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On sandwich theorems for p-valent functions involving a new generalized differential operator

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Abstract. A new differential operator $F^m_{\alpha,\beta,\lambda}f(z)$ is introduced for functions of the form $f(z) = z^p + \sum_{n=2}^{\infty} a_n z^n$ which are p-valent in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. The main object of this paper is to derive some subordination and superordination results involving differential operator $F^m_{\alpha,\beta,\lambda}f(z)$.

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1. Introduction

Let $H(\mathbb{U})$ denote the class of analytic functions in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$$

and let $H[a, b]$ denote the subclass of the functions $f \in H(\mathbb{U})$ of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + ... \quad (a \in \mathbb{C};\ p \in \mathbb{N} = \{1, 2, ...\}). \tag{1.1}$$

For simplicity $H[a] = H[a, 1]$. Also, let $\mathcal{A}(p)$ be the subclass of $H(\mathbb{U})$ consisting of functions of the form:

$$f(z) = z^p + \sum_{n=2}^{\infty} a_n z^n, \qquad (a_n \geq 0; p \in \mathbb{N} := \{1, 2, 3, ...\}), \tag{1.2}$$

which are p-valent in $\mathbb{U}$. If $f,\ g \in H(\mathbb{U})$, we say that f is subordinate to g or g is subordinate to f, written $f(z) \prec g(z)$, if there exists an analytic function w on $\mathbb{U}$ such that $w(0) = 0$, $|w(z)| < 1$, such that $g(z) = h(w(z))$ for $z \in \mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see [5] and [13]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the second-order differential subordination:

$$\phi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z), \tag{1.3}$$

then $p(z)$ is a solution of the differential subordination (1.3). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.3) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.3). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (1.3) is called the best dominant. If $p(z)$ and $\phi\left(p(z), zp'(z); z\right)$ are univalent in $\mathbb{U}$ and if $p(z)$ satisfies second-order differential superordination:

$$h(z) \prec \phi\left(p(z), zp'(z), z^2p''(z); z\right), \tag{1.4}$$

then $p(z)$ is a solution of the differential supordination (1.4). An univalent function $q(z)$ is called a subordinant of the solutions of the differential superordination (1.4) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.4). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (1.4) is called the best subordinant. Using the results of Miller and Mocanu [14], Bulboaca [4] considered certain classes of first-order differential superordinations as well as superordination-preserving integral operators [5]. Ali et al. [1], have used the results of Bulboaca [4] to obtain sufficient conditions for normalized analytic functions $f \in \mathcal{A}(1)$ to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent normalized functions in $\mathbb{U}$ with $q_1(0) = q_2(0) = 1$.

Also, Tuneski [23] obtained a sufficient condition for starlikeness of $f \in \mathcal{A}(1)$ in terms of the quantity $\frac{f''(z)f(z)}{(f'(z))^2}$.

Recently, Shanmugam et al. [18], [19] and [21] obtained sufficient conditions for the normalized analytic function $f \in \mathcal{A}(1)$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2f'(z)}{f^2(z)} \prec q_2(z).$$

Recently, Shanmugam et al. [19] obtained the such called sandwich results for certain classes of analytic functions.

For the function $f \in \mathcal{A}(p)$, we define the following new differential operator:

$$\begin{aligned} F^0 f(z) &= f(z); \\ F^1_{\alpha,\beta,\lambda} f(z) &= (1 - p\beta(\lambda-\alpha))f(z) + \beta(\lambda-\alpha)zf'(z); \\ F^2_{\alpha,\beta,\lambda} f(z) &= (1 - p\beta(\lambda-\alpha))(F^1_{\alpha,\beta,\lambda} f(z)) + \beta(\lambda-\alpha)z(F^1_{\alpha,\beta,\lambda} f(z))' \end{aligned}$$

and for $m = 1, 2, 3, ...$

$$\begin{aligned} F^m_{\alpha,\beta,\lambda} f(z) &= (1 - p\beta(\lambda - \alpha))(F^{m-1}_{\alpha,\beta,\lambda} f(z)) + \beta(\lambda - \alpha)z(F^{m-1}_{\alpha,\beta,\lambda} f(z))' \\ &= F^1_{\alpha,\beta,\lambda}(F^{m-1}_{\alpha,\beta,\lambda} f(z)) \\ &= z^p + \sum_{n=2}^{\infty} [1 + \beta(\lambda - \alpha)(n - p)]^m a_n z^n, \end{aligned} \tag{1.5}$$

for $\alpha \geq 0, \beta \geq 0, \lambda \geq 0$, and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

It easily verified from (1.5) that

$$\beta(\lambda - \alpha)z(F^m_{\alpha,\beta,\lambda} f(z))' = F^{m+1}_{\alpha,\beta,\lambda} f(z) - (1 - p\beta(\lambda - \alpha))F^m_{\alpha,\beta,\lambda} f(z). \tag{1.6}$$

Remark 1.1. (i) When $\delta = 0$ and $p = 1$, we have the operator introduced and studied by Rabha (see [7]).

(ii) When $\alpha = 0$ and $\beta = p = 1$, we have the operator introduced and studied by Al-Oboudi (see [3]).

(iii) And when $\alpha = 0$ and $\lambda = \beta = p = 1$, we have the operator introduced and studied by Sălăgean (see [17]).

In this paper, we will derive several subordination results, superordination results and sandwich results involving the operator $F^m_{\lambda,p} f(z)$.

2. Definitions and preliminaries

In order to prove our subordinations and superordinations, we need the following definition and lemmas.

Definition 2.1. [14] *Denote by Q, the set of all functions f that are analytic and injective on $\overline{\mathbb{U}} \backslash E(f)$, where*

$$E(f) = \left\{ \zeta \in \partial\mathbb{U}\colon \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \backslash E(f)$.

Lemma 2.2. [14] *Let $q(z)$ be univalent in $\mathbb{U}$ and let θ and φ be analytic in a domain D containing $q(\mathbb{U})$, with $\varphi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set $\psi(z) = zq'(z)\varphi(q(z))$ and $h(z) = \theta(q(z)) + \psi(z)$. Suppose that*

(i) ψ is a starlike function in $\mathbb{U}$,

(ii) $\mathrm{Re}\left\{\frac{zh'(z)}{\psi(z)}\right\} > 0$, $z \in \mathbb{U}$.

If $p(z)$ is a analytic in $\mathbb{U}$ with $p(0) = q(0)$, $p(\mathbb{U}) \subset D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.1}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant of (2.1).

Lemma 2.3. [4] *Let $q(z)$ be convex univalent in $\mathbb{U}$ and let ϑ and ϕ be analytic in a domain D containing $q(\mathbb{U})$. Suppose that*

(i) $Re\left\{\frac{\vartheta'(q(z))}{\phi(q(z))}\right\} > 0$, $z \in \mathbb{U}$,

(ii) $\Psi(z) = zq'(z)\phi(q(z))$ is starlike univalent in $\mathbb{U}$.

If $p(z) \in H[q(0), 1] \cap Q$, with $p(\mathbb{U}) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\phi(p(z))$ is univalent in $\mathbb{U}$ and

$$\vartheta(q(z)) + zq'(z)\phi(q(z)) \prec \vartheta(p(z)) + zp'(z)\phi(p(z)), \tag{2.2}$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant of (2.2).

3. Subordination and superordination for p-valent functions

We begin with the following result involving differential subordination between analytic functions.

Theorem 3.1. *Let $q(z)$ be univalent in $\mathbb{U}$ with $q(0) = 1$, Further, assume that*

$$\mathrm{Re}\left\{\frac{2(\delta+\alpha)q(z)}{\delta} + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0. \tag{3.1}$$

If $f \in \mathcal{A}(p)$ satisfy the following subordination condition:

$$\Upsilon(m, \lambda, p, \delta; z) \prec \delta zq'(z) + (\delta+\alpha)\left(q(z)\right)^2, \tag{3.2}$$

where

$$\Upsilon(m, \lambda, p, \delta; z) = \frac{\delta F_{\lambda,p}^{m+2} f(z)}{\beta(\lambda-\alpha) F_{\lambda,p}^{m} f(z)} + \left(\delta + \alpha - \frac{\delta}{\beta(\lambda-\alpha)}\right) \frac{\left(F_{\lambda,p}^{m+1} f(z)\right)^2}{\left(F_{\lambda,p}^{m} f(z)\right)^2}, \tag{3.3}$$

then

$$\frac{F_{\lambda,p}^{m+1} f(z)}{F_{\lambda,p}^{m} f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Define a function $p(z)$ by

$$p(z) = \frac{F_{\lambda,p}^{m+1} f(z)}{F_{\lambda,p}^{m} f(z)} \quad (z \in \mathbb{U}). \tag{3.4}$$

Then the function $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = 1$. Therefore, differentiating (3.4) logarithmically with respect to z and using the identity (1.6) in the resulting equation, we have

$$\frac{\delta F_{\lambda,p}^{m+2} f(z)}{\beta(\lambda-\alpha) F_{\lambda,p}^{m} f(z)} + \left(\delta + \alpha - \frac{\delta}{\beta(\lambda-\alpha)}\right) \frac{\left(F_{\lambda,p}^{m+1} f(z)\right)^2}{\left(F_{\lambda,p}^{m} f(z)\right)^2} = (\delta+\alpha)\left(p(z)\right)^2 + \delta zp'(z), \tag{3.5}$$

that is,

$$(\delta+\alpha)\left(p(z)\right)^2 + \delta zp'(z) \prec (\delta+\alpha)\left(q(z)\right)^2 + \delta zq'(z).$$

Therefore, Theorem 3.1 now follows by applying Lemma 2.2 by setting

$$\theta(w) = (\delta+\alpha)w^2 \text{ and } \varphi(w) = \delta.$$

□

Corollary 3.2. *Let* $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ *in Theorem* 3.1*, further assuming that (*3.1*) holds.*

If $f \in \mathcal{A}(p)$ *satisfy the following subordination condition:*

$$\Upsilon(m, \lambda, p, \delta; z) \prec \frac{\delta(A-B)z}{(1+Bz)^2} + (\delta+\alpha)\left(\frac{1+Az}{1+Bz}\right)^2,$$

then

$$\frac{F_{\lambda,p}^{m+1}f(z)}{F_{\lambda,p}^{m}f(z)} \prec \frac{1+Az}{1+Bz}$$

and the function $\frac{1+Az}{1+Bz}$ *is the best dominant.*

In particular, if $q(z) = \frac{1+z}{1-z}$*, then for* $f \in \mathcal{A}(p)$ *we have,*

$$\Upsilon(m, \lambda, p, \delta; z) \prec \frac{2\delta z}{(1-z)^2} + (\delta+\alpha)\left(\frac{1+z}{1-z}\right)^2,$$

then

$$\frac{F_{\lambda,p}^{m+1}f(z)}{F_{\lambda,p}^{m}f(z)} \prec \frac{1+z}{1-z}$$

and the function $\frac{1+z}{1-z}$ *is the best dominant.*

Furthermore, if we take $q(z) = \left(\frac{1+z}{1-z}\right)^{\mu}$, $(0 < \mu \leq 1)$*, then for* $f \in \mathcal{A}(p)$ *we have,*

$$\Upsilon(m, \lambda, p, \delta; z) \prec \frac{2\delta\mu z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\mu-1} + (\delta+\alpha)\left(\frac{1+z}{1-z}\right)^{2\mu},$$

then

$$\frac{F_{\lambda,p}^{m+1}f(z)}{F_{\lambda,p}^{m}f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\mu}$$

and the function $\left(\frac{1+z}{1-z}\right)^{\mu}$ *is the best dominant.*

Next, by applying Lemma 2.3 we prove the following.

Theorem 3.3. *Let* $q(z)$ *be convex univalent in* $\mathbb{U}$ *with* $q(0) = 1$*. Assume that*

$$\operatorname{Re}\left\{\frac{2(\delta+\alpha)q(z)q'(z)}{\delta}\right\} > 0. \tag{3.6}$$

Let $f \in \mathcal{A}(p)$ *such that* $\frac{F_{\lambda,p}^{m+1}f(z)}{F_{\lambda,p}^{m}f(z)} \in H[q(0), 1] \cap Q$*,* $\Upsilon(m, \lambda, p, \delta; z)$ *is univalent in* $\mathbb{U}$ *and the following superordination condition*

$$(\delta+\alpha)(q(z))^2 + \delta z q'(z) \prec \Upsilon(m, \lambda, p, \delta; z) \tag{3.7}$$

holds, then

$$q(z) \prec \frac{F_{\lambda,p}^{m+1}f(z)}{F_{\lambda,p}^{m}f(z)} \tag{3.8}$$

and $q(z)$ *is the best subordinant.*

Proof. Let the function $p(z)$ be defined by

$$p(z) = \frac{F_{\lambda,p}^{m+1} f(z)}{F_{\lambda,p}^{m} f(z)}.$$

Then from the assumption of Theorem 3.3, the function $p(z)$ is analytic in $\mathbb{U}$ and (3.5) holds. Hence, the subordination (3.7) is equivalent to

$$(\delta + \alpha)(q(z))^2 + \delta z q'(z) \prec (\delta + \alpha)(p(z))^2 + \delta z p'(z)$$

The assertion (3.8) of Theorem 3.3 now follows by an application of Lemma 2.3. □

Corollary 3.4. *Let* $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ *in Theorem* 3.3*, further assuming that (*3.6*) holds.*

If $f \in \mathcal{A}(p)$ such that $\frac{F_{\lambda,p}^{m+1} f(z)}{F_{\lambda,p}^{m} f(z)} \in H[q(0), 1] \cap Q$, $\Upsilon(m, \lambda, p, \delta; z)$ is univalent in $\mathbb{U}$ and the following superordination condition

$$\frac{\delta(A-B)z}{(1+Bz)^2} + (\delta + \alpha)\left(\frac{1+Az}{1+Bz}\right)^2 \prec \Upsilon(m, \lambda, p, \delta; z)$$

holds, then

$$\frac{1+Az}{1+Bz} \prec \frac{F_{\lambda,p}^{m+1} f(z)}{F_{\lambda,p}^{m} f(z)}$$

and $q(z)$ is the best subordinant.

Also, let $q(z) = \frac{1+z}{1-z}$, then for $f \in \mathcal{A}(p)$ we have,

$$\frac{2\delta z}{(1-z)^2} + (\delta + \alpha)\left(\frac{1+z}{1-z}\right)^2 \prec \Upsilon(m, \lambda, p, \delta; z),$$

then

$$\frac{1+z}{1-z} \prec \frac{F_{\lambda,p}^{m+1} f(z)}{F_{\lambda,p}^{m} f(z)}$$

and the function $\frac{1+z}{1-z}$ is the best subordinant.

Finally, by taking $q(z) = \left(\frac{1+z}{1-z}\right)^{\mu}$, $(0 < \mu \le 1)$, then for $f \in \mathcal{A}(p)$ we have,

$$\frac{2\delta\mu z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\mu-1} + (\delta + \alpha)\left(\frac{1+z}{1-z}\right)^{2\mu} \prec \Upsilon(m, \lambda, p, \delta; z),$$

then

$$\left(\frac{1+z}{1-z}\right)^{\mu} \prec \frac{F_{\lambda,p}^{m+1} f(z)}{F_{\lambda,p}^{m} f(z)}$$

and the function $\left(\frac{1+z}{1-z}\right)^{\mu}$ is the best subordinant.

Combining Theorem 3.1 and Theorem 3.3, we get the following sandwich theorem.

Theorem 3.5. *Let* q_1 *and* q_2 *be convex univalent in* $\mathbb{U}$ *with* $q_1(0) = q_2(0) = 1$ *and satisfies* (3.1) *and* (3.6) *respectively. If* $f \in \mathcal{A}(p)$ *such that* $\frac{F_{\lambda,p}^{m+1}f(z)}{F_{\lambda,p}^{m}f(z)} \in H[q(0),1] \cap Q$, $\Upsilon(m,\lambda,p,\delta;z)$ *is univalent in* $\mathbb{U}$ *and*

$$(\delta+\alpha)\left(q_1(z)\right)^2 + \delta z q_1'(z) \prec \Upsilon(m,\lambda,p,\delta;z) \prec (\delta+\alpha)\left(q_2(z)\right)^2 + \delta z q_2'(z),$$

holds, then $q_1(z) \prec \frac{F_{\lambda,p}^{m+1}f(z)}{F_{\lambda,p}^{m}f(z)} \prec q_2(z)$ *and* $q_1(z)$ *and* $q_2(z)$ *are, respectively, the best subordinant and the best dominant.*

Corollary 3.6. *Let* $q_i(z) = \frac{1+A_i z}{1+B_i z}$ $(i = 1,2; -1 \le B_2 < B_1 < A_1 \le A_2 \le 1)$ *in Theorem* 3.5. *If* $f \in \mathcal{A}(p)$ *such that* $\frac{F_{\lambda,p}^{m+1}f(z)}{F_{\lambda,p}^{m}f(z)} \in H[q(0),1] \cap Q$, $\Upsilon(m,\lambda,p,\delta;z)$ *is univalent in* $\mathbb{U}$ *and*

$$\frac{\delta(A_1-B_1)z}{(1+B_1z)^2} + (\delta+\alpha)\left(\frac{1+A_1z}{1+B_1z}\right)^2 \prec \Upsilon(m,\lambda,p,\delta;z) \prec \frac{\delta(A_2-B_2)z}{(1+B_2z)^2} + (\delta+\alpha)\left(\frac{1+A_2z}{1+B_2z}\right)^2$$

holds, then $\frac{1+A_1z}{1+B_1z} \prec \frac{F_{\lambda,p}^{m+1}f(z)}{F_{\lambda,p}^{m}f(z)} \prec \frac{1+A_2z}{1+B_2z}$ *and* $\frac{1+A_1z}{1+B_1z}$ *and* $\frac{1+A_2z}{1+B_2z}$ *are, respectively, the best subordinant and the best dominant.*

Remarks. Other works related to differential subordination or superordination can be found in [2], [6], [8]-[12], [15], [16], [20], [22].

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