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Some new integral inequalities of Hermite-Hadamard type for $(\log, (\alpha, m))$-convex functions on co-ordinates

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Abstract. In the paper, the authors introduce a new concept "$(\log, (\alpha, m))$-convex functions on the co-ordinates on the rectangle of the plane" and establish some new integral inequalities of Hermite-Hadamard type for $(\log, (\alpha, m))$-convex functions on the co-ordinates on the rectangle from the plane.

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1. Introduction

The following definitions are well known in the literature.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, +\infty) \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.2. If a positive function $f : I \subseteq \mathbb{R} \to (0, \infty)$ satisfies

$$f(\lambda x + (1 - \lambda)y) \le f^{\lambda}(x)f^{1-\lambda}(y),$$

for all $x, y \in I$ and $\lambda \in [0, 1]$, then we call f a logarithmically convex function on I.

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Definition 1.3 ([8])**.** For $f:[0,b]\to\mathbb{R}$ and $m\in(0,1]$, if

$$f(tx+m(1-t)y)\le tf(x)+m(1-t)f(y)$$

is valid for all $x,y\in[0,b]$ and $t\in[0,1]$, then we say that f is an m-convex function on $[0,b]$.

Definition 1.4. [(9)] For $f:[0,b]\to\mathbb{R}$ and $(\alpha,m)\in(0,1]\times(0,1]$, if

$$f(tx+m(1-t)y)\le t^\alpha f(x)+m(1-t^\alpha)f(y)$$

is valid for all $x,y\in[0,b]$ and $t\in[0,1]$, then we say that f is an (α,m)-convex function on $[0,b]$.

Definition 1.5 ([4, 5])**.** A function $f:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$, with $a<b$ and $c<d$, is said to be convex on the co-ordinates on Δ if the partial functions

$$f_y:[a,b]\to\mathbb{R},\quad f_y(u)=f(u,y)\quad\text{and}\quad f_x:[c,d]\to\mathbb{R},\quad f_x(v)=f(x,v)$$

are convex for all $x\in(a,b)$ and $y\in(c,d)$.

Definition 1.6 ([4, 5])**.** A function $f:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$, with $a<b$ and $c<d$, is said to be convex on the co-ordinates on Δ if the partial functions

$$\begin{aligned}&f(tx+(1-t)z,\lambda y+(1-\lambda)w)\\&\quad\le t\lambda f(x,y)+t(1-\lambda)f(x,w)+(1-t)\lambda f(z,y)+(1-t)(1-\lambda)f(z,w)\end{aligned}$$

holds for all $t,\lambda\in[0,1],(x,y),(z,w)\in\Delta$.

Definition 1.7 ([3])**.** For some $(\alpha_1,m_1),(\alpha_2,m_2)\in(0,1]^2$, a function $f:[0,b]\times[0,d]\to\mathbb{R}$ is said to be (α_1,m_1)-(α_2,m_2)-convex on the co-ordinates on $[0,b]\times[0,d]$, if

$$\begin{aligned}f(ta+m_1(1-t)b,\lambda c+m_2(1-\lambda)d)&\le t^{\alpha_1}\lambda^{\alpha_2}f(a,c)+m_2t^{\alpha_1}(1-\lambda^{\alpha_2})f(a,d)\\&\quad+m_1(1-t^{\alpha_1})\lambda^{\alpha_2}f(b,c)+m_1m_2(1-t^{\alpha_1})(1-\lambda^{\alpha_2})f(b,d)\end{aligned}\tag{1.1}$$

holds for all $t,\lambda\in[0,1]$ and $(x,y),(z,w)\in[0,b]\times[0,d]$.

Now we recite some integral inequalities of Hermite-Hadamard type for the above-mentioned convex functions.

Theorem 1.1 ([6])**.** *Let $f:\mathbb{R}_0\to\mathbb{R}$ be m-convex and $m\in(0,1]$. If $f\in L([a,b])$ for $0\le a<b<\infty$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\le\min\biggl\{\frac{f(a)+mf(b/m)}{2},\frac{mf(a/m)+f(b)}{2}\biggr\}.$$

Theorem 1.2 ([7, Theorem 3.1])**.** *Let $I\supseteq\mathbb{R}_0$ be an open real interval and let $f:I\to\mathbb{R}$ be a differentiable function on I such that $f'\in L([a,b])$ for $0\le a<b<\infty$. If $[f'(x)]^q$ is (α,m)-convex on $[a,b]$ for some given numbers $\alpha,m\in(0,1]$ and $q\ge1$, then*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{b-a}{2}\left(\frac12\right)^{1-1/q}\min\biggl\{\biggl[v_1[f'(a)]^q\\+v_2m\biggl[f'\biggl(\frac{b}{m}\biggr)\biggr]^q\biggr]^{1/q},\biggl[v_2m\biggl[f'\biggl(\frac{a}{m}\biggr)\biggr]^q+v_1[f'(b)]^q\biggr]^{1/q}\biggr\},\end{aligned}$$

where

$$v_1 = \frac{1}{(\alpha+1)(\alpha+2)}\left(\alpha + \frac{1}{2^\alpha}\right)$$

and

$$v_2 = \frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{\alpha^2+\alpha+2}{2} - \frac{1}{2^\alpha}\right).$$

Theorem 1.3 **([4, 5, Theorem 2.2]).** *Let $f : \Delta = [a,b] \times [c,d]$ be convex on the co-ordinates on Δ with $a < b$ and $c < d$. Then*

$$\begin{aligned}
f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) \,\mathrm{d}\,x \right. \\
&\quad \left. + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) \,\mathrm{d}\,y\right] \\
&\le \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x,y) \,\mathrm{d}\,y \,\mathrm{d}\,x \\
&\le \frac{1}{4}\left[\frac{1}{b-a}\left(\int_a^b f(x,c) \,\mathrm{d}\,x + \int_a^b f(x,d) \,\mathrm{d}\,x\right)\right. \\
&\quad \left. + \frac{1}{d-c}\left(\int_c^d f(a,y) \,\mathrm{d}\,y + \int_c^d f(b,y) \,\mathrm{d}\,y\right)\right] \\
&\le \frac{1}{4}\big[f(a,c) + f(b,c) + f(a,d) + f(b,d)\big].
\end{aligned}$$

For more information on this topic, please refer to [1, 2, 10, 11, 12, 13, 14, 15] and closely related references therein.

In this paper, we will introduce a new concept "$(\log, (\alpha, m))$-convex function on the co-ordinates" and establish some integral inequalities of Hermite-Hadamard type for functions whose derivatives are of "co-ordinated $(\log, (\alpha, m))$-convexity".

2. A definition and a lemma

Motivated by Definitions 1.2 to 1.4, we introduce the notion "co-ordinated $(\log, (\alpha, m))$-convex function".

Definition 2.1. A mapping $f : [0,b] \times [c,d] \to \mathbb{R}_+ = (0, +\infty)$ is called co-ordinated $(\log, (\alpha, m))$-convex on $[0,b] \times [c,d]$ for $b > 0$ and $c, d \in \mathbb{R}$ with $c < d$, if

$$\begin{aligned}
f(tx + (1-t)z, \lambda y + m(1-\lambda)w) &\le [\lambda^\alpha f(x,y) \\
&\quad + m(1-\lambda^\alpha) f(x,w)]^t [\lambda^\alpha f(z,y) + m(1-\lambda^\alpha) f(z,w)]^{1-t} \qquad (2.1)
\end{aligned}$$

holds for all $t, \lambda \in [0,1]$, for all $(x,y), (z,w) \in [0,b] \times [c,d]$, and for all $m, \alpha \in (0,1]$.

Remark 2.1. It is clear that, for all $t, \lambda \in [0,1]$ and $(x,y), (z,w) \in [0,b] \times [c,d]$ and for some $m, \alpha \in (0,1]$,

$$\begin{aligned}
&[\lambda^\alpha f(x,y) + m(1-\lambda^\alpha) f(x,w)]^t [\lambda^\alpha f(z,y) + m(1-\lambda^\alpha) f(z,w)]^{1-t} \\
&\le t\lambda^\alpha f(x,y) + mt(1-\lambda^\alpha) f(x,w) + (1-t)\lambda^\alpha f(z,y) + m(1-t)(1-\lambda^\alpha) f(z,w).
\end{aligned}$$

If the function f is co-ordinated $(\log,(\alpha,m))$-convex on $[0,b]\times[c,d]$, then, by taking $(\alpha_1,m_1)=(1,1)$ and $(\alpha_2,m_2)=(\alpha,m)$ in Definition 1.7, we easily see that it is also co-ordinated $(1,1)$-(α,m)-convex on $[0,b]\times[c,d]$.

In order to prove our main results, we need the following lemma.

Lemma 2.1. *Let $f:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$ have partial derivatives of the second order. If $\frac{\partial^2 f}{\partial x\partial y}\in L(\Delta)$, then*

$$\begin{aligned}S(f)\triangleq{}&\frac{4}{(b-a)(d-c)}\biggl\{\frac{9f(a,c)-3f(a,d)-3f(b,c)+f(b,d)}{4}\\&-\frac{1}{2(b-a)}\int_a^b[3f(x,c)-f(x,d)]\,\mathrm{d}\,x-\frac{1}{2(d-c)}\int_c^d[3f(a,y)-f(b,y)]\,\mathrm{d}\,y\\&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}\,y\,\mathrm{d}\,x\biggr\}\\&=\int_0^1\int_0^1(1+2t)(1+2\lambda)f''_{xy}(ta+(1-t)b,\lambda c+(1-\lambda)d)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda.\end{aligned}\tag{2.2}$$

Proof. By integration by parts, we have

$$\begin{aligned}&\int_0^1\int_0^1(1+2t)(1+2\lambda)f''_{xy}(ta+(1-t)b,\lambda c+(1-\lambda)d)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\\&=-\frac{1}{b-a}\int_0^1(1+2\lambda)\biggl[(1+2t)f'_y(ta+(1-t)b,\lambda c+(1-\lambda)d)\Big|_{t=0}^{t=1}\\&\quad-2\int_0^1 f'_y(ta+(1-t)b,\lambda c+(1-\lambda)d)\,\mathrm{d}\,t\biggr]\,\mathrm{d}\,\lambda\\&=-\frac{1}{b-a}\biggl\{\int_0^1\Bigl[3(1+2\lambda)f'_y(a,\lambda c+(1-\lambda)d)\\&\quad-(1+2\lambda)f'_y(b,\lambda c+(1-\lambda)d)\Bigr]\,\mathrm{d}\,\lambda\\&\quad-2\int_0^1\int_0^1(1+2\lambda)f'_y(ta+(1-t)b,\lambda c+(1-\lambda)d)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\biggr\}\\&=\frac{1}{(b-a)(d-c)}\biggl\{3(1+2\lambda)f(a,\lambda c+(1-\lambda)d)\\&\quad-(1+2\lambda)f(b,\lambda c+(1-\lambda)d)\Big|_{\lambda=0}^{\lambda=1}\\&\quad-6\int_0^1 f(a,\lambda c+1-\lambda)d)\,\mathrm{d}\,\lambda+2\int_0^1 f(b,\lambda c+(1-\lambda)d)\,\mathrm{d}\,\lambda\\&\quad-2\int_0^1(1+2\lambda)f(ta+(1-t)b,\lambda c+(1-\lambda)d)\Big|_{\lambda=0}^{\lambda=1}\mathrm{d}\,t\\&\quad+4\int_0^1\int_0^1 f(ta+(1-t)b,\lambda c+(1-\lambda)d)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\biggr\}\end{aligned}$$

$$= \frac{1}{(b-a)(d-c)}\bigg[9f(a,c)-3f(b,c)-3f(a,d)+f(b,d) \\ -6\int_0^1 f(a,\lambda c+(1-\lambda)d)\,\mathrm{d}\,\lambda+2\int_0^1 f(b,\lambda c+(1-\lambda)d)\,\mathrm{d}\,\lambda \\ -6\int_0^1 f(ta+(1-t)b,c)\,\mathrm{d}\,t+2\int_0^1 f(ta+(1-t)b,d)\,\mathrm{d}\,t \\ +4\int_0^1\int_0^1 f(ta+(1-t)b,\lambda c+(1-\lambda)d)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\bigg].$$

After further making use of the substitutions $x = ta+(1-t)b$ and $y = \lambda c+(1-\lambda)d$ for $t,\lambda \in [0,1]$, we obtain (2.2). Lemma 2.1 is thus proved. □

3. Some integral inequalities of Hermite-Hadamard type

Now we turn our attention to establish inequalities of Hermite-Hadamard type for $(\log,(\alpha,m))$-convex functions on the co-ordinates.

Theorem 3.1. *Let $f:\mathbb{R}_0\times\mathbb{R}\to\mathbb{R}$ be a partial differentiable mapping on $\mathbb{R}_0\times\mathbb{R}$ and $f''_{xy}\in L_1([a,\frac{b}{m}]\times[c,d])$ with $0\le a<b$ and $c<d$ for some fixed $m\in(0,1]$. If $\left|f''_{xy}\right|^q$ is co-ordinated $(\log,(\alpha,m))$-convex on $[0,\frac{b}{m}]\times[c,d]$ for $q\ge 1$ and $\alpha\in(0,1]$, then*

$$|S(f)| \le \frac{2^{2(1-1/q)}}{[6(\alpha+1)(\alpha+2)]^{1/q}}\bigg[7(3\alpha+4)\left|f''_{xy}(a,c)\right|^q+7m\alpha(2\alpha \\ +3)\left|f''_{xy}\left(a,\frac{d}{m}\right)\right|^q+5(3\alpha+4)\left|f''_{xy}(b,c)\right|^q+5m(2\alpha+3)\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|^q\bigg]^{1/q}.$$

Proof. By Lemma 2.1, Hölder's integral inequality, the $(\log,(\alpha,m))$-convexity of $\left|f''_{xy}\right|^q$, and the GA-inequality, we obtain

$$|S(f)| \le \int_0^1\int_0^1(1+2t)(1+2\lambda)\left|f''_{xy}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|\mathrm{d}\,t\,\mathrm{d}\,\lambda \\ \le \left(\int_0^1\int_0^1(1+2t)(1+2\lambda)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\right)^{1-1/q}\bigg[\int_0^1\int_0^1(1+2t)(1+2\lambda) \\ \times\left|f''_{xy}f(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q\mathrm{d}\,t\,\mathrm{d}\,\lambda\bigg]^{1/q} \\ \le 2^{2(1-1/q)}\bigg\{\int_0^1\int_0^1(1+2t)(1+2\lambda)\bigg[\lambda^\alpha\left|f''_{xy}(a,c)\right|^q \\ +m(1-\lambda^\alpha)\left|f''_{xy}\left(a,\frac{d}{m}\right)\right|^q\bigg]^t\bigg[\lambda^\alpha\left|f''_{xy}(b,c)\right|^q \\ +m(1-\lambda^\alpha)\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|^q\bigg]^{1-t}\mathrm{d}\,t\,\mathrm{d}\,\lambda\bigg\}^{1/q} \\ \le 2^{2(1-1/q)}\bigg\{\int_0^1\int_0^1(1+2t)(1+2\lambda)\bigg[t\lambda^\alpha\left|f''_{xy}(a,c)\right|^q$$

$$
\begin{aligned}
&+mt(1-\lambda^\alpha)\left|f''_{xy}\left(a,\frac{d}{m}\right)\right|^q+(1-t)\lambda^\alpha\left|f''_{xy}(b,c)\right|^q\\
&+m(1-t)(1-\lambda^\alpha)\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|^q\Bigg]\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\Bigg\}^{1/q}\\
&=\frac{2^{2(1-1/q)}}{[6(\alpha+1)(\alpha+2)]^{1/q}}\bigg[7(3\alpha+4)\left|f''_{xy}(a,c)\right|^q\\
&+7m\alpha(2\alpha+3)\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^q\\
&+5(3\alpha+4)\left|f''_{xy}(a,d)\right|^q+5m(2\alpha+3)\left|f''_{xy}\left(\frac{b}{m},d\right)\right|^q\bigg]^{1/q}.
\end{aligned}
$$

This completes the proof of Theorem 3.1. □

Corollary 3.1.1. *Under the assumptions of Theorem 3.1, if $q=1$, we have*

$$
\begin{aligned}
|S(f)|\le\frac{1}{6(\alpha+1)(\alpha+2)}\bigg[&7(3\alpha+4)\left|f''_{xy}(a,c)\right|+7m\alpha(2\alpha+3)\left|f''_{xy}\left(a,\frac{d}{m}\right)\right|\\
&+5(3\alpha+4)\left|f''_{xy}(b,c)\right|+5m(2\alpha+3)\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|\bigg].
\end{aligned}
$$

Corollary 3.1.2. *Under the assumptions of Corollary 3.1.1,*

1. *if $m=1$, then*

$$
\begin{aligned}
|S(f)|\le\frac{1}{6(\alpha+1)(\alpha+2)}\Big[&7(3\alpha+4)\left|f''_{xy}(a,c)\right|+7\alpha(2\alpha+3)\left|f''_{xy}(a,d)\right|\\
&+5(3\alpha+4)\left|f''_{xy}(b,c)\right|+5(2\alpha+3)\left|f''_{xy}(b,d)\right|\Big];
\end{aligned}
$$

2. *if $\alpha=1$, then*

$$
\begin{aligned}
|S(f)|\le\frac{1}{36}\bigg[&49\left|f''_{xy}(a,c)\right|+35m\left|f''_{xy}\left(a,\frac{d}{m}\right)\right|\\
&+35\left|f''_{xy}(b,c)\right|+25m\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|\bigg];
\end{aligned}
$$

3. *if $m=\alpha=1$, then*

$$
|S(f)|\le\frac{1}{36}\Big[49\left|f''_{xy}(a,c)\right|+35\left|f''_{xy}(a,d)\right|+35\left|f''_{xy}(b,c)\right|+25\left|f''_{xy}(b,d)\right|\Big].
$$

Corollary 3.1.3. *Under the assumptions of Theorem 3.1,*

1. *if $m=1$, then*

$$
\begin{aligned}
|S(f)|\le\frac{2^{2(1-1/q)}}{[6(\alpha+1)(\alpha+2)]^{1/q}}\Big[&7(3\alpha+4)\left|f''_{xy}(a,c)\right|^q\\
&+7\alpha(2\alpha+3)\left|f''_{xy}(b,c)\right|^q+5(3\alpha+4)\left|f''_{xy}(a,d)\right|^q+5(2\alpha+3)\left|f''_{xy}(b,d)\right|^q\Big]^{1/q};
\end{aligned}
$$

2. *if* $\alpha = 1$, *then*

$$|S(f)| \le \frac{4}{12^{2/q}}\left[49\left|f''_{xy}(a,c)\right|^q + 35m\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^q + 35\left|f''_{xy}(a,d)\right|^q + 25m\left|f''_{xy}\left(\frac{b}{m},d\right)\right|^q\right]^{1/q}.$$

Theorem 3.2. *Let* $f:\mathbb{R}_0\times\mathbb{R}\to\mathbb{R}$ *be a partial differentiable mapping on* $\mathbb{R}_0\times\mathbb{R}$ *and* $f''_{xy}\in L_1([a,\frac{b}{m}]\times[c,d])$ *for* $0\le a<b$, $c<d$ *and some fixed* $m\in(0,1]$. *If* $\left|f''_{xy}\right|^q$ *is co-ordinated* $(\log,(\alpha,m))$*-convex on* $[0,\frac{b}{m}]\times[c,d]$ *for* $q>1$ *and some* $\alpha\in(0,1]$ *with* $q\ge r>-1$, *then*

$$\begin{aligned}|S(f)| \le{}& \left[\frac{(3^{(2q-r-1)/(q-1)}-1)(q-1)}{2q-r-1}\right]^{1-1/q} \\ &\times\left[\frac{1}{4(\alpha+1)(\alpha+2)(r+1)(r+2)}\right]^{1/q}\Big\{(2r3^{r+1}+3^{r+1}+1) \\ &\times\left[(3\alpha+4)\left|f''_{xy}(a,c)\right|^q + m\alpha(2\alpha+3)\left|f''_{xy}\left(a,\frac{d}{m}\right)\right|^q\right] + (3^{r+2}-5 \\ &-2r)\left[(3\alpha+4)\left|f''_{xy}(b,c)\right|^q + m(2\alpha+3)\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|^q\right]\Big\}^{1/q}.\end{aligned}$$

Proof. By Lemma 2.1, Hölder's integral inequality, the $(\log,(\alpha,m))$-convexity of $\left|f''_{xy}\right|^q$, and the well known GA-inequality, we obtain

$$\begin{aligned}|S(f)| \le{}& \int_0^1\int_0^1(1+2t)(1+2\lambda)\left|f''_{xy}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|\,\mathrm{d}\,t\,\mathrm{d}\,\lambda \\ \le{}& \left(\int_0^1\int_0^1(1+2t)^{(q-r)/(q-1)}(1+2\lambda)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\right)^{1-1/q}\Big[\int_0^1\int_0^1(1+2t)^r \\ &\times(1+2\lambda)\left|f''_{xy}f(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\Big]^{1/q} \\ \le{}& \left(\frac{(3^{(2q-r-1)/(q-1)}-1)(q-1)}{2q-r-1}\right)^{1-1/q}\Big\{\int_0^1\int_0^1(1+2t)^r(1+2\lambda) \\ &\times\left[\lambda^\alpha\left|f''_{xy}(a,c)\right|^q + m(1-\lambda^\alpha)\left|f''_{xy}\left(a,\frac{d}{m}\right)\right|^q\right]^t \\ &\times\left[\lambda^\alpha\left|f''_{xy}(b,c)\right|^q + m(1-\lambda^\alpha)\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|^q\right]^{1-t}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\Big\}^{1/q} \\ \le{}& \left(\frac{(3^{(2q-r-1)/(q-1)}-1)(q-1)}{2q-r-1}\right)^{1-1/q}\Big\{\int_0^1\int_0^1(1+2t)^r \\ &\times(1+2\lambda)\Big[t\lambda^\alpha\left|f''_{xy}(a,c)\right|^q + mt(1-\lambda^\alpha)\left|f''_{xy}\left(a,\frac{d}{m}\right)\right|^q\end{aligned}$$

$$+(1-t)\lambda^\alpha\big|f''_{xy}(b,c)\big|^q+m(1-t)(1-\lambda^\alpha)\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|^q\Bigg]\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\Bigg\}^{1/q}$$

$$=\left(\frac{(3^{(2q-r-1)/(q-1)}-1)(q-1)}{2q-r-1}\right)^{1-1/q}$$

$$\times\left(\frac{1}{4(\alpha+1)(\alpha+2)(r+1)(r+2)}\right)^{1/q}\Bigg[(2r3^{r+1}+3^{r+1}+1)$$

$$\times\left((3\alpha+4)\big|f''_{xy}(a,c)\big|^q+m\alpha(2\alpha+3)\left|f''_{xy}\left(a,\frac{d}{m}\right)\right|^q\right)+(3^{r+2}-5$$

$$-2r)\left((3\alpha+4)\big|f''_{xy}(b,c)\big|^q+m(2\alpha+3)\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|^q\right)\Bigg]^{1/q}.$$

The proof of Theorem 3.2 is complete. □

Corollary 3.2.1. *Under the conditions of Theorem 3.2, if $r=0$, we have*

$$|S(f)|\le\left(\frac{(3^{(2q-1)/(q-1)}-1)(q-1)}{2q-1}\right)^{1-1/q}\left(\frac{1}{2(\alpha+1)(\alpha+2)}\right)^{1/q}$$
$$\times\Bigg[(3\alpha+4)\big|f''_{xy}(a,c)\big|^q+m\alpha(2\alpha+3)\left|f''_{xy}\left(a,\frac{d}{m}\right)\right|^q$$
$$+(3\alpha+4)\big|f''_{xy}(b,c)\big|^q+m(2\alpha+3)\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|^q\Bigg]^{1/q}.$$

Corollary 3.2.2. *Under the conditions of Theorem 3.2,*

1. *if $m=1$, then*

$$|S(f)|\le\left[\frac{(3^{(2q-r-1)/(q-1)}-1)(q-1)}{2q-r-1}\right]^{1-1/q}$$
$$\times\left[\frac{1}{4(\alpha+1)(\alpha+2)(r+1)(r+2)}\right]^{1/q}\Big\{(2r3^{r+1}+3^{r+1}+1)\Big[(3\alpha$$
$$+4)\big|f''_{xy}(a,c)\big|^q+\alpha(2\alpha+3)\big|f''_{xy}(a,d)\big|^q\Big]$$
$$+(3^{r+2}-5-2r)\Big[(3\alpha+4)\big|f''_{xy}(b,c)\big|^q+(2\alpha+3)\big|f''_{xy}(b,d)\big|^q\Big]\Big\}^{1/q};$$

2. *if $\alpha=1$, then*

$$|S(f)|\le\left(\frac{(3^{(2q-r-1)/(q-1)}-1)(q-1)}{2q-r-1}\right)^{1-1/q}\left(\frac{1}{24(r+1)(r+2)}\right)^{1/q}$$
$$\times\Bigg\{(2r3^{r+1}+3^{r+1}+1)\Bigg[7\big|f''_{xy}(a,c)\big|^q+5m\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^q\Bigg]$$
$$+(3^{r+2}-5-2r)\Bigg[7\big|f''_{xy}(a,d)\big|^q+5m\left|f''_{xy}\left(\frac{b}{m},d\right)\right|^q\Bigg]\Bigg\}^{1/q};$$

3. *if* $m=\alpha=1$, *then*

$$|S(f)| \le \left(\frac{(3^{(2q-r-1)/(q-1)}-1)(q-1)}{2q-r-1}\right)^{1-1/q}\left(\frac{1}{24(r+1)(r+2)}\right)^{1/q}$$
$$\times\left\{(2r3^{r+1}+3^{r+1}+1)\left[7\left|f''_{xy}(a,c)\right|^q+5\left|f''_{xy}(b,c)\right|^q\right]\right.$$
$$\left.+(3^{r+2}-5-2r)\left[7\left|f''_{xy}(a,d)\right|^q+5\left|f''_{xy}(b,d)\right|^q\right]\right\}^{1/q}.$$

Theorem 3.3. *Let* $f:\mathbb{R}_0\times\mathbb{R}\to\mathbb{R}$ *be a partial differentiable mapping on* $\mathbb{R}_0\times\mathbb{R}$ *and* $f''_{xy}\in L_1([a,\frac{b}{m}]\times[c,d])$ *for* $0\le a<b$, $c<d$ *and some fixed* $m\in(0,1]$. *If* $\left|f''_{xy}\right|^q$ *is co-ordinated* $(\log,(\alpha,m))$*-convex on* $[0,\frac{b}{m}]\times[c,d]$ *for* $q>1$ *and some* $\alpha\in(0,1]$, *then*

$$|S(f)| \le \left(\frac{(3^{(2q-1)/(q-1)}-1)(q-1)}{2q-1}\right)^{2(1-1/q)}\left(\frac{1}{2(\alpha+1)}\right)^{1/q}\left[\left|f''_{xy}(a,c)\right|^q\right.$$
$$\left.+m\alpha\left|f''_{xy}\left(a,\frac{b}{m}\right)\right|^q+\left|f''_{xy}(b,c)\right|^q+m\alpha\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|^q\right]^{1/q}.$$

Proof. By Lemma 2.1, Hölder's integral inequality, the $(\log,(\alpha,m))$-convexity of $\left|f''_{xy}\right|^q$, and the GA-inequality, we obtain

$$|S(f)| \le \int_0^1\int_0^1(1+2t)(1+2\lambda)\left|f''_{xy}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|\,\mathrm{d}\,t\,\mathrm{d}\,\lambda$$
$$\le\left(\int_0^1\int_0^1(1+2t)^{q/(q-1)}(1+2\lambda)^{q/(q-1)}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\right)^{1-1/q}$$
$$\times\left[\int_0^1\int_0^1\left|f''_{xy}f(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\right]^{1/q}$$
$$\le\left(\frac{(3^{(2q-1)/(q-1)}-1)(q-1)}{2q-1}\right)^{2(1-1/q)}$$
$$\times\left\{\int_0^1\int_0^1\left[\lambda^\alpha\left|f''_{xy}(a,c)\right|^q+m(1-\lambda^\alpha)\left|f''_{xy}\left(a,\frac{d}{m}\right)\right|^q\right]^t\right.$$
$$\left.\times\left[\lambda^\alpha\left|f''_{xy}(b,c)\right|^q+m(1-\lambda^\alpha)\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|^q\right]^{1-t}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\right\}^{1/q}$$
$$\le\left(\frac{(3^{(2q-1)/(q-1)}-1)(q-1)}{2q-1}\right)^{2(1-1/q)}$$
$$\times\left\{\int_0^1\int_0^1\left[t\lambda^\alpha\left|f''_{xy}(a,c)\right|^q+mt(1-\lambda^\alpha)\left|f''_{xy}\left(a,\frac{d}{m}\right)\right|^q\right.\right.$$
$$\left.\left.+(1-t)\lambda^\alpha\left|f''_{xy}(b,c)\right|^q+m(1-t)(1-\lambda^\alpha)\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|^q\right]\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\right\}^{1/q}$$
$$=\left(\frac{(3^{(2q-1)/(q-1)}-1)(q-1)}{2q-1}\right)^{2(1-1/q)}\left(\frac{1}{2(\alpha+1)}\right)^{1/q}\left[\left|f''_{xy}(a,c)\right|^q\right.$$

$$+m\alpha\left|f''_{xy}\left(a,\frac{b}{m}\right)\right|^q+\left|f''_{xy}(b,c)\right|^q+m\alpha\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|^q\Bigg]^{1/q}.$$

The proof of Theorem 3.3 is complete. □

Corollary 3.3.1. *Under the conditions of Theorem 3.3, if $m=\alpha=1$, then*

$$|S(f)|\le\left(\frac{(3^{(2q-1)/(q-1)}-1)(q-1)}{2q-1}\right)^{2(1-1/q)}\left(\frac{1}{4}\right)^{1/q}\Big[\left|f''_{xy}(a,c)\right|^q+\left|f''_{xy}(a,d)\right|^q+\left|f''_{xy}(b,c)\right|^q+\left|f''_{xy}(b,d)\right|^q\Big]^{1/q}.$$

Theorem 3.4. *Let $f:\mathbb{R}_0\times\mathbb{R}\to\mathbb{R}_+$ be integrable on $[0,\frac{b}{m^2}]\times[c,d]$ for $0\le a<b$, $c<d$, and some $m\in(0,1]$. If f is co-ordinated $(\log,(\alpha,m))$-convex on $[0,\frac{b}{m^2}]\times[c,d]$ for $\alpha\in(0,1]$, then*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\
&\le\frac{1}{2(\alpha+1)(b-a)}\int_a^b\left[f(x,c)+m(2^\alpha-1)f\left(x,\frac{d}{m}\right)\right]\mathrm{d}\,x+\frac{1}{2^{\alpha+1}(d-c)}\\
&\quad\times\int_c^d L\left(f(a,y)+m(2^\alpha-1)f\left(a,\frac{y}{m}\right),f(b,y)+m(2^\alpha-1)f\left(b,\frac{y}{m}\right)\right)\mathrm{d}\,y\\
&\le\frac{1}{2(\alpha+1)(b-a)}\int_a^b\left[f(x,c)+m(2^\alpha-1)f\left(x,\frac{d}{m}\right)\right]\mathrm{d}\,x+\frac{1}{2^{\alpha+2}(d-c)}\\
&\quad\times\int_c^d\left\{f(a,y)+f(b,y)+m(2^\alpha-1)\left[f\left(a,\frac{y}{m}\right)+f\left(b,\frac{y}{m}\right)\right]\right\}\mathrm{d}\,y\\
&\le\frac{1}{2^\alpha(\alpha+1)}\left\{L\left(f(a,c)+m(2^\alpha-1)f\left(a,\frac{c}{m}\right),f(b,c)\right.\right.\\
&\quad\left.+m(2^\alpha-1)f\left(b,\frac{c}{m}\right)\right)+m(2^\alpha-1)L\left(f\left(a,\frac{d}{m}\right)+m(2^\alpha-1)f\left(a,\frac{d}{m^2}\right),\right.\\
&\quad\left.\left.f\left(b,\frac{d}{m}\right)+m(2^\alpha-1)f\left(b,\frac{d}{m^2}\right)\right)\right\}\le\frac{1}{2^{\alpha+1}(\alpha+1)}\\
&\quad\times\left\{f(a,c)+f(b,c)+m(2^\alpha-1)\left[f\left(a,\frac{c}{m}\right)+f\left(b,\frac{c}{m}\right)\right]+m(2^\alpha-1)\right.\\
&\quad\left.\times\left[f\left(a,\frac{d}{m}\right)+f\left(b,\frac{d}{m}\right)+m(2^\alpha-1)\left(f\left(a,\frac{d}{m^2}\right)+f\left(b,\frac{d}{m^2}\right)\right)\right]\right\},
\end{aligned}$$

where $L(u,v)$ is the logarithmic mean defined by

$$L(u,v)=\begin{cases}\dfrac{v-u}{\ln v-\ln u}, & u\ne v,\\ u, & u=v.\end{cases}$$

Proof. Putting $y = \lambda c + (1-\lambda)d$ for $0 \le \lambda \le 1$ and using the $(\log, (\alpha, m))$-convexity of f, we have

$$f(x,y) = f(x, \lambda c + (1-\lambda)d) \le \lambda^\alpha f(x,c) + m(1-\lambda^\alpha) f\left(x, \frac{d}{m}\right) \tag{3.1}$$

for all $(x,y) \in [a,b] \times [c,d]$, $t = \frac{1}{2}$, and $0 \le \lambda \le 1$.

Similarly, setting $x = ta + (1-t)b$ for $0 \le t \le 1$ and using the $(\log, (\alpha, m))$-convexity of f with $0 \le t \le 1$ and $\lambda = \frac{1}{2}$ in (2.1), we obtain

$$\begin{aligned} f(x,c) &= f(ta + (1-t)b, c) \\ &\le \frac{1}{2^\alpha}\left[f(a,c) + m(2^\alpha - 1) f\left(a, \frac{c}{m}\right)\right]^t \left[f(b,c) + m(2^\alpha - 1) f\left(b, \frac{c}{m}\right)\right]^{1-t} \end{aligned}$$

and

$$\begin{aligned} f\left(x, \frac{d}{m}\right) &= f\left(ta + (1-t)b, \frac{d}{m}\right) \le \frac{1}{2^\alpha}\left[f\left(a, \frac{d}{m}\right)\right. \\ &\left. + m(2^\alpha - 1) f\left(a, \frac{d}{m^2}\right)\right]^t \left[f\left(b, \frac{d}{m}\right) + m(2^\alpha - 1) f\left(b, \frac{d}{m^2}\right)\right]^{1-t}. \end{aligned} \tag{3.2}$$

From inequalities (3.1) to (3.2), we have

$$\begin{aligned} &\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x,y) \,\mathrm{d}\,x \,\mathrm{d}\,y \le \frac{1}{b-a} \int_0^1 \int_a^b \Big[\lambda^\alpha f(x,c) + m(1 \\ &- \lambda^\alpha) f\left(x, \frac{d}{m}\right)\Big] \,\mathrm{d}\,x \,\mathrm{d}\,\lambda = \frac{1}{(\alpha+1)(b-a)} \int_a^b \Big[f(x,c) + m(2^\alpha - 1) \\ &\times f\left(x, \frac{d}{m}\right)\Big] \,\mathrm{d}\,x \le \frac{1}{2^\alpha(\alpha+1)} \int_0^1 \bigg\{\left[f(a,c) + m(2^\alpha - 1) f\left(a, \frac{c}{m}\right)\right]^t \\ &\times \left[f(b,c) + m(2^\alpha - 1) f\left(b, \frac{c}{m}\right)\right]^{1-t} + m(2^\alpha - 1)\Big[f\left(a, \frac{d}{m}\right) + m(2^\alpha \\ &- 1) f\left(a, \frac{d}{m^2}\right)\Big]^t \times \left[f\left(b, \frac{d}{m}\right) + m(2^\alpha - 1) f\left(b, \frac{d}{m^2}\right)\right]^{1-t}\bigg\} \,\mathrm{d}\,t. \end{aligned} \tag{3.3}$$

It is obvious that

$$\int_0^1 u^t v^{1-t} \,\mathrm{d}\,t = L(u,v) \quad \text{and} \quad L(u,v) \le \frac{u+v}{2}. \tag{3.4}$$

By (3.3) and (3.4), we acquire

$$\begin{aligned} &\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x,y) \,\mathrm{d}\,x \,\mathrm{d}\,y \\ &\le \frac{1}{2^\alpha(\alpha+1)} \bigg\{L\bigg(f(a,c) + m(2^\alpha - 1) f\left(a, \frac{c}{m}\right), f(b,c) + m(2^\alpha - 1) f\left(b, \frac{c}{m}\right)\bigg) \\ &+ m(2^\alpha - 1) L\bigg(f\left(a, \frac{d}{m}\right) + m(2^\alpha - 1) f\left(a, \frac{d}{m^2}\right), f\left(b, \frac{d}{m}\right) + m(2^\alpha - 1) f\left(b, \frac{d}{m^2}\right)\bigg)\bigg\} \end{aligned}$$

$$\le \frac{1}{2^{\alpha+1}(\alpha+1)}\bigg\{f(a,c)+f(b,c)+m(2^\alpha-1)\bigg[f\bigg(a,\frac{c}{m}\bigg)+f\bigg(b,\frac{c}{m}\bigg)\bigg]$$
$$+m(2^\alpha-1)\bigg[f\bigg(a,\frac{d}{m}\bigg)+f\bigg(b,\frac{d}{m}\bigg)+m(2^\alpha-1)\bigg(f\bigg(a,\frac{d}{m^2}\bigg)+f\bigg(b,\frac{d}{m^2}\bigg)\bigg)\bigg]\bigg\}.$$

Similarly, one has

$$\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y=\frac{1}{d-c}\int_c^d\int_0^1 f(ta+(1-t)b,y)\,\mathrm{d}\,t\,\mathrm{d}\,y$$
$$\le \frac{1}{2^\alpha(d-c)}\int_c^d\int_0^1\bigg[f(a,y)+m(2^\alpha-1)f\bigg(a,\frac{y}{m}\bigg)\bigg]^t\bigg[f(b,y)+m(2^\alpha-1)$$
$$\times f\bigg(b,\frac{y}{m}\bigg)\bigg]^{1-t}\,\mathrm{d}\,t\,\mathrm{d}\,y=\frac{1}{2^\alpha(d-c)}\int_c^d L\bigg(f(a,y)+m(2^\alpha-1)f\bigg(a,\frac{y}{m}\bigg),$$
$$f(b,y)+m(2^\alpha-1)f\bigg(b,\frac{y}{m}\bigg)\bigg)\,\mathrm{d}\,y\le\frac{1}{2^{\alpha+1}(d-c)}\int_c^d\bigg\{f(a,y)+f(b,y)$$
$$+m(2^\alpha-1)\bigg[f\bigg(a,\frac{y}{m}\bigg)+f\bigg(b,\frac{y}{m}\bigg)\bigg]\bigg\}\mathrm{d}\,y\le\frac{1}{2^{\alpha+1}}\int_0^1\bigg\{\bigg[\lambda^\alpha f(a,c)$$
$$+m(2^\alpha-1)f\bigg(a,\frac{d}{m}\bigg)+\lambda^\alpha f(b,c)+m(1-\lambda^\alpha)f\bigg(b,\frac{d}{m}\bigg)+m(2^\alpha-1)$$
$$\times\bigg[\lambda^\alpha f\bigg(a,\frac{c}{m}\bigg)+m(1-\lambda^\alpha)f\bigg(a,\frac{d}{m^2}\bigg)+\lambda^\alpha f\bigg(b,\frac{c}{m}\bigg)+m(1-\lambda^\alpha)$$
$$\times f\bigg(b,\frac{d}{m^2}\bigg)\bigg]\bigg\}\,\mathrm{d}\,\lambda=\frac{1}{2^{\alpha+1}(\alpha+1)}\bigg\{f(a,c)+f(b,c)+m(2^\alpha$$
$$-1)\bigg[f\bigg(a,\frac{c}{m}\bigg)+f\bigg(b,\frac{c}{m}\bigg)\bigg]+m(2^\alpha-1)\bigg[f\bigg(a,\frac{d}{m}\bigg)+f\bigg(b,\frac{d}{m}\bigg)$$
$$+m(2^\alpha-1)\bigg(f\bigg(a,\frac{d}{m^2}\bigg)+f\bigg(b,\frac{d}{m^2}\bigg)\bigg)\bigg]\bigg\}.$$

Theorem 3.4 is thus proved. □

Theorem 3.5. *Let* $f:[0,\frac{b}{m}]\times[c,d]\subseteq\mathbb{R}_0\times\mathbb{R}\to\mathbb{R}_+$ *be integrable on* $[0,\frac{b}{m^2}]\times[c,d]$ *for* $0\le a<b$, $c<d$, *and some fixed* $m\in(0,1]$. *If* f *is co-ordinated* $(\log,(\alpha,m))$*-convex on* $[0,\frac{b}{m^2}]\times[c,d]$ *for* $\alpha\in(0,1]$, *then*

$$f\bigg(\frac{a+b}{2},\frac{c+d}{2}\bigg)\le\frac{1}{2^{\alpha+1}(b-a)}\int_a^b\bigg[f\bigg(x,\frac{c+d}{2}\bigg)+m(2^\alpha-1)f\bigg(x,\frac{c+d}{2m}\bigg)\bigg]^{1/2}$$
$$\times\bigg[f\bigg(a+b-x,\frac{c+d}{2}\bigg)+m(2^\alpha-1)f\bigg(a+b-x,\frac{c+d}{2m}\bigg)\bigg]^{1/2}\,\mathrm{d}\,x$$
$$+\frac{1}{2^{\alpha+1}(d-c)}\int_c^d\bigg[f\bigg(\frac{a+b}{2},y\bigg)+m(2^\alpha-1)f\bigg(\frac{a+b}{2},\frac{y}{m}\bigg)\bigg]\,\mathrm{d}\,y$$
$$\le\frac{1}{2^{\alpha+1}(b-a)}\int_a^b\bigg[f\bigg(x,\frac{c+d}{2}\bigg)+m(2^\alpha-1)f\bigg(x,\frac{c+d}{2m}\bigg)\bigg]\,\mathrm{d}\,x$$

$$+\frac{1}{2^{\alpha+1}(d-c)}\int_c^d\left[f\left(\frac{a+b}{2},y\right)+m(2^\alpha-1)f\left(\frac{a+b}{2},\frac{y}{m}\right)\right]\mathrm{d}\,y$$
$$\leq\frac{1}{2^{2\alpha+1}(b-a)(d-c)}\int_c^d\int_a^b\Bigg\{f(x,y)+2m(2^\alpha-1)f\left(x,\frac{y}{m}\right)$$
$$+m^2(2^\alpha-1)^2f\left(x,\frac{y}{m^2}\right)+\left[f(x,y)+m(2^\alpha-1)f\left(x,\frac{y}{m}\right)\right]^{1/2}$$
$$\times\left[f(a+b-x,y)+m(2^\alpha-1)f\left(a+b-x,\frac{y}{m}\right)\right]^{1/2}$$
$$+m(2^\alpha-1)\left[f\left(x,\frac{y}{m}\right)+m(2^\alpha-1)f\left(x,\frac{y}{m^2}\right)\right]^{1/2}$$
$$\times\left[f\left(a+b-x,\frac{y}{m}\right)+m(2^\alpha-1)f\left(a+b-x,\frac{y}{m^2}\right)\right]^{1/2}\Bigg\}\mathrm{d}\,x\,\mathrm{d}\,y$$
$$\leq\frac{1}{2^{2\alpha}(b-a)(d-c)}\int_c^d\int_a^b\Bigg[f(x,y)+2m(2^\alpha-1)f\left(x,\frac{y}{m}\right)$$
$$+m^2(2^\alpha-1)^2f\left(x,\frac{y}{m^2}\right)\Bigg]\mathrm{d}\,x\,\mathrm{d}\,y.$$

Proof. Using the $(\log,(\alpha,m))$-convexity of f, we have

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)=f\left(\frac{1}{2}(ta+(1-t)b+(1-t)a+tb),\frac{c+d}{2}\right)$$
$$\leq\frac{1}{2^\alpha}\left[f\left(ta+(1-t)b,\frac{c+d}{2}\right)+m(2^\alpha-1)f\left(ta+(1-t)b,\frac{c+d}{2m}\right)\right]^{1/2}$$
$$\times\left[f\left((1-t)a+tb,\frac{c+d}{2}\right)+m(2^\alpha-1)f\left((1-t)a+tb,\frac{c+d}{2m}\right)\right]^{1/2}$$

for all $t\in[0,1]$.

Integrating on both sides of the above inequality on $[0,1]$, from the GA-inequality, and by the $(\log,(\alpha,m))$-convexity of f, we reveals

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)=\int_0^1 f\left(\frac{1}{2}(ta+(1-t)b+(1-t)a+tb),\frac{c+d}{2}\right)\mathrm{d}\,t$$
$$\leq\frac{1}{2^\alpha}\int_0^1\left[f\left(ta+(1-t)b,\frac{c+d}{2}\right)+m(2^\alpha-1)f\left(ta+(1-t)b,\frac{c+d}{2m}\right)\right]^{1/2}$$
$$\times\left[f\left((1-t)a+tb,\frac{c+d}{2}\right)+m(2^\alpha-1)f\left((1-t)a+tb,\frac{c+d}{2m}\right)\right]^{1/2}\mathrm{d}\,t$$
$$=\frac{1}{2^\alpha(b-a)}\int_a^b\left[f\left(x,\frac{c+d}{2}\right)+m(2^\alpha-1)f\left(x,\frac{c+d}{2m}\right)\right]^{1/2}$$
$$\times\left[f\left(a+b-x,\frac{c+d}{2}\right)+m(2^\alpha-1)f\left(a+b-x,\frac{c+d}{2m}\right)\right]^{1/2}\mathrm{d}\,x$$

$$
\begin{aligned}
&\le \frac{1}{2^\alpha(b-a)}\int_a^b\left[f\left(x,\frac{c+d}{2}\right)+m(2^\alpha-1)f\left(x,\frac{c+d}{2m}\right)\right]\mathrm{d}\,x\\
&=\frac{1}{2^\alpha(b-a)}\int_0^1\int_a^b\left[f\left(x,\frac{1}{2}[\lambda c+(1-\lambda)d+(1-\lambda)c+\lambda d]\right)\right.\\
&\left.+m(2^\alpha-1)f\left(x,\frac{1}{2m}[\lambda c+(1-\lambda)d+(1-\lambda)c+\lambda d]\right)\right]\mathrm{d}\,x\,\mathrm{d}\,\lambda\\
&\le\frac{1}{2^{2\alpha}(b-a)}\int_0^1\int_a^b\left\{f(x,\lambda c+(1-\lambda)d)+m(2^\alpha-1)f\left(x,\frac{(1-\lambda)c+\lambda d}{m}\right)\right.\\
&\left.+m(2^\alpha-1)\left[f\left(x,\frac{\lambda c+(1-\lambda)d}{m}\right)+m(2^\alpha-1)f\left(x,\frac{(1-\lambda)c+\lambda d}{m^2}\right)\right]\right\}\mathrm{d}\,x\,\mathrm{d}\,\lambda\\
&=\frac{1}{2^{2\alpha}(b-a)(d-c)}\int_c^d\int_a^b\left[f(x,y)\right.\\
&\left.+2m(2^\alpha-1)f\left(x,\frac{y}{m}\right)+m^2(2^\alpha-1)^2f\left(x,\frac{y}{m^2}\right)\right]\mathrm{d}\,x\,\mathrm{d}\,y.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\le\frac{1}{2^\alpha}\int_0^1\left[f\left(\frac{a+b}{2},\lambda c+(1-\lambda)d\right)\right.\\
&\left.+m(2^\alpha-1)f\left(\frac{a+b}{2},\frac{\lambda c+(1-\lambda)d}{m}\right)\right]\mathrm{d}\,\lambda\\
&=\frac{1}{2^\alpha(d-c)}\int_c^d\left[f\left(\frac{a+b}{2},y\right)+m(2^\alpha-1)f\left(\frac{a+b}{2},\frac{y}{m}\right)\right]\mathrm{d}\,y\\
&\le\frac{1}{2^{2\alpha}(d-c)}\int_c^d\int_0^1\left\{\left[f(ta+(1-t)b,y)+m(2^\alpha-1)f\left(ta+(1-t)b,\frac{y}{m}\right)\right]^{1/2}\right.\\
&\times\left[f((1-t)a+tb,y)+m(2^\alpha-1)f\left((1-t)a+tb,\frac{y}{m}\right)\right]^{1/2}\\
&+m(2^\alpha-1)\left[f\left(ta+(1-t)b,\frac{y}{m}\right)+m(2^\alpha-1)f\left(ta+(1-t)b,\frac{y}{m^2}\right)\right]^{1/2}\\
&\left.\times\left[f\left((1-t)a+tb,\frac{y}{m}\right)+m(2^\alpha-1)f\left((1-t)a+tb,\frac{y}{m^2}\right)\right]^{1/2}\right\}\mathrm{d}\,t\,\mathrm{d}\,y\\
&=\frac{1}{2^{2\alpha}(b-a)(d-c)}\int_c^d\int_a^b\left\{\left[f(x,y)+m(2^\alpha-1)f\left(x,\frac{y}{m}\right)\right]^{1/2}\right.\\
&\times\left[f(a+b-x,y)+m(2^\alpha-1)f\left(a+b-x,\frac{y}{m}\right)\right]^{1/2}\\
&+m(2^\alpha-1)\left[f\left(x,\frac{y}{m}\right)+m(2^\alpha-1)f\left(x,\frac{y}{m^2}\right)\right]^{1/2}\\
&\left.\times\left[f\left(a+b-x,\frac{y}{m}\right)+m(2^\alpha-1)f\left(a+b-x,\frac{y}{m^2}\right)\right]^{1/2}\right\}\mathrm{d}\,x\,\mathrm{d}\,y
\end{aligned}
$$

$$\leq \frac{1}{2^{2\alpha}(b-a)(d-c)} \int_c^d \int_a^b \bigg[f(x,y) + 2m(2^\alpha - 1) f\bigg(x, \frac{y}{m}\bigg) + m^2(2^\alpha - 1)^2 f\bigg(x, \frac{y}{m^2}\bigg)\bigg] \,\mathrm{d}\,x \,\mathrm{d}\,y.$$

The proof of Theorem 3.5 is complete. □

Corollary 3.5.1. *Under the conditions of Theorems 3.4 and 3.5, if $m = 1$, then*

$$\begin{aligned}
f\bigg(\frac{a+b}{2}, \frac{c+d}{2}\bigg) &\leq \frac{1}{2}\bigg\{\frac{1}{b-a}\int_a^b \bigg[f\bigg(x, \frac{c+d}{2}\bigg) f\bigg(a+b-x, \frac{c+d}{2}\bigg)\bigg]^{1/2} \,\mathrm{d}\,x \\
&\quad + \frac{1}{d-c}\int_c^d f\bigg(\frac{a+b}{2}, y\bigg) \,\mathrm{d}\,y\bigg\} \\
&\leq \frac{1}{2}\bigg[\frac{1}{b-a}\int_a^b f\bigg(x, \frac{c+d}{2}\bigg) \,\mathrm{d}\,x + \frac{1}{d-c}\int_c^d f\bigg(\frac{a+b}{2}, y\bigg) \,\mathrm{d}\,y\bigg] \\
&\leq \frac{1}{(b-a)(d-c)}\int_c^d \int_a^b f(x,y) \,\mathrm{d}\,x \,\mathrm{d}\,y \\
&\leq \frac{1}{2(\alpha+1)(b-a)}\int_a^b \big[f(x,c) + (2^\alpha - 1) f(x,d)\big] \,\mathrm{d}\,x \\
&\quad + \frac{1}{2(d-c)}\int_c^d L\big(f(a,y), f(b,y)\big) \,\mathrm{d}\,y \\
&\leq \frac{1}{2(\alpha+1)(b-a)}\int_a^b \big[f(x,c) + (2^\alpha - 1) f(x,d)\big] \,\mathrm{d}\,x \\
&\quad + \frac{1}{4(d-c)}\int_c^d \big[f(a,y) + f(b,y)\big] \,\mathrm{d}\,y \\
&\leq \frac{1}{2(\alpha+1)}\Big\{L\big(f(a,c), f(b,c)\big) + (2^\alpha - 1) L\big(f(a,d), f(b,d)\big) \\
&\quad + f(a,c) + f(b,c) + (2^\alpha - 1)\big[f(a,d) + f(b,d)\big]\Big\} \\
&\leq \frac{1}{2(\alpha+1)}\Big\{f(a,c) + f(b,c) + (2^\alpha - 1)\big[f(a,d) + f(b,d)\big]\Big\}.
\end{aligned}$$

If $m = \alpha = 1$, then

$$\begin{aligned}
f\bigg(\frac{a+b}{2}, \frac{c+d}{2}\bigg) &\leq \frac{1}{2}\bigg\{\frac{1}{b-a}\int_a^b \bigg[f\bigg(x, \frac{c+d}{2}\bigg) f\bigg(a+b-x, \frac{c+d}{2}\bigg)\bigg]^{1/2} \,\mathrm{d}\,x \\
&\quad + \frac{1}{d-c}\int_c^d f\bigg(\frac{a+b}{2}, y\bigg) \,\mathrm{d}\,y\bigg\} \\
&\leq \frac{1}{2}\bigg[\frac{1}{b-a}\int_a^b f\bigg(x, \frac{c+d}{2}\bigg) \,\mathrm{d}\,x + \frac{1}{d-c}\int_c^d f\bigg(\frac{a+b}{2}, y\bigg) \,\mathrm{d}\,y\bigg] \\
&\leq \frac{1}{(b-a)(d-c)}\int_c^d \int_a^b f(x,y) \,\mathrm{d}\,x \,\mathrm{d}\,y
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{4(b-a)}\int_a^b \big[f(x,c)+f(x,d)\big]\,\mathrm{d}\,x+\frac{1}{2(d-c)}\int_c^d L\big(f(a,y),f(b,y)\big)\,\mathrm{d}\,y\\
&\le \frac{1}{4}\bigg\{\frac{1}{b-a}\int_a^b \big[f(x,c)+f(x,d)\big]\,\mathrm{d}\,x+\frac{1}{d-c}\int_c^d \big[f(a,y)+f(b,y)\big]\,\mathrm{d}\,y\bigg\}\\
&\qquad\le \frac{1}{4}\Big\{f(a,c)+f(a,d)+f(b,c)+f(b,d)\\
&\qquad+L\big(f(a,c),f(b,c)\big)+L\big(f(a,d),f(b,d)\big)\Big\}\\
&\qquad\le \frac{1}{4}\big[f(a,c)+f(a,d)+f(b,c)+f(b,d)\big].
\end{aligned}
$$

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