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Curves with constant geodesic curvature in the Bolyai-Lobachevskian plane

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Abstract. The aim of this note is to present the curves with constant geodesic curvature of the Bolyai-Lobachevskian hyperbolic plane. By using the Lobachevskian metric the equations of the circle, paracycloid and hipercycloid are given. Furthermore, we determine a new family of curves with constant curvature which was not emphasized before. During the analysis we use Cartesian and polar coordinates.

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1. General formulas in Cartesian coordinates

We consider the Lobachevskian metric

$$ds^2 = \cosh^2 \frac{y}{k} dx^2 + dy^2 \ , \tag{1.1}$$

where k is the parameter of the two-dimensional hyperbolic plane.

Among the geodesics we can find so-called Euclidean lines too, which can be used as coordinate axes. Therefore we can define a Cartesian coordinate system in the hyperbolic plane. If $dx = 0$, then $ds^2 = dy^2$, thus we can use the euclidean method when determining the value of y. The x-value of a point can only be determined by the x-axis, because when $dy = 0$, then the formula $ds^2 = dx^2$ can only be used in the case of $y = 0$. Now let us consider a point $P(x, y)$ in the hyperbolic plane. The foot of the perpendicular from P to the x-axis is denoted by $P_1(x, 0)$. Then distance $\overline{OP_1}$ corresponds with the x-coordinate of P, while the length of $\overline{PP_1}$ equals the y-coordinate of P.

As the reflection over the coordinate axes is a symmetry operation, during the analysis we will consider only the first quadrant of the plane. Note that the lines we obtain in the first quadrant have segments in the other quadrants, too. From metric (1.1) it follows that the translation of the origin along the direction of the x-axis is also a symmetry operation.

If we use s as variable, the functions characterizing the geodesics are of form $x = x(s), y = y(s)$. The geodesic lines are determined by the following differential equations:

$$\frac{d^2x}{ds^2} + \frac{2}{k}\tanh\frac{y}{k}\frac{dx}{ds}\frac{dy}{ds} = 0, \tag{1.2}$$

$$\frac{d^2y}{ds^2} - \frac{1}{k}\sinh\frac{y}{k}\cosh\frac{y}{k}\left(\frac{dx}{ds}\right)^2 = 0. \tag{1.3}$$

If we replace variable s with x, equation (1.2) can be written in the following equivalent form:

$$\cosh^2\frac{y}{k}\frac{dx}{ds} = C_1, \tag{1.4}$$

where C_1 is constant. Using (1.3) and (1.4) we obtain the

$$\frac{d^2\tanh\frac{y}{k}}{dx^2} - \frac{1}{k^2}\tanh\frac{y}{k} = 0 \tag{1.5}$$

differential equation. If we use variable x, we only need to determine the constants a_1 and a_2 which appear in the equation

$$\tanh\frac{y}{k} = a_1\cosh\frac{x}{k} + a_2\sinh\frac{x}{k}. \tag{1.6}$$

By using (1.1), (1.4) and (1.6), for the value of C_1 we get

$$\frac{1}{C_1^2} = 1 - \tanh^2\frac{y}{k} + k^2\left(\frac{d\tanh\frac{y}{k}}{dx}\right)^2 = 1 - a_1^2 + a_2^2. \tag{1.7}$$

In order to determine the geodesic curvature, we use the formula given by Schlesinger:

$$\frac{1}{r_g} = \\ = \cosh\frac{y}{k}\left\{\left(\frac{d^2x}{ds^2} + \frac{2}{k}\tanh\frac{y}{k}\frac{dx}{ds}\frac{dy}{ds}\right)\frac{dy}{ds} - \left[\frac{d^2y}{ds^2} - \frac{1}{k}\sinh\frac{y}{k}\cosh\frac{y}{k}\left(\frac{dx}{ds}\right)^2\right]\frac{dx}{ds}\right\}. \tag{1.8}$$

From (1.2), (1.3) and (1.8) it follows that

$$\frac{1}{r_g} = 0. \tag{1.9}$$

Metric (1.1) can also be obtained by using the metric

$$ds^2 = dx_1^2 + dx_2^2 - dx_0^2 \tag{1.10}$$

defined in the three-dimensional pseudo-Euclidean space, with the help of the following formulas:

$$x_1 = k\sinh\frac{x}{k}\cosh\frac{y}{k}, \quad x_2 = k\sinh\frac{y}{k}, \quad x_0 = k\cosh\frac{x}{k}\cosh\frac{y}{k}. \tag{1.11}$$

2. General formulas in polar coordinates

Using (1.10) and equalities

$$x_1 = k\cos\varphi\sinh\frac{\rho}{k},\quad x_2 = k\sin\varphi\sinh\frac{\rho}{k},\quad x_0 = k\cosh\frac{\rho}{k}, \tag{2.1}$$

we obtain the metric

$$ds^2 = d\rho^2 + k^2\sinh^2\frac{\rho}{k}d\varphi^2, \tag{2.2}$$

where ρ and φ represent polar coordinates.

If we use s as variable, we obtain the following differential equations which determine the lines of the hyperbolic plane.

$$\frac{d^2\varphi}{ds^2} + \frac{2}{k}\coth\frac{\rho}{k}\frac{d\varphi}{ds}\frac{d\rho}{ds} = 0, \tag{2.3}$$

$$\frac{d^2\rho}{ds^2} - k\sinh\frac{\rho}{k}\cosh\frac{\rho}{k}\left(\frac{d\rho}{ds}\right)^2 = 0. \tag{2.4}$$

If we replace variable s with φ, equation (2.3) can be written in the following equivalent form:

$$\sinh^2\frac{\rho}{k}\frac{d\varphi}{ds} = C_2. \tag{2.5}$$

Also, from (2.4) we get

$$\frac{d^2\coth\frac{\rho}{k}}{d\varphi^2} + \coth\frac{\rho}{k} = 0. \tag{2.6}$$

The geodesics satisfy

$$\coth\frac{\rho}{k} = b_1\sin\varphi + b_2\cos\varphi, \tag{2.7}$$

where b_1 and b_2 are constant values. From (1.6) and (2.7) it follows that the values x_1, x_2 and x_0 admit a linear connection.

Using (2.2), (2.5) and (2.7), we get for the value of C_2

$$\frac{1}{C_2^2} = k^2\left[\coth^2\frac{\rho}{k} - 1 + \left(\frac{d\coth\frac{\rho}{k}}{d\varphi}\right)^2\right] = k^2\left(b_1^2 + b_2^2 - 1\right). \tag{2.8}$$

The geodesic curvature verifies the formula given by Schlesinger:

$$\frac{1}{r_g} = k\sinh\frac{\rho}{k}\left\{\left(\frac{d^2\varphi}{ds^2} + \frac{2}{k}\coth\frac{\rho}{k}\frac{d\varphi}{ds}\frac{d\rho}{ds}\right)\frac{d\rho}{ds} - \left[\frac{d^2\rho}{ds^2} - k\sinh\frac{\rho}{k}\cosh\frac{\rho}{k}\left(\frac{d\varphi}{ds}\right)^2\right]\frac{d\varphi}{ds}\right\}. \tag{2.9}$$

Using (1.11) and (2.1) it follows that the connection between the Cartesian and polar coordinates is determined by the following equations:

$$\sinh\frac{x}{k}\cosh\frac{y}{k} = \cos\varphi\sinh\frac{\rho}{k}, \tag{2.10}$$

$$\sinh\frac{y}{k} = \sin\varphi\sinh\frac{\rho}{k}, \tag{2.11}$$

$$\cosh\frac{x}{k}\cosh\frac{y}{k} = \cosh\frac{\rho}{k}. \tag{2.12}$$

Furthermore, from (2.10) and (2.12) we obtain

$$\tanh\frac{x}{k}=\cos\varphi\tanh\frac{\rho}{k}. \tag{2.13}$$

3. The Bolyai-Lobachevskian lines

As the geometry in discussion is based on metric (1.1), we differentiate four types of lines. The first family contains lines crossing the origin. The second set consists of lines which cross the x-axis, while the lines of the third family do not cross the x-axis. Then there are the lines which are parallel to the x-axis.

a) Based on (1.6), the lines crossing the origin satisfy

$$\tanh\frac{y}{k}=a_2\sinh\frac{x}{k}.$$

Let us consider a point $P(x,y)$ on a line in question, then the tangent vector to the line in P admits

$$\tan\alpha=\frac{1}{\cosh\frac{y}{k}}\frac{dy}{dx}.$$

In this case

$$\tan\alpha=a_2\cosh\frac{x}{k}\cosh\frac{y}{k},$$

thus the value of a_2 determines the tangent vector in the origin. The angle of intersection between the line and the x-axis is denoted by φ, which verifies

$$\tanh\frac{y}{k}=\tan\varphi\sinh\frac{x}{k}. \tag{3.1}$$

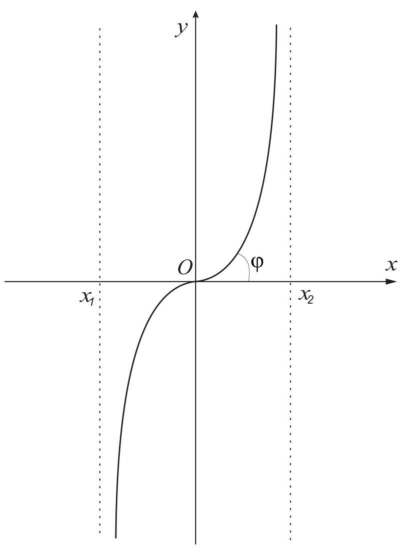

FIGURE 1

Obviously, φ is the polar angle of point P. If φ is set as constant, we get $ds^2 = d\rho^2$, thus the value of ρ determines the distance between the origin and point P measured along the geodesic.

The values of x_1 and x_2 represented in figure 1 are determined by equation (3.1), when $y \longrightarrow -\infty$ and $y \longrightarrow \infty$.

Using equation (1.7), we obtain

$$C_1 = \cos\varphi.$$

b) From (1.6) it follows that the lines crossing the x-axis and passing through points $P_0(a,0)$ and $P_1(0,b)$ verify

$$\tanh\frac{y}{k} = \tanh\frac{b}{k}\left(\cosh\frac{x}{k} - \coth\frac{a}{k}\sinh\frac{x}{k}\right). \tag{3.2}$$

The values x_1 and x_2 are determined by equation (3.1) as $y \longrightarrow -\infty$ and $y \longrightarrow \infty$ (figure 2).

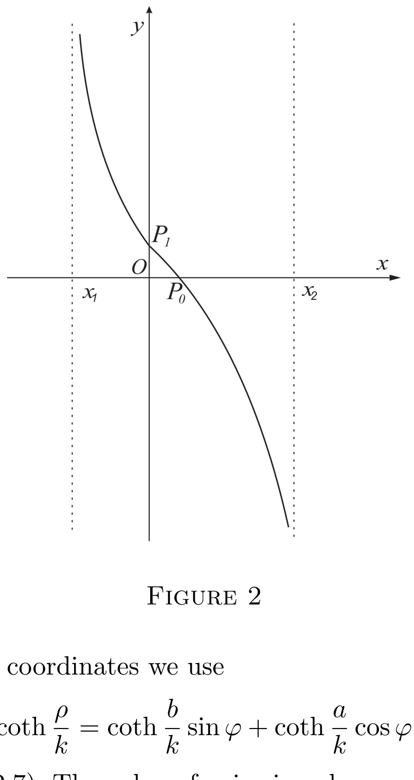

FIGURE 2

In the case of polar coordinates we use

$$\coth\frac{\rho}{k} = \coth\frac{b}{k}\sin\varphi + \coth\frac{a}{k}\cos\varphi \tag{3.3}$$

obtained from formula (2.7). The value of φ is given by equation (3.3) as $\rho \longrightarrow \infty$.

Using (2.6) and (2.8), we get for the constants C_1 and C_2 the following formulas:

$$\frac{1}{C_1} = \sqrt{1 + \frac{\tanh^2\frac{b}{k}}{\sinh^2\frac{a}{k}}}, \quad \frac{1}{C_2} = k\sqrt{\coth^2\frac{b}{k} + \coth^2\frac{a}{k} - 1}. \tag{3.4}$$

c) Figure 3 illustrates that the line passing through the y-axis in point $P_1(0,b)$ while not crossing the x-axis has a minimum point.

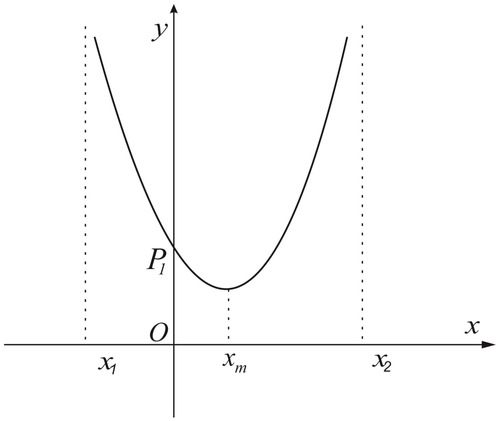

FIGURE 3

The line admits

$$\tanh\frac{y}{k}=\tanh\frac{b}{k}\left(\cosh\frac{x}{k}-\tanh\frac{x_m}{k}\sinh\frac{x}{k}\right), \tag{3.5}$$

where x_m denotes the value of x determined by the minimum point.

The domain of the line is determined by equation (3.5) as $y\longrightarrow\infty$.

In the case of polar coordinates, by using equations (2.9), (2.10), (2.11) and (2.12), we get

$$\coth\frac{\rho}{k}=\coth\frac{b}{k}\sin\varphi+\tanh\frac{x_m}{k}\cos\varphi. \tag{3.6}$$

For the values of C_1 and C_2, we obtain formulas

$$\frac{1}{C_1}=\sqrt{1-\frac{\tanh^2\frac{b}{k}}{\cosh^2\frac{x_m}{k}}}\quad\text{and}\quad\frac{1}{C_2}=k\sqrt{\coth^2\frac{b}{k}+\tanh^2\frac{x_m}{k}-1}. \tag{3.7}$$

If $x_m=0$, the line admits

$$\tanh\frac{y}{k}=\tanh\frac{b}{k}\cosh\frac{x}{k},\quad\coth\frac{\rho}{k}=\coth\frac{b}{k}\sin\varphi, \tag{3.8}$$

while the values C_1 and C_2 verify

$$C_1=\cosh\frac{b}{k},\quad C_2=\frac{1}{k}\sinh\frac{b}{k}. \tag{3.9}$$

d) If $a\longrightarrow\infty$ and $x_m\longrightarrow\infty$, we obtain the line parallel to the x-axis, passing through $P_1(0,b)$. The lines of this family satisfy

$$\tanh\frac{y}{k}=\tanh\frac{b}{k}e^{-\frac{x}{k}},\quad\coth\frac{\rho}{k}=\coth\frac{b}{k}\sin\varphi+cos\varphi, \tag{3.10}$$

while $C_1=1,\quad C_2=\frac{1}{k}\tanh\frac{b}{k}$.

4. The orthogonal curves

If in the formulas obtained in the previous section we set a variable as a varying parameter, we obtain families of lines. Each family of lines admits orthogonal lines. Let us denote the original lines by the index 1. The line passing through point $P(x, y)$ admits the following orthogonality condition:

$$\cosh^2 \frac{y}{k} dx dx_1 + dy dy_1 = 0. \tag{4.1}$$

The lines verify

$$y_1 = y_1(x_1, p), \quad p = p(x_1, y_1), \tag{4.2}$$

where the parameter is denoted by p. By deriving this equation with respect to variable x_1, we obtain

$$\frac{dy_1}{dx_1} = f(x_1, p). \tag{4.3}$$

Using (4.2) and (4.3), we eliminate the parameter, thus we get

$$\frac{dy_1}{dx_1} = f[x_1, p(x_1, y_1)] = F(x_1, y_1). \tag{4.4}$$

From (4.1) and (4.4) it follows that the orthogonal lines verify

$$\cosh^2 \frac{y}{k} dx + F(x, y) dy = 0. \tag{4.5}$$

In the case of polar coordinates we use the formula

$$\rho_1 = \rho_1(\varphi_1, p).$$

Hence we get

$$\frac{d\rho_1}{d\varphi_1} = g(\varphi_1, p) = G(\rho_1, \varphi_1). \tag{4.6}$$

By applying the orthogonality condition

$$d\rho d\rho_1 + k^2 \sinh^2 \frac{\rho}{k} d\varphi d\varphi_1 = 0, \tag{4.7}$$

we get for the orthogonal lines

$$G(\rho, \varphi) d\rho + k^2 \sinh^2 \frac{\rho}{k} d\varphi = 0. \tag{4.8}$$

Now let us consider the distance along the line between points $P_1(0, b)$ and $P(x, y)$ represented in figure 4.

We denote the length of $\overline{PP_1}$ by d. The distance from the origin to P equals ρ. In figure 4 a right-angled triangle is formed, where the hypotenuse is equal to ρ and the other two sides are x and y. Equality (2.12) gives a formula considering these values.

Let P_3 be the foot of the perpendicular from P_1 to the line OP. Then two right-angled triangles, namely OP_3P_1 and P_1P_3P are formed.

In OP_3P_1 the length of the hypotenuse $\overline{OP_1}$ is denoted by b, while the legs $\overline{OP_3}$ and $\overline{P_1P_3}$ are denoted by ρ_0 and c. Therefore we can write

$$\cosh \tfrac{b}{k} = \cosh \tfrac{\rho_0}{k} \cosh \tfrac{c}{k}.$$

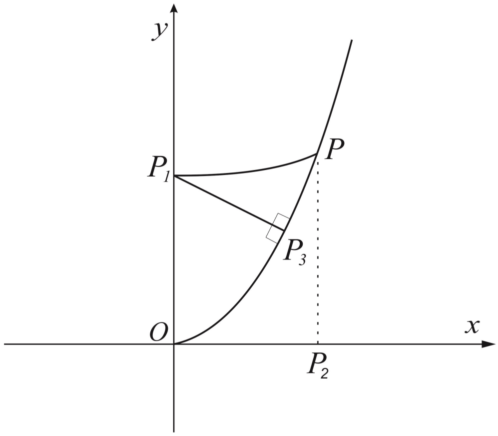

FIGURE 4

In triangle P_1P_3P the hypotenuse is d, while the value of the legs are $\rho - \rho_0$ and c, thus we get

$$\cosh\frac{d}{k} = \cosh\frac{\rho - \rho_0}{k}\cosh\frac{c}{k}.$$

If we eliminate c, we obtain

$$\cosh\frac{d}{k} = \cosh\frac{b}{k}\left(\cosh\frac{\rho}{k} - \tanh\frac{\rho_0}{k}\sinh\frac{\rho}{k}\right). \tag{4.9}$$

In triangle OP_3P_1 the angle between OP_1 and OP_3 is equal to $\frac{\pi}{2} - \varphi$. Hence, from (2.13) we can write

$$\tanh\frac{\rho_0}{k} = \sin\varphi\tanh\frac{b}{k}. \tag{4.10}$$

Applying (4.9) and (4.10), we get

$$\cosh\frac{d}{k} = \cosh\frac{b}{k}\cosh\frac{\rho}{k} - \sin\varphi\sinh\frac{b}{k}\sinh\frac{\rho}{k}. \tag{4.11}$$

If we use Cartesian coordinates, from (4.11), (2.12) and (2.11) it follows that

$$\cosh\frac{d}{k} = \cosh\frac{b}{k}\cosh\frac{x}{k}\cosh\frac{y}{k} - \sinh\frac{b}{k}\sinh\frac{y}{k}. \tag{4.12}$$

In the following sections we determine the orthogonal lines. During the analysis our choice of coordinates may vary depending on the form of calculations.

We will prove that the curvature of the orthogonal lines is constant. Moreover, any two orthogonal lines from the same family are parallel. We define parallelism in the following way: let us consider a geodesic and its two orthogonal lines passing through the line in two different points. If the distance between the two points of intersection is constant in the case of any geodesic, we say that the two orthogonal lines are parallel.

5. The orthogonal curves of the radial lines

Radial lines are lines which pass through a common point. We will consider four family of lines.

a) The first family consists of lines crossing the origin, which are determined by equation (3.1). Here we use

$$p = \tan\varphi$$

as parameter. In this case we can write

$$F(x,y) = \coth\frac{x}{k}\sinh\frac{y}{k}\cosh\frac{y}{k}.$$

Therefore, by using (4.5) we get

$$\frac{\cosh\frac{y}{k}}{\sinh\frac{x}{k}}d\left(\cosh\frac{x}{k}\cosh\frac{y}{k}\right) = 0.$$

The expression in bracket is constant. From equation (2.12) it follows that the points of an orthogonal curve are always at the same distance from the origin. Hence we obtain a circle with center O.

$$\cosh\frac{x}{k}\cosh\frac{y}{k} = \cosh\frac{R}{k}, \quad \rho = R, \tag{5.1}$$

where R denotes the radius of the circle.

If we use polar coordinates, from (5.1) and (2.2) it follows that

$$\frac{d\varphi}{ds} = \frac{1}{k\sinh\frac{R}{k}}.$$

From (2.9) we obtain the formula characterizing the curvature:

$$\frac{1}{r_g} = \frac{1}{k}\coth\frac{R}{k}. \tag{5.2}$$

b) Now we determine the orthogonal curves of lines crossing point $P_0(a,0) \in Ox$. As the translation of the origin along the direction of the x-axis into point P_0 is a symmetry operation, we obtain circles with center P_0, which verify

$$\cosh\frac{x-a}{k}\cosh\frac{y}{k} = \cosh\frac{R}{k}. \tag{5.3}$$

The curvature of the orthogonal lines is determined by formula (5.2).

If we use polar coordinates, from (5.3), (2.10) and (2.12) it follows that the circles verify

$$\cosh\frac{a}{k}\cosh\frac{\rho}{k} - \cos\varphi\sinh\frac{a}{k}\sinh\frac{\rho}{k} = \cosh\frac{R}{k}.$$

c) The lines parallel to the x-axis admit the

$$p = \tanh\frac{b}{k}$$

parameter and satisfy equation (3.10).

As

$$F = -\sinh\frac{y}{k}\cosh\frac{y}{k},$$

from (4.5) it follows that

$$d\left(\ln\cosh\frac{y}{k}-\frac{x}{k}\right)=0.$$

The orthogonal curve crossing point $P(x_0,0)$ is called paracycloid, which verifies

$$\cosh\frac{y}{k}=e^{\frac{x-x_0}{k}}. \tag{5.4}$$

We can also determine the equation of the paracycloid by the following way. If $a>x_0$, the circle with center $P_0(a,0)$ passing through $P(x_0,0)$ has the radius $R=a-x_0$. If $a\longrightarrow\infty$, from (5.3) we obtain formula (5.4). Thus the paracycloid can be considered a semicircle with infinite radius.

d) The orthogonal curves of lines passing through point $P_1(0,b)$ are lines which cross or do not cross the x-axis. By the use of polar coordinates we obtain

$$\coth\frac{\rho}{k}=\coth\frac{b}{k}\sin\varphi+p\cos\varphi. \tag{5.5}$$

Here the parameters are given by

$$\coth\frac{a}{b}\quad\text{and}\quad\tanh\frac{x_m}{k}.$$

The orthogonal lines admit the following formula:

$$G(\rho,\varphi)=-\frac{k}{cos\varphi}\sinh^2\frac{\rho}{k}\left(\coth\frac{b}{k}-\sin\varphi\coth\frac{\rho}{k}\right).$$

Using (4.8), we get

$$-\frac{1}{cos\varphi}\left(\coth\frac{b}{k}-\sin\varphi\coth\frac{\rho}{k}\right)d\rho+kd\varphi=$$

$$=-\frac{k}{\cos\varphi\sinh\frac{b}{k}\sinh\frac{\rho}{k}}d\left(\cosh\frac{b}{k}\cosh\frac{\rho}{k}-\sin\varphi\sinh\frac{b}{k}\sinh\frac{\rho}{k}\right)=0.$$

Hence, by using (4.11) it follows that the orthogonal lines are circles with center $P_1(0,b)$, which verify the following equations:

$$\cosh\frac{b}{k}\cosh\frac{\rho}{k}-\sin\varphi\sinh\frac{b}{k}\sinh\frac{\rho}{k}=\cosh\frac{R}{k}, \tag{5.6}$$

$$\cosh\frac{b}{k}\cosh\frac{x}{k}\cosh\frac{y}{k}-\sinh\frac{b}{k}\sinh\frac{y}{k}=\cosh\frac{R}{k}. \tag{5.7}$$

We obtain the curvature by considering (2.2) and using the formulas below:

$$\sin\varphi=f(\rho),\quad f(\rho)=\coth\frac{b}{k}\coth\frac{\rho}{k}-\frac{\cosh\frac{R}{k}}{\sinh\frac{b}{k}\sinh\frac{\rho}{k}}. \tag{5.8}$$

Hence we obtain

$$\frac{d\rho}{ds}=\frac{\sinh\frac{b}{k}}{\sinh\frac{R}{k}}\sqrt{1-f^2}. \tag{5.9}$$

From (5.8) we get

$$\cos\varphi d\varphi=\sqrt{1-f^2}d\varphi=\frac{df}{d\rho}d\rho.$$

Thus by using (5.9) we obtain

$$\frac{d\varphi}{ds} = \frac{\sinh\frac{b}{k}}{\sinh\frac{R}{k}}\frac{df}{d\rho}. \tag{5.10}$$

The derivatives of the second kind are as follows:

$$\frac{d^2\rho}{ds^2} = -\frac{\sinh^2\frac{b}{k}}{\sinh^2\frac{R}{k}} f\frac{df}{d\rho}, \quad \frac{d^2\varphi}{ds^2} = \frac{\sinh^2\frac{b}{k}}{\sinh^2\frac{R}{k}}\sqrt{1-f^2}\frac{d^2 f}{d\rho^2}.$$

Hence, from (2.9) and (5.5) it follows that the curvature is given by formula (5.2).

If we consider two circles, the distance between the intersections with a geodesic is equal to the difference of the two radiuses, which is a constant value. This proves that the orthogonal lines determined above are parallel.

6. The orthogonal curves of lines not having common point

We will consider two different cases.

a) The first family consists of lines being parallel to the y-axis. The orthogonal lines are called hipercycloids having

$$y = b, \tag{6.1}$$

where b is a constant value. In the upper half-plane $b > 0$, while below the x-axis $b < 0$. The hipercycloids satisfy the orthogonality condition (4.1), because in the case of the geodesic satisfying $dx_1 = 0$ its orthogonal curve verifies $dy = 0$.

From (2.2) and (6.1) we obtain

$$\frac{dx}{ds} = \frac{1}{\cosh\frac{b}{k}}. \tag{6.2}$$

Using (6.1), (6.2) and (1.8), we get for the curvature

$$\frac{1}{r_g} = \frac{1}{k}\tanh\frac{b}{k}.$$

b) The lines of the second family do not cross the x-axis and have minimum point on the y-axis. By using (3.8), these lines verify

$$\tanh\frac{y}{k} = p\cosh\frac{x}{k}, \quad p = \tanh\frac{b}{k}, \tag{6.3}$$

where b can be either positive or negative value.

In this case

$$F(x, y) = \tanh\frac{x}{k}\sinh\frac{y}{k}\cosh\frac{y}{k}.$$

From the (4.5) orthogonality condition we get

$$\tanh\frac{x}{k}\sinh\frac{y}{k}dy + \cosh\frac{y}{k}dx = \frac{k}{\cosh\frac{x}{k}}d\left(\sinh\frac{x}{k}\cosh\frac{y}{k}\right) = 0.$$

Thus the orthogonal curve passing through point $P(x_0, 0)$ verifies

$$\sinh\frac{x}{k}\cosh\frac{y}{k} = \sinh\frac{x_0}{k}. \tag{6.4}$$

Figure 5 illustrates the geodesics determined by b and $-b$, and their orthogonal curves passing through points $P(x_0, 0)$ and $P'(-x_0, 0)$.

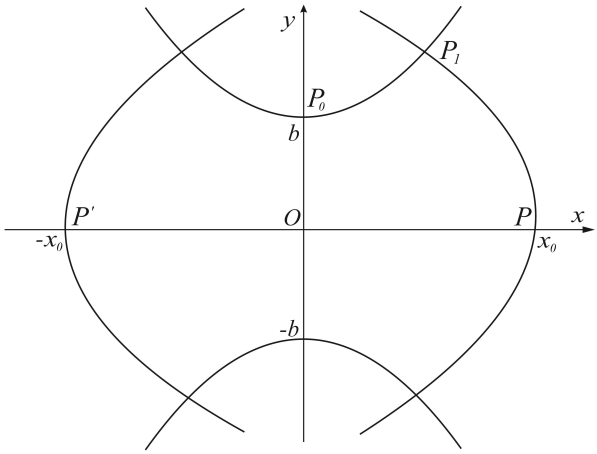

FIGURE 5

The tangent field of the orthogonal line admits

$$\tan\alpha = \frac{1}{\cosh\frac{y}{k}}\frac{dy}{dx} = -\sqrt{\coth^2\frac{x_0}{k}\coth^2\frac{y}{k} - 1}.$$

Thus as $y \longrightarrow \infty$, we obtain

$$\tan\alpha = -\frac{1}{\sinh\frac{x_0}{k}}.$$

We use polar coordinates in order to determine the curvature of the lines. From (6.4) and (2.10) we obtain

$$\cos\varphi\sinh\frac{\rho}{k} = \sinh\frac{x_0}{k}. \tag{6.5}$$

Also, from (2.2) and (6.5) it follows that

$$\frac{d\rho}{ds} = \sqrt{1 - \tanh^2\frac{x_0}{k}\coth^2\frac{\rho}{k}} = g(\rho).$$

Hence, by using equation (6.5) we obtain

$$\frac{d\varphi}{ds} = \frac{g(\rho)}{k}\cot\varphi\coth\frac{\rho}{k} = \frac{1}{k}\tanh\frac{x_0}{k}\frac{\cosh\frac{\rho}{k}}{\sinh^2\frac{\rho}{k}}.$$

The derivatives of the second kind are as follows:

$$\frac{d^2\rho}{ds^2} = \frac{1}{2}\frac{dg^2}{d\rho},$$

$$\frac{d^2\varphi}{ds^2} = \frac{\tanh\frac{x_0}{k}}{k^2 \sinh\frac{\rho}{k}} \left(1 - 2\coth^2\frac{\rho}{k}\right) g.$$

From (2.9) we obtain for the curvature the following formula:

$$\frac{1}{r_g} = \frac{1}{k} \tanh\frac{x_0}{k}. \tag{6.6}$$

The orthogonal curves are parallel to the y-axis. We will prove this by determining the distance between the points $P_0(0, b)$ and $P_1(x_1, y_1)$, illustrated on figure 5. Point P_1 is the intersection point of the geodesic and its orthogonal curve. The coordinates are given by the formulas

$$\tanh\frac{y_1}{k} = \tanh\frac{b}{k}\cosh\frac{x_1}{k}, \quad \sinh\frac{x_1}{k}\cosh\frac{y_1}{k} = \sinh\frac{x_0}{k}.$$

Hence we obtain

$$\sinh\frac{y_1}{k} = \sinh\frac{b}{k}\cosh\frac{x_0}{k}, \quad \cosh\frac{x_1}{k} = \frac{\tanh\frac{y_1}{k}}{\tanh\frac{b}{k}}.$$

From (4.12) we get

$$\cosh\frac{d}{k} = \cosh\frac{b}{k}\cosh\frac{x_1}{k}\cosh\frac{y_1}{k} - \sinh\frac{b}{k}\sinh\frac{y_1}{k} = \cosh\frac{x_0}{k},$$

thus

$$d = x_0.$$

The distance of two orthogonal lines is given by the difference of the values x_0, which proves the parallelism of the orthogonal curves.

These orthogonal lines are the duals of the hipercycloids, fact which is illustrated also by the curvatures determined above. This family of lines was not considered in the past.

Note that any orthogonal line can be described by formula (1.11). In the case of hipercycloids the value of x_2, while in the case of (6.4) the value of x_1 is constant. Along the circles (5.1), (5.3) and (5.7) the following values are constant:

$$\cosh\frac{a}{k}x_0 - \sinh\frac{a}{k}x_1, \quad \cosh\frac{b}{k}x_0 - \sinh\frac{b}{k}x_2.$$

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