Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 4, 493–507

$\aleph_1$-A-coseperable groups

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Abstract. Let A be a countable self-small Abelian group with a right Noetherian right hereditary endomorphism ring. We show that the question whether strongly-$\aleph_1$-A-generated groups are $\aleph_1$-A-coseparable is undecidable in ZFC. Our main focus is on the algebraic aspect of the proof, not on the underlying set-theory.

Mathematics Subject Classification (2010): 20K20, 20K40.

Keywords: $\aleph_1$-a-coseparable groups, Martin's axiom, endomorphism rings.

1. Introduction

Let A be an Abelian group with endomorphism ring $E = E(A)$. Associated with A are the functors $H_A(.) = \mathrm{Hom}(A, .)$ and $T_A(.) = . \otimes_E A$ which induce natural maps $\theta_G : T_A H_A(G) \to G$ and $\phi_M : M \to H_A T_A(M)$ defined by $\theta_G(\alpha \otimes a) = \alpha(a)$ and $[\phi_M(x)](a) = x \otimes a$ for all $\alpha \in H_A(G)$, $x \in M$ and $a \in A$. The *A-solvable groups* are the Abelian groups G such that θ_G is an isomorphism. Finally, a sequence $0 \to G \to H \to L \to 0$ of Abelian group is *A-balanced* if the induced sequence $0 \to H_A(G) \to H_A(H) \to H_A(L) \to 0$ of right E-modules is exact

An important class of A-solvable groups are the *(finitely) A-projective groups*, i.e. groups which are isomorphic to a direct summand of $\oplus_I A$ for some (finite) index-set I. Finitely A-projective groups are always A-solvable [8], and the same holds for arbitrary A-projective groups [9] if A is *self-small*, i.e. if H_A preserves direct sums of copies of A. Arnold and Murley showed in [9, Corollary 2.3] that a countable Abelian group is self-small if and only if E is countable.

Epimorphic images of A-projective groups are called *A-generated*, but need not be A-solvable. It is easy to see that a group G is A-generated if and only if θ_G is onto. Moreover, if A is self-small, then a group G is A-solvable if and only if there is an A-balanced exact sequence $0 \to U \to F \to G \to 0$ in which F is A-projective and U is A-generated [3]. Finally, G is *A-torsion-free* if every finitely A-generated subgroup of G is isomorphic to a subgroup of a finitely A-projective group, and an A-generated subgroup U of an A-torsion-free group G is *A-pure* if $(U + P)/U$ is A-torsion-free for all finitely A-generated subgroups P of G. If A is flat as an E-module, then A-torsion-free groups are A-solvable [4]. We want to remind the reader that a right E-module M

is *non-singular* if $xI \neq 0$ for all non-zero x in M and all essential right ideals I of E. The ring R is *right non-singular* if R_R is a non-singular module. If U is a submodule of a non-singular right E-module M, then the S-closure of U in M consists of all $x \in M$ such that $xI \subseteq M$ for some essential right ideal I of E [14]. Non-singularity is closely related to A-torsion-freeness whenever A is a self-small Abelian group whose endomorphism ring is right non-singular [5]:

a) If an A-generated group G is A-torsion-free, then $H_A(G)$ is non-singular.
b) An A-generated subgroup U of an A-torsion-free group G is contained in a smallest A-pure subgroup V of G which is obtained as $\theta_G(T_A(W))$ where W is the S-closure of $H_A(U)$ in $H_A(G)$.

The focus of this paper are A-torsion-free groups G such that all A-generated subgroups U of G with $|U| < |G|$ are A-projective. Since A-generated subgroups of A-projective groups need not be A-projective in general ([4] and [8]), some immediate restrictions on A are needed to guarantee the existence of non-trivial groups with the above property.

2. Hereditary Endomorphism Rings and κ-A-projective groups

An Abelian group is *κ-A-generated*, where κ is an infinite cardinal, if it is an epimorphic image of $\oplus_I A$ for some index-set I with $|I| < \kappa$. The $\aleph_0$-A-generated groups are referred to as *finitely A-generated groups*. An A-generated group G is *κ-A-projective* if every κ-A-generated subgroup U of G is A-projective. If $|A| < \kappa$, then this is equivalent to the condition that all A-generated subgroups U with $|U| < \kappa$ are A-projective. Since every finitely A-generated subgroup of a κ-A-projective group G is A-projective, G is A-solvable. In particular, an A-generated group G is $\aleph_0$-A-projective if every finitely A-generated subgroup is A-projective. If A is faithfully flat as a left E-module, then finitely A-generated A-projective groups are finitely A-projective [4].

Theorem 2.1. *The following conditions are equivalent for a self-small torsion-free Abelian group A:*

a) i) *A-projective groups are κ-A-projective for all infinite cardinals κ.*
   ii) *Every exact sequence $0 \to U \to G \to H \to 0$, in which G and H is κ-A-projective for some infinite cardinal κ, is A-balanced.*

b) *E is a right hereditary ring.*

In this case, A is faithfully flat as an E-module.

Proof. *a)* $\Rightarrow$ *b)*: To see that A is flat as an E-module, observe that A^n is $\aleph_0$-A-projective for all $n < \omega$, from which we obtain that $G = \alpha(A^n)$ is A-projective for all $\alpha : A^n \to A$. By a.ii), the exact sequence $0 \to U \to A^n \to G \to 0$ with $U = \ker \alpha$ is A-balanced which yields the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_AH_A(U) & \longrightarrow & T_AH_A(A^n) & \longrightarrow & T_AH_A(G) & \longrightarrow & 0 \\ & & \downarrow \theta_U & & \wr\downarrow \theta_{A^n} & & \wr\downarrow \theta_G & & \\ 0 & \longrightarrow & U & \longrightarrow & A^n & \longrightarrow & G & \longrightarrow & 0. \end{array}$$

Thus, θ_U is an isomorphism. By Ulmer's Theorem [17], A is E-flat.

Consider a right ideal I of E. Because A is E-flat, $T_A(I) \cong IA \subseteq A$. Since IA is an A-generated subgroup of A, and A is $|IA|^+$-A-projective by a.i), IA is A-projective. Thus, $H_AT_A(I)$ is a projective module fitting into the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & H_AT_A(I) & \longrightarrow & H_AT_A(E) \\ & & \uparrow \phi_I & & \wr\uparrow \phi_E \\ 0 & \longrightarrow & I & \longrightarrow & E \end{array}$$

from which we obtain that ϕ_I is one-to-one.

On the other hand, consider an exact sequence $0 \to V \to F \to I \to 0$ where F is a free right E-module. It induces the exact sequence

$$0 \to T_A(V) \to T_A(F) \to T_A(I) \to 0.$$

The latter sequence is A-balanced by a.ii). Hence, the top-row in the commutative diagram

$$\begin{array}{ccccc} H_AT_A(F) & \longrightarrow & H_AT_A(I) & \longrightarrow & 0 \\ \wr\uparrow \phi_F & & \uparrow \phi_I & & \\ F & \longrightarrow & I & \longrightarrow & 0 \end{array}$$

is exact, which yields that ϕ_I is onto. Consequently, I is projective, and E is right hereditary.

b) $\Rightarrow$ *a)*: Let M be a right E-module. Since E is right hereditary, we can find an exact sequence $0 \to P \to F \to M \to 0$ in which P and F are projective. It induces exact sequence

$$0 = \mathrm{Tor}_1^R(F, A) \to \mathrm{Tor}_1^R(M, A) \to T_A(P) \to T_A(F) \to T_A(M) \to 0.$$

We obtain the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_A(\mathrm{Tor}_1^R(M, A)) & \longrightarrow & H_AT_A(P) & \longrightarrow & H_AT_A(F) \\ & & & & \wr\uparrow \phi_P & & \wr\uparrow \phi_F \\ & & 0 & \longrightarrow & P & \longrightarrow & F. \end{array}$$

Therefore, $H_A(\mathrm{Tor}_1^R(M, A)) = 0$ for all right R-modules M.
If M^+ is torsion-free, then it is isomorphic to a submodule of $\mathbb{Q}M = \mathbb{Q} \otimes_{\mathbb{Z}} M$. Since $\mathrm{Tor}_1^R(\mathbb{Q}M, A)$ is torsion-free and divisible, $H_A(\mathrm{Tor}_1^R(\mathbb{Q}M, A)) = 0$ is only possible if $\mathrm{Tor}_1^R(\mathbb{Q}M, A) = 0$. However, because E is right hereditary, we have the exact sequence $0 \to \mathrm{Tor}_1^R(M, A) \to \mathrm{Tor}_1^R(\mathbb{Q}M, A) = 0$, and $\mathrm{Tor}_1^R(M, A) = 0$.

If M^+ is torsion, then we select an exact sequence $0 \to U \to F_1 \to A \to 0$ in which F_1 is a free left E-module. It induces

$$0 = \mathrm{Tor}_1^R(M, F_1) \to \mathrm{Tor}_1^R(M, A) \to M \otimes_E V.$$

Since $M \otimes_E V$ is torsion, the same holds for $\mathrm{Tor}_1^R(M, A)$. But, the latter also is isomorphic to a subgroup of the torsion-free group $T_A(P)$. Thus, $\mathrm{Tor}_1^R(M, A) = 0$.

For an arbitrary M, we consider the exact sequence

$$0 = \operatorname{Tor}_1^R(tM, A) \to \operatorname{Tor}_1^R(M, A) \to \operatorname{Tor}_1^R(M/tM, A) = 0$$

where the first and the last term vanish but what has already been shown. Thus, A is E-flat.

To show that A is faithful as a left E-module, suppose that $T_A(M) = 0$. The sequence $0 \to P \to F \to M \to 0$ yields the exact sequence

$$0 \to T_A(P) \to T_A(F) \to T_A(M) = 0$$

since A is flat as an E-module. Hence, the top-row of the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_AT_A(P) & \longrightarrow & H_AT_A(F) & \longrightarrow & 0 & & \\
 & & \wr\big\uparrow \phi_P & & \wr\big\uparrow \phi_F & & & & \\
0 & \longrightarrow & P & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

is exact. A simple diagram chase shows $M = 0$.

Finally, A-generated subgroups of A-projective groups are A-projective if A is faithfully flat and E is right hereditary [4], and a.i) holds. Finally, a.ii) is a direct consequence of [6] since κ-A-projective groups are A-solvable. □

In particular, the last result shows that A-generated subgroups of self-small groups with right hereditary endomorphism ring are A-projective. Our next results summarizes other properties of such groups which we use frequently in this paper:

Corollary 2.2. *Let A be a self-small torsion-free Abelian group whose endomorphism ring is right hereditary:*

a) *Every exact sequence $G \to P \to 0$ such that G is A-generated and P is A-projective splits.*
b) *An A-generated group is A-torsion-free if and only if it is $\aleph_0$-A-projective.*
c) *An A-generated subgroup of an A-torsion-free group is A-pure if and only if U is a direct summand of $U + V$ for all finitely A-generated subgroups V of G.*

Proof. a) follows directly from the fact that A is faithfully flat which was established in Theorem 2.1.

b) It remains to show that A-torsion-free groups are $\aleph_0$-A-projective. Suppose that G is A-torsion-free, and let U be a finitely A-generated subgroup of G. Then U can be embedded into an A-projective group, and thus is A-projective by Theorem 2.1.

c) Let U be an A-pure subgroup of an A-torsion-free group G. If V is a finitely A-generated subgroup of G, then $(U + V)/U$ can be embedded into an A-projective group by Theorem 2.1. Thus, $(U+V)/U$ is A-projective. By a), U is a direct summand of $U + V$. □

However, the S-closure of a countable submodule of a non-singular module does not need to be countable even if R is countable. For instance, if $Q = \mathbb{Q}^\omega$ and $R = \mathbb{Z}1_S + \mathbb{Z}^{(\omega)}$, then Q is the maximal ring of quotients of R and $|Q| > |R|$ although Q is an essential extension of R. We want to remind the reader that a right E-module M has *Goldie-dimension* $m < \infty$ if it contains an essential submodule which is the direct sum of m non-zero uniform submodules where a module $X \neq 0$ is uniform if all its non-zero submodules are essential.

Proposition 2.3. *Let R be a countable right non-singular ring which has finite right Goldie-dimension. The S-closure of a countable submodule U of a non-singular right R-module M is countable.*

Proof. Let V be S-closure of U, and assume that V is uncountable. Let

$$\mathcal{F} = \{X \subseteq R \mid |X| < \infty \text{ and } \sum_{x \in X} xR \text{ is an essential right ideal}\}.$$

Then, $V = \{y \in M \mid yX \subseteq U \text{ for some } x \in \mathcal{F}\}$ since R has finite right Goldie-dimension. Since V is uncountable and $\mathcal{F}$ is countable, we can find $X_0 \in \mathcal{F}$ such that $yX_0 \subseteq U$ for uncountably many $y \in V$. Let $Y_0 = \{y \in V \mid yX_0 \subseteq U\}$. Write $X_0 = \{x_1, \ldots, x_n\}$, and consider $Y_0 x_1 \subseteq U$. There is an uncountable subset Y_1 of Y_0 such that $yx_1 = y'x_1$ for all $y, y' \in Y_1$ since U is countable. Repeating this argument with x_2 and Y_1 yields an uncountable subset Y_2 of Y_1 such that $yx_2 = y'x_2$ for all $y, y' \in Y_2$. By induction, we can find an uncountable subset Y_n of Y_0 such that $yx_i = y'x_i$ for all $i = 1, \ldots, n$ and all $y, y' \in Y_n$. Thus, $(y - y')(x_1R + \ldots + x_nR) = 0$ for all $y, y' \in Y_n$ which contradicts the fact that M is non-singular because $x_1R + \ldots + x_nR$ is essential. Thus V has to be countable. □

By Sandomierski's Theorem [11], a right finite dimensional, right hereditary ring is right Noetherian.

Corollary 2.4. *The following conditions are equivalent for a self-small torsion-free Abelian group A whose endomorphism ring is right hereditary:*

a) *E is right Noetherian.*
b) *An A-generated subgroup U of a finitely A-projective group G is finitely A-projective.*

Proof. *a) ⇒ b)*: Suppose that U is an A-generated subgroup of a finitely A-projective group P. Then $H_A(U)$ is a submodule of $H_A(P)$, and hence a finitely generated projective module. By Theorem 2.1, U is A-solvable, and $U \cong T_A H_A(U)$ is finitely A-projective.

b) ⇒ a): Let I be a right ideal of E. Arguing as in the proof of Theorem 2.1, ϕ_I is an isomorphism. Moreover $T_A(I) \cong IA$ since A is flat as an E-module. By b), IA is finitely A-projective, from which we obtain that I is finitely generated. □

In view of the results of this section, we assume from this point on that A is a self-small torsion-free group with a right Noetherian right hereditary endomorphism ring. Huber and Warfield showed in [16] that E is a right and left Noetherian ring whenever A is a torsion-free reduced group of finite rank with a right hereditary endomorphism ring. Moreover, no generality is lost if we restrict our discussion to the case that κ is a regular cardinal because Shelah's singular compactness theorem applies to A-projective groups [2].

3. $\aleph_1$-A-Coseparable Groups

Let $\kappa > \aleph_0$ be a regular cardinal, and A be a torsion-free Abelian group with $|A| < \kappa$. An A-projective subgroup U of an $\aleph_0$-A-projective group G is *κ-A-closed* provided that $(U+V)/U$ is A-projective for all κ-A-generated subgroups V of G. If $|U| < \kappa$, then this is equivalent to saying that G/U is κ-A-projective. The group G is *strongly κ-A-projective* if it is κ-A-projective and every κ-A-generated subgroup U of G is contained in an κ-A-generated, κ-A-closed subgroup V of G. By [1], $S_A(A^I)$ is $\aleph_1$-A-projective, but not strongly $\aleph_1$-A-projective since $\oplus_I A$ is not an $\aleph_1$-A-closed subgroup.

In the following we focus on the case $\kappa = \aleph_1$ since we are mainly interested in the algebraic aspects instead of the underlying set-theory. However, most results of this section carry over to the general case. In order to avoid immediate difficulties, we restrict our discussion to the case that A is countable.

Lemma 3.1. *Let A be a self-small countable torsion-free group with a right Noetherian right hereditary endomorphism ring.*

a) *If G is $\aleph_1$-A-projective, then G/U is $\aleph_1$-A-projective for all $\aleph_1$-A-closed subgroups U of G.*

b) *If G is strongly $\aleph_1$-A-projective, then G/U is strongly $\aleph_1$-A-projective for all countable $\aleph_1$-A-closed subgroups U of G.*

Proof. a) Let $\{\phi_n | n < \omega\} \subseteq H_A(G/U)$. Since $\Sigma_{n<\omega}\phi_n(A)$ is countable, there is a countable subgroup K of G such that $\Sigma_{n<\omega}\phi_n(A) \subseteq (K+U)/U$. Because A is countable, we can choose K to be A-generated. Since U is $\aleph_1$-A-closed in G, the group $(K+U)/U$ is U-projective, and the same holds $\Sigma_{n<\omega}\phi_n(A)$. Therefore, G/U is $\aleph_1$-A-projective.

b) Let V/U be a countable A-generated subgroup of G/U. Without loss of generality, we may assume that V is A-generated. Then, V is contained in an $\aleph_1$-A-closed subgroup W is a $\aleph_1$-A-closed subgroup of G. Since U is countable this means that G/W is $\aleph_1$-A-projective. Since $G/W \cong (G/U)/(W/U)$ and G/U is $\aleph_1$-A-projective, we obtain that G/U is strongly $\aleph_1$-A-projective. □

An A-generated group G $\aleph_1$-*A-coseparable* if it is $\aleph_1$-A-projective and every A-generated subgroup U of G such that G/U is countable contains a direct summand V of G such that G/V is countable. Our next results describes $\aleph_1$-A-coseparable group. Although our arguments follow the general outline of [13], significant modifications are necessary in our setting.

Theorem 3.2. *Let A be a self-small countable torsion-free group with a right Noetherian right hereditary endomorphism ring. A group G is $\aleph_1$-A-coseparable if and only if G is A-solvable and every exact sequence*

$$0 \to P \to X \to G \to 0$$

with P a direct summand of $\oplus_\omega A$ and X A-generated splits.

Proof. Suppose that G is $\aleph_1$-A-coseparable, and consider an exact sequence

$$0 \to P \overset{\alpha}{\to} X \overset{\beta}{\to} G \to 0$$

with P a direct summand of $\oplus_\omega A$ and X A-generated. Since A is faithfully flat, X is A-generated and G is A-solvable, the induced sequence

$$0 \to H_A(P) \xrightarrow{\alpha} H_A(X) \xrightarrow{H_A(\beta)} H_A(G) \to 0$$

of right E-modules is exact by Theorem 2.1. Since $H_A(G)$ is non-singular by the remarks in the introduction, the same holds for $H_A(X)$. Observe that $H_A(P)$ is countable since it is a direct summand of $H_A(\oplus_\omega A)$, and the latter is countable because A is self-small. We choose a complement W of $im(H_A(\alpha))$ in $H_A(X)$, and observe that $H_A(X)/W$ is nonsingular. Since

$$M = H_A(X)/(im(H_A(\alpha) \oplus W) \cong [H_A(X)/W][(im(H_A(\alpha) \oplus W)/W]$$

is singular and $(im(H_A(\alpha) \oplus W)/W$ is countable, $H_A(X)/W$ is countable as the S-closure of a countable submodule by Proposition 2.3 because E is right Noetherian and countable. Applying T_A yields the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_AH_A(P) & \xrightarrow{T_AH_A(\alpha)} & T_AH_A(X) & \xrightarrow{T_AH_A(\beta)} & T_AH_A(G) & \longrightarrow & 0 \\
 & & \wr\downarrow \theta_P & & \downarrow \theta_X & & \wr\downarrow \theta_G & & \\
0 & \longrightarrow & P & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & G & \longrightarrow & 0.
\end{array}$$

Therefore, X is A-solvable, and $U = \theta_X(T_A(W))$ is an A-generated subgroup of X such that $\alpha(P) \cap X = 0$ and

$$X/[\alpha(P) \oplus U] \cong T_A(M)$$

is countable. If $H = \beta(U)$, then $\beta|U$ is one-to-one. Since $\beta(U) \cong U \cong T_A(W)$ is A-generated and $G/\beta(U)$ is countable, there is a subgroup K of U such that $G = \beta(K) \oplus B$ for some countable subgroup B of G using the fact that G is $\aleph_1$-A-coseparable. Select a subgroup V of X containing $\alpha(P)$ such that $\beta(V) = B$. Clearly, V is countable.

To show $X = K \oplus V$, take $x \in X$ and write $\beta(x) = \beta(k) + \beta(v)$ for some $k \in K$ and $v \in V$. Then $x - k - b \in \alpha(P) \subseteq V$. On the other hand, suppose that $y \in K \cap V$. Then $\beta(y) \in \beta(K) \cap B = 0$, from which we obtain

$$y \in \alpha(P) \cap K \subseteq \alpha(P) \cap U = 0.$$

Moreover, V is A-generated since it is a direct summand of X, while $\beta(V) \cong V/\alpha(P)$ is A-projective as a countable subgroup of G. Therefore, $\alpha(P)$ is a direct summand of V.

Conversely, assume that G is an A-solvable group such that every exact sequence $0 \to P \to X \to G \to 0$ with P a direct summand of $\oplus_\omega A$ and X A-generated splits. Suppose that G contains a countable A-generated subgroup U which is not A-projective. Since U is A-solvable because A is E-flat by Theorem 2.1, $H_A(U)$ is not projective. Looking at projective resolutions of $H_A(U)$, we can find a countable projective module P with $\mathrm{Ext}^1_E(H_A(U), P) \neq 0$. Since E is right hereditary, we have an exact sequence

$$\mathrm{Ext}^1_E(H_A(G), P) \to \mathrm{Ext}^1_E(H_A(U), P) \to 0.$$

Thus, there is a non-splitting sequence $0 \to P \to M \to H_A(G) \to 0$ which induces $0 \to T_A(P) \to T_A(M) \to T_AH_A(G) \to 0$ which splits since $G \cong T_AH_A(G)$. We therefore obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_AT_A(P) & \longrightarrow & H_AT_A(M) & \longrightarrow & H_AT_AH_A(G) & \longrightarrow & 0 \\ & & \wr\uparrow \phi_P & & \uparrow \phi_M & & \wr\uparrow \phi_{H_A(G)} & & \\ 0 & \longrightarrow & P & \longrightarrow & M & \longrightarrow & H_A(G) & \longrightarrow & 0 \end{array}$$

in which ϕ_M is an isomorphism by the 3-Lemma. Since the top-row splits, the same has to hold for the bottom, which contradicts its choice. Therefore, G is $\aleph_1$-A-projective.

Consider an A-generated subgroup C of G such that G/C is countable. We can find a countable subgroup B such that $G = C + B$, and no generality is lost if we assume in addition that B is A-generated. By what was shown in the last paragraph, B is A-projective. We consider the exact sequence $0 \to K \to B \oplus C \xrightarrow{\pi} G \to 0$ with $\pi(b, c) = b + c$. Since G is A-solvable, and C is an A-generated subgroup of G, the group $B \oplus C$ is A-solvable. By Theorem 2.1, $K = \{(b, -b) \mid b \in B \cap C\}$ is A-generated, and hence A-solvable since A is E-flat. Since K is isomorphic to a subgroup of the countable A-projective group B, another application of Theorem 2.1 yields that K is a countable A-projective group. By our hypotheses, the map π splits, say $\pi\delta = 1_G$ for some homomorphism $\delta : G \to B \oplus C$. Let $\rho : B \oplus C \to B$ be the projection onto B with kernel C, and consider $D = \ker(\rho\delta)$. Since G/D is A-generated and isomorphic to a subgroup of the countable A-projective group B, it is A-projective itself. By Theorem 2.1, D is a direct summand of G. Moreover, every $d \in D$ satisfies $\delta(d) = (0, c)$ for some $c \in C$ since $\rho\delta(d) = 0$ yields $\delta(d) \in \ker\ \rho = C$. Then $d = \pi\delta(d) = \pi(0, c) = c$, and $D \subseteq C$. □

A group W is an *A-Whitehead group* if it admits an A-balanced exact sequence $0 \to U \to F \to W \to 0$ in which F is A-projective and U is A-generated with the property that

$$0 \to \mathrm{Hom}(W, A) \to \mathrm{Hom}(F, A) \to Hom(U, A) \to 0$$

is exact.

Corollary 3.3. *Let A be a self-small countable torsion-free group with a right Noetherian right hereditary endomorphism ring.*

a) *Every $\aleph_1$-A-coseparable group W is an A-Whitehead group.*
b) *It is consistent with ZFC that there exists a strongly $\aleph_1$-A-projective group G which is not $\aleph_1$-A-coseparable.*

Proof. a) By [7], an A-solvable group W is an A-Whitehead group if every exact sequence $0 \to A \to X \to W \to 0$ with $S_A(X) = X$ splits which is satisfied by W because of Theorem 3.2.

b) If we assume $V = L$, then all A-Whitehead groups are A-projective. However, there exist strongly $\aleph_1$-A-projective group G with $\mathrm{Hom}(G, A) = 0$ [7]. □

4. Strongly $\aleph_1$-A-Projective Groups and Martin's Axiom

We use the formulation of Martin's Axiom given in [13, Definition VI.4.2]. A partially ordered set $(P, \leq)$ satisfies the *countable chain condition (ccc)* if any antichain in $(P, \leq)$ is countable. An *antichain* is a subset A of P such that any two distinct members of A are *incompatible*, i.e., whenever $p, q \in A$, then there does not exist $r \in P$ such that $r \geq p$ and $r \geq q$. A subset D of P is dense if, for every $p \in P$ there is $q \in D$ such that $p \leq q$. Finally, a subset $\mathcal{F}$ of P is directed, if, for all $p, q \in \mathcal{F}$, there is $r \in \mathcal{F}$ such that $r \geq p$ and $r \geq q$.

For a cardinal κ, let $\mathrm{MA}(\kappa)$ denote the statement:

Let $(P, \leq)$ be a partially ordered set satisfying the *countable chain condition (ccc)*. For every family $\mathcal{D} = \{D_\alpha \mid \alpha < \kappa\}$ of dense subsets of P, there is a directed subset $\mathcal{F}$ of P such that $\mathcal{F} \cap D_\alpha \neq \emptyset$ for all α, i.e. $\mathcal{F}$ is *$\mathcal{D}$-generic.*

Martin's axiom (MA) stipulates that $\mathrm{MA}(\kappa)$ holds for every $\kappa < 2^{\aleph_0}$ [13].

Theorem 4.1. *($MA + \aleph_1 < 2^{\aleph_0}$) Let A be a self-small countable torsion-free group with a right Noetherian right hereditary endomorphism ring. If G is a strongly $\aleph_1$-A-projective group and $0 \to U \to \oplus_I A \to G \to 0$ is an A-balanced exact sequence such that $S_A(U) = U$ and $|I| < 2^{\aleph_0}$, then the induced sequence*

$$0 \to \mathrm{Hom}(G, B) \to Hom(\oplus_I A, B) \to \mathrm{Hom}(G, B) \to 0$$

is exact for all countable A-solvable group B.

Proof. We consider an A-balanced exact sequence $0 \to U \to \oplus_I A \to G \to 0$ where $U \to \oplus_I A$ is the inclusion map. Let $\mathcal{P}(U)$ be the collection of A-generated A-pure subgroups V of $F = \oplus_I A$ containing U such that V/U is finitely A-projective. Since V is A-generated and A is faithfully flat, U is a direct summand of V by Corollary 2.2, say $V = U \oplus R_V$ for some finitely A-projective group R_V.

To show that the sequence $Hom(\oplus_I A, B) \to \mathrm{Hom}(G, B) \to 0$ is exact whenever B is a countable A-solvable group, let $\phi \in \mathrm{Hom}(U, B)$, and consider

$$P = \{(V, \psi) \mid V \in \mathcal{P}(U), \psi \in \mathrm{Hom}(V, B), \text{ and } \psi|U = \phi\}.$$

We partially order P by $(V_1, \psi_1) \geq (V_2, \psi_2)$ if and only if $V_2 \subseteq V_1$ and $\psi_1|V_2 = \psi_2$. Once we have shown that P and $\mathcal{D} = \{D(J)|J \subseteq I \text{ finite}\}$ satisfy the hypotheses of Martin's Axiom, then we can find a $\mathcal{D}$-generic directed directed subset $\mathcal{F}$ of P. Define a map $\psi : \oplus_I A \to B$ as follows. For $x \in \oplus_I A$, choose a finite subset J of I such that $x \in \oplus_J A$. Since $\mathcal{F}$ is $\mathcal{D}$-generic, there is $(V, \delta) \in D(J) \cap F$ with $x \in V$. Define $\psi(x) = \delta(x)$. Moreover, if (V_1, δ_1) and (V_2, δ_2) are two choices, then there is $(V_3, \delta_3) \in \mathcal{F}$ such that $(V_i, \delta_i) \leq (V_3, \delta_3)$ for $i = 1, 2$ since $\mathcal{F}$ is directed. Thus, $\delta_1(x) = \delta_3(x) = \delta_2(x)$. □

The key towards showing that $(P, \leq)$ satisfies the countable chain condition is

Theorem 4.2. *Every uncountable subset P' of P contains an uncountable subset P'' for which we can find an A-pure A-projective subgroup X of F containing U as a direct summand such that $V \subseteq X$ whenever $(V, \psi) \in P''$.*

Proof. We may assume that $P' = \{(V_\nu, \psi_\nu) | \nu < \omega_1\}$. Since U is a direct summand of V_ν, we obtain that $H_A(V_\nu/U) \cong H_A(V_\nu)/H_A(U)$ is a finitely generated right E-module. In particular, it has finite right Goldie dimension since E is right Noetherian. Therefore, no generality is lost if we assume that there is $m < \omega$ such that $H_A(V_\nu/U)$ has Goldie dimension m for all $\nu < \omega_1$.

Let $0 \leq k \leq m$ be maximal with respect to the property that there exists an A-pure A-projective subgroup T of F containing U such that $H_A(T/U)$ has Goldie dimension k and T is contained in uncountable many V_ν. This k exists since U is the choice for T in the case $k = 0$. Observe that T/U is A-solvable as an A-generated subgroup of the A-solvable group $G = F/U$. Thus, $0 \to U \to T \to T/U \to 0$ is A-balanced, and $H_A(T/U) \cong H_A(T)/H_A(U)$ has finite Goldie-dimension and is non-singular. Thus, it contains a finitely generated essential submodule. Since E is right Noetherian and countable, any essential extension of a non-singular finite dimensional right E-module is countable by Proposition 2.3. In particular, $H_A(T/U)$ is countable, and hence $T/U \cong T_A H_A(T/U)$ is countable. Since G is $\aleph_1$-A-projective, T/U is A-projective, and $T = U \oplus W$ because A is faithfully flat by Corollary 2.2.

Suppose that T' is an A-generated subgroup of F containing T such that $T \neq T'$. There exists $\alpha \in H_A(T')$ with $\alpha(A) \not\subseteq T$. Since T is A-pure in F, we obtain $T + \alpha(A) = T \oplus C$ with $C \neq 0$. Thus, the Goldie-dimension of $H_A(T')$ is at least $k+1$, and T' is contained in only countably many of the V_ν. No generality is lost if we assume that T is contained in V_ν for all ν. Since T is A-pure in F and $V_\nu = U \oplus R_{V_\nu} = T + R_{V_\nu}$ for some finitely A-projective subgroup R_{V_ν} of F, we obtain decompositions $V_\nu = T \oplus W_\nu$. Observe that W_ν is finitely A-projective.

We construct X as the union of a smooth ascending chain $\{X_\nu | \nu < \omega_1\}$ of A-pure A-projective subgroups of F containing T and an ascending chain of ordinals $\{\sigma_\nu | \nu < \omega_1\}$ such that $X_{\nu+1}/X_\nu$ is A-projective, $W_{\sigma_{\nu+1}} \subseteq X_{\nu+1}$, and X_ν/U is an image of $\oplus_\omega A$ for all $\nu < \omega_1$. We set $X_0 = T$, and $X_\alpha = \cup_{\nu<\alpha} X_\nu$ if α is a limit ordinal. Then, X_α/U is a countable subgroup of F/U, and hence A-projective. Set $\sigma_\alpha = sup(\sigma_\nu | \nu < \alpha)$.

If $\alpha = \nu + 1$, then there exists a subgroup C_ν of F containing X_ν such that the group C_ν/U is an A-projective countable $\aleph_1$-A-closed subgroup of F/U since F/U is strongly $\aleph_1$-A-projective. In particular, $F/C_\nu \cong (F/U)/(C_\nu/U)$ is A-solvable. Since A is flat, C_ν is A-generated by Theorem 2.1. If K is a countable A-generated subgroup of F, then $(K+U)/U$ is a countable subgroup of F/U. Hence,

$$(K + C_\nu)/C_\nu \cong [(K + C_\nu)/U]/[C_\nu/U]$$

is A-projective.

To construct σ_α, assume $W_\mu \cap C_\nu \neq 0$ for all $\mu > \sigma_\nu$. Then,

$$W_\mu/(W_\mu \cap C_\nu) \cong (W_\mu + C_\nu)/C_\nu$$

is A-projective by the last paragraph. Since A is faithfully flat, $W_\mu \cap C_\nu$ is A-generated, and there is a map $0 \neq \alpha_\mu \in H_A(W_\mu \cap C_\nu) \subseteq H_A(C_\nu)$. Since C_ν/U is a countable subgroup of F/U, it is A-projective, and $C_\nu = U \oplus P_\nu$ since A is faithfully flat. Observe that P_ν is countable and A-projective. Write $\alpha_\mu = \beta_\mu + \epsilon_\mu$ with $\beta_\mu \in H_A(U)$ and $\epsilon_\mu \in H_A(P_\nu)$. Since E is countable, the same holds for $H_A(P_\nu)$, and there is

$\epsilon \in H_A(P_\nu)$ such that $\epsilon_\mu = \epsilon$ for uncountably many μ. For all these μ, we have $\epsilon(A) \subseteq W_\mu + U \subseteq V_\mu$. Hence, $T + \epsilon(A) \subseteq V_\mu$ for uncountably many μ. However, this is only possible if $\epsilon(A) \subseteq T$. But then, $\alpha_\mu(A) \subseteq T \cap W_\mu = 0$, a contradiction.

Therefore, we can find an ordinal $\sigma_\alpha > \sigma_\nu$ with $C_\nu \cap W_{\sigma_\alpha} = 0$. In particular, $X_\nu \subseteq C_\nu$ yields $X_\nu \cap W_{\sigma_\alpha} = 0$. Let Y be the S-closure of

$$H_A(X_\nu \oplus W_{\sigma_\alpha}) = H_A(X_\nu \oplus H_A(W_{\sigma_\alpha}) \supseteq H_A(U)$$

in $H_A(F)$ and let $X_\alpha = \theta_F(Y \otimes A) = YA$. As an A-generated subgroup of F, X_α is A-solvable. Then, the inclusion $Y \subseteq H_A(X_\alpha)$ induces the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_A(Y) & \longrightarrow & T_AH_A(X_\alpha) & \longrightarrow & T_A(H_A(X_\alpha)/Y) & \longrightarrow & 0 \\ & & \wr\big\downarrow \theta_F|T_A(Y) & & \wr\big\downarrow \theta_{X_\alpha} & & & & \\ 0 & \longrightarrow & YA & \xrightarrow{1_{YA}} & YA & \longrightarrow & 0 & & \end{array}$$

from which we get $T_A(H_A(X_\alpha)/Y) = 0$. Since A is faithfully flat, $Y = H_A(X_\alpha)$, and $H_A(X_\alpha)/[H_A(X_\nu) \oplus H_A(W_{\sigma_\alpha})]$ is singular.

Observe that $Y/H_A(U)$ is the S-closure of $[H_A(X_\nu) + H_A(W_{\sigma_\alpha})]/H_A(U)$ in $H_A(F)/H_A(U)$ because

$$H_A(F)/Y \cong [H_A(F)/H_A(U)]/[Y/H_A(U)]$$

is non-singular and

$$Y/H_A(X_\nu \oplus W_{\sigma_\alpha}) \cong [Y/H_A(U)]/[H_A((X_\nu \oplus W_{\sigma_\alpha})/H_A(U)]$$

is singular. Since F/U is A-solvable, and X_ν/U is countable, $H_A(X_\nu)/H_A(U)$ is countable. Moreover, W_μ is finitely A-projective. Hence, the E-module $H_A(W_{\sigma_\alpha})$ is countable too, and

$$[H_A(X_\nu) + H_A(W_{\sigma_\alpha})]/H_A(U)$$

is countable. Thus, $Y/H_A(U)$ is an essential extension of a countable non-singular right E-module. By Proposition 2.3, we obtain that $Y/H_A(U)$ is countable. Thus, there is a countable submodule Y' of Y with $Y = Y' + H_A(U)$. Then X_α/U is countable and $X_\alpha = Y'A + X_\nu$, and. Consequently, X_α/U has to be A-projective, and the same holds for $X_\alpha \cong X_\alpha/U \oplus U$.

It remains to show that X_α/X_ν is A-projective. For this, observe that the group

$$X_\alpha/(X_\alpha \cap C_\nu) \cong (X_\alpha + C_\nu)/C_\nu$$

is countable since it is an epimorphic image of $(X_\alpha + C_\nu)/U$ which is countable because X_α and C_ν/U are countable. Since C_ν/U is $\aleph_1$-A-closed in F/U, we have that

$$X_\alpha/(X_\alpha \cap C_\nu) \cong [(X_\alpha + C_\nu)/U]/[C_\nu/U]$$

is A-projective. Since A is flat, $X_\alpha \cap C_\nu$ is A-generated as in Theorem 2.1. For τ in $H_A(X_\alpha \cap C_\nu)$, choose a regular element $c \in E$ such that $\tau c \in H_A(X_\nu) \oplus H_A(W_{\sigma_\alpha})$, say $\tau c = \beta + \gamma$ for some $\beta \in H_A(X_\nu)$ and $\gamma \in H_A(W_{\sigma_\alpha})$. Then

$$\gamma = \tau c - \beta \in H_A(W_{\sigma_\alpha}) \cap H_A(C_\nu) = 0.$$

Hence, $\tau c \in H_A(X_\nu)$. Since X_ν is A-pure in F, we obtain $\tau \in H_A(X_\nu)$. Therefore, $H_A(X_\alpha \cap C_\nu) \subseteq H_A(X_\nu)$, and $X_\alpha \cap C_\nu \subseteq X_\nu$. Since X_ν is contained in X_α and in

C_ν, we obtain $X_\alpha \cap C_\nu = X_\nu$. Then $X_\alpha/X_\nu \cong (X_\alpha + C_\nu)/C_\nu$ is A-projective by what we have already shown. In particular, X_ν is a direct summand of X_α.

Consequently, $X = \cup_{\nu<\omega_1} X_\nu$ is A-pure and A-projective. Because

$$X_{\nu+1}/X_\nu \cong [X_{\nu+1}/T]/[X_\nu/T]$$

is A-projective for all ν, the group X/T is A-projective. This yields $X = T \oplus S$. However, $T = U \oplus W$, so that $X = U \oplus W \oplus S$. Finally,

$$V_{\sigma_{\nu+1}} = T \oplus W_{\sigma_{\nu+}} \subseteq X_{\nu+1} \subseteq X$$

for all $\nu < \omega_1$. Let $P'' = \{(V_{\sigma_{\nu+1}}, \psi_{\sigma_{\nu+1}})|\nu < \omega_1\}$. □

Corollary 4.3. *P satisfies the countable chain condition.*

Proof. Since B is a countable A-solvable group, there is an exact sequence

$$0 \to V \to \oplus_\omega A \to B \to 0$$

which is A-balanced by Theorem 2.1. Thus, $H_A(B)$ is countable as an epimorphic image of $H_A(\oplus_\omega A) \cong \oplus_\omega E$ using the self-smallness of A.

Let P' be an uncountable subset of P. By the previous Lemma, we may assume $P' = \{(V_\nu, \psi_\nu)|\nu < \omega_1\}$ such that there is an A-pure A-projective subgroup X containing U as a direct summand satisfying $V_\nu \subseteq X$ for all $\nu < \omega_1$. We can write $X = U \oplus Y$ and $Y = \oplus_J Y_j$ where each Y_j is isomorphic to a subgroup of A. This is possible since E is hereditary.

For $\nu < \omega_1$, we have $V_\nu = U \oplus (Y \cap V_\nu)$. Since $Y \cap V_\nu$ is finitely A-projective, there is a finite subset J_ν of J such that $H_A(Y \cap V_\nu) \subseteq H_A(\oplus_{J_\nu} Y_j)$, and $Y \cap V_\nu \subseteq \oplus_{J_\nu} Y_j$. Therefore, V_ν is an A-pure subgroup of

$$V_\nu + (\oplus_{J_\nu} Y_j) = U \oplus \ (\oplus_{J_\nu} Y_j).$$

Because $\oplus_{J_\nu} Y_j$ is finitely A-generated, V_ν is a direct summand of $U \oplus (\oplus_{J_\nu} Y_j)$, say $V_\nu + (\oplus_{J_\nu} Y_j) = V_\nu \oplus X_\nu$. Since $V_\nu + (\oplus_{J_\nu} Y_j)$ is A-projective, the same holds for X_ν. Thus, X_ν is isomorphic to a direct summand of $\oplus_{J_\nu} Y_j$. Moreover, $\psi_\nu : V_\nu \to B$ extends to a map $\lambda_\nu : U \oplus (\oplus_{J_\nu} Y_j) \to B$. By the Adjoint-Functor-Theorem,

$$\mathrm{Hom}(\oplus_{J_\nu} Y_j, B) \cong \mathrm{Hom}_E(H_A(\oplus_{J_\nu} Y_j), H_A(B))$$

is countable since $H_A(B)$ is countable as was shown in the first paragraph of the proof and J_ν is finite. Consequently, there are at most countably many different extensions of ϕ to $U \oplus (\oplus_{J_\nu})$.

If there are only countably many different J_ν's, then there is ν_0 such that $J_{\nu_0} = J_\mu$ for uncountable μ. Thus, there are μ_1 and μ_2 with $J_{\nu_0} = J_{\mu_1} = J_{\mu_2}$ and $\lambda_{\mu_1} = \lambda_{\mu_2}$. Thus, ψ_{μ_1} and ψ_{μ_2} have a common extension. Therefore, $P' = \{(V_\nu, \psi_\nu)|\nu < \omega_1\}$ cannot be an antichain. On the other hand, if there are uncountably many J_ν's, then we may assume without loss of generality that $J_\nu \neq J_\mu$ for $\mu \neq \nu$. Finally, we can impose the requirement that all the J_ν have the same order. Thus, J_ν cannot be contained in J_μ for $\mu \neq \nu$. Since $(V_\nu, \psi_\nu) \leq (V_\nu \oplus X_\nu, \lambda_\nu)$, we may assume that $V_\nu = U \oplus (\oplus_{J_\nu} Y_j)$ and $\lambda_\nu = \psi_\nu$.

There is a subset T of J which is maximal with respect to the property that it is contained in uncountably many of the J_ν. We may assume that T is actually contained in all of the J_ν. Observe that T is finite and a proper subset of all the J_ν. Otherwise, all the J_ν would have to coincide with T since they have the same finite order. Since $\mathrm{Hom}(\oplus_T Y_j, B) \cong \mathrm{Hom}_E(H_A(\oplus_T Y_j), H_A(B))$ is countable by the Adjoint-Functor-Theorem, there are uncountably many ψ_ν which have the same restriction to $W = U \oplus (\oplus_T Y_j)$. Without loss of generality, we may assume that this happens for all ν.

Let $j \in J_0 \setminus T$. The maximality of T guarantees that j is contained in only countably many of the J_ν. Since $J_0 \setminus T$ is finite, there is $\mu < \omega_1$ with $J_\mu \cap J_0 = T$. The maps ψ_μ and ψ_0 have a common extension $\sigma : U \oplus (\oplus_{J_0 \cup J_\nu} Y_j) \to B$ since they coincide on W. Since $U \oplus (\oplus_{J_0 \cup J_\nu} Y_j)$ is a direct summand of X, and X is A-pure in F, we have that $U \oplus (\oplus_{J_0 \cup J_\nu} Y_j)$ is A-pure in F. Because $J_0 \cup J_\nu$ is finite,

$$(U \oplus (\oplus_{J_0 \cup J_\nu} Y_j), \sigma) \in P.$$

Thus,

$$(U \oplus (\oplus_{J_0 \cup J_\nu} Y_j), \sigma) \geq (V_\mu, \psi_\mu), (V_0, \psi_0).$$

Consequently, P' cannot be an anti-chain. □

For every finite subset J of I, let $D(J) = \{(V, \psi) \in P \mid \oplus_J A \subseteq V\}$.

Proposition 4.4. *P and $\mathcal{D} = \{D(J) | J \subseteq I$ finite$\}$ satisfy the hypotheses of Martin's Axiom.*

Proof. By Corollary 4.3, it remains to show that $D(J)$ is dense in P. For this, let $(V, \psi) \in P$. We have to find $(W, \alpha) \in P$ such that $\oplus_J A$ and V are contained in W and $\alpha | V = \psi$. Since V/U is finitely A-projective and $G \cong F/U$ is strongly $\aleph_1$-A-projective, there is a subgroup X of F containing V and $\oplus_J A$ such that X/U is a $\aleph_1$-A-closed, A-projective countable subgroup of F/U. Since

$$[F/U]/[X/U] \cong F/X$$

is $\aleph_1$-A-projective, it is A-solvable by Theorem 2.1. Using the same result once more, we obtain that the sequence $0 \to X \to F \to F/X \to 0$ is A-balanced. In particular, $S_A(X) = X$ and X is A-projective. Moreover,

$$H_A(F)/H_A(X) \cong H_A(F/X) \cong H_A([F/U]/[X/U]) \cong H_A(F/U)/H_A(X/U)$$

since X in F and X/U in F/U are A-balanced by the faithful flatness of A. But the latter is non-singular, since $[F/U]/[X/U]$ is $\aleph_1$-Aprojective. Therefore, X is A-pure in F.

Since the group X/U is A-projective, we have a decomposition $X = U \oplus P$. Hence, $V = U \oplus (V \cap P)$ and $V \cap P$ is finitely A-projective. In the same way,

$$(\oplus_J A) + U = U \oplus [((\oplus_J A) + U) \cap P]$$

yields that $((\oplus_J A) + U) \cap P$ is A-generated and an image of $\oplus_J A$.
Therefore, $((\oplus_J A) + U) \cap P$ and $V \cap P$ are finitely A-projective subgroups of P. Thus, $H_A(((\oplus_J A) + U) \cap P)$ and $H_A(V \cap P)$ are finitely generated submodule of $H_A(P)$. Since E is right hereditary, $H_A(P)$ is a direct sum of right ideals of E, which yields that $H_A(((\oplus_J A) + U) \cap P)$ and $H_A(V \cap P)$ are contained in a finitely generated direct summand of $H_A(P)$. Hence, there is a finitely A-projective summand D of P

which contains $V \cap P$ and $((\oplus_J A) + U) \cap P$. Since $U \oplus D = V + D$ and V is A-pure in F, we obtain that V is a direct summand of $U \oplus D$. Thus, ψ extends to a map $\alpha : U \oplus D \to B$. Clearly, $(U \oplus D, \alpha) \in P$ and $(U \oplus D, \alpha) \geq (V, \psi)$. □

An A-generated group G $\aleph_1$-*A-separable* if every countable subset of G is contained in an A-projective direct summand of G.

Corollary 4.5. $(MA + \aleph_1 < 2^{\aleph_0})$ *If A is a self-small countable torsion-free group with a right Noetherian right hereditary endomorphism ring, then every strongly $\aleph_1$-A-projective group is $\aleph_1$-A-separable and $\aleph_1$-A-coseparable.*

Proof. By Theorem 3.2 and Theorem 4.1, a strongly $\aleph_1$-A-projective group G is $\aleph_1$-A-coseparable. It remains to show that is $\aleph_1$-A-separable too. For a countable subset X of G select a countable $\aleph_1$-A-closed subgroup U of G containing X. Since G/U is strongly $\aleph_1$-A-projective, the sequence $0 \to U \to G \to G/U \to 0$ splits by Theorem 4.1. □

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