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Multisymplectic connections on supermanifolds

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Abstract. In this paper we show that on any multisymplectic supermanifold there exist a connection compatible to the multisymplectic form.

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1. Introduction

Multisymplectic structures in field theory play a role similar to that of symplectic structures in classical mechanics. In the other hand supergeometry plays an important role in physics. In [2] and [3], the authors studied geometry of symplectic connections and in [1], the author studied symplectic connections on supermanifold. In this paper we study multisymplectic connections on supermanifolds.
A supermanifold $\mathcal{M}$ of dimension $n|m$ is a pair $(M, \mathcal{O}_\mathcal{M})$, where M is a Hausdorff topological space and $\mathcal{O}_\mathcal{M}$ is a sheaf of commutative superalgebras with unity over $\mathbb{R}$ locally isomorphic to $\mathbb{R}^{m|n} = (\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n} \otimes \Lambda_{\eta^1,\dots,\eta^m})$, where $\mathcal{O}_{\mathbb{R}^n}$ is the sheaf of smooth functions on $\mathbb{R}^n$ and $\Lambda_{\eta^1,\dots,\eta^m}$ is the grassmann superalgebra of m generators (for more details see [5]).

If $\mathcal{M}$ is a supermanifold of dimension $n|m$, we define the tangent sheaf as follows,

$$\mathcal{T}_\mathcal{M}(U) = Der(\mathcal{O}_\mathcal{M}(U)),$$

the $\mathcal{O}_\mathcal{M}(U)$-supermodule of derivations of $\mathcal{O}_\mathcal{M}(U)$. $\mathcal{T}_\mathcal{M}$ is locally free of dimension $n|m$. The sections of $\mathcal{T}_\mathcal{M}$ are called vector fields.

Definition 1.1. *If ξ be a locally free sheaf of $\mathcal{O}_\mathcal{M}$-supermodules on $\mathcal{M}$, a connection on ξ is a morphism $\nabla : \mathcal{T}_\mathcal{M} \otimes_\mathbb{R} \xi \to \xi$ of sheaves of supermodules over $\mathbb{R}$ such that*

$$\nabla_{fX} v = f\nabla_X v,\ \nabla_X fv = (Xf) + (-1)^{\widetilde{X}\widetilde{f}} f\nabla_X v \text{ and } \widetilde{\nabla_X v} = \widetilde{v} + \widetilde{X},$$

for all homogeneous function f, vector fields X and section v of ξ. (In the case $\xi = \mathcal{T}_\mathcal{M}$ we speak of a connection on $\mathcal{M}$).

We define the torsion of a connection ∇ on $\mathcal{T}_{\mathcal{M}}$ by

$$T(X,Y)=\nabla_X Y-(-1)^{\widetilde{X}\widetilde{Y}}\nabla_Y X-[X,Y].$$

Definition 1.2. *A graded Riemannian metric on supermanifold $\mathcal{M}$ is a graded-symmetric non-degenerate $\mathcal{O}_{\mathcal{M}}$-linear morphism of sheaves*

$$g:\mathcal{T}_{\mathcal{M}}\otimes\mathcal{T}_{\mathcal{M}}\to\mathcal{O}_{\mathcal{M}}.$$

A supermanifold equipped with graded Riemannian metric is called a Riemannian supermanifold. If $\mathcal{M}$ is a Riemannian supermanifold with Riemannian metric g, we call a connection ∇ metric if $\nabla g=0$.

On a suppermanifold M with a Riemannian metric g, there exist a unique torsion free and metric connection ∇^0, which will be called the Levi-Civita connection of the metric(see [4]).

2. Multisymplectic connections on supermanifolds

Let us consider a multisymplectic supermanifold of degree k $(\mathcal{M},\omega)$, i.e. a supermanifold $\mathcal{M}$ with a closed non-degenerate graded differential k-form ω.

Definition 2.1. *A multisymplectic connection on $\mathcal{M}$ is a connection for which:*
i) The torsion tensor vanishes, i.e.

$$\nabla_X Y-(-1)^{\widetilde{X}\widetilde{Y}}\nabla_Y X=[X,Y].$$

ii) It is compatible to the multisymplectic form, i.e. $\nabla\omega=0$.

To prove the existence of such a connection, take ∇^0 to be the Levi-Civita connection associated to a metric g on $\mathcal{M}$. Consider tensor N on $\mathcal{M}$ defined by

$$\nabla^0_{Y_0}\omega(Y_1,Y_2,...,Y_k)=(-1)^{\widetilde{\omega}\widetilde{Y_0}}\omega(N(Y_0,Y_1),Y_2,...,Y_k).$$

We shall proof some properties of N.

Lemma 2.2. *We have*
i) $\omega(N(Y_0,Y_1),Y_2,...,Y_k)=-(-1)^{\widetilde{Y_1}\widetilde{Y_2}}\omega(N(Y_0,Y_2),Y_1,...,Y_k)$;
ii) $\omega(N(Y_0,Y_1),Y_2,...,Y_k)+\Sigma_{i=1}^k(-1)^{i+\sum_{p<i}\widetilde{Y_p}\widetilde{Y_i}}\omega(N(Y_i,Y_0),Y_1,...,\hat{Y_i},...,Y_k)=0$,
where the hats indicate omitted arguments.

Proof. We first prove (i)

$$\begin{aligned}\omega(N(Y_0,Y_1),Y_2,...,Y_k)&=(-1)^{\widetilde{Y_0}\widetilde{\omega}}\nabla^0_{Y_0}\omega(Y_1,Y_2,...,Y_k)\\&=-(-1)^{\widetilde{Y_0}\widetilde{\omega}+\widetilde{Y_1}\widetilde{Y_2}}\nabla^0_{Y_0}\omega(Y_2,Y_1,...,Y_k)\\&=-(-1)^{\widetilde{Y_1}\widetilde{Y_2}}\omega(N(Y_0,Y_2),Y_1,...,Y_k).\end{aligned}$$

For proof (ii) we know $d\omega=0$ so

$$0=d\omega(Y_0,Y_1,...,Y_k)=\sum_{i=0}^k(-1)^{i+\widetilde{Y_i}(\widetilde{w}+\sum_{p<i}\widetilde{Y_p})}Y_i(\omega(Y_0,...,\hat{Y_i},...,Y_k))$$

$$+\sum_{i<j}(-1)^{j+\sum_{i<p<j}\widetilde{Y_j}\widetilde{Y_p}}\omega(Y_0,...,Y_{i-1},[Y_i,Y_j],Y_{i+1},...,\hat{Y}_j,...,Y_k)$$

$$=\sum_{i=0}^{k}(-1)^{i+\widetilde{Y}_i(\widetilde{w}+\sum_{p<i}\widetilde{Y_p})}Y_i(\omega(Y_0,...,\hat{Y}_i,...,Y_k))$$

$$+\sum_{i<j}(-1)^{j+\sum_{i<p<j}\widetilde{Y_j}\widetilde{Y_p}}\omega(Y_0,...,Y_{i-1},\nabla^0_{Y_i}Y_j-(-1)^{\widetilde{Y_i}\widetilde{Y_j}}\nabla^0_{Y_j}Y_i,Y_{i+1},...,\hat{Y}_j,...,Y_k)$$

$$=\sum_{i=0}^{k}(-1)^{i+\widetilde{Y}_i(\widetilde{w}+\sum_{p<i}\widetilde{Y_p})}Y_i(\omega(Y_0,...,\hat{Y}_i,...,Y_k))$$

$$+\sum_{i<j}(-1)^{j+\sum_{i<p<j}\widetilde{Y_j}\widetilde{Y_p}}\omega(Y_0,...,Y_{i-1},\nabla^0_{Y_i}Y_j,Y_{i+1},...,\hat{Y}_j,...,Y_k)$$

$$-\sum_{i<j}(-1)^{j+\sum_{i\leq p<j}\widetilde{Y_j}\widetilde{Y_p}}\omega(Y_0,...,Y_{i-1},\nabla^0_{Y_j}Y_i,Y_{i+1},...,\hat{Y}_j,...,Y_k)$$

$$=\sum_{i=0}^{k}(-1)^{i+\widetilde{Y}_i(\widetilde{w}+\sum_{p<i}\widetilde{Y_p})}Y_i(\omega(Y_0,...,\hat{Y}_i,...,Y_k))$$

$$+\sum_{i<j}(-1)^{j+\sum_{i<p<j}\widetilde{Y_j}\widetilde{Y_p}}\omega(Y_0,...,Y_{i-1},\nabla^0_{Y_i}Y_j,Y_{i+1},...,\hat{Y}_j,...,Y_k)$$

$$-\sum_{j<i}(-1)^{i+\sum_{j\leq p<i}\widetilde{Y_i}\widetilde{Y_p}}\omega(Y_0,...,Y_{j-1},\nabla^0_{Y_i}Y_j,Y_{j+1},...,\hat{Y}_i,...,Y_k)$$

$$=\sum_{i=0}^{k}(-1)^{i+\widetilde{Y}_i(\widetilde{w}+\sum_{p<i}\widetilde{Y_p})}Y_i(\omega(Y_0,...,\hat{Y}_i,...,Y_k))$$

$$-\sum_{i<j}(-1)^{i+\sum_{i<p<j}\widetilde{Y_i}\widetilde{Y_p}}\omega(Y_0,...,Y_{i-1},\hat{Y}_i,...,Y_{j-1},\nabla^0_{Y_i}Y_j,Y_{j+1},...,Y_k)$$

$$-\sum_{j<i}(-1)^{i+\sum_{j\leq p<i}\widetilde{Y_i}\widetilde{Y_p}}\omega(Y_0,...,Y_{j-1},\nabla^0_{Y_i}Y_j,Y_{j+1},...,\hat{Y}_i,...,Y_k)$$

$$=\sum_{i=0}^{k}(-1)^{i+\widetilde{Y}_i(\widetilde{w}+\sum_{p<i}\widetilde{Y_p})}(Y_i(\omega(Y_0,....\hat{Y}_i,...,Y_k))$$

$$-\sum_{j}(-1)^{\widetilde{Y}_i(\widetilde{\omega}+\sum_{p<i}\widetilde{Y_p})}\omega(Y_0,...,Y_{j-1},\nabla^0_{Y_i}Y_j,...,\hat{Y}_i,...,Y_k)$$

$$=\sum_{i=0}^{k}(-1)^{i+\widetilde{Y}_i(\widetilde{w}+\sum_{p<i}\widetilde{Y_p})}\nabla^0_{Y_i}\omega(Y_0,...,\hat{Y}_i,...,Y_k)$$

$$=(-1)^{\widetilde{Y_0}\widetilde{\omega}}\nabla^0_{Y_0}\omega(Y_1,...,Y_k)$$

$$+\sum_{i=1}^{k}(-1)^{i+\widetilde{Y}_i(\widetilde{\omega}+\sum_{p<i}\widetilde{Y_p})}\nabla^0_{Y_i}\omega(Y_0,...,\hat{Y}_i,...,Y_k)$$

$$=\omega(N(Y_0,Y_1),Y_2,...,Y_k)+\Sigma_{i=1}^{k}(-1)^{i+\sum_{p<i}\widetilde{Y_p}\widetilde{Y_i}}\omega(N(Y_i,Y_0),Y_1,...,\hat{Y}_i,...,Y_k).\qquad\square$$

Now we show that on any multisymplectic supermanifold there exist a connection compatible to the multisymplectic form.

Theorem 2.3. *Let $(\mathcal{M},\omega)$ be a multisymplectic supermanifold. Then on $\mathcal{M}$ there is at least a multisymplectic connection.*

Proof. We define now a new connection ∇ as follows

$$\nabla_X Y = \nabla^0_X Y + \frac{1}{k+1}N(X,Y) + \frac{(-1)^{\widetilde{X}\widetilde{Y}}}{k+1}N(Y,X).$$

It is easy to show that ∇ is a torsion free connection. We show that the connection is compatible with the multisymplectic form ω, i.e. $\nabla\omega = 0$. We have

$$\nabla_{Y_0}\omega(Y_1,...,Y_k) = Y_0(\omega(Y_1,...,Y_k))$$

$$-\sum_{i=1}^{k}(-1)^{\widetilde{Y_0}(\widetilde{\omega}+\sum_{p<i}\widetilde{Y_p})}\omega(Y_1,...,Y_{i-1},\nabla_{Y_0}Y_i,Y_{i+1},...,Y_k)$$

$$= Y_0(\omega(Y_1,...,Y_k)) - \sum_{i=1}^{k}(-1)^{\widetilde{Y_0}(\widetilde{\omega}+\sum_{p<i}\widetilde{Y_p})}\omega(Y_1,...,Y_{i-1},\nabla^0_{Y_0}Y_i$$

$$+\frac{1}{k+1}N(Y_0,Y_i) + \frac{(-1)^{\widetilde{Y_0}\widetilde{Y_i}}}{k+1}N(Y_i,Y_0),Y_{i+1},...,Y_k)$$

$$= Y_0(\omega(Y_1,...,Y_k)) - \sum_{i=1}^{k}(-1)^{\widetilde{Y_0}(\widetilde{\omega}+\sum_{1\le p<i}\widetilde{Y_p})}\omega(Y_1,...,Y_{i-1},\nabla^0_{Y_0}Y_i,Y_{i+1},...,Y_k)$$

$$-\frac{1}{k+1}\sum_{i=1}^{k}(-1)^{\widetilde{Y_0}(\widetilde{\omega}+\sum_{1\le p<i}\widetilde{Y_p})}\omega(Y_1,...,Y_{i-1},N(Y_0,Y_i),Y_{i+1},...,Y_k)$$

$$-\frac{1}{k+1}\sum_{i=1}^{k}(-1)^{\widetilde{Y_0}(\widetilde{\omega}+\sum_{1\le p\le i}\widetilde{Y_p})}\omega(Y_1,...,Y_{i-1},N(Y_i,Y_0),Y_{i+1},...,Y_k)$$

$$= \nabla^0_{Y_0}\omega(Y_1,...,Y_k)$$

$$-\frac{1}{k+1}\sum_{i=1}^{k}(-1)^{i-1}(-1)^{\widetilde{Y_0}\widetilde{\omega}+\widetilde{Y_i}\sum_{1\le p<i}\widetilde{Y_p}}\omega(N(Y_0,Y_i),Y_1,...,\hat{Y_i},..,Y_k)$$

$$-\frac{1}{k+1}\sum_{i=1}^{k}(-1)^{i-1}(-1)^{\widetilde{Y_0}\widetilde{\omega}+\widetilde{Y_i}\sum_{0\le p<i}\widetilde{Y_p}}\omega(N(Y_i,Y_0),Y_1,...,\hat{Y_i},...,Y_k)$$

$$= (-1)^{\widetilde{Y_0}\widetilde{\omega}}\omega(N(Y_0,Y_1),Y_2,...,Y_k) - \frac{k}{k+1}(-1)^{\widetilde{Y_0}\widetilde{\omega}}\omega(N(Y_0,Y_1),Y_2,...,Y_k)$$

$$+\frac{1}{k+1}\sum_{i=1}^{k}(-1)^{i+\widetilde{Y_0}\widetilde{\omega}+\widetilde{Y_i}\sum_{0\le p<i}\widetilde{Y_p}}\omega(N(Y_i,Y_0),Y_1,...,\hat{Y_i},...,Y_k)$$

$$= \frac{1}{k+1}(-1)^{\widetilde{Y_0}\widetilde{\omega}}(\omega(N(Y_0,Y_1),Y_2,...,Y_k)$$

$$+\sum_{i=1}^{k}(-1)^{i+\widetilde{Y_i}\sum_{p<i}\widetilde{Y_p}}\omega(N(Y_i,Y_0),Y_1,...,\hat{Y_i},...,Y_k))=0. \qquad \square$$

Let now ∇ be a multisymplectic connection and $\nabla'_X Y=\nabla_X Y+S(X,Y)$, where S is a tensor field on $\mathcal{M}$. We have

Theorem 2.4. *∇' is a multisymplelectic connection if and only if S is supersymmetric and*

$$\sum_i(-1)^{\sum_{p<i}\widetilde{Y_0}\widetilde{Y_p}}\omega(Y_1,...,Y_{i-1},S(Y_0,Y_i),Y_{i+1},...,Y_k)=0.$$

Proof. If we want ∇' to be torsion free then

$$\nabla_Y X+S(X,Y)-(-1)^{\widetilde{X}\widetilde{Y}}\nabla_Y X-(-1)^{\widetilde{X}\widetilde{Y}}S(Y,X)=[X,Y].$$

So $S(X,Y)=-(-1)^{\widetilde{X}\widetilde{Y}}S(Y,X)$. If ∇' be compatible to the multisymplectic form ω. We have

$$0=\nabla'_{Y_0}\omega(Y_1,...,Y_k)=Y_0(\omega(Y_1,...,Y_k))$$
$$-\sum_i(-1)^{\widetilde{Y_0}(\widetilde{\omega}+\sum_{p<i}\widetilde{Y_p})}\omega(Y_1,...,Y_{i-1},\nabla'_{Y_0}Y_i,Y_{i+1},..,Y_k)$$
$$=\nabla_{Y_0}\omega(Y_1,...,Y_k)-(-1)^{\widetilde{Y_0}\widetilde{\omega}}(\Sigma_i(-1)^{\sum_{p<i}\widetilde{Y_0}\widetilde{Y_p}}\omega(Y_1,...,Y_{i-1},S(Y_0,Y_i),Y_{i+1},...,Y_k)).$$

So

$$\sum_i(-1)^{\sum_{p<i}\widetilde{Y_0}\widetilde{Y_p}}\omega(Y_1,...,Y_{i-1},S(Y_0,Y_i),Y_{i+1},...,Y_k)=0. \qquad \square$$

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