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Helicoidal surfaces with $\Delta^J r = Ar$ in 3-dimensional Euclidean space

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Abstract. In this paper we study the helicoidal surfaces in the 3-dimensional Euclidean space under the condition $\Delta^J r = Ar$; $J = I, II, III$, where $A = (a_{ij})$ is a constant 3×3 matrix and Δ^J denotes the Laplace operator with respect to the fundamental forms I, II and III.

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1. Introduction

Let $r = r(u, v)$ be an isometric immersion of a surface M^2 in the Euclidean space $\mathbb{E}^3$.

The inner product on $\mathbb{E}^3$ is

$$g(X, Y) = x_1 y_1 + x_2 y_2 + x_3 y_3,$$

where $X = (x_1, x_2, x_3), Y = (y_1, y_2, y_3) \in \mathbb{R}^3$. The Euclidean vector product $X \wedge Y$ of X and Y is defined as follows:

$$X \wedge Y = \big(x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1\big).$$

The notion of finite type immersion of submanifolds of a Euclidean space has been widely used in classifying and characterizing well known Riemannian submanifolds [6]. B.-Y. Chen posed the problem of classifying the finite type surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$. An Euclidean submanifold is said to be of Chen finite type if its coordinate functions are a finite sum of eigenfunctions of its Laplacian Δ [6]. Further, the notion of finite type can be extended to any smooth functions on a submanifold of a Euclidean space or a pseudo-Euclidean space. Since then the theory of submanifolds of finite type has been studied by many geometers.

A well known result due to Takahashi [18] states that minimal surfaces and spheres are the only surfaces in $\mathbb{E}^3$ satisfying the condition

$$\Delta r = \lambda r, \ \ \lambda \in \mathbb{R}.$$

In [10] Ferrandez, Garay and Lucas proved that the surfaces of $\mathbb{E}^3$ satisfying

$$\Delta H = AH,\ A \in Mat(3,3)$$

are either minimal, or an open piece of sphere or of a right circulaire cylindre.

In [7] M. Choi and Y. H. Kim characterized the minimal helicoid in terms of pointwise 1-type Gauss map of the first kind. In [2] M. Bekkar and H. Zoubir classified the surfaces of revolution with non zero Gaussian curvature K_G in the 3-dimensional Lorentz-Minkowski space $\mathbb{E}^3_1$, whose component functions are eigenfunctions of their Laplace operator, i.e.

$$\Delta^{II} r_i = \lambda_i r_i,\ \lambda_i \in \mathbb{R}.$$

In [9] F. Dillen, J. Pas and L. Verstraelen proved that the only surfaces in $\mathbb{E}^3$ satisfying

$$\Delta r = Ar + B,\ A \in Mat(3,3),\ B \in Mat(3,1),$$

are the minimal surfaces, the spheres and the circular cylinders.

In [1] Ch. Baba-Hamed and M. Bekkar studied the helicoidal surfaces without parabolic points in $\mathbb{E}^3_1$, which satisfy the condition

$$\Delta^{II} r_i = \lambda_i r_i,$$

where Δ^{II} is the Laplace operator with respect to the second fundamental form.

In [13] G. Kaimakamis and B.J. Papantoniou classified the first three types of surfaces of revolution without parabolic points in the 3-dimensional Lorentz–Minkowski space, which satisfy the condition

$$\Delta^{II} r = Ar,\ A \in Mat(3,3).$$

We study helicoidal surfaces M^2 in $\mathbb{E}^3$ which are of finite type in the sense of B.-Y. Chen with respect to the fundamental forms I, II and III, i.e., their position vector field $r(u,v)$ satisfies the condition

$$\Delta^J r = Ar;\ J = I, II, III, \tag{1.1}$$

where $A = (a_{ij})$ is a constant 3×3 matrix and Δ^J denotes the Laplace operator with respect to the fundamental forms I, II and III. Then we shall reduce the geometric problem to a simpler ordinary differential equation system.

In [14] G. Kaimakamis, B.J. Papantoniou and K. Petoumenos classified and proved that such surfaces of revolution in the 3-dimensional Lorentz-Minkowski space $\mathbb{E}^3_1$ satisfying

$$\Delta^{III} \overrightarrow{r} = A\overrightarrow{r}$$

are either minimal or Lorentz hyperbolic cylinders or pseudospheres of real or imaginary radius, where Δ^{III} is the Laplace operator with respect to the third fundamental form. S. Stamatakis and H. Al-Zoubi in [17] classified the surfaces of revolution with non zero Gaussian curvature in $\mathbb{E}^3$ under the condition

$$\Delta^{III} r = Ar,\ A \in Mat(3, \mathbb{R}).$$

On the other hand, a helicoidal surface is well known as a kind of generalization of some ruled surfaces and surfaces of revolution in a Euclidean space $\mathbb{E}^3$ or a Minkowski space $\mathbb{E}^3_1$ ([5], [8], [12]).

2. Preliminaries

Let $\gamma : I \subset \mathbb{R} \to P$ be a plane curve in $\mathbb{E}^3$ and let l be a straight line in P which does not intersect the curve γ (axis). A helicoidal surface in $\mathbb{E}^3$ is a surface invariant by a uniparametric group $G_{L,c} = \{g_v / \ g_v : \mathbb{E}^3 \to \mathbb{E}^3;\ v \in \mathbb{R}\}$ of helicoidal motions. The motion g_v is called a helicoidal motion with axis l and pitch c. If we take $c = 0$, then we obtain a rotations group about the axis l.

A helicoidal surface in $\mathbb{E}^3$ which is spanned by the vector $(0,0,1)$ and with pitch $c \in \mathbb{R}^*$ as follows:

$$r(u,v) = \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ 0 \\ \varphi(u) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ cv \end{pmatrix}, c \in \mathbb{R}^*.$$

Next, we will use the parametrization of the profile curve γ as follows:

$$\gamma(u) = (u, 0, \varphi(u)).$$

Therefore, the surface M^2 may be parameterized by

$$r(u,v) = \big(u\cos v, u\sin v, \varphi(u) + cv\big) \tag{2.1}$$

in $\mathbb{E}^3$, where $(u,v) \in I \times [0, 2\pi]$, $c \in \mathbb{R}^*$.

A surface M^2 is said to be of finite type if each component of its position vector field r can be written as a finite sum of eigenfunctions of the Laplacian Δ of M^2, that is, if

$$r = r_0 + \sum_{i=1}^{k} r_i,$$

where r_i are $\mathbb{E}^3$ -valued eigenfunctions of the Laplacian of (M^2, r): $\Delta r_i = \lambda_i r_i$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, .., k$ [6]. If λ_i are different, then M^2 is said to be of k-type.
The coefficients of the first fundamental form and the second fundamental form are

$$\begin{aligned} E &= g_{11} = g(r_u, r_u),\ F = g_{12} = g(r_u, r_v),\ G = g_{22} = g(r_v, r_v); \\ L &= h_{11} = g(r_{uu}, \mathbf{N}),\ M = h_{12} = g(r_{uv}, \mathbf{N}),\ N = h_{22} = g(r_{vv}, \mathbf{N}), \end{aligned}$$

where $\mathbf{N}$ is the unit normal vector to M^2.

The Laplace-Beltrami operator of a smooth function

$$\varphi : M^2 \to \mathbb{R}, (u,v) \mapsto \varphi(u,v)$$

with respect to the first fundamental form of the surface M^2 is the operator Δ^I, defined in [15] as follows:

$$\Delta^I \varphi = \frac{-1}{\sqrt{|EG - F^2|}} \left[\frac{\partial}{\partial u} \left(\frac{G\varphi_u - F\varphi_v}{\sqrt{|EG - F^2|}} \right) - \frac{\partial}{\partial v} \left(\frac{F\varphi_u - E\varphi_v}{\sqrt{|EG - F^2|}} \right) \right]. \tag{2.2}$$

The second differential parameter of Beltrami of a function

$$\varphi : M^2 \to \mathbb{R}, (u,v) \longmapsto \varphi(u,v)$$

with respect to the second fundamental form of M^2 is the operator Δ^{II} which is defined by [15]

$$\Delta^{II}\varphi = \frac{-1}{\sqrt{|LN-M^2|}}\left[\frac{\partial}{\partial u}\left(\frac{N\varphi_u - M\varphi_v}{\sqrt{|LN-M^2|}}\right)+\frac{\partial}{\partial v}\left(\frac{L\varphi_v - M\varphi_u}{\sqrt{|LN-M^2|}}\right)\right], \quad (2.3)$$

where $LN - M^2 \neq 0$ since the surface has no parabolic points.

In the classical literature, one write the third fundamental form as

$$III = e_{11}du^2 + 2e_{12}dudv + e_{22}dv^2.$$

The second Beltrami differential operator with respect to the third fundamental form III is defined by

$$\Delta^{III} = \frac{-1}{\sqrt{|e|}}\left(\frac{\partial}{\partial x^i}\left(\sqrt{|e|}e^{ij}\frac{\partial}{\partial x^j}\right)\right), \quad (2.4)$$

where $e = \det(e_{ij})$ and e^{ij} denote the components of the inverse tensor of e_{ij}.

If $r = r(u,v) = (r_1 = r_1(u,v), r_2 = r_2(u,v), r_3 = r_3(u,v))$ is a function of class C^2 then we set

$$\Delta^J r = (\Delta^J r_1, \Delta^J r_2, \Delta^J r_3);\ J = I, II, III.$$

The mean curvature H and the Gauss curvature K_G are, respectively, defined by

$$H = \frac{1}{2(EG-F^2)}\big(EN + GL - 2FM\big)$$

and

$$K_G = \frac{LN - M^2}{EG - F^2}.$$

Suppose that M^2 is given by (2.1).

3. Helicoidal surfaces with $\Delta^I r = Ar$ in $\mathbb{E}^3$

The main result of this section states that the only helicoidal surfaces M^2 of $\mathbb{E}^3$ satisfying the condition

$$\Delta^I r = Ar \quad (3.1)$$

on the Laplacian are open pieces of helicoidal minimal surfaces.

The coefficients of the first and the second fundamental forms are:

$$E = 1 + \varphi'^2,\ F = c\varphi',\ G = c^2 + u^2; \quad (3.2)$$

$$L = \frac{u\varphi''}{W},\ M = -\frac{c}{W},\ N = \frac{u^2\varphi'}{W}, \quad (3.3)$$

where $W = \sqrt{EG - F^2} = \sqrt{u^2(1+\varphi'^2) + c^2}$ and the prime denotes derivative with respect to u.

The unit normal vector of M^2 is given by

$$\mathbf{N} = \frac{1}{W}(u\varphi'\cos v - c\sin v, c\cos v + u\varphi'\sin v, -u).$$

From these we find that the mean curvature H and the curvature K_G of (3.2) are given by

$$\begin{aligned} H &= \frac{1}{2W^3}\big(u^2\varphi'(1+\varphi'^2) + 2c^2\varphi' + u\varphi''(c^2+u^2)\big) \\ &= \frac{1}{2u}\left(\frac{u^2\varphi'}{W}\right)' \end{aligned}$$

and

$$K_G = \frac{1}{W^4}\big(u^3\varphi'\varphi'' - c^2\big). \tag{3.4}$$

If a surface M^2 in $\mathbb{E}^3$ has no parabolic points, then we have

$$u^3\varphi'\varphi'' - c^2 \neq 0.$$

The Laplacian Δ^I of M^2 can be expressed as follows:

$$\begin{aligned} \Delta^I &= -\frac{1}{W^2}((c^2+u^2)\frac{\partial^2}{\partial u^2} - 2c\varphi'\frac{\partial^2}{\partial u \partial v} + (1+\varphi'^2)\frac{\partial^2}{\partial v^2}) \\ &\quad -\frac{1}{W^4}(u^3(1+\varphi'^2) + c^2u(1-\varphi'^2) - u^2\varphi'\varphi''(c^2+u^2))\frac{\partial}{\partial u} \\ &\quad -\frac{1}{W^4}(-c\varphi''(c^2+u^2) + cu\varphi'(1+\varphi'^2))\frac{\partial}{\partial v}. \end{aligned}$$

Accordingly, we get

$$\Delta^I r = -2H\mathbf{N}. \tag{3.5}$$

The equation (3.1) by means of (3.2) and (3.5) gives rise to the following system of ordinary differential equations

$$(u\varphi'A(u) - a_{11}u)\cos v - (cA(u) + a_{12}u)\sin v = a_{13}(\varphi + cv) \tag{3.6}$$

$$(u\varphi'A(u) - a_{22}u)\sin v + (cA(u) - a_{21}u)\cos v = a_{23}(\varphi + cv) \tag{3.7}$$

$$-uA(u) = a_{31}u\cos v + a_{32}u\sin v + a_{33}(\varphi + cv), \tag{3.8}$$

where

$$A(u) = \frac{2H}{W}. \tag{3.9}$$

On differentiating (3.6), (3.7) and (3.8) twice with respect to v we have

$$a_{13} = a_{23} = a_{33} = 0,\ A(u) = 0. \tag{3.10}$$

From (3.10) we obtain

$$\begin{aligned} -a_{11}u\cos v - a_{12}u\sin v &= 0 \\ -a_{22}u\sin v - a_{21}u\cos v &= 0 \\ a_{31}u\cos v + a_{32}u\sin v &= 0. \end{aligned} \tag{3.11}$$

But cos and sin are linearly independent functions of v, so we finally obtain $a_{ij} = 0$. From (3.9) we obtain $H = 0$. Consequently M^2, being a minimal surface.

Theorem 3.1. *Let $r : M^2 \to \mathbb{E}^3$ be an isometric immersion given by (2.1). Then $\Delta^I r = Ar$ if and only if M^2 has zero mean curvature.*

4. Helicoidal surfaces with $\Delta^{II} r = Ar$ in $\mathbb{E}^3$

In this section we are concerned with non-degenerate helicoidal surfaces M^2 without parabolic points satisfying the condition

$$\Delta^{II} r = Ar. \tag{4.1}$$

By a straightforward computation, the Laplacian Δ^{II} of the second fundamental form II on M^2 with the help of (3.3) and (2.3) turns out to be

$$\begin{aligned}\Delta^{II} &= -\frac{W}{R}\left(u^2\varphi'\frac{\partial^2}{\partial u^2} + u\varphi''\frac{\partial^2}{\partial v^2} + 2c\frac{\partial^2}{\partial u \partial v}\right)\\ &\quad -\frac{W}{2R^2}u\big(-\varphi'(\varphi'\varphi''' - \varphi''^2)u^4 + \varphi'^2\varphi''u^3 - 2c^2\varphi''u - 4c^2\varphi'\big)\frac{\partial}{\partial u}\\ &\quad +\frac{W}{2R^2}cu^2\big((\varphi'\varphi''' + \varphi''^2)u + 3\varphi'\varphi''\big)\frac{\partial}{\partial v},\end{aligned}$$

where $R = u^3\varphi'\varphi'' - c^2$.

Accordingly, we get

$$\Delta^{II} r(u,v) = \begin{pmatrix} (u\varphi'\cos v - c\sin v)P(u) \\ (u\varphi'\sin v + c\cos v)P(u) \\ u\varphi'^2 P(u) - u^2 Q(u) \end{pmatrix}, \tag{4.2}$$

where

$$P(u) = \frac{W}{2R^2}\big((\varphi''^2 + \varphi'\varphi''')u^4 - \varphi'\varphi''u^3 + 4c^2\big) \tag{4.3}$$

$$Q(u) = \frac{W}{2R^2}\big(4\varphi'^2\varphi''^2u^3 - c^2(\varphi''^2 + \varphi'\varphi''')u - 7c^2\varphi'\varphi''\big). \tag{4.4}$$

Therefore, the problem of classifying the helicoidal surfaces M^2 given by (2.1) and satisfying (4.1) is reduced to the integration of this system of ordinary differential equations

$$\begin{aligned}(u\varphi'P(u) - a_{11}u)\cos v - (cP(u) + a_{12}u)\sin v &= a_{13}(\varphi + cv)\\ (u\varphi'P(u) - a_{22}u)\sin v + (cP(u) - a_{21}u)\cos v &= a_{23}(\varphi + cv)\\ u\varphi'^2P(u) - u^2Q(u) &= a_{31}u\cos v + a_{32}u\sin v + a_{33}(\varphi + cv).\end{aligned}$$

Remark 4.1. We observe that

$$c^2P(u) + u^3Q(u) = 2W. \tag{4.5}$$

But $\cos v$ and $\sin v$ are linearly independent functions of v, so we finally obtain

$$a_{32} = a_{31} = a_{33} = a_{13} = a_{23} = 0.$$

We put $a_{11} = a_{22} = \alpha$ and $a_{21} = -a_{12} = \beta$, $\alpha, \beta \in \mathbb{R}$. Therefore, this system of equations is equivalently reduced to

$$\begin{cases} \varphi'P(u) = \alpha \\ cP(u) = \beta u \\ \varphi'^2P(u) - uQ(u) = 0. \end{cases} \tag{4.6}$$

Now, let us examine the system of equations (4.6) according to the values of the constants α and β.

Case 1. Let $\alpha = 0$ and $\beta \neq 0$.

In this case the system (4.6) is reduced equivalently to

$$\begin{cases} \varphi' = 0 \\ cP(u) = -\beta u \\ Q(u) = 0. \end{cases} \tag{4.7}$$

From (4.7) we have $P''(u) = 0$. From (4.5) and the fact that $c \neq 0$ we have a contradiction. Hence there are no helicoidal surfaces of $\mathbb{E}^3$ in this case which satisfy (4.1).

Case 2. Let $\alpha \neq 0$ and $\beta = 0$.

In this case the system (4.6) is reduced equivalently to

$$\begin{cases} \varphi' P(u) = \alpha \\ P(u) = 0. \end{cases}$$

But this is not possible. So, in this case there are no helicoidal surfaces of $\mathbb{E}^3$.

Case 3. Let $\alpha = \beta = 0$.

In this case the system (4.6) is reduced equivalently to

$$\begin{cases} P(u) = 0 \\ Q(u) = 0. \end{cases}$$

From (4.5) we have $W = 0$, which is a contradiction. Consequently, there are no helicoidal surfaces of $\mathbb{E}^3$ in this case.

Case 4. Let $\alpha \neq 0$ and $\beta \neq 0$.

In this case the system (4.6) is reduced equivalently to

$$\varphi(u) = \frac{\alpha c}{\beta} \ln(u) + k, \quad k \in \mathbb{R}. \tag{4.8}$$

By using (4.6) and (4.8), we obtain

$$\begin{cases} P(u) = \frac{\beta}{c} u \\ Q(u) = \frac{\alpha^2 c}{\beta u^2}. \end{cases} \tag{4.9}$$

Substituting (4.9) into (4.5), we get

$$\frac{c^2(\alpha^2+\beta^2)^2}{\beta^2} u^2 = 4(u^2 + \frac{c^2\alpha^2}{\beta^2} + c^2).$$

Then

$$\begin{cases} c^2(\alpha^2+\beta^2) = 0 \\ c^2(\alpha^2+\beta^2)^2 = 4\beta^2. \end{cases}$$

From the first equation we have $\alpha = \beta = 0$, which is a contradiction. Hence, there are no helicoidal surfaces of $\mathbb{E}^3$ in this case.

Consequently, we have:

Theorem 4.2. *Let $r : M^2 \to \mathbb{E}^3$ be an isometric immersion given by (2.1). There are no helicoidal surfaces in $\mathbb{E}^3$ without parabolic points, satisfying the condition $\Delta^{II} r = Ar$.*

Theorem 4.3. *If* $K_G = a \in \mathbb{R}\backslash\{0\}$, *then*

$$\Delta^{II} r(u,v) = -2N. \tag{4.10}$$

Proof. If $K_G = a \in \mathbb{R}\backslash\{0\}$, then $\frac{\partial K_G}{\partial u} = 0$.

From (3.4) we obtain

$$\begin{aligned}-\varphi'\varphi''u^4 + \varphi''^2u^5 + 7c^2\varphi'\varphi''u^2 + c^2\varphi''^2u^3 - 3\varphi'^2\varphi''^2u^5 + 4c^2u - \varphi'^3\varphi''u^4 + 4c^2\varphi'^2u \\ = -(\varphi'\varphi''' + \varphi'^3\varphi''')u^5 - c^2\varphi'\varphi'''u^3 \end{aligned} \tag{4.11}$$

By using (4.3), (4.4) and (4.11) we get

$$\begin{aligned} uP(u) - u^2Q(u) &= \frac{W}{2R^2}\big(\varphi'^3\varphi''u^4 - 4c^2\varphi'^2u - \varphi'^3\varphi'''u^5 - \varphi'^2\varphi''^2u^5\big) \\ &= -\varphi'^2uP(u). \end{aligned} \tag{4.12}$$

From (4.2) and (4.12) we deduce that

$$\Delta^{II} r(u,v) = WP(u)N. \tag{4.13}$$

From (4.5) and (4.12) we have that

$$P(u) = \frac{2}{W}. \tag{4.14}$$

By using (4.13) and (4.14) we get (4.10). □

5. Helicoidal surfaces with $\Delta^{III} r = Ar$ in $\mathbb{E}^3$

In this section we are concerned with non-degenerate helicoidal surfaces M^2 without parabolic points satisfying the condition

$$\Delta^{III} r = Ar. \tag{5.1}$$

The components of the third fundamental form of the surface M^2 is given by

$$\begin{aligned} e_{11} &= \frac{1}{W^4}(c^2(\varphi' + u\varphi'')^2 + c^2 + u^4\varphi''^2), \\ e_{12} &= -\frac{c}{W^2}(\varphi' + u\varphi''), \\ e_{22} &= \frac{1}{W^2}(c^2 + u^2\varphi'^2), \end{aligned} \tag{5.2}$$

hence

$$e = \frac{1}{W^6}(u^3\varphi'\varphi'' - c^2)^2.$$

The Laplacian of M^2 can be expressed as follows:

$$\begin{aligned} \Delta^{III} = & -\frac{1}{\sqrt{|e|}}\Big(W\big(\frac{c^2 + u^2\varphi'^2}{c^2 - u^3\varphi'\varphi''}\big)\frac{\partial^2}{\partial u^2} + 2cW\big(\frac{\varphi' + u\varphi''}{c^2 - u^3\varphi'\varphi''}\big)\frac{\partial^2}{\partial u \partial v} + \\ & \frac{1}{W}\big(\frac{c^2(\varphi' + u\varphi'')^2 + c^2 + u^4\varphi''}{c^2 - u^3\varphi'\varphi''}\big)\frac{\partial^2}{\partial v^2} + \frac{d}{du}W\big(\frac{c^2 + u^2\varphi'^2}{c^2 - u^3\varphi'\varphi''}\big)\frac{\partial}{\partial u} \\ & + c\frac{d}{du}W\big(\frac{\varphi' + u\varphi''}{c^2 - u^3\varphi'\varphi''}\big)\frac{\partial}{\partial v}\Big). \end{aligned} \tag{5.3}$$

By using (5.1) and (5.3) we get

$$\begin{cases} \Delta^{III}(u\cos v) = -u\varphi' Q(u)\cos v - cQ(u)\sin v \\ \Delta^{III}(u\sin v) = cQ(u)\cos v - u\varphi' Q(u)\sin v \\ \Delta^{III}(\varphi(u)+cv) = P(u). \end{cases}$$

Hence

$$\Delta^{III} r(u,v) = \begin{pmatrix} -u\varphi' Q(u)\cos v - cQ(u)\sin v \\ cQ(u)\cos v - u\varphi' Q(u)\sin v \\ P(u) \end{pmatrix}, \tag{5.4}$$

where

$$\begin{aligned} Q(u) &= \frac{W^2}{(c^2 - u^3\varphi'\varphi'')^3}\big(W^2u^2(c^2+u^2\varphi'^2)\varphi''' + 3c^2u^2\varphi' + 3c^2u^2\varphi'^3 \\ &\quad +7c^2u^3\varphi'^2\varphi'' + 5c^2u^3\varphi'' + c^2u^4\varphi'\varphi''^2 + 4c^4u\varphi'' - u^6\varphi'\varphi''^2 \\ &\quad +u^7\varphi''^3 + c^2u^5\varphi''^3 + 2c^4\varphi' - 2u^6\varphi'^3\varphi''^2\big), \end{aligned} \tag{5.5}$$

$$\begin{aligned} P(u) &= \frac{-W^2}{(c^2 - u^3\varphi'\varphi'')^3}\big(W^2u(c^2+u^2\varphi'^2)^2\varphi''' + 4c^4u^2\varphi'' + \\ &\quad 7c^2u^4\varphi'^2\varphi'' - 2u^7\varphi'^3\varphi''^2 + 3c^6\varphi'' + 15c^4u^2\varphi'^2\varphi'' \\ &\quad -3c^2u^5\varphi'^3\varphi''^2 + 9c^2u^4\varphi'^4\varphi'' - 3u^7\varphi'^5\varphi''^2 + 2c^4u\varphi' \\ &\quad +4c^4u\varphi'^3 + 3c^2u^3\varphi'^3 + 3c^2u^3\varphi'^5 \\ &\quad +3c^4u^3\varphi'\varphi''^2 + c^2u^5\varphi'\varphi''^2 + c^2u^6\varphi''^3 + c^4u^4\varphi''^3\big). \end{aligned} \tag{5.6}$$

From (5.5) and (5.6) we have

$$\begin{aligned} P(u) &= \frac{-u}{W}L(u) - \left(\frac{c^2+u^2\varphi'^2}{u^3\varphi'\varphi''-c^2}\right) W L'(u) \\ Q(u) &= \frac{-1}{W}L(u) + \left(\frac{u}{u^3\varphi'\varphi''-c^2}\right) W L'(u), \end{aligned} \tag{5.7}$$

where $L(u) = h_{11}e^{11} + 2h_{12}e^{12} + h_{22}e^{22} = \frac{2H}{K_G}$.

Remark 5.1. We observe that

$$uP(u) + (c^2+u^2\varphi'^2)Q(u) = -W\left(\frac{2H}{K_G}\right). \tag{5.8}$$

The equation (5.1) by means of (2.1) and (5.4) gives rise to the following system of ordinary differential equations

$$\begin{cases} -u\varphi' Q(u)\cos v - cQ(u)\sin v = a_{11}u\cos v + a_{12}u\sin v + a_{13}(\varphi+cv) \\ cQ(u)\cos v - u\varphi' Q(u)\sin v = a_{21}u\cos v + a_{22}u\sin v + a_{23}(\varphi+cv) \\ P(u) = a_{31}u\cos v + a_{32}u\sin v + a_{33}(\varphi+cv). \end{cases}$$

But $\cos v$ and $\sin v$ are linearly independent functions of v, so we finally obtain

$$a_{32} = a_{31} = a_{33} = a_{13} = a_{23} = 0.$$

We put $-a_{11} = -a_{22} = \lambda_1$ and $a_{21} = -a_{12} = \lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$. Therefore, this system of equations is equivalently reduced to

$$\begin{cases} \varphi' Q(u) = \lambda_1 \\ cQ(u) = \lambda_2 u \\ P(u) = 0. \end{cases} \tag{5.9}$$

Therefore, the problem of classifying the surfaces M^2 given by (2.1) and satisfying (5.1) is reduced to the integration of this system of ordinary differential equations.

Case 1. Let $\lambda_1 = 0$ and $\lambda_2 \neq 0$.

In this case the system (5.9) is reduced equivalently to

$$\begin{cases} \varphi' Q(u) = 0 \\ cQ(u) = \lambda_2 u \\ P(u) = 0. \end{cases} \tag{5.10}$$

Differentiating (5.10), we obtain $P''(u) = 0$, which is a contradiction. Hence there are no helicoidal surfaces of $\mathbb{E}^3$ in this case which satisfy (5.1).

Case 2. Let $\lambda_1 \neq 0$ and $\lambda_2 = 0$.

In this case the system (5.9) is reduced equivalently to

$$\begin{cases} \varphi' Q(u) = \lambda_1 \\ cQ(u) = 0 \\ P(u) = 0. \end{cases}$$

But this is not possible. So, in this case there are no helicoidal surfaces of $\mathbb{E}^3$.

Case 3. Let $\lambda_1 = \lambda_2 = 0$.

In this case the system (5.9) is reduced equivalently to

$$\begin{cases} \varphi' Q(u) = 0 \\ Q(u) = 0. \end{cases}$$

From (5.8) we have $H = 0$. Consequently M^2, being a minimal surface.

Case 4. Let $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.

In this case the system (5.9) is reduced equivalently to

$$\varphi(u) = \frac{\lambda_1 c}{\lambda_2} \ln(u) + a, \quad a \in \mathbb{R}. \tag{5.11}$$

If we substitute (5.11) in (5.5) we get $Q(u) = 0$. So we have a contradiction and therefore, in this case there are no helicoidal surfaces of $\mathbb{E}^3$.

Consequently, we have:

Theorem 5.2. *Let $r : M^2 \to \mathbb{E}^3$ be an isometric immersion given by (2.1). Then $\Delta^{III} r = Ar$ if and only if M^2 has zero mean curvature.*

Theorem 5.3. *If $\frac{2H}{K_G} = \alpha \in \mathbb{R} \backslash \{0\}$, then*

$$\Delta^{III} r(x, y) = -\frac{2H}{K_G} N.$$

Proof. From (5.7) we have

$$P(u) = uQ(u). \tag{5.12}$$

Finally, (5.12) and (5.4) give

$$\begin{aligned}\Delta^{III} r(u,v) &= Q(u)(-u\varphi'\cos v + c\sin v, -c\cos v - u\varphi'\sin v, u)\\ &= \frac{-1}{W}(\frac{2H}{K_G})(-u\varphi'\cos v + c\sin v, -c\cos v - u\varphi'\sin v, u)\\ &= -\frac{2H}{K_G}N.\end{aligned}$$

□

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