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Book reviews

Thomas Bedürftig und Roman Murawski, Philosophie der Mathematik, 3rd expanded and revised edition, xvii+465 pp, Walter de Gruyter, Berlin, 2015, ISBN: 978-3-11-033117-2/pbk; 978-3-11-033118-9/ebook.

Any person, well acquainted with the history of mathematics, could grasp the fact that the development of mathematics is not a linear one. Indisputably, the most notable conceptual changing happened in the 19 century, a moment in which the analysis of the real numbers, the set theory and the axiomatic approach impose a new paradigm, in which much of the old problems get a mathematical solution and in which new topics do generate new problems. This is the conceptual ground on which, in a very inspired and extremely flexible mode, the mathematics and the philosophy encounter each other in an excellent book on the *philosophy of mathematics.*

A "philosophy of mathematics" in a time in which the mathematics is separated from philosophy? What is the good of it? The answer to this question is also the leading idea of the whole analysis carried out in the chapters of the book and that can be outlined in the following terms: the mathematical solutions to some problems do not necessarily dissolve the philosophical solutions; "we come to decisions but the philosophical questions are not answered. They remained and will remain" (372). This is the key subject organizing in a perfectly coherent totality the six chapters of the book.

Although the starting point, Chapter 1, *On the road to the real numbers*, is a usual one in the present mathematical practice, that of the real number, the analysis reveals a whole range of fundamental philosophical problems connected to this notion: irrationality of a number, incommensurability of some quantities, the geometrical continuum and the real line, the infinite. Surely, since in the new paradigm the set R of the real numbers gives the solutions, these are the problems mathematically decided, but philosophically some of them are still important problems. According to Cantor and Dedekind, for example, the linear continuum is R, but can the set R be the linear continuum? Mathematically, the small infinity isn't a problem any more, but if R doesn't express the intuitive, geometrical continuum, then does not the intuitive meaning be captured, at least partially, by resorting to the classical notion of infinitesimals? The actual infinity, stipulated by the axiom of infinity in the set theory, claims to work with infinite sets as given mathematical objects, but can be understood an infinite series like an usual given object? Moreover, why such abstract, theoretical constructions, as actual infinity and *eo ipso* real numbers, find their applications in the solving of a number of *concrete* problems?

All these matters are the subject of a historical analysis in Chapter 2, *From the history of philosophy and mathematics*, from the antiquity to the recent realism-antirealism controversy in the philosophy of mathematics. This analysis paves the way to some possible answers to the above questions.

The effectiveness of the concept "philosophy of mathematics", proposed by the authors, is masterly illustrated in the next chapters. No doubt, the standard is given by the analysis of the real number, based on the idea of continuum and on the related notion of infinity, the topic of Chapter 3, *About fundamental questions of the philosophy of mathematics*. Sketched, the idea is this. The actual understanding of the continuum (Cantor/Dedekind) is a form of the atomism, the lines and their parts being uncountable sets of points, different from the infinitesimal understanding (Leibniz), in which the lines are made up of an infinite number of infinitely small segments. Is this identification *Continuum* = R the answer to the question about the very nature of the continuum? Mathematically, yes! Philosophically, no, if by "philosophical" we understand the capture of the intuitive meaning of this notion, in which the continuum (e.g. the line) has parts and the parts obtained by division are again the continua. Since the points have no parts, they represent the idea of discontinuity and can form only something discontinuous. Therefore, R is not the intuitive continuum, a notion that, given the above identification, did not survive mathematically, but that is still living philosophically. Can this intuitive meaning be recovered also in a formal mathematical way? On the authors view, the solution is given by the vindication of the infinitesimals, in a non-standard analysis (D.Laugwitz, C.Schmieden, A.Robinson), in which the notion of real number is extended to a non-Archimedean field of the "hyperreal numbers" with infinitely large and infinitely small numbers. In this field besides the real "standard" numbers, at an infinite distance there are "nonstandard" numbers; and this new understanding occurs by reintroducing Leibniz's infinitesimal quantities.

Undoubtedly, one key problem of the philosophy of mathematics is that concerning the *nature* of mathematical objects. Since the mathematical concepts can be defined in set-theoretic terms, they are reducible to the notion of set. And, in this way, the question referring to the nature of mathematical objects (e.g. real number) reduces to the question of what kind of entities the sets are. Chapter 4, *The sets and the theories of sets*, integrates therefore in the structure of the book, in a perfectly coherent way, the *ontological* problems of what actually the sets are (hard to say!) in the terms of the way they are defined, i.e. in the terms of the axioms describing the relations between them. Two axiomatic theories of sets (Zermelo-Frenkel and Neuman-Bernays-Gödel), in an elegant comparative analysis, characterize ontological this central notion of mathematical thinking that, by integration of the notion of set, becomes set-theoretic thinking.

Certainly, the new methodological paradigm of the present mathematics is represented by the set theory and logic, *axiomatically organized*. Then, in a natural way, the analysis continues with a distinct Chapter 5, *The axiomatic and the logic*, focused on the new mathematical and philosophical problems, as the problem of the consistency of mathematics, that of completeness, the problem of the nature of mathematical truth and the re-thinking of the continuum. Directly or indirectly, all these problems are originated in the axiomatic setting of the present mathematical theories. They are those which reveals the “distance” between the provability and the truth and, at last, the limits of this kind of mathematical thinking.

A *Retrospection*, in Chapter 6, about the topics developed in the book and some technical considerations on *To think and to calculate infinitesimally*, concludes this excellent book on the philosophy of mathematics.

The historical, mathematical and philosophical perspectives from which the key notions are investigated are very different. But the authors avoid to give “definitive” solutions to these problems. Often we find the expressions of the following form: ”We don’t know what is the point” (203), ”R is not the continuum, R is a model” (264), “What exactly are the sets, nobody knows” (310), “We do not find an effective answer to the question about numbers” (377), “For the set theory, for the theory of infinite, the infinite remains enigmatical” (314), “Even mathematically, the continuum remains transcendent” (227). Therefore, the fundamental pieces of the philosophical and mathematical investigation “remain transcendent” (386). In a perfect consonance with the project of the construction of the book, the source of the transcendence is to be viewed in the essential difference between a philosophical notion and its mathematical description, and from here the necessity of the *philosophical analysis* of the fundamental mathematical meanings.

The text of the book has an exemplary clearness, all the key notions are indicated in the distinct paragraphs and the bibliographical list of the papers to which the authors are referred is impressively vast and relevant.

The conceptual strictness, the quality of argumentation and the originality of the analysis make of this book one of the profoundest and more elegant conceptual construction in the philosophy of mathematics.

Virgil Drăghici

Gerard Walschap, Multivariable Calculs and Differential Geometry, Walter de Gruyter (De Gruyter Graduate), 2015, ix+355 pages, Paperback, ISBN 978–3-11-036949-6.

The book under review is a rigorous introduction to differential geometry (mainly of hypersurfaces in a Euclidean space) based on a solid foundation of calculus and linear algebra.

I give below a short description of the contents. The first chapter is a short introduction to the topology of a Euclidean space. The second chapter (called, generically, *Differentiation*) is concerned with a study of the mappings between two Euclidean spaces, including, among the classical stuff (Taylor series, implicit functions), a thorough introduction to vector field and Lie brackets, as well as the partition of unity on an open set of a Euclidean space. What these two chapter have in common is the consistent use of the tools of linear algebra (for instance, à la manière de *Calculus on Manifolds*, Addison Wesley, 1965, by Michael Spivak). This language is not, usually, the favorite of the analysts, although we can hardly call it "modern" (the book of Spivak has been published fifty years ago). The third chapter is the first one with a geometric flavor. Although the name of the chapter is, simply, *Manifolds*, in reality, it is a short introduction to the geometry of the manifolds (well, actually submanifolds of a Euclidean space). It includes, beside the standard notions of differential topology (manifolds, Lie groups, maps, vector fields, the tangent bundle), also differential geometric concepts as: covariant derivative, geodesics, the second fundamental form, curvatures, isometries (with respect to the Euclidean scalar product on the ambient Euclidean spaces). The following chapter is devoted to the theory of integration on a Euclidean space. It includes the classical subjects (the definition of the integral and of the integrability, the Fubini theorem, the classical integral theorems), but also some applications to physics and the Sard theorem for Euclidean spaces, a quite unexpected (but not inappropriate) presence here. The chapter five, *Differential forms*, has, actually, as subject the theory of integration on manifolds (tensors and forms, differential forms on manifolds, integration on manifolds and manifolds with boundary, Stokes' theorem). The author takes time to discuss the connection between the Stokes' theorem and the classical theorems of integral calculus. The chapter ends with a short look at the de Rham cohomology. For the last two chapters, the author returns to geometry. Thus, chapter six treats standard subjects from Riemannian geometry (extremal properties of geodesics, Jacobi fields, the variation of the length functional, Hopf-Rinow theorem, comparison theorems a.o.), while the final chapter focuses on the geometry of hypersurfaces in a Euclidean space, with the induced Riemannian metric (orientation, Gauss map, curvature a.o.). In this context, the general concepts from manifold theory become more intuitive. In particular, there are also discussed some classical examples of surfaces (ruled surface, surfaces of revolution).

Multivariable calculus and geometry always meet (well, at least in textbooks). The question is *where* do they meet. I could mention several such intersection points:

- analytical geometry that goes together with calculus (this is common, especially, in the American universities);
- the calculus as a prerequisite to a textbook of manifolds;
- the other way around: (sub)manifolds used as a foundation for multiple integration (for instance).

What I haven't seen elsewhere (or, if I have, I don't remember) is a marriage between calculus and differential geometry, where the two subjects to be, more or less, on equal footing. From this point of view, the author did an excellent job. The book can be used as a textbook for a course in differential geometry for advanced undergraduate or beginning graduate student in mathematics or physics, or for self-study.

The book includes a large number of worked examples and exercises, as well as a number of excellent drawings that improve the presentation.

Paul A. Blaga

The Princeton Companion to Applied Mathematics, Edited by Nicholas J. Higham, Associate Editors: Mark R. Dennis, Paul Glendinning, Paul A. Martin, Fadil Santosa and Jared Tanner, Princeton University Press, Princeton, NJ, 2015, xvii + 994 pp, ISBN: 978-0-691-15039-0/hbk; 978-1-4008-7447-7/ebook.

As the editor writes in the Preface "*The Princeton Companion to Applied Mathematics* describes what applied mathematics is about, why it is important, its connections with other disciplines, and some of the main areas of current research." We also reproduce here the nice words of Paul Halmos quoted in Editor's (NJH) article *What is applied mathematics*?

> Pure mathematics can be practically useful and applied mathematics can be artistically elegant... Just as pure mathematics can be useful, applied mathematics can be more beautifully useless than is sometimes recognized...

On the other side, it is not easy to give a precise definition of applied mathematics and, in some cases, it is difficult to say whether a specific domain (or topic) belongs to pure or to applied mathematics. Also, over time, a domain of pure mathematics finds its applications, as, e.g., the applications of number theory to cryptography. Pure mathematics was presented in another Princeton volume *The Princeton companion to mathematics* (Editors: Timothy Gowers, June Barrow-Green, Imre Leader; Princeton University Press, 2008), which contains some topics (Mathematics and Chemistry, Mathematical Biology, Mathematical Statistics, Optimization and Lagrange Multipliers) which can be considered to belong to applied mathematics and could be included in the present volume as well. In order to avoid overlapping the topics presented in the previous Companion are excluded from the present one, and in the case of some crucial concepts (e.g. algebraic geometry, fast Fourier transform), the approach here is different, with emphasis on applications and computational aspects. In some cases, particular aspects of topics treated in the previous companion are included here.

The book is divided into eight parts.

Part I, *Introduction to Applied Mathematics*, contains some general results about applied mathematics as: what is it, the language, algorithms, goals and history.

Part II, *Concepts*, contains short articles explaining specific concepts as convexity, chaos, floating-point arithmetic, Markov chains, etc. This part is not a comprehensive, other concepts being presented in other articles.

Part III, *Equations, Laws, and Functions of Applied Mathematics*, contains short presentations of some functions and equations encountered in applied mathematics, as, e.g., Bessel functions, Mathieu functions, Euler functions, Black-Scholes law, Hooke's law, the equations of Cauchy-Riemann, Laplace, Korteweg-de Vries, Dirac, etc.

Part IV, *Areas of Applied Mathematics*, contains longer articles giving an overview of some domains of applied mathematics as complex analysis, ordinary and partial differential equations, data mining, random matrices, control theory, information theory, etc.

Part V, *Modeling*, presents some mathematical models in chemistry, biology, financial mathematics, wheather prediction, etc.

Part VI, *Example Problems*, contains short articles on various interesting problems as bubbles, foams, the inverted pendulum, robotics, random number generation, etc.

Part VII, *Application Areas* contains articles on connections of mathematics with other domains as aircraft noise, social networks, chip design, digital imaging, medical imaging, radar imaging, etc.

The final part, VIII, *Final Perspectives*, contains some longer articles of general interest on mathematical writing, the reading of mathematical papers, teaching applied mathematics, mathematics in the media, mathematics and policy (how mathematicians can influence political decisions).

In the last years, partly due to high performance computers, the area of applied mathematics enlarged considerably, so that the present volume is a welcome addition to the existing publications and a guide for the researchers interested to apply mathematics in their domains (an unavoidable option, as it can be seen).

Together with its elder brother, *The Princeton companion to mathematics*, which turned to be a very successful enterprise, the present volume covers a lot of topics in applied mathematics, and surely it will also have great success and impact. It is dedicated to students (starting with the undergraduate level), teachers, researchers in various areas and specialists in various domains desiring to know what mathematics can offer and, as it is proved in this volume, it has a lot to offer.

S. Cobzaş