Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 4, 583–595

Lines in the three-dimensional Bolyai-Lobachevskian hyperbolic geometry

Zoltán Gábos and Ágnes Mester

Abstract. The purpose of this paper is to describe the geodesics of the three-dimensional Bolyai-Lobachevskian hyperbolic space. We also determine the equation of the orthogonal surfaces and the scalar curvature of the surfaces of revolution. The metric applied is the Lobachevskian metric extended into three dimensions. During the analysis we use Cartesian and cylindrical coordinates. This article is a continuation of the paper [4].

Mathematics Subject Classification (2010): 53A35.

Keywords: Hyperbolic geometry, geodesics, orthogonal surfaces, cylindrical coordinates.

1. General context

In the literature exists several models for hyperbolic geometry, see [1]-[10]. The aim of this paper is to present a three dimensional model using [8] to describe some classical and new properties.

We consider the following metric

$$ds^2 = \cosh^2 \frac{z}{k} \left(\cosh^2 \frac{y}{k} dx^2 + dy^2 \right) + dz^2, \tag{1.1}$$

where k is the parameter of the three-dimensional hyperbolic space, and x, y, z are the Cartesian coordinates of any $P(x, y, z)$ point. The usage of Cartesian coordinates is justified by the existence of such hyperbolic lines which can also be considered Euclidean lines. These lines include the coordinate axes illustrated in figure 1. Note that the x-value can only be determined by axis Ox. Figure 1 also represents how the coordinates of any $P(x, y, z)$ point are determined: $x = \overline{OP_2},\ y = \overline{P_1P_2},\ z = \overline{PP_1}$.

From metric (1.1) we can obtain two possible symmetry operations. These consist of the reflections across the coordinate planes and the translation of the origin along the direction of the x-axis (the values y and z are not modified).

Based on metric (1.1), the geodesic lines verify

$$\cosh^2\frac{z}{k}\cosh^2\frac{y}{k}\frac{dx}{ds}=C_1, \tag{1.2}$$

where C_1 is constant. From this we obtain

$$\frac{d}{ds}\left(\cosh^2\frac{z}{k}\frac{dy}{ds}\right)-\frac{1}{k}\cosh^2\frac{z}{k}\sinh\frac{y}{k}\cosh\frac{y}{k}\left(\frac{dx}{ds}\right)^2=0, \tag{1.3}$$

$$\frac{d^2z}{ds^2}-\frac{1}{k}\sinh\frac{z}{k}\cosh\frac{z}{k}\left[\cosh^2\frac{y}{k}\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2\right]=0. \tag{1.4}$$

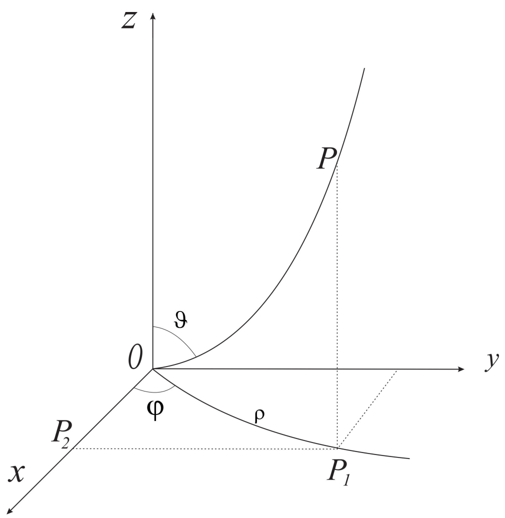

FIGURE 1

If we use x instead of variable s in (1.2), we can write equations (1.3) and (1.4) in the following form:

$$\frac{d^2\tanh\frac{y}{k}}{dx^2}-\frac{1}{k^2}\tanh\frac{y}{k}=0, \tag{1.5}$$

$$\frac{d}{dx}\left(\frac{1}{\cosh^2\frac{y}{k}}\frac{d\tanh\frac{z}{k}}{dx}\right)-\frac{1}{k^2}\left[1+k^2\cosh^2\frac{y}{k}\left(\frac{d\tanh\frac{y}{k}}{dx}\right)^2\right]\tanh\frac{z}{k}=0. \tag{1.6}$$

If we use variable x, we can apply the results obtained in the hyperbolic plane by determining the function $y=y(x)$. Moreover, we claim that the projections of the geodesics in the three-dimensional space to the xOy plane are geodesics of the two-dimensional plane.

Using (1.1) and (1.2), we get

$$\frac{1}{C_1^2}=\frac{1}{\cosh^2\frac{z}{k}}\left[\frac{1}{\cosh^2\frac{y}{k}}+k^2\left(\frac{d\tanh\frac{y}{k}}{dx}\right)^2\right]+k^2\frac{1}{\cosh^4\frac{y}{k}}\left(\frac{d\tanh\frac{z}{k}}{dx}\right)^2. \tag{1.7}$$

The curvature of the geodesics equals zero,

$$\frac{1}{r_g} = 0. \tag{1.8}$$

Metric (1.1) can also be obtained by using the metric

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - dx_0^2 \tag{1.9}$$

defined in the four-dimensional pseudo-Euclidean space, with the help of the following equations:

$$x_1^2 + x_2^2 + x_3^2 - x_0^2 = -k^2, \tag{1.10}$$

$$\begin{aligned} &x_1 = k \sinh \frac{x}{k} \cosh \frac{y}{k} \cosh \frac{z}{k}, \quad x_2 = k \sinh \frac{y}{k} \cosh \frac{z}{k}, \\ &x_3 = k \sinh \frac{z}{k}, \quad x_0 = k \cosh \frac{x}{k} \cosh \frac{y}{k} \cosh \frac{z}{k}. \end{aligned} \tag{1.11}$$

If we use equations

$$\begin{aligned} &x_1 = k \cos\varphi \sinh \frac{\rho}{k} \cosh \frac{z}{k}, \quad x_2 = k \sin\varphi \sinh \frac{\rho}{k} \cosh \frac{z}{k}, \\ &x_3 = k \sinh \frac{z}{k}, \quad x_0 = k \cosh \frac{\rho}{k} \cosh \frac{z}{k}, \end{aligned} \tag{1.12}$$

we obtain metric

$$ds^2 = \cosh^2 \frac{z}{k} \left(d\rho^2 + k^2 \sinh^2 \frac{\rho}{k} d\varphi^2 \right) + dz^2, \tag{1.13}$$

where ρ, φ and z represent cylindrical coordinates (figure 1).

Metric (1.13) justifies that the rotation around axis Oz (the constant choices for ρ and z) is a symmetry operation.

By choosing s as variable, the geodesic lines verify

$$\sinh^2 \frac{\rho}{k} \cosh^2 \frac{z}{k} \frac{d\rho}{ds} = C_2, \tag{1.14}$$

where C_2 is constant. We can also write

$$\frac{d^2\rho}{ds^2} + \frac{2}{k} \tanh \frac{z}{k} \frac{d\rho}{ds} \frac{dz}{ds} - k \sinh \frac{\rho}{k} \cosh \frac{\rho}{k} \left(\frac{d\varphi}{ds} \right)^2 = 0, \tag{1.15}$$

$$\frac{d^2 z}{ds^2} - \frac{1}{k} \sinh \frac{z}{k} \cosh \frac{z}{k} \left[\left(\frac{d\rho}{ds} \right)^2 + k^2 \sinh^2 \frac{\rho}{k} \left(\frac{d\varphi}{ds} \right)^2 \right] = 0. \tag{1.16}$$

If we consider φ as variable, we will use the following differential equations:

$$\frac{d^2 \coth \frac{\rho}{k}}{d\varphi^2} + \coth \frac{\rho}{k} = 0, \tag{1.17}$$

$$\frac{d}{d\varphi} \left(\frac{1}{\sinh^2 \frac{\rho}{k}} \frac{d \tanh \frac{z}{k}}{d\varphi} \right) - \tanh \frac{z}{k} \left[1 + k^2 \left(\frac{d \coth \frac{\rho}{k}}{d\varphi} \right)^2 \right] = 0. \tag{1.18}$$

Using (1.13) and (1.14), we obtain

$$\frac{1}{C_2^2} = k^2 \left\{ \frac{1}{\cosh^2 \frac{z}{k}} \left[\frac{1}{\sinh^2 \frac{\rho}{k}} + k^2 \left(\frac{d \coth \frac{\rho}{k}}{d\varphi} \right)^2 \right] + \frac{1}{\sinh^4 \frac{\rho}{k}} \left(\frac{d \tanh \frac{z}{k}}{d\varphi} \right)^2 \right\}. \tag{1.19}$$

In the following sections we describe the different types of lines. Note that each line verifies the differential equations which characterize the geodesics. Also, C_1 and C_2 are constant values. During the analysis our choice of coordinates may vary depending on the form of calculations.

2. Lines crossing the origin

Let us consider the line passing through points O and P represented in figure 1, where ϑ is the angle of intersection with axis Oz. In the OP_1P right triangle we can write

$$\tanh\frac{z}{k} = \cot\vartheta\sinh\frac{\rho}{k}. \tag{2.1}$$

The projection of line OP onto the xOy plane satisfies the following equation:

$$\tanh\frac{y}{k} = \tan\varphi\sinh\frac{x}{k}. \tag{2.2}$$

Using (1.11) and (1.12), we obtain

$$\cosh\frac{x}{k}\cosh\frac{y}{k} = \cosh\frac{\rho}{k}.$$

These formulas imply

$$\tanh\frac{z}{k} = \frac{\cot\vartheta\sinh\frac{x}{k}}{\cos\varphi\sqrt{1-\tan^2\varphi\sinh^2\frac{x}{k}}}. \tag{2.3}$$

The lines verifying equations (2.2) and (2.3) also satisfy the (1.5) and (1.6) differential equations. Using (1.7), we get

$$C_1 = \cos\varphi\sin\vartheta$$

constant. Therefore, the lines crossing the origin satisfy the conditions mentioned in the previous section.

In the two-dimensional hyperbolic plane the orthogonal curves of lines crossing the origin are circles. Based on the rotational symmetry operation, we claim that in the three-dimensional case the orthogonal surfaces are spheres. By the use of cylindrical coordinates we can write

$$\cosh\frac{\rho}{k}\cosh\frac{z}{k} = \cosh\frac{R}{k}. \tag{2.4}$$

In order to determine the curvature of the sphere surface, we use the metric

$$ds^2 = E(\rho)d\rho^2 + G(\rho)d\varphi^2$$

obtained from equations (1.13) and (2.4), where

$$E(\rho) = \frac{\sinh^2\frac{R}{k}\cosh^2\frac{R}{k}}{\cosh^2\frac{\rho}{k}\left(\cosh^2\frac{R}{k} - \cosh^2\frac{\rho}{k}\right)}, \quad G(\rho) = k^2\cosh^2\frac{R}{k}\tanh^2\frac{\rho}{k}.$$

The Christoffel symbols of the second kind are as follows:

$$\Gamma^1_{11} = \frac{2\sinh\frac{\rho}{k}\cosh\frac{\rho}{k} - \cosh^2\frac{R}{k}\tanh\frac{\rho}{k}}{k\left(\cosh^2\frac{R}{k} - \cosh^2\frac{\rho}{k}\right)}, \quad \Gamma^2_{12} = \frac{1}{k\sinh\frac{\rho}{k}\cosh\frac{\rho}{k}},$$

$$\Gamma^1_{22} = -\frac{k}{\sinh^2\frac{R}{k}\cosh^2\frac{R}{k}}\tanh\frac{\rho}{k}\left(\cosh^2\frac{R}{k} - \cosh^2\frac{\rho}{k}\right),$$

where we used index 1 for ρ and index 2 for φ.

The components of the Ricci curvature tensor are

$$R_{11} = \frac{d\Gamma^2_{12}}{d\rho} + \Gamma^2_{12}\left(\Gamma^2_{12} - \Gamma^1_{11}\right), \quad R_{22} = -\frac{d\Gamma^1_{22}}{d\rho} + \Gamma^1_{22}\left(\Gamma^2_{12} - \Gamma^1_{11}\right).$$

Using the expressions above, we obtain for the scalar curvature

$$R = \frac{1}{E}R_{11} + \frac{1}{G}R_{22} = -\frac{2}{k^2\sinh^2\frac{R}{k}}. \tag{2.5}$$

3. Lines crossing the x-axis

Let us consider the line passing through points $P_0(a,0,0)$ and $P_1(0,b,c)$ illustrated in figure 2. If we project this line onto the xOy plane, we get the line passing through points $P_0(a,0,0)$ and $P_2(0,b,0)$, which verifies

$$\tanh\frac{y}{k} = \tanh\frac{b}{k}\frac{\sinh\frac{a-x}{k}}{\sinh\frac{a}{k}}. \tag{3.1}$$

The angle of intersection between the lines P_0P_1 and P_0P_2 is denoted by δ.

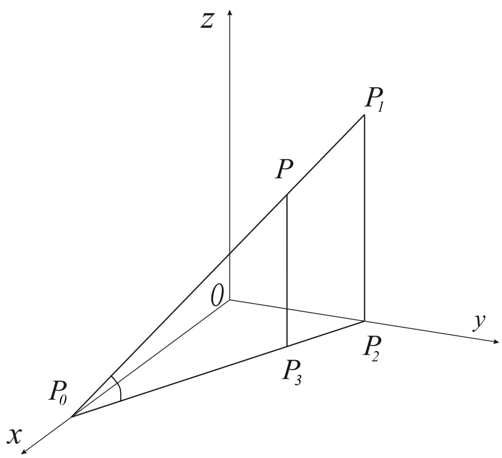

FIGURE 2

We can obtain the distance d between the points $P_0(a,0,0)$ and $P_2(0,b,0)$ from

$$\cosh\frac{d}{k} = \cosh\frac{a}{k}\cosh\frac{b}{k}.$$

Furthermore, distance d_1 between $P_0(a,0,0)$ and $P_3(x,y,0)$ verifies

$$\cosh\frac{d_1}{k}=\cosh\frac{a-x}{k}\cosh\frac{y}{k}.$$

If we consider the P_0PP_3 right triangle, we can write

$$\tanh\frac{z}{k}=\tan\delta\sinh\frac{d_1}{k},$$

while in the right triangle $P_0P_1P_2$

$$\tanh\frac{c}{k}=\tan\delta\sinh\frac{d}{k}.$$

These formulas imply

$$\tanh\frac{z}{k}=\tanh\frac{c}{k}\frac{B\sinh\frac{a-x}{k}}{A\sqrt{1-\tanh^2\frac{b}{k}\frac{\sinh^2\frac{a-x}{k}}{\sinh^2\frac{a}{k}}}},\tag{3.2}$$

where

$$B=\sqrt{1+\frac{\tanh^2\frac{b}{k}}{\sinh^2\frac{a}{k}}},\quad A=\sqrt{\cosh^2\frac{a}{k}\cosh^2\frac{b}{k}-1}.$$

Using (1.5) and (1.6) one can easily prove that equations (3.1) and (3.2) determine geodesic lines. Also, formula (1.13) implies

$$\frac{1}{C_1^2}=\left(1+\frac{\tanh^2\frac{b}{k}}{\sinh^2\frac{a}{k}}\right)\left(1+\frac{\tanh^2\frac{c}{k}}{\cosh^2\frac{a}{k}\cosh^2\frac{b}{k}-1}\right),\tag{3.3}$$

thus C_1 is constant.

Now we determine the orthogonal surface of the family of lines crossing point $P_0\in Ox$. As the translation of the origin along the direction of the x-axis into point P_0 is a symmetry operation, we obtain spheres with center P_0. If we use Cartesian coordinates, these spheres verify

$$\cosh\frac{x-a}{k}\cosh\frac{y}{k}\cosh\frac{z}{k}=\cosh\frac{R}{k}.\tag{3.4}$$

The curvature of the orthogonal surface is determined by formula (2.5).

As $a\longrightarrow\infty$, we obtain lines being parallel to the x-axis:

$$\tanh\frac{y}{k}=\tanh\frac{b}{k}e^{-\frac{x}{k}},\quad\tanh\frac{z}{k}=\frac{\tanh\frac{c}{k}}{\sqrt{\cosh^2\frac{b}{k}e^{2x}-\sinh^2\frac{b}{k}}}.$$

Thus we get

$$C_1=1.$$

If

$$R=a,\tag{3.5}$$

by applying equation (3.4), we obtain the equation of a parasphere containing the origin:

$$\cosh\frac{y}{k}\cosh\frac{z}{k}=e^{\frac{x}{k}}.$$

Therefore the parasphere which contains point $P(x_0, 0, 0)$ verifies

$$\cosh \frac{y}{k} \cosh \frac{z}{k} = e^{\frac{x-x_0}{k}}. \tag{3.6}$$

By using condition (3.5) and equation (2.5), the curvature of the parasphere becomes

$$R = 0.$$

This implies that we can use Euclidean geometry in order to study the surface of the parasphere, fact which was also mentioned by Bolyai in his main work [1].

4. Lines crossing the z-axis

If we consider the set of lines crossing the z-axis, we can differentiate three types of lines. The first set contains lines crossing the xOy plane, the second set consists of lines which do not cross the xOy plane, finally, the lines of the third family are parallel to the xOy plane.

In each case the projections of the lines contain the origin. Note that the rotation around the z-axis is a symmetry operation. Therefore, we can determine the relevant lines by using surfaces of revolution which are created by rotating the curves around the z-axis in the xOy plane (the role of x is taken by ρ). On the other hand, the lines which cross a projected line onto the xOy plane while being parallel to the z-axis determine an orthogonal surface perpendicular to the xOy plane. The intersection of this orthogonal surface and the surface of revolution determines the lines in question.

For fixed φ we obtain from metric (1.13)

$$ds^2 = \cosh^2 \frac{z}{k} d\rho^2 + dz^2, \tag{4.1}$$

which describes the orthogonal surfaces.

If we use s as variable, we can write

$$\cosh^2 \frac{z}{k} \frac{d\rho}{ds} = C, \quad \frac{d^2 z}{ds^2} - \frac{1}{k} \sinh \frac{z}{k} \cosh \frac{z}{k} \left(\frac{d\rho}{ds} \right)^2 = 0. \tag{4.2}$$

Then by substituting s with ρ, we obtain the following differential equation:

$$\frac{d^2 \tanh \frac{z}{k}}{d\rho^2} - \frac{1}{k^2} \tanh \frac{z}{k} = 0. \tag{4.3}$$

From (4.1) and (4.2) we get

$$\frac{1}{C^2} = \frac{1}{\cosh^2 \frac{z}{k}} + k^2 \left(\frac{d \tanh \frac{z}{k}}{d\rho} \right)^2.$$

Applying (2.1), we obtain the condition

$$C = \sin \vartheta$$

for the lines passing through the origin.

a. Lines parallel to the xOy plane and lines crossing the xOy plane

Using the rotational symmetry operation, we obtain for the line passing through points $P_1(0,0,z_0)$ and $P_2(a,b,0)$

$$\tanh\frac{z}{k}=\tanh\frac{z_0}{k}\frac{\sinh\frac{\rho_0-\rho}{k}}{\sinh\frac{\rho_0}{k}}. \tag{4.4}$$

This formula satisfies equation (4.3), where

$$\cosh\frac{\rho_0}{k}=\cosh\frac{a}{k}\cosh\frac{b}{k}.$$

Also

$$\frac{1}{C^2}=1+\frac{\tanh^2\frac{z_0}{k}}{\sinh^2\frac{\rho_0}{k}}. \tag{4.5}$$

If $\rho_0\longrightarrow\infty$, it follows that

$$\tanh\frac{z}{k}=\tanh\frac{z_0}{k}e^{-\frac{\rho}{k}} \tag{4.6}$$

and

$$C=1.$$

b. Lines not crossing the xOy plane

In this case the lines have a minimum point. If we apply the rotational symmetry, we obtain

$$\tanh\frac{z}{k}=\tanh\frac{z_0}{k}\frac{\cosh\frac{\rho_m-\rho}{k}}{\cosh\frac{\rho_m}{k}}, \tag{4.7}$$

where ρ_m denotes the value of ρ determined by the minimum point. For the value of C we have

$$\frac{1}{C^2}=1-\frac{\tanh^2\frac{z_0}{k}}{\cosh^2\frac{\rho_m}{k}}. \tag{4.8}$$

If $\rho_m=0$, which means that the intersection coincides with the minimum point, we can write

$$\tanh\frac{z}{k}=\tanh\frac{z_0}{k}\cosh\frac{\rho}{k} \tag{4.9}$$

and

$$C=\cosh\frac{z_0}{k}.$$

If we consider the lines passing through $P(0,0,z_0)$, we get for the orthogonal curves circles with center P in the xOz plane. Therefore, because of the rotational symmetry, the orthogonal surfaces of the lines containing $P(0,0,z_0)$ are spheres with center P, which verify

$$\cosh\frac{z_0}{k}\cosh\frac{z}{k}\cosh\frac{\rho}{k}-\sinh\frac{z_0}{k}\sinh\frac{z}{k}=\cosh\frac{R}{k}. \tag{4.10}$$

Now let us consider the orthogonal surface which is perpendicular to the xOy plane and contains the projected line. Here we use variables ρ and z. Furthermore, we will use indexes 1 and 2 for two arbitrary lines which intersect in point $P(\rho,z)$ on this surface. Thus we obtain

$$\left(\frac{dz_1}{d\rho_1}\right)_P\left(\frac{dz_2}{d\rho_2}\right)_P+\cosh^2\frac{z}{k}=0,$$

where the lower index P means that we need to substitute the coordinates of the intersection.

Using equations (4.4), (4.6) and (4.7), we get for the lines crossing the z-axis

$$\tanh\frac{z_1}{k} = \tanh\frac{z_0}{k}\left(\cosh\frac{\rho_1}{k} - p\sinh\frac{\rho_1}{k}\right). \tag{4.11}$$

In the three different cases (lines crossing the xOy plane, lines not crossing the xOy plane, lines parallel to the xOy plane) the required values are as follows:

$$\coth\frac{\rho_0}{k}, \quad \tanh\frac{\rho_m}{k} \text{ and } \quad 1.$$

By deriving equation (4.11) we obtain

$$\frac{dz_1}{d\rho_1} = \tanh\frac{z_0}{k}\cosh^2\frac{z_1}{k}\left(\sinh\frac{\rho_1}{k} - p\cosh\frac{\rho_1}{k}\right). \tag{4.12}$$

Then, using (4.11) and (4.12), we eliminate variable p. Thus we get

$$\frac{dz_1}{d\rho_1} = \frac{\tanh\frac{z_0}{k}\cosh^2\frac{z_1}{k}}{\sinh\frac{\rho_1}{k}}\left(\coth\frac{z_0}{k}\tanh\frac{z_1}{k}\cosh\frac{\rho_1}{k} - 1\right).$$

After differentiating equation (4.10) we obtain

$$\frac{dz_2}{d\rho_2} = -\frac{\coth\frac{z_0}{k}\sinh\frac{\rho_2}{k}}{\coth\frac{z_0}{k}\tanh\frac{z_2}{k}\coth\frac{\rho_2}{k} - 1}.$$

In the point of intersection we have $\rho_1 = \rho_2 = \rho$ and $z_1 = z_2 = z$. Thus the orthogonality condition holds, which proves the validity of equation (4.10).

Equation (4.10) can be written in the following form:

$$\cos\frac{\rho}{k} = \frac{\cosh\frac{R}{k} + \sinh\frac{z_0}{k}\sinh\frac{z}{k}}{\cosh\frac{z_0}{k}\cosh\frac{z}{k}} = F(\rho). \tag{4.13}$$

Furthermore, equation (1.13) yields metric

$$ds^2 = \frac{1 - F^2 + \cosh^2\frac{z}{k}\left(\frac{dF}{d\rho}\right)^2}{F^2 - 1}d\rho^2 + k^2\cosh^2\frac{z}{k}\left(F^2 - 1\right)d\varphi^2. \tag{4.14}$$

Using metric (4.14) and formula (4.13), we can obtain the curvature of the orthogonal surface. The Ricci scalar is determined by formula (2.5).

5. Family of lines not having common point

In this section we consider two sets of lines.

a. Lines parallel to the z-axis

In this case, on the orthogonal surfaces the value of z is constant, $z = z_0$. Indeed, the lines verify $d\rho_1 = 0$, while on the orthogonal surface $dz_2 = 0$. Thus we obtain the following orthogonality condition:

$$\cosh\frac{z}{k}d\rho_1 d\rho_2 + dz_1 dz_2 = 0.$$

The orthogonal surface called hypersphere verifies

$$ds^2 = \cosh^2 \frac{z_0}{k} \left(d\rho^2 + k^2 \sinh^2 \frac{\rho}{k} d\varphi^2 \right), \tag{5.1}$$

the scalar curvature is

$$R = \frac{2}{k^2}.$$

b. Lines having minimum point on the z-axis

Here we use formula (4.9), where the parameter is $\tanh \frac{z_0}{k}$.

By deriving (4.9) and eliminating the parameter, we obtain

$$\frac{dz_1}{d\rho_1} = \sinh \frac{z_1}{k} \cosh \frac{z_1}{k} \tanh \frac{\rho_1}{k}.$$

The rotational symmetry operation induces for the orthogonal surface equation

$$\sinh \frac{\rho}{k} \cosh \frac{z}{k} = \sinh \frac{\rho_0}{k}, \tag{5.2}$$

where ρ_0 is constant. Hence we get

$$\frac{dz_2}{d\rho_2} = -\coth \frac{z_2}{k} \coth \frac{\rho_2}{k}.$$

This and the orthogonality condition proves formula (5.2).
From equations (1.13) and (5.2) we obtain metric

$$ds^2 = \sinh^2 \frac{\rho_0}{k} \frac{1 + \coth^2 \frac{\rho}{k}}{\sinh^2 \frac{\rho}{k}} d\rho_0 + k^2 \sinh^2 \frac{\rho_0}{k} d\varphi^2.$$

Hence the scalar curvature of the orthogonal surface is

$$R = 0.$$

This means that this orthogonal surface is the dual of the parasphere.

6. Surfaces with constant curvature

For lines crossing axis Oz we applied equations of type

$$\tanh \frac{z}{k} = \Phi(\rho), \tag{6.1}$$

which were as follows: equation (2.2), (4.4), (4.7) and (4.6).

Using formulas (5.1) and (6.1), we obtain

$$ds^2 = E(\rho) d\rho^2 + G(\rho) d\varphi^2$$

for the metric, where

$$E(\rho) = \cosh^2 z(\rho) + \left(\frac{dz}{d\rho} \right)^2 = \frac{A}{(1 - \Phi^2)^2},$$

$$G(\rho) = k^2 \cosh^2 z(\rho) \sinh^2 \frac{\rho}{k} = k^2 \frac{\sinh^2 \frac{\rho}{k}}{1 - \Phi^2}.$$

In the four different cases (lines crossing the origin, lines crossing the xOy plane, lines not crossing the xOy plane, lines parallel to the xOy plane) the values of the constant A are as follows:

$$\frac{1}{\sin^2\vartheta}, \quad 1+\frac{\tanh^2\frac{z_0}{k}}{\sinh^2\frac{z_0}{k}}, \quad 1-\frac{\tanh^2\frac{z_0}{k}}{\cosh^2\frac{\rho_m}{k}} \text{ and } \quad 1.$$

The Christoffel symbols of the second kind are as follows:

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2E}\frac{dE}{d\rho} = 2\frac{\Phi\frac{d\Phi}{d\rho}}{1-\Phi^2},\\
\Gamma^1_{22} &= -\frac{1}{2E}\frac{dG}{d\rho} = -\frac{k}{A}\sinh\frac{\rho}{k}\cosh\frac{\rho}{k}\left(1-\Phi^2\right) - \frac{k^2}{A}\sinh^2\frac{\rho}{k}\Phi\frac{d\Phi}{d\rho},\\
\Gamma^2_{12} &= \frac{1}{2G}\frac{dG}{d\rho} = \frac{1}{1-\Phi^2}\left[\frac{1}{k}\left(1-\Phi^2\right)\coth\frac{\rho}{k} + \Phi\frac{d\Phi}{d\rho}\right],
\end{aligned}$$

while the components of the Ricci curvature tensor are

$$\begin{aligned}
R_{11} &= \frac{d\Gamma^2_{12}}{d\rho} + \Gamma^2_{12}\left(\Gamma^2_{12}-\Gamma^1_{11}\right) = \frac{1}{k^2\left(1-\Phi^2\right)^2}\left[1-\Phi^2+k^2\left(\frac{d\Phi}{d\rho}\right)^2\right],\\
R_{22} &= -\frac{d\Gamma^1_{22}}{d\rho} + \Gamma^1_{22}\left(\Gamma^2_{12}-\Gamma^1_{11}\right) = \frac{\sinh^2\frac{\rho}{k}}{A\left(1-\Phi^2\right)}\left[1-\Phi^2+k^2\left(\frac{d\Phi}{d\rho}\right)^2\right].
\end{aligned}$$

Therefore the scalar curvature is

$$R = \frac{1}{E}R_{11} + \frac{1}{G}R_{22} = \frac{2}{k^2}$$

constant for all surfaces.

7. Lines not crossing the z-axis and the xOy plane

As the projection of these lines to the xOy plane verifies

$$\tanh\frac{y}{k} = \tanh\frac{b}{k}\cosh\frac{x}{k}, \tag{7.1}$$

from equations (7.1) and (4.9) it follows that

$$\tanh\frac{z}{k} = \tanh\frac{z_0}{k}\frac{\cosh\frac{x}{k}}{\sqrt{1-\tanh^2\frac{b}{k}\cosh^2\frac{x}{k}}}. \tag{7.2}$$

By deriving (7.2) we obtain

$$\frac{d\tanh\frac{z}{k}}{dx} = \frac{\tanh\frac{z_0}{k}}{k}\frac{\sinh\frac{x}{k}}{\left(1-\tanh^2\frac{b}{k}\cosh^2\frac{x}{k}\right)^{\frac{3}{2}}}. \tag{7.3}$$

Using (1.7) and (7.3), we get for the value of C_1

$$\frac{1}{C_1^2} = \frac{1}{\cosh^2\frac{b}{k}} - \tanh^2\frac{z_0}{k}.$$

C_1 is real if and only if $\cosh\frac{b}{k} \leq \coth\frac{z_0}{k}$. Indeed, using formula (4.9) as $z_0 \longrightarrow \infty$, we get for the maximal value of ρ

$$\cosh\frac{\rho_m}{k} = \coth\frac{z_0}{k}.$$

From

$$\frac{d}{dx}\left(\frac{1}{\cosh^2\frac{y}{k}}\frac{d\tanh\frac{z}{k}}{dx}\right) = \frac{\cosh\frac{x}{k}}{k^2\left(1-\tanh\frac{y_0}{k}\cosh^2\frac{x}{k}\right)^{\frac{3}{2}}\cosh^2\frac{y_0}{k}}$$

and

$$1+k^2\cosh^2\frac{y}{k}\left(\frac{d\tanh\frac{z}{k}}{dx}\right)^2 = \frac{1}{\cosh^2\frac{y_0}{k}\left(1-\tanh\frac{y_0}{k}\cosh^2\frac{x}{k}\right)}$$

it follows that the lines verifying (7.1) and (7.2) satisfy differential equations (1.5) and (1.6).

If we use cylindrical coordinates, from

$$\coth\frac{\rho}{k} = \coth\frac{b}{k}\sin\varphi \tag{7.4}$$

and equation (4.9) we get

$$\tanh\frac{z}{k} = \tanh\frac{z_0}{k}\coth\frac{b}{k}\frac{\sin\varphi}{\sqrt{\coth^2\frac{b}{k}\sin^2\varphi-1}}. \tag{7.5}$$

These lines verify differential equations (1.17) and (1.18). Applying (1.19), we get for the value of C_2

$$\frac{1}{C_2^2} = \frac{k^2}{\sinh^2\frac{b}{k}}\left(1-\tanh^2\frac{z_0}{k}\cosh^2\frac{b}{k}\right).$$

Thus C_2 is real if and only if $\cosh\frac{b}{k} \leq \coth\frac{z_0}{k}$.

If we use equation (4.9) and formula

$$\coth\frac{\rho}{k} = \coth\frac{a}{k}\left(\sin\varphi+\cos\varphi\right), \tag{7.6}$$

we obtain a different line which satisfies

$$\tanh\frac{z}{k} = \tanh\frac{z_0}{k}\coth\frac{a}{k}\frac{\sin\varphi+\cos\varphi}{\sqrt{\coth^2\frac{a}{k}\left(\sin\varphi+\cos\varphi\right)^2-1}}. \tag{7.7}$$

The curves verifying (7.6) and (7.7) also satisfy the differential equations (1.17) and (1.18). Also, from

$$\frac{1}{C_2^2} = k^2\frac{\cosh^2\frac{a}{k}+\cosh^2\frac{z_0}{k}}{\sinh^2\frac{a}{k}\cosh^2\frac{z_0}{k}}$$

we obtain a constant value for C_2. Thus these lines are lines of the hyperbolic space.

If we use Cartesian coordinates, instead of (7.6) and (7.7) we may write

$$\tanh\frac{y}{k} = \tanh\frac{a}{k}\cosh\frac{x}{k} - \sinh\frac{x}{k}$$

and

$$\tanh\frac{z}{k} = \tanh\frac{z_0}{k}\frac{\cosh\frac{x}{k}}{\sqrt{1-\left(\tanh\frac{a}{k}\cosh\frac{x}{k}-\sinh\frac{x}{k}\right)^2}}.$$

For each surface of revolution the scalar curvature equals the curvature of the xOy plane and xOz is a plane of symmetry. Hence we obtain surfaces on the left and the right side of the xOz plane. However, only equation (4.9) provides a necessary condition. Let us consider a line crossing axis Oz, which connects two distinct surfaces. The transitions between the line and the surfaces are smooth (the tangent vector field is continuous) only in the case of (4.9). Therefore, new lines can only be derived by the surface of revolution (4.9).

References

[1] Bolyai, J., Kárteszi, F., *Appendix, a tér tudománya*, Akadémiai Kiadó, Budapest, 1977.

[2] Dávid L., *Bolyai geometria az Appendix alapján*, Kolozsvár, 1944.

[3] Dubrovin, B.A., Fomenko, A.T., Novikov, S.P., *Modern Geometry - Methods and Applications*, Springer, New York, 1984.

[4] Gábos, Z., Mester, Á., *Curves with constant geodesic curvature in the Bolyai-Lobachevskian plane*, Stud. Univ. Babeş-Bolyai Math., **60**(2015), no. 3, 449-462.

[5] Liebmann, H., *Nichteuklidische Geometrie*, Leipzig, Berlin, 1923.

[6] Lobacsevszkij., N.I., Kárteszi F., *Geometriai vizsgálatok a párhuzamosok elméletének köréből*, Akadémiai Kiadó, Budapest, 1951.

[7] Lobacsevszkij, N.I., *Pangeometrie*, Leibzig, 1902.

[8] Sanielevici, S., *Opera matematică* (Romanian), [Mathematical Works], Editura Academiei Republicii Socialiste România, Bucharest, 1968.

[9] Szász, P., *Bevezetés a Bolyai-Lobacsevszkij-féle geometriába*, Akadémiai Kiadó, Budapest, 1973.

[10] Thorpe, J. A., *Elementary Topics in Differential Geometry*, Springer, New York, 1994.

Zoltán Gábos
Babeş-Bolyai University, Faculty of Physics
Cluj-Napoca, Romania
e-mail: zoltan.gabos@gmail.com

Ágnes Mester
Babeş-Bolyai University, Faculty of Mathematics and Computer Science
Cluj-Napoca, Romania
e-mail: mester.agnes@yahoo.com