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On a functional differential inclusion

Aurelian Cernea

Abstract. We consider a Cauchy problem associated to a nonconvex functional differential inclusion and we prove a Filippov type existence result. This result allows to obtain a relaxation theorem for the problem considered.

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1. Introduction

In this note we study functional differential inclusions of the form

$$x'(t) \in F(t, x(t), x(\lambda t)), \quad x(0) = x_0, \tag{1.1}$$

where $F(.,.,.) : [0, T] \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map with non-empty values, $\lambda \in (0, 1)$ and $x_0 \in \mathbf{R}$. The present note is motivated by a recent paper [5], where it was studied problem (1.1) with F single valued and several results were obtained using fixed point techniques: existence, uniqueness and differentiability with respect with the delay of the solutions. The study in [5] contains, as a particular case, the problem

$$x'(t) = -ax(t) + a\lambda x(\lambda t), \quad x(0) = x_0,$$

which appears in the radioactive propagation theory ([2]).

The aim of this note is to consider the multivalued framework and to show that Filippov's ideas ([3]) can be suitably adapted in order to obtain the existence of solutions of problem (1.1). We recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values Filippov's theorem ([3]) consists in proving the existence of a solution starting from a given "quasi" solution. Moreover, the result provides an estimate between the starting "quasi" solution and the solution of the differential inclusion.

As an application of our main result we obtain a relaxation theorem for the problem considered. Namely, we prove that the solution set of the problem (1.1) is dense in the set of the relaxed solutions; i.e. the set of solutions of the differential inclusion whose right hand side is the convex hull of the original set-valued map.

The paper is organized as follows: in Section 2 we briefly recall some preliminary results that we will use in the sequel and in Section 3 we prove the main results of the paper.

2. Preliminaries

In this section we sum up some basic facts that we are going to use later.

Let (X,d) be a metric space. The Pompeiu-Hausdorff distance of the closed subsets $A,B\subset X$ is defined by

$$d_H(A,B)=\max\{d^*(A,B),d^*(B,A)\},\ \ d^*(A,B)=\sup\{d(a,B);\, a\in A\},$$

where $d(x,B)=\inf\{d(x,y); y\in B\}$. Let $T>0,\ I:=[0,T]$ and denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I. Denote by $\mathcal{P}(\mathbf{R})$ the family of all nonempty subsets of $\mathbf{R}$ and by $\mathcal{B}(\mathbf{R})$ the family of all Borel subsets of $\mathbf{R}$. For any subset $A\subset\mathbf{R}$ we denote by $\mathrm{cl}A$ the closure of A and by $\overline{\mathrm{co}}(A)$ the closed convex hull of A.

As usual, we denote by $C(I,\mathbf{R})$ the Banach space of all continuous functions $x(.):I\to\mathbf{R}$ endowed with the norm

$$|x|_C=\sup_{t\in I}|x(t)|$$

and by $L^1(I,\mathbf{R})$ the Banach space of all integrable functions $x(.):I\to\mathbf{R}$ endowed with the norm

$$|x|_1=\int_0^T|x(t)|\mathrm{d}t.$$

The Banach space of all absolutely continuous functions $x(.):I\to\mathbf{R}$ will be denoted by $AC(I,\mathbf{R})$. We recall that for a set-valued map $U:I\to\mathcal{P}(\mathbf{R})$ the Aumann integral of U, denoted by $\int_I U(t)\mathrm{d}t$, is the set

$$\int_I U(t)\mathrm{d}t=\left\{\int_I u(t)\mathrm{d}t;\ u(.)\in L^1(I,\mathbf{R}),\ u(t)\in U(t)\, a.e.\,(I)\right\}.$$

We recall two results that we are going to use in the next section. The first one is a selection result (e.g., [1]) which is a version of the celebrated Kuratowski and Ryll-Nardzewski selection theorem. The proof of the second one may be found in [4].

Lemma 2.1. *Consider X a separable Banach space, B is the closed unit ball in X, $H:I\to\mathcal{P}(X)$ is a set-valued map with nonempty closed values and $g:I\to X, L:I\to\mathbf{R}_+$ are measurable functions. If*

$$H(t)\cap(g(t)+L(t)B)\neq\emptyset\quad a.e.(I),$$

then the set-valued map $t\to H(t)\cap(g(t)+L(t)B)$ has a measurable selection.

Lemma 2.2. *Let $U:I\to\mathcal{P}(\mathbf{R})$ be a measurable set-valued map with closed nonempty images and having at least one integrable selection. Then*

$$cl\left(\int_0^T\overline{co}U(t)\,dt\right)=cl\left(\int_0^T U(t)\,dt\right).$$

3. The main results

In what follows we assume the following hypotheses.

Hypothesis. i) $F(.,.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R} \times \mathbf{R})$ measurable.

ii) There exist $l_1(.), l_2(.) \in L^1(I, \mathbf{R}_+)$ such that, for almost all $t \in I$,

$$\mathrm{d}_H(F(t, x_1, y_1), F(t, x_2, y_2)) \leq l_1(t)|x_1 - x_2| + l_2(t)|y_1 - y_2| \; \forall x_1, x_2, y_1, y_2 \in \mathbf{R}.$$

Theorem 3.1. *Assume that Hypothesis is satisfied and* $|l_1|_1 + |l_2|_1 < 1$. *Let* $y(.) \in AC(I, \mathbf{R})$ *be such that there exists* $p(.) \in L^1(I, \mathbf{R}_+)$ *verifying*

$$d(y(t), F(t, y(t), y(\lambda t))) \leq p(t) \text{ a.e. } (I).$$

Then there exists $x(.)$ *a solution of problem (1.1) satisfying for all* $t \in I$

$$|x - y|_C \leq \frac{1}{1 - (|l_1|_1 + |l_2|_1)}(|x_0 - y(0)| + |p|_1). \tag{3.1}$$

Proof. We set $x_0(.) = y(.)$, $f_0(.) = y'(.)$. It follows from Lemma 2.1 and Hypothesis that there exists a measurable function $f_1(.)$ such that $f_1(t) \in F(t, x_0(t), x_0(\lambda t))$ a.e. (I) and, for almost all $t \in I$, $|f_1(t) - y'(t)| \leq p(t)$. Define

$$x_1(t) = x_0 + \int_0^t f_1(s)ds$$

and one has

$$|x_1(t) - y(t)| \leq |x_0 - y(0)| + \int_0^t p(s)ds \leq |x_0 - y(0)| + |p|_1.$$

Thus $|x_1 - y|_C \leq |x_0 - y(0)| + |p|_1$.

From Lemma 2.1 and Hypothesis we deduce the existence of a measurable function $f_2(.)$ such that $f_2(t) \in F(t, x_1(t), x_1(\lambda t))$ a.e. (I) and for almost all $t \in I$

$$|f_1(t) - f_2(t)| \leq d(f_1(t), F(t, x_1(t), x_1(\lambda t))) \leq d_H(F(t, x_0(t), x_0(\lambda t)),$$
$$F(t, x_1(t), x_1(\lambda t))) \leq l_1(t)|x_1(t) - x_2(t)| + l_2(t)|x_1(\lambda t) - x_2(\lambda t)|.$$

Define

$$x_2(t) = x_0 + \int_0^t f_2(s)ds$$

and one has

$$|x_1(t) - x_2(t)| \leq \int_0^t |f_1(s) - f_2(s)|ds$$
$$\leq \int_0^t [l_1(s)|x_1(s) - x_2(s)| + l_2(s)|x_1(\lambda s) - x_2(\lambda s)|]ds$$
$$\leq (|l_1|_1 + |l_2|_1)|x_1 - x_2|_C \leq (|l_1|_1 + |l_2|_1)(|x_0 - y(0)| + |p|_1).$$

Assume that for some $p \geq 1$ we have constructed $(x_i)_{i=1}^p$ with x_p satisfying

$$|x_p - x_{p-1}|_C \leq (|l_1|_1 + |l_2|_1)^p(|x_0 - y(0)| + |p|_1).$$

Using Lemma 2.1 and Hypothesis we deduce the existence of a measurable function $f_{p+1}(.)$ such that $f_{p+1}(t) \in F(t, x_p(t), x_p(\lambda t))$ a.e. (I) and for almost all $t \in I$

$$|f_{p+1}(t) - f_p(t)| \leq d(f_{p+1}(t), F(t, x_{p-1}(t), x_{p-1}(\lambda t)))$$

$$\leq d_H(F(t, x_p(t), x_p(\lambda t)), F(t, x_{p-1}(t), x_{p-1}(\lambda t)))$$

$$\leq l_1(t)|x_p(t) - x_{p-1}(t)| + l_2(t)|x_p(\lambda t) - x_{p-1}(\lambda t)|.$$

Define

$$x_{p+1}(t) = x_0 + \int_0^t f_{p+1}(s)ds. \tag{3.2}$$

We have

$$|x_{p+1}(t) - x_p(t)| \leq \int_0^t |f_{p+1}(s) - f_p(s)|ds$$

$$\leq \int_0^t [l_1(s)|x_p(s) - x_{p-1}(s)| + l_2(s)|x_p(\lambda s) - x_{p-1}(\lambda s)|]ds$$

$$\leq (|l_1|_1 + |l_2|_1)|x_p - x_{p-1}|_C \leq (|l_1|_1 + |l_2|_1)^p(|x_0 - y(0)| + |p|_1).$$

Therefore $(x_p(.))_{p \geq 0}$ is a Cauchy sequence in the Banach space $C(I, \mathbf{R})$, so it converges to $x(.) \in C(I, \mathbf{R})$. Since, for almost all $t \in I$, we have

$$|f_{p+1}(t) - f_p(t)| \leq l_1(t)|x_p(t) - x_{p-1}(t)| + l_2(t)|x_p(\lambda t) - x_{p-1}(\lambda t)|$$

$$\leq [l_1(t) + l_2(t)]|x_p - x_{p-1}|_C,$$

$\{f_p(.)\}$ is a Cauchy sequence in the Banach space $L^1(I, \mathbf{R})$ and thus it converges to $f(.) \in L^1(I, \mathbf{R})$. Passing to the limit in (3.2) and using Lebesgue's dominated convergence theorem we get $x(t) = x_0 + \int_0^t f(s)ds$, which shows, in particular, that $x(.)$ is absolutely continuous.

Moreover, since the values of $F(., ., .)$ are closed and $f_{p+1}(t) \in F(t, x_p(t), x_p(\lambda t))$ passing to the limit we obtain $f(t) \in F(t, x(t), x(\lambda t))$ a.e. (I).

It remains to prove the estimate (3.2). One has

$$|x_p - x_0|_C \leq |x_p - x_{p-1}|_C + ... + |x_2 - x_1|_C + |x_1 - x_0|_C$$

$$\leq (|l_1|_1 + |l_2|_1)^p(|x_0 - y(0)| + |p|_1) + ... + (|l_1|_1 + |l_2|_1)(|x_0 - y(0)| + |p|_1) + (|x_0 - y(0)| + |p|_1)$$

$$\leq \frac{1}{1 - (|l_1|_1 + |l_2|_1)}(|x_0 - y(0)| + |p|_1).$$

Passage to the limit in the last inequality completes the proof. □

Remark 3.2. a) If we consider the space $C(I, \mathbf{R})$ endowed with a Bielecki type norm of the form $|x|_B = \sup_{t \in I} e^{-at}|x(t)|$ with an appropriate choice of $a \in \mathbf{R}$, the condition $|l_1|_1 + |l_2|_1 < 1$ can be removed from the assumptions of Theorem 3.1.

b) The statement in Theorem 3.1 remains valid for the more general problem

$$x'(t) \in F(t, x(t), x(g(t))), \quad x(0) = x_0,$$

with $g(.) : I \to I$ a continuous function.

As we already pointed out, Theorem 3.1 allows to obtain a relaxation theorem for problem (1.1). In what follows, we are concerned also with the convexified (relaxed) problem

$$x'(t) \in \overline{\mathrm{co}}F(t, x(t), x(\lambda t)), \quad x(0) = x_0. \tag{3.3}$$

Note that if $F(.,.,.)$ satisfies Hypothesis, then so does the set-valued map

$$(t, x, y) \to \overline{\mathrm{co}}F(t, x, y).$$

Theorem 3.3. *We assume that Hypothesis is satisfied and* $|l_1|_1 + |l_2|_1 < 1$. *Let* $\overline{x}(.) : I \to \mathbf{R}$ *be a solution to the relaxed inclusion (3.3) such that the set-valued map* $t \to F(t, \overline{x}(t), \overline{x}(\lambda t))$ *has at least one integrable selection.*

Then for every $\varepsilon > 0$ *there exists* $x(.)$ *a solution of problem (1.1) such that*

$$|x - \overline{x}|_C < \varepsilon.$$

Proof. Since $\overline{x}(.)$ is a solution of the relaxed inclusion (3.3), there exists $\overline{f}(.) \in L^1(I, \mathbf{R})$, $\overline{f}(t) \in \overline{\mathrm{co}}F(t, \overline{x}(t), \overline{x}(\lambda t))$ a.e. (I) such that

$$\overline{x}(t) = x_0 + \int_0^t \overline{f}(s)ds.$$

From Lemma 2.2, for $\delta > 0$, there exists $\tilde{f}(t) \in F(t, \overline{x}(t), \overline{x}(\lambda t))$ a.e. (I) such that

$$\sup_{t \in I} \left| \int_0^t (\tilde{f}(s) - \overline{f}(s))ds \right| \leq \delta.$$

Define

$$\tilde{x}(t) = x_0 + \int_0^t \tilde{f}(s)ds.$$

Therefore, $|\tilde{x} - \overline{x}|_C \leq \delta$.

We apply, now, Theorem 3.1 for the "quasi" solution $\tilde{x}(.)$ of (1.1). One has

$$p(t) = d(\tilde{f}(t), F(t, \tilde{x}(t), \tilde{x}(\lambda t))) \leq d_H(F(t, \overline{x}(t), \overline{x}(\lambda t)),$$

$$F(t, \tilde{x}(t), \tilde{x}(\lambda t))) \leq l_1(t)|\overline{x}(t) - \tilde{x}(t)| + l_2(t)|\overline{x}(\lambda t) - \tilde{x}(\lambda t)|$$

$$\leq l_1(t)|\tilde{x} - \overline{x}|_C + l_2(t)|\tilde{x} - \overline{x}|_C \leq (l_1(t) + l_2(t))\delta,$$

which shows that $p(.) \in L^1(I, \mathbf{R})$.

From Theorem 3.1 there exists $x(.)$ a solution of (1.1) such that

$$|x - \tilde{x}|_C \leq \frac{1}{1 - (|l_1|_1 + |l_2|_1)}|p|_1 \leq \frac{|l_1|_1 + |l_2|_1}{1 - (|l_1|_1 + |l_2|_1)}\delta.$$

It remains to take $\delta = [1 - (|l_1|_1 + |l_2|_1)]\varepsilon$ and to deduce that

$$|x - \overline{x}|_C \leq |x - \tilde{x}|_C + |\tilde{x} - \overline{x}|_C \leq \varepsilon.$$ □

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Aurelian Cernea
University of Bucharest
Faculty of Mathematics and Computer Sciences
14, Academiei Street, 010014 Bucharest, Romania
e-mail: acernea@fmi.unibuc.ro