)d\tau, \tag{2.7}$$

where M is a two-dimensional monospline of the form (1.2), N_l and L_k are one-dimensional monosplines of the form (2.4). Thus the lemma is proved.

Let $f \in W^{(r,s)} L^2(D)$. From (2.7), we obtain

$$\sup_{f\in W^{(r,s)}L_2(D)} |R\ (f)| \leqslant \left(\int_0^1\int_0^1 M^2(t\tau,\ dtd\tau\right)^{1/2} + \sum_{l=0}^{s-1}\frac{1}{(l+1)!}\left(\int_0^1 N_l^2(t)dt\right)^{1/2} +$$

$$+ \sum_{k=0}^{r-1}\frac{1}{(k+1)!}\left(\int_0^1 L_k^2(\tau)d\tau\right)^{1/2}.$$

Because there exists a function $f_0 \in W^{(r,s)}L_2(D)$ for which

$$R(f_0) = \left(\int_0^1\int_0^1 M^2(t,\tau)dtd\tau\right)^{1/2} + \sum_{l=0}^{s-1}\frac{1}{(e+1)!}\left(\int_0^1 N_l^2(t)dt\right)^{1/2} +$$

$$+ \sum_{k=0}^{r-1}\frac{1}{(k+1)!}\left(\int_0^1 L_k^2(\tau)d\tau\right)^{1/2},$$

it follows that

$$\sup_{f\in W^{(r,s)}L_2(D)} |R_Q(f)| = \left(\int_0^1\int_0^1 M^2(t,\tau)dtd\tau\right)^{1/2} + \tag{2.8}$$

$$+\sum_{l=0}^{s-1}\frac{1}{(l+1)!}\left(\int_0^1 N_l^2(t)dt\right)^{1/2} + \sum_{k=0}^{r-1}\frac{1}{(k+1)!}\left(\int_0^1 L_k^2(\tau)d\tau\right)^{1/2}.$$

Taking into account the lemma, we have reduced the problem of determining the cubature formula which is optimal in $W^{(r,s)}\,L_2(D)$ to the problem of determining the monosplines $M \in \mathfrak{M}_{rs}$ and $N_q \in \mathfrak{N}_r$, $L_p \in \mathfrak{L}_s$, $(p = 0,\ldots, r-1;\ q = 0,\ldots,s-1)$ which, on D respectively $[0,1]$, are of least deviation fom zero, in the square mean.

THEOREM 2. *Let m, n, r, s be given natural numbers. Then in the set $\mathcal{F}_{rs}(m, n, r-1, s-1)$ there exists a unique cubature formula $\bar{F}$ which is optimal in the class of functionas $W^{(r,s)}\,L_2(D)$. Its knots and its coefficients have respectively the expresions* (2.12) *and* (2.13), *while the remainder has the expresion* (2.15)

Proof. Taking into account (2.8) and the above lemma it follows that the optimal formula corresponds to the monosplines $M \in \mathfrak{M}_{rs}$, $L_p \in \mathfrak{L}_s$, $N \in \mathfrak{N}_r$, $(p = 0,\ldots, r-1;\ q = 0,\ldots, s-1)$ which, on the corresponding domains, are of least deviation from zero, in the square mean.

From (2.7) taking $\mu = r-1$, $\nu = s-1$ and using the notations

$$1-t = u,\ 1-\tau = v,\ A_{m-i,\,n-j}^{r-p-1,\,s-q-1} = B_{ij}^{pq},\ 1-x_{m-i} = u_i,\ 1-y_{n-j} = v_j, \tag{2.9}$$

$(i = 0,\ldots, m-1;\ j=0,\ldots,n-1;\ p = 0,\ldots, r-1;\ q = 0,\ldots, s-1)$, we obtain

$$(-1)^{r+s}M(u,v) = \frac{u^r v^s}{r!\,s!} - \sum_{i,j=0}^{m-1,n-1}\sum_{p,q=0}^{r-1,s-1} B_{ij}^{pq}\frac{(u-u_i)_+^p}{p!}\frac{(v-v_j)_+^q}{q!},$$

which is a two-dimensional monospline from the set $\mathfrak{M}_{rs}(m, n, r-1, s-1)$. By the theorem 1 we know that $\|M\|_{L_2(D)}$ takes the minimal value for

$$u_i = \bar{u}_i = \frac{i}{m},\quad (i = 0,\ldots, m-1)$$

$$v_j = \bar{v}_j = \frac{j}{n},\quad (j = 0,\ldots, n-1) \tag{2.10}$$

and

$$B_{ij}^{pq} = \bar{B}_{ij}^{pq} = \bar{C}_i^p\,\bar{D}_j^q,\quad \begin{pmatrix} i = 0,\ldots, m-1, & p = 0,\ldots, r-1 \\ j = 0,\ldots, n-1 & q = 0,\ldots, s-1\end{pmatrix},$$

where

$$\overline{C}_o^p = \frac{(-1)^{r-p}}{r!} h^{r-p} X_r^{(p)}(1), \quad \overline{C}_i^p = \frac{1-(-1)^{r-p}}{r!} h^{r-p} X_r^{(p)}(1),$$
$$\overline{D}_0^q = \frac{(-1)^{s-q}}{s!} g^{s-q} X_s^{(q)}(1), \quad \overline{D}_j^q = \frac{1-(-1)^{s-q}}{s!} g^{s-q} X_s^{(q)}(1). \tag{2.11}$$

From (2.9) — (2.11) it follows that

$$\bar{x}_i = \frac{i}{m}, \; \bar{y}_j = \frac{j}{n}, \; (i = 0, \ldots, m; \; j = 0, \ldots, n) \tag{2.12}$$

and

$$\bar{A}_{ij}^{pq} = \alpha_i^p \beta_j^q, \quad \begin{pmatrix} i = 0, \ldots, m, & p = 0, \ldots, r-1 \\ j = 0, \ldots, n & q = 0, \ldots, s-1 \end{pmatrix}, \tag{2.13}$$

where

$$\alpha_i^p = \overline{C}_{m-i}^{r-p-1}, \quad \beta_j^q = \overline{D}_{n-j}^{s-q-1}.$$

Furthermore, $\bar{x}_i$, $\bar{y}_j$ and $\bar{A}_{ij}^{pq}$ are the unique values of the parameters x_i, y_j respectively A_{ij}^{pq} such that $||N_l||_{L_2[0,1]}$ and $||L_k||_{L_2[0,1]}$ take the minimal value.

Indeed, from (2.7), we obtain

$$(-1)^{r+l}\overline{N}_l(t) = \frac{(1-t)^r}{r!} - \left[(l+1)!\sum_{j=0}^{n}\sum_{q=0}^{s-1}\beta_j^q \frac{\bar{y}_j^l - q}{(e-q)!}\right]\left[\sum_{i=0}^{m}\sum_{p=0}^{r-1}\alpha_i^p \frac{(\bar{x}_i - t)^{r-p-1}}{(r-p-1)!}\right]$$

$$(-1)^{s+k}\overline{L}_k(\tau) = \frac{(1-\tau)^s}{s!} - \left[(k+1)!\sum_{i=0}^{m}\sum_{p=0}^{r-1}\alpha_i^p \frac{\bar{x}_i^{k-p}}{(k-p)!}\right]\left[\sum_{j=0}^{n}\sum_{q=0}^{s-1}\beta_j^q \frac{(\bar{y}_j - \tau)_+^{s-q-1}}{(s-q-1)!}\right].$$

Because

$$(l+1)!\sum_{j=0}^{n}\sum_{q=0}^{s-1}\beta_j^q \frac{\bar{y}_j^{l-q}}{(e-q)!} = 1, \quad (k+1)!\sum_{i=0}^{m}\sum_{p=0}^{r-1}\alpha_i^p \frac{\bar{x}_i^{k-p}}{(k-p)!} = 1,$$

it follows that

$$(-1)^{r+l}\overline{N}_l(t) = \frac{(1-t)^r}{r!} - \sum_{i=0}^{m}\sum_{p=0}^{r-1}\alpha_i^p \frac{(\bar{x}_i - t)_+^{r-p-1}}{(r-p-1)!},$$

$$(-1)^{s+k}\overline{L}_k(\tau) = \frac{(1-\tau)^s}{s!} - \sum_{j=0}^{n}\sum_{q=0}^{s-1}\beta_j^q \frac{(\bar{y}_j - \tau)_+^{s-q-1}}{(s-q-1)!},$$

which (see [1]) are the minimal monosplines in L_2 [0,1], and

$$\begin{aligned} \|\overline{N}_l\|_{L_2[0,1]} &= \frac{r!}{(2r)!\, m\sqrt{2r+1}} = \lambda \\ \|\overline{L}_k\|_{L_2[0,1]} &= \frac{s!}{(2s)!\, n\sqrt{2s+1}} = \delta. \end{aligned} \tag{2.14}$$

According to the theorem 1, (2.8) and (2.14), we finally obtain

$$R_{\overline{Q}}(f) = \left(\frac{\lambda^2}{(2s+1)(s!)^2} + \frac{\delta^2}{(2r+1)(r!)^2} - \lambda^2\delta^2\right)^{1/2} + \lambda\sum_{p=0}^{r-1}\frac{1}{(p+1)!} + \delta\sum_{q=0}^{s-1}\frac{1}{(q+1)!}. \tag{2.15}$$

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MONOSPLINE BIDIMENSIONALE ŞI FORMULE OPTIMALE DE CUBATURĂ

(Rezumat)

În această lucrare se introduce noţiunea de funcţie monsplină bidimensională şi se studiază monosplinul de tipul introdus care se abate, în metrica lui L_2, cel mai puţin de la zero pe domeniul considerat. Rezultatele obţinute sînt aplicate la construirea formulelor optimale de cubatură.

ДВУХМЕРНЫЕ МОНОСПЛАЙНЫ И ОПТИМАЛЬНЫЕ КУБАТУРНЫЕ ФОРМУЛЫ

(Резюме)

В работе вводится понятие двухмерного моносплайна и изучается моносплайн введенного типа, отклоняющийся в метрике L_2 меньше всего от нуля в рассматриваемой области. Полученные результаты применены к построению оптимальных кубатурных формул.

ASUPRA CONDIȚIILOR DE CONVERGENȚĂ LA METODA COARDEI ÎN SPAȚII SUPERMETRICE

SEVER GROZE

Considerăm ecuația operațională

$$P(x) = \theta \tag{1}$$

unde operatorul $P(x)$, neliniar și continuu, transformă spațiul supermetric complet X, în spațiul de același tip Y, $\theta \in Y$ fiind elementul nul al spațiului.

Pentru obținerea soluției ecuației (1) folosim metoda iterativă

$$x_{n+1} = x_n - \Lambda_n P(x_n) \tag{2}$$

unde $\Lambda_n = + [P_{x_n, x_{n-1}}]^{-1}$, $n = 0, 1, \ldots$, cunoscută sub denumirea de „metoda nemodificată a coardei".

În lucrarea [1] sînt date condiții suficient pentru convergența șirului $\{x_n\}$ furnizat de algoritmul (2) către soluția ecuației (1), condiții care îmbunătățeasc pe cele date în lucrările [2] și [3].

În lucrarea de față ne propunem să demonstrăm teoreme privitoare la existența și unicitatea soluției ecuației (1) obținută cu ajutorul algoritmului (2), slăbind condițiile date în [1]. Vom presupune că operatorul $P(x)$ admite diferențe divizate de ordinul I și II.

TEOREMA 1. *Dacă sînt îndeplinite condițiile*

1° *Există* $\Lambda = [P_{x^{(1)}, x^{(2)}}]^{-1}$, *oricare ar fi* $x^{(i)} \in S(x_0, r)$

$$i = 1, 2 \text{ și } \rho_{Y,X}(\Lambda) \leqslant B.$$

2° *Pentru aproximațiile inițiale* x_0, $x_{-1} \in S$ *avem*

$$\rho_Y(P(x_i)) \leqslant \eta_i, \quad i = 0, -1, \quad \eta_0 < \eta_{-1}$$

3° $\rho_{X^2, Y}(P_{x^{(1)}, x^{(2)}, x^{(3)}}) \leqslant k$, oricare ar fi $x^{(i)} \in S \quad i = 1, 2, 3$

4° $h_0 = B^2 k\, \eta_{-1} < 1$,

atunci ecuația (1) *admite o soluție* $x^* = \lim_{n\to\infty} x$, *șirul* $\{x_n\}$ *fiind construit cu ajutorul algoritmului* (2), *soluție ce aparține sferei* $S(x_0, r)$ *unde*

$$r = max\left\{\rho_X(x_0 - x_{-1}),\ B\eta_0 \sum_{n=0}^{\infty} h_0^{s_n}\right\}$$

s_n *fiind termenul general al șirului sumelor parțiale ale seriei Fibonacci* $(u_1 = u_2 = 1)$, *rapiditatea convergenței fiind dată de*

$$\rho_X(x^* - x_0) \leqslant Bh_0^{s_n}\eta_0 \sum_{n=0}^{\infty} h_0^{s_n}.$$

Demonstrație. Vom arăta că trecînd de la perechea de elemente x_0, x_{-1} la perechea x_1, x_0, condițiile 2° și 4° ale teoremei sînt satisfăcute, condițiile 1° și 3° fiind evident verificate.

Se observă că în baza condiției 1°, algoritmul (2) este aplicabil pentru $n = 0$ și x_1 astfel determinat aparține sferei $S(x_0, r)$ și avem

$$\rho_X(x_1 - x_0) \leqslant B\,\eta_0 \tag{4}$$

$$\rho_X(x_1 - x_{-1}) \leqslant B\,\eta_{-1}.$$

Considerăm egalitatea

$$P_{x_0, x_{-1}} - P_{x_0, x_1} = - P_{x_0, x_1, x_{-1}}(x_1 - x_0)$$

care înmulțită cu $x_1 - x_{-1}$, ținînd seama de (2), ne conduce la

$$P(x_1) = P_{x_0, x_1, x_{-1}}(x_1 - x_0)(x_1 - x_{-1}).$$

Ținînd seama de (4) se găsește că

$$\rho_Y(P(x_1)) \leqslant kB^2\eta_{-1}\,\eta_0 = h_0\,\eta_0 \equiv \eta_1 < \eta_0.$$

Avînd și $\rho_Y(P(x)) \leqslant \eta_0$ rezultă că este îndeplinită condiția 2° din teoremă și de către perechea de elemente x_1, x_0.

Avem deasemenea

$$h_1 = B^2k\,\eta_0 \leqslant B^2k\,\eta_{-1} = h_0 < 1$$

deci și condiția 4° este verificată.

Algoritmul (2) ne permite să determinăm apoi elementul $x_2 \in S(x_0, r)$ pentru care avem

$$\rho_X(x_2 - x_1) \leqslant B\,\eta_1 \tag{5}$$

$$\rho_X(x_2 - x_0) \leqslant B\,\eta_0.$$

Făcînd un raționament analog cu precedentul se poate arăta că și pentru perechea de elemente x_2, x_1 condițiile 2° și 4° ale teoremei sînt verificate.

Prin inducție se arată că oricare ar fi x_n construit cu ajutorul lui (2) condițiile 2° și 4° sînt îndeplinite și au loc relațiile

$$\rho_X(x_{n+1} - x_n) \leqslant B\,\eta_n \tag{6}$$

$$\rho_Y(P(x_n)) \leqslant \eta_n = h_{n-1}\,\eta_{n-1} \tag{7}$$

$$h_n = h_{n-1}\,h_{n-2} \leqslant h_0^{u_n+1} \tag{8}$$

unde u_n este termenul general al șirului Fibonacci.

Din (7) și (8) se deduce

$$\eta_n \leqslant \eta_0\,h_0^{s_n} \tag{9}$$

s_n fiind termenul general al șirului sumelor parțiale ale seriei Fibonacci.

Folosind relația (9), deducem

$$\rho_X(x_{n+p} - x_n) \leqslant B(\eta_n + \eta_{n+1} + \ldots + \eta_{n+p-1}) \leqslant$$

$$\leqslant B\,\eta_0 h_0^{s_n}(1 + h_0^{u_n} + h_0^{u_n+u_{n+1}} + \ldots + h_0^{u_n+\ldots+u_{n+p-2}}). \tag{10}$$

Deoarece pentru $n \to \infty$, $\rho_X(x_{n+p} - x_n) \to 0$ și avînd în vedere că spațiul supermetric X este complet, rezultă existența limitei x^*,

În (10) făcînd $p \to \infty$, se regăsește relația (3) care caracterizează rapiditatea convergenței procedeului, iar din

$$\rho_X(x_n - x_0) \leqslant B\,\eta_0(1 + h_0 + h_0^2 + \ldots + h_0^{s_n} + \ldots) =$$

$$= B\,\eta_0 \sum_{n=0}^{\infty} h_0^{s_n}$$

unde

$s_0 = 0$, dovedește că aproximațiile $x_n \in S(x_0, r)$.

Rămîne să arătăm că $x^* = \lim\limits_{n\to\infty} x_n$ este soluție a ecuației (1). Pentru aceasta considerăm relația (7)

$$\rho_Y\,P(x_n) \leqslant \eta_n.$$

Deoarece $\eta_n \to 0$ pentru $n \to \infty$, iar $n_n \to x^*$ pentru $n \to \infty$, în baza continuității operatorului $P(x)$, se deduce

$$P(x_n) \to P(x^*) \quad \text{pentru} \quad n \to \infty.$$

Observație. În cazul spațiilor Banach o teoremă analogă a fost demonstrată de către *M*. Balázs [4].

TEOREMA 2. *Dacă condițiile* 1° — 4° *ale teoremei* 1 *sînt verificate, atunci în sfera* $S(x_0, r)$ *ecuația* (1) *admite o singură soluție, limita șirului dat de algoritmul* (2).

Demonstrație. Fie $\bar{x}^*$ o soluție oarecare a ecuației operaționale (1), soluție ce aparține sferei S.

Dacă x_n este termenul general al șirului aproximațiilor succesive construit cu ajutorul algoritmului (2), ținînd seama de condiția 1° rezultă existența operatorului invers

$$\Lambda = [P_{\bar{x}^*, x_n}]^{-1}$$

a cărui normă generalizată verifică relația

$$\rho_{Y,X}(\Lambda) \leqslant B.$$

Considerînd relația evidentă

$$\bar{x}^* - x_n = [P_{\bar{x}^*, x_n}]^{-1} P_{\bar{x}^*, x_n}(\bar{x}^* - x_n),$$

și ținînd seama de definiția dată diferențelor divizate [5] și de ipoteza că $\bar{x}^*$ este o soluție a ecuației (1), vom avea

$$\bar{x}^* - x_n = - [P_{\bar{x}^*, x_n}]^{-1} \quad P(x_n) \tag{11}$$

și atunci, în baza relațiilor (7) și (8), avem

$$\rho_X(\bar{x}^* - x_n) \leqslant B \, \eta_0 \, h_0^{s_n},$$

ceea ce înseamnă că $\lim\limits_{n\to\infty} x_n = \bar{x}^*$. Din unicitatea limitei șirului $\{x_n\}$ rezultă că

$$\bar{x}^* = x^*$$

teorema fiind astfel demonstrată.

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ОБ УСЛОВИЯХ СХОДИМОСТИ МЕТОДА ХОРДЫ В СВЕРХМЕТРИЧЕСКИХ ПРОСТРАНСТВАХ

(Резюме)

Автор доказывает некоторые теоремы, касающиеся существования и единственности решения уравнения (1), используя алгоритм (2). Улучшаются результаты работ [1], [2] и [3], считая более общие пространства, а также изменяя гипотезы теорем.

SUR LES CONDITIONS DE CONVERGENCE AVEC LA MÉTHODE DE LA CORDE DANS LES ESPACES SUPERMETRIQUES

(Résumé)

L'auteur démontre le théorème concernant l'existence et l'unicité de la solution de l'équation (1), en employant l'algorithme (2) et en améliorant les résultats de [1], [2] et [3] tant par l'extension de l'espace que par la modification des hypothèses des théorèmes.

PROCÉDÉS DE TYPE RUNGE-KUTTA-FEHLBERG POUR L'APPROXIMATION DE LA SOLUTION DE L'ÉQUATION INTÉGRODIFFÉRENTIELLE DE PREMIER ORDRE DE TYPE VOLTERRA

MARIA MICULA

La méthode de Runge-Kutta pour l'intégration numérique des équations differentielles s'élargit, avec quelques modifications, à la résolution numérique des équations intégrales. Pour la première fois cette question a été étudiée par E. Aparo [1] et améliorée par H. Oulès [5] et P. Pouzet [6].

A. N. Lomakovich [4] a résolu numériquement le problème de Cauchy pour les équations intégrodifférentielles non-linéaires de type Volterra, en utilisant des procédés de type Runge-Kutta-Fehlberg, avec l'erreur d'ordre de $O(h^7)$ qui nécessite neuf substitutions dans l'équation intégrodifférentielle transformée.

Dans ce travail on donne une extension de la transformation utilisée par Lomakovich, obtenant des procédés de type Runge-Kutta-Fehlberg avec l'erreur d'ordre de $O(h^8)$ et $O(h^9)$, avec le même nombre de substitutions, dans l'équation intégrodifférentielle transformée.

1. Procédés d'ordre sept d'exactitude

§ **1.1.** Soit l'équation non-linéaire intégrodifférentielle de premier ordre, de type Volterra

$$z'(x) = G\left[x, z(x), \int_{x_0}^{x} g_1[x, s, z(s), z'(s)]\, ds\right] \tag{1}$$

avec la condition initiale

$$z(x_0) = z_0 \tag{2}$$

où x, s appartiennent à l'intervalle $[x_0, x_0 + M]$, $(M > 0)$, G, g_1 sont continues sur $D: x_0 \leqslant x \leqslant x_0 + M$, $x_0 \leqslant s \leqslant x$, $|z| < +\infty$, avec leurs dérivées jusqu'à l'ordre 9 inclusivement. L'équation intégrodifférentielle (1) peut être écrite sous la forme

$$\psi(x) = \int_{x_0}^{x} g[x, s, z(s), \psi(s)]ds, \tag{3}$$

$$z'(x) = G[x, z(x), \psi(x)] \tag{4}$$

avec la condition initiale (2).

En lieu des fonctions ψ et z, on introduit les fonctions φ et y avec les conditions :

1°. φ et y satisfont le système

$$\varphi(x) = \int_{x_0}^{x} f[x, s, y(s), \varphi(s)]\, ds \tag{5}$$

$$y'(x) = F[x, y(x), \varphi(x)]$$

et

$$y(x_0) = y_0 = z_0, \quad \varphi(x_0) = \psi(x_0) = 0 \tag{6}$$

2°.

$$\begin{gathered} \varphi^{(k)}(x_0) = 0, \\ k = 1, 2, 3, 4 \\ y^{(k)}(x_0) = 0 \end{gathered} \tag{7}$$

3°.

$$\left(\frac{\partial F}{\partial y}\right)_0 = 0 \qquad (\text{pour } x = s = x_0) \tag{8}$$

Les conditions (6) — (7) sont vérifiées par les transformations :

$$\psi(x) = \psi(x, \varphi) = \varphi(x) + \sum_{k=1}^{4} \frac{(x - x_0)^k}{k\,!} \left(\frac{\partial^{k-1} g[x, s, z, \psi]}{\partial s^{k-1}}\right)_{s=x_0} \tag{9}$$

$$z(x) = z(x, y) = y(x) + \sum_{k=1}^{4} \frac{(x - x_0)^k}{k\,!}\, z_0^{(k)} + A(x - x_0)(y - y_0) \tag{10}$$

La détermination du A se fait en satisfaisant la condition (8), avec la méthode exposée par A. Coţiu [3] pour les équations différentielles. On obtient $A = \left(\frac{\partial G}{\partial z}\right)_0$. Nous avons donc

$$\varphi(x) = \int_{x_0}^{x} f[x, s, y(s), \varphi(s)]ds = \int_{x_0}^{x} \{g[x, s, z(s, y), \psi(s, \varphi)] -$$

$$- \sum_{k=1}^{4} \frac{(s - x_0)^{k-1}}{(k-1)!} \left(\frac{\partial^{k-1} g[x, s, z, \psi]}{\partial s^{k-1}}\right)_{s=x_0}\Bigg\} ds \tag{11}$$

$$y'(x) = F[x, y(x), \varphi(x)] = \frac{1}{1 + \left(\frac{\partial G}{\partial z}\right)_0 (x - x_0)} \{G[x, z(x, y), \psi(x, \varphi)] -$$

$$- \sum_{k=1}^{4} \frac{(x - x_0)^{k-1}}{(k-1)!} z_0^{(k)} - \left(\frac{\partial G}{\partial z}\right)_0 (y - y_0)\Bigg\} \tag{12}$$

avec la condition initiale $y(x_0) = y_0$.

La résolution de l'équation intégrodifférentielle (1) avec la condition (2) s'est réduite à la résolution du système (11), (12). On montre facilement que les conditions (6)—(8) attirent aussi les conditions: $f_{x^k s^p}(x, x_0, y_0, \varphi_0) = 0$, $k = \overline{0, 6 - p}$, $p = \overline{0,3}$, $(F_{x^k})_0 = 0$, $k = \overline{0,3}$.

§ 1.2. *Formules pour passer de* x_0 *à* $x_0 + h$. On divise l'intervalle $[x_0, x_0 + M]$ en n parties égales, avec le pas h, $x_i = x_0 + ih$, $x_n = x_0 + M$, $i = \overline{0, n}$.

Pour calculer la valeur de la solution du système (11), (12), pour le noeud $x_0 + h$, on développe en série de Taylor, par rapport à h, les fonctions φ et y. En écrivant les conditions nécessaires pour que les développements de Taylor de φ et y coïncident avec celles de leurs approximations $\tilde{\varphi}$ et $\tilde{y}$ données par

$$\tilde{\varphi}(x) = \sum_{i=1}^{3} A_{3i} k_i, \quad \tilde{y}(x) = y_0 + \sum_{i=1}^{3} B_{3i} h_i \tag{13}$$

où

$$\begin{aligned} k_i &= hf(x_0 + h, x_0 + \alpha_i h, y_0 + B_{i-11} h_1 + B_{i-12} h_2, \varphi_0 + A_{i-11} k'_{i-1} + A_{i-12} k'_i) \\ k'_1 &= hf(x_0 + \alpha_2 h, x_0 + \alpha_1 h, y_0, \varphi_0) \\ k'_2 &= hf(x_0 + \alpha_3 h, x_0 + \alpha_1 h, y_0, \varphi_0) \\ k'_3 &= hf(x_0 + \alpha_3 h, x_0 + \alpha_2 h y_0 + B_{11} h_1, \varphi_0 + A_{11} k'_1) \\ h_i &= hF(x_0 + \beta_i h, y_0 + B_{i-11} h_1 + B_{i-12} h_2, \varphi_0 + A_{i-11} k'_{i-1} + A_{i-12} k'_i) \\ &(i = 1, 2, 3), \ (A_{01} = A_{02} = A_{12} = B_{01} = B_{02} = B_{12} = 0), \end{aligned} \tag{14}$$

jusqu'à l'ordre de h^7 inclusivement, on obtient un système algébrique non-linéaire de 7 équations avec 9 inconnues. On voit donc que nous avons une classe des procédés.

Prenant $\alpha_3 = 1$, $\alpha_1 = \frac{2}{5}$ on obtient la solution suivante:

$$A_{31} = B_{31} = \frac{625}{1344},\ A_{32} = B_{32} = \frac{6875}{21504},\ A_{33} = B_{33} = \frac{2}{35},\ A_{11} = B_{11} = \frac{32}{11}$$

$$A_{21} = B_{21} = -\frac{4375}{384} A_{22} = B_{22} = \frac{6875}{6144},\ \alpha_1 = \frac{4}{5} \tag{15}$$

On a obtenu un procédé avec une erreur d'ordre de $O(h^8)$, avec 9 substitutions, les fonctions $\widetilde{\varphi}$ et $\widetilde{y}$ étant données par les formules

$$\widetilde{\varphi}(x) = \frac{1}{107520}(50000\, k_1 + 34375\, k_2 + 6144\, k_3)$$

$$\widetilde{y}(x) = \frac{1}{107520}(50000\, h_1 + 34375\, h_2 + 6144\, h_3) + y_0 \tag{16}$$

où h_i, k_i sont données dans (14) pour les valeurs des constantes de (15). Ainsi on a trouvé les approximations ψ et z en $x_0 + h$, par les relations (9), (10), avec l'erreur d'ordre de $O(h^8)$.

§ **1.3.** *Formules pour passer de* x_{n-1} *à* x_n $(n \geqslant 2)$. L'équation intégrale (3) peut être écrite sous la forme

$$\psi(x) = \sum_{k=1}^{n-1} \int_{x_{k-1}}^{x_k} g[x, s, z(s), \psi(s)]ds + \int_{x_{n-1}}^{x} g[x, s, z(s), \psi(s)]ds \tag{17}$$

Introduisant les notations

$$\psi_{n-1}(x) = \int_{x_{n-1}}^{x} g[x, s, z(s), \psi(s)]ds$$

$$\mu_k(x) = \int_{x_{k-1}}^{x_k} g[x, s, z(s), \psi(s)]ds$$

$$\tau_{n-1}(x) = \sum_{k=1}^{n-1} \mu_k(x),\ \tau_0(x) = 0,$$

l'équation (17) devient

$$\psi(x) = \tau_{n-1}(x) + \psi_{n-1}(x) \tag{18}$$

On obtient ainsi le système

$$\psi_{n-1}(x) = \int_{x_{n-1}}^{x} g[x, s, z(s), \tau_{n-1}(s) + \psi_{n-1}(s)]ds$$

$$z'(x) = G[x, z(x), \tau_{n-1}(x) + \psi_{n-1}(x)].$$

Nous supposons connues les valeurs des solutions ψ et z dans l'intervalle $[x_0, x_{n-1}]$. Pour avoir l'approximation ψ en x_n, il faut connaître les approximations pour τ_{n-1} et ψ_{n-1}en x_n. Soit $\overline{\mu}_k$ les approximations de μ_k $(k = \overline{1, n-1})$, pour $x \in [x_0, x_0 + M]$. Alors

$$\overline{\tau}_{n-1}(x) = \sum_{k=1}^{n-1} \overline{\mu}_k(x)$$

et par conséquent

$$\psi^*_{n-1}(x) = \int_{x_{n-1}}^{x} g[x, s, z^*(s), \overline{\tau}_{n-1}(s) + \psi^*_{n-1}(s)]ds \tag{19}$$

$$z^{*\prime}(x) = G[x, z^*(x), \overline{\tau}_{n-1}(x) + \psi^*_{n-1}(x)] \tag{20}$$

un système des équations intégrodifferentielles dont nous voulons déterminer la solution avec la condition initiale $z^*(x_{n-1}) = \tilde{z}(x_{n-1})$. Suivant le raisonnement de § 1.1, on introduit les fonctions φ^*_{n-1} et y^* qui satisfont les conditions (5) — (8) dans $x = x_{n-1}$, par les relations

$$\psi^*_{n-1}(x) = \varphi^*_{n-1}(x) + \sum_{k=1}^{4} \frac{(x - x_{n-1})^k}{k!} \left(\frac{\partial^{k-1} g[x, s, z^*, \psi^*_{n-1}]}{\partial s^{k-1}} \right)_{s=x_{n-1}} \tag{21}$$

$$z^*(x) = y^*(x) + \sum_{k=1}^{4} \frac{(x - x_{n-1})^k}{k!} z^{(k)}_{n-1} + \left(\frac{\partial G}{\partial z^*}\right)_{x=s=x_{n-1}} (y^* - y^*_{n-1})(x - x_{n-1}) \tag{22}$$

À l'aide des formules (16) on obtient les approximations correspondantes pour φ^*_{n-1}et y^*, dans x_n, qui remplacées dans les formules (21) — (22) nous donnent les approximations ψ^*_{n-1} et $\tilde{z}$ dans $x = x_n$. Il faut déterminer les approximations $\overline{\mu}_n$. Ayant en vue l'expression de ψ de (18) nous pouvons écrire

$$\mu_n(x) = \int_{x_{n-1}}^{x_n} g[x, s, z(s), \overline{\tau}_{n-1}(s) + \psi^*_{n-1}(s)]ds$$

Nous introduisons les notations

$$\mu_n^*(x) = \int_{x_{n-1}}^{x_n} g[x, s, z^*(s), \bar{\tau}_{n-1}(s) + \psi_{n-1}^*(s)]ds$$

$$\sigma_x^*(x) = \int_{x_{n-1}}^{x_n} f[x, s, y^*(s), \varphi_{n-1}^*(s)]ds$$

Tenant compte des relations (21), (22), on obtient

$$\sigma_n^*(x) = \int_{x_{n-1}}^{x_n} g[x, s, z^*(s), \bar{\tau}_{n-1}(s) + \psi_{n-1}^*(s)]ds - \sum_{k=1}^{4} \frac{(x_n - x_{n-1})^k}{k!} \cdot$$

$$\left(\frac{\partial^{k-1} g[x, s, z^*, \bar{\tau}_{n-1} + \psi_{n-1}^*]}{\partial s^{k-1}}\right)_{s=x_{n-1}}$$

ou

$$\mu_n^*(x) = \sigma_n^*(x) + \sum_{k=1}^{4} \frac{h^k}{k!}\left(\frac{\partial^{k-1} g[x, s, z^*, \bar{\tau}_{n-1} + \psi_{n-1}^*]}{\partial s^{k-1}}\right)_{s=x_{n-1}}$$

σ_n^* on obtient par (16) pour x_{n-1}. Ainsi on a obtenu les approximations μ_n. Dans [2] est considérée une autre méthode de détermination de μ_n.

2. Procédés d'ordre huit d'exactitude

§ **2.1.** On déterminera, d'abord, la transformation qui permet la détermination des procédés d'ordre 8 d'exactitude.

On introduit les fonctions φ et y tel que

1°. $$\varphi(x) = \int_{x_0}^{x} f(x, s, y(s), \varphi(s))\, ds \tag{23}$$

$$y'(x) = F[x, y(x), \varphi(x)]$$

et $$y(x_0) = y_0 = z_0,\ \varphi(x_0) = \psi(x_0) = 0 \tag{24}$$

2°. $$\varphi^{(k)}(x_0) = 0,\ y^{(k)}(x_0) = 0,\ k = \overline{1,5} \tag{25}$$

3°. $$\left(\frac{\partial F}{\partial y}\right)_0 = 0 \tag{26}$$

La transformation qui vérifie les conditions ci-dessus est

$$\psi(x) = \psi(x, \varphi) = \varphi(x) + \sum_{k=1}^{5} \frac{(x - x_0)^k}{k!} \left(\frac{\partial^{k-1} g[x, s, z, \psi]}{\partial s^{k-1}}\right)_{s=x_0}$$

$$z(x) = z(x, y) = y(x) + \sum_{k=1}^{5} \frac{(x - x_0)^k}{k!} z_0^{(k)} + \left(\frac{\partial G}{\partial z}\right)_0 (x - x_0)(y - y_0)$$

On obtient

$$\varphi(x) = \int_{x_0}^{x} f(x, s, y(s), \varphi(s))\, ds = \int_{x_0}^{x} \Bigg\{ g[x, s, z(s, y), \psi(s, \varphi)] -$$

$$- \sum_{k=1}^{5} \frac{(s - x_0)^k}{(k-1)!} \left(\frac{\partial^{k-1} g[x, s, z, \psi]}{\partial s^{k-1}}\right)_{s=x_0} \Bigg\}\, ds$$

$$y'(x) = F[x, y(x), \varphi(x)] = \frac{1}{1 + \left(\frac{\partial G}{\partial z}\right)_0 (x - x_0)} \Bigg\{ G[x, z(x, y), \psi(x, \varphi)] -$$

$$- \sum_{k=1}^{5} \frac{(x - x_0)^k}{k!} z_0^{(k)} - \left(\frac{\partial G}{\partial z}\right)_0 (y - y_0) \Bigg\}$$

avec la condition intiale $y(x_0) = y_0$.
Les conditions (24) — (26) attirent les relations

$$f_{x^k s^p}(x, x_0, y_0, \varphi_0) = 0, \quad k = \overline{0,7 - p}, \quad p = \overline{0,4}, \quad (F_x k)_0 = 0, \quad k = \overline{0,4}.$$

§ **2.2.** *Formules pour passer de* x_0 *à* $x_0 + h$. Suivant le raisonnement de § 1.2., on obtient

$$\varphi(x_0 + h) = \sum_{k=6}^{8} \frac{h^k}{k!} \varphi_0^{(k)} + 0(h^9) \tag{27}$$

$$y(x_0 + h) = \sum_{k=6}^{8} \frac{h^k}{k!} y_0^{(k)} + y_0 + 0(h^9) \tag{28}$$

Le schéma de type Runge-Kutta que nous avons utilisé est le même que celui de § 1.2. Par identification des coefficients des même puissances de h du développement en série Taylor des fonctions φ, et y, au voisinage de x_0, données respectivement par (27), (28) et $\widetilde{\varphi}$, $\widetilde{y}$, on obtient un système algébrique non-linéaire de 7 équatons avec 9 inconnues. Donc nous avons une classe de procédés. Prenant $\alpha_3 = 1$, $\alpha_1 = \frac{2}{3}$, on obtient la solution

$$A_{31} = B_{31} = \frac{15625}{3225}, \quad A_{32} = B_{32} = \frac{15625}{49152}, \quad A_{33} = B_{33} = \frac{29}{504} \tag{29}$$

$$A_{11} = B_{11} = \frac{32}{7}, \quad A_{21} = B_{21} = -\frac{9375}{464}, \quad A_{22} = B_{22} = \frac{65625}{59392} \quad \alpha_2 = \frac{4}{5}$$

On a obtenu un procédé avec l'erreur de l'ordre de $0(h^9)$, les fonctions $\tilde{\varphi}$ et $\tilde{y}$ étant données par les formules (13), (14) avec les coefficients (29).

§ 2.3. *Formules pour passer de x_{n-1} à x_n $(n \geqslant 2)$.* La déduction des formules pour passer de x_{n-1} à x_n se fait comme dans § 1.3, en tenant compte de formules obtenues dans § 2.2.

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PROCEDEE DE TIP RUNGE-KUTTA-FEHLBERG PENTRU APROXIMAREA SOLUŢIEI ECUAŢIEI INTEGRO-DIFERENŢIALE DE ORDINUL ÎNTÎI DE TIP VOLTERRA

(R e z u m a t)

În această lucrare se dă o extindere a transformării folosite de L o m a k o v i c i, obţinîndu-se procedee de tip Runge-Kutta-Fehlberg cu eroarea de ordinul lui $O(h^8)$ şi $O(h^9)$ cu acelaşi număr de substituţii în ecuaţia integro-diferenţială transformată.

МЕТОДЫ ТИПА РУНГЕ-КУТТА-ФЕЛЬБЕРГА ДЛЯ ПРИБЛИЖЕНИЯ РЕШЕНИЯ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ТИПА ВОЛЬТЕРРА ПЕРВОГО РОДА

(Р е з ю м е)

В статье даётся одно обобщение преобразования, использованного Л о м а к о в и ч е м и получаются методы типа Рунге-Кутта-Фельберга с погрешностью порядка $O(h^8)$ и $O(h^9)$ с одним и тем же числом подстановок в преобразованное интегро-дифференциальное уравнение.

CONVERGENȚA ȘIRURILOR VALORILOR UNOR OPERATORI CONVOLUTIVI ÎN R^m

GRIGOR MOLDOVAN

1. Problema convergenței operatorilor liniari pozitivi relativi la funcții de mai multe variabile i-a preocupat pe mulți matematicieni. Aceasta a revenit, de fapt, la extinderea teoremei lui B o h m a n-K o r o v k i n la funcții de mai multe variabile. Vom da formularea acestei teoreme, aici într-un caz mai general, cum se face în [2].

Fie A un spațiu topologic Husdorff compact și fie funcțiile reale continue pe $A : f_1, f_2, \ldots, f_s$. Presupunem că ele au următoarea proprietate:

există funcțiile continue $a_i(y)$, $y \in A$, $i = 1, 2, \ldots, s$ astfel ca

$$P_y(x) = \sum_{i=1}^{s} a_i(y) f_i(x) \tag{1}$$

să fie pozitivă și egală cu zero dacă și numai dacă $x = y$.

Notăm cu $C[A]$ mulțimea funcțiilor reale continue pe compactul A.

TEOREMA 1. *Dacă funcțiile $f_1, f_2, \ldots, f_s$ satisfac proprietatea de mai sus (1) și dacă (L_n) este un șir de operatori liniari pozitivi care aplică $C[A]$ în ea însăși și satisface condițiile*

$$L_i(f_i, x) \rightrightarrows f_i(x), \ x \in A, \ i = 1, 2, \ldots, s$$

atunci

$$L_n(f; x) \rightrightarrows f, \ \forall f \in C[A].$$

Observații. Dacă $A = [a, b]$ atunci $P_y(x) = (y - x)^2$ și sistemul de funcții $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2$ are proprietatea (1). Numărul acestor funcții de probă este minim. Dacă $A = [a_1, b] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$ atunci $P_y(x) = \sum_{i=1}^{m} (y_i - x_i)^2$ și astfel avem $2m + 1$ funcții de probă: $1, x_i, x_i^2, i = 1, 2, \ldots, m$. În acest caz numărul funcțiilor de probă considerat aici nu este minim. În cazul funcțiilor de două variabile, de exemplu, după cum demonstrează V. I. Volkov [8], drept funcţii de probă se pot considera funcţiile 1, x_1, x_2, $x_1^2 + x_2^2$.

2. Vom da în cele ce urmează definiţia operatorilor convolutivi în R^m [5], care sînt generalizări ale cazului unidimensional [3, 4].

Notăm cu P_n, $n = (n_1, n_2, \ldots, n_m)$ un polinom în m variabile de grad efectiv n_i în raport cu a i-a variabilă, $i = 1, 2, \ldots, m$, iar prin $P_{\bar{n}}$ vom înţelege un polinom în m variabile, de grad efectiv $\bar{n}$ în raport cu fiecare dintre variabile. Dacă $n = (n_1, n_2, \ldots, n_m)$ atunci prin $\bar{n}$ înţelegem $\bar{n} = \sum_{i=1}^{m} n_i$. Prin $\mathfrak{L}^m(E)$ notăm mulţimea tuturor şirurilor de polinoame în m variabile $P = (P_n)_{n \in N^m}$, $P_n : E \to R$, $E \subseteq R^m$.

Vom folosi notaţiile:

$$\sum_{k=0}^{n} = \sum_{k_1=0}^{n_1} \cdots \sum_{k_m=0}^{n_m}, \qquad n \in N^m; \tag{2}$$

$$\sum_{k<n} = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \cdots \sum_{k_m=0}^{n-k_1-\ldots-k_m-1}, \qquad n \in N; \tag{3}$$

unde $N = \{0, 1, 2, \ldots\}$;

$\mathfrak{F}(A)$ — mulţimea funcţiilor reale definite pe A.

Definiţia 1. Relaţiile

$$Q_n(u, v) = \sum_{k=0}^{n} P_k^{(1)}(u)\, P_{n-k}^{(2)}(v), \; k, n \in N^m, \tag{4}$$

$$u, v, \in E, \quad P^{(1)}, \quad P^{(2)} \in \mathfrak{L}^m(E), \quad E \subseteq R^m,$$

$$Q'_n(u, v) = \sum_{\bar{k}<n} P_k^{(1)}(u)\, P_{n-k}^{(2)}(v), \; n \in N, \; k \in N^m; \tag{5}$$

$$u \in E, \; v \in G; \quad P^{(1)} \in \mathfrak{L}^m(E), \quad P^{(2)} \in \mathfrak{L}(G); \quad E \subseteq R^m, \; G \subseteq R$$

determinăm două tipuri de operaţii

$$*_1 : \mathfrak{L}^m(E) \times \mathfrak{L}^m(E) \to \mathfrak{F}(E^2), \; E \subseteq R^m$$

$$*_2 : \mathfrak{L}^m(E) \times \mathfrak{L}(G) \to \mathfrak{F}(E \times G), \; E \subseteq R^m, \; G \subseteq R,$$

numite *convoluţii discrete*.

Definiţia 2. Numim *convoluţie binomă* o convoluţie care se bucură de proprietatea

$$(P *_1 P)(u, v) = P(u + v) \tag{6}$$

sau

$$(P *_2 Q)(u, v) = Q(\bar{u} + v) \tag{7}$$

pentru anumite şiruri de polinoame.

Cele două operații „$*_1$", „$*_2$" conduc la două tipuri de operatori convolutivi în R^m.

Fie $u: X \to E$, $v: X \to E$, $X, E \subseteq R^m$ două funcții vectoriale de o variabilă vectorială. Alegem două elemente $P^{(1)}$ și $P^{(2)}$ din $\mathfrak{L}^m(E)$ pe care le fixăm și să considerăm R^X mulțimea funcțiilor reale definite pe X. Definim operatorul

$$L_n(\cdot\,; P^{(1)}, P^{(2)}; x): R^X \to R^X, \quad n \in N^m$$

astfel

$$L_n(f; P^{(1)}, P^{(2)}; x) = L_n(f; x) = A_n(x) \sum_{k=0}^{n} f(x_k^{(n)})\, P_k^{(1)}(u(x))\, P_{n-k}^{(2)}(v(x)), \tag{8}$$

unde

$$A_n(x) = [Q_n(u(x), v(x))]^{-1} < \infty. \tag{9}$$

Analog se definesc operatorii

$$L'_n(f; P^{(1)}, P^{(2)}; x): R^X \to R^X, \quad n \in N$$

astfel

$$L'_n(f; P^{(1)}, P^{(2)}; x) = L'_n(f; x) = A'_n(x) \sum_{\overline{k} \leqslant n} f(x_k^{(n)})\, P_k^{(1)}(u(x))\, P_{n-k}^{(2)}(v(x)) \tag{10}$$

unde

$$A'_n(x) = [Q'_n(u(x), v(x))]^{-1} < \infty \tag{11}$$

și $u: X \to E$, $u = u(x)$ este o funcție vectorială de o variabilă vectorială, iar $v: X \to G$ este o funcție reală de o variabilă vectorială.

Operatorii convolutivi (8) și (10) corespunzători celor două tipuri de convoluții sînt operatori liniari. Dacă ei sînt pozitivi, atunci pentru stabilirea convergenței lor uniforme către f se aplică teorema 1, sau consecințe ale ei. În cazul cînd sînt numai continui atunci se poate aplica teorema lui Banach-Steinhaus.

Avînd în vedere particularitatea operatorilor considerați, aceea de a fi convolutivi, se pot pune în evidență anumite proprietăți speciale, mai cu seamă luînd în considerare exemplele de astfel de operatori [5].

3. Ne vom referi întîi la operatori convolutivi pozitivi, definiți cu ajutorul operației de convoluție de primul tip. Convergența acestor operatori se reduce la convergența unor operatori unidimensionali în cazul cînd mulțimea $\mathfrak{L}^m(E)$, $E \subseteq R^m$, deci operatorii L_n, prezintă anumite particularități.

Notăm cu

$$X_i = \{x_j \in R \mid \exists x_i \in R,\ j = 1, 2, \ldots, i-1,\ i+1, \ldots, m;\ (x_1, \ldots, x_m) \in X\}. \tag{12}$$

Evident, avem $X = \bigtimes_{i=1}^{m} X_i$.

Pentru studiul convergenței operatorilor convolutivi vom considera cazul cînd X_i, $i = 1, 2, \ldots, m$, X sînt mulțimi compacte, iar $R^X = C[X]$, $R^{X_i} = C[X_i]$, $i = 1, 2, \ldots, m$ unde prin $C[X]$ am notat mulțimea funcțiilor continue pe compactul X.

Avem

TEOREMA 2. *Fie $L_n : C[X] \to C[X]$ un operator convolutiv pozitiv m-dimensional, iar $L_{n_i} : C[X_i] \to C[X_i]$, $n_i \in N$, $i = 1, 2, \ldots, m$ operatori convolutivi pozitivi unidimensionali.*

Dacă L_n admite descompunerea în operatori unidimensionali

$$L_n = L_{n_1} L_{n_2} \ldots L_{n_m} \tag{13}$$

atunci din convergența uniformă a șirului valorilor operatorilor $L_{n_i} h$ către h $\forall h \in C[X_i]$, rezultă convergența uniformă și șirului valorilor operatorilor $L_n f$ către f, $\forall f \in C[X]$.

Demonstrație. Descompunerea (13) este înțeleasă în sensul următor:

$$L_n(f; x) = L_{n_1, \ldots, n_m} f = L_{n_1} (L_{n_2} \ldots (L_{n_m} f) \ldots) \tag{14}$$

unde

$$L_{n_i} f = L_{n_i}(f(x_1, \ldots, x_{i-1}, t_i, x_{i+1}, \ldots, x_m) ; x_i) \tag{15}$$

iar aici variabilele $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m$ sînt considerate ca parametri.

Operatorul L_n este un operator liniar pozitiv, deci putem aplica teorema 1 pentru stabilirea convergenței lui. După cum am menționat mai sus, e suficient să dovedim această convergență pentru sistemul de funcții 1, x_i, x_i^2, $i = 1, 2, \ldots, m$. Or, întîi din faptul că

$$L_{n_i}(1 ; x_i) = 1, \quad i = 1, 2, \ldots, m \tag{16}$$

rezultă

$$L_n(1 ; x) = 1. \tag{17}$$

De fapt, această identitate rezultă din însăși definiția operatorilor convolutivi.

Apoi în baza relației

$$L_{n_i}(t_j^p ; x_i) = t_j^p, \ i \neq j, \ p \in R ; i, j = 1, 2, \ldots, m \tag{18}$$

avem

$$L_n(t_j^p ; x_1, \ldots, x_m) = L_{n_j}(t_j^p ; x_j), p = 1, 2 ; j = 1, 2, \ldots, m. \tag{19}$$

Ținînd seama de descompunerea (13), rezultă relația (19). Acum, folosind teorema 1, rezultă concluzia teoremei enunțate.

În lucrarea [5] am dat cîteva exemple de operatori convolutivi pozitivi relativi la funcţii de mai multe variabile.

Considerăm următorii operatori convolutivi pozitivi:

$$L_{n_1, \ldots, n_m}^{\begin{bmatrix} s_1, \ldots, s_m \\ \alpha_1, \ldots, \alpha_m \\ \beta_1, \ldots, \beta_m \end{bmatrix}}(f; x_1, \ldots, x_m) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_m=0}^{n_m} f(x_{k_1}^{(n_1)}, \ldots, x_{k_m}^{(n_m)}) \cdot \tag{20}$$

$$\cdot \prod_{i=1}^{m} \binom{n_i}{k_i} \frac{s_i(x_i) \prod_{\nu=1}^{k_i-1} [s_i(x_i) + k_i \beta_i + \nu\alpha_i] \prod_{\mu=1}^{n_i-k_i-1} [1 - s_i(x_i) + (n_i - k_i)\beta_i + \mu\alpha_i]}{\prod_{\tau=0}^{n_i-1} (1 + n_i \beta_i + \tau\alpha_i)}$$

$$\overline{L}_{n_1, \ldots, n_m}^{\begin{bmatrix} s_1, \ldots, s_m \\ \alpha_1, \ldots, \alpha_m \\ \beta_1, \ldots, \beta_m \end{bmatrix}}(f; x_1, \ldots, x_m) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_m=0}^{n_m} f(x_{k_1}^{(n_1)}), \ldots, x_{k_m}^{(n_m)}) \cdot \tag{21}$$

$$\cdot \prod_{i=1}^{m} \binom{n_i}{k_i} \frac{s_i(x_i) \prod_{\nu=1}^{k_i-1} [s_i(x_i) + \beta_i k_i + \alpha_i\nu] \cdot [1 - s_i(x_i)] \prod_{\mu=1}^{n_i-k_i-1} [1 - s_i(x_i) + \beta_i(n_i - k_i) + \mu\alpha_i]}{\prod_{\tau=1}^{n_i-1} (1 + n_i \beta_i + \tau\alpha_i)}$$

Aceşti operatori constituie generalizări ale operatorilor consideraţi de autor în lucrările [3] şi [4], pentru funcţii de mai multe variabile.

Să considerăm cazul cînd $x_{k_i}^{(n_i)} = \frac{k_i}{n_i}$, $s_i(x_i) = x_i$, $i = 1, 2, \ldots, m$ şi $X = [0, 1]^m$. Pentru prescurtare să notăm operatorii (20) cu $L_n^{[s, \alpha, \beta]}$, operatorii (21) cu $\overline{L}_n^{[s, \alpha, \beta]}$, iar în ipotezele de mai sus, asupra nodurilor şi funcţiilor s, operatorii corespunzători să-i notăm cu $L_n^{[\alpha, \beta]}$, respectiv $\overline{L}_n^{[\alpha, \beta]}$. Se constată uşor că operatorul $L_n^{[0, 0]}$ este operatorul lui Bernstein [2] că $L_n^{[\alpha, 0]}$ este operatorul lui Stancu [6] şi că $L_n^{[0, \beta]}$ este operatorul lui Cheney-Sharma generalizat [5]. Particularizări analoage se obţin şi pentru operatorul $\overline{L}_n^{[\alpha, \beta]}$. Vom nota

$L_n^{[0, 0]}f = \overline{L}_n^{[0, 0]}f = B_n f$, operatorul lui Bernstein;
$L_n^{[\alpha, 0]}f = \overline{L}_n^{[\alpha, 0]}f = P_n^{[\alpha]}f$, operatorul lui Stancu;
$L_n^{[0, \beta]}f = P_n f$, operatorul lui Cheney-Sharma;
$\overline{L}_n^{[0, \beta]}f = Q_n f$, operatorul lui Cheney-Sharma.

Ca o consecinţă a teoremei 2 şi a teoremelor din lucrarea [4] avem următorul corolar:

COROLAR 1. Fie $X = [0,1]^m$, *nodurile* $x_{k_i}^{(n_i)} = \frac{k_i}{n_i}$, $k_i = 0,1 \ldots, n_i$, $s_i(x_i) = x_i$, $i = 1, 2, \ldots, m$. *Atunci pentru* $\forall f \in C[X]$ *și* $n \in N^m$ *sînt adevărate următoarele afirmații:*

a) $B_n f \rightrightarrows f$;

b) *Dacă* $0 \leqslant \alpha = \alpha_n$, $\lim\limits_{n\to\infty} \alpha_n = 0$, $\alpha \in R^m$, *atunci* $P_n^{[\alpha]} f \rightrightarrows f$;

c) *Dacă* $0 \leqslant \beta = o(n^{-1})$ *atunci* $P_n f \rightrightarrows f$ *și* $Q_n f \rightrightarrows f$;

d) *Dacă* $\alpha \geqslant 0$, $\beta \geqslant 0$, $\alpha = \alpha_n$, $\beta = \beta_n$ *și dacă*

(I) $\alpha \to 0$, $n\beta \to 0, \frac{n\beta}{\alpha} \to 0$, *cînd* $n \to \infty$,

sau

(II) $n\alpha \to 0$, $n\beta \to 0$, $\frac{n\beta}{\alpha} \to c (c > 0)$, *cînd* $n \to \infty$

atunci

$$L_n^{[\alpha, \beta]} f \rightrightarrows f, \ \bar{L}_n^{[\alpha, \beta]} f \rightrightarrows f, \ \alpha, \beta \in R^m.$$

Toți operatorii considerați în acest corolar sînt operatori polinomiali, iar parametrii α, β sînt niște parametri mici.

Să considerăm acum cazul cînd operatorii convolutivi pozitivi se reprezintă prin intermediul funcțiilor s. Tot ca o consecință a teoremei 2 și a teoremelor din lucrarea [4] avem

COROLAR 2. *Fie* $X = [0, 1]^m$, *nodurile* $x_{k_i}^{(n_i)} = \frac{k_i}{n_i}$, $k_i = 0, 1, \ldots, n_i$; $i = 1, 2, \ldots, m$, *iar* $s_i : [0, 1] \to [0, 1]$, $s_i \in C[0, 1]$, $i = 1, 2, \ldots, m$. *Fie* $s_i(x_i) = x_i + \omega_{n_i}(x_i)$ *unde* $\lim\limits_{n_i\to\infty} \omega_{n_i}(x_i) = 0$ *uniform*, $i = 1, 2, \ldots, m$.

În aceste ipoteze afirmațiile a), b), c), d), din corolarul precedent pentru operatorii

$$B_n^{[s]}, \ P_n^{[s, \alpha]}, \ P_n^{[s]}, \ Q_n^{[s]}, \ L_n^{[s, \alpha, \beta]}, \ \bar{L}_n^{[s, \alpha, \beta]}$$

rămîn adevărate pentru orice $f \in C[X]$, $n \in N^m$.

Tot ca o consecință a teoremei 2 și a observațiilor în legătură cu convergența operatorilor L_n, $\bar{L}_n$, relativi la funcții de o singură variabilă, putem extinde corolarele de mai sus.

Observație. Concluziile corolarelor 1 și 2 au loc și pentru noduri de o formă mai generală decît $\frac{k_i}{n_i}$ și anume

$$x_{k_i}^{(n_i)} = k_i a(n_i) + b(n_i),$$

unde a, $b : N \to R$, $0 \leqslant k_i a(n_i) + b(n_i) \leqslant 1$, iar $\lim\limits_{n_i\to\infty} n_i a(n_i) = 1$ și $\lim\limits_{n_i\to\infty} b(n_i) = 0$; peste tot $i = 1, 2, \ldots, m$.

4. În cele ce urmează acum, ne vom referi la operatori convolutivi ce se construiesc cu ajutorul operației de convoluție de al doilea tip. Spre deosebire de cazul precedent, acești operatori, în ce privește convergența uniformă, se vor lega direct de relațiile convolutive care conduc la exemple de operatori convolutivi binomiali pozitivi [5].

Înainte de a ne ocupa de convergența acestor șiruri de operatori, să facem cîteva considerații pregătitoare. Să ne referim la relația convolutivă

$$Q_n(\bar{n} + u_{m+1} ; y) = \sum_{k \leqslant n} P_k(u ; y) Q_{n-\bar{k}}(u_{m+1} ; y) \tag{22}$$

unde $y \in R$, iar $P_k(u ; y) = P_{k_1}(u_1 ; y) \ldots P_{k_m}(u_m ; y)$. Dacă aici punem $Q_n = P_n$ obținem o nouă relație.

Vom arăta o proprietate de care se bucură convoluțiile (22).

LEMA 1. *Dacă* P_k *sînt soluții ale ecuației* (22) *atunci are loc și următoarea relație*

$$\sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \ldots \sum_{k_m=0}^{n-k_1-\ldots-k_{m-1}} [x_{k_i}]^p P_{k_1}(u_1 ; y) \ldots P_{k_m}(u_m ; y) \cdot$$

$$\cdot Q_{n-k_1-\ldots-k_m}(u_{m+1} ; y) = \sum_{k_i=0}^{n} [x_{k_i}]^p P_{k_i}(u_i ; y) \cdot$$

$$\cdot Q_{n-k_i}(u_1 + \ldots + u_{i-1} + u_{i+1} + \ldots + u_{m+1} ; y). \tag{23}$$

Demonstrație. Ordinea de însumare în raport cu indicii $k_1, k_2, \ldots, k_m$ este independentă. De aceea e suficient să demonstrăm proprietatea de mai sus pentru $i = 1$.

Avem

$$\sum_{k \leqslant n} [x_{k_i}]^p P_k(u ; y) Q_{n-\bar{k}}(u_{m+1} ; y) =$$

$$= \sum_{k_1=0}^{n} [x_{k_1}]^p P_{k_1}(u_1 ; y) \sum_{k_2=0}^{n-k_1} \sum_{k_m=0}^{n-k_1-\ldots-k_{m-1}} P_{k_2}(u_2 ; y) \ldots P_{k_m}(u_m ; y) \cdot$$

$$\cdot Q_{n-k_1-\ldots-k_m}(u_{m+1} ; y) = \sum_{k_1=0}^{n} [x_{k_1}]^p P_{k_1}(u_1 ; y) Q_{n-k_1}(u_2 + \ldots + u_{m+1} ; y).$$

Cu aceasta lema este demonstrată.

Relația (23) rămîne adevărată dacă înlocuim pe Q cu P.

Să notăm cu L_n, $n \in N$ operatorii proveniți din relația convolutivă (22). Considerăm X compact, $R^X = C[X]$ și $\sum_{i=1}^{m+1} u_i = C$. Atunci

$$u_1 + \ldots + u_{i-1} + u_{i+1} + \ldots + u_{m+1} = C - u_i.$$

Exemplele de operatori convolutivi, corespunzători operației de convoluție de tipul al doilea, s-au construit în ipoteza că $C = 1$ și $C = -\frac{1}{\alpha}$.

Fie $X_i = pr_i X$ și să notăm cu

$$L_n^{(i)} : C[X_i] \to C[X_i]. \tag{24}$$

Avem următoarea

LEMA 2. *Dacă* $f(x_1, \ldots, x_m) = \sum_{i=1}^{m} f_i(x_i)$ *unde* $f_i : X_i \to R$, *atunci*

$$L_n(f; x) = \sum_{i=1}^{m} L_n^{(i)}(f_i; x_i). \tag{25}$$

Demonstrație. Ținînd seama de liniaritatea operatorului L_n și de proprietatea (23), rezultă relația din enunțul lemei. Operatorii $L_n^{(i)}$ au forma

$$L_n^{(i)}(f_i; x_i) = A \sum_{k_i=0}^{n} f_i(x_{k_i}^{(n)}) P_{k_i}(u_i(x_i)) \cdot Q_{n-k_i}(C - u_i(x_i))$$

unde $A_n = [Q_n(C; y)]^{-1}$. Aici am considerat $u_i = u_i(x_i)$.

Relația (25) constituie o formulă de descompunere a unui operator m-dimensional în operatori unidimensionali.

Cele două leme servesc pentru demonstrarea următoarei:

TEOREMA 3. *Fie* $L_n : C[X] \to C[X]$, $n \in N$, *un operator convolutiv pozitiv m-dimensional binomial, iar* $L_n^{(i)} : C[X_i] \to C[X_i]$, $i = 1, 2, \ldots, m$, *operatori convolutivi pozitivi unidimensionali binomiali.*

Dacă L_n *admite descompunerea* (25) *atunci din convergența uniformă a șirului valorilor operatorilor* $L_n^{(i)}h$ *către* h, $\forall h \in C[X_i]$, $i = 1, 2, \ldots, m$ *rezultă convergența uniformă a șirului valorilor operatorului* $L_n f$ *către* , $\forall f \in C[X]$.

Demonstrație. Operatorii L_n fiind operatori liniari și pozitivi, vom aplica teorema lui Bohman-Korovkin. Avem

$$L_n(1; x) = 1$$

din însăși definiția operatorilor L_n. Apoi

$$L_n(t_j^p; x_1, \ldots, x_m) = L_n^{(j)}(t_j^p; x_j),\ p = 1, 2, \ldots, m.$$

Această relație este o consecință a relației (25). Cu aceasta teorema este demonstrată.

Ca exemple, avem următorii operatori convolutivi considerați în [5]:

$$\mathfrak{L}_n^{[s,\alpha,\beta]}(f;\, x_1, \ldots, x_m) = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \cdots \sum_{k_m=0}^{n-k_1-\ldots-k_{m-1}} f(x_{k_1}^{(n)}), \ldots, x_{k_m}^{(n)}).$$

$$\frac{\binom{n}{k_1}\binom{n-k_1}{k_2}\cdots\binom{n-k_1-\ldots-k_{m-1}}{k_m}}{\prod\limits_{\tau=0}^{n-1}(1+n\beta+\tau\alpha)} \prod_{i=1}^{m} s_i(x_i) \prod_{\nu=1}^{k_i-1} [s_i(x_i) + \beta k_i + \alpha\nu].$$

$$\prod_{\mu=0}^{n-\sum\limits_{i=1}^{m} k_i} \left[1 - \sum_{i=1}^{m} s_i(x_i) + \beta\left(n - \sum_{i=1}^{m} k_i\right) + \mu\alpha\right];$$

$$\overline{\mathfrak{L}}_n^{[s,\alpha,\beta]}(f;\, x_1, \ldots, x_m) = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \cdots \sum_{k_m=0}^{n-k_1-\ldots-k_{m-1}} f(x_{k_1}^{(n)}, \ldots, x_{k_m}^{(n)})$$

$$\frac{\binom{n}{k_1}\binom{n-k_1}{k_2}\cdots\binom{n-k_1-\ldots-k_{m-1}}{k_m}}{\prod\limits_{\tau=1}^{n-1}(1+n\beta+\tau\alpha)} \prod_{i=1}^{m} s_i(x_i) \prod_{\nu=1}^{k_i-1} [s_i(x_i) + \beta k_i + \alpha\nu].$$

$$\cdot\left[1 - \sum_{i=1}^{m} s_i(x_i)\right] \prod_{\mu=1}^{n-\sum\limits_{1}^{m} k_i - 1} \left[1 - \sum_{i=1}^{m} s_i(x_i) + \left(n - \sum_{i=1}^{m} k_i\right)\beta + \alpha\mu\right].$$

Fie $s_i(x_i) = x_i$ și $x_{k_i}^{(n)} = \frac{k_i}{n}$ și

$$X = \left\{(x_1, \ldots, x_m) \in R^m \,\middle|\, x_i \geqslant 0,\ i = 1, 2, \ldots, m;\ \sum_{i=1}^{m} x_i \leqslant 1\right\};$$

atunci cei doi operatori îi vom nota cu $\mathfrak{L}_n^{[\alpha,\beta]}$, $\overline{\mathfrak{L}}_n^{[\alpha,\beta]}$ și $\mathfrak{L}_n^{[0,0]}$ este operatorul lui Bernstein [2], $\mathfrak{L}_n^{[\alpha,0]}$ este operatorul lui Stancu [7], iar $\mathfrak{L}_n^{[0,\beta]}$ este o nouă generalizare a operatorului lui Cheney-Sharma.

Vom enunța ca și în cazul precedent sub formă de corolare rezultatele privind convergența uniformă a acestor operatori.

COROLAR 3. *Fie* $X = \left\{(x_1, \ldots, x_m) \in R^m \,\middle|\, x_i \geqslant 0,\ i = 1, 2, \ldots, m;\ \sum_{i=1}^{m} x_i \leqslant 1\right\}$, *nodurile* $x_{k_i}^{(n)} = \frac{k_i}{n}$, $k_i = 0, 1, \ldots, n - k_1 - \ldots - k_{i-1}$, $s_i(x_i) = x_i$, $i = 1, 2 \ldots, m$. *Atunci* pentru orice $f \in C[X]$ și $n \in N$ *sînt adevărate următoarele afirmații*

a) $B_n f \rightrightarrows f$;

b) *Dacă* $0 \leqslant \alpha = \alpha_n$, $\lim\limits_{n\to\infty} \alpha_n = 0$, $\alpha \in R$ *atunci* $P_n^{[\alpha]} f \rightrightarrows f$, *operatorul lui Stancu*;

c) *Dacă* $0 \leqslant \beta \leqslant o(n^{-1})$, $\beta \in R$ *atunci* $\bar{P}_n f \rightrightarrows f$, *operatorul lui Cheney-Sharma*;

$\bar{Q}_n f \rightrightarrows f$, *operatorul lui Cheney-Sharma*;

d) *Dacă* $0 \leqslant \alpha = \alpha_n$, $0 \leqslant \beta = \beta_n$, $\alpha, \beta \in R$ *şi dacă*

(*I*) $\alpha \to 0$, $n\beta \to 0$, $\frac{n\beta}{\alpha} \to 0$, *cînd* $n \to \infty$

sau

(II) $n\alpha \to 0$, $n\beta \to 0$, $\frac{n\beta}{\alpha} \to c$ $(c > 0)$, *cînd* $n \to \infty$

atunci

$$\mathfrak{L}_n^{[\alpha,\beta]} f \rightrightarrows f, \qquad \bar{\mathfrak{L}}_n^{[\alpha,\beta]} f \rightrightarrows f.$$

Operatorii consideraţi în acest corolar sînt operatori polinomiali, iar parametrii $\alpha, \beta \in R$ sînt parametri mici.

Considerăm acum cazul cînd în reprezentarea operatorilor consideraţi mai înainte, intervin funcţiile s.

COROLAR 4. *Fie* $X = \left\{(x_1, \ldots, x_m) \in R^m \,\middle|\, x_i \geqslant 0,\ i = 1, 2, \ldots, m;\ \sum\limits_{i=1}^{m} x_i \leqslant 1\right\}$, *nodurile* $x_{k_i}^{(n)} = \frac{k_i}{n}$, $k_i = 0, 1, \ldots, n - k_1 - \ldots - k_{i-1}$, *iar* $s_i : [0, 1] \to [0, 1]$, $s_i \in C[0, 1]$, $i = 1, 2, \ldots, m$. Fie $s_i(x_i) = x_i + \omega_n(x_i)$ *unde* $\lim\limits_{n\to\infty} \omega_n(x_i) = 0$, *uniform*, $i = 1, 2, \ldots, m$.

În aceste ipoteze afirmaţiile a), b), c), d) din corolarul precedent 3 pentru operatorii

$$B_n^{[s]},\quad P_n^{[s,\alpha]},\quad \bar{P}_n^{[s,\beta]},\quad \bar{Q}_n^{[s,\beta]},\quad \mathfrak{L}_n^{[s,\alpha,\beta]},\quad \bar{\mathfrak{L}}_n^{[s,\alpha,\beta]}.$$

rămîn adevărate pentru orice $f \in C[X]$, $n \in N$.

De asemenea putem face următoarea

Observaţie. Corolarele 3 şi 4 rămîn adevărate şi pentru noduri mai generale. Anume:

$$x_{k_i}^{(n)} = k_i\, a(n) + b(n),\ i = 1, 2, \ldots, m$$

unde $a, b : N \to R$, $0 \leqslant k_i\, a(n) + b(n) \leqslant 1$, iar

$$\lim_{n\to\infty} n a(n) = 1 \text{ şi } \lim_{n\to\infty} b(n) = 0.$$

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СХОДИМОСТЬ ПОСЛЕДОВАТЕЛЬНОСТЕЙ ЗНАЧЕНИЙ НЕКОТОРЫХ КОНВОЛЮТИВНЫХ ОПЕРАТОРОВ В R^m

(Резюме)

Изучается сходимость последовательностей значений некоторых положительных конволютивных операторов в R^m. Результаты, полученные автором в [4], распространяются на функции со многими переменными. Пусть $X \subseteq R_i$ — компакт и пусть X^m, определённое соотношением (12). Доказывается следующая теорема:

Теорема. Пусть $L_n: C[X] \to C[X]$ — m-мерный положительный конволютивный оператор, а $L_{n_i}: C[X_i] \to C[X_i]$, $n_i \in N$, $i = 1, 2, \ldots, m$ положительные одномерные конволютивные операторы.

Если L_n допускает разложение на одномерные операторы

$$L_n = L_{n_1} L_{n_2} \ldots L_{n_m}$$

то из равномерной сходимости последовательности значений операторов $L_{n_i} h$ к h, $\forall h \in C[X_i]$ следует равномерная сходимость последовательности значений операторов $L_n f$ к $f C \in [X]$.

Рассматриваемые автором примеры операторов $L_n^{[s,\alpha,\beta]}$ (20) и $\overline{L}_n^{[s,\alpha,\beta]}$ обладают свойством разложения из теоремы. Формулируется следствие 1, в котором содержатся условия равномерной сходимости конволютивных операторов, рассматриваемых как примеры.

Во второй части работы изучается сходимость положительных конволютивных операторов вида (10).

LA CONVERGENCE DES SERIES DES VALEURS D'OPERATEURS CONVOLUTIFS EN R^m

(Résumé)

L'auteur de l'article étudie la convergence des séries des valeurs d'opérateurs convolutifs positifs, en R^m. Il étend les résultats de son travail [4] aux fonctions de plusieurs variables. Soit $X \subseteq R^m$ — un compact et soit X_i défini par (12). On démontre le théorème:

Théorème. Soit $L_n : C[X] \to C[X]$ un opérateur convolutif positif m-dimensionnel, et $L_{n_i} : C[X_i] \to C[X_i]$, $n_i \in N$, $i = 1, 2, \ldots, m$ opérateurs convolutifs positifs unidimensionnels.

Si L_n admet la décomposition en opérateurs unidimensionnels

$$L_n = L_{n_1} L_{n_2} \ldots L_{n_m}$$

alors, de la convergence uniforme de la série des valeurs des opérateurs $L_{n_i} h$ vers h, $\forall h \in C[X_i]$, résulte la convergence uniforme de la série des valeurs des opérateurs $L_n f$ vers f, $f \in C[X]$.

Les exemples d'opérateurs considérés par l'auteur $L_n^{[s, \alpha, \beta]}$(20) et $\overline{L}_n^{[s, \alpha, \beta]}$ jouissent de la propriété de décomposition énoncée du théorème. On énonce le corollaire 1. qui comprend des conditions de convergence uniforme d'opérateurs involutifs, considérés comme exemples.

Dans la seconde partie on étudie la convergence des opérateurs convolutifs positifs de la forme (10).

MÉTHODES STATISTIQUES POUR L'APPROXIMATION DES FONCTIONS

ELENA FRĂȚILĂ

1. Le problème d'approximation des fonctions par des méthodes statistiques a été traité par A. S. Novik, V. E. Samenski [2] et, avec la méthode Monte Carlo, par J. M. Hammersley, D. C. Handscomb [1].

Le travail présente deux procédés statistiques d'approximation :

— l'un de type Monte Carlo, pour les fonctions de plusieurs variables, en construisant un polynôme d'interpolation qui peut être réalisé comme la valeur moyenne d'une certaine variable aléatoire ;

— et l'autre pour les fonction d'une seule variable, fondé sur la méthode des moindres carrés.

2. Soit la fonction $f(x_1, \ldots, x_n)$ pour laquelle on connaît ses valeurs pour les points $M_i(\varepsilon_l^i, \ldots, \varepsilon_n^i)$, $1 \leqslant i \leqslant \overline{N}$, dans l'espace euclidien n-dimensionnel E^n, où ε_k^i peut prendre l'une des valeurs 0, a_k, a_k^2, $\ldots$, a_k^d, $d \leqslant 1$, $0 < a_k < 1$, $k = 1, \ldots, n$. On suppose que pour les points M_i avec toutes les coordonnées différentes de zéro, les valeurs de la fonction f sont nulles, à l'exception de $\overline{M}(a_1, \ldots, a_n)$.

On demande d'interpoler la fonction f sur un point $X(X_1, \ldots, X_n)$ de l'intérieur du domaine parallélépipédique D, déterminé par l'origine, les points $M_j(0, \ldots, a_j, \ldots, 0)$, $j = 1., \ldots, n$ et $\overline{M}$.

On considère le polynôme d'interpolation

$$P(X_1, \ldots, X_n) = \Sigma_{j=l}^{N} r_1 \ldots r_n f(\varepsilon_l^j, \ldots, \varepsilon_n^j) \cdot d^{tj}, \tag{1}$$

où t_j est le nombre des valeurs ε_k^j de $f(\varepsilon_k^j, \ldots, \varepsilon_n^j)$ qui sont différentes de zéro, N le nombre des termes de (1) différents de zéro, et

$$r_i = \begin{cases} \frac{1}{d}\,\frac{X_i}{a_i} = p_i, \text{ si } \varepsilon_i^j = a_i \\ \frac{1}{d}\,\frac{X_i^2}{a_i^2} = q_i, \text{ si } \varepsilon_i^j = a_i^2 \\ \ldots \quad \ldots \quad \ldots \\ \frac{1}{d}\,\frac{X_i^d}{a_i^d} = s_i, \text{ si } \varepsilon_i^j = a_i^d \\ 1 - p_i - \ldots - s_i, \text{ si } \varepsilon_i^j = 0. \end{cases} \qquad \begin{matrix} i = 1, \ldots, n \\ j = 1, \ldots, N \end{matrix}$$

Ce polynôme coïncide avec la fonction f sur les points donnés, a N termes et le degré $\frac{d(d+3)}{2}$ en $X_1, \ldots, X_n$.

Soit maintenant les variables aléatoires

$$\xi_i \begin{Bmatrix} a_i, & a_i^2, & \ldots, & a_i^d, & 0 \\ p_i, & q_i, & \ldots, & s_i, & 1 - p_i - \ldots - s_i \end{Bmatrix}, \quad 1 \leqslant i \leqslant n, \tag{2}$$

avec $p_i, q_i, \ldots, s_i$, données ci dessus, et $\zeta = f(\xi_1, \ldots, \xi_n)$ avec la distribution

$$\begin{Bmatrix} \dfrac{f(\varepsilon, \ldots, \varepsilon_n^j)d^{t_j}}{r_1 \ldots r_n} \end{Bmatrix} 1 \leqslant i \leqslant \overline{N}.$$

On remarque que

$$E\zeta = P(X_1, \ldots, X_n).$$

Par conséquent on a:

THÉORÈME 1. *La valeur moyenne d'échantillon*

$$\overline{\zeta} = \frac{\sum_{j=1}^{m} f(\varepsilon_l^j, \ldots, \varepsilon_n^j)d^{t_j}}{m} \tag{3}$$

où $(\varepsilon_l^j, \ldots, \varepsilon_n^j)$, $1 \leqslant j \leqslant m$, *est un échantillon de volume* m *effectué sur une variable aléatoire n-dimensionelle* $(\xi_1, \ldots, \xi_n)$, ξ_i, $1 \leqslant i \leqslant n$, *avec la distribution* (2), *est une estimation non-biaisée et consistante de* $P[X_1, \ldots, X_n)$.

Cette estimation peut être calculée facilement en réalisant l'échantillon $(\varepsilon_l^j, \ldots, \varepsilon_n^j)$, $1 \leqslant j \leqslant m$, avec une suite de nombres pseudo-aléatoires $\alpha_1, \ldots, \alpha_m$, qui sont uniformes sur $[0, 1]$. C'est à dire que, si :

$0 < \alpha_j \leqslant p_i$ alors ξ_i prend la valeur a_i,

$p_i < \alpha_j \leqslant p_i + q_i$, alors ξ_i prend la valeur a_i^2,

... ...

$p_i + \ldots + s_i < \alpha_j \leqslant 1$, alors ξ_i prend la valeur o, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, en obtenant ainsi l'échantillon $(\varepsilon_l^j, \ldots, \varepsilon_n^j)$. Cet échantillon détermine les points $M_i(\varepsilon_l^i, \ldots, \varepsilon_n^i)$ pour lesquels on connaît les valeurs de la fonction f, qu'on introduit dans (3).

L'erreur de ce procédé Monte Carlo peut être évaluée en conformité avec [1], par la variance de $\overline{\zeta}$:

$$D^2\overline{\zeta} = \frac{1}{m} D^2\zeta = \frac{1}{m} \Sigma \left[f(\varepsilon_l^j, \ldots, \varepsilon_n^j) d^{t_j} - P(X_1, \ldots, X_n)\right]^2 r_1 \ldots r_n.$$

Puisque chaque $r_k \leqslant 1$, $1 \leqslant k \leqslant n$, et que les valeurs

$$|f(\varepsilon_l^i, \ldots, \varepsilon_n^j) d^{t_j} - P(X)|, \; X = (X_1, \ldots, X_n),$$

sont finies, en notant

$$R(X) = \max_j |f(\varepsilon_l^j, \ldots, \varepsilon_n^j) d^{t_j} - P(X)|,$$

on a

$$D^2\overline{\zeta} \leqslant \frac{1}{m} \overline{N} \cdot R^2(X).$$

Et par conséquent :

THÉORÈME 2. *L'erreur*

$$\delta_m(X) = |P(X) - \overline{\zeta}|,$$

avec une probabilité plus grande que $1 - \varepsilon$, $\varepsilon > 0$, *et suffisamment petit, ne dépasse pas*

$$\sqrt{\frac{\overline{N}}{\varepsilon m}}\, R(X),$$

où $\overline{N}$ *est le nombre des points donnés,* m *le volume d'échantillon,* X *le point d'interpolation fixé.*

Remarque. Le cas $a_i \geqslant 1$, $1 \leqslant i \leqslant n$, se traite de façon analogue, en considérant comme point M le point de coordonnées $(a_l^d, \ldots, a_n^d)$.

Cas particuliers.

1. Si $a_i = 1$, $1 \leqslant i \leqslant n$, $d = 1$, le domaine D est le cube unité de l'espace E^n et on obtient le cas traité dans [1].

2. Si $a_i \neq 1$, $1 \leqslant i \leqslant n$, $d = 1$, le domaine D est un parallélépipède de E^n, le nombre de points étant $\overline{N} = 2^n$.

3. Si $a_i > 0$, $1 \leqslant i \leqslant n$, $d = 1$, mais on considère seulement les points qui ont une seule coordonnée différente de zéro et l'origine, le domaine D est polyédrique, le procédé Monte Carlo d'interpolation donné par la formule (3) reste valable, $\overline{N} = n + 1$.

4. Si on considère $a_i = x_i$, $a_i^2 = - x_i$, $1 \leqslant i \leqslant n$, $d = 2$, $x_i > 0$, alors le domaine D est un parallélépipede dans E^n, centré sur l'origine, le procédé d'interpolation étant aussi applicable, dans ce cas.

3. Soit la fonction $f(x) \in C[a, b]$, pour laquelle on connaît les valeurs $f(x_1), \ldots, f(x_n)$, sur les noeuds $x_1, \ldots, x_n$ appartenant à l'intervalle $[a, b,]$ qui représente un échantillon de volume n effectué sur une variable aléatoire ξ définie sur $[a, b]$. On approxime la fonction inconnue $f(x)$ par le polynôme

$$P_{n-1}(x) = A_0 + A_1 x + \ldots + A_{n-1} x^{n-1}$$

de sorte que

$$E\{f(\xi) - (A_0 + A_1 \xi + \ldots + A_{n-1} \xi^{n-1})\}^2 = min. \tag{4}$$

Cette condition permet de déterminer les coefficients A_i, $0 \leqslant i \leqslant n - 1$, en fonction de $x_1, \ldots, x_n$ et $f(x_i)$, $1 \leqslant i \leqslant n$.

De fait, de (4) il résulte le système d'équations

$$\begin{cases} Ef(\xi) - A_0 - A_1 E\xi - \ldots - A_{n-1} E\xi^{n-k-1} = 0 \\ Ef(\xi)\xi^k - A_0 E\xi^k - \ldots - A_{n-1} E\xi^{n-k-1} = 0, \ 1 \leqslant k \leqslant n - 1. \end{cases} \tag{5}$$

La valeur moyenne $E\xi^k$ et $E\xi^k f(\xi)$ peut être évaluée par la moyenne d'échantillon correspondante, en supposant n suffisamment grand :

$$E\xi^k = \frac{\sum_l^n x_i^k}{n}$$

$$Ef(\xi)\xi^k = \frac{\sum_l^n f(x_i) \cdot x_i^k}{n}.$$

On observe que (5) est un système Cramer, car

$$D = \begin{vmatrix} 1 & E\xi & \ldots & E\xi^{n-1} \\ E\xi & E\xi^2 & \ldots & E\xi^n \\ & \ldots & \ldots & \\ E\xi^{n-1}, & E\xi^n & \ldots & E\xi^{2(n-1)} \end{vmatrix} \neq 0,$$

et

$$A_i = \frac{D_{i+1}}{D},\ 0 \leqslant i \leqslant n-1,$$

où

$$D_{i+1} = \begin{vmatrix} 1 & E\xi & \ldots & Ef(\xi) & \ldots & E\xi^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ E\xi^{n-1}, & E\xi^n & \ldots & Ef(\xi)\xi^{n-1} & \ldots & E\xi^{2(n-1)} \end{vmatrix}$$

Par conséquent

$$P^!_{n-1}(x) = D^{-1}(D_1 + D_2 x + \ldots + D_n x^{n-1}). \tag{6}$$

On peut montrer que la solution du système (5) correspond à un minimum de (4).

Remarques. a. La formule (6) correspond à une erreur moyenne quadratique minime sur les noeuds donnés. Mais pour obtenir une bonne approximation de la fonction f par le polynôme $P_{n-1}(x)$, il est nécessaire que le nombre des noeuds soit suffisamment grand. Cette formule peut être utilisée aussi en considérant au lieu du polynôme $P_{n-1}(x)$, un polynôme de degré $h < n-1$. Pour déterminer les coefficients A_i, $0 \leqslant i \leqslant h$, en utilisant pour l'évaluation des moyennes $Ef(\xi)\xi^k$ et $E\xi^k$, $0 \leqslant k \leqslant h$, toutes les valeurs données x_i, $1 \leqslant i \leqslant n$.

b. Le cas où la valeur du minimum (4) serait nulle implique le fait que, sur les noeuds donnés, la valeur de la fonction $f(x)$ coïncide avec la valeur du polynôme d'approximation $P_{n-1}(x)$. Cette restriction attachée au système (5) conduit à une condition de compatibilité du nouveau système, qui peut être accomplie pour une certaine classe de fonctions ou pour un certain échantillon, déterminé dans ce sens.

c. Le reste de la formule d'interpolation obtenue est délimité par la valeur du minimum de (4) et peut être déterminée sur les points x_i, après le calcul des coefficients A_i.

L'erreur du procédé par lequel on a calculé les coefficients A_i, procédé de type Monte Carlo, provient de la substitution aux moyennes $E\xi^k$, $Ef(\xi)\xi^k$, des moyennes d'échantillon correspondantes, qui sont des estimations consistantes et nonbiaisées pour les moyennes théoriques.

Il résulte de là que les valeurs ainsi obtenues pour A_i convergent en probabilité, pour $n \to \infty$, vers les vraies valeurs A_i.

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METODE STATISTICE PENTRU APROXIMAREA FUNCŢIILOR

(Rezumat)

Articolul prezintă două procedee pentru aproximarea funcţiilor:

— unul, de tip Monte-Carlo, pentru funcţiile de mai multe variabile, construind un polinom de interpolare care poate fi realizat ca valoarea medie a unei variabile aleatoare;

— şi altul, pentru funcţiile de o singură variabilă, bazat pe metoda celor mai mici patrate.

СТАТИСТИЧЕСКИЕ МЕТОДЫ ПРИБЛИЖЕНИЯ ФУНКЦИЙ

(Резюме)

Автор приводит два метода приближения функций:

— метод типа Монте Карло для функций с многими переменными, посредством построения интерполяционного полинома, который может быть получен при помощи среднего значения одной случайной переменной;

— метод для функций с одной переменной, основанный на методе наименьших квадратов.

NATURA SISTEMULUI BINAR *VW CYGNI*

VASILE URECHE

1. Introducere. Importanța determinării elementelor absolute ale componentelor sistemelor binare este bine cunoscută [5], atît pentru cercetările cu caracter statistic (precizarea relațiilor de stare), cît și pentru fundamentarea teoriilor de evoluție stelară. Numărul sistemelor binare strînse la care astfel de determinări absolute se pot face nu este prea mare. Catalogul lui S v e c i n i k o v [11] conține 197 astfel de sisteme. Pe baza datelor din acest catalog, Svecinikov clasifică sistemele binare strînse în șase tipuri, clasificarea sa urmînd, în esență, pe cea a lui K o p a l [3,4].

În catalogul lui Svecinikov, sistemul dublu *VW Cygni* este considerat ca făcînd parte din sistemele conținînd subgigante detașate (sisteme în care una, sau ambele componente, sînt subgigante detașate de suprafața critică Roche). Însă se arată că este posibil ca raza r_2 a componentei secundare să fie subestimată, iar în realitate sistemul să fie apropiat de sistemele semidetașate. În cele ce urmează vom arăta, pe baza unor date mai noi, că sistemul *VW Cygni* este îndr-adevăr semidetașat (deși posedă unele caracteristici fizice asemănătoare cu cele ale sistemelor formate din subgigante detașate).

Elementele fotometrice ale sistemului *VW Cygni* au fost redeterminate recent de L a c y [6] din observații fotografice (măsurări pe plăci Kodak). Astfel Lacy a obținut următoarele valori (notațiile fiind cele curente, indicele 1 se referă la componenta principală — cu masă mai mare și mai luminoasă — iar indicele 2 la componenta secundară) :

$$\begin{array}{lll} P = 8^{z}\!\!.43\,; & r_1 = 0{,}099\,; & r_2 = 0{,}290 \\ i = 80^{\circ}\!\!.8\,; & L_1 = 0{,}955\,; & L_2 = 0{,}045 \end{array} \qquad (1)$$

2. Raportul maselor. Utilizînd elementele fotometrice determinate de Lacy și elementele spectroscopice determinate de S t r u v e [10] (din observarea unui singur spectru, al componentei principale), am redeterminat raportul maselor componentelor, $q = \mathfrak{M}_2/\mathfrak{M}_1$. În acest scop am aplicat metoda lui Kopal [5], considerînd că steaua mai masivă (componenta principală a sistemului, de tip spectral A3) este o stea normală care satisface o relaţie masă-luminozitate de forma

$$\log \mathfrak{M}_1 = \alpha - \beta M_{b_1}, \tag{2}$$

unde M_{b_1} este magnitudinea absolută bolometrică a acestei stele. Din orbita spectroscopică (bazată pe observarea unui singur spectru) rezultă mărimile

$$f \equiv f(\mathfrak{M}) = \frac{\mathfrak{M}_2^3 \sin^3 i}{(\mathfrak{M}_1 + \mathfrak{M}_2)^2}, \tag{3}$$

$$g = \mathcal{a}_1 \sin i, \tag{4}$$

notaţiile fiind cele uzuale.

Utilizînd raportul maselor $q = \mathfrak{M}_2/\mathfrak{M}_1$, din (3) şi (4) se obţin următoarele expresii pentru masa şi raza componentei principale (cu notaţiile lui Svecinikov)

$$\mathfrak{M}_1 = \frac{1}{q}\left(1 + \frac{1}{q}\right)^2 f \operatorname{cosec}^3 i \tag{5}$$

$$R_1 = \left(1 + \frac{1}{q}\right) g\, r_1 \operatorname{cosec} i \tag{6}$$

Mai departe, introducînd expresiile (5) şi (6) în ecuaţia (2), în care utilizăm formula magnitudinilor absolute bolometrice

$$M_b = M_{b\odot} - 5 \log \frac{R}{R_\odot} - 10 \log \frac{T}{T_\odot} \tag{7}$$

unde indicele $\odot$ se referă la datele corespunzătoare Soarelui, obţinem ecuaţia

$$\log \frac{1}{q} + A \log \left(1 + \frac{1}{q}\right) = B \tag{8}$$

unde

$$A = 2 - 5\beta \tag{9}$$

$$B = \alpha - \beta\left[M_{b\odot} - 5 \log (g r_1 \operatorname{cosec} i) - 10 \log \frac{T_1}{T_\odot}\right] - \log (f \operatorname{cosec}^3 i), \tag{10}$$

g fiind exprimat în raze solare.

Deoarece componenta principală a sistemului *VW Cygni* este de tip spectral A3 este de aşteptat ca $M_{b_1} > 0$. De aceea am utilizat valorile $\alpha = 0{,}3980$; $\beta = 0{,}0784$ pentru coeficienţii din relaţia masă-luminozitate (ecuaţia (12), pag. 129 din lucrarea lui S v e c i n i k o v [11].) Adoptînd scara temperaturilor efective dedusă de H a r r i s [1] rezultă: $T_{\odot} = 5784°K$, $T_1 = 9260°K$. Pentru magnitudinea absolută bolometrică a Soarelui, Harris dă valoarea $M_{b\odot} = 4{,}69$. Din orbita spectroscopică a lui Struve [10] rezultă $f \equiv f(\mathfrak{M}) = 0{,}033\ \mathfrak{M}_{\odot}$, $g = 3{,}88 \cdot 10^6$ km $= 5{,}5771\ R_{\odot}$ (considerînd $R_{\odot} = 6{,}957 \cdot 10^5$ km [8]). Cu valorile numerice date mai sus şi cu elementele fotometrice (1), rezultă $A = 1{,}6080$; $B = 1{,}5563$; iar pentru ecuaţia (8) se obţine soluţia

$$q = 0{,}30 \tag{11}$$

Cu valoarea $q = 0{,}30$, din tabelele lui P l a v e c şi K r a t o c h v i l [7] obţinem razele critice $r_{1\,\text{crit}} = 0{,}490$; $r_{2\,\text{crit}} = 0{,}272$ (identificînd razele critice cu ordonatele y_{11}, respectiv y_{12}).

Comparînd razele componentelor din (1) cu razele critice se observă că în timp ce $r_1 \leqslant r_{1\,\text{crit}}$, $r_2 > r_{2\,\text{crit}}$, adică *sistemul VW Cygni este un sistem semidetaşat.*

3. Elementele absolute. Din elementele fotometrice (1), ecuaţiile (4) — (6) şi valoarea (11) a raportului maselor obţinem:

Masele componentelor: $\mathfrak{M}_1 = 2{,}2\mathfrak{M}_{\odot}$; $\mathfrak{M}_2 = 0{,}7\ \mathfrak{M}_{\odot}$

Separarea componentelor: $\mathcal{A} = 24{,}6\ R_{\odot}$

Razele componentelor: $R_1 = 2{,}4\ R_{\odot}$; $R_2 = 7{,}1\ R_{\odot}$

Formula (7) ne dă pentru magnitudinea absolută bolometrică a componentei principale valoarea $M_{b_1} = 0{,}7$. Valoarea obţinută confirmă ipoteza făcută la început: $M_{b_1} > 0$.

Pentru a deduce magnitudinea absolută bolometrică a componenţei secundare, avem nevoie de temperatura acesteia, care se poate deduce din raportul intensităţilor medii pe discurile celor două componente, considerînd că acestea radiază ca nişte corpuri negre (după legea lui Planck). Avem

$$\frac{J_1}{J_2} = \frac{r_2^2}{r_1^2} \cdot \frac{L_1}{L_2} = \frac{e^{\frac{c_2}{\lambda T_2}} - 1}{e^{\frac{c_2}{\lambda T_1}} - 1} \tag{12}$$

Utilizînd valorile numerice date mai înainte şi luînd $\lambda = \lambda_{pg} = 4250$ Å [8], rezultă $T_2 \cong 4000°K$, corespunzîndu-i un „spectru calculat" (K4 IV) pentru componenta secundară. Mai departe formula (7) ne dă $M_{b_2} = 2{,}2$.

Tabela de mai jos conţine elementele absolute ale sistemului *VW Cygni* obţinute mai sus, în comparaţie cu cele date de S v e c i n i k o v.

Elemente absolute

Elementele	Svecinikov	Lucrarea prezentă
Spectrele	$A3_1$ + [KI IV]	$A3_1$ + [K4 IV]
q	0,28	0,30
$\mathfrak{M}_1$	2,5	2,2
$\mathfrak{M}_2$	0,7	0,7
$\mathfrak{A}$	26	24,6
R_1	2,9	2,4
R_2	5,7	7,1
M_{b1}	0,3	0,7
M_{b2}	1,7	2,2

Se observă că elementele absolute obţinute nu diferă prea mult de cele date de Svecinikov, cu excepţia razei componentei secundare, care în lucrarea prezentă este sensibil mai mare. Chiar datorită acestui fapt sistemul *VW Cygni* se prezintă ca un sistem semidetaşat.

4. Consideraţii asupra naturii sistemului VW Cygni. Din compararea dimensiunilor componentelor cu dimensiunile suprafeţei echipotenţiale critice Roche, am văzut că sistemul *VW Cygni* este un sistem semidetaşat. Mai detarte, compararea elementelor absolute obţinute mai sus cu datele catalogului lui Svecinikov (mai ales diagrama H—R a sistemelor semidetaşate) arată că sistemul considerat aparţine grupei a III-a a sistemelor semidetaşate. Poziţia componentelor sistemului *VW Cygni* în diagrama H—R se vede în fig. 1 (linia continuă reprezintă Secvenţa Principală de Vîrstă Zero, după Svecinikov).

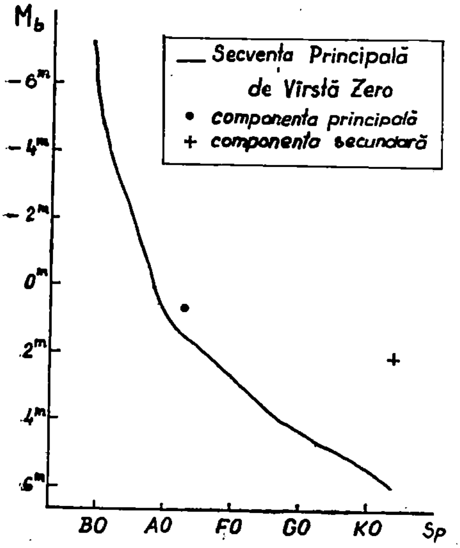

Fig. 1.

Dacă se compară fig. 1, cu „drumurile" teoretice de evoluţie ale sistemelor binare strînse (vezi de exemplu P a c z y n s k i [9]) se observă că steaua actualmente principală este practic neevoluată, fiind apropiată de Secvenţa Principală de Vîrstă Zero, în timp ce steaua secundară a evoluat, ajungînd în domeniul subgigantelor. Aceasta din urmă a fost iniţial componentă principală; avînd masă mai mare a evoluat mai repede; trecînd în faza de expansiune şi atingînd limita Roche, a pierdut o parte din masă care a fost transferată la celealaltă componentă neevoluată. Astfel componenta iniţial secundară a devenit componentă principală. Procesul schiţat mai sus este aşa numitul proces de „schimbare a rolurilor" (în cursul procesului este posibil ca o fracțiune din masa inițială a sistemului să se elimine în afara lui). Aceasta este schema general acceptată de evoluție a sistemelor semidetașate, căreia se pare că i se supune și sistemul *VW Cygni*.

Considerînd, însă, alte caracteristici fizice ale sistemului *VW Cygni*, atunci se observă că acesta se aseamănă mult cu sistemele conținînd subgigante detașate. Astfel, perioada sistemului ($P > 8^z$) este neobișnuit de mare pentru sistemele semidetașate (dintre sistemele semidetașate considerate de Svecinikov, numai DN Orionis are perioadă mai mare). Asemenea perioade se întîlnesc de obicei la sistemele conținînd subgigante detașate. (Perioade mari se întîlnesc și la sistemele supermasive de clase spectrale timpurii, ca și la sistemele gigante și supergigante, dar sistemul VW Cygni nu face parte în nici un caz din aceste tipuri).

Pe de altă parte, calculînd excesul de luminozitate pentru componenta secundară, conform definiției

$$\Delta M_b = M_{b\ \mathrm{calc}} - M_{b\ \mathrm{obs}}\ ; \tag{13}$$

unde $M_{b\ \mathrm{obs}}$ este magnitudinea absolută bolometrică dedusă din observații pentru steaua considerată, iar $M_{b\ \mathrm{calc}}$ este magnitudinea absolută bolometrică pe care ar avea-o steaua dacă ar satisface relația masă-luminozitate (2), obținem $\Delta M_{b_2} = +4{,}9$. În fig. 2 este dat excesul de luminozitate al componentelor secundare ale sistemelor semidetașate și ale sistemelor conținînd subgigante detașate, în funcție de masa corespunzătoare, după Svecinikov.

Se observă că sistemul *VW Cygni* se apropie mai mult de sistemele conținînd subgigante detașate (totuși trebuie arătat că dispersia în diagrama lui Svecinikov este destul de mare).

În sfîrșit, să considerăm diagrama perioadă-raportul maselor. După cum arată Svecinikov, din teoriile de evoluție stelară rezultă că dacă pierderea de masă (după atingerea limitei Roche) are loc prin transfer de la o componentă la ceealaltă, atunci între perioada —P și rapor-

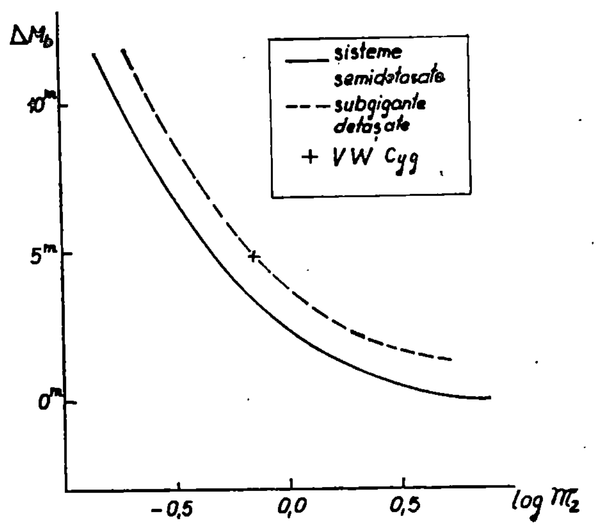

Fig. 2.

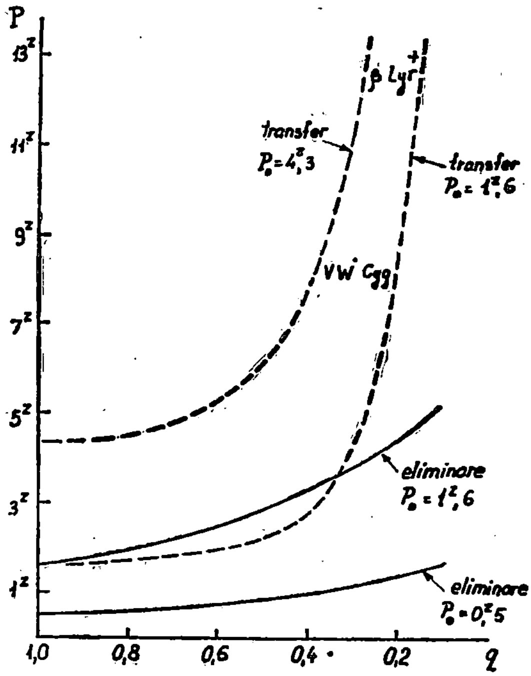

Fig. 3.

tul maselor $-q$, avem relaţia

$$P(q) = P_0 \frac{(1+q)^6}{2^6\, q^3}, \qquad (14)$$

iar dacă are loc eliminarea de masă din sistem, atunci

$$P(q) = P_0 \frac{4}{(1+q)^2} \qquad (15)$$

în ambele cazuri P_0 fiind valoarea lui P pentru $q = 1$. În fig. 3 liniile continue reprezintă relaţia (15) (pentru $P_0 = 0^z{,}5$ şi $P_0 = 1^z{,}6$) iar curbele întrerupte-relaţia (14) (Pentru $P_0 = 1^z{,}6$ şi $P_0 = 4^z{,}3$). Svecinikov arată că sistemele semidetaşate se aşează între curbele descrise de ecuaţia (15), sugerînd evoluţia lor prin eliminare de masă, în timp ce sistemele conţinînd subgigante detaşate se aşează între curbele descrise de ecuaţia (14), sugerînd evoluţia acestora prin transfer de masă. Tot în fig. 3 este reprezentată poziţia sistemului *VW Cygni* în diagrama corespunzătoare a lui Svecinikov, după care s-ar părea să sistemul *VW Cygni* se comportă ca sistemele formate din subgigante detaşate, evoluînd prin transfer de masă.

După cele spuse mai sus rezultă că sistemeul *VW Cygni* este un sistem semidetaşat, care se pare că evoluează prin transfer de masă, fiind în faza de transformare într-un sistem cu subgigante detaşate. Nu este exclus ca această transformare să se facă şi prin eliminare de masă, observînd că poziţia lui *VW Cygni* din fig. 3 poate fi atinsă pe o curbă (15) cu $P_0 \cong 3^z{,}6$.

Pentru o mai bună precizare a stadiului evolutiv în care se găseşte sistemul *VW Cygni*, ar fi necesară compararea datelor observaţionale cu „drumurile" teoretice de evoluţie stelară, însă, din păcate, pînă în prezent încă nu s-au calculat asemenea „drumuri" pentru sisteme binare strînse cu mase relativ mici, de tipul celui considerat în prezenta lucrare [2].

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ПРИРОДА ДВОЙНОЙ СИСТЕМЫ *VW CYGNI*

(Резюме)

Используя новые фотометрические элементы Л а с и [6] и спектроскопические элементы С т р у в е [10], автор определил вновь абсолютные элементы двойной системы *VW Cygni*. Отношение масс вычислено методом К о п а л а [5] на основе соотношения масс-светимостей, полученного С в е ч н и к о в ы м [11].

Тесная двойная система *VW Cygni* является полуразделённой.

Работа заканчивается некоторыми соображениями о природе и эволюции двойной системы *VW Cygni*, которая будучи полуразделенной, обладает некоторыми физическими свойствами, сходными со свойствами систем, содержащих в качестве компонентов разделенные звезды-субгиганты [11] („субнормальные" звёзды-субгиганты [5]). Автор высказывает предположение о том, что система находится в эволюционной фазе преобразования из полуразделённой системы в систему с одной разделённой звездой-субгигантом.

THE NATURE OF THE BINARY SYSTEM *VW CYGNI*

(Summary)

Using the new photometric elements of L a c y [6] and the spectroscopic elements of S t r u v e [10], the absolute elements of the binary system *VW Cygni* are redetermined. On the basis of the mass-luminosity relation obtained by S v e t c h n i k o v [11], the mass ratio has been computed with K o p a l's method [5].

The close binary system *VW Cygni* is semidetached.

The paper is concluded with some considerations about the nature and the evolution of the binary system *VW Cygni*, which, being semidetached, has some physical properties similar to those of the systems containing detached subgiants [11] ("undersized" subgiants [5]). It is suggested that the system is in an evolution phase of transformation from a semidetached system to a system containing one detached subgiant.

RECENZII

W i l l i a m   B u l g r e n, **A Computer Assisted Approach to Elementary Statistics. Examples and Problems.**

L'ouvrage contient une ample liste d'exercices de statistique élémentaire, de pratique courante, destinée à un séminaire-laboratoire. Les exercices proposés poursuivent la compréhension de la signification statistique des différentes notions, des méthodes et des résultats statistiques, de l'analyse statistique.

La technique du travail, fondée sur la simulation des exercices avec les nombres pseudo-aléatoires et l'emploi des machines de calcul, permet d'introduire le lecteur dans la pratique de la programmation et de l'ajustage.

Le travail peut être utilisé après une connaissance préalable de la problématique statistique à laquelle se rapportent les exercices proposés; il contient huit chapitres et quatre appendices. Dans chaque chapitre il s'agit des principaux concepts et des principes de statistique, qui sont présentés succinctement dans la partie introductive, puis discutés dans les exercices proposés.

Après le chapitre introductif (1), suit (2): description des distributions de mesurages; (3): probabilité — concepts de base; (4): variables aléatoires et distributions de probabilité; (5): statistique inductive, méthode d'échantillon d'un grand volume; (6): distributions exactes, méthodes d'estimation, tests des hypothèses statistiques; (7) régression linéaire; (8): analyse de la: Variance, et quatre appendices complémentaires: (I) éléments de Fortran IV; (II) générateurs de nombres pseudo-aléatoires; (III) tables de nombres pseudo-aléatoires normaux; (IV) réponses à quelques exercices.

Chaque chapitre contient une partie introductive de présentation succincte des notions auxquelles se réfèrent les exercices suivants. Quoique les exercices soient en apparence simples, ils ne sont aucunement formels; ils poursuivent par des discussions et questions complémentaires l'approfodissement du sens statistique des différentes notions. Les questions posées conduisent à une analyse judicieuse des effets produits par le changement des données d'échantillon, le volume d'échantillon, le coefficient de confiance ou le niveau de signification, et permettent de compléter le contenu du concept respectif, et aussi de comparer les résultats expérimentaux et les théoriques.

Une liste bibliographique présentée dans l'introduction donne des indications relatives aux problèmes de statistique traités dans les exercices, au langage Fortran IV de programmation de ceux-ci, enfin aux machines à calcul.

L'ouvrage est recommandable pour un séminaire pratique de statistique élémentaire courante, dans les facultés et les écoles qui peuvent utiliser effectivement les calculateurs, car il répond au triple but: d'apprendre activement le contenu de chaque notion traitée, d'initier à une pratique de programmation, et aussi de compréhension du sens de la méthode de l'ajustage statistique.

Ainsi présenté sous l'aspect d'un cours pratique de statistique élémentaire, le travail est très utile grâce à l'approfondissement, d'intention pratique évidente, de toutes les notions et méthodes abordées, illustrant les qualités du profeseur expérimenté qu'est l'auteur et la rigueur scientifique des matériaux mis en oeuvre.

ELENA FRĂȚILĂ

Georgi E. Shilov, **Linear algebra,** revised, English edition, translated and edited by Richard A. Silverman; Prentice-Hall Inc., Englewood Cliffs N. J., 1971.

The book is intended as a text for undergraduate students majorizing in mathematics and physics and largely covers linear algebra courses in use in most faculties of mathematics. The starting point is the theory of determinants. Then after studying linear spaces, the main results of the theory of systems of linear equations over numerical fields are deduced by means of the properties of these spaces. Following chapters discuss linear, bilinear and quadratic functions, functors, canonical forms of Jordan and also Euclidean and unitary spaces. The last chapter is devoted to finite dimensional algebras and their representations. There is also an appendix dealing with categories of finite-dimensional spaces based on a paper written by the author and I. M. Gelfand.

In order to facilitate understanding, each new concept is accompanied by examples and all chapters end with exercises. There are hints and answers at the end of the book. The careful and detailed treatment of all subjects as well as its easy and atractive style will geatly facilitate the task of many students and professors.

M. FRODA-SCHECHTER

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