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On some generalized integral inequalities for φ-convex functions

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Abstract. The main goal of the paper is to state and prove some new general inequalities for φ-convex function.

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1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g.,[4], [8, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The inequality (1.1) has evoked the interest of many mathematicians. Especially in the last three decades numerous generalizations, variants and extensions of this inequality have been obtained, to mention a few, see ([3]-[15]) and the references cited therein.

Let us consider a function $\varphi : [a, b] \to [a, b]$ where $[a, b] \subset \mathbb{R}$. Youness have defined the φ-convex functions in [16], but we work here with the improved definition, according to [1]:

Definition 1.1. *A function $f : [a, b] \to \mathbb{R}$ is said to be φ- convex on $[a, b]$ if for every two points $x, y \in [a, b]$ and $t \in [0, 1]$ the following inequality holds:*

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t)f(\varphi(y)).$$

Obviously, if function φ is the identity, then the classical convexity is obtained from the previous definition. Many properties of the φ-convex functions can be found, for instance, in [1], [2], [16], [17], [18].

Moreover in [2], Cristescu have presented a version Hermite-Hadamard type inequality for the φ-convex functions as follows:

Theorem 1.2. *If a function* $f:[a,b]\to\mathbb{R}$ *is* φ*- convex for the continuous function* $\varphi:[a,b]\to[a,b]$*, then*

$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\leq\frac{1}{\varphi(b)-\varphi(a)}\int\limits_{\varphi(a)}^{\varphi(b)}f(x)dx\leq\frac{f(\varphi(a))+f(\varphi(b))}{2}. \tag{1.2}$$

In this article, we will consider two parts which within the first section we give some new general inequalities for φ-convex function. In the second part, using functions whose derivatives absolute values are φ-convex function, we obtained new inequalities related to the left and the right sides of Hermite-Hadamard inequality are given with (2.1).

2. Hermite-Hadamard type inequality for φ-convex function

Theorem 2.1. *Let* J *be an interval* $a,b\in J$ *with* $a<b$ *and* $\varphi:J\to\mathbb{R}$ *a continuous increasing function. Let* $f:I\subseteq\mathbb{R}\to\mathbb{R}$ *be a* φ*-convex function on* $I=[a,b]$*, then we have*

$$\begin{aligned}f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)&\leq\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}f(\varphi(x))\,d\varphi(x)\\&\leq\frac{f(\varphi(a))+f(\varphi(b))}{2}.\end{aligned} \tag{2.1}$$

Proof. By definition of the φ-convex function

$$\begin{aligned}f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)&=\int_0^1 f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)dt\\&=\int_0^1 f\left(\frac{(1-t)\varphi(a)+t\varphi(b)+t\varphi(a)+(1-t)\varphi(b)}{2}\right)dt\\&\leq\frac{1}{2}\int_0^1\left[f((1-t)\varphi(a)+t\varphi(b))+f(t\varphi(a)+(1-t)\varphi(b))\right]dt.\end{aligned}$$

Using the change of the variable in last integrals, we get

$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\leq\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}f(\varphi(x))\,d\varphi(x). \tag{2.2}$$

By similar way, we have

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi a)}^{\varphi(b)} f(\varphi(x))\,d\varphi(x) = \int_0^1 f((1-t)\varphi(a)+t\varphi(b))\,dt$$
$$\leq \int_0^1 [(1-t)f(\varphi(a))+tf(\varphi(b))]\,dt$$
$$= \frac{f(\varphi(a))+f(\varphi(b))}{2}. \tag{2.3}$$

From (2.2) and (2.3), it is obtained desired result. □

Remark 2.2. If we choose $\varphi(x) = x$ for all $x \in [a,b]$ in Theorem 2.1, the (2.1) inequality reduce to the inequality (1.1).

Theorem 2.3. *Let J be an interval $a,b \in J$ with $a < b$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let f be a φ-convex function on $I = [a,b]$ and let $w : [\varphi(a), \varphi(b)] \to \mathbb{R}$ be nonnegative, integrable and symmetric about $\frac{\varphi(a)+\varphi(b)}{2}$. Then*

$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\int_{\varphi(a)}^{\varphi(b)} w(\varphi(x))\,d\varphi(x) \leq \int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,w(\varphi(x))\,d\varphi(x)$$

$$\leq \frac{f(\varphi(a))+f(\varphi(b))}{2}\int_{\varphi(a)}^{\varphi(b)} w(\varphi(x))\,d\varphi(x). \tag{2.4}$$

Proof. Since f be a φ-convex function and $w : [\varphi(a), \varphi(b)] \to \mathbb{R}$ be nonnegative, integrable and symmetric about $\frac{\varphi(a)+\varphi(b)}{2}$, then we obtain

$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\int_{\varphi(a)}^{\varphi(b)} w(\varphi(x))\,d\varphi(x) = \int_{\varphi(a)}^{\varphi(b)} f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) w(\varphi(x))\,d\varphi(x)$$

$$\leq \frac{1}{2}\int_{\varphi(a)}^{\varphi(b)} [f(\varphi(a)+\varphi(b)-\varphi(x))+f(\varphi(x))]\,w(\varphi(x))\,d\varphi(x)$$

$$= \int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,w(\varphi(x))\,d\varphi(x)$$

$$=\frac{1}{2}\int_{\varphi(a)}^{\varphi(b)} [f(\varphi(a)+\varphi(b)-\varphi(x))]\,w(\varphi(x))\,d\varphi(x)+\frac{1}{2}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,w(\varphi(x))\,d\varphi(x)$$

$$\leq \frac{1}{2}\int_{\varphi(a)}^{\varphi(b)} [f(\varphi(a))+f(\varphi(b))]\,w(\varphi(x))\,d\varphi(x)$$

$$= \frac{f(\varphi(a))+f(\varphi(b))}{2}\int_{\varphi(a)}^{\varphi(b)} w(\varphi(x))\,d\varphi(x)$$

which completes the proof of Theorem 2.3. □

Corollary 2.4. *Under the same assumptions of Theorem* 2.3 *with* $\varphi(x) = x$ *for all* $x \in [a,b]$, *we have*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \leq \int_a^b f(x)\,w(x)\,dx \leq \frac{f(a)+f(b)}{2}\int_a^b w(x)dx.$$

Theorem 2.5. *Let* J *be an interval* $a,b \in J$ *with* $a < b$ *and* $\varphi : J \to \mathbb{R}$ *a continuous increasing function. Let* $f, w : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a* φ*-convex and nonnegative function on* $I = [a,b]$. *Then, for all* $t \in [0,1]$, *we have*

$$\begin{aligned} 2f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)w\left(\frac{\varphi(a)+\varphi(b)}{2}\right) &\leq \frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,w(\varphi(x))\,d\varphi(x) \\ &\leq \frac{1}{6}M(\varphi(a),\varphi(b)) + \frac{1}{3}N(\varphi(a),\varphi(b)) \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} M(\varphi(a),\varphi(b)) &= f(\varphi(a))\,w(\varphi(a)) + f(\varphi(b))\,w(\varphi(b)), \\ N(\varphi(a),\varphi(b)) &= f(\varphi(a))\,w(\varphi(b)) + f(\varphi(b))\,w(\varphi(a)). \end{aligned} \tag{2.6}$$

Proof. Since f and w be φ-convex functions, then we have

$$\begin{aligned} f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)w\left(\frac{\varphi(a)+\varphi(b)}{2}\right) &= f\left(\frac{t\varphi(a)+(1-t)\varphi(b)+(1-t)\varphi(a)+t\varphi(b)}{2}\right) \\ &\quad\times w\left(\frac{t\varphi(a)+(1-t)\varphi(b)+(1-t)\varphi(a)+t\varphi(b)}{2}\right) \\ &\leq \frac{1}{2}\left[f(t\varphi(a)+(1-t)\varphi(b)) + f((1-t)\varphi(a)+t\varphi(b))\right] \\ &\quad\times\frac{1}{2}\left[w(t\varphi(a)+(1-t)v(b)) + w((1-t)\varphi(a)+t\varphi(b))\right] \\ &\leq \frac{1}{4}\{2t(1-t)f(\varphi(a))w(\varphi(a)) + 2t(1-t)f(\varphi(b))w(\varphi(b)) \\ &\quad+\left(t^2+(1-t)^2\right)[f(\varphi(a))w(\varphi(b)) + f(\varphi(b))w(\varphi(a)]\right\}. \end{aligned}$$

Integrating with respect to on $[0,1]$, we get

$$\begin{aligned} & f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)w\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \\ \leq\ & \frac{1}{4}\left[\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,w(\varphi(x))\,d\varphi(x)\right] \\ & +\frac{1}{2}\left[\frac{1}{6}M(\varphi(a),\varphi(b)) + \frac{1}{3}N(\varphi(a),\varphi(b))\right] \end{aligned}$$

which completes the proof of Theorem 2.5. □

Remark 2.6. If we choose $\varphi(x) = x$ for all $x \in [a,b]$ in Theorem 2.5, the inequality (2.5) reduce to the inequality

$$2f\left(\frac{a+b}{2}\right) w\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\, w(x)\, dx \leq \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b)$$

which is proved by Cristescu in [2].

Theorem 2.7. *Let J be an interval $a,b \in J$ with $a<b$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f, w : I \subseteq \mathbb{R} \to \mathbb{R}$ be a φ-convex on and nonnegative function on $I = [a,b]$. If w is symmetric about $\frac{\varphi(a)+\varphi(b)}{2}$, then, for all $t \in [0,1]$, we have*

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\, w(\varphi(x))\, d\varphi(x) \leq \frac{1}{6}M(\varphi(a),\varphi(b)) + \frac{1}{3}N(\varphi(a),\varphi(b))$$

where $M(\varphi(a),\varphi(b))$ and $N(\varphi(a),\varphi(b))$ are given by (2.6).

Proof. Since w is symmetric about $\frac{\varphi(a)+\varphi(b)}{2}$, and f, w be φ-convex functions, then we have

$$\begin{aligned}
&\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\, w(\varphi(x))\, d\varphi(x) \\
=\ &\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\, w(\varphi(a)+\varphi(b)-\varphi(x))\, d\varphi(x) \\
=\ &\int_0^1 f(t\varphi(a)+(1-t)\varphi(b))w((1-t)\varphi(a)+t\varphi(b))dt \\
\leq\ &\int_0^1 \left[tf(\varphi(a))+(1-t)f(\varphi(b))\right]\left[(1-t)w(\varphi(a))+tw(\varphi(b))\right]dt \\
=\ &\int_0^1 \{t(1-t)\left[f(\varphi(a))w(\varphi(a))+f(\varphi(b))w(\varphi(b))\right] \\
&+ t^2 f(\varphi(a))w(\varphi(b)) + (1-t)^2 f(\varphi(b))w(\varphi(a)\}\, dt \\
=\ &\frac{1}{6}M(\varphi(a),\varphi(b)) + \frac{1}{3}N(\varphi(a),\varphi(b)).
\end{aligned}$$

This completes the proof. □

Remark 2.8. If we choose $\varphi(x) = x$ for all $x \in [a,b]$ in Theorem 2.7, the inequality (2.5) reduce to the inequality

$$\frac{1}{b-a}\int_a^b f(x)\, w(x)\, dx \leq \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b)$$

which is proved by Cristescu in [2].

3. The left and right sides of Hermite-Hadamard type inequality

In order to prove our results, we need the following lemma (see, [11]):

Lemma 3.1. *Let J be an interval $a,b\in J$ with $a<b$ and $\varphi:J\to\mathbb{R}$ a continuous increasing function. Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differantiable function on I° (the interior I). If $f'\in L_1[\varphi(a),\varphi(b)]$ for $\varphi(a),\varphi(b)\in I$, then the following equality holds:*

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,d\varphi(x)-f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \tag{3.1}$$

$$=\frac{\varphi(b)-\varphi(a)}{2}\int_0^1 p(t)f'(t\varphi(a)+(1-t)\varphi(b))\,dt$$

where

$$p(t)=\begin{cases} t, & t\in\left[0,\frac{1}{2}\right) \\ t-1, & t\in\left[\frac{1}{2},1\right]. \end{cases}$$

Proof. A simple proof of the equality can be done by performing integration by parts. □

Let us begin with the following Theorem.

Theorem 3.2. *Let J be an interval $a,b\in J$ with $a<b$ and $\varphi:J\to\mathbb{R}$ a continuous increasing function. Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I° (the interior I) and $f'\in L_1[\varphi(a),\varphi(b)]$ for $\varphi(a),\varphi(b)\in I$. If $|f'|$ is the φ- convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,d\varphi(x)-f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right| \tag{3.2}$$

$$\leq\frac{(\varphi(b)-\varphi(a))}{8}\left[|f'(\varphi(a))|+|f'(\varphi(b))|\right].$$

Proof. The proof of this Theorem is done with a similar method of proof Noor et al. in [11]. □

Remark 3.3. If we take $\varphi(x)=x$ for all $x\in[a,b]$, then inequality (3.2) coincide with the left sides of Hermite-Hadamard inequality proved by Kirmanci in [10].

Theorem 3.4. *Let J be an interval $a,b\in J$ with $a<b$ and $\varphi:J\to\mathbb{R}$ a continuous increasing function. Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I° (the interior*

I) and $f' \in L_1[\varphi(a),\varphi(b)]$ for $\varphi(a),\varphi(b) \in I$. If $|f'|^q$ is the φ- convex on $[a,b]$, $q>1$, then the following inequalities hold:

$$\begin{aligned}
&\left|\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,d\varphi(x) - f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right| \\
\leq\ & \frac{(g(b)-g(a))}{4(p+1)^{\frac{1}{p}}}\left[\left(\frac{|f'(\varphi(a))|^q + 3|f'(\varphi(b))|^q}{8}\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\frac{3|f'(\varphi(a))|^q + |f'(\varphi(b))|^q}{8}\right)^{\frac{1}{q}}\right] \qquad (3.3)\\
\leq\ & \frac{\varphi(b)-\varphi(a)}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{8}\right)^{\frac{1}{q}}\left(|f'(\varphi(a))| + |f'(\varphi(b))|\right),
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$

Proof. From Lemma 3.1 , using Hölder's inequality and the φ-convexity of $|f'|^q$, we find

$$\begin{aligned}
&\left|\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,d\varphi(x) - f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right| \\
\leq\ & \frac{\varphi(b)-\varphi(a)}{2}\left\{\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} |f'(t\varphi(a)+(1-t)\varphi(b))|^q\,dt\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\int_{\frac{1}{2}}^1 (1-t)^p\,dt\right)\left(\int_{\frac{1}{2}}^1 |f'(t\varphi(a)+(1-t)\varphi(b))|^q\,dt\right)^{\frac{1}{q}}\right\} \\
\leq\ & \frac{(\varphi(b)-\varphi(a))}{4(p+1)^{\frac{1}{p}}}\left\{\left(\int_0^{\frac{1}{2}} \left[t|f'(\varphi(a))|^q + (1-t)|f'(\varphi(b))|^q\right]dt\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\int_{\frac{1}{2}}^1 \left[t|f'(\varphi(a))|^q + (1-t)|f'(b)|^q\right]dt\right)^{\frac{1}{q}}\right\} \\
\leq\ & \frac{\varphi(b)-\varphi(a)}{4(p+1)^{\frac{1}{p}}} \\
&\times\left\{\left(\frac{|f'(\varphi(a))|^q + 3|f'(\varphi(b))|^q}{8}\right)^{\frac{1}{q}} + \left(\frac{3|f'(\varphi(a))|^q + |f'(\varphi(b))|^q}{8}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Let $a_1 = |f'(a)|^q$, $b_1 = 3|f'(b)|^q$, $a_2 = 3|f'(a)|^q$, $b_2 = |f'(b)|^q$. Here , $0 < \frac{1}{q} < 1$ for $q > 1$. Using the fact that,

$$\sum_{k=1}^{n} (a_k + b_k)^s \leq \sum_{k=1}^{n} a_k^s + \sum_{k=1}^{n} b_k^s.$$

For $(0 \leq s < 1)$, $a_1, a_2, ..., a_n \geq 0$, $b_1, b_2, ..., b_n \geq 0$, we obtain

$$\begin{aligned} &\left| \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\, d\varphi(x) - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right| \\ &\leq \frac{\varphi(b) - \varphi(a)}{4(p+1)^{\frac{1}{p}}} \left(\frac{1}{8}\right)^{\frac{1}{q}} \left[\left(|f'(\varphi(a))| + 3^{\frac{1}{q}} |f'(\varphi(b))| \right) + \left(3^{\frac{1}{q}} |f'(\varphi(a))| + |f'(\varphi(b))| \right) \right] \\ &= \frac{\varphi(b) - \varphi(a)}{4(p+1)^{\frac{1}{p}}} \left(\frac{1}{8}\right)^{\frac{1}{q}} \left[\left(1 + 3^{\frac{1}{q}}\right) \left(|f'(\varphi(a))| + |f'(\varphi(b))| \right) \right] \\ &\leq \frac{\varphi(b) - \varphi(a)}{(p+1)^{\frac{1}{p}}} \left(\frac{1}{8}\right)^{\frac{1}{q}} \left(|f'(\varphi(a))| + |f'(\varphi(b))| \right). \end{aligned}$$

This completes the proof. □

Remark 3.5. If we thake $\varphi(x) = x$ for all $x \in [a, b]$, then inequality (3.3) coincide with the left sides of Hermite-Hadamard inequality proved by Kirmanci in [10].

Lemma 3.6. *Let J be an interval $a, b \in J$ with $a < b$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differantiable function on I° (the interior I). If $f' \in L_1[\varphi(a), \varphi(b)]$ for $\varphi(a), \varphi(b) \in I$, then the following equality holds:*

$$\frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\, d\varphi(x) \tag{3.4}$$

$$= \frac{\varphi(b) - \varphi(a)}{2} \int_0^1 (2t - 1) \left[f'(t\varphi(b) + (1 - t)\varphi(a)) \right] dt.$$

Proof. A simple proof of the equality can be done by performing integration by parts. □

Let us begin with the following Theorem.

Theorem 3.7. *Let J be an interval $a, b \in J$ with $a < b$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differantiable function on I° (the interior I) and $f' \in L_1[\varphi(a), \varphi(b)]$ for $\varphi(a), \varphi(b) \in I$. If $|f'|$ is the φ- convex on $[a, b]$, then* *the following inequaliy holds:*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int\limits_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,d\varphi(x)\right| \tag{3.5}$$
$$\leq \frac{\varphi(b)-\varphi(a)}{4}\left(\frac{|f'(\varphi(b))|+|f'(\varphi(a))|}{2}\right).$$

Proof. From Lemma 3.6 and by using $\varphi-$convexity function of $|f'|$, we have

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int\limits_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,d\varphi(x)\right|$$
$$\leq \frac{\varphi(b)-\varphi(a)}{2}\int\limits_{0}^{1}|2t-1|\,|f'(t\varphi(b)+(1-t)\varphi(a))|\,dt$$
$$\leq \frac{\varphi(b)-\varphi(a)}{2}\int\limits_{0}^{1}|2t-1|\,[t\,|f'(\varphi(b))|+(1-t)\,|f'(\varphi(a))|]\,dt$$
$$= \frac{\varphi(b)-\varphi(a)}{2}\left[\frac{|f'(\varphi(b))|+|f'(\varphi(a))|}{4}\right]$$

which completes the proof. □

Remark 3.8. If we thake $\varphi(x)=x$ for all $x\in[a,b]$, then inequality (3.5) coincide with the right sides of Hermite-Hadamard inequality proved by Dragomir and Agarwal in [5].

Theorem 3.9. *Let J be an interval $a,b\in J$ with $a<b$ and $\varphi: J\to\mathbb{R}$ a continuous increasing function. Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differantiable function on I° (the interior I) and $f'\in L_1[\varphi(a),\varphi(b)]$ for $\varphi(a),\varphi(b)\in I$. If $|f'|^q$ is the φ- convex on $[a,b]$, $q>1$, then the following inequaliy holds:*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int\limits_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,d\varphi(x)\right| \tag{3.6}$$
$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(\varphi(b))|^q+|f'(\varphi(a))|^q}{2}\right)^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 3.6 and by using Hölder's integral inequality, we have

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,d\varphi(x)\right|$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left(\int_0^1 |2t-1|^p\,dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 |f'(t\varphi(b)+(1-t)\varphi(a))|^q\,dt\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is $\varphi-$convex on $[a,b]$, we get

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,d\varphi(x)\right|$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 \left[t\,|f'(\varphi(b))|^q+(1-t)\,|f'(\varphi(a))|^q\right]dt\right)^{\frac{1}{q}}$$

which completes the proof. □

Remark 3.10. If we thake $\varphi(x)=x$ for all $x\in[a,b]$, then inequality (3.6) coincide with the right sides of Hermite-Hadamard inequality proved by Dragomir and Agarwal in [5].

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