Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 3, 421–430

Some extensions of the Open Door Lemma

Ming Li and Toshiyuki Sugawa

Abstract. Miller and Mocanu proved in their 1997 paper a greatly useful result which is now known as the Open Door Lemma. It provides a sufficient condition for an analytic function on the unit disk to have positive real part. Kuroki and Owa modified the lemma when the initial point is non-real. In the present note, by extending their methods, we give a sufficient condition for an analytic function on the unit disk to take its values in a given sector.

Mathematics Subject Classification (2010): 30C45, 30C80.

Keywords: Open door function, subordination.

1. Introduction

We denote by $\mathcal{H}$ the class of holomorphic functions on the unit disk

$$\mathbb{D} = \{z : |z| < 1\}$$

of the complex plane $\mathbb{C}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a,n]$ denote the subclass of $\mathcal{H}$ consisting of functions h of the form $h(z) = a + c_n z^n + c_{n+1} z^{n+1} + \cdots$. Here, $\mathbb{N} = \{1, 2, 3, \dots\}$. Let also $\mathcal{A}_n$ be the set of functions f of the form $f(z) = zh(z)$ for $h \in \mathcal{H}[1,n]$.

A function $f \in \mathcal{A}_1$ is called *starlike* (resp. *convex*) if f is univalent on $\mathbb{D}$ and if the image $f(\mathbb{D})$ is starlike with respect to the origin (resp. convex). It is well known (cf. [1]) that $f \in \mathcal{A}_1$ is starlike precisely if $q_f(z) = zf'(z)/f(z)$ has positive real part on $|z| < 1$, and that $f \in \mathcal{A}_1$ is convex precisely if $\varphi_f(z) = 1+zf''(z)/f'(z)$ has positive real part on $|z| < 1$. Note that the following relation holds for those quantities:

$$\varphi_f(z) = q_f(z) + \frac{zq_f'(z)}{q_f(z)}.$$

It is geometrically obvious that a convex function is starlike. This, in turn, means the implication

$$\mathrm{Re}\left[q(z) + \frac{zq'(z)}{q(z)}\right] > 0 \text{ on } |z| < 1 \quad \Rightarrow \quad \mathrm{Re}\, q(z) > 0 \text{ on } |z| < 1$$

The authors were supported in part by JSPS Grant-in-Aid for Scientific Research (B) 22340025.

for a function $q \in \mathcal{H}[1,1]$. Interestingly, it looks highly nontrivial. Miller and Mocanu developed a theory (now called *differential subordination*) which enables us to deduce such a result systematically. See a monograph [4] written by them for details.

The set of functions $q \in \mathcal{H}[1,1]$ with $\operatorname{Re} q > 0$ is called the Carathéodory class and will be denoted by $\mathcal{P}$. It is well recognized that the function

$$q_0(z) = (1+z)/(1-z)$$

(or its rotation) maps the unit disk univalently onto the right half-plane and is extremal in many problems. One can observe that the function

$$\varphi_0(z) = q_0(z) + \frac{zq_0'(z)}{q_0(z)} = \frac{1+z}{1-z} + \frac{2z}{1-z^2} = \frac{1+4z+z^2}{1-z^2}$$

maps the unit disk onto the slit domain $V(-\sqrt{3}, \sqrt{3})$, where

$$V(A,B) = \mathbb{C} \setminus \{iy : y \le A \text{ or } y \ge B\}$$

for $A, B \in \mathbb{R}$ with $A < B$. Note that $V(A,B)$ contains the right half-plane and has the "window" (Ai, Bi) in the imaginary axis to the left half-plane. The Open Door Lemma of Miller and Mocanu asserts for a function $q \in \mathcal{H}[1,1]$ that, if $q(z) + zq'(z)/q(z) \in V(-\sqrt{3}, \sqrt{3})$ for $z \in \mathbb{D}$, then $q \in \mathcal{P}$. Indeed, Miller and Mocanu [3] (see also [4]) proved it in a more general form. For a complex number c with $\operatorname{Re} c > 0$ and $n \in \mathbb{N}$, we consider the positive number

$$C_n(c) = \frac{n}{\operatorname{Re} c}\left[|c|\sqrt{\frac{2\operatorname{Re} c}{n} + 1} + \operatorname{Im} c\right].$$

In particular, $C_n(c) = \sqrt{n(n+2c)}$ when c is real. The following is a version of the Open Door Lemma modified by Kuroki and Owa [2].

Theorem A (Open Door Lemma). *Let c be a complex number with positive real part and n be an integer with $n \ge 1$. Suppose that a function $q \in \mathcal{H}[c,n]$ satisfies the condition*

$$q(z) + \frac{zq'(z)}{q(z)} \in V(-C_n(c), C_n(\bar{c})), \quad z \in \mathbb{D}.$$

Then $\operatorname{Re} q > 0$ on $\mathbb{D}$.

Remark 1.1. In the original statement of the Open Door Lemma in [3], the slit domain was erroneously described as $V(-C_n(c), C_n(c))$. Since $C_n(\bar{c}) < C_n(c)$ when $\operatorname{Im} c > 0$, we see that $V(-C_n(\bar{c}), C_n(\bar{c})) \subset V(-C_n(c), C_n(\bar{c})) \subset V(-C_n(c), C_n(c))$ for $\operatorname{Im} c \ge 0$ and the inclusions are strict if $\operatorname{Im} c > 0$. As the proof will suggest us, seemingly the domain $V(-C_n(c), C_n(\bar{c}))$ is maximal for the assertion, which means that the original statement in [3] and the form of the associated open door function are incorrect for a non-real c. This, however, does not decrease so much the value of the original article [3] by Miller and Mocanu because the Open Door Lemma is mostly applied when c is real. We also note that the Open Door Lemma deals with the function $p = 1/q \in \mathcal{H}[1/c, n]$ instead of q. The present form is adopted for convenience of our aim.

The Open Door Lemma gives a sufficient condition for $q \in \mathcal{H}[c,n]$ to have positive real part. We extend it so that $|\arg q| < \pi\alpha/2$ for a given $0 < \alpha \le 1$. First we note that the Möbius transformation

$$g_c(z) = \frac{c + \bar{c}z}{1 - z}$$

maps $\mathbb{D}$ onto the right half-plane in such a way that $g_c(0) = c$, where c is a complex number with $\operatorname{Re} c > 0$. In particular, one can take an analytic branch of $\log g_c$ so that $|\operatorname{Im}\log g_c| < \pi/2$. Therefore, the function $q_0 = g_c^\alpha = \exp(\alpha \log g_c)$ maps $\mathbb{D}$ univalently onto the sector $|\arg w| < \pi\alpha/2$ in such a way that $q_0(0) = c^\alpha$. The present note is based mainly on the following result, which will be deduced from a more general result of Miller and Mocanu (see Section 2).

Theorem 1.2. *Let c be a complex number with* $\operatorname{Re} c > 0$ *and α be a real number with $0 < \alpha \le 1$. Then the function*

$$R_{\alpha,c,n}(z) = g_c(z)^\alpha + \frac{n\alpha z g_c'(z)}{g_c(z)} = \left(\frac{c + \bar{c}z}{1 - z}\right)^\alpha + \frac{2n\alpha(\operatorname{Re} c)z}{(1 - z)(c + \bar{c}z)}$$

is univalent on $|z| < 1$. If a function $q \in \mathcal{H}[c^\alpha, n]$ satisfies the condition

$$q(z) + \frac{zq'(z)}{q(z)} \in R_{\alpha,c,n}(\mathbb{D}), \quad z \in \mathbb{D},$$

then $|\arg q| < \pi\alpha/2$ on $\mathbb{D}$.

We remark that the special case when $\alpha = 1$ reduces to Theorem A (see the paragraph right after Lemma 3.3 below. Also, the case when $c = 1$ is already proved by Mocanu [5] even under the weaker assumption that $0 < \alpha \le 2$ (see Remark 3.6). Since the shape of $R_{\alpha,c,n}(\mathbb{D})$ is not very clear, we will deduce more concrete results as corollaries of Theorem 1.2 in Section 3. This is our principal aim in the present note.

2. Preliminaries

We first recall the notion of subordination. A function $f \in \mathcal{H}$ is said to be *subordinate* to $F \in \mathcal{H}$ if there exists a function $\omega \in \mathcal{H}[0,1]$ such that $|\omega| < 1$ on $\mathbb{D}$ and that $f = F \circ \omega$. We write $f \prec F$ or $f(z) \prec F(z)$ for subordination. When F is univalent, $f \prec F$ precisely when $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$.

Miller and Mocanu [3, Theorem 5] (see also [4, Theorem 3.2h]) proved the following general result, from which we will deduce Theorem 1.2 in the next section.

Lemma 2.1 (Miller and Mocanu). *Let $\mu, \nu \in \mathbb{C}$ with $\mu \ne 0$ and n be a positive integer. Let $q_0 \in \mathcal{H}[c,1]$ be univalent and assume that $\mu q_0(z) + \nu \ne 0$ for $z \in \mathbb{D}$ and* $\operatorname{Re}(\mu c + \nu) > 0$. *Set $Q(z) = zq_0'(z)/(\mu q_0(z) + \nu)$, and*

$$h(z) = q_0(z) + nQ(z) = q_0(z) + \frac{nzq_0'(z)}{\mu q_0(z) + \nu}. \tag{2.1}$$

Suppose further that

(a) $\operatorname{Re}[zh'(z)/Q(z)] = \operatorname{Re}[h'(z)(\mu q_0(z) + \nu)/q_0'(z)] > 0$, *and*
(b) *either h is convex or Q is starlike.*

If $q \in \mathcal{H}[c,n]$ *satisfies the subordination relation*

$$q(z) + \frac{zq'(z)}{\mu q(z) + \nu} \prec h(z), \tag{2.2}$$

then $q \prec q_0$*, and* q_0 *is the best dominant. An extremal function is given by*

$$q(z) = q_0(z^n).$$

In the investigation of the generalized open door function $R_{\alpha,c,n}$, we will need to study the positive solution to the equation

$$x^2 + Ax^{1+\alpha} - 1 = 0, \tag{2.3}$$

where $A > 0$ and $0 < \alpha \le 1$ are constants. Let $F(x) = x^2 + Ax^{1+\alpha} - 1$. Then $F(x)$ is increasing in $x > 0$ and $F(0) = -1 < 0$, $F(+\infty) = +\infty$. Therefore, there is a unique positive solution $x = \xi(A,\alpha)$ to the equation. We have the following estimates for the solution.

Lemma 2.2. *Let* $0 < \alpha \le 1$ *and* $A > 0$*. The positive solution* $x = \xi(A,\alpha)$ *to equation* (2.3) *satisfies the inequalities*

$$(1+A)^{-1/(1+\alpha)} \le \xi(A,\alpha) \le (1+A)^{-1/2} \ (< 1).$$

Here, both inequalities are strict when $0 < \alpha < 1$*.*

Proof. Set $\xi = \xi(A,\alpha)$. Since the above $F(x)$ is increasing in $x > 0$, the inequalities $F(x_1) \le 0 = F(\xi) \le F(x_2)$ imply $x_1 \le \xi \le x_2$ for positive numbers x_1, x_2 and the inequalities are strict when $x_1 < \xi < x_2$. Keeping this in mind, we now show the assertion. First we put $x_2 = (1+A)^{-1/2}$ and observe

$$F(x_2) = \frac{1}{1+A} + \frac{A}{(1+A)^{(1+\alpha)/2}} - 1 \ge \frac{1}{1+A} + \frac{A}{1+A} - 1 = 0,$$

which implies the right-hand inequality in the assertion.

Next put $x_1 = (1+A)^{-1/(1+\alpha)}$. Then

$$F(x_1) = \frac{1}{(1+A)^{2/(1+\alpha)}} + \frac{A}{1+A} - 1 \le \frac{1}{1+A} + \frac{A}{1+A} - 1 = 0,$$

which implies the left-hand inequality. We note also that $F(x_1) < 0 < F(x_2)$ when $\alpha < 1$. The proof is now complete. □

3. Proof and corollaries

Theorem 1.2 can be rephrased in the following.

Theorem 3.1. *Let* c *be a complex number with* $\operatorname{Re} c > 0$ *and* α *be a real number with* $0 < \alpha \le 1$*. Then the function*

$$R_{\alpha,c,n}(z) = g_c(z)^\alpha + \frac{n\alpha z g_c'(z)}{g_c(z)}$$

is univalent on $|z| < 1$. *If a function* $q \in \mathcal{H}[c^\alpha, n]$ *satisfies the subordination condition*

$$q(z) + \frac{zq'(z)}{q(z)} \prec R_{\alpha,c,n}(z)$$

on $\mathbb{D}$, *then* $q(z) \prec g_c(z)^\alpha$ *on* $\mathbb{D}$. *The function* g_c^α *is the best dominant.*

Proof. We first show that the function $Q(z) = \alpha z g_c'(z)/g_c(z)$ is starlike. Indeed, we compute

$$\frac{zQ'(z)}{Q(z)} = 1 - \frac{\bar{c}z}{c+\bar{c}z} + \frac{z}{1-z} = \frac{1}{2}\left[\frac{c-\bar{c}z}{c+\bar{c}z} + \frac{1+z}{1-z}\right].$$

Thus we can see that $\operatorname{Re}[zQ'(z)/Q(z)] > 0$ on $|z| < 1$. Next we check condition (a) in Lemma 2.1 for the functions $q_0 = g_c^\alpha, h = R_{\alpha,c,n}$ with the choice $\mu = 1, \nu = 0$. We have the expression

$$\frac{zh'(z)}{Q(z)} = q_c(z)^\alpha + n\frac{zQ'(z)}{Q(z)}.$$

Since both terms in the right-hand side have positive real part, we obtain (a). We now apply Lemma 2.1 to obtain the required assertion up to univalence of $h = R_{\alpha,c,n}$. In order to show the univalence, we have only to note that the condition (a) implies that h is close-to-convex, since Q is starlike. As is well known, a close-to-convex function is univalent (see [1]), the proof has been finished. □

We now investigate the shape of the image domain $R_{\alpha,c,n}(\mathbb{D})$ of the generalized open door function $R_{\alpha,c,n}$ given in Theorem 1.2. Let $z = e^{i\theta}$ and $c = re^{it}$ for $\theta \in \mathbb{R}, r > 0$ and $-\pi/2 < t < \pi/2$. Then we have

$$\begin{aligned} R_{\alpha,c,n}(e^{i\theta}) &= \left(\frac{re^{it} + re^{-it}e^{i\theta}}{1-e^{i\theta}}\right)^\alpha + \frac{2n\alpha e^{i\theta}\cos t}{(1-e^{i\theta})(e^{it}+e^{-it}e^{i\theta})} \\ &= \left(\frac{r\cos(t-\theta/2)}{\sin(\theta/2)}i\right)^\alpha + \frac{i}{2}\cdot\frac{n\alpha\cos t}{\sin(\theta/2)\cos(t-\theta/2)} \\ &= r^\alpha e^{\pi\alpha i/2}(\cos t\cot(\theta/2)+\sin t)^\alpha + \frac{i}{2}\cdot\frac{n\alpha(1+\cot^2(\theta/2))\cos t}{\cos t\cot(\theta/2)+\sin t}. \end{aligned}$$

Let $x = \cot(\theta/2)\cos t + \sin t$. When $x > 0$, we write $R_{\alpha,c,n}(e^{i\theta}) = u_+(x) + iv_+(x)$ and get the expressions

$$\begin{cases} u_+(x) = a(rx)^\alpha, \\ v_+(x) = b(rx)^\alpha + \dfrac{n\alpha}{2\cos t}\left(x - 2\sin t + \dfrac{1}{x}\right), \end{cases}$$

where

$$a = \cos\frac{\alpha\pi}{2} \quad \text{and} \quad b = \sin\frac{\alpha\pi}{2}.$$

Taking the derivative, we get

$$v_+'(x) = \frac{n\alpha}{2x^2\cos t}\left[x^2 + \frac{2br^\alpha\cos t}{n}x^{\alpha+1} - 1\right].$$

Hence, the minimum of $v_+(x)$ is attained at $x = \xi(A, \alpha)$, where $A = 2br^\alpha n^{-1}\cos t$. By using the relation (2.3), we obtain

$$\begin{aligned}\min_{0<x} v_+(x) = v_+(\xi) &= \frac{n}{2\cos t}\left(A\xi^\alpha + \alpha\xi + \frac{\alpha}{\xi}\right) - n\alpha\tan t \\ &= \frac{n}{2\cos t}\left((\alpha-1)\xi + \frac{\alpha+1}{\xi}\right) - n\alpha\tan t = U(\xi),\end{aligned}$$

where

$$U(x) = \frac{n}{2\cos t}\left((\alpha-1)x + \frac{\alpha+1}{x}\right) - n\alpha\tan t.$$

Since the function $U(x)$ is decreasing in $0 < x < 1$, Lemma 2.2 yields the inequality

$$\begin{aligned}v_+(\xi) = U(\xi) &\geq U((1+A)^{-1/2}) \\ &= \frac{n}{2\cos t}\left(\frac{\alpha-1}{\sqrt{1+A}} + (\alpha+1)\sqrt{1+A}\right) - n\alpha\tan t.\end{aligned}$$

We remark here that

$$U((1+A)^{-1/2}) > U(1) = \frac{n\alpha(1-\sin t)}{\cos t} > 0;$$

namely, $v_+(x) > 0$ for $x > 0$. When $x < 0$, letting $y = -x = -\cot(\theta/2)\cos t - \sin t$, we write $R_{\alpha,c,n}(e^{i\theta}) = u_-(y) + iv_-(y)$. Then, with the same a and b as above,

$$\begin{cases} u_-(y) = a(ry)^\alpha, \\ v_-(y) = -b(ry)^\alpha - \dfrac{n\alpha}{2\cos t}\left(y + 2\sin t + \dfrac{1}{y}\right), \end{cases}$$

We observe here that $u_+ = u_- > 0$ and, in particular, we obtain the following.

Lemma 3.2. *The left half-plane $\Omega_1 = \{w : \operatorname{Re} w < 0\}$ is contained in $R_{\alpha,c,n}(\mathbb{D})$.*

We now look at $v_-(y)$. Since

$$v_-'(y) = -\frac{n\alpha}{2y^2\cos t}\left[y^2 + \frac{2br^\alpha\cos t}{n}y^{\alpha+1} - 1\right],$$

in the same way as above, we obtain

$$\begin{aligned}\max_{0<y} v_-(y) = v_-(\xi) &= -\frac{n}{2\cos t}\left((\alpha-1)\xi + \frac{\alpha+1}{\xi}\right) - n\alpha\tan t \\ &\leq -\frac{n}{2\cos t}\left(\frac{\alpha-1}{\sqrt{1+A}} + (\alpha+1)\sqrt{1+A}\right) - n\alpha\tan t,\end{aligned}$$

where $\xi = \xi(A, \alpha)$ and $A = 2br^\alpha n^{-1}\cos t$. Note also that $v_-(y) < 0$ for $y > 0$.

Since the horizontal parallel strip $v_-(\xi) < \operatorname{Im} w < v_+(\xi)$ is contained in the image domain $R_{\alpha,c,n}(\mathbb{D})$ of the generalized open door function, we obtain the following.

Lemma 3.3. *The parallel strip Ω_2 described by*

$$|\operatorname{Im} w + n\alpha\tan t| < \frac{n}{2\cos t}\left(\frac{\alpha-1}{\sqrt{1+A}} + (\alpha+1)\sqrt{1+A}\right)$$

is contained in $R_{\alpha,c,n}(\mathbb{D})$. Here, $t = \arg c \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $A = \frac{2}{n}|c|^\alpha \sin\frac{\pi\alpha}{2}\cos t$.

When $\alpha = 1$, we have $u_\pm = 0$, that is, the boundary is contained in the imaginary axis. Since $\xi(A, 1) = (1 + A)^{-1/2}$ by Lemma 2.2, the above computations tell us

$$\min v_+ = (n/\cos t)(\sqrt{1 + A} - \sin t) = C_n(\bar{c}).$$

Similarly, we have

$$\max v_- = -(n/\cos t)(\sqrt{1 + A} + \sin t) = -C_n(c).$$

Therefore, we have

$$R_{1,c,n}(\mathbb{D}) = V(-C_n(c), C_n(\bar{c})).$$

Note that the open door function then takes the following form

$$\begin{aligned} R_{1,c,n}(z) &= \frac{c + \bar{c}z}{1 - z} + \frac{2n(\operatorname{Re} c)z}{(1 - z)(c + \bar{c}z)} \\ &= \frac{2\operatorname{Re} c + n}{1 + cz/\bar{c}} - \frac{n}{1 - z} - \bar{c}, \end{aligned}$$

which is the same as given by Kuroki and Owa [2, (2.2)]. In this way, we see that Theorem 1.2 contains Theorem A as a special case.

Remark 3.4. In [2], they proposed another open door function of the form

$$R(z) = \frac{2n|c|}{\operatorname{Re} c}\sqrt{\frac{2\operatorname{Re} c}{n} + 1}\,\frac{(\zeta - z)(1 - \bar{\zeta}z)}{(1 - \bar{\zeta}z)^2 - (\zeta - z)^2} - \frac{\operatorname{Im} c}{\operatorname{Re} c}i,$$

where

$$\zeta = 1 - \frac{2}{\omega}, \quad \omega = \frac{c}{|c|}\sqrt{\frac{2\operatorname{Re} c}{n} + 1} + 1.$$

It can be checked that $R(z) = R_{1,c,n}(-\omega z/\bar{\omega})$. Hence, R is just a rotation of $R_{1,c,n}$.

We next study the argument of the boundary curve of $R_{\alpha,c,n}(\mathbb{D})$. We will assume that $0 < \alpha < 1$ since we have nothing to do when $\alpha = 1$.

As we noted above, the boundary is contained in the right half-plane $\operatorname{Re} w > 0$. When $x > 0$, we have

$$\frac{v_+(x)}{u_+(x)} = \frac{b}{a} + \frac{n\alpha}{2ar^\alpha x^\alpha \cos t}\left[x + \frac{1}{x} - 2\sin t\right].$$

We observe now that $v_+(x)/u_+(x) \to +\infty$ as $x \to 0+$ or $x \to +\infty$. We also have

$$\left(\frac{v_+}{u_+}\right)'(x) = \frac{n\alpha}{2ar^\alpha x^{\alpha+2}\cos t}\left[(1 - \alpha)x^2 + 2\alpha x \sin t - (1 + \alpha)\right].$$

Therefore, $v_+(x)/u_+(x)$ takes its minimum at $x = \xi$, where

$$\xi = \frac{-\alpha \sin t + \sqrt{1 - \alpha^2 \cos^2 t}}{1 - \alpha}$$

is the positive root of the equation $(1-\alpha)x^2+2\alpha x\sin t-(1+\alpha)=0$. It is easy to see that $1<\xi$ and that

$$T_+ := \min_{0<x}\frac{v_+(x)}{u_+(x)}=\frac{v_+(\xi)}{u_+(\xi)}=\frac{b}{a}+\frac{n\alpha}{2ar^\alpha\xi^\alpha\cos t}\left[\xi+\frac{1}{\xi}-2\sin t\right]$$
$$=\tan\frac{\pi\alpha}{2}+\frac{n(\xi-\xi^{-1})}{2ar^\alpha\xi^\alpha\cos t}.$$

When $x=-y<0$, we have

$$\frac{v_-(y)}{u_-(y)}=-\frac{b}{a}-\frac{n\alpha}{2ar^\alpha y^\alpha\cos t}\left[y+\frac{1}{y}+2\sin t\right]$$

and

$$\left(\frac{v_-}{u_-}\right)'(y)=\frac{-n\alpha}{2ar^\alpha y^{\alpha+2}\cos t}\left[(1-\alpha)y^2-2\alpha y\sin t-(1+\alpha)\right].$$

Hence, $v_-(y)/u_-(y)$ takes its maximum at $y=\eta$, where

$$\eta=\frac{\alpha\sin t+\sqrt{1-\alpha^2\cos^2 t}}{1-\alpha}.$$

Note that

$$T_- := \max_{0<y}\frac{v_-(y)}{u_-(y)}=\frac{v_-(\eta)}{u_-(\eta)}=-\tan\frac{\pi\alpha}{2}-\frac{n(\eta-\eta^{-1})}{2ar^\alpha\eta^\alpha\cos t}.$$

Therefore, the sector $\{w : T_- < \arg w < T_+\}$ is contained in the image $R_{\alpha,c,n}(\mathbb{D})$. It is easy to check that $T_- < -\tan(\pi\alpha/2) < \tan(\pi\alpha/2) < T_+$. In particular $T_- < \arg c^\alpha = \alpha t < T_+$. We summarize the above observations, together with Theorem 1.2, in the following form.

Corollary 3.5. *Let $0<\alpha<1$ and $c=re^{it}$ with $r>0, -\pi/2<t<\pi/2$, and n be a positive integer. If a function $q\in\mathcal{H}[c^\alpha,n]$ satisfies the condition*

$$-\Theta_- < \arg\left(q(z)+\frac{zq'(z)}{q(z)}\right) < \Theta_+$$

on $|z|<1$, then $|\arg q|<\pi\alpha/2$ on $\mathbb{D}$. Here,

$$\Theta_\pm=\arctan\left[\tan\frac{\pi\alpha}{2}+\frac{n(\xi_\pm-\xi_\pm^{-1})}{2r^\alpha\xi_\pm^\alpha\cos(\pi\alpha/2)\cos t}\right],$$

and

$$\xi_\pm=\frac{\mp\alpha\sin t+\sqrt{1-\alpha^2\cos^2 t}}{1-\alpha}.$$

It is a simple task to check that $x^{1-\alpha}-x^{-1-\alpha}$ is increasing in $0<x$. When $\operatorname{Im} c>0$, we see that $\xi_- > \xi_+$ and thus $\Theta_- > \Theta_+$. It might be useful to note the estimates $\xi_- < \sqrt{(1+\alpha)/(1-\alpha)} < \xi_+$ and $\xi_- < 1/\sin t$ for $\operatorname{Im} c>0$.

Remark 3.6. When $c=1$ and $n=1$, we have

$$\xi := \xi_\pm = \sqrt{(1+\alpha)/(1-\alpha)},\ \xi-\xi^{-1}=2\alpha/\sqrt{1-\alpha^2},$$

and thus

$$\begin{aligned}\Theta_{\pm} &= \arctan\left[\tan\frac{\pi\alpha}{2} + \frac{\xi - \xi^{-1}}{2\xi^{\alpha}\cos\frac{\pi\alpha}{2}}\right] \\ &= \arctan\left[\tan\frac{\pi\alpha}{2} + \frac{\alpha}{\cos\frac{\pi\alpha}{2}(1-\alpha)^{\frac{1-\alpha}{2}}(1+\alpha)^{\frac{1+\alpha}{2}}}\right] \\ &= \frac{\pi\alpha}{2} + \arctan\left[\frac{\alpha\cos\frac{\pi\alpha}{2}}{(1-\alpha)^{\frac{1-\alpha}{2}}(1+\alpha)^{\frac{1+\alpha}{2}} + \alpha\sin\frac{\pi\alpha}{2}}\right].\end{aligned}$$

Therefore, the corollary gives a theorem proved by Mocanu [6].

Since the values Θ_+ and Θ_- are not given in an explicitly way, it might be convenient to have a simpler sufficient condition for $|\arg q| < \pi\alpha/2$.

Corollary 3.7. *Let $0 < \alpha \le 1$ and c with $\operatorname{Re} c > 0$ and n be a positive integer. If a function $q \in \mathcal{H}[c^\alpha, n]$ satisfies the condition*

$$q(z) + \frac{zq'(z)}{q(z)} \in \Omega,$$

then $|\arg q| < \pi\alpha/2$ on $\mathbb{D}$. Here, $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$, and Ω_1 and Ω_2 are given in Lemmas 3.2 and 3.3, respectively, and $\Omega_3 = \{w \in \mathbb{C} : |\arg w| < \pi\alpha/2\}$.

Proof. Lemmas 3.2 and 3.3 yield that $\Omega_1 \cup \Omega_2 \subset R_{\alpha,c,n}(\mathbb{D})$. Since $\Theta_{\pm} > \pi\alpha/2$, we also have $\Omega_3 \subset R_{\alpha,c,n}(\mathbb{D})$. Thus $\Omega \subset R_{\alpha,c,n}(\mathbb{D})$. Now the result follows from Theorem 1.2. □

See Figure 1 for the shape of the domain Ω together with $R_{\alpha,c,n}(\mathbb{D})$. We remark that $\Omega = R_{\alpha,c,n}(\mathbb{D})$ when $\alpha = 1$.

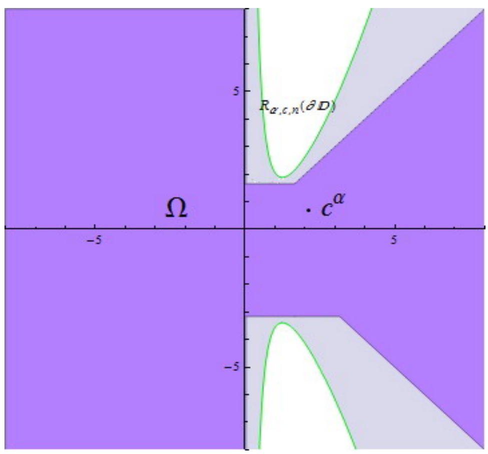

FIGURE 1. The image $R_{\alpha,c,n}(\mathbb{D})$ and Ω for $\alpha = 1/2, c = 4 + 3i, n = 2$.

References

[1] Duren, P.L., *Univalent Functions*, Springer-Verlag, 1983.

[2] Kuroki, K., Owa, S., *Notes on the open door lemma*, Rend. Semin. Mat. Univ. Politec. Torino, **70**(2012), 423434.

[3] Miller, S.S., Mocanu, P.T., *Briot-Bouquet differential equations and differential subordinations*, Complex Variables Theory Appl., **33**(1997), 217–237.

[4] Miller, S.S., Mocanu, P.T., *Differential subordinations. Theory and applications*, Marcel Dekker, Inc., New York, 2000.

[5] Mocanu, P.T., *On strongly-starlike and strongly-convex functions*, Studia Univ. Babeş-Bolyai Math., **31**(1986), 16–21.

[6] Mocanu, P.T., *Alpha-convex integral operator and strongly starlike functions*, Studia Univ. Babeş-Bolyai, Math., **34**(1989), 18–24.

Ming Li
Graduate School of Information Sciences
Tohoku University
Aoba-ku, Sendai 980-8579, Japan
e-mail: `li@ims.is.tohoku.ac.jp`

Toshiyuki Sugawa
Graduate School of Information Sciences
Tohoku University
Aoba-ku, Sendai 980-8579, Japan
e-mail: `sugawa@math.is.tohoku.ac.jp`