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On applications of Andrica-Badea and Nagy inequalities in spectral graph theory

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Abstract. Applications of Andrica-Badea and Nagy inequalities for determining bounds of graph invariants of undirected, connected graphs are investigated. We consider bounds of the following invariants: the first Zagreb index, general Randić index, Laplacian linear spread and normalized Laplacian spread of graphs.

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1. Introduction

Andrica and Badea (see [1]) have proved the following result.

Let $p_1, p_2, \ldots, p_n$ be non-negative real numbers and $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ real numbers with the properties

$$0 < r_1 \le a_i \le R_1 < +\infty \qquad \text{and} \qquad 0 < r_2 \le b_i \le R_2 < +\infty$$

for each $i = 1, 2, \ldots, n$. Further, let S be a subset of $I_n = \{1, 2, \ldots, n\}$ which minimizes the expression

$$\left| \sum_{i \in S} p_i - \frac{1}{2} \sum_{i=1}^{n} p_i \right| . \tag{1.1}$$

Then

$$\left| \sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \right|$$

$$\le (R_1 - r_1)(R_2 - r_2) \sum_{i \in S} p_i \left(\sum_{i=1}^{n} p_i - \sum_{i \in S} p_i \right) \tag{1.2}$$

In [17] Nagy has proved the following result:

Let $a_1, a_2, \dots, a_n$ are real numbers with the property $r \le a_i \le R$, for each $i = 1, 2, \dots, n$. Then

$$n\sum_{i=1}^{n} a_i^2 - \left(\sum_{i=1}^{n} a_i\right)^2 \ge \frac{n}{2}(R-r)^2. \tag{1.3}$$

In this paper we consider bounds of some graph invariants and prove that they are direct corollaries of inequalities (1.2) and (1.3). Some of the obtained bounds are better than those obtained in the literature so far.

In the next section we recall some results from spectral graph theory needed for our work.

2. Laplacian and normalized laplacian spectrum of graph

Let $G = (V, E)$, $V = \{1, 2, \dots, n\}$, be undirected connected graph with n vertices and m edges, with sequence of vertex degrees $d_1 \ge d_2 \ge \cdots \ge d_n > 0$, $d_i = d(i)$, $i = 1, 2, \dots, n$. Denote with $\mathbf{A}$ adjacency matrix of G. Its eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ represent ordinary eigenvalues of graph G. If $\mathbf{D} = \operatorname{diag}(d_1, d_2, \dots, d_n)$ is diagonal matrix of vertex degrees, then $\mathbf{L} = \mathbf{D} - \mathbf{A}$ is the Laplacian matrix of the G. Eigenvalues of $\mathbf{L}$, $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_{n-1} > \mu_n = 0$ represent Laplacian eigenvalues of graph G. The main properties of these eigenvalues are (see [3, 7, 8, 15])

$$\sum_{i=1}^{n-1} \mu_i = \sum_{i=1}^{n} d_i = 2m \quad \text{and} \quad \sum_{i=1}^{n-1} \mu_i^2 = \sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i = M_1 + 2m, \tag{2.1}$$

where $M_1 = \sum_{i=1}^{n} d_i^2$ is the first Zagreb index (see [13]).

Because the graph G is assumed to be connected, it has no isolated vertices and therefore the matrix $\mathbf{D}^{-1/2}$ is well–defined. Then $\mathbf{L}^* = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$ is the normalized Laplacian matrix of the graph G. Its eigenvalues are $\rho_1 \ge \rho_2 \ge \cdots \ge \rho_{n-1} > \rho_n = 0$. Main properties of these eigenvalues are given by (see [19])

$$\sum_{i=1}^{n-1} \rho_i = n \qquad \text{and} \qquad \sum_{i=1}^{n-1} \rho_i^2 = n + 2R_{-1}, \tag{2.2}$$

where $R_{-1} = \sum_{\{i,j\}\in E}(d_i d_j)^{-1}$ is the general Randić index (see [6, 18]).

3. Main result

In the following theorem we prove the inequality that establishes lower and upper bounds for invariant M_1 in terms of parameters n, m, d_1 and d_n.

Theorem 3.1. *Let $G = (V, E)$ be undirected connected graph with $n, n \ge 2$, vertices and m edges. Then*

$$\frac{4m^2}{n} + \frac{1}{2}(d_1 - d_n)^2 \le M_1 \le \frac{4m^2}{n} + n\alpha(n)(d_1 - d_n)^2, \tag{3.1}$$

where

$$\alpha(n) = \frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\right) = \frac{1}{4}\left(1-\frac{(-1)^{n+1}+1}{2n^2}\right) = \begin{cases}\frac{1}{4}, & \textit{if } n \textit{ is even}\\ \frac{n^2-1}{4n^2}, & \textit{if } n \textit{ is odd}\end{cases}$$

Equality holds if and only if G is regular graph.

Proof. For $a_i = d_i$, $i = 1, 2, \ldots, n$, $R = d_1$ and $r = d_n$, the inequality (1.3) transforms into

$$n\sum_{i=1}^{n} d_i^2 - \left(\sum_{i=1}^{n} d_i\right)^2 \geq \frac{n}{2}(d_1 - d_n)^2,$$

i.e. according to (2.1), into

$$nM_1 - 4m^2 \geq \frac{n}{2}(d_1 - d_n)^2, \tag{3.2}$$

wherefrom the left part of inequality (3.1) is obtained.

For $p_i = 1$, $i = 1, 2, \ldots, n$ and $S = \{1, 2, \ldots, k\} \subset I_n$, the expression (1.1) reaches the minimum for $k = \left\lfloor\frac{n}{2}\right\rfloor$. Now for $S = \{1, 2, \ldots, \lfloor\frac{n}{2}\rfloor\}$, $p_i = 1$, $a_i = b_i = d_i$, $i = 1, 2, \ldots, n$, $r_1 = r_2 = d_n$ and $R_1 = R_2 = d_1$, the inequality (1.2) becomes

$$n\sum_{i=1}^{n} d_i^2 - \left(\sum_{i=1}^{n} d_i\right)^2 \leq (d_1 - d_n)^2\left\lfloor\frac{n}{2}\right\rfloor\left(n - \left\lfloor\frac{n}{2}\right\rfloor\right)$$

i.e.

$$nM_1 - 4m^2 \leq n^2(d_1 - d_n)^2\alpha(n) \tag{3.3}$$

where

$$\alpha(n) = \frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\right) = \frac{1}{4}\left(1-\frac{(-1)^{n+1}+1}{2n^2}\right) = \begin{cases}\frac{1}{4}, & \text{if } n \text{ is even}\\ \frac{n^2-1}{4n^2}, & \text{if } n \text{ is odd}\end{cases}.$$

From (3.3) right side of inequality (3.1) immediately follows.

Equalities in (3.2) and (3.3) hold if and only if $d_1 = d_2 = \cdots = d_n$, so the equalities in (3.1) hold if and only if G is a regular graph. □

Remark 3.2. Since $(d_1 - d_n)^2 \geq 0$, left inequality in (3.1) is stronger than

$$M_1 \geq \frac{4m^2}{n} \tag{3.4}$$

which was proved in [9].

Remark 3.3. In [2] the invariant $irr_{EB}(G) = \sqrt{\frac{nM_1}{4m^2} - 1}$ as the irregularity measure of graph was introduced. In [11] another irregularity measure $irr_g(G) = \frac{d_1}{d_n} - 1$ was defined. According to (3.1) we can establish the following relationship between these two measures

$$\sqrt{\frac{nd_n}{8m^2}}irr_g(G) \leq irr_{EB}(G) \leq \sqrt{\frac{n^2d_n\alpha(n)}{4m^2}}irr_g(G).$$

The linear Laplacian spread of graph G is defined as $\mu_1 - \mu_{n-1}$. The following theorem establishes lower and upper bounds for this invariant in terms of parameters n, m and M_1.

Theorem 3.4. *Let $G = (V, E)$ be undirected connected graph with $n, n \geq 2$, vertices and m edges. Then*

$$\sqrt{\frac{(n-1)(M_1+2m)-4m^2}{(n-1)^2\alpha(n-1)}} \leq \mu_1 - \mu_{n-1} \leq \sqrt{\frac{2((n-1)(M_1+2m)-4m^2)}{n-1}}. \quad (3.5)$$

where

$$\alpha(n-1) = \frac{1}{n-1}\left\lfloor\frac{n-1}{2}\right\rfloor\left(1-\frac{1}{n-1}\left\lfloor\frac{n-1}{2}\right\rfloor\right) = \frac{1}{4}\left(1-\frac{(-1)^n+1}{2(n-1)^2}\right).$$

Equalities hold if and only if G is a complete graph, $G \cong K_n$.

Proof. For $n := n-1$, $a_i = \mu_i$, $i = 1, 2, \ldots, n-1$, $R = \mu_1$ and $r = \mu_{n-1}$, the inequality (1.3) becomes

$$(n-1)\sum_{i=1}^{n-1}\mu_i^2 - \left(\sum_{i=1}^{n-1}\mu_i\right)^2 \geq \frac{(n-1)}{2}(\mu_1-\mu_{n-1})^2$$

i.e. according to (2.1) we have

$$(n-1)(M_1+2m)-4m^2 \geq \frac{(n-1)}{2}(\mu_1-\mu_{n-1})^2 \quad (3.6)$$

wherefrom right side of (3.5) is obtained.

For $n := n-1$, $p_i = 1$, $a_i = b_i = \mu_i$, $i = 1, 2, \ldots, n-1$, $r_1 = r_2 = \mu_{n-1}$ and $R_1 = R_2 = \mu_1$, the inequality (1.2) transforms into

$$(n-1)\sum_{i=1}^{n-1}\mu_i^2 - \left(\sum_{i=1}^{n-1}\mu_i\right)^2 \leq (\mu_1-\mu_{n-1})^2\left\lfloor\frac{n-1}{2}\right\rfloor\left(n-1-\left\lfloor\frac{n-1}{2}\right\rfloor\right)$$

i.e.

$$(n-1)(M_1+2m)-4m^2 \leq (\mu_1-\mu_{n-1})^2(n-1)^2\alpha(n), \quad (3.7)$$

where

$$\alpha(n-1) = \frac{1}{n-1}\left\lfloor\frac{n-1}{2}\right\rfloor\left(1-\frac{1}{n-1}\left\lfloor\frac{n-1}{2}\right\rfloor\right) = \begin{cases}\frac{1}{4} & \text{if } n \text{ is even}\\ \frac{n(n-2)}{4(n-1)^2}, & \text{if } n \text{ is odd}\end{cases}.$$

From (3.7) left part of inequality (3.5) is directly obtained.

Equalities (3.6) and (3.7) hold if and only if $\mu_1 = \mu_2 = \cdots = \mu_{n-1}$, hence equalities in (3.5) hold if and only if G is a complete graph, $G \cong K_n$. $\square$

Remark 3.5. Right side of inequality (3.5) was proved in [16]. Since $\alpha(n-1) \leq \frac{1}{4}$, for each n, left side of inequality (3.5) is stronger than

$$\mu_1 - \mu_{n-1} \geq \frac{2}{n-1}\sqrt{(n-1)(M_1+2m)-4m^2},$$

for even n. The above inequality was proved in [10] and [20].

From inequality (3.4) and $M_1 \leq m\left(\frac{2m}{n-1}+(n-2)\right)$, proved in [5], and inequality $M_1 \geq \frac{4m^2}{n}$, proved in [9], the following corollary of Theorem 3.4 holds.

Corollary 3.6. *Let $G=(V,E)$ be undirected connected graph with $n, n\geq 2$, vertices and m edges. Then*

$$\frac{1}{n-1}\sqrt{\frac{2m(n(n-1)-2m)}{n\alpha(n-1)}} \leq \mu_1-\mu_{n-1} \leq \sqrt{\frac{2m(n(n-1)-2m)}{n-1}}.$$

Equalities hold if and only if G is a complete graph, $G\cong K_n$.

In the following theorem we determine lower and upper bounds for graph invariant R_{-1} in terms of parameters n, ρ_1 and ρ_{n-1}.

Theorem 3.7. *Let $G=(V,E)$ be undirected connected graph with $n, n\geq 2$, vertices and m edges. Then*

$$\frac{n}{2(n-1)}+\frac{1}{4}(\rho_1-\rho_{n-1})^2 \leq R_{-1} \leq \frac{n}{2(n-1)}+\frac{(n-1)\alpha(n-1)}{2}(\rho_1-\rho_{n-1})^2. \quad (3.8)$$

Equalities hold if and only if G is a complete graph, $G\cong K_n$.

Proof. for $n:=n-1$, $a_i=\rho_i$, $i=1,2,\ldots,n-1$, $r=\rho_{n-1}$ and $R=\rho_1$ the inequality (1.3) becomes

$$(n-1)\sum_{i=1}^{n-1}\rho_i^2-\left(\sum_{i=1}^{n-1}\rho_i\right)^2 \geq \frac{n-1}{2}(\rho_1-\rho_{n-1})^2$$

i.e. according to (2.2)

$$(n-1)(n+2R_{-1})-n^2 \geq \frac{n-1}{2}(\rho_1-\rho_{n-1})^2, \quad (3.9)$$

wherefrom left side of inequality (3.8) is obtained.

For $n:=n-1$, $p_i=1$, $a_i=b_i=\rho_i$, $i=1,2,\ldots,n-1$, $r_1=r_2=\rho_{n-1}$ and $R_1=R_2=\rho_1$, inequality (1.3) transforms into

$$(n-1)\sum_{i=1}^{n-1}\rho_i^2-\left(\sum_{i=1}^{n-1}\rho_i\right)^2 \leq (\rho_1-\rho_{n-1})^2\left\lfloor\frac{n-1}{2}\right\rfloor\left(n-1-\left\lfloor\frac{n-1}{2}\right\rfloor\right),$$

i.e.

$$(n-1)(n+2R_{-1})-n^2 \leq (n-1)^2\alpha(n-1)(\rho_1-\rho_{n-1})^2, \quad (3.10)$$

wherefrom right part of inequality (3.8) is obtained.

Equalities in (3.9) and (3.10) hold if and only if $\rho_1=\rho_2=\cdots=\rho_{n-1}$, therefore equalities in (3.8) hold if and only if G is complete graph, $G\cong K_n$. ☐

Remark 3.8. Since $(\rho_1-\rho_{n-1})^2\geq 0$, it follows that left side of inequality (3.8) is stronger than inequality

$$R_{-1}\geq\frac{n}{2(n-1)}$$

which was proved in [14].

Remark 3.9. Inequalities in (3.8) can be presented in an equivalent form as

$$\sqrt{\frac{2(n-1)R_{-1}-n}{(n-1)^2\alpha(n-1)}} \leq \rho_1 - \rho_{n-1} \leq \sqrt{\frac{2(2(n-1)R_{-1}-n)}{n-1}}. \tag{3.11}$$

For even n, left side of inequality (3.11) is stronger than inequality

$$\rho_1 - \rho_{n-1} \geq \frac{2}{n-1}\sqrt{2(n-1)R_{-1}-n},$$

which was proved in [4].

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